Differential Geometry

Brayden Letwin

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What is Differential Geometry? In the sequence of mathematics one deals first with the standard calculus of $\mathbb{R}$, and then a generalization is made towards the vector calculus of $\mathbb{R}^n$. A further generalization of vector calculus can be made on so called "smooth manifolds." In these notes I will assume comfortable knowledge of the theory of vector spaces, multivariable calculus, and topological spaces (though I will introduce the basic definitions of topological spaces at the start). Throughout the notes we will view $\mathbb{R}^n$ as the n-dimensional vector space under the point-wise addition and scalar multiplication operations.

Definition 1 *Let X be a set. A metric on X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:*

- $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2 *A metric space is an ordered pair (X, d) where X is a set and d is a metric on X.*

Lemma 1 *Define $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d(x, y) = \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2}$. Then $(\mathbb{R}^n, d)$ is a metric space (we call this space the "Euclidean" space).*

Proof. Exercise. ■

Definition 3 *Let X be a set. A topology on X is a set $\tau \subset \mathcal{P}(X)$, where elements of τ are called open sets such that:*

- $\emptyset \in \tau, X \in \tau$.
- *If $\mathcal{I}$ is an arbitrary index set and $(a_i)_{i \in \mathcal{I}}$ is an arbitrary sequence of elements of $\mathcal{I}$, then $\bigcup_{i \in \mathcal{I}} a_i \in \tau$.*
- *If $a_1, \ldots, a_n \in \tau$, then $\bigcap_{i=1}^{n} a_i \in \tau$.*

Definition 4 *A topological space is an ordered pair (X, τ) where X is a set and τ is a topology on X.*

Definition 5 *Let (X, τ) be a topological space. A subset $C \subset X$ is called closed if $X \setminus C \in \tau$.*

Example 1 *Given the Euclidean space $(\mathbb{R}^n, d)$, one can define a set $K \subset \mathbb{R}^n$ to be open if for all $x \in K$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \subset K$. The collection of open sets of $\mathbb{R}^n$, denoted by τ is a topology on $\mathbb{R}^n$ and (R^n, τ) forms a topological space (we call this topology the "usual" or "Euclidean" topology).*

Definition 6 *Let (X, τ) be a topological space. Let $\mathcal{B} \subset \mathcal{P}(X)$. Then $\mathcal{B}$ is called a basis for τ if and only if $\mathcal{B} \subset \tau$ and for all $U \in \tau$ and $x \in U$ there is a $B \in \mathcal{B}$ such that $x \in B \subset U$.*

Definition 7 *Let (X, τ) be a topological space. Then (X, τ) is said to be Hausdorff if for all $x, y \in X$ such that $x \neq y$ there are disjoint sets $K, L \in \tau$ such that $x \in K$ and $y \in L$.*

Definition 8 *Let (X, τ) be a topological space. Then (X, τ) is said to be second-countable (or completely separable) if τ has a countable basis.*

Example 2 *Consider $(\mathbb{R}^n, \tau)$ where τ is the usual topology. One can then verify that $(\mathbb{R}^n, \tau)$ is Hausdorff and $\mathcal{B} = \{B_\epsilon(x) : x \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}\}$ is a countable basis, implying that $(\mathbb{R}^n, \tau)$ is second-countable.*

It is often a pain of notation to write (X, τ) for a topological space, so from now on we will write $X := (X, \tau)$ and specify the topology τ beforehand (or leave the topology τ to be arbitrary if we don't specify or if $X = \mathbb{R}^n$ we endow the usual topology). For saying a specific set U lies in τ we will simply say U is open. Similarly for metric spaces, we will write $X := (X, d)$ and specify the metric beforehand (or leave the metric d to be arbitary if we don't specify or if $X = \mathbb{R}^n$ we will endow the Euclidean metric).

Definition 9 *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a function where U is open. Let $r \in \mathbb{Z}_{\geq 0}$. If all k-th order partial derivatives of f exist and are continuous for $k \leq r$, then we say that $f \in C^r(U)$. We say that $f \in C^\infty(U)$ (or smooth) if $f \in \bigcap_{i=1}^\infty C^i(U)$. If f is real analytic (i.e. f can be expressed by a power series) then we say that $f \in C^\omega(U)$.*

Definition 10 *Let X, Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for all O open in Y we have $f^{-1}(O)$ is open in X (w.r.t. their own topologies).*

Lemma 2 *Let X, Y be topological spaces. Let $f : X \to Y$ be a function. The following are equivalent:*

1. *f is continuous,*
2. *For all O closed in Y we have $f^{-1}(O)$ is closed in X (w.r.t. their own topologies).*

Proof. Note if O is any subset of Y we have $X \setminus f^{-1}(O) = f^{-1}(Y \setminus O)$. The result follows by applying the defintion of continuity and closedness. ■

Definition 11 *Let X, Y be topological spaces. A map $f : X \to Y$ is called a homeomorphism if f is continuous, bijective, and $f^{-1} : Y \to X$ is continuous. X and Y are said to be homeomorphic, write $X \cong Y$.*

Remark 1 $\cong$ is reflexive, symmetric, and transitive but does not form an equivalence relation on the set of all topological spaces (why?), but rather an equivalence relation between topological spaces.

