ON A GENERALIZATION OF GRÜNBAUM'S INEQUALITY AND A VOLUME BOUND FOR SECTIONS OF CONVEX BODIES

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Abstract

For any convex body K with centroid at the origin and $\xi \in S^{n-1}$, Grünbaum's inequality states that

$$\left(\frac{n}{n+1}\right)^n \leq \frac{\mathrm{vol}_n(K \cap \xi^+)}{\mathrm{vol}_n(K)} \leq 1 - \left(\frac{n}{n+1}\right)^n,$$

We generalize this, establishing bounding constants for hyperplanes that do not necessarily pass through the centroid. As a result, we obtain an inequality comparing volumes of maximal and centroid-distanced sections.

1. INTRODUCTION

Consider a convex body $K \subseteq \mathbb{R}^n$. The centroid of K is defined as the point

$$g(K) = \frac{1}{\mathrm{vol}_n(K)} \int_K x dx,$$

where $\mathrm{vol}_n(K)$ signifies the n-dimensional volume of K, and the integration is performed with respect to the Lebesgue measure. An established result by Grünbaum [5] states that for all convex bodies $K \subseteq \mathbb{R}^n$ with the centroid at the origin and for any $\xi \in S^{n-1}$ we have that

$$\left(\frac{n}{n+1}\right)^n \leq \frac{\mathrm{vol}_n(K \cap \xi^+)}{\mathrm{vol}_n(K)} \leq 1 - \left(\frac{n}{n+1}\right)^n, \tag{1}$$

where ξ^+ represents the positive halfspace given by the hyperplane orthogonal to ξ. These bounds, as encapsulated by (1), are sharp. The equalities occur when K is a cone with base lying in and centered at the origin of an (affine) hyperplane orthogonal to ξ.

Recent advancements in Grünbaum-type inequalities, encompassing sections and projections of convex bodies, can be explored in the following papers: [3], [6], [7], [10].

Suppose instead we define our positive halfspace by an orthogonal translation of $\xi^\perp$. Can an analogous inequality to (1), complete with sharp bounds, be established? This paper offers an affirmative answer to this inquiry. Let $K \subseteq \mathbb{R}^n$ be a convex body with its centroid located at the origin. Let $-1 < \alpha < n$ and let $\xi \in S^{n-1}$. Consider the (affine) hyperplane

$$H_\alpha = \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle = \alpha h_K(-\xi)\},$$

where h_K is the support function for K. Then there are sharp constants $C_1(\alpha)$ and $C_2(\alpha)$ such that

$$C_1(\alpha) \leq \frac{\mathrm{vol}_n(K \cap H_\alpha^+)}{\mathrm{vol}_n(K)} \leq C_2(\alpha). \tag{2}$$

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The bounds in (2) were established by preceding work in $\mathbb{R}^2$ (refer to [9]). In this paper, we will adopt a similar methodology within the context of $\mathbb{R}^n$. The exact values of $C_1(\alpha)$ and $C_2(\alpha)$ are clarified in Theorem 4, which also presents scenarios where equality is observed. The special case where $\alpha \in (-1, 0]$ is also covered in Theorem 3, where a simple proof is given. It is noteworthy that within (1), the demonstration of just one inequality is sufficient, as the other immediately follows suit. Conversely, within (2), both inequalities necessitate separate validation.

The subsequent section of this paper delves into a targeted application of Theorem 4. It seeks to address the following inquiry: How can we compare the volumes of sections within a convex body to the volume of its maximal section(s)? This problem is rigorously formulated and resolved in Theorem 5.

2. Preliminaries

First, we will discuss some preliminaries that we will use throughout the paper.

Definition 1. The support function $h_K : S^{n-1} \to \mathbb{R}$ for a convex body K is

$$h_K(\xi) = \sup\{\langle \xi, x\rangle \mid x \in K\}.$$

From a geometric standpoint, when a convex body's centroid is situated at the origin, its support function signifies the distance from the centroid to the supporting hyperplane in the direction of ξ. A well-established result attributed to Minkowski and Radon [1, p. 58] states that, for any convex body $K \subseteq \mathbb{R}^n$ with its centroid at the origin, and for all $\xi \in S^{n-1}$, we have that

$$\frac{1}{n} h_K(\xi) \leq h_K(-\xi) \leq n h_K(\xi). \tag{3}$$

Take note that the choice of bounds for α in Theorem 3 - 5 are a result of (3).

Definition 2. Let $\xi \in S^{n-1}$. The parallel section function $A_{K,\xi} : \mathbb{R} \to \mathbb{R}$ for a convex body $K \subseteq \mathbb{R}^n$ is

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}\Big(K \cap (\xi^\perp + t\xi)\Big).$$

If $K \cap (\xi^\perp + t\xi)$ forms an $(n-1)$-dimensional Euclidean ball, then $A_{K,\xi}^{\frac{1}{n-1}}(t)$ provides the radius of this Euclidean ball, up to a constant.

Definition 3. Let $\xi \in S^{n-1}$. The halfspace intersection function $A^*_{K,\xi} : \mathbb{R} \to \mathbb{R}$ for a convex body $K \subseteq \mathbb{R}^n$ is

$$A^*_{K,\xi}(t) = \int_t^\infty \mathrm{vol}_{n-1}\Big(K \cap (\xi^\perp + x\xi)\Big) dx.$$

Lemma 1. *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Then the parallel section function $A_{K,\xi}$ is $\frac{1}{n-1}$ concave on its support.*

Proof. See [11, p. 299] $\square$

Lemma 2. *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Then the halfspace intersection function $A^*_{K,\xi}$ is $\frac{1}{n}$ concave on its support.*

Proof. Let $\lambda \in [0,1]$ and $x, y \in \operatorname{supp}\left({A^*_{K,\xi}}^{\frac{1}{n}}\right)$. The trick is to observe that

$$\begin{aligned}&\lambda\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq x\}\Big) + (1-\lambda)\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq y\}\Big)\\ &\subseteq \Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq \lambda x + (1-\lambda)y\}\Big).\end{aligned}$$

