A GENERALIZATION OF GRÜNBAUM'S INEQUALITY

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Abstract. Grünbaum's inequality gives sharp bounds between the volume of a convex body and its part cut off by a hyperplane through the centroid of the body. We provide a generalization of this inequality for hyperplanes that do not necessarily contain the centroid. As an application, we arrive at a sharp inequality that compares sections of a convex body to the maximal section parallel to it.

1. Introduction

A *convex body* K is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. As usual, we write $\langle \cdot, \cdot \rangle$ for the Euclidean inner product with $\|\cdot\|_2$ the induced Euclidean norm. We also denote by $S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ the unit sphere in $\mathbb{R}^n$. The *centroid* (otherwise called the *center of mass*, or *barycenter*) of K is the point

$$g(K) = \frac{1}{|K|} \int_K x \, dx.$$

Here and throughout the paper, $|A|$ denotes the k-dimensional Lebesgue measure of a k-dimensional set A. An inequality of Grünbaum [5] states if K is a convex body with centroid at the origin then

$$\left(\frac{n}{n+1}\right)^n \leq \frac{|K \cap \xi^+|}{|K|} \leq 1 - \left(\frac{n}{n+1}\right)^n \quad \text{for all } \xi \in S^{n-1}. \tag{1}$$

Here $\xi^+ = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq 0\}$. The bounds in (1) are sharp and equality occurs when $K = \operatorname{conv}(B, v)$ is a cone with its base B being an $(n-1)$-dimensional convex body lying parallel to $\xi^\perp$ in ξ^+ and vertex v lying in $\xi^- = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq 0\}$. Note that in Grünbaum's proof of (1), it is enough to obtain one of the bounds and the other follows immediately by symmetry. For recent advancements in Grünbaum-type inequalities for sections and projections of convex bodies see [3], [8], [9], [12].

In light of (1), the goal of this paper is to establish a similar result with hyperplanes that do not necessarily contain the centroid. Let K be a convex body with centroid at the origin, $\alpha \in (-1, n)$, and $\xi \in S^{n-1}$. Consider the halfspace

$$H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq \alpha h_K(-\xi)\},$$

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where h_K is the support function for K (see Section 2 for the precise definition). Are there absolute constants C_1, $C_2 > 0$ such that

$$C_1 \leq \frac{|K \cap H_\alpha^+|}{|K|} \leq C_2? \tag{2}$$

We give an affirmative answer to this question. The two constants C_1, C_2 depend only on α and n, i.e., $C_1 = C_1(\alpha, n)$, $C_2 = C_2(\alpha, n)$. Both bounds are sharp, and the exact values of $C_1(\alpha, n)$ and $C_2(\alpha, n)$ are presented in Theorem 3, which also discusses the equality cases. The 2-dimensional version of Theorem 3 was obtained earlier in [11], where it was used to prove a discrete version of Grünbaum's inequality. Recall that in (1) one bound automatically determines the other bound. On the other hand, the bounds in (2) need to be shown seperately.

As an application of (2) we obtain a generalization of the following result of Makai and Martini [7]; see also [2]. Let K be a convex body with centroid at the origin, then

$$\left|K \cap \xi^\perp\right| \geq \left(\frac{n}{n+1}\right)^{n-1} \sup_{t\in\mathbb{R}} \left|K \cap \left(\xi^\perp + t\xi\right)\right| \quad \text{for all } \xi \in S^{n-1}.$$

In this paper, we establish an analogue of the inequality above for sections that do not necessarily pass through the centroid of K. Let $\alpha \in (-1, n)$, $\xi \in S^{n-1}$, and consider the hyperplane

$$H_\alpha = \{x \in \mathbb{R}^n : \langle x, \xi\rangle = \alpha h_K(-\xi)\}.$$

Then

$$|K \cap H_\alpha| \geq D \sup_{t\in\mathbb{R}} \left|K \cap \left(\xi^\perp + t\xi\right)\right|,$$

where $D = D(\alpha, n)$ is an absolute constant depending on only α and n. The inequality is sharp, and the exact value of D, along with equality cases are discussed in Theorem 4.

2. Preliminaries

Given $\xi \in S^{n-1}$, we will denote $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi\rangle = 0\}$. The *support function* $h_K : \mathbb{R}^n \longrightarrow \mathbb{R}$ for a convex body K is

$$h_K(\xi) = \sup\{\langle x, \xi\rangle : x \in K\}.$$

If $\xi \in S^{n-1}$ then $h_K(\xi)$ gives the signed distance from the origin to the supporting hyperplane for K in the direction of ξ. A result of Minkowski and Radon [1, p. 58] says if K is a convex body with centroid at the origin and $\xi \in S^{n-1}$ then

$$\frac{1}{n} h_K(\xi) \leq h_K(-\xi) \leq n h_K(\xi). \tag{3}$$

Note that the choice of bounds for α in Theorems 3 and 4 is a result of (3).

Let $\xi \in S^{n-1}$. The *parallel section function* $A_{K,\xi} : \mathbb{R} \longrightarrow \mathbb{R}$ for a convex body K is

$$A_{K,\xi}(t) = \left|K \cap \left(\xi^\perp + t\xi\right)\right|.$$

If $K \cap \left(\xi^\perp + t\xi\right)$ is an $(n-1)$-dimensional Euclidean ball, then $A_{K,\xi}^{1/(n-1)}(t)$ gives the radius of $K \cap \left(\xi^\perp + t\xi\right)$ up to a constant.

Lemma 1. *Let K be a convex body. Then the function $A_{K,\xi}$ is $1/(n-1)$ concave on its support, for every $\xi \in S^{n-1}$.*

For the proof of Lemma 1, refer to [6, p. 18].

