AN AFFINE GENERALIZATION OF GRÜNBAUM'S INEQUALITY

BRAYDEN LETWIN AND VLADYSLAV YASKIN

Abstract

Grünbaum's inequality states that for any convex body $K \subseteq \mathbb{R}^n$ with centroid at the origin and for any $\xi \in S^{n-1}$ we have that

$$\left(\frac{n}{n+1}\right)^n \leq \frac{\mathrm{vol}_n\left(K \cap \xi^+\right)}{\mathrm{vol}_n(K)} \leq 1 - \left(\frac{n}{n+1}\right)^n,$$

where $\xi^+ = \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle \geq 0\}$. We generalize this inequality by establishing sharp bounding constants $C_1(\alpha)$ and $C_2(\alpha)$ such that

$$C_1(\alpha) \leq \frac{\mathrm{vol}_n\left(K \cap \left(\xi^+ + \alpha h_K(-\xi)\xi\right)\right)}{\mathrm{vol}_n(K)} \leq C_1(\alpha),$$

where $-1 < \alpha < n$ and h_K is the support function for K.

1. INTRODUCTION

Let $K \subseteq \mathbb{R}^n$ be a convex body. The centroid of K is defined as the point

$$g(K) = \frac{1}{\mathrm{vol}_n(K)} \int_K x \, dx,$$

where $\mathrm{vol}_n(K)$ denotes the n-dimensional volume of K, and the integration is performed with respect to the Lebesgue measure. An established result by Grünbaum [5] states that for any convex body $K \subseteq \mathbb{R}^n$ with centroid at the origin and for any $\xi \in S^{n-1}$ we have that

$$\left(\frac{n}{n+1}\right)^n \leq \frac{\mathrm{vol}_n\left(K \cap \xi^+\right)}{\mathrm{vol}_n(K)} \leq 1 - \left(\frac{n}{n+1}\right)^n, \tag{1}$$

where $\xi^+ = \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle \geq 0\}$ is the positive half-space induced by the $(n-1)$-dimensional subspace orthogonal to ξ. These bounds, as encapsulated by (1), are sharp. The equalities occur when K is a cone with base lying in and centered at the origin of an affine hyperplane orthogonal to ξ.

Recent advancements in Grünbaum-type inequalities, encompassing sections and projections of convex bodies, can be explored in the following papers: [3], [6], [7], [10].

Suppose instead we consider affine translations of ξ^+. Can an analogous inequality to (1), complete with sharp bounds, be established? This paper offers an affirmative answer to this inquiry. Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid located at the origin. Let $-1 < \alpha < n$ and let $\xi \in S^{n-1}$. Then there are sharp constants $C_1(\alpha)$ and $C_2(\alpha)$ such that

$$C_1(\alpha) \leq \frac{\mathrm{vol}_n\left(K \cap \left(\xi^+ + \alpha h_K(-\xi)\xi\right)\right)}{\mathrm{vol}_n(K)} \leq C_2(\alpha). \tag{2}$$

where h_K is the support function for K.

The bounds in (2) were established by preceding work in $\mathbb{R}^2$ (refer to [9]). In this paper, we will adopt a similar methodology within the context of $\mathbb{R}^n$. The exact values of $C_1(\alpha)$ and $C_2(\alpha)$ are clarified in Theorem 3, which also presents scenarios where equality is observed. The special case where $\alpha \in (-1, 0]$ is also covered in special detail, where a simple proof is given. It is noteworthy that within (1), the demonstration of just one inequality is sufficient, as the other immediately follows suit. Conversely, within (2), both inequalities necessitate separate validation.

The subsequent section of this paper delves into a targeted application of Theorem 3. It seeks to address the following inquiry: How can we compare the volume of any section of a convex body to the volume of its maximal section(s)? This problem is rigorously formulated and resolved in Theorem 4.

2. PRELIMINARIES

First, we will discuss some preliminaries that we will use throughout the paper. Given some affine hyperplane $H \subset \mathbb{R}^n$ (i.e. an $(n-1)$-dimensional affine subspace), we will denote H^+ and H^- to be the respective affine positive and negative half-spaces induced by H. Given some $\xi \in \mathbb{R}^n$, we will denote $\xi^\perp$ to be the orthogonal compliment of span$\{\xi\}$. We will denote ξ^+ and ξ^- to be the respective positive and negative half-spaces induced by $\xi^\perp$.

Definition 1. The *support function* $h_K : S^{n-1} \to \mathbb{R}$ for a convex body K is

$$h_K(\xi) = \sup\{\langle \xi, x \rangle \mid x \in K\}.$$

From a geometric standpoint, when a convex body's centroid is situated at the origin, its support function signifies the distance from the centroid to the supporting hyperplane in the direction of ξ. A well-established result attributed to Minkowski and Radon [1, p. 58] states that, for any convex body $K \subseteq \mathbb{R}^n$ with its centroid at the origin, and for all $\xi \in S^{n-1}$, we have that

$$\frac{1}{n} h_K(\xi) \le h_K(-\xi) \le n h_K(\xi). \tag{3}$$

Take note that the choice of bounds for α in Theorem 3 - 4 are a result of (3).

Definition 2. Let $\xi \in S^{n-1}$. The *parallel section function* $A_{K,\xi} : \mathbb{R} \to \mathbb{R}$ for a convex body $K \subseteq \mathbb{R}^n$ is

$$A_{K,\xi}(t) = \text{vol}_{n-1}\Big(K \cap \big(\xi^\perp + t\xi\big)\Big).$$

If $K \cap \big(\xi^\perp + t\xi\big)$ forms an $(n-1)$-dimensional Euclidean ball, then $A_{K,\xi}^{\frac{1}{n-1}}(t)$ provides the radius of this Euclidean ball, up to a constant.