Then this implies that

$$\begin{aligned}&\mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq \lambda x + (1-\lambda)y\}\Big)^{\frac{1}{n}}\\ &\geq \mathrm{vol}_n\Big(\lambda\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq x\}\Big) + (1-\lambda)\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq y\}\Big)\Big)^{\frac{1}{n}},\end{aligned}$$

and then by applying the Brunn-Minkowski inequality

$$\geq \lambda \mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq x\}\Big)^{\frac{1}{n}} + (1-\lambda)\mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq y\}\Big)^{\frac{1}{n}}.$$

Finally, applying Fubini's theorem yields

$$\begin{aligned}&\int_{\lambda x+(1-\lambda)y}^{\infty} \mathrm{vol}_{n-1}\Big(K \cap (\xi^\perp + t\xi)\Big)dt\\ &\geq \lambda \int_x^\infty \mathrm{vol}_{n-1}\Big(K \cap (\xi^\perp + t\xi)\Big)dt + (1-\lambda)\int_y^\infty \mathrm{vol}_{n-1}\Big(K \cap (\xi^\perp + t\xi)\Big)dt.\end{aligned}$$

Which proves the result. □

Definition 4. The Schwartz symmetral of a convex body $K \subseteq \mathbb{R}^n$ along the direction ξ is the convex body $\bar{K}$, which is constructed as follows: Over the interval $-h_K(-\xi) \leq t \leq h_K(\xi)$, $\bar{K} \cap \left(\xi^\perp + t\xi\right)$ forms an $(n-1)$-dimensional Euclidean ball positioned within $\xi^\perp$ and centered at its origin with

$$\mathrm{vol}_{n-1}\Big(K \cap (\xi^\perp + t\xi)\Big) = \mathrm{vol}_{n-1}\Big(\bar{K} \cap (\xi^\perp + t\xi)\Big). \tag{4}$$

As a result of this transformation, it becomes evident that

$$h_{\bar{K}}(\pm\xi) = h_K(\pm\xi), \tag{5}$$

and

$$\mathrm{vol}_n\Big(\bar{K} \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq t\}\Big) = \mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq t\}\Big), \tag{6}$$

for all $t \in \mathbb{R}$. The centroid of $\bar{K}$ resides on $\mathrm{span}\{\xi\}$ due to the rotational symmetry of $\bar{K}$ around ξ.

Now, in the forthcoming central theorems of the paper, we establish bounding constants for volume ratios between hyperplane subdivisions of a convex body K and K itself, where these subdivisions are defined by centroid-based distances. First, we will consider the case where $\alpha \in (-1, 0]$. This case gives rise to a simple proof that we will cover first in Theorem 3. We will cover the general case in Theorem 4.

Theorem 3. *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Let $-1 < \alpha \leq 0$ and let $\xi \in S^{n-1}$. Consider the (affine) hyperplane:*

$$H_\alpha = \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle = \alpha h_K(-\xi)\}.$$

Then

$$\left(\frac{n-\alpha}{n+1}\right)^n \leq \frac{\mathrm{vol}_n(K \cap H_\alpha^+)}{\mathrm{vol}_n(K)} \leq 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n.$$

The lower bounds and upper bounds are sharp. The equality cases are discussed in the proof below.

Proof. We will prove the upper bound first. Observe that

$$\begin{aligned} &\mathrm{vol}_n(K \cap H_\alpha^+) \\ &= \mathrm{vol}_n(K) - \mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \leq \alpha h_K(-\xi)\}\Big). \end{aligned}$$

Now by dilating $K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \leq 0\}$ by $\frac{1}{\alpha+1}$ we see that the following inequalities hold

$$\begin{aligned} &\mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \leq \alpha h_K(-\xi)\}\Big) \\ &\geq (\alpha+1)^n \mathrm{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \leq 0\}\Big) \\ &\geq (\alpha+1)^n \left(\frac{n}{n+1}\right)^n \mathrm{vol}_n(K), \end{aligned}$$

which implies for $\alpha \in (-1, 0]$ that

$$\frac{\mathrm{vol}_n(K \cap H_\alpha^+)}{\mathrm{vol}_n(K)} \leq 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n.$$

Now we will prove the lower bound. By the $\frac{1}{n}$ concavity of $A^*_{K,\xi}$ on its support we have

$$\begin{aligned} &\mathrm{vol}_n(K \cap H_\alpha^+)^{\frac{1}{n}} \\ &= {A^*_{K,\xi}}^{\frac{1}{n}}(\alpha h_K(-\xi)) \\ &\geq -\alpha {A^*_{K,\xi}}^{\frac{1}{n}}(-h_K(-\xi)) + (1+\alpha){A^*_{K,\xi}}^{\frac{1}{n}}(0). \end{aligned}$$

Then by Grünbaum's inequality

$$\geq -\alpha \mathrm{vol}_n(K)^{\frac{1}{n}} + (1+\alpha)\left(\frac{n}{n+1}\right)\mathrm{vol}_n(K)^{\frac{1}{n}},$$

which implies for $\alpha \in (-1, 0]$ that

$$\left(\frac{n-a}{n+1}\right)^n \leq \frac{\mathrm{vol}_n(K \cap H_\alpha^+)}{\mathrm{vol}_n(K)}.$$

Thus we now have our bounds for $\alpha \in (-1, 0]$. One can then see the equality cases hold precisely when K is a cone with its base lying in and centered at the origin of $\xi^\perp$ and vertex at $-h_K(\xi)\xi$. $\square$

Now we will consider the general case in Theorem 4. We will again cover the case where $\alpha \in (-1, 0]$ in Theorem 4, but expand to the case where $\alpha \in (0, n)$.

