Features

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Records and embedding

Raw data

- *raw data* pairs are (u, v), with $u \in \mathcal{U}$, $v \in \mathcal{V}$
- $\mathcal{U}$ is set of all possible input values
- $\mathcal{V}$ is set of all possible output values
- each u is called a *record*
- typically a record is a tuple, or list, $u = (u_1, u_2, \ldots, u_r)$
- each u_i is a *field* or *component*, which has a *type*, *e.g.*, real number, Boolean, categorical, ordinal, word, text, audio, image, parse tree (more on this later)
- *e.g.*, a record for a house for sale might consist of

  (address, photo, description, house/apartment?, lot size, ..., # bedrooms)

Feature map

- learning algorithms are applied to (x, y) pairs,

$$x = \phi(u), \qquad y = \psi(v)$$

- $\phi : \mathcal{U} \to \mathbf{R}^d$ is the *feature map* for u
- $\psi : \mathcal{V} \to \mathbf{R}$ is the *feature map* for v
- feature maps transform *records* into *vectors*
- feature maps usually work on each field separately,

$$\phi(u_1, \ldots, u_r) = (\phi_1(u_1), \ldots, \phi_r(u_r))$$

- ϕ_i is an *embedding* of the type of field i into a vector

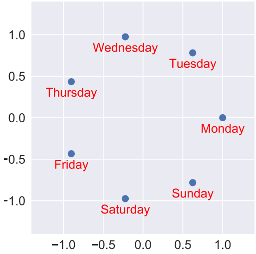
Embeddings

- embedding puts the different field types on an equal footing, *i.e.*, vectors
- some embeddings are simple, *e.g.*,
  - for a number field ($\mathcal{U} = \mathbf{R}$), $\phi_i(u_i) = u_i$
  - for a Boolean field, $\phi_i(u_i) = \begin{cases} 1 & u_i = \text{TRUE} \\ -1 & u_i = \text{FALSE} \end{cases}$
- others are more sophisticated
  - text to TFID histogram
  - `word2vec` (maps words into vectors)
  - pre-trained ImageNet NN (maps images into vectors)

(more on these later)

More embeddings

- color to (R, G, B)
- geolocation data: $\phi(u)$ =(Lat,Long) in $\mathbf{R}^2$ or embed in $\mathbf{R}^3$ (if data points are spread over planet)
- day of week:



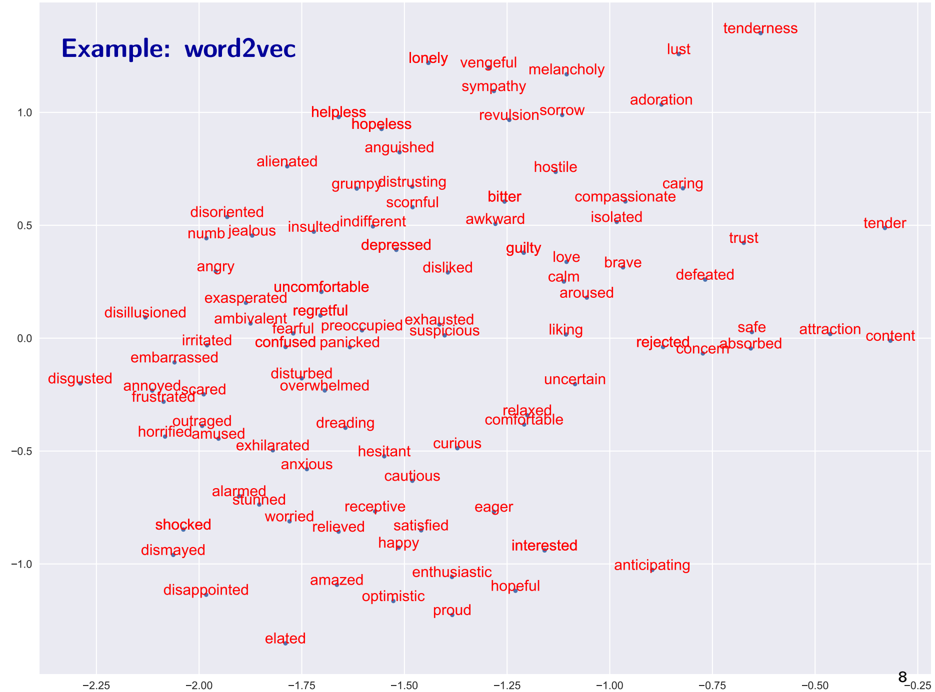
Faithful embeddings

a *faithful* embedding satisfies

- $\phi(u)$ is near $\phi(\tilde{u})$ when u and $\tilde{u}$ are 'similar'
- $\phi(u)$ is not near $\phi(\tilde{u})$ when u and $\tilde{u}$ are 'dissimilar'

- lefthand concept is *vector distance*
- righthand concept depends on field type, application

- interesting examples: names, professions, companies, countries, languages, ZIP codes, cities, songs, movies
- we will see later how such embeddings can be constructed


