A GENERALIZATION OF GRÜNBAUM'S INEQUALITY

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Abstract

Grünbaum's inequality gives sharp volume bounds between a convex body and a division of the body by a hyperplane through its centroid. We provide a generalization of this inequality by looking at divisions of the body by a hyperplane that does not necessarily contain the centroid. As an application, we arrive at a sharp inequality that compares the maximal section(s) of a convex body to any section.

1. Introduction

A convex body K in $\mathbb{R}^n$ is a convex and compact set in $\mathbb{R}^n$ with non-empty interior. The centroid of K is the point

$$g(K) = \frac{1}{|K|}\int_K x\,dx.$$

Here and throughout the paper, $|A|_k$ denotes the k-dimensional Lebesgue measure of a set A. If the dimension is clear, we will simply write $|A|$. A classical inequality of Grünbaum [4] states for a convex body K in $\mathbb{R}^n$ with centroid at the origin that

$$\left|K \cap \xi^+\right| \geq \left(\frac{n}{n+1}\right)^n |K| \quad \text{for all } \xi \in S^{n-1}, \tag{1}$$

where $\xi^+ = \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq 0\}$. The bounds in (1) are sharp and equality occurs when K is a cone. For recent advancements in Grünbaum-type inequalities for sections and projections of convex bodies, see [2], [7], [8], [11].

Suppose we consider translations of ξ^+. Can an analogous inequality to (1), with sharp bounds be found? This paper provides an affirmative answer to this question. Let K be a convex body in $\mathbb{R}^n$ with centroid at the origin. Let $-1 < \alpha < n$ and $\xi \in S^{n-1}$. Consider the half-space

$$H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \alpha h_K(-\xi)\}.$$

Then there are sharp constants $C_1(\alpha, n)$ and $C_2(\alpha, n)$ such that

$$C_1(\alpha, n)\,|K| \leq \left|K \cap H_\alpha^+\right| \leq C_2(\alpha, n)\,|K|\,. \tag{2}$$

where h_K is the support function for K. The exact values of $C_1(\alpha, n)$ and $C_2(\alpha, n)$ are presented in Theorem 3, which also discusses the equality cases. It is noteworthy that within (1), the demonstration of the lower bound automatically determines the upper bound. On the other hand, both sides of the inequality in (2) need to be shown separately. As an application of (2) we obtain a generalization of the following result of Makai and Martini [6]. Let K be a convex body in $\mathbb{R}^n$ with centroid at the origin. Then

$$\left|K \cap \xi^\perp\right| \geq \left(\frac{n}{n+1}\right)^{n-1} \sup_{t\in\mathbb{R}} \left|K \cap \left(\xi^\perp + t\xi\right)\right|.$$

In this paper, we establish an analog of this inequality for sections that do not pass through the centroid of K.

2. Preliminaries

Throughout the paper, we will be working in the Euclidean space $\mathbb{R}^n$. We will denote $\langle\cdot,\cdot\rangle$ to be the inner product. We will denote $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ to be the n-dimensional Euclidean norm. We will assume all n-dimensional Euclidean balls with radius $r > 0$ and center $x_0 \in \mathbb{R}^n$ are of the form $\{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$. Given some $\xi \in \mathbb{R}^n$, we will denote $\xi^\perp = \{x \in \mathbb{R}^n \mid \langle\xi, x\rangle = 0\}$. We will denote $\xi^+ = \{x \in \mathbb{R}^n \mid \langle\xi, x\rangle \geq 0\}$ and $\xi^- = \{x \in \mathbb{R}^n \mid \langle\xi, x\rangle \leq 0\}$.

Definition 1. The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ for a convex body K in $\mathbb{R}^n$ is

$$h_K(\xi) = \sup\{\langle\xi, x\rangle \mid x \in K\}.$$

If $\xi \in S^{n-1}$ then $h_K(\xi)$ gives the distance from the origin to the supporting hyperplane for K in the direction of ξ. A result from Minkowski and Radon [1, p. 58] states for any convex body K in $\mathbb{R}^n$ with its centroid at the origin, and for all $\xi \in S^{n-1}$ we have

$$\frac{1}{n} h_K(\xi) \leq h_K(-\xi) \leq n h_K(\xi). \tag{3}$$

Note that the choice of bounds for α in Theorems 3 and 4 is a result of (3).