Theorem 4. *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Let $-1 < \alpha < n$ and let $\xi \in S^{n-1}$. Consider the (affine) hyperplane:*

$$H_\alpha = \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle = \alpha h_K(-\xi)\}.$$

Then

$$C_1(\alpha) \leq \frac{\mathrm{vol}_n(K \cap H_\alpha^+)}{\mathrm{vol}_n(K)} \leq C_2(\alpha),$$

where

$$C_1(\alpha) = \begin{cases} \left(\frac{n-\alpha}{n+1}\right)^n & \alpha \in (-1, 0], \\ \left(\frac{n}{n+1}\right)^n (a+1)^{n-1}(1-\alpha n) & \alpha \in (0, 1/n), \\ 0 & \alpha \in [1/n, n), \end{cases} \tag{7}$$

and

$$C_2(\alpha) = \begin{cases} 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n & \alpha \in (-1, 0], \\ \tau_{\alpha,n} & \alpha \in (0, n). \end{cases} \tag{8}$$

The value of $\tau_{\alpha,n}$ is not explicit for general dimension because finding the value involves finding the roots of a high degree rational function. The lower bounds and upper bounds are sharp. The equality cases are discussed in the proof below.

Proof. We will prove the upper bound first. We can assume, without loss of generality, after applying a suitable linear transformation $T \in GL_n$ that $\xi = e_1$, and subsequently focus on the Schwartz symmetral of the convex body K along the direction e_1, because of the conditions in (4) - (6). Following this, we will perform a translation of K to create a new convex body $\hat{K} = K + h_K(-e_1)e_1$. It's important to note that $g(\hat{K}) = h_K(-e_1)e_1$, and further observe that

$$H_\alpha = \{x \in \mathbb{R}^n \mid \langle e_1, x\rangle = (\alpha + 1) \langle e_1, g(\hat{K})\rangle\}.$$

Once again, with a slight departure from strict notation, we will denote the translated body as K. To establish the upper bound, our approach involves constructing an $(n-1)$-dimensional Euclidean ball B_{n-1} situated within and centered at the origin of $e_1^\perp$, satisfying the following conditions:

- $\mathrm{vol}_{n-1}(K \cap e_1^\perp) \leq \mathrm{vol}_{n-1}(B_{n-1})$,
- $\mathrm{vol}_n\Big(\mathrm{conv}\,(B_{n-1}, K \cap H_\alpha)\Big) = \mathrm{vol}_n\,(K \cap H_\alpha^-)$.

Denote $\mathrm{conv}(B_{n-1}, K \cap H_\alpha)$ by K^-. Now, there exists a unique μ in $\mathrm{span}\{e_1\}$, enabling the construction of another $(n-1)$-dimensional Euclidean ball D_{n-1} lying in and centered at the origin of $\mu^\perp + \mu$ so that the following conditions hold:

- $\alpha < \langle e_1, \mu \rangle < h_K(e_1)$,
- $\operatorname{vol}_{n-1}\Big(K \cap (\mu^\perp + \mu)\Big) \leq \operatorname{vol}_{n-1}(D_{n-1})$,
- $\operatorname{vol}_n\Big(\operatorname{conv}(D_{n-1}, K \cap H_\alpha)\Big)$
  $= \operatorname{vol}_n\Big(K \cap H_\alpha^+ \cap \{x \in \mathbb{R}^n \mid \langle e_1, x \rangle = h_K(e_1)\}\Big)$,
- $\operatorname{conv}(D_{n-1}, K \cap H_\alpha) \cup K^- = \operatorname{conv}(B_{n-1}, D_{n-1})$.

To gain a geometric insight into the construction conditions shown above, refer to Figure 1.

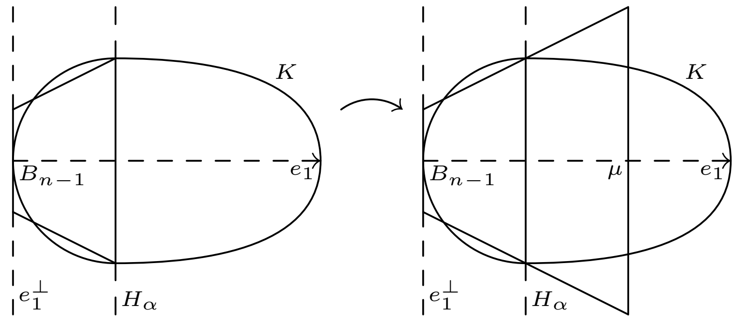

Figure 1: Constructing B_{n-1} and D_{n-1}

Denote $\operatorname{conv}(D_{n-1}, K \cap H_\alpha)$ by K^+. By virtue of the conditions above, it becomes evident that

$$\operatorname{vol}_n(K^- \cup K^+) = \operatorname{vol}_n(K).$$

Additionally, the conditions above and an application of Lemma 1 yields the following inequality:

$$\langle e_1, g(K^- \cup K^+) \rangle \leq \langle e_1, g(K) \rangle.$$

This is because the concavity of the radius function r_K on its support implies that r_K can intersect any affine function on its support at most twice. r_{K^-} is affine on its support, and therefore, by the conditions it is ensured that r_K intersects r_{K^-} exactly twice on its support. Similarly, r_{K^+} is affine on its support, so the conditions above establish that both K and $K^- \cup K^+$ are bodies of revolution about $\operatorname{span}\{e_1\}$ with equal volumes. Therefore, it suffices to show that

$$\int_0^{h_K(e_1)} x r_{K^- \cup K^+}(x) dx \leq \int_0^{h_K(e_1)} x r_K(x) dx,$$

which is clear.