Definition 3. Let $\xi \in S^{n-1}$. The *half-space intersection function* $A_{K,\xi}^* : \mathbb{R} \to \mathbb{R}$ for a convex body $K \subseteq \mathbb{R}^n$ is

$$A_{K,\xi}^*(t) = \int_t^\infty \text{vol}_{n-1}\Big(K \cap \big(\xi^\perp + x\xi\big)\Big) dx.$$

Lemma 1. *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Then the parallel section function $A_{K,\xi}$ is $1/(n-1)$ concave on its support.*

Proof. See [11, p. 299] $\square$

2

Lemma 2. *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Then the half-space intersection function $A_{K,\xi}^*$ is $1/n$ concave on its support.*

Proof. Let $\lambda \in [0,1]$ and $x, y \in \operatorname{supp}\left(A_{K,\xi}^{*\frac{1}{n}}\right)$. The trick is to observe that

$$\lambda\Big(K \cap \{t \in \mathbb{R}^n \mid \langle \xi, t \rangle \geq x\}\Big) + (1-\lambda)\Big(K \cap \{t \in \mathbb{R}^n \mid \langle \xi, t \rangle \geq y\}\Big)$$
$$\subseteq \Big(K \cap \{t \in \mathbb{R}^n \mid \langle \xi, t \rangle \geq \lambda x + (1-\lambda)y\}\Big).$$

Then this implies that

$$\operatorname{vol}_n\Big(K \cap \{t \in \mathbb{R}^n \mid \langle \xi, t \rangle \geq \lambda x + (1-\lambda)y\}\Big)^{\frac{1}{n}}$$
$$\geq \operatorname{vol}_n\Big(\lambda\big(K \cap \{t \in \mathbb{R}^n \mid \langle \xi, t \rangle \geq x\}\big) + (1-\lambda)\big(K \cap \{t \in \mathbb{R}^n \mid \langle \xi, t \rangle \geq y\}\big)\Big)^{\frac{1}{n}},$$

and then by applying the Brunn-Minkowski inequality

$$\geq \lambda\operatorname{vol}_n\Big(K \cap \{t \in \mathbb{R}^n \mid \langle \xi, t \rangle \geq x\}\Big)^{\frac{1}{n}} + (1-\lambda)\operatorname{vol}_n\Big(K \cap \{t \in \mathbb{R}^n \mid \langle \xi, t \rangle \geq y\}\Big)^{\frac{1}{n}}.$$

Finally, applying Fubini's theorem yields

$$\int_{\lambda x + (1-\lambda)y}^{\infty} \operatorname{vol}_{n-1}\Big(K \cap \big(\xi^\perp + t\xi\big)\Big) dt$$
$$\geq \lambda \int_x^\infty \operatorname{vol}_{n-1}\Big(K \cap \big(\xi^\perp + t\xi\big)\Big) dt + (1-\lambda) \int_y^\infty \operatorname{vol}_{n-1}\Big(K \cap \big(\xi^\perp + t\xi\big)\Big) dt.$$

Which proves the result. $\qquad\square$

Definition 4. The Schwartz symmetral of a convex body $K \subseteq \mathbb{R}^n$ along the direction ξ is the convex body $\bar{K}$, which is constructed as follows: Over the interval $-h_K(-\xi) \leq t \leq h_K(\xi)$, $\bar{K} \cap \big(\xi^\perp + t\xi\big)$ forms an $(n-1)$-dimensional Euclidean ball positioned within $\xi^\perp$ and centered at its origin with

$$\operatorname{vol}_{n-1}\Big(K \cap \big(\xi^\perp + t\xi\big)\Big) = \operatorname{vol}_{n-1}\Big(\bar{K} \cap \big(\xi^\perp + t\xi\big)\Big). \tag{4}$$

As a result of this transformation, it becomes evident that

$$h_{\bar{K}}(\pm\xi) = h_K(\pm\xi), \tag{5}$$

and

$$\operatorname{vol}_n\Big(\bar{K} \cap \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle \geq t\}\Big) = \operatorname{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle \geq t\}\Big), \tag{6}$$

for all $t \in \mathbb{R}$. The centroid of $\bar{K}$ resides on $\operatorname{span}\{\xi\}$ due to the rotational symmetry of $\bar{K}$ around ξ.

Now, in the forthcoming central theorem of the paper, we establish bounding constants for the volume ratio between any affine positive half-space intersection with K, and K itself, where these half-spaces are classified by a distance from the centroid.

Theorem 3. *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Let $-1 < \alpha < n$ and let $\xi \in S^{n-1}$. Consider the hyperplane:*

$$H_\alpha = \xi^\perp + \alpha h_K(-\xi)\xi.$$

Then

$$C_1(\alpha) \leq \frac{\mathrm{vol}_n\big(K \cap H_\alpha^+\big)}{\mathrm{vol}_n(K)} \leq C_2(\alpha),$$

where

$$C_1(\alpha) = \begin{cases} \left(\frac{n-\alpha}{n+1}\right)^n & \alpha \in (-1, 0], \\ \left(\frac{n}{n+1}\right)^n (a+1)^{n-1}(1-\alpha n) & \alpha \in (0, 1/n), \\ 0 & \alpha \in [1/n, n), \end{cases} \tag{7}$$

and

$$C_2(\alpha) = \begin{cases} 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n & \alpha \in (-1, 0], \\ \tau_{\alpha,n} & \alpha \in (0, n). \end{cases} \tag{8}$$

The value of $\tau_{\alpha,n}$ is not explicit for general dimension because finding the value involves finding the roots of a high degree rational function. The lower bounds and upper bounds are sharp. The equality cases are discussed in the proof below.

Proof. For $-1 < \alpha \leq 0$ there is a very simple proof to Theorem 3, which we will cover first below. A proof for general $-1 < \alpha < n$ will be covered right after. We will prove the upper bound first. Observe that

$$\mathrm{vol}_n\big(K \cap H_\alpha^+\big)$$
$$= \mathrm{vol}_n(K) - \mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle \leq \alpha h_K(-\xi)\}\Big).$$

Now dilating $K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle \leq 0\}$ by $1/(\alpha+1)$ we see that the following inequalities hold

$$\mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle \leq \alpha h_K(-\xi)\}\Big)$$
$$\geq (\alpha+1)^n \mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle \leq 0\}\Big)$$
$$\geq (\alpha+1)^n \left(\frac{n}{n+1}\right)^n \mathrm{vol}_n(K),$$

which implies for $\alpha \in (-1, 0]$ that

$$\frac{\mathrm{vol}_n\big(K \cap H_\alpha^+\big)}{\mathrm{vol}_n(K)} \leq 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n.$$

Now we will prove the lower bound. By the $1/n$ concavity of $A_{K,\xi}^*$ on its support we have

$$\mathrm{vol}_n\big(K \cap H_\alpha^+\big)^{\frac{1}{n}}$$
$$= A_{K,\xi}^{*\,\frac{1}{n}}\big(\alpha h_K(-\xi)\big)$$
$$\geq -\alpha A_{K,\xi}^{*\,\frac{1}{n}}\big(-h_K(-\xi)\big) + (1+\alpha)A_{K,\xi}^{*\,\frac{1}{n}}(0).$$

4

Then by Grünbaum's inequality

$$\geq -\alpha \mathrm{vol}_n(K)^{\frac{1}{n}} + (1 + \alpha)\left(\frac{n}{n+1}\right)\mathrm{vol}_n(K)^{\frac{1}{n}},$$

which implies for $\alpha \in (-1, 0]$ that

$$\left(\frac{n-a}{n+1}\right)^n \leq \frac{\mathrm{vol}_n\big(K \cap H_\alpha^+\big)}{\mathrm{vol}_n(K)}.$$

Thus we now have our bounds for $\alpha \in (-1, 0]$. By the implications of Grünbaum's inequality, one can then see the equality cases hold precisely when K is a cone with its base lying in and centered at the origin of $\xi^\perp$ and vertex at $-h_K(\xi)\xi$.