Let $\xi \in S^{n-1}$. The *volume cut-off function* $V_{K,\xi} : \mathbb{R} \longrightarrow \mathbb{R}$ for a convex body K is

$$V_{K,\xi}(t) = \int_t^\infty A_{K,\xi}(s)\, ds.$$

The following result is also well-known, but we include a proof for completeness.

Lemma 2. *Let K be a convex body. Then the function $V_{K,\xi}$ is $1/n$-concave on its support, for every $\xi \in S^{n-1}$.*

Proof. Let $\lambda \in [0,1]$ and $s, t \in \operatorname{supp}(V_{K,\xi})$. Note that

$$\lambda\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq s\}\Big) + (1-\lambda)\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t\}\Big) \subset \Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \lambda s + (1-\lambda)t\}\Big).$$

This, together with the Brunn-Minkowski inequality, implies that

$$\begin{aligned}
&\Big|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \lambda s + (1-\lambda)t\}\Big|^{1/n} \\
&\quad\geq \Big|\lambda\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq s\}\Big) + (1-\lambda)\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t\}\Big)\Big|^{1/n} \\
&\quad\geq \lambda\Big|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq s\}\Big|^{1/n} + (1-\lambda)\Big|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t\}\Big|^{1/n},
\end{aligned}$$

which proves the result. □

Let K be a convex body and $\xi \in S^{n-1}$. The *Schwartz symmetral* of K with respect to the line $\ell = \{x \in \mathbb{R}^n : x = t\xi,\, t \in \mathbb{R}\}$ is the convex body $\bar{K}$ constructed as follows: For each $t \in [-h_K(-\xi), h_K(\xi)]$, $\bar{K} \cap (\xi^\perp + t\xi)$ is an $(n-1)$-dimensional Euclidean ball centered at $t\xi$ with

$$\left|K \cap \left(\xi^\perp + t\xi\right)\right| = \left|\bar{K} \cap \left(\xi^\perp + t\xi\right)\right|.$$

By construction we get

$$h_K(\pm\xi) = h_{\bar{K}}(\pm\xi) \text{ and } V_{K,\xi}(t) = V_{\bar{K},\xi}(t),$$

for all $t \in \mathbb{R}$. Note that the centroid of $\bar{K}$ lies on ℓ due to the rotational symmetry of $\bar{K}$ about ℓ.

3. Main Results

We will now prove our main result.

Theorem 3. *Let K be a convex body with centroid at the origin. Let $\alpha \in (-1, n)$ and $\xi \in S^{n-1}$. Consider the halfspace*

$$H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \alpha h_K(-\xi)\}.$$

Then

$$C_1(\alpha, n) \leq \frac{|K \cap H_\alpha^+|}{|K|} \leq C_2(\alpha, n).$$

where

$$C_1(\alpha,n)=\begin{cases}\left(\frac{n-\alpha}{n+1}\right)^n & \alpha\in(-1,0],\\ \left(\frac{n}{n+1}\right)^n(\alpha+1)^{n-1}(1-\alpha n) & \alpha\in(0,1/n),\\ 0 & \alpha\in[1/n,n),\end{cases}$$

and

$$C_2(\alpha,n)=\begin{cases}1-\left(\frac{n(\alpha+1)}{n+1}\right)^n & \alpha\in(-1,0],\\ \tau(\alpha,n) & \alpha\in(0,n).\end{cases}$$

$\tau(\alpha,n)$ is a constant depending on α and n that is not explicit for general dimension because finding the value involves finding the roots of a high-degree rational function. The lower bounds and upper bounds are sharp. The equality cases are discussed in the proof below.

Proof. We will start by applying the Schwartz symmetrization to K with respect to the line $\ell=\{x\in\mathbb{R}^n : x=t\xi,\, t\in\mathbb{R}\}$. Abusing notation, we will denote the new body again by K. Note that the symmetrization does not change the centroid of K, the value of $h_K(-\xi)$, and $|K\cap H_\alpha^+|$.

First consider the case $\alpha\in(-1,0]$. We will start with the upper bound. Observe that

$$\left|K\cap H_\alpha^+\right|=|K|-\left|K\cap H_\alpha^-\right|,$$

where $H_\alpha^-=\{x\in\mathbb{R}^n:\langle x,\xi\rangle\le\alpha h_K(-\xi)\}$.

We will now translate K and $H_\alpha=\{x\in\mathbb{R}^n:\langle x,\xi\rangle=\alpha h_K(-\xi)\}$ by $h_K(-\xi)\xi$, and denote $\bar K=K+h_K(-\xi)\xi$ and $\bar H_\alpha^-=\{x\in\mathbb{R}^n:\langle x,\xi\rangle\le(\alpha+1)h_K(-\xi)\}$. Then by translation invariance we obtain

$$\left|K\cap H_\alpha^-\right|=\left|\bar K\cap\bar H_\alpha^-\right|.$$

Observe that the origin is located on the boundary of $\bar K$. Let $\frac{1}{\alpha+1}\bar K$ denote the dilation of $\bar K$ with respect to the origin by a factor of $1/(\alpha+1)>1$. Using that $\frac{1}{\alpha+1}\bar K\subset\bar K$ and $\frac{1}{\alpha+1}\bar H_\alpha^-=\{x\in\mathbb{R}^n:\langle x,\xi\rangle\le h_K(-\xi)\}$, we get

$$\begin{aligned}\left|K\cap H_\alpha^-\right|&=(\alpha+1)^n\left|\frac{1}{\alpha+1}\bar K\cap\frac{1}{\alpha+1}\bar H_\alpha^-\right|\\ &\ge(\alpha+1)^n\left|\bar K\cap\{x\in\mathbb{R}^n:\langle x,\xi\rangle\le h_K(-\xi)\}\right|\\ &=(\alpha+1)^n\left|K\cap\{x\in\mathbb{R}^n:\langle x,\xi\rangle\le 0\}\right|\ge(\alpha+1)^n\left(\frac{n}{n+1}\right)^n|K|,\end{aligned}$$