As a consequence, we have successfully constructed a convex body $F = \operatorname{conv}(B_{n-1}, D_{n-1})$, which satisfies the following conditions:

$$\operatorname{vol}_n(K) = \operatorname{vol}_n(F),$$

and

$$\begin{aligned}
&\operatorname{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle e_1, g(K)\rangle\}\Big) \\
&\leq \operatorname{vol}_n\Big(F \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle e_1, g(F)\rangle\}\Big).
\end{aligned}$$

Hence, our objective narrows down to establishing a maximum volume ratio within the scope of such constructed convex bodies. Without loss of generality, we can perform a suitable linear transformation $T \in GL_n$, and assume that $\mu = 1$. Additionally, for the interval $0 \leq t \leq 1$, we can assume that

$$F \cap (e_1^\perp + te_1) = (mt+b)(F \cap e_1^\perp), \tag{9}$$

where $b \geq 0$ and either (1) $m \geq 0$, or (2) $m < 0$ and $m + b \geq 0$. For now, our focus will be on the scenario where $m \neq 0$, while we will address the case of $m = 0$ later. Then by Fubini's theorem and (9):

$$\begin{aligned}
\operatorname{vol}_n(F) &= \int_F dx \\
&= \int_{x_1=0}^{1} \operatorname{vol}_{n-1}\Big(F \cap (e_1^\perp + x_1 e_1)\Big) dx_1 \\
&= \operatorname{vol}_{n-1}(F \cap e_1^\perp)\frac{(b+m)^n - b^n}{mn}.
\end{aligned}$$

Additionally, determining the x_1 coordinate of the centroid of F can be accomplished through a straightforward application of Fubini's theorem and (9):

$$\begin{aligned}
\langle e_1, g(F)\rangle &= \frac{1}{\operatorname{vol}_n F}\int_F x_1 dx \\
&= \frac{1}{\operatorname{vol}_n F}\int_{x_1=0}^{1} x_1 \operatorname{vol}_{n-1}\Big(F \cap (e_1^\perp + te_1)\Big) dx_1 \\
&= \frac{b^{n+1} + (mn-b)(b+m)^n}{m(n+1)\Big((b+m)^n - b^n\Big)}.
\end{aligned}$$

Given that F takes the form of a body of revolution around the x_1 axis, it naturally ensures that all remaining coordinates of the centroid of F are equal to zero.

Now we can compute

$$\begin{aligned}
\frac{\operatorname{vol}_n(F \cap H_\alpha^+)}{\operatorname{vol}_n(F)} &= \frac{1}{\operatorname{vol}_n(F)}\int_{F \cap H_\alpha^+} dx \\
&= \frac{(b+m)^n - \left(b + m(\alpha+1)\left(\frac{b^{n+1}+(mn-b)(b+m)^n}{m(n+1)\left((b+m)^n - b^n\right)}\right)\right)^n}{(b+m)^n - b^n}.
\end{aligned}$$

Introducing the variable $z = \frac{b}{m}$, we arrive at:

$$\langle e_1, g(F)\rangle = (\alpha+1)\frac{z\Big(1+(\frac{n}{z}-1)(\frac{1}{z}+1)^n\Big)}{(n+1)\Big((\frac{1}{z}+1)^n-1\Big)},$$

and

$$\frac{\mathrm{vol}_n(F\cap H_\alpha^+)}{\mathrm{vol}_n(F)} = \frac{(1+\frac{1}{z})^n - \left(1+\frac{1}{z}(\alpha+1)\left(\frac{z\left(1+(\frac{1}{z}-1)(\frac{1}{z}+1)^n\right)}{(n+1)\left((\frac{1}{z}+1)^n-1\right)}\right)\right)^n}{(1+\frac{1}{z})^n-1}.$$

Letting ϕ denote the latter function of z, our goal is to determine the supremum of ϕ within the domain $z \in (-\infty,-1]\cup[0,\infty)$. For values of $\alpha \in (-1,0]$, ϕ exhibits strict decreasing behavior as a function of z across its entire domain. Therefore, the supremum of ϕ is attained either as z approaches $-\infty$ or when $z=0$. Then it follows that

$$\lim_{z\to-\infty}\phi(z) = \frac{1-\alpha}{2} \le \phi(0) = 1 - \left(\frac{(\alpha+1)\,n}{n+1}\right)^n.$$

However, for $\alpha \in (0,n)$, the function ϕ lacks the characteristics of monotone increase or decrease. Moreover, no stable intervals of concavity or convexity persist across all dimensions. Consequently, determining the upper bound $\tau_{\alpha,n}$ becomes a complex task, as solving for $\tau_{\alpha,n}$ involves handling high-degree rational functions.

Lastly, let's revisit the case where $m=0$. It becomes evident that our body F is a cylinder. Thus, we have that

$$\frac{\mathrm{vol}_n(F\cap H_\alpha^+)}{\mathrm{vol}_n F} = \frac{1-\alpha}{2}.$$

For $\alpha \in (-1,0]$ this ratio is smaller than than over our existing ratio and for $\alpha \in (0,n)$, it is clear that

$$\tau_{\alpha,n} \ge \frac{1-\alpha}{2}.$$

Thus, we now see that for any convex body K, we have the following bounds as seen in (8):

$$\frac{\mathrm{vol}_n(K\cap H_\alpha^+)}{\mathrm{vol}_n K} \le C_2(\alpha).$$

Now, let's delve into the discussion of equality cases. For $\alpha \in (-1,0]$, the configuration of our body K involves the convex hull of two $(n-1)$-dimensional Euclidean balls (which may yield a cone). The supremum of ϕ is realized at $z=0$, implying $b=0$, and thus the extreme shapes correspond to cones with a vertex at the origin and a base lying in the hyperplane $\{x\in\mathbb{R}^n \mid x_1 = 1\}$.

As we move on to $\alpha \in (0, n)$, the structure of our body K remains the convex hull of two $(n-1)$-dimensional Euclidean balls (which may lead to a cone). In the case of $\alpha = 1/n$, the body becomes a cylinder. However, in general, when $\alpha \in (0, n)$, we face difficulties in determining the precise values at which the supremum of ϕ is attained. Consequently, the equality bodies cannot be consistently determined in all cases. Refer to Figure 2 for an illustrative depiction of how shapes evolve as α transitions from -1 to n.