Now, we will move to the general proof for $-1 < \alpha < n$. We will prove the upper bound first. We can assume, without loss of generality, after applying a suitable linear transformation $T \in \mathrm{GL}_n(\mathbb{R})$ that $\xi = e_1$, and subsequently focus on the Schwartz symmetral of the convex body K along the direction e_1, because of the conditions in (4) - (6). Following this, we will perform a translation of K to create a new convex body $\hat{K} = K + h_K(-e_1)e_1$. It's important to note that $g(\hat{K}) = h_K(-e_1)e_1$, and further observe that $H_\alpha = e_1^\perp + (\alpha + 1)h_K(-e_1)\xi$ is our new hyperplane of interest. Once again, with a slight departure from strict notation, we will denote the translated body as K.

To establish the upper bound, our approach involves constructing an $(n-1)$-dimensional Euclidean ball B_{n-1} situated within and centered at the origin of $e_1^\perp$, satisfying the following conditions:

- $\mathrm{vol}_{n-1}\big(K \cap e_1^\perp\big) \leq \mathrm{vol}_{n-1}(B_{n-1})$,

- $\mathrm{vol}_n\Big(\mathrm{conv}\big(B_{n-1}, K \cap H_\alpha\big)\Big) = \mathrm{vol}_n\big(K \cap H_\alpha^-\big)$.

Denote $\mathrm{conv}\big(B_{n-1}, K \cap H_\alpha\big)$ by K^-. Now, there exists a unique μ in $\mathrm{span}\{e_1\}$, enabling the construction of another $(n-1)$-dimensional Euclidean ball D_{n-1} lying in and centered at the origin of $\mu^\perp + \mu$ so that the following conditions hold:

- $(\alpha + 1) < \langle e_1, \mu \rangle < h_K(e_1)$,

- $\mathrm{vol}_{n-1}\Big(K \cap \big(\mu^\perp + \mu\big)\Big) \leq \mathrm{vol}_{n-1}(D_{n-1})$,

- $\mathrm{vol}_n\Big(\mathrm{conv}\big(D_{n-1}, K \cap H_\alpha\big)\Big)$
  $= \mathrm{vol}_n\Big(K \cap H_\alpha^+ \cap \{x \in \mathbb{R}^n \mid \langle e_1, x \rangle \leq h_K(e_1)\}\Big),$

- $\mathrm{conv}\big(D_{n-1}, K \cap H_\alpha\big) \cup K^- = \mathrm{conv}\big(B_{n-1}, D_{n-1}\big).$

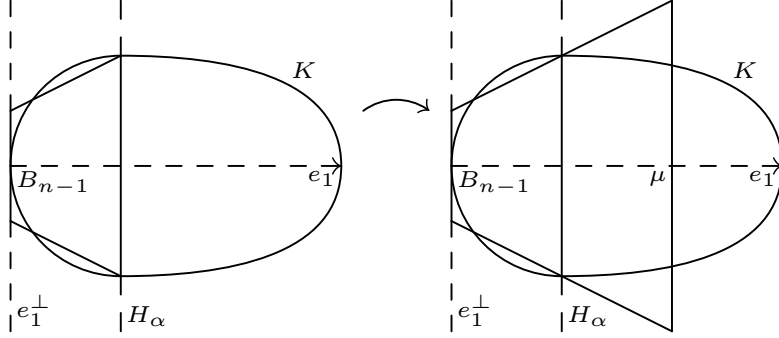To gain a geometric insight into the construction conditions shown above, refer to Figure 1.

Figure 1: Constructing B_{n-1} and D_{n-1}

Denote $\mathrm{conv}\big(D_{n-1}, K \cap H_\alpha\big)$ by K^+. By virtue of the conditions above, it becomes evident that

$$\mathrm{vol}_n\big(K^- \cup K^+\big) = \mathrm{vol}_n(K).$$

Additionally, the conditions above and an application of Lemma 1 yields the following inequality:

$$\langle e_1, g\big(K^- \cup K^+\big)\rangle \leq \langle e_1, g(K)\rangle.$$

This is because the $1/(n-1)$ concavity of the parallel section function $A_{K,\xi}$ on its support implies that $A_{K,\xi}^{\frac{1}{n-1}}$ can intersect any affine function on its support at most twice. $A_{K^-,\xi}^{\frac{1}{n-1}}$ is affine on its support, and therefore, by the conditions above it is ensured that $A_{K,\xi}^{\frac{1}{n-1}}$ intersects $A_{K^-,\xi}^{\frac{1}{n-1}}$ exactly twice on its support. Similarly, $A_{K^+,\xi}^{\frac{1}{n-1}}$ is affine on its support, so the conditions above establish that both K and $K^- \cup K^+$ are bodies of revolution about $\mathrm{span}\{e_1\}$ with equal volumes. Therefore, it suffices to show that

$$\int_0^\infty x A_{K^- \cup K^+,\xi}^{\frac{1}{n-1}}(x)dx \leq \int_0^\infty x A_{K,\xi}^{\frac{1}{n-1}}(x)dx,$$

which is clear.

As a consequence, we have successfully constructed a convex body $F = \mathrm{conv}\big(B_{n-1}, D_{n-1}\big)$, which satisfies the following conditions:

$$\mathrm{vol}_n(K) = \mathrm{vol}_n(F),$$

and

$$\mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle e_1, g(K)\rangle\}\Big)$$
$$\leq \mathrm{vol}_n\Big(F \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle e_1, g(F)\rangle\}\Big).$$

Hence, our objective narrows down to establishing a maximum volume ratio within the scope of such constructed convex bodies. Without loss of generality,

we can perform a suitable linear transformation $T \in \mathrm{GL}_n(\mathbb{R})$, and assume that $\mu = e_1$. Additionally, for the interval $0 \le t \le 1$, we can assume that

$$F \cap \left(e_1^\perp + te_1\right) = (mt + b)\left(F \cap e_1^\perp\right), \tag{9}$$

where $b \ge 0$ and either (1) $m \ge 0$, or (2) $m < 0$ and $m + b \ge 0$. For now, our focus will be on the scenario where $m \ne 0$, while we will address the case of $m = 0$ later. Then by Fubini's theorem and (9):

$$\begin{aligned}
\mathrm{vol}_n(F) &= \int_F dx \\
&= \int_{x_1=0}^1 \mathrm{vol}_{n-1}\left(F \cap \left(e_1^\perp + x_1 e_1\right)\right) dx_1 \\
&= \mathrm{vol}_{n-1}\left(F \cap e_1^\perp\right) \frac{(b+m)^n - b^n}{mn}.
\end{aligned}$$