where we used Grünbaum's inequality (1). Therefore, for $\alpha\in(-1,0]$ we get

$$\frac{|K\cap H_\alpha^+|}{|K|}\le1-\left(\frac{n(\alpha+1)}{n+1}\right)^n.$$

Now we will obtain the lower bound. By the $1/n$-concavity of $V_{K,\xi}$ on its support we obtain

$$\begin{aligned}\left|K\cap H_\alpha^+\right|^{1/n}&=V_{K,\xi}{}^{1/n}\big(\alpha h_K(-\xi)\big)=V_{K,\xi}{}^{1/n}\big(-\alpha(-h_K(-\xi))+(1+\alpha)0\big)\\ &\ge-\alpha V_{K,\xi}{}^{1/n}\big(-h_K(-\xi)\big)+(1+\alpha)V_{K,\xi}{}^{1/n}(0).\end{aligned}$$

Using Grünbaum's inequality and the observation that $V_{K,\xi}\big(-h_K(-\xi)\big) = |K|$, we get

$$\left|K \cap H_\alpha^+\right|^{1/n} \geq -\alpha \left|K\right|^{1/n} + (1+\alpha)\left(\frac{n}{n+1}\right)\left|K\right|^{1/n},$$

which implies for $\alpha \in (-1, 0]$:

$$\left(\frac{n-\alpha}{n+1}\right)^n \leq \frac{|K \cap H_\alpha^+|}{|K|}.$$

Thus, we have shown our bounds for $\alpha \in (-1, 0]$.

Now we will move to the case $\alpha \in (0, n)$. As before, after applying the Schwartz symmetrization, we can assume that K is a body of revolution. Without loss of generality we can choose the x_1-axis to be the axis of revolution. We will apply a translation to K and H_α to make computations easier. Denote $h = h_K(-e_1)$ and $\bar{K} = K + he_1$. Note that $g(\bar{K}) = he_1$. Shifting H_α as well we get $\bar{H}_\alpha = e_1^\perp + (\alpha+1)he_1$. Abusing notation we will denote the translated body again by K and the hyperplane by H_α. Now K lies entirely in $\{x \in \mathbb{R}^n : x_1 \geq 0\}$ and $\{x \in \mathbb{R}^n : x_1 = 0\}$ is a supporting hyperplane for K.

We will prove the upper bound first. Let B_{n-1} be the unit $(n-1)$-dimensional Euclidean ball in $e_1^\perp$ centered at the origin. By continuity there is $r_1 \geq 0$ such that

$$\left|\mathrm{conv}\big(r_1 B_{n-1}, K \cap H_\alpha\big)\right| = \left|K \cap H_\alpha^-\right|.$$

Note that $K \cap e_1^\perp \subset r_1 B_{n-1}$. Denote $\mathrm{conv}\big(r_1 B_{n-1}, K \cap H_\alpha\big)$ by L^-. Then there are unique μ and r_2 with $(\alpha+1)h < \mu < h$ such that

$$\left|\mathrm{conv}\big(r_2 B_{n-1} + \mu e_1, K \cap H_\alpha\big)\right| = \left|K \cap H_\alpha^+\right|$$

and

$$\mathrm{conv}\big(K \cap H_\alpha, r_2 B_{n-1} + \mu e_1\big) \cup L^- = \mathrm{conv}\big(r_1 B_{n-1}, r_2 B_{n-1} + \mu e_1\big).$$

Note that

$$A_{K,e_1}(\mu) \leq \left|r_2 B_{n-1}\right|.$$

Refer to Figure 1 for a picture of the construction.

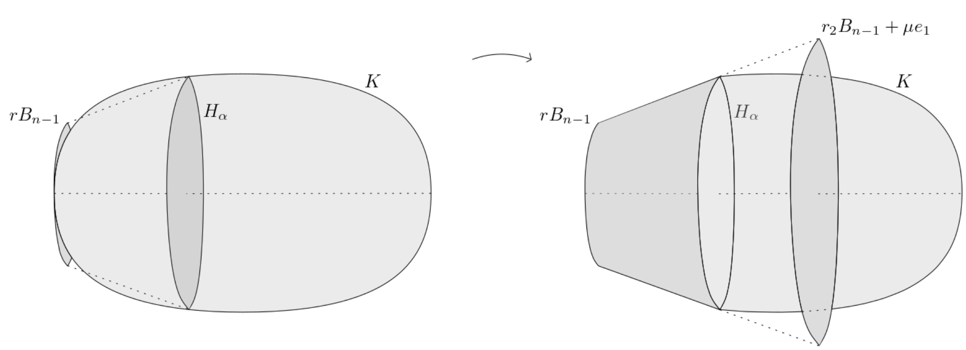

FIGURE 1. Constructing $r_1 B_{n-1}$ and $r_2 B_{n-1} + \mu e_1$

Denote $\mathrm{conv}\big(K \cap H_\alpha, r_2 B_{n-1} + \mu e_1\big)$ by L^+. Then $L = L^- \cup L^+$ is a convex body of revolution, which is a truncated cone. Note that $\langle g(L^+), e_1\rangle \le \langle g(K\cap H_\alpha^+), e_1\rangle$ and $\langle g(L^-), e_1\rangle \le \langle g(K\cap H_\alpha^-), e_1\rangle$, and thus

$$\langle g(L), e_1\rangle \le \langle g(K), e_1\rangle.$$

By construction we get,

$$|L| = |K|$$

and

$$\begin{aligned} &|K\cap\{x\in\mathbb{R}^n : x_1 \ge (\alpha+1)\langle g(K), e_1\rangle\}| \\ &\qquad = |L\cap\{x\in\mathbb{R}^n : x_1 \ge (\alpha+1)\langle g(K), e_1\rangle\}| \\ &\qquad \le |L\cap\{x\in\mathbb{R}^n : x_1 \ge (\alpha+1)\langle g(L), e_1\rangle\}|\,. \end{aligned}$$