Definition 2. Let $\xi \in S^{n-1}$. The *parallel section function* $A_{K,\xi} : \mathbb{R} \to \mathbb{R}$ for a convex body K in $\mathbb{R}^n$ is

$$A_{K,\xi}(t) = \left|K \cap \left(\xi^\perp + t\xi\right)\right|.$$

If $K \cap \left(\xi^\perp + t\xi\right)$ is an $(n-1)$-dimensional Euclidean ball, then $A_{K,\xi}^{\frac{1}{n-1}}(t)$ provides the radius of $K \cap \left(\xi^\perp + t\xi\right)$, up to a constant.

Definition 3. Let $\xi \in S^{n-1}$. The *cut-off volume function* $V_{K,\xi} : \mathbb{R} \to \mathbb{R}$ for a convex body K in $\mathbb{R}^n$ is

$$V_{K,\xi}(t) = \int_t^\infty \left|K \cap \left(\xi^\perp + s\xi\right)\right| ds.$$

Lemma 1. *Let K be a convex body in $\mathbb{R}^n$. Then the function $A_{K,\xi}$ is $1/(n-1)$ concave on its support.*

Proof. See [5, p. 18] □

The following result is also well-known, but we could not find a reference specific to our setting. Therefore we decided to include a proof for completeness.

Lemma 2. *Let K be a convex body in $\mathbb{R}^n$ be a convex body. Then the function $V_{K,\xi}$ is $1/n$ concave on its support.*

Proof. Let $\lambda \in [0,1]$ and $s, t \in \operatorname{supp} V_{K,\xi}$. Observe that

$$\begin{aligned}&\lambda\Big(K \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \geq s\}\Big) + (1-\lambda)\Big(K \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \geq t\}\Big)\\ &\subseteq \Big(K \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \geq \lambda s + (1-\lambda)t\}\Big).\end{aligned}$$

This, together with the Brunn-Minkowski inequality, implies that

$$\begin{aligned}
&\left|K \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \geq \lambda s + (1-\lambda)t\}\right|^{\frac{1}{n}} \\
&\geq \left|\lambda\Big(K \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \geq s\}\Big) + (1-\lambda)\Big(K \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \geq t\}\Big)\right|^{\frac{1}{n}}, \\
&\geq \lambda\left|K \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \geq s\}\right|^{\frac{1}{n}} + (1-\lambda)\left|K \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \geq t\}\right|^{\frac{1}{n}},
\end{aligned}$$

which proves the result. □

Definition 4. The Schwartz symmetral of a convex body K in $\mathbb{R}^n$ along the direction ξ is the convex body $\bar{K}$, which is constructed as follows: Over the interval $-h_K(-\xi) \leq t \leq h_K(\xi)$, $\bar{K} \cap (\xi^\perp + t\xi)$ forms an $(n-1)$-dimensional Euclidean ball positioned in and centered at the origin of $\xi^\perp$ with

$$\left|K \cap (\xi^\perp + t\xi)\right| = \left|\bar{K} \cap (\xi^\perp + t\xi)\right|. \tag{4}$$

As a result of this transformation, it follows that

$$h_K(\pm\xi) = h_{\bar{K}}(\pm\xi), \tag{5}$$

and

$$V_{K,\xi}(t) = V_{\bar{K},\xi}(t), \tag{6}$$

for all $t \in \mathbb{R}$. Note that the centroid of $\bar{K}$ resides on the line $\ell = \{x \in \mathbb{R}^n \mid x = t\xi,\ t \in \mathbb{R}\}$ due to the rotational symmetry of $\bar{K}$ around ξ.

We will now prove our main result.

Theorem 3. *Let K be a convex body in $\mathbb{R}^n$ with centroid at the origin. Let $-1 < \alpha < n$ and let $\xi \in S^{n-1}$. Consider the hyperplane*

$$H_\alpha = \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle = \alpha h_K(-\xi)\}.$$

Then

$$C_1(\alpha, n)\,|K| \leq \left|K \cap H_\alpha^+\right| \leq C_2(\alpha, n)\,|K|.$$

where

$$C_1(\alpha, n) = \begin{cases} \left(\frac{n-\alpha}{n+1}\right)^n & \alpha \in (-1, 0], \\ \left(\frac{n}{n+1}\right)^n (\alpha+1)^{n-1}(1-\alpha n) & \alpha \in (0, 1/n), \\ 0 & \alpha \in [1/n, n), \end{cases} \tag{7}$$

and

$$C_2(\alpha, n) = \begin{cases} 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n & \alpha \in (-1, 0], \\ \tau(\alpha, n) & \alpha \in (0, n). \end{cases} \tag{8}$$

$\tau(\alpha, n)$ is a constant that is not explicit for general dimension because finding the value involves finding the roots of a high-degree rational function. The lower bounds and upper bounds are sharp. The equality cases are discussed in the proof below.