Definition 12 *Let M be a topological space. We say that M is a manifold if M is Hausdorff, second-countable, and for all $x \in M$ there is a neighbourhood U of x such that $U \cong \mathbb{R}^n$ for some $n \in \mathbb{Z}_{\geq 1}$. We say the dimension of M is n.*

Definition 13 *Let M be a manifold of dimension n. Let U be open in M. Suppose $\varphi : U \subset M \to \mathbb{R}^n$ is a homeomorphism. We call (U, φ) a chart of M. We call U the coordinate neighbourhood of φ. If $x \in U$, we define the local coordinates of x to be the coordinates of $\varphi(x) \in \mathbb{R}^n$.*

Definition 14 *Let M be a manifold. Suppose I is an arbitary index set. An atlas $\{(U_i, \varphi_i) : i \in I\}$ on M is an indexed family of charts on M such that the indexed family of coordinate neighbourhoods $\{U_i : i \in I\}$ covers M.*

Lemma 3 *Let M be a manifold. Suppose $(U_1, \varphi_1), (U_2, \varphi_2)$ are two charts of M such that $U_1 \cap U_2 \neq \emptyset$. Then $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$ are non-empty open sets in $\mathbb{R}^n$, and the map*

$$\varphi_1 \circ \varphi_2^{-1}|_{U_1 \cap U_2} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2), \tag{1}$$

is a homeomorphism with inverse

$$\varphi_2 \circ \varphi_1^{-1}|_{U_1 \cap U_2} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2). \tag{2}$$

We call these homeomorphisms transition maps between the charts $(U_1, \varphi_1), (U_2, \varphi_2)$.

Proof. Since φ_i has a continuous inverse it follows that φ_i is an open map. The intersection of two open sets is open and thus $\varphi_i(U_1 \cap U_2)$ is open. The composition of two continuous functions are continuous so $\varphi_1 \circ \varphi_2^{-1}$ is continuous. The restriction of two continuous functions are continuous so $\varphi_1 \circ \varphi_2^{-1}|_{U_1 \cap U_2}$ is continuous. To check bijectivity it suffices to check $\varphi_1 \circ \varphi_2^{-1} \circ \varphi_2 \circ \varphi_1^{-1} = \mathrm{id}_{\mathbb{R}^n}$. ■

Remark 2 Transition maps are homeomorphisms between two subsets of $\mathbb{R}^n$. They induce so called coordinate transformations that map coordinates from the one subset of $\mathbb{R}^n$ to the other subset of $\mathbb{R}^n$. The coordinates in each subset are called local coordinates for the coordinates in the other subset.

Definition 15 *Let M be a manifold. Two charts $(U_1, \varphi_1), (U_2, \varphi_2)$ of M are called C^k-compatible (resp. C^∞-compatible, C^ω-compatible) if their respective transition maps $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are $C^k(D)$ (resp. $C^\infty(D), C^\omega(D)$) where D is the domain of the specific transition map. This definition only makes sense when $U_1 \cap U_2 \neq \emptyset$.*

Definition 16 *Let M be a manifold and $\mathcal{A}$ be an atlas on M. A chart (U, φ) is called C^k-admissable (resp. C^∞-admissable, C^ω-admissable) to $\mathcal{A}$ if it is C^k-compatible (resp. C^∞-compatible, C^ω-compatible) with every chart in $\mathcal{A}$.*

Definition 17 *A smooth (C^∞) atlas is called maximal if it contains all of its C^∞-admissable charts.*

Definition 18 *Let M be a manifold. A C^k-differentiable (resp. C^∞-differentiable, C^ω-differentiable) structure on M is an atlas $\mathcal{A}$ on M such that any two charts are C^k-compatible (resp. C^∞-compatible, C^ω-compatible) and $\mathcal{A}$ is maximal.*