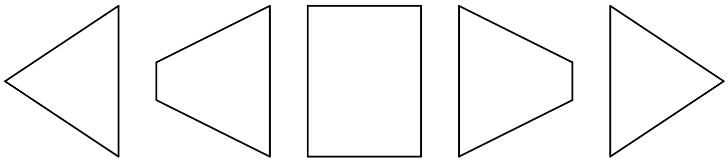

Figure 2: Extreme shapes for the upper bound

Let us now shift our focus to establishing the lower bound. Similarly to the upper bound, we can simplify the problem by applying a suitable linear transformation $T \in GL_n$. Without loss of generality we'll begin by assuming $\xi = e_1$ and apply the same initial transformations as we did to prove the upper bound. In other words, we will assume that K is in its Schwartz symmetral form about e_1 and that $h_K(-e_1) = 0$. Now, there exists a unique $\rho \in \operatorname{span}\{e_1\}$ such that the following conditions are met:

- $h_K(e_1) \le \langle e_1, \rho \rangle$,
- $\operatorname{vol}_n\Big(\operatorname{conv}(K \cap H_\alpha, \rho e_1)\Big) = \operatorname{vol}_n(K \cap H_\alpha^+)$.

Let's denote $\operatorname{conv}(K \cap H_\alpha, \rho e_1)$ as K^+. Now, there exists another unique $\beta \in \operatorname{span}\{e_1\}$ that allows us to construct a $(n-1)$-dimensional Euclidean ball B_{n-1}. This ball lies within and is centered at the origin of $\beta^\perp + \beta$, satisfying the subsequent conditions:

- $0 \le \langle e_1, \beta \rangle < \alpha$,
- $\operatorname{vol}_{n-1}\Big(K \cap (\beta^\perp + \beta)\Big) \le \operatorname{vol}_{n-1}(B_{n-1})$,
- $\operatorname{conv}(B_{n-1}, K^+) = \operatorname{conv}(B_{n-1}, \rho)$,
- $\operatorname{vol}_n\Big(\operatorname{conv}(0, B_{n-1}, \rho)\Big) = \operatorname{vol}_n(K)$.

For a geometrical description of the construction conditions as seen above, see Figure 3.

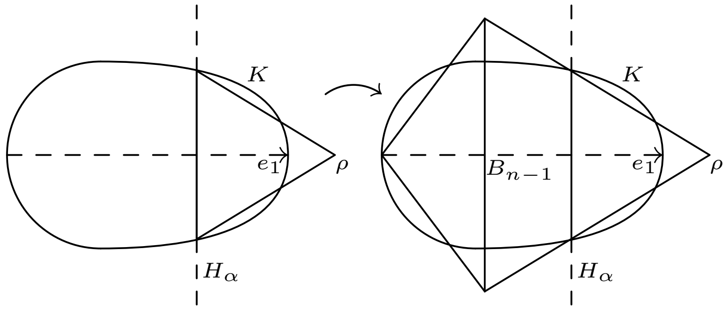

Figure 3: Constructing B_{n-1} and ρ

Denote $\text{conv}(0, B_{n-1}, \rho)$ as B and $\text{conv}(0, B_{n-1})$ as K^-. From the conditions above, we can deduce that B is formed by the union of two cones. These cones have vertices at 0 and ρ, and they share a common base located at β. Utilizing the conditions above and making use of Lemma 1, we obtain the following inequality:

$$\langle e_1, g(B)\rangle \geq \langle e_1, g(K)\rangle.$$

This stems from the fact that the concavity of the radius function r_K implies that r_K can intersect any affine function on its support at most twice. However, both r_{K^-} and r_{K^+} are affine on their support and intersect r_K. Therefore, by utilizing conditions above, we establish that K and B have the same volume and are bodies of revolution about $\text{span}\{e_1\}$. Thus, it is equivalent to show that

$$\int_0^\rho x r_B(x) dx \geq \int_0^\rho x r_K(x) dx,$$

which is evidently true.

As a result, we have constructed a convex body B such that:

$$\text{vol}_n(K) = \text{vol}_n(B),$$

and

$$\begin{aligned}&\text{vol}_n\Big(K \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle e_1, g(K)\rangle\}\Big)\\ &\geq \text{vol}_n\Big(B \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle e_1, g(B)\rangle\}\Big).\end{aligned}$$

Hence, our task reduces to demonstrating a minimum volume ratio within the class of such constructed convex bodies.

We can proceed without loss of generality by employing an appropriate linear transformation $T \in GL_n$. We then consider the scenario where $h_B(e_1) = 1$, and the base of B is a $(n-1)$-dimensional Euclidean ball B_{n-1} centered at $t \in [0, 1]$ with a unit radius. Define

$$C_1 = B \cap \{x \in \mathbb{R}^n \mid x_1 \leq t\},$$

and

$$C_2 = B \cap \{x \in \mathbb{R}^n \mid x_1 \geq t\}.$$

Then, by Fubini's theorem:

$$\begin{aligned}
\mathrm{vol}_n(B) &= \int_B dx \\
&= \int_{C_1} dx + \int_{C_2} dx \\
&= \int_{x_1=0}^{t} \mathrm{vol}_{n-1}(\frac{x_1}{t}B_{n-1})dx_1 + \int_{x_1=t}^{1} \mathrm{vol}_{n-1}((\frac{-x_1+1}{1-t})B_{n-1})dx_1 \\
&= \frac{\mathrm{vol}_{n-1}(B_{n-1})}{n}.
\end{aligned}$$

Next, we can compute the x_1 coordinate of the centroid of B. Employing Fubini's theorem, we can determine this value as follows:

$$\begin{aligned}
\langle e_1, g(B)\rangle &= \frac{1}{\mathrm{vol}_n(B)}\int_B x_1 dx \\
&= \frac{1}{\mathrm{vol}_n(B)}(\int_{C_1} dx + \int_{C_2} dx) \\
&= \frac{1}{\mathrm{vol}_n(B)}(\int_{x_1=0}^{t} x_1\mathrm{vol}_{n-1}(\frac{x_1}{t}B_{n-1})dx_1 + \int_{x_1=t}^{1} x_1\mathrm{vol}_{n-1}((\frac{-x_1+1}{1-t})B_{n-1})dx_1) \\
&= \frac{t(n-1)+1}{n+1}.
\end{aligned}$$

Once again, due to the rotational symmetry of B about e_1, it's evident that all other coordinates of the centroid are zero.