Proof. We will start by applying the Schwarz symmetrization to the body K with respect to the line $\ell = \{x \in \mathbb{R}^n : x = t\xi, t \in \mathbb{R}\}$. Abusing notation, we will denote the new body again by K. Note that the symmetrization does not change the location of the centroid of K, the value of $h_K(-\xi)$, and the position of the half-space H_α^+.

First consider the case $-1 < \alpha \leq 0$. We will start with the upper bound. Observe that

$$\left|K \cap H_\alpha^+\right| = |K| - |K \cap \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \leq \alpha h_K(-\xi)\}|.$$

We will now translate K and H_a by $h_K(-\xi)\xi$, and denote $\widetilde{K} = K + h_K(-\xi)\xi$ and $\widetilde{H}_\alpha^- = \{x \in \mathbb{R}^n \mid \langle \xi, x\rangle \leq (\alpha+1)h_K(-\xi)\}$. Clearly,

$$\left|K \cap H_\alpha^-\right| = \left|\widetilde{K} \cap \widetilde{H}_\alpha^-\right|$$

Observe that the origin is located on the boundary of $\widetilde{K}$. Let $\frac{1}{\alpha+1}\widetilde{K}$ denote the dilation of $\widetilde{K}$ with respect to the origin by a factor of $1/(\alpha+1) > 1$.

Performing a dilation with respect to the origin by a factor of $1/(\alpha+1) > 1$, and using that $\frac{1}{\alpha+1}\widetilde{K} \supset \tilde{K}$ and $\frac{1}{\alpha+1}\widetilde{H}_\alpha^- = \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \leq h_K(-\xi)\}$, we get

$$\begin{aligned}
\left|K \cap H_\alpha^-\right| &= (\alpha+1)^n \left|\frac{1}{\alpha+1}\widetilde{K} \cap \frac{1}{\alpha+1}\widetilde{H}_\alpha^-\right| \\
&\geq (\alpha+1)^n \left|\widetilde{K} \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \leq h_K(-\xi)\}\right| \\
&= (\alpha+1)^n \left|K \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \leq 0\}\right| \\
&\geq (\alpha+1)^n \left(\frac{n}{n+1}\right)^n |K|,
\end{aligned}$$

where we used Grünbaum's inequality (1).

Therefore, for $\alpha \in (-1, 0]$ we have

$$\frac{|K \cap H_\alpha^+|}{|K|} \leq 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n.$$

Now we will obtain the lower bound. By the $1/n$ concavity of $V_{K,\xi}$ on its support we have

$$\begin{aligned}
\left|K \cap H_\alpha^+\right|^{\frac{1}{n}} &= V_{K,\xi}\big(\alpha h_K(-\xi)\big)^{\frac{1}{n}} = V_{K,\xi}\big(-\alpha(-h_K(-\xi)) + (1+\alpha)0\big)^{\frac{1}{n}} \\
&\geq -\alpha V_{K,\xi}{}^{\frac{1}{n}}\big(-h_K(-\xi)\big) + (1+\alpha)V_{K,\xi}{}^{\frac{1}{n}}(0).
\end{aligned}$$

Using Grünbaum's inequality and the observation that $V_{K,\xi}\big(-h_K(-\xi)\big) = |K|$, we get

$$\left|K \cap H_\alpha^+\right|^{\frac{1}{n}} \geq -\alpha\,|K|^{\frac{1}{n}} + (1+\alpha)\left(\frac{n}{n+1}\right)|K|^{\frac{1}{n}},$$

which implies for $\alpha \in (-1, 0]$:

$$\left(\frac{n-a}{n+1}\right)^n \leq \frac{|K \cap H_\alpha^+|}{|K|}.$$

Thus, we have shown our bounds for $\alpha \in (-1, 0]$. By Grünbaum's inequality, the equality cases hold precisely when K is a cone with its base orthogonal to $\xi^\perp$.