Additionally, determining the x_1 coordinate of the centroid of F can be accomplished through a straightforward application of Fubini's theorem and (9):

$$\begin{aligned}
\langle e_1, g(F)\rangle &= \frac{1}{\mathrm{vol}_n(F)} \int_F x_1 dx \\
&= \frac{1}{\mathrm{vol}_n(F)} \int_{x_1=0}^1 x_1 \mathrm{vol}_{n-1}\left(F \cap \left(e_1^\perp + te_1\right)\right) dx_1 \\
&= \frac{b^{n+1} + (mn-b)(b+m)^n}{m(n+1)\left((b+m)^n - b^n\right)}.
\end{aligned}$$

Given that F takes the form of a body of revolution around the x_1 axis, it naturally ensures that all remaining coordinates of the centroid of F are equal to zero.

Now we can compute

$$\begin{aligned}
\frac{\mathrm{vol}_n\left(F \cap H_\alpha^+\right)}{\mathrm{vol}_n(F)} &= \frac{1}{\mathrm{vol}_n(F)} \int_{F \cap H_\alpha^+} dx \\
&= \frac{(b+m)^n - \left(b + m(\alpha+1)\left(\frac{b^{n+1}+(mn-b)(b+m)^n}{m(n+1)\left((b+m)^n-b^n\right)}\right)\right)^n}{(b+m)^n - b^n}.
\end{aligned}$$

Introducing the variable $z = \frac{b}{m}$, we arrive at

$$\langle e_1, g(F)\rangle = (\alpha+1) \frac{z\left(1 + (\frac{n}{z} - 1)(\frac{1}{z}+1)^n\right)}{(n+1)\left((\frac{1}{z}+1)^n - 1\right)},$$

and

$$\frac{\mathrm{vol}_n\left(F \cap H_\alpha^+\right)}{\mathrm{vol}_n(F)} = \frac{\left(1+\frac{1}{z}\right)^n - \left(1 + \frac{1}{z}(\alpha+1)\left(\frac{z\left(1+(\frac{1}{z}-1)(\frac{1}{z}+1)^n\right)}{(n+1)\left((\frac{1}{z}+1)^n-1\right)}\right)\right)^n}{\left(1+\frac{1}{z}\right)^n - 1}.$$

Letting ϕ denote the latter function of z, our goal is to determine the supremum of ϕ within the domain $z \in (-\infty, -1] \cup [0, \infty)$. For values of $\alpha \in (-1, 0]$, one can observe that ϕ exhibits strict decreasing behavior as a function of z across its entire domain. Therefore, the supremum of ϕ is attained either as z approaches $-\infty$ or when $z = 0$. Then it follows that

$$\lim_{z \to -\infty} \phi(z) = \frac{1 - \alpha}{2} \leq \phi(0) = 1 - \left(\frac{(\alpha + 1)\, n}{n + 1} \right)^n.$$

However, for $\alpha \in (0, n)$, the function ϕ lacks the characteristics of monotone increase or decrease. Moreover, no stable intervals of concavity or convexity persist across all dimensions. Consequently, determining the upper bound $\tau_{\alpha,n}$ becomes a complex task, as solving for $\tau_{\alpha,n}$ involves handling high-degree rational functions.

Lastly, let's revisit the case where $m = 0$. It becomes evident that our body F is a cylinder. Thus, we have that

$$\frac{\mathrm{vol}_n\left(F \cap H_\alpha^+ \right)}{\mathrm{vol}_n(F)} = \frac{1 - \alpha}{2}.$$

For $\alpha \in (-1, 0]$ this ratio is smaller than than over our existing ratio and for $\alpha \in (0, n)$, it is clear that

$$\tau_{\alpha,n} \geq \frac{1 - \alpha}{2}.$$

Thus, we now see that for any convex body K, we have the following bounds as seen in (8):

$$\frac{\mathrm{vol}_n\left(F \cap H_\alpha^+ \right)}{\mathrm{vol}_n(K)} \leq C_2(\alpha).$$

Now, let's delve into the discussion of equality cases. For $\alpha \in (-1, 0]$, the configuration of our body K involves the convex hull of two $(n-1)$-dimensional Euclidean balls (which may yield a cone). The supremum of ϕ is realized at $z = 0$, implying $b = 0$, and thus the extreme shapes correspond to cones with a vertex at the origin and a base lying in the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 1\}$.

As we move on to $\alpha \in (0, n)$, the structure of our body K remains the convex hull of two $(n - 1)$-dimensional Euclidean balls (which may lead to a cone). In the case of $\alpha = 1/n$, the body becomes a cylinder. However, in general, when $\alpha \in (0, n)$, we face difficulties in determining the precise values at which the supremum of ϕ is attained. Consequently, the equality bodies cannot be consistently determined in all cases. Refer to Figure 2 for an illustrative depiction of how shapes evolve as α transitions from -1 to n.
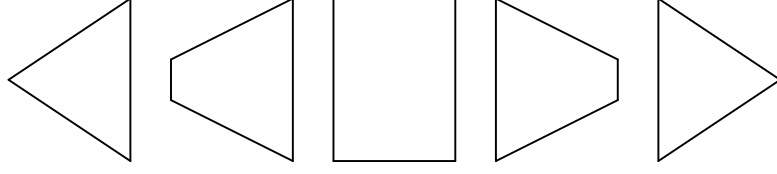
Figure 2: Extreme shapes for the upper bound

Let us now shift our focus to establishing the lower bound. Similarly to the upper bound, we can simplify the problem by applying a suitable linear transformation $T \in \mathrm{GL}_n(\mathbb{R})$. Without loss of generality we'll begin by assuming $\xi = e_1$ and apply the same initial transformations as we did to prove the upper bound. In other words, we will assume that K is in its Schwartz symmetral form about e_1 and that $h_K(-e_1) = 0$. Now, there exists a unique $\rho \in \mathrm{span}\{e_1\}$ such that the following conditions are met:

- $h_K(e_1) \leq \langle e_1, \rho \rangle$,

- $\mathrm{vol}_n\Big(\mathrm{conv}\big(K \cap H_\alpha, \rho\big)\Big)$
  $= \mathrm{vol}_n\big(K \cap H_\alpha^+\big).$