Hence, our task reduces to finding maximizers for such constructed convex bodies. Without loss of generality, we can assume that $\mu = 1$. Additionally, for $0\le t\le 1$, we can assume that

$$A_{L,e_1}(t) = (mt+b)^{n-1}, \tag{4}$$

where either $m=0$ and $b>0$, or $b\ge 0$ and either (1) $m>0$, or (2) $m<0$ and $m+b\ge 0$. For now, we will focus on $m\ne 0$, and address the case of $m=0$ later. By Fubini's theorem and (4) we get

$$|L| = \int_0^1 A_{L,e_1}(x_1)\,dx_1 = \frac{(b+m)^n - b^n}{mn}.$$

Similarly, the x_1 coordinate of the centroid of L is given by

$$\begin{aligned} \langle g(L), e_1\rangle &= \frac{1}{|L|}\int_L x_1\,dx \\ &= \frac{1}{|L|}\int_0^1 x_1 A_{L,e_1}(x_1)\,dx_1 \\ &= \frac{b^{n+1} + (mn-b)(b+m)^n}{m(n+1)\Big((b+m)^n - b^n\Big)}. \end{aligned}$$

Given that L is a body of revolution around the x_1 axis, one can note that all remaining coordinates of the centroid of L vanish. Denote $g = (\alpha+1)\langle g_L, e_1\rangle$. Now we can compute

$$\begin{aligned} \frac{|L\cap H_\alpha^+|}{|L|} &= \frac{1}{|L|}\int_{L\cap H_\alpha^+} dx = \frac{1}{|L|}\int_g^1 A_{L,e_1}(x_1)\,dx_1 \\ &= \frac{(b+m)^n - (b+mg)^n}{(b+m)^n - b^n}. \end{aligned}$$

Now suppose $m=0$. Then $A_{L,e_1}(t) = b^{n-1}$ and in similar vein of the computations above we get $|L| = b^{n-1}$, $\langle g(L), e_1\rangle = 1/2$, and

$$\frac{|L\cap H_\alpha^+|}{|L|} = \frac{1-\alpha}{2}.$$

Let ϕ_1 denote the above function of m and b for when $m\ne 0$, and ϕ_2 denote the above two function of m and b for when $m=0$. For fixed b we have $\lim_{m\to 0}\phi_1(m,b)$ is finite (in fact equal to $\phi_2(m,b)$ for all b), and also ϕ_1 is continuous, so the supremum of ϕ_1 exists. Our goal is to find the supremum of ϕ_1 and ϕ_2, the bigger of those two will yield the upper bound $\tau(\alpha, n)$. We can ignore $\alpha \in (-1, 0]$ since the case was discussed above. For $\alpha \in (0, n)$, it isn't immediately clear whether ϕ_1 has nice properties, so as a result determining $\tau(\alpha, n)$ becomes an unfeasible task, as solving for $\tau(\alpha, n)$ involves solving for roots of high-degree rational functions. When $n = 2$ one can explicitly solve for $\tau(\alpha, n)$ (see [11] for the derivation):

$$\tau(\alpha, 2) = \begin{cases} \frac{5-3\alpha}{9(\alpha+1)} & \alpha \in (0, 1), \\ \frac{1}{9}(2-\alpha)^2 & \alpha \in [1, 2). \end{cases}$$

Thus, we now see that for any convex body K, we have the following upper bound:

$$\frac{|K \cap H_\alpha^+|}{|K|} \leq C_2(\alpha, n).$$

Let us now discuss the lower bound. Note for $\alpha \in [1/n, n)$ there are cases for which $|K \cap H_\alpha^+| = 0$. Here one uses the equality cases of (3) to construct such examples. Therefore we can not do better than

$$\frac{|K \cap H_\alpha^+|}{|K|} \geq 0$$

for $\alpha \in [1/n, n)$. Now assume without loss of generality that $\alpha \in (0, 1/n)$, since the case $\alpha \in (-1, 0]$ was discussed above. Similarly to the upper bound, we will assume that K is a body of revolution about e_1 and that $h = h_K(-e_1) = 0$. By continuity there is a unique $\rho \geq h_K(e_1)$ so that

$$\left|\mathrm{conv}\big(K \cap H_\alpha, \rho e_1\big)\right| = \left|K \cap H_\alpha^+\right|.$$

Denote $\mathrm{conv}\big(K \cap H_\alpha, \rho e_1\big)$ by M^+. Then there is a unique $0 < t < (\alpha + 1)h$ and $r > 0$ so that

$$\mathrm{conv}\big(rB_{n-1} + te_1, M^+\big) = \mathrm{conv}\big(rB_{n-1} + te_1, \rho e_1\big),$$

and

$$\left|\mathrm{conv}\big(0, rB_{n-1} + te_1, K \cap H_\alpha\big)\right| = \left|K \cap H_\alpha^-\right|.$$

Note that

$$K \cap \left(e_1^\perp + te_1\right) \subset rB_{n-1} + te_1.$$

Denote $\mathrm{conv}(0, rB_{n-1} + te_1, K \cap H_\alpha)$ by M^-. Then $M = M^+ \cup M^-$ is a convex body formed by the union of two cones with a common base contained in $e_1^\perp + te_1$. Refer to Figure 2 for a picture of the construction above.