For our ratio computations, let's analyze the positive halfspace defined by the equation $x_1 \geq \frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}$. This analysis involves two distinct cases:

1. $0 \leq t \leq \frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}$,
2. $\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1} \leq t \leq 1$.

These two cases are respectively equivalent to

$$0 \leq t \leq \frac{\alpha+1}{2-(n-1)\alpha}$$

and

$$\frac{\alpha+1}{2-(n-1)\alpha} \leq t \leq 1.$$

In the first case, we have the following:

$$\begin{aligned}
\mathrm{vol}_n(B\cap H_\alpha^+) &= \int_{B\cap H_\alpha^+} dx \\
&= \int_{x_1=\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}}^{1} \mathrm{vol}_{n-1}\Big((\frac{-x_1+1}{1-t})B_{n-1}\Big)dx_1 \\
&= \mathrm{vol}_{n-1}(B_{n-1})\frac{\left(1-\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}\right)^n}{(1-t)^{n-1}n}.
\end{aligned}$$

Then it follows that

$$\frac{\mathrm{vol}_n(B\cap H_\alpha^+)}{\mathrm{vol}_n(B)}=\frac{\left(1-\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}\right)^n}{(1-t)^{n-1}}. \tag{10}$$

In the second case, we have:

$$\mathrm{vol}_n(B\cap H_\alpha^+)=\int_{B\cap H_\alpha^+}dx$$
$$=\int_{x_1=\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}}^{t}\mathrm{vol}_{n-1}\Big(\frac{x_1}{t}B_{n-1}\Big)dx_1+\int_{x_1=t}^{1}\mathrm{vol}_{n-1}\Big(\Big(\frac{-x_1+1}{1-t}\Big)B_{n-1}\Big)dx_1$$
$$=\mathrm{vol}_{n-1}(B_{n-1})\Big(\frac{t^n-\left(\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}\right)^n}{t^{n-1}n}+\frac{(1-t)}{n}\Big).$$

Then it follows that

$$\frac{\mathrm{vol}_n(B\cap H_\alpha^+)}{\mathrm{vol}_n(B)}=\frac{t^n-\left(\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}\right)^n}{t^{n-1}}+1-t. \tag{11}$$

Denoting the piecewise function of t in (10) and (11) as ψ, our objective is to determine the infimum of ψ over the interval on which it is defined. For the first case when

$$0\le t\le\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1},$$

The function ψ exhibits strict convexity across the entire interval $t\in[0,1]$. Therefore, it attains a sole unique infimum, which is $\left(\frac{n-\alpha}{n+1}\right)^n$, when $\alpha\in(-1,0]$. This is because ψ is strictly increasing within this interval of α, which implies that the infimum exists at $t=0$. Moreover, for $\alpha\in(0,1/n)$, strict convexity establishes that ψ possesses a singular critical point within the domain [0, 1] at $t=\frac{(n+1)a}{a+1}$. Upon evalutating the boundary points and the critical value, it becomes apparent that the infimum of ψ is $\left(\frac{n}{n+1}\right)^n(a+1)^{n-1}(1-\alpha n)$, for all α in the interval $(0,1/n)$. In instances where α lies within the range $[1/n,n)$, there are specific convex bodies represented as B that exhibit the property that $B\cap H_\alpha^+=\emptyset$ (refer to (3) to construct easy examples), therefore the infimum of ψ on this interval is 0. Now for the to the second case when

$$\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}\le t\le 1,$$

we observe that ψ is strictly concave for all $t\in[0,1]$. As a result, the infimum of ψ can only be achieved at its boundary points, namely when $t=\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}$ and when $t=1$. After a brief calculation, we find that either of these values are not relevant within any interval of α. As a result, we can disregard the second case and deduce that our proper lower bounds in $C_1(\alpha)$ are established as presented in (7).

Lastly, we address the equality cases for our lower bound. When $\alpha \in (-1, 0]$, the infimum is attained at $t = 0$, leading to a scenario where our body transforms into a cone with its base positioned in the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 0\}$ and its vertex at e_1. As α increases from 0 towards $1/n$, the parameter t ranges from 0 to 1. Thus, the base shifts from the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 0\}$ to the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 1\}$, while the vertices remain fixed at 0 and e_1. To observe this evolution of shapes, refer to Figure 4 which illustrates the changes as α varies from -1 to n. $\square$

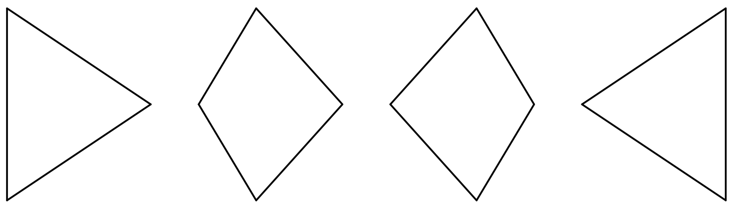

Figure 4: Extreme shapes for the lower bound

Moving forward, we delve into an immediate application of Theorem 4. Specifically, we focus on the comparison between the $(n-1)$-dimensional volume of maximal section(s) of a convex body and the $(n-1)$-dimensional volume of a section located at a certain distance from its centroid. This extension builds upon the work done by Fradelizi in [2].

Theorem 5. *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Let $\xi \in S^{n-1}$ and let $-1 < \alpha < n$. Consider the hyperplane:*

$$H = \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle = \alpha h_K(-\xi)\}.$$

Then

$$\mathrm{vol}_{n-1}(K \cap H) \geq c_n(\alpha) \sup_{t \in \mathbb{R}} \mathrm{vol}_{n-1}\Big(K \cap (\xi^{\perp} + t\xi)\Big),$$

where

$$c_n(\alpha) = \begin{cases} \left(\frac{n(a+1)}{n+1}\right)^{n-1} & \alpha \in (-1, 0], \\ \left(\left(\frac{n}{n+1}\right)^n (a+1)^{n-1}(1-\alpha n)\right)^{\frac{n-1}{n}} & \alpha \in (0, 1/n), \\ 0 & \alpha \in [1/n, n). \end{cases} \tag{12}$$

The bounds in (12) are sharp. The equality cases are discussed in the proof below.