Let's denote $\mathrm{conv}\big(K \cap H_\alpha, \rho\big)$ as K^+. Now, there exists another unique $\beta \in \mathrm{span}\{e_1\}$ that allows us to construct a $(n-1)$-dimensional Euclidean ball B_{n-1}. This ball lies within and is centered at the origin of $\beta^\perp + \beta$, satisfying the subsequent conditions:

- $0 \leq \langle e_1, \beta \rangle < (\alpha + 1)$,

- $\mathrm{vol}_{n-1}\Big(K \cap \big(\beta^\perp + \beta\big)\Big) \leq \mathrm{vol}_{n-1}(B_{n-1})$,

- $\mathrm{conv}\big(B_{n-1}, K^+\big) = \mathrm{conv}\big(B_{n-1}, \rho\big)$,

- $\mathrm{vol}_n\Big(\mathrm{conv}\big(0, B_{n-1}, \rho\big)\Big) = \mathrm{vol}_n(K).$

For a geometrical description of the construction conditions as seen above, see Figure 3.
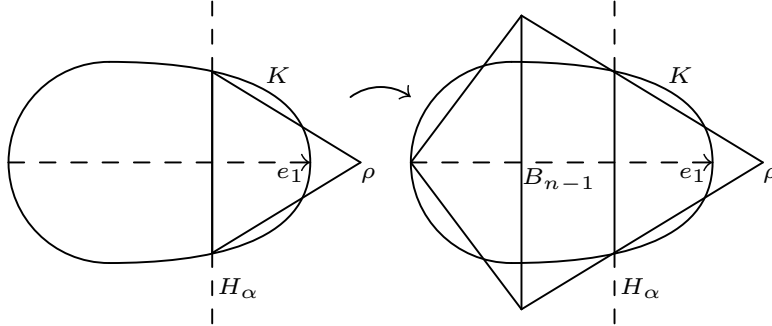

Figure 3: Constructing B_{n-1} and ρ

Denote $\text{conv}(0, B_{n-1}, \rho)$ as B and $\text{conv}(0, B_{n-1})$ as K^-. From the conditions above, we can deduce that B is formed by the union of two cones. These cones have vertices at 0 and ρ, and they share a common base located at β. Utilizing the conditions above and making use of Lemma 1, we obtain the following inequality:

$$\langle e_1, g(B) \rangle \geq \langle e_1, g(K) \rangle.$$

This stems from the fact that the $1/(n-1)$ concavity of the parallel section function $A_{K,\xi}$ implies that $A_{K,\xi}^{\frac{1}{n-1}}$ can intersect any affine function on its support at most twice. However, both $A_{K^-,\xi}^{\frac{1}{n-1}}$ and $A_{K^+,\xi}^{\frac{1}{n-1}}$ are affine on their support and intersect $A_{K,\xi}^{\frac{1}{n-1}}$. Therefore, by utilizing conditions above, we establish that K and B have the same volume and are bodies of revolution about $\text{span}\{e_1\}$. Thus, it is equivalent to show that

$$\int_0^\infty x A_{B,\xi}^{\frac{1}{n-1}}(x)dx \geq \int_0^\infty x A_{K,\xi}^{\frac{1}{n-1}}(x)dx,$$

which is evidently true.

As a result, we have constructed a convex body B such that:

$$\text{vol}_n(K) = \text{vol}_n(B),$$

and

$$\text{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle e_1, g(K) \rangle\}\Big)$$
$$\geq \text{vol}_n\Big(B \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle e_1, g(B) \rangle\}\Big).$$

Hence, our task reduces to demonstrating a minimum volume ratio within the class of such constructed convex bodies.

We can proceed without loss of generality by employing an appropriate linear transformation $T \in \text{GL}_n(\mathbb{R})$. We then consider the scenario where $h_B(e_1) = 1$, and the base of B is a $(n-1)$-dimensional Euclidean ball B_{n-1} with unit radius, and centered at $t \in \text{span}\{e_1\}$ with $0 \leq \langle e_1, t \rangle \leq 1$. Define

$$C_1 = B \cap \{x \in \mathbb{R}^n \mid \langle e_1, x \rangle \leq \|t\|\},$$

and

$$C_2 = B \cap \{x \in \mathbb{R}^n \mid \langle e_1, x \rangle \geq \|t\|\}.$$

Then, by Fubini's theorem:

$$\text{vol}_n(B) = \int_B dx$$
$$= \int_{C_1} dx + \int_{C_2} dx$$
$$= \int_{x_1=0}^{\|t\|} \text{vol}_{n-1}(\frac{x_1}{\|t\|}B_{n-1})dx_1 + \int_{x_1=\|t\|}^1 \text{vol}_{n-1}((\frac{-x_1+1}{1-\|t\|})B_{n-1})dx_1$$
$$= \frac{\text{vol}_{n-1}(B_{n-1})}{n}.$$

Next, we can compute the x_1 coordinate of the centroid of B. Employing Fubini's theorem, we can determine this value as follows:

$$\langle e_1, g(B) \rangle = \frac{1}{\text{vol}_n(B)} \int_B x_1 dx$$

$$= \frac{1}{\text{vol}_n(B)} \left(\int_{C_1} dx + \int_{C_2} dx \right)$$

$$= \frac{1}{\text{vol}_n(B)} \left(\int_{x_1=0}^{\|t\|} x_1 \text{vol}_{n-1} \left(\frac{x_1}{\|t\|} B_{n-1} \right) dx_1 + \int_{x_1=\|t\|}^{1} x_1 \text{vol}_{n-1} \left(\left(\frac{-x_1+1}{1-\|t\|} \right) B_{n-1} \right) dx_1 \right)$$

$$= \frac{\|t\|(n-1)+1}{n+1}.$$

Once again, due to the rotational symmetry of B about e_1, it's evident that all other coordinates of the centroid are zero.

For our ratio computations, let's analyze the positive half-space defined by the equation $x_1 \geq \frac{(\alpha+1)\left(\|t\|(n-1)+1 \right)}{n+1}$. This analysis involves two distinct cases:

1. $0 \leq \|t\| \leq \frac{(\alpha+1)\left(\|t\|(n-1)+1 \right)}{n+1}$,

2. $\frac{(\alpha+1)\left(\|t\|(n-1)+1 \right)}{n+1} \leq \|t\| \leq 1$.