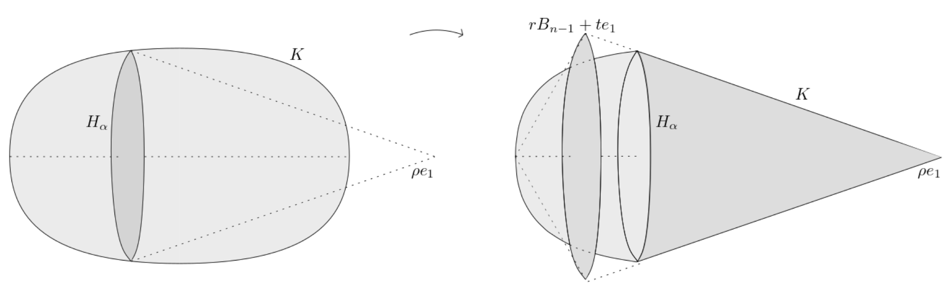

Figure 2. Constructing ρ and t

Note that by construction $\langle g(M^+), e_1\rangle \geq \langle g(K\cap H_\alpha^+), e_1\rangle$ and $\langle g(M^-), e_1\rangle \geq \langle g(K\cap H_\alpha^-), e_1\rangle$, and therefore

$$\langle g(M), e_1\rangle \geq \langle g(K), e_1\rangle.$$

As a result we have constructed a convex body M such that:

$$|K| = |M|,$$

and

$$\begin{aligned}|K\cap\{x\in\mathbb{R}^n : x_1 \geq (\alpha+1)\langle g(K), e_1\rangle\}|\\ &= |M\cap\{x\in\mathbb{R}^n : x_1 \geq (\alpha+1)\langle g(K), e_1\rangle\}|\\ &\geq |M\cap\{x\in\mathbb{R}^n : x_1 \geq (\alpha+1)\langle g(M), e_1\rangle\}|.\end{aligned}$$

Hence, our task reduces to finding minimizers for such constructed convex bodies. Without loss of generality (after applying an appropriate linear transformation), we can assume $h_M(e_1) = 1$, and $|rB_{n-1}| = n$. Define

$$M_1 = M\cap\{x\in\mathbb{R}^n \mid \langle x, e_1\rangle \leq t\},$$

and

$$M_2 = M\cap\{x\in\mathbb{R}^n \mid \langle x, e_1\rangle \geq t\}$$

to be the cones forming M. Since the heights of M_1 and M_2 are equal to t and $1-t$ respectively, and the $(n-1)$-dimensional volume of their common base is n, we obtain that $|M_1| = t$, $|M_2| = 1-t$, and $|M| = 1$. It is a well-known fact that the centroid of a cone in $\mathbb{R}^n$ divides its height in the ratio $[1 : n]$ and therefore $\langle g(M_1), e_1\rangle = \frac{tn}{n+1}$ and $\langle g(M_2), e_1\rangle = \frac{tn+1}{n+1}$. Then it follows that

$$\begin{aligned}\langle g(M), e_1\rangle &= |M_1|\,\langle g(M_1), e_1\rangle + |M_2|\,\langle g(M_2), e_1\rangle\\ &= t\frac{tn}{n+1} + (1-t)\frac{tn+1}{n+1} = \frac{t(n-1)+1}{n+1}.\end{aligned}$$

Denote $g = (\alpha+1)\langle g(M), e_1\rangle$. Recall that we are interested in computing the volume of the intersection of M with the halfspace $H_\alpha^+ = \{x\in\mathbb{R}^n : x_1 \geq g\}$. We will consider two cases, first when $0\leq t\leq g$, and then when $g\leq t\leq 1$. These cases are respectively equivalent to $0\leq t\leq a_0$ and $a_0\leq t\leq 1$, where $a_0 = \frac{-\alpha-1}{(n-1)\alpha-2}$. In the first case, $M\cap H_\alpha^+$ is a cone homothetic to M_2 with the homothety coefficient equal to $\frac{1-g}{1-t}$. Therefore,

$$\left|M\cap H_\alpha^+\right| = \left(\frac{1-g}{1-t}\right)^n(1-t) = \frac{(1-g)^n}{(1-t)^{n-1}}.$$

In the second case, $M\cap H_\alpha^-$ is a cone homothetic to M_1 with the homothety coefficient equal to $\frac{g}{t}$. Thus,

$$\begin{aligned}\left|M\cap H_\alpha^+\right| &= 1 - \left|M\cap H_\alpha^-\right| = 1 - \left(\frac{g}{t}\right)^n t\\ &= 1 - \frac{g^n}{t^{n-1}}.\end{aligned}$$

Summarizing, $|M\cap H_\alpha^+|$ is equal to the following differentiable piecewise function

$$\psi(t) = \begin{cases}\frac{(1-g)^n}{(1-t)^{n-1}}, & 0\leq t\leq a_0,\\ 1-\frac{g^n}{t^{n-1}}, & a_0\leq t\leq 1.\end{cases}$$

Our goal is to find the infimum of ψ on $[0,1]$ when $\alpha \in (0, 1/n)$. This infimum will yield the lower bound for $\alpha \in (0, 1/n)$. Simple calculations show that the derivative of ψ vanishes at $t_0 = \frac{(n+1)\alpha}{\alpha+1} \in (0, \alpha_0)$, and ψ is decreasing on $[0, t_0)$ and increasing on $(t_0, 1]$. Thus the infimum of ψ is

$$\psi(t_0) = \left(\frac{n}{n+1}\right)^n (\alpha+1)^{n-1}(1-\alpha n).$$

Now we will discuss the equality cases. Recall that in both the upper bound construction and lower bound construction, we performed operations such as Schwartz symmetrization to transform the sections of K in the direction of ξ into $(n-1)$-dimensional Euclidean balls. We also performed scalings and translations. If we have an equality body K for either bound under these operations, then we can undo these operations to produce a new body whose sections are no longer $(n-1)$-dimensional Euclidean balls but instead $(n-1)$-dimensional convex bodies homothetic to each other. This preserves equality by the equality case of the Brunn-Minkowski inequality and therefore extends the equality cases for the upper and lower bound.