Example: word2vec
tenderness
lust
lonely
vengeful
melancholy
sympathy
adoration
helpless
sorrow
revulsion
hopeless
anguished
alienated
hostile
grumpy
distrusting
caring
bitter
compassionate
scornful
disoriented
isolated
awkward
insulted
indifferent
tender
numb
jealous
trust
depressed
guilty
love
brave
angry
disliked
calm
defeated
uncomfortable
aroused
exasperated
disillusioned
regretful
exhausted
ambivalent
preoccupied
safe
attraction
fearful
liking
suspicious
content
irritated
rejected
absorbed
confused
panicked
concern
embarrassed
disturbed
uncertain
disgusted
annoyed
overwhelmed
scared
frustrated
relaxed
outraged
dreading
comfortable
horrified
amused
exhilarated
curious
hesitant
anxious
cautious
alarmed
stunned
receptive
eager
worried
shocked
relieved
satisfied
happy
interested
dismayed
anticipating
enthusiastic
amazed
hopeful
disappointed
optimistic
proud
elated
1.0
0.5
0.0
−0.5
−1.0
−2.25
−2.00
−1.75
−1.50
−1.25
−1.00
−0.75
−0.50
−0.25

Standardized embeddings

usually assume that an embedding is *standardized*

- entries of $\phi(u)$ are centered around 0
- entries of $\phi(u)$ have RMS value around 1
- roughly speaking, entries of $\phi(u)$ ranges over ± 1

- with standarized embeddings, entries of feature map

$$\phi(u_1, \ldots, u_r) = (\phi_1(u_1), \ldots, \phi_r(u_r))$$

  are all comparable, *i.e.*, centered around zero, standard deviation around one
- $\mathrm{rms}(\phi(u) - \phi(\tilde{u}))$ is reasonable measure of how close records u and $\tilde{u}$ are

Standardization or z-scoring

- suppose $\mathcal{U} = \mathbf{R}$ (field type is real numbers)
- for data set $u^1, \ldots, u^n \in \mathbf{R}$

$$\bar{u} = \frac{1}{n}\sum_{i=1}^{n} u^i \qquad \mathrm{std}(u) = \left(\frac{1}{n}\sum_{i=1}^{n}(u^i - \bar{u})^2\right)^{\frac{1}{2}}$$

- the *z-score* or *standardization* of u is the embedding

$$x = \mathrm{zscore}(u) = \frac{1}{\mathrm{std}(u)}(u - \bar{u})$$

- ensures that embedding values are centered at zero, with standard deviation one
- z-scored features are very easy to interpret: $x = \phi(u) = +1.3$ means that u is 1.3 standard deviations above the mean value

Standardized data matrix

- suppose all d (real) features have been standardized
- columns of $n \times d$ feature matrix X have zero mean, RMS value one
- $(1/n)X^T X = \Sigma$ is the *feature correlation matrix*
- $\Sigma_{ii} = 1$ (since each column of X has RMS value 1, and so norm $\sqrt{n}$)
- Σ_{ij} is *correlation coefficient* of ith and jth raw features

Log transform

- old school rule-of-thumb: if field u is positive and ranges over wide scale, embed as $\phi(u) = \log u$ (or $\log(1+u)$) (and then standarize)
- examples: web site visits, ad views, company capitalization
- interpretation as faithful embedding:
  - 20 and 22 are similar, as are 1000 and 1100
  - but 20 and 120 are not similar
  - *i.e.*, you care about fractional or relative differences between raw values

  (here, log embedding is faithful, affine embedding is not)
- can also apply to output or label field, *i.e.*, $y = \psi(v) = \log v$ if you care about percentage or fractional errors; recover $\hat{v} = \exp(\hat{y})$

Example: House price prediction

- we want to predict house selling price v from record $u = (u_1, u_2)$
  - u_1 = area (sq. ft.)
  - u_2 = # bedrooms
- we care about relative error in price, so we embed v as $\psi(v) = \log v$ (and then standardize)
- we standardize fields u_1 and u_2

$$x_1 = \frac{u_1 - \mu_1}{\sigma_1}, \qquad x_2 = \frac{u_2 - \mu_2}{\sigma_2}$$

  - $\mu_1 = \bar{u}_1$ is mean area
  - $\mu_2 = \bar{u}_2$ is mean number of bedrooms
  - $\sigma_1 = \mathrm{std}(u_1)$ is std. dev. of area
  - $\sigma_2 = \mathrm{std}(u_2)$ is std. dec. of # bedrooms

(means and std. dev. are over our data set)

Example: House price regression model

- regression model: $\hat{y} = \theta_1 + \theta_2 x_1 + \theta_3 x_2$
- in terms of original raw data:

$$\hat{v} = \exp\left(\theta_1 + \theta_2 \frac{u_1 - \mu_1}{\sigma_1} + \theta_3 \frac{u_2 - \mu_2}{\sigma_2}\right)$$