Now we will move to the case $0 < \alpha < n$. As before, after applying the Schwarz symmetrization, we can assume that K is a body of revolution. Without loss of generality we can choose the x_1-axis to be the axis of revolution. Denote $h = h_K(e_1)$ and let $\widetilde{K} = K + he_1$. Note that $g(\widetilde{K}) = he_1$, and further observe that $H_\alpha = e_1^\perp + (\alpha+1)he_1$ is our new hyperplane of interest. Abusing notation we will denote the translated body again by K.

We will prove the upper bound first. Let B_{n-1} be the unit $(n-1)$-dimensional Euclidean ball in $e_1^\perp$ centered at the origin. By continuity there is $r_1 \geq 0$ such that

$$\left|\operatorname{conv}\left(r_1 B_{n-1}, K \cap H_\alpha\right)\right| = \left|K \cap H_\alpha^-\right|.$$

Observe that $K \cap e_1^\perp \subset r_1 B_{n-1}$. Denote $\operatorname{conv}\left(r_1 B_{n-1}, K \cap H_\alpha\right)$ by L^-. There exist unique μ and r_2 with $\alpha + 1 < \mu < h_K(e_1)$ such that

$$\left|\operatorname{conv}\left(r_2 B_{n-1} + \mu e_1, K \cap H_\alpha\right)\right| = \left|K \cap H_\alpha^+\right|$$

and

$$\operatorname{conv}\left(K \cap H_\alpha, r_2 B_{n-1} + \mu e_1\right) \cup L^- = \operatorname{conv}\left(r_1 B_{n-1}, r_2 B_{n-1} + \mu e_1\right).$$

Note that

$$\left|K \cap \left(e_1^\perp + \mu e_1\right)\right| \leq \left|r_2 B_{n-1}\right|,$$

To gain a geometric insight into the construction conditions shown above, refer to Figure 1.

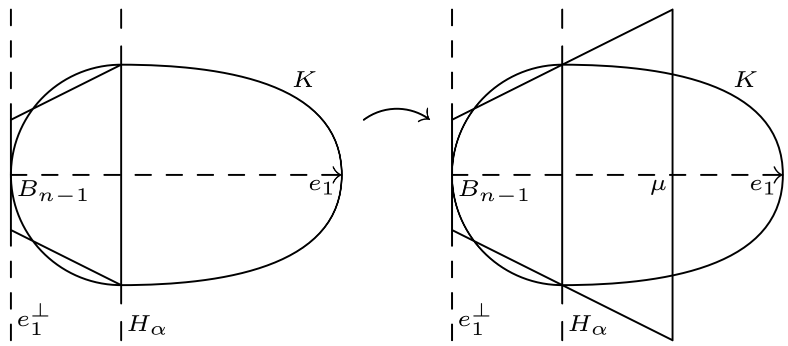

Figure 1: Constructing $r_1 B_{n-1}$ and $r_2 B_{n-1} + \mu e_1$

Denote $\operatorname{conv}\big(K \cap H_\alpha, r_2 B_{n-1} + \mu e_1\big)$ by L^+. Then $L = L^- \cup L^+$ is a convex body. It is a truncated cone of revolution or possibly a cylinder. By construction,

$$|L| = |K|$$

and

$$\langle g(L), e_1 \rangle \leq \langle g(K), e_1 \rangle,$$

since L is obtained from K by shifting mass to the left.

In particular,

$$\begin{aligned} &|K \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle e_1, g(K)\rangle\}| \\ &\leq |L \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle e_1, g(L)\rangle\}| \,. \end{aligned}$$

Hence, our objective is to estimate the latter from above. Without loss of generality, we can assume that $\mu = e_1$. Additionally, for the interval $0 \leq t \leq 1$, we can assume that

$$\big|L \cap \big(e_1^\perp + te_1\big)\big| = (mt+b)^{n-1}, \tag{9}$$

where $b \geq 0$ and either (1) $m \geq 0$, or (2) $m < 0$ and $m + b \geq 0$. For now, we will focus on $m \neq 0$, and address the case of $m = 0$ later. By Fubini's theorem and (9) we have

$$|L| = \int_0^1 \big|L \cap \big(e_1^\perp + x_1 e_1\big)\big|\, dx_1 = \frac{(b+m)^n - b^n}{mn}.$$