Definition 19 *Let M be a manifold. Let $\mathcal{A}$ be a C^k-differentiable (resp. C^∞-differentiable, C^ω-differentiable) structure on M. Then $(M, \mathcal{A})$ is a C^k-differentiable (resp. C^∞-differentiable, C^ω-differentiable) manifold.*

It is silly to write $(M, \mathcal{A})$ for a C^k-differentiable (resp. C^∞-differentiable, C^ω-differentiable) manifold, so from now on we will write $M := (M, \mathcal{A})$ and specify the C^k-differentiable (resp. C^∞-differentiable, C^ω-differentiable) structure $\mathcal{A}$ on M beforehand (or leave the structure $\mathcal{A}$ to be C^∞-differentiable if we don't specify). We will always assume that manifolds have a C^∞ structure attached unless otherwise stated.

Definition 20 *Let M be a manifold. A local coordinate system about $x \in M$ is an admissable chart (U, φ) such that $x \in U$.*

Example 3 *Let $M = \mathbb{R}^n$. Then $(M, \mathrm{id}_{\mathbb{R}^n})$ is a covering of M by a single chart, so $\{(M, \mathrm{id}_{\mathbb{R}^n})\}$ is an atlas on M. This is the "usual" differentiable structure on $\mathbb{R}^n$.*

Example 4 *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then $\Gamma(f) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in U,\ y = f(x)\}$ is a manifold (we call this manifold the graph of the function f).*

Example 5 *Consider $S^n = \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle = 1\}$. Consider the atlas $\mathcal{A}$ consisting of 2^{n+1} different charts $\mathcal{A} = \{(S_1^+, \gamma_1^+(x)), (S_1^-, \gamma_1^-(x)), \ldots, (S_{n+1}^+, \gamma_{n+1}^+(x)), (S_{n+1}^-, \gamma_{n+1}^-(x))\}$ where $S_i^+ = \{x \in S^n : x^i > 0\}$, $S_i^- = \{x \in S^n : x^i < 0\}$ and $\gamma_i^\pm : S_i^\pm \to \mathbb{R}^n$ are the clear homeomorphisms.*

On a smooth manifold, the concept of a smooth real-valued function is well defined. Let M be a smooth manifold of dimension n. Let $f : M \to \mathbb{R}$ be a real-valued function. If $x \in M$ and (U, φ_U) is a compatible chart containing x, then $f \circ \varphi_U^{-1}$ is a real-valued function defined on the domain $\varphi_U(U) \subset \mathbb{R}^n$. If $f \circ \varphi_U^{-1}$ is C^∞ at a point $\varphi_U(x) \in \mathbb{R}^n$, we say that f is C^∞ at $x \in M$. The differentiability of the function f at $x \in M$ is independent of the choice of the compatible coordinate chart containing x. In fact, for another compatible coordinate chart (V, φ_V) containing x such that $U \cap V \neq \emptyset$, we have

$$f \circ \varphi_V^{-1} = (f \circ \varphi_U^{-1}) \circ (\varphi_U \circ \varphi_V^{-1}).$$

Since $\varphi_U \circ \varphi_V^{-1}$ is smooth, we see that $f \circ \varphi_V^{-1}$ and $f \circ \varphi_U^{-1}$ are both differentiable at the same points in their domain. If $f : M \to \mathbb{R}$ is C^∞ at every $x \in M$ we say f is smooth on M. The set of all smooth real-valued functions on M is denoted by $C^\infty(M)$.

Definition 21 *Suppose $f : M \to N$ is a continuous map between one smooth manifold M and another N where the dimension of M (resp. N) is m (resp. n). If there is compatible coordinate charts (U, φ_U) at the point $x \in M$ and (V, φ_V) at $f(x) \in N$ such that the map:*

$$\varphi_V \circ f \circ \varphi_U^{-1} : \varphi_U(U) \to \varphi_V(V)$$

is C^∞ at the point $\varphi_U(x)$, then the map f is called C^∞ at x. If the map f is C^∞ at every point $x \in M$, then we say that f is a smooth map from M to N.

Remark 3 Since $\varphi_V \circ f \circ \varphi_U^{-1}$ is a continuous map from an open set $\varphi_U(U) \subset \mathbb{R}^m$ to $\varphi_V(V) \subset \mathbb{R}^n$, it follows that its differentiability at the point $\varphi_U(x)$ is defined. Obviously the differentiability of f at x is independent of the choice of compatible coordinate charts (U, φ_U) and (V, φ_V).

Definition 22 *In the case where* $\dim N = \dim M$, *if $f : M \to N$ is a homeomorphism and f, f^{-1} are both smooth maps, then we call $f : M \to N$ a diffeomorphism.*