These two cases are respectively equivalent to

$$0 \leq \|t\| \leq \frac{\alpha+1}{2-(n-1)\alpha}$$

and

$$\frac{\alpha+1}{2-(n-1)\alpha} \leq \|t\| \leq 1.$$

In the first case, we have the following:

$$\text{vol}_n \left(B \cap H_\alpha^+ \right) = \int_{B \cap H_\alpha^+} dx$$

$$= \int_{x_1=\frac{(\alpha+1)\left(\|t\|(n-1)+1 \right)}{n+1}}^{1} \text{vol}_{n-1} \left(\left(\frac{-x_1+1}{1-\|t\|} \right) B_{n-1} \right) dx_1$$

$$= \text{vol}_{n-1}(B_{n-1}) \frac{\left(1 - \frac{(\alpha+1)\left(\|t\|(n-1)+1 \right)}{n+1} \right)^n}{(1-\|t\|)^{n-1} n}.$$

Then it follows that

$$\frac{\text{vol}_n \left(B \cap H_\alpha^+ \right)}{\text{vol}_n(B)} = \frac{\left(1 - \frac{(\alpha+1)\left(\|t\|(n-1)+1 \right)}{n+1} \right)^n}{(1-\|t\|)^{n-1}}. \tag{10}$$

11

In the second case, we have:

$$\text{vol}_n\big(B \cap H_\alpha^+\big) = \int_{B \cap H_\alpha^+} dx$$

$$= \int_{x_1 = \frac{(\alpha+1)\big(\|t\|(n-1)+1\big)}{n+1}}^{\|t\|} \text{vol}_{n-1}\Big(\frac{x_1}{\|t\|} B_{n-1}\Big) dx_1 + \int_{x_1 = \|t\|}^{1} \text{vol}_{n-1}\Big(\Big(\frac{-x_1+1}{1-\|t\|}\Big) B_{n-1}\Big) dx_1$$

$$= \text{vol}_{n-1}(B_{n-1})\Bigg(\frac{\|t\|^n - \Big(\frac{(\alpha+1)\big(\|t\|(n-1)+1\big)}{n+1}\Big)^n}{\|t\|^{n-1} n} + \frac{(1-\|t\|)}{n}\Bigg).$$

Then it follows that

$$\frac{\text{vol}_n\big(B \cap H_\alpha^+\big)}{\text{vol}_n(B)} = \frac{\|t\|^n - \Big(\frac{(\alpha+1)\big(\|t\|(n-1)+1\big)}{n+1}\Big)^n}{\|t\|^{n-1}} + 1 - \|t\|. \qquad (11)$$

Denoting the piecewise function of $\|t\|$ in (10) and (11) as ψ, our objective is to determine the infimum of ψ over the interval on which it is defined. For the first case when

$$0 \le \|t\| \le \frac{(\alpha+1)\Big(\|t\|(n-1)+1\Big)}{n+1},$$

The function ψ exhibits strict convexity across the entire interval when $\|t\| \in (0,1)$. Therefore, it attains a sole unique infimum, which is $\Big(\frac{n-\alpha}{n+1}\Big)^n$, when $\alpha \in (-1, 0]$. This is because ψ is strictly increasing within this interval of α, which implies that the infimum exists as $\|t\|$ approaches 0. Moreover, for $\alpha \in (0, 1/n)$, strict convexity establishes that ψ possesses a singular critical point within the domain $[0, 1]$ at $\|t\| = \frac{(n+1)a}{a+1}$. Upon evaluating the boundary points and the critical value, it becomes apparent that the infimum of ψ is $\Big(\frac{n}{n+1}\Big)^n (a+1)^{n-1}(1-\alpha n)$, for all α in the interval $(0, 1/n)$. In instances where α lies within the range $[1/n, n)$, there are specific convex bodies represented as B that exhibit the property that $B \cap H_\alpha^+ = \emptyset$ (refer to (3) to construct easy examples), therefore the infimum of ψ on this interval is 0. Now for the to the second case when

$$\frac{(\alpha+1)\Big(\|t\|(n-1)+1\Big)}{n+1} \le \|t\| \le 1,$$

we observe that ψ is strictly concave for all $\|t\| \in [0, 1]$. As a result, the infimum of ψ can only be achieved at its boundary points, namely when $\|t\| = \frac{(\alpha+1)\Big(\|t\|(n-1)+1\Big)}{n+1}$ and in the limit when $\|t\|$ approaches 1. After a brief calculation, we find that either of these values are not relevant within any interval of α. As a result, we can disregard the second case and deduce that our proper lower bounds in $C_1(\alpha)$ are established as presented in (7).

Lastly, we address the equality cases for our lower bound. When $\alpha \in (-1, 0]$, the infimum is attained in the limit when $\|t\|$ approaches 0, leading to a scenario where our body transforms into a cone with its base positioned in the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 0\}$ and its vertex at e_1. As α increases from 0 towards $1/n$, the

parameter $\|t\|$ ranges from 0 to 1. Thus, the base shifts from the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 0\}$ to the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 1\}$, while the vertices remain fixed at 0 and e_1. To observe this evolution of shapes, refer to Figure 4 which illustrates the changes as α varies from -1 to n. $\qquad\square$
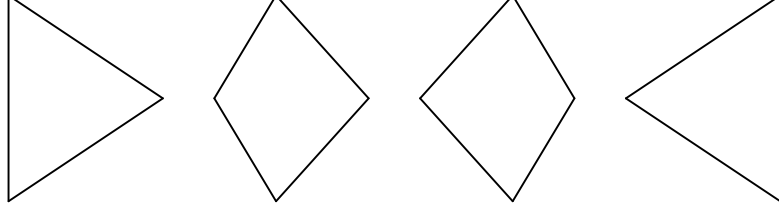

Figure 4: Extreme shapes for the lower bound

Moving forward, we delve into an immediate application of Theorem 3. Specifically, we focus on the comparison between the $(n-1)$-dimensional volume of maximal section(s) of a convex body and the $(n-1)$-dimensional volume of a section located at a certain distance from its centroid. This extension builds upon the work done by Fradelizi in [2].

Theorem 4. *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Let $\xi \in S^{n-1}$ and let $-1 < \alpha < n$. Consider the hyperplane:*

$$H = \xi^\perp + \alpha h_K(-\xi)\xi.$$

Then

$$\mathrm{vol}_{n-1}\big(K \cap H\big) \geq c(\alpha) \sup_{t \in \mathbb{R}} \mathrm{vol}_{n-1}\Big(K \cap \big(\xi^\perp + t\xi\big)\Big),$$

where

$$c(\alpha) = \begin{cases} \left(\frac{n(a+1)}{n+1}\right)^{n-1} & \alpha \in (-1, 0], \\ \left(\left(\frac{n}{n+1}\right)^n (a+1)^{n-1}(1 - \alpha n)\right)^{\frac{n-1}{n}} & \alpha \in (0, 1/n), \\ 0 & \alpha \in [1/n, n). \end{cases} \qquad (12)$$

The bounds in (12) are sharp. The equality cases are discussed in the proof below.