Similarly, the x_1 coordinate of the centroid of L is given by

$$\begin{aligned} \langle e_1, g(L)\rangle &= \frac{1}{|L|}\int_L x_1 dx \\ &= \frac{1}{|L|}\int_0^1 x_1 \big|L \cap \big(e_1^\perp + te_1\big)\big|\, dx_1 \\ &= \frac{b^{n+1} + (mn-b)(b+m)^n}{m(n+1)\Big(\big(b+m\big)^n - b^n\Big)}. \end{aligned}$$

Given that L is a body of revolution around the x_1 axis, one can see that all remaining coordinates of the centroid of F are equal to zero.

Now we can compute

$$\begin{aligned} \frac{|L \cap H_\alpha^+|}{|L|} &= \frac{1}{|L|}\int_{L \cap H_\alpha^+} dx \\ &= \frac{(b+m)^n - \left(b + m(\alpha+1)\left(\frac{b^{n+1}+(mn-b)(b+m)^n}{m(n+1)\left((b+m)^n - b^n\right)}\right)\right)^n}{(b+m)^n - b^n}. \end{aligned}$$

Introducing the variable $z = \frac{b}{m}$, we arrive at

$$\langle e_1, g(L)\rangle = (\alpha+1)\frac{z\Big(1 + (\frac{n}{z} - 1)(\frac{1}{z}+1)^n\Big)}{(n+1)\Big(\big(\frac{1}{z}+1\big)^n - 1\Big)},$$

and

$$\frac{|L \cap H_\alpha^+|}{|L|} = \frac{\left(1 + \frac{1}{z}\right)^n - \left(1 + \frac{1}{z}(\alpha+1)\left(\frac{z\left(1+(\frac{1}{z}-1)(\frac{1}{z}+1)^n\right)}{(n+1)\left((\frac{1}{z}+1)^n-1\right)}\right)\right)^n}{\left(1+\frac{1}{z}\right)^n - 1}.$$

Letting ϕ denote the latter function of z, our goal is to determine the supremum of ϕ within the domain $z \in (-\infty, -1] \cup [0, \infty)$. For values of $\alpha \in (-1, 0]$, one can observe ϕ exhibits strict decreasing behavior as a function of z across its entire domain. Therefore, the supremum of ϕ is attained either as z approaches $-\infty$ or when $z = 0$. Then it follows that

$$\lim_{z \to -\infty} \phi(z) = \frac{1-\alpha}{2} \leq \phi(0) = 1 - \left(\frac{(\alpha+1)\, n}{n+1}\right)^n.$$

However, for $\alpha \in (0, n)$, the function ϕ lacks the characteristics of monotone increase or decrease. Moreover, no stable intervals of concavity or convexity persist all dimensions. Consequently, determining the upper bound $\tau_{\alpha,n}$ becomes a complex task, as solving for $\tau_{\alpha,n}$ involves handling high-degree rational functions.

Lastly, let's revisit the case where $m = 0$. It becomes evident that our body F is a cylinder. Thus, we have

$$\frac{|L \cap H_\alpha^+|}{|L|} = \frac{1-\alpha}{2}.$$

For $\alpha \in (-1, 0]$ this ratio is smaller than our existing ratio and for $\alpha \in (0, n)$, it is clear that

$$\tau_{\alpha,n} \geq \frac{1-\alpha}{2}.$$

Thus, we now see that for any convex body K, we have the following bounds as seen in (8):

$$\frac{|L \cap H_\alpha^+|}{|K|} \leq C_2(\alpha).$$

Now, let's dive into the discussion of equality cases. For $\alpha \in (-1, 0]$, the configuration of our body K involves the convex hull of two $(n-1)$-dimensional Euclidean balls (which may yield a cone). The supremum of ϕ is attained at $z = 0$, implying $b = 0$, and thus the extreme shapes correspond to cones with a vertex at the origin and a base lying in the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 1\}$.