- exp undoes log embedding of house price

Vector embeddings

Vector embeddings for real field

- we can embed a field u into a vector $x = \phi(u) \in \mathbf{R}^k$
- useful even when $\mathcal{U} = \mathbf{R}$ (real field)
- polynomial embedding:

$$\phi(u) = (1, u, u^2, \ldots, u^d)$$

- piecewise linear embedding:

$$\phi(u) = (1, (u)_-, (u)_+)$$

  where $(u)_- = \min(u, 0)$, $(u)_+ = \max(u, 0)$
- regression with these features yield polynomial and piecewise linear predictors

Whitening

- analog of standardization for raw data $\mathcal{U} = \mathbf{R}^d$
- start with raw data, $n \times d$ matrix U
- $\bar{u} = U^T \mathbf{1}/n$ is vector of column means
- $\tilde{U} = U - \mathbf{1}\bar{u}^T$ is de-meaned data matrix
- $\tilde{U} = QR$ is its QR factorization
- $X = \sqrt{n}Q = \sqrt{n}\tilde{U}R^{-1}$ defines embedding $x^i = \phi(u^i)$
    - columns of X have zero mean and RMS value one
    - columns of X are orthogonal
    - features are uncorrelated
    - feature correlation matrix is $\Sigma = I$

Categorical data

- data field is *categorical* if it only takes a finite number of values
- *i.e.*, $\mathcal{U}$ is a finite set $\{\alpha_1, \ldots, \alpha_k\}$
- examples:
  - $\textsc{true}/\textsc{false}$ (two values, also called Boolean)
  - $\textsc{apple}$, $\textsc{orange}$, $\textsc{banana}$ (three values)
  - $\textsc{monday}, \ldots, \textsc{sunday}$ (seven values)
  - ZIP code (40000 values)
- *one-hot embedding for categoricals*: $\phi(\alpha_i) = e_i \in \mathbf{R}^k$

$$\phi(\textsc{apple}) = (1, 0, 0), \quad \phi(\textsc{orange}) = (0, 1, 0), \quad \phi(\textsc{banana}) = (0, 0, 1)$$

- standardizing these features handles *unbalanced* data

Ordinal data

- ordinal data is categorical, with an order
- example: *Likert scale*, with values

  STRONGLY DISAGREE, DISAGREE, NEUTRAL, AGREE, STRONGLY AGREE

- can embed into **R** with values $-2, -1, 0, 1, 2$
- or treat as categorical, with one-hot embedding into $\mathbf{R}^5$
- example: number of bedrooms in house
  - can be treated as a real number
  - or as an ordinal with (say) values $1, \ldots, 6$

Feature engineering

How feature maps are constructed

- start by embedding each field

$$\phi(u_1, \ldots, u_r) = (\phi_1(u_1), \ldots, \phi_r(u_r))$$

- can then standardize, if needed
- use *feature engineering* to create new features from existing ones

Creating new features

- product features: $x_{\mathrm{new}} = x_i x_j$ (models *interactions* between features)
- max features: $x_{\mathrm{new}} = \max(x_i, x_j)$ (can also use min)
- positive/negative parts:

$$x_{\mathrm{new}+} = (x_i)_+ = \max(x_i, 0), \qquad x_{\mathrm{new}-} = (x_i)_- = \min(x_i, 0)$$

- random features:
  - choose random matrix R
  - new features are $(Rx)_+$ or $(Rx)_-$

Un-embedding

Un-embedding

- we embed v as $y = \psi(v)$, $\psi : \mathcal{V} \to \mathbf{R}$
- we need to 'invert' this operation, and go from $\hat{y}$ to $\hat{v}$
- when the inverse function exists, we use $\psi^{-1} : \mathbf{R} \to \mathcal{V}$
- example: log embedding $y = \log v$ has inverse $v = \exp y$
- prediction stack:
  1. *embed*: given record u, feature vector is $x = \phi(u)$
  2. *predict*: $\hat{y} = g(x)$
  3. *un-embed:* $\hat{v} = \psi^{-1}(\hat{y})$
- final predictor is $\hat{v} = \psi^{-1}(g(\phi(u)))$

Un-embedding

- in many cases, the inverse of ψ function doesn't exist
- for example, embedding a Boolean or ordinal into $\mathbf{R}$
- for the purposes of un-embedding, we define

$$\psi^{-1}(y) = \underset{v \in \mathcal{V}}{\operatorname{argmin}} \|y - \psi(v)\|$$

  i.e., we choose the value of v for which $\psi(v)$ is closest to y
- example: embed $\textsc{true} \mapsto 1$ and $\textsc{false} \mapsto -1$
- un-embed via

$$\psi^{-1}(y) = \begin{cases} \textsc{true} & \text{if } y > 0 \\ \textsc{false} & \text{otherwise} \end{cases}$$

Example: Un-embedding one-hot

- *one-hot embedding*: $\phi(u) = e_u$ for $\mathcal{U} = \{1, \dots, d\}$
- un-embed

$$\phi^{-1}(x) = \operatorname*{argmin}_u ||x - e_u||_2 = \operatorname*{argmax}_u x_u$$