We will start with classifying equality cases for the upper bound. Recall in the upper bound construction, the extremal body L is, up to translation, the convex hull of a $(n-1)$-dimensional convex body B lying parallel to $\xi^\perp$ in ξ^+, and a homothetic copy of B lying in $\xi^- = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq 0\}$. For $\alpha \in (-1, 0]$, we have equality from the equality conditions of Grünbaum's theorem, in other words $L = \mathrm{conv}(B, v)$ is a cone with its base B being an $(n-1)$-dimensional convex body lying parallel to $\xi^\perp$ in ξ^+ and vertex v lying in ξ^-. For $\alpha \in (0, n)$, L is still the convex hull of a $(n-1)$-dimensional convex body B and a homothetic copy of B, but there is no information on an explicit maximum, so we can not determine all equality cases.

Now we will classify equality cases for the lower bound. Recall in the lower bound construction, the extremal body M is, up to translation, the union of two cones which share the same base B (where one of the two cones are possibly degenerate). For $\alpha \in (-1, 0]$, we have again equality from the equality conditions of Grünbaum's theorem, in other words $M = \mathrm{conv}(B, v)$ is a cone with its base B being an $(n-1)$-dimensional convex body lying parallel to $\xi^\perp$ in ξ^- and vertex v lying in ξ^+. For $\alpha \in (0, 1/n)$, recall that we have a minimum for ψ at $t_0 = \frac{(n+1)\alpha}{\alpha+1}$. As α increases from 0 towards $1/n$, t_0 increases from 0 to 1, so B shifts in the direction of ξ. When $\alpha \in [1/n, n)$ we have equality in the limiting case for $\alpha \in (0, 1/n)$, $\alpha \longrightarrow 1/n$; see Figure 3. $\square$

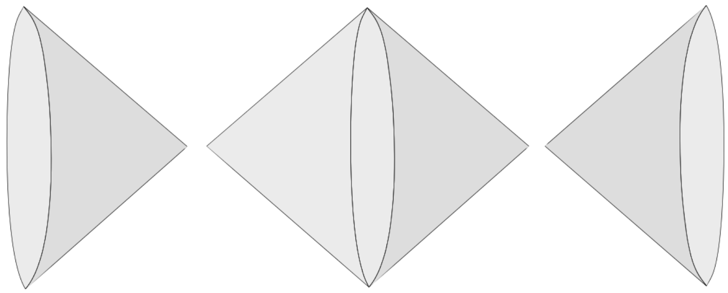

FIGURE 3. Extreme shapes for the lower bound

As an application of Theorem 3 we obtain a generalization of the result of Makai and Martini [7] stated in the introduction.

Theorem 4. *Let K be a convex body with centroid at the origin. Let $\xi \in S^{n-1}$ and $\alpha \in (-1, n)$. Consider the hyperplane*

$$H_\alpha = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = \alpha h_K(-\xi)\}.$$

Then

$$|K \cap H_\alpha| \geq D(\alpha, n) \sup_{t \in \mathbb{R}} \left| K \cap \left(\xi^\perp + t\xi\right) \right|,$$

where

$$D(\alpha, n) = \begin{cases} \left(\frac{n(\alpha+1)}{n+1}\right)^{n-1} & \alpha \in (-1, 0], \\ \left(\frac{n}{n+1}\right)^{n-1} (\alpha+1)^{\frac{(n-1)^2}{n}} (1-\alpha n)^{\frac{n-1}{n}} & \alpha \in (0, 1/n), \\ 0 & \alpha \in [1/n, n). \end{cases}$$

The bound is sharp and equality cases are discussed in the proof below.

Proof. Note that for $\alpha \in (1/n, n)$ there are cases where the intersection $K \cap H_\alpha$ is empty. Therefore for such α we cannot do better than $D(\alpha, n) = 0$.

Now we will consider $\alpha \in (-1, 0]$. We can assume that

$$|K \cap H_\alpha| < \sup_{t \in \mathbb{R}} \left| K \cap \left(\xi^\perp + t\xi\right) \right|,$$

otherwise the theorem follows immediately.

Apply Schwartz symmetrization to K with respect to the line $\ell = \{x \in \mathbb{R}^n : x = t\xi,\ t \in \mathbb{R}\}$. Abusing notation, we will denote the new body again by K. Let $t_0 = \min\{t \in \mathbb{R} : A_{K,\xi}(t) = \max_{t \in \mathbb{R}} A_{K,\xi}(t)\}$. Since $0 < |K \cap H_\alpha| < \left|K \cap \left(\xi^\perp + t_0\xi\right)\right|$ we can find a cone with base equal to $K \cap \left(\xi^\perp + t_0\xi\right)$ and section equal to $K \cap H_\alpha$. Such a cone is uniquely determined by these two sections. Denote this cone by N_1. Let $\gamma\xi$ be the vertex of N_1, for some number γ (either positive or negative). Due to the convexity of K, $\gamma\xi$ lies outside of K. Define N_2 to be the cone with base equal to $K \cap H_\alpha$ and vertex $\gamma\xi$. Finally, we will let H_α^* be the halfspace bounded by the hyperplane H_α that contains N_2.