As we move on to $\alpha \in (0, n)$, the structure of our body K remains as the convex hull of two $(n-1)$-dimensional Euclidean balls (which may lead to a cone). In the case of $\alpha = 1/n$, the body becomes a cylinder. However, in general, when $\alpha \in (0, n)$, we face difficulties in determining the precise values at which the supremum of ϕ is attained. Consequently, the equality bodies cannot be consistently determined in all cases. Refer to Figure 2 for an illustration of how shapes evolve as α transitions from -1 to n. Let us now discuss the lower bound. Similarly to the upper bound, we will assume that K is a body of revolution about e_1 and that $h_K(-e_1) = 0$. By continuity there exists a unique $\rho \geq h_K(e_1)$ such that

$$\left|\operatorname{conv}\left(K \cap H_\alpha, \rho e_1\right)\right| = \left|K \cap H_\alpha^+\right|.$$

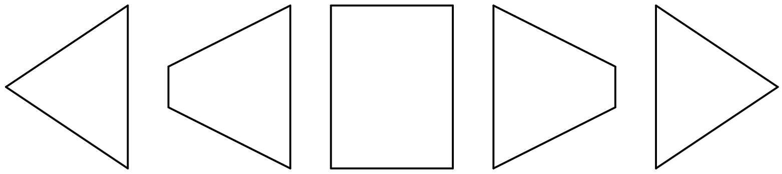

Figure 2: Extreme shapes for the upper bound

Let us denote $\operatorname{conv}\big(K \cap H_\alpha, \rho e_1\big)$ by M^+. There exists unique $t \in (0, \alpha+1)$ and $r > 0$ such that

$$\operatorname{conv}\big(rB_{n-1} + te_1, M^+\big) = \operatorname{conv}\big(rB_{n-1} + te_1, \rho e_1\big),$$

and

$$\big|\operatorname{conv}\big(0, rB_{n-1} + te_1, K \cap H_\alpha\big)\big| = \big|K \cap H_\alpha^-\big|.$$

Observe that

$$K \cap \big(e_1^\perp + te_1\big) \subset rB_{n-1} + te_1.$$

For a geometric description of the construction conditions as seen above, see Figure 3.

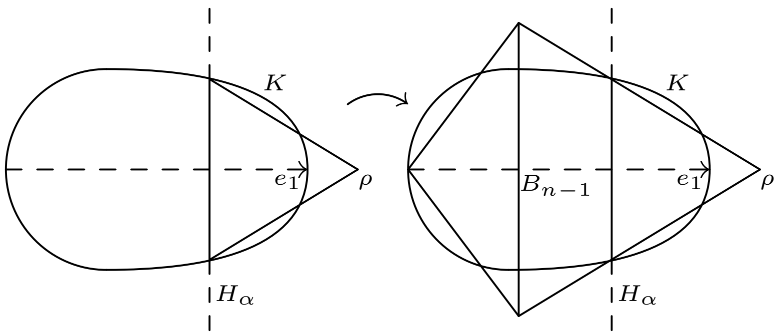

Figure 3: Constructing ρ and t

Denote $\operatorname{conv}(0, rB_{n-1}+te_1, K\cap H_\alpha)$ as M^-. Then $M = M^- \cup M^+$ is a convex body formed by two cones with a common base contained in $e_1^\perp + te_1$. Observe that by construction $\langle g(M \cap H_\alpha^+), e_1\rangle \geq \langle g(K \cap H_\alpha^+), e_1\rangle$ and $\langle g(M \cap H_\alpha^-), e_1\rangle \geq \langle g(K \cap H_\alpha^-), e_1\rangle$, and therefore

$$\langle g(M), e_1\rangle \geq \langle g(K), e_1\rangle.$$

As a result, we have constructed a convex body M such that:

$$|K| = |M|,$$

and

$$\begin{aligned}
&|K \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle g(K), e_1\rangle\}| \\
&= |M \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle g(K), e_1\rangle\}| \\
&\geq |M \cap \{x \in \mathbb{R}^n \mid x_1 \geq (\alpha+1)\langle g(M), e_1\rangle\}|.
\end{aligned}$$

Hence, our task reduces to demonstrating a minimum volume ratio within the class of such constructed convex bodies.

Without loss of generality (after applying an appropriate affine transformation), we can assume $h_M(e_1) = 1$, and $|rB_{n-1}| = n$. Define

$$M_1 = M \cap \{x \in \mathbb{R}^n \mid \langle x, e_1 \rangle \le t\},$$

and

$$M_2 = M \cap \{x \in \mathbb{R}^n \mid \langle x, e_1 \rangle \ge t\}$$

to be the cones forming M.