Proof. To prove the theorem, we consider two cases:

1. $\mathrm{vol}_{n-1}(K \cap H) = \sup_{t \in \mathbb{R}} \mathrm{vol}_{n-1}\Big(K \cap (\xi^{\perp} + t\xi)\Big)$,
2. $\mathrm{vol}_{n-1}(K \cap H) \neq \sup_{t \in \mathbb{R}} \mathrm{vol}_{n-1}\Big(K \cap (\xi^{\perp} + t\xi)\Big)$.

The first case is straightforward, yielding $c_n = 1$. Shifting our attention to the second case, we can assume without loss of generality that:

$$\mathrm{vol}_{n-1}(K \cap H) < \sup_{t \in \mathbb{R}} \mathrm{vol}_{n-1}\Big(K \cap (\xi^{\perp} + t\xi)\Big).$$

Let t_o denote the argument that maximizes the volume of sections of K along ξ. To proceed, we apply the Schwartz symmetrization along the direction ξ to the body K, yielding a new body denoted as $\bar{K}$. This transformation can be carried out without affecting the volume of any such section. For convenience, we will use the notation K to refer to the Schwartz symmetral body $\bar{K}$ as well. Now there exists a $\gamma \in \mathrm{span}\{\xi\}$ such that the following conditions are satisfied:

- $\gamma \notin \mathrm{conv}(K \cap H, K \cap (\xi^{\perp} + t_o\xi))$,
- $\mathrm{conv}\Big(K \cap H, K \cap (\xi^{\perp} + t_o\xi), \gamma\Big) = \mathrm{conv}\Big(K \cap (\xi^{\perp} + t_o\xi), \gamma\Big)$.

Under these given conditions, the value of γ is uniquely determined. In general, γ resides in $\mathbb{R}^n \setminus \mathrm{int}K$. Additionally, γ lies on the boundary ∂K if and only if $K \cap H^*$ takes the form of a cone, where H^* represents the halfspace defined by H that excludes $K \cap (\xi^{\perp} + t_o\xi)$. As a result of this construction, two cones are produced:

$$C = \mathrm{conv}\Big(K \cap (\xi^{\perp} + t_o\xi), \gamma\Big) \text{ and } G = \mathrm{conv}(K \cap H, \gamma).$$

For a geometrical description of the construction conditions, refer to Figure 5.

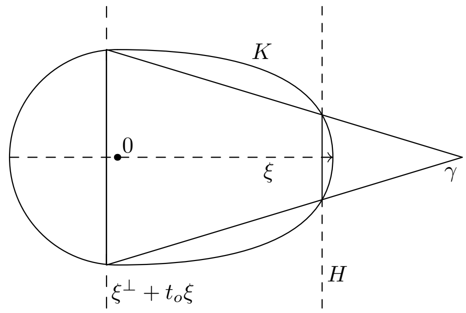

Figure 5: Constructing γ

Let's consider the two cases separately: when $H^* = H^+$ and when $H^* = H^-$. In the case where $H^* = H^+$, the following inequalities hold:

$$\begin{aligned}&\mathrm{vol}_{n-1}(K \cap H)\\ &= \frac{\mathrm{vol}_n(G)n}{\|\gamma - \alpha h_K(-\xi)\xi\|}\\ &\geq \frac{\mathrm{vol}_n(K \cap H^+)n}{\|\gamma - \alpha h_K(-\xi)\xi\|}.\end{aligned}$$

Then, by Theorem 4, we can leverage the bounds to produce the following inequalities:

$$\begin{aligned}&\geq C_1(\alpha)\frac{\mathrm{vol}_n(K)n}{\|\gamma-\alpha h_K(-\xi)\xi\|}\\&\geq\left(\frac{C_1(\alpha)}{1-C_1(\alpha)}\right)\frac{\mathrm{vol}_n(K\cap H^-)n}{\|\gamma-\alpha h_K(-\xi)\xi\|}\\&\geq\left(\frac{C_1(\alpha)}{1-C_1(\alpha)}\right)\frac{\mathrm{vol}_n(C\setminus G)n}{\|\gamma-\alpha h_K(-\xi)\xi\|}.\end{aligned}$$

Returning to the context of $(n-1)$-dimensional section volumes:

$$=\left(\frac{C_1(\alpha)}{1-C_1(\alpha)}\right)\frac{\mathrm{vol}_{n-1}\Big(K\cap(\xi^\perp+t_o\xi)\Big)\|\gamma-t_o\xi\|}{\|\gamma-\alpha h_K(-\xi)\xi\|}-\left(\frac{C_1(\alpha)}{1-C_1(\alpha)}\right)\mathrm{vol}_{n-1}(K\cap H).$$

Then, because C and G are homothetic, we arrive at

$$\begin{aligned}\mathrm{vol}_{n-1}(K\cap H)&\geq\left(\frac{\left(\frac{C_1(\alpha)}{1-C_1(\alpha)}\right)}{1+\left(\frac{C_1(\alpha)}{1-C_1(\alpha)}\right)}\right)\mathrm{vol}_{n-1}\Big(K\cap(\xi^\perp+t_o\xi)\Big)\left(\frac{\mathrm{vol}_{n-1}\Big(K\cap(\xi^\perp+t_o\xi)\Big)}{\mathrm{vol}_{n-1}(K\cap H)}\right)^{\frac{1}{n-1}}\\&=C_1(\alpha)\mathrm{vol}_{n-1}\Big(K\cap(\xi^\perp+t_o\xi)\Big)\left(\frac{\mathrm{vol}_{n-1}\Big(K\cap(\xi^\perp+t_o\xi)\Big)}{\mathrm{vol}_{n-1}(K\cap H)}\right)^{\frac{1}{n-1}},\end{aligned}$$

which ultimately implies

$$\mathrm{vol}_{n-1}(K\cap H)\geq C_1(\alpha)^{\frac{n-1}{n}}\mathrm{vol}_{n-1}\Big(K\cap(\xi^\perp+t_o\xi)\Big).\tag{13}$$