Proof. To prove the theorem, we consider two cases:

1. $\mathrm{vol}_{n-1}\big(K \cap H\big) = \sup_{t \in \mathbb{R}} \mathrm{vol}_{n-1}\Big(K \cap \big(\xi^\perp + t\xi\big)\Big),$

2. $\mathrm{vol}_{n-1}\big(K \cap H\big) \neq \sup_{t \in \mathbb{R}} \mathrm{vol}_{n-1}\Big(K \cap \big(\xi^\perp + t\xi\big)\Big).$

The first case is straightforward, yielding $c(\alpha) = 1$. Shifting our attention to the second case, we can assume without loss of generality that:

$$\mathrm{vol}_{n-1}\big(K \cap H\big) < \sup_{t \in \mathbb{R}} \mathrm{vol}_{n-1}\Big(K \cap \big(\xi^\perp + t\xi\big)\Big).$$

13

Let t_o denote the argument that maximizes the volume of sections of K along ξ. To proceed, we apply the Schwartz symmetrization along the direction ξ to the body K, yielding a new body denoted as $\bar{K}$. This transformation can be carried out without affecting the volume of any such section. For convenience, we will use the notation K to refer to the Schwartz symmetral body $\bar{K}$ as well. Now there exists a $\gamma \in \mathrm{span}\{\xi\}$ such that the following conditions are satisfied:

- $\gamma \notin \mathrm{conv}\Big(K \cap H, K \cap \big(\xi^\perp + t_o\xi\big)\Big),$

- $\mathrm{conv}\Big(K \cap H, K \cap \big(\xi^\perp + t_o\xi\big), \gamma\Big) = \mathrm{conv}\Big(K \cap \big(\xi^\perp + t_o\xi\big), \gamma\Big).$

Under these given conditions, the value of γ is uniquely determined. In general, γ resides in $\mathbb{R}^n \setminus \mathrm{int} K$. Additionally, γ lies on the boundary ∂K if and only if $K \cap H^*$ takes the form of a cone, where H^* represents the half-space defined by H that excludes $K \cap \big(\xi^\perp + t_o\xi\big)$. As a result of this construction, two cones are produced:

$$C = \mathrm{conv}\Big(K \cap \big(\xi^\perp + t_o\xi\big), \gamma\Big) \text{ and } G = \mathrm{conv}\big(K \cap H, \gamma\big).$$

For a geometrical description of the construction conditions, refer to Figure 5.
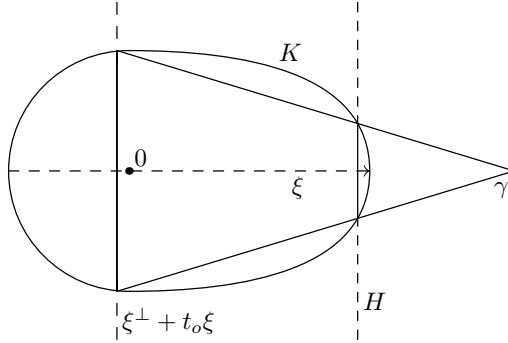

Figure 5: Constructing γ

Let's consider the two cases separately: when $H^* = H^+$ and when $H^* = H^-$. In the case where $H^* = H^+$, the following inequalities hold:

$$
\begin{aligned}
\mathrm{vol}_{n-1}&\big(K \cap H\big) \\
&= \frac{\mathrm{vol}_n(G)n}{\|\gamma - \alpha h_K(-\xi)\xi\|} \\
&\geq \frac{\mathrm{vol}_n\big(K \cap H^+\big)n}{\|\gamma - \alpha h_K(-\xi)\xi\|}.
\end{aligned}
$$

Then, by Theorem 3, we can leverage the bounds to produce the following inequalities:

$$\geq C_1(\alpha)\frac{\text{vol}_n(K)n}{\|\gamma - \alpha h_K(-\xi)\xi\|}$$

$$\geq \left(\frac{C_1(\alpha)}{1 - C_1(\alpha)}\right)\frac{\text{vol}_n\left(K \cap H^-\right)n}{\|\gamma - \alpha h_K(-\xi)\xi\|}$$

$$\geq \left(\frac{C_1(\alpha)}{1 - C_1(\alpha)}\right)\frac{\text{vol}_n(C \setminus G)n}{\|\gamma - \alpha h_K(-\xi)\xi\|}.$$

Returning to the context of $(n-1)$-dimensional section volumes:

$$= \left(\frac{C_1(\alpha)}{1 - C_1(\alpha)}\right)\frac{\text{vol}_{n-1}\left(K \cap \left(\xi^\perp + t_o\xi\right)\right)\|\gamma - t_o\xi\|}{\|\gamma - \alpha h_K(-\xi)\xi\|} - \left(\frac{C_1(\alpha)}{1 - C_1(\alpha)}\right)\text{vol}_{n-1}\left(K \cap H\right).$$

Then, because C and G are homothetic, we arrive at

$$\text{vol}_{n-1}\left(K \cap H\right) \geq \left(\frac{\left(\frac{C_1(\alpha)}{1 - C_1(\alpha)}\right)}{1 + \left(\frac{C_1(\alpha)}{1 - C_1(\alpha)}\right)}\right)\text{vol}_{n-1}\left(K \cap \left(\xi^\perp + t_o\xi\right)\right)\left(\frac{\text{vol}_{n-1}\left(K \cap \left(\xi^\perp + t_o\xi\right)\right)}{\text{vol}_{n-1}\left(K \cap H\right)}\right)^{\frac{1}{n-1}}$$

$$= C_1(\alpha)\text{vol}_{n-1}\left(K \cap \left(\xi^\perp + t_o\xi\right)\right)\left(\frac{\text{vol}_{n-1}\left(K \cap \left(\xi^\perp + t_o\xi\right)\right)}{\text{vol}_{n-1}\left(K \cap H\right)}\right)^{\frac{1}{n-1}},$$

which ultimately implies

$$\text{vol}_{n-1}\left(K \cap H\right) \geq C_1(\alpha)^{\frac{n-1}{n}}\text{vol}_{n-1}\left(K \cap \left(\xi^\perp + t_o\xi\right)\right). \tag{13}$$

Now, suppose $H^* = H^-$. Then the following inequalities hold:

$$\text{vol}_{n-1}\left(K \cap H\right)$$

$$= \frac{\text{vol}_n(G)n}{\|\gamma - \alpha h_K(-\xi)\xi\|}$$

$$\geq \frac{\text{vol}_n\left(K \cap H^-\right)n}{\|\gamma - \alpha h_K(-\xi)\xi\|}.$$