We will consider two cases: $H_\alpha^* = H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq \alpha h_K(-\xi)\}$ and $H_\alpha^* = H_\alpha^- = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq \alpha h_K(-\xi)\}$. Denote $h = \alpha h_K(-\xi)$. When

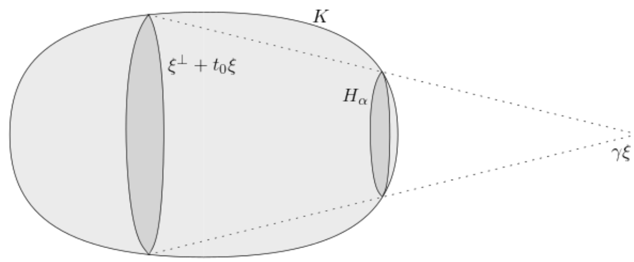

FIGURE 4. Constructing cones N_1 and N_2

$H_\alpha^* = H_\alpha^+$ the following inequality holds:

$$|K \cap H_\alpha| = \frac{|N_2|\, n}{|\gamma - h|} \geq \frac{|K \cap H_\alpha^+|\, n}{|\gamma - h|}.$$

Then by Theorem 3 and using that $|K \cap H_\alpha^+| + |K \cap H_\alpha^-| = |K|$ we note that $\big(1 - C_1(\alpha, n)\big)|K| \geq |K \cap H_\alpha^-| = |K| - |K \cap H_\alpha^+|$. Then we arrive at the following estimates

$$\begin{aligned} \frac{|K \cap H_\alpha^+|\, n}{|\gamma - h|} &\geq C_1(\alpha, n)\frac{|K|\, n}{|\gamma - h|} \\ &\geq \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\right)\frac{|K \cap H_\alpha^-|\, n}{|\gamma - h|} \geq \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\right)\frac{|N_1 \setminus N_2|\, n}{|\gamma - h|}. \end{aligned}$$

Expressing the volumes of N_1 and N_2 in terms of their bases, we get

$$\begin{aligned} |K \cap H_\alpha| &\geq \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\right)\frac{|N_1 \setminus N_2|\, n}{|\gamma - h|} = \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\right)\frac{(|N_1| - |N_2|)n}{|\gamma - h|} \\ &= \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\right)\frac{\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|\, |\gamma - t_0|}{|\gamma - h|} - \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\right)|K \cap H_\alpha|. \end{aligned}$$

And so,

$$\begin{aligned} |K \cap H_\alpha| &\geq \frac{\left(\frac{C_1(\alpha,n)}{1 - C_1(\alpha,n)}\right)}{\left(1 + \frac{C_1(\alpha,n)}{1 - C_1(\alpha,n)}\right)}\frac{|K \cap (\xi^\perp + t_0\xi)||\gamma - t_0|}{|\gamma - h|} \\ &= C_1(\alpha, n)\frac{|K \cap (\xi^\perp + t_0\xi)||\gamma - t_0|}{|\gamma - h|} \end{aligned}$$

Because N_1 is a homothetic copy of N_2, we can write

$$\frac{|\gamma - t_0|}{|\gamma - h|} = \frac{\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|^{1/(n-1)}}{|K \cap H_\alpha|^{1/(n-1)}}.$$

Thus,

$$|K \cap H_\alpha| \geq C_1(\alpha, n)\left|K \cap \left(\xi^\perp + t_0\xi\right)\right| \frac{\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|^{1/(n-1)}}{|K \cap H_\alpha|^{1/(n-1)}},$$

which implies

$$|K \cap H_\alpha| \geq C_1(\alpha, n)^{\frac{n-1}{n}}\left|K \cap \left(\xi^\perp + t_0\xi\right)\right|. \tag{5}$$

Now suppose $H_\alpha^* = H_\alpha^-$. Then the following inequality holds

$$|K \cap H_\alpha| = \frac{|N_2|\, n}{|\gamma - h|} \geq \frac{|K \cap H_\alpha^-|\, n}{|\gamma - h|}.$$

Similarly, by Theorem 3 we get $\big((1 - C_2(\alpha, n)\big)|K| \leq |K \cap H_\alpha^-|$ and so the following inequalities hold

$$\begin{aligned}\frac{|K \cap H_\alpha^-|\, n}{|\gamma - h|} &\geq \big(1 - C_2(\alpha, n)\big)\frac{|K|\, n}{|\gamma - h|}\\ &\geq \left(\frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\right)\frac{|K \cap H_\alpha^+|\, n}{|\gamma - h|} \geq \left(\frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\right)\frac{|N_1 \setminus N_2|\, n}{|\gamma - h|}.\end{aligned}$$

Expressing the volumes of N_1 and N_2 in terms of their bases, we get

$$\begin{aligned}|K \cap H_\alpha| &\geq \left(\frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\right)\frac{|N_1 \setminus N_2|\, n}{|\gamma - h|} = \left(\frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\right)\frac{(|N_1| - |N_2|)n}{|\gamma - h|}\\ &= \left(\frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\right)\frac{\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|\, |\gamma - t_0|}{|\gamma - h|} - \left(\frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\right)|K \cap H_\alpha|\,.\end{aligned}$$

So,

$$\begin{aligned}|K \cap H_\alpha| \geq \frac{\left(\frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\right)}{\left(1 + \frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\right)}\frac{|K \cap (\xi^\perp + t_0\xi)||\gamma - t_0|}{|\gamma - h|}&\\ = (1 - C_2(\alpha, n))\,\frac{|K \cap (\xi^\perp + t_0\xi)||\gamma - t_0|}{|\gamma - h|}&\end{aligned}$$

Again using that N_1 and N_2 are homothetic we arrive at

$$|K \cap H_\alpha| \geq \big(1 - C_2(\alpha, n)\big)\, \big|K \cap \big(\xi^\perp + t_0\xi\big)\big|\, \frac{\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|^{1/(n-1)}}{|K \cap H_\alpha|^{1/(n-1)}},$$

which implies

$$|K \cap H_\alpha| \geq \big(1 - C_2(\alpha, n)\big)^{\frac{n-1}{n}}\, \big|K \cap \big(\xi^\perp + t_0\xi\big)\big|\,. \tag{6}$$

Now to determine $D(\alpha, n)$ we need to find the minimum of the two constants in equations (5) and (6) for fixed α. Note that $n\alpha \leq -\alpha$ for $\alpha \in (-1, 0]$. Then it follows that

$$\big(1 - C_2(\alpha, n)\big)^{\frac{n-1}{n}} = \left(\frac{n(\alpha + 1)}{n + 1}\right)^{n-1} \leq \left(\frac{n - \alpha}{n + 1}\right)^{n-1} = C_1(\alpha, n)^{\frac{n-1}{n}},$$

for all $\alpha \in (-1, 0]$.