Now, suppose $H^*=H^-$. Then the following inequalities hold:

$$\begin{aligned}&\mathrm{vol}_{n-1}(K\cap H)\\&=\frac{\mathrm{vol}_n(G)n}{\|\gamma-\alpha h_K(-\xi)\xi\|}\\&\geq\frac{\mathrm{vol}_n(K\cap H^-)n}{\|\gamma-\alpha h_K(-\xi)\xi\|}.\end{aligned}$$

Thus, by Theorem 4, the subsequent inequalities hold:

$$\begin{aligned}&\geq\Big(1-C_2(\alpha)\Big)\frac{\mathrm{vol}_n(K)n}{\|\gamma-\alpha h_K(-\xi)\xi\|}\\&\geq\left(\frac{1-C_2(\alpha)}{C_2(\alpha)}\right)\frac{\mathrm{vol}_n(K\cap H^+)n}{\|\gamma-\alpha h_K(-\xi)\xi\|}\\&\geq\left(\frac{1-C_2(\alpha)}{C_2(\alpha)}\right)\frac{\mathrm{vol}_n(C\setminus G)n}{\|\gamma-\alpha h_K(-\xi)\xi\|},\end{aligned}$$

Translating back to $(n-1)$-dimensional volumes of sections:

$$=\left(\frac{1-C_2(\alpha)}{C_2(\alpha)}\right)\frac{\mathrm{vol}_{n-1}\Big(K\cap(\xi^\perp+t_o\xi)\Big)\|\gamma-t_o\xi\|}{\|\gamma-\alpha h_K(-\xi)\xi\|}-\left(\frac{1-C_2(\alpha)}{C_2(\alpha)}\right)\mathrm{vol}_{n-1}(K\cap H).$$

As a result, because C and G are homothetic, we arrive at

$$\mathrm{vol}_{n-1}(K\cap H)\geq\left(\frac{\left(\frac{1-C_2(\alpha)}{C_2(\alpha)}\right)}{1+\left(\frac{1-C_2(\alpha)}{C_2(\alpha)}\right)}\right)\mathrm{vol}_{n-1}\Big(K\cap(\xi^{\perp}+t_o\xi)\Big)\left(\frac{\mathrm{vol}_{n-1}\Big(K\cap(\xi^{\perp}+t_o\xi)\Big)}{\mathrm{vol}_{n-1}(K\cap H)}\right)^{\frac{1}{n-1}}$$

$$=\Big(1-C_2(\alpha)\Big)\mathrm{vol}_{n-1}\Big(K\cap(\xi^{\perp}+t_o\xi)\Big)\left(\frac{\mathrm{vol}_{n-1}\Big(K\cap(\xi^{\perp}+t_o\xi)\Big)}{\mathrm{vol}_{n-1}(K\cap H)}\right)^{\frac{1}{n-1}},$$

which ultimately implies

$$\mathrm{vol}_{n-1}(K\cap H)\geq\Big(1-C_2(\alpha)\Big)^{\frac{n-1}{n}}\mathrm{vol}_{n-1}\Big(K\cap(\xi^{\perp}+t_o\xi)\Big).\tag{14}$$

For all values of α, it holds that $c_n(\alpha)\leq 1$, which renders the straightforward case irrelevant. Upon comparing the two constants seen in (13) and (14), we find that for $\alpha\in(-1,0]$:

$$\Big(1-C_2(\alpha)\Big)^{\frac{n-1}{n}}=\left(\frac{n(a+1)}{n+1}\right)^{n-1}\leq C_1(\alpha)^{\frac{n-1}{n}}=\left(\frac{n-a}{n+1}\right)^{n-1}.$$

Unfortunately, a direct comparison of our constants is not possible for $\alpha\in(0,n)$. Nonetheless, given that $C_2(\alpha)\leq 1$, we can deduce that for $\alpha\in[1/n,n)$:

$$C_1(\alpha)^{\frac{n-1}{n}}=0\leq\Big(1-C_2(\alpha)\Big)^{\frac{n-1}{n}}.$$

Hence, we only need to show:

$$C_1(\alpha)^{\frac{n-1}{n}}\leq\Big(1-C_2(\alpha)\Big)^{\frac{n-1}{n}},$$

for all $\alpha\in(0,1/n)$. This conclusion is established because for $\alpha\in(0,1/n)$ we have:

$$C_2(\alpha)\leq 1-\left(\frac{n}{n+1}\right)^{n}=C_2(0).$$

Consequently, it directly follows that:

$$C_1(\alpha)^{\frac{n-1}{n}}\leq\left(\frac{n}{n+1}\right)^{n-1}\leq\Big(1-C_2(\alpha)\Big)^{\frac{n-1}{n}}.$$

As a result, we have now shown the bounds as depicted by $c_n(\alpha)$ as seen in (12). For all $\alpha\in(-1,n)$, equality within the bounds is precisely achieved when K is any convex body that adheres to the equality cases of the lower bound in Theorem 4. This implies that the maximal section is the one that serves as the base of the equality bodies. For a geometrical example of the equality case, refer to Figure 6. □

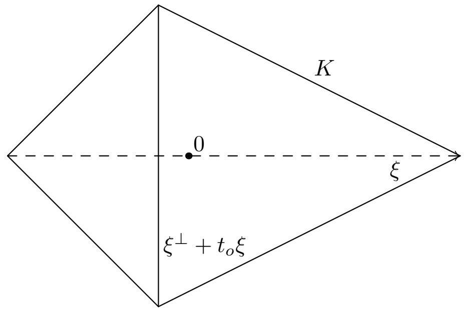

Figure 6: An example of the equality case in Theorem 5

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