Thus, by Theorem 3, the subsequent inequalities hold:

$$\geq \left(1 - C_2(\alpha)\right)\frac{\text{vol}_n(K)n}{\|\gamma - \alpha h_K(-\xi)\xi\|}$$

$$\geq \left(\frac{1 - C_2(\alpha)}{C_2(\alpha)}\right)\frac{\text{vol}_n\left(K \cap H^+\right)n}{\|\gamma - \alpha h_K(-\xi)\xi\|}$$

$$\geq \left(\frac{1 - C_2(\alpha)}{C_2(\alpha)}\right)\frac{\text{vol}_n(C \setminus G)n}{\|\gamma - \alpha h_K(-\xi)\xi\|},$$

Translating back to $(n-1)$-dimensional volumes of sections:

$$= \left(\frac{1 - C_2(\alpha)}{C_2(\alpha)}\right)\frac{\text{vol}_{n-1}\left(K \cap \left(\xi^\perp + t_o\xi\right)\right)\|\gamma - t_o\xi\|}{\|\gamma - \alpha h_K(-\xi)\xi\|} - \left(\frac{1 - C_2(\alpha)}{C_2(\alpha)}\right)\text{vol}_{n-1}\left(K \cap H\right).$$

15

As a result, because C and G are homothetic, we arrive at

$$\text{vol}_{n-1}\big(K\cap H\big) \geq \left(\frac{\left(\frac{1-C_2(\alpha)}{C_2(\alpha)}\right)}{1+\left(\frac{1-C_2(\alpha)}{C_2(\alpha)}\right)}\right)\text{vol}_{n-1}\Big(K\cap\big(\xi^\perp+t_o\xi\big)\Big)\left(\frac{\text{vol}_{n-1}\Big(K\cap\big(\xi^\perp+t_o\xi\big)\Big)}{\text{vol}_{n-1}\big(K\cap H\big)}\right)^{\frac{1}{n-1}}$$

$$= \Big(1-C_2(\alpha)\Big)\text{vol}_{n-1}\Big(K\cap\big(\xi^\perp+t_o\xi\big)\Big)\left(\frac{\text{vol}_{n-1}\Big(K\cap\big(\xi^\perp+t_o\xi\big)\Big)}{\text{vol}_{n-1}\big(K\cap H\big)}\right)^{\frac{1}{n-1}},$$

which ultimately implies

$$\text{vol}_{n-1}\big(K\cap H\big) \geq \Big(1-C_2(\alpha)\Big)^{\frac{n-1}{n}}\text{vol}_{n-1}\Big(K\cap\big(\xi^\perp+t_o\xi\big)\Big). \qquad (14)$$

For all values of α, it holds that $c(\alpha) \leq 1$, which renders the straightforward case irrelevant. Upon comparing the two constants seen in (13) and (14), we find that for $\alpha \in (-1,0]$:

$$\Big(1-C_2(\alpha)\Big)^{\frac{n-1}{n}} = \left(\frac{n(a+1)}{n+1}\right)^{n-1} \leq C_1(\alpha)^{\frac{n-1}{n}} = \left(\frac{n-a}{n+1}\right)^{n-1}.$$

Unfortunately, a direct comparison of our constants is not possible for $\alpha \in (0,n)$. Nonetheless, given that $C_2(\alpha) \leq 1$, we can deduce that for $\alpha \in [1/n,n)$:

$$C_1(\alpha)^{\frac{n-1}{n}} = 0 \leq \Big(1-C_2(\alpha)\Big)^{\frac{n-1}{n}}.$$

Hence, we only need to show:

$$C_1(\alpha)^{\frac{n-1}{n}} \leq \Big(1-C_2(\alpha)\Big)^{\frac{n-1}{n}},$$

for all $\alpha \in (0,1/n)$. This conclusion is established because for $\alpha \in (0,1/n)$ we have:

$$C_2(\alpha) \leq 1 - \left(\frac{n}{n+1}\right)^n = C_2(0).$$

Consequently, it directly follows that:

$$C_1(\alpha)^{\frac{n-1}{n}} \leq \left(\frac{n}{n+1}\right)^{n-1} \leq \Big(1-C_2(\alpha)\Big)^{\frac{n-1}{n}}.$$

As a result, we have now shown the bounds as depicted by $c(\alpha)$ as seen in (12). For all $\alpha \in (-1,n)$, equality within the bounds is precisely achieved when K is any convex body that adheres to the equality cases of the lower bound in Theorem 3. This implies that the maximal section is the one that serves as the base of the equality bodies. For a geometrical example of the equality case, refer to Figure 6. $\qquad\square$
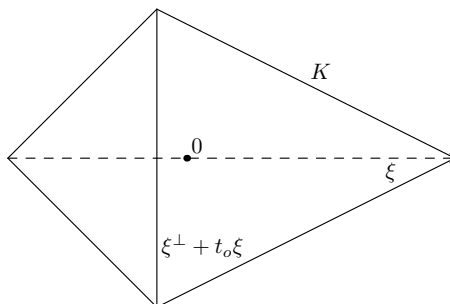
Figure 6: An example of the equality case in Theorem 4

References

[1] T. Bonnesen and W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, ID, 1987.

[2] M. Fradelizi, *Sections of convex bodies through their centroid*, Proc. Arch. Math. **69** (1997), 515-522.

[3] M. Fradelizi, M. Meyer, and V. Yaskin, *On the volume of sections of a convex body by cones*, Proc. Amer. Math. Soc. 145 (2017), 3153–3164.

[4] R. J. Gardner, *Geometric Tomography*, second edition, Cambridge University Press, New York, 2006.

[5] B. Grünbaum, *Partitions of mass-distributions and of convex bodies by hyperplanes*, Pacific J. Math. (1960), no. 10, 1257–1261.

[6] M. Meyer, F. Nazarov, D. Ryabogin, *Grünbaum-type inequality for log-concave functions*, Bull. Lond. Math. Soc. 50 (2018), no. 4, 745–752.

[7] S. Myroshnychenko, M. Stephen, and N. Zhang, *Grünbaum's inequality for sections*, J. Funct. Anal. **275** (2018), no. 9, 2516–2537.

[8] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, second expanded edition. Encyclopedia of Mathematics and its Applications, **151**. Cambridge University Press, Cambridge, 2014.

[9] A. Shyntar and V. Yaskin, *A generalization of Winternitz's theorem and its discrete version*, Proc. Amer. Math. Soc. **149** (2021), no. 7, 3089-3104.

[10] M. Stephen and N. Zhang, *Grünbaum's inequality for projections*, J. Funct. Anal. **272** (2017), no. 6, 2628–2640.

[11] R. Webster, *Convexity*, Oxford University Press, Oxford, 1994.

Brayden Letwin, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada
E-mail address: bletwin@ualberta.ca

Yladyslav Yaskin, Department of Mathematical & Statistical Sciences, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada
E-mail address: yaskin@ualberta.ca