Now consider the case $\alpha \in (0, 1/n)$. Let A be a convex body that gives an equality case for the upper bound in Theorem 3. Then

$$C_2(\alpha, n)|A| = |A \cap H_\alpha^+| \leq |A \cap H_0^+| \leq C_2(0, n)|A|.$$

Thus, for $\alpha \in (0, 1/n)$ it follows that

$$C_2(\alpha, n) \leq C_2(0, n) = 1 - \left(\frac{n}{n + 1}\right)^n,$$

and hence,

$$1 - C_2(\alpha, n) \geq \left(\frac{n}{n+1}\right)^n.$$

Now observe that the function $f(\alpha) = (\alpha+1)^{n-1}(1-\alpha n)$ is decreasing on $(0, 1/n)$. Thus $(\alpha+1)^{n-1}(1-\alpha n) \leq 1$ so it follows that

$$\begin{aligned} C_1(\alpha, n)^{\frac{n-1}{n}} = \left(\frac{n}{n+1}\right)^{n-1} (\alpha+1)^{\frac{(n-1)^2}{n}} (1-\alpha n)^{\frac{n-1}{n}} &\leq \left(\frac{n}{n+1}\right)^{n-1} \\ &\leq \big(1 - C_2(\alpha, n)\big)^{\frac{n-1}{n}}. \end{aligned}$$

Our argument for $\alpha = 1/n$ does not naturally extend from either case of $\alpha \in (0, 1/n)$ or $\alpha \in (1/n, n)$ so we will now prove $D(\alpha, n) = 0$ for $\alpha = 1/n$. Consider the sequence of convex bodies $(K_i)_{i \in \mathbb{N}}$ where

$$K_i = \operatorname{conv}(-\xi, B_{n-1}, \xi/i)$$

and B_{n-1} is the unit $(n-1)$-dimensional Euclidean ball centered in $\{x \in \mathbb{R}^n : x_1 = 0\}$. Then B_{n-1} is the maximal section for each K_i in the direction ξ and for all $i \geq 1$ we have $G_i = K_i - g(K_i)$ is a convex body with centroid at the origin and maximal section $B_{n-1} - g(K_i)$. Let $H = \{x \in \mathbb{R}^n : \langle x, \xi\rangle = (1/n) h_{G_i}(-\xi)\}$. Calculations similar to the lower bound derivation in Theorem 3 show that for all $i \geq 1$ we have

$$\lim_{i\to\infty} |G_i \cap H| = 0,$$

and so this implies $c(1/n, n) = 0$.

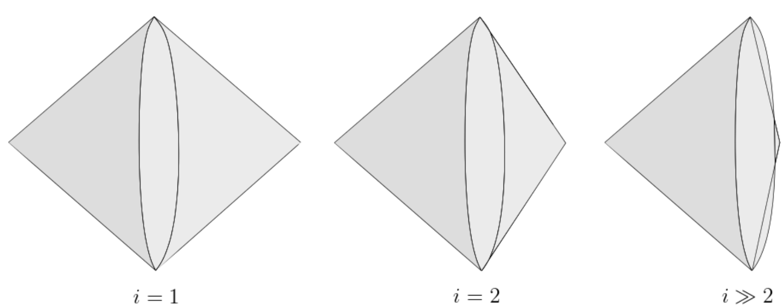

FIGURE 5. The sequence of convex bodies $(K_i)_{i=1}^{\infty}$

Therefore, we have now shown the values for $D(\alpha, n)$. Discussing the equality cases for $\alpha \in (-1, 1/n)$ we note that $D(\alpha, n)$ was derived from the constants $C_1(\alpha, n)$, $C_2(\alpha, n)$ in Theorem 3. Therefore, all inequalities with $D(\alpha, n)$ become equalities under the equality cases in Theorem 3. For $\alpha = 1/n$ we have equality in the limit of sequence of convex bodies G_i from the above construction. For $\alpha \in (1/n, n)$ one can observe we have equality when K is a cone with base centered in $\{x \in \mathbb{R}^n : x_1 = 1\}$ and vertex at 0 since $|K \cap H_\alpha| = 0$. See the Figure 5 below for an illustration of equality cases for $\alpha \neq 1/n$ and Figure 6 for an illustration of the limiting equality case for $\alpha = 1/n$. □

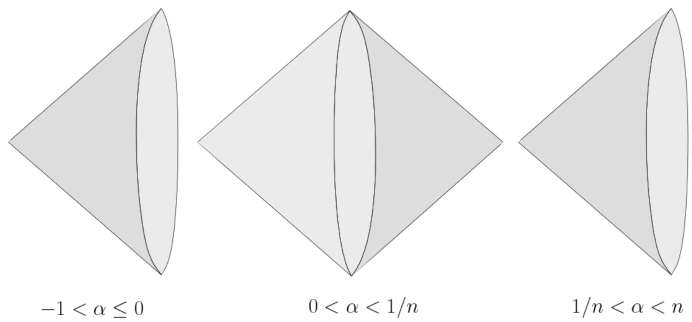

FIGURE 6. Equality cases for Theorem 4

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