Since the heights of M_1 and M_2 are equal to t and $1-t$ respectively, and the $(n-1)$-dimensional volume of their common base is n, we obtain that $|M_1| = t$, $|M_2| = 1 - t$, and $|M| = 1$. It is a well-known fact that the centroid of a cone in $\mathbb{R}^n$ divides its height in the ratio $1 : n$. Thus, $\langle g(M_1), e_1 \rangle = \frac{tn}{n+1}$ and $\langle g(M_2), e_1 \rangle = \frac{tn+1}{n+1}$.

Therefore,

$$\begin{aligned}\langle g(M), e_1 \rangle &= |M_1| \langle g(M_1), e_1 \rangle + |M_2| \langle g(M_2), e_1 \rangle \\ &= t\frac{tn}{n+1} + (1-t)\frac{tn+1}{n+1} = \frac{t(n-1)+1}{n+1}.\end{aligned}$$

Recall that we are interested in the volume of the intersection of M with the half-space $H_\alpha^+ = \{x \in \mathbb{R}^n : x_1 \ge \frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}\}$. We will consider two cases:

1. $0 \le t \le \frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}$,
2. $\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1} \le t \le 1$.

These two cases are respectively equivalent to

$$0 \le t \le \alpha_0 \quad \text{and} \quad \alpha_0 \le t \le 1,$$

where $\alpha_0 = \frac{\alpha+1}{2-(n-1)\alpha}$.

In the first case, $M \cap H_\alpha^+$ is a cone homothetic to M_2 with the homothety coefficient equal to $\left(1 - \frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}\right)\frac{1}{1-t}$. Therefore,

$$\begin{aligned}\left|M \cap H_\alpha^+\right| &= \left(\frac{1 - \frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}}{1-t}\right)^n (1-t) \\ &= \frac{\left(1 - \frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}\right)^n}{(1-t)^{n-1}}. \qquad (10)\end{aligned}$$

In the second case, $M \cap H_\alpha^-$ is a a cone homothetic to M_1 with the homothety coefficient equal to $\frac{(\alpha+1)\big(t(n-1)+1\big)}{n+1}\frac{1}{t}$. Thus,

$$\begin{aligned} \left|M \cap H_\alpha^+\right| &= 1 - \left|M \cap H_\alpha^-\right| = 1 - \left(\frac{(\alpha+1)\big(t(n-1)+1\big)}{(n+1)t}\right)^n t \\ &= 1 - \frac{(\alpha+1)^n\big(t(n-1)+1\big)^n}{(n+1)^n t^{n-1}}. \end{aligned} \tag{11}$$

Summarizing, $|M \cap H_\alpha^+|$ is equal to the following piecewise function

$$\psi(t) = \begin{cases} \frac{\left(1-\frac{(\alpha+1)\left(t(n-1)+1\right)}{n+1}\right)^n}{(1-t)^{n-1}}, & 0 \le t \le \alpha_0, \\ 1 - \frac{(\alpha+1)^n\left(t(n-1)+1\right)^n}{(n+1)^n t^{n-1}}, & \alpha_0 \le t \le 1. \end{cases}$$

Our goal is to find the minimum of ψ on $[0,1]$. Elementary calculations show that the derivative of ψ is zero at $t_0 = \frac{(n+1)\alpha}{\alpha+1} \in [0, \alpha_0]$, and ψ is decreasing on $(0, t_0)$ and increasing on (t_0, α_0). Also observe that ψ is decreasing on $(\alpha_0, 1)$. Thus the minimum of ψ is the smaller of the numbers

$$\psi(t_0) = \left(\frac{n}{n+1}\right)^n (\alpha+1)^{n-1}(1-\alpha n) \quad \text{and} \quad \psi(1) = 1 - \frac{(\alpha+1)^n n^n}{(n+1)^n}.$$

It is not difficult to see that $\psi(t_0) \le \psi(1)$ for all n and all $\alpha \in (0, 1/n)$. Indeed, to show that

$$\left(\frac{n}{n+1}\right)^n (\alpha+1)^{n-1}(1-\alpha n) + \frac{(\alpha+1)^n n^n}{(n+1)^n} \le 1,$$

it is enough to observe that the left-hand side is an increasing function of $\alpha \in (0, 1/n)$ and it is equal to the right-hand side when $\alpha = 1/n$.

Lastly, we address the equality cases for our lower bound. When $\alpha \in (-1, 0]$, the infimum is attained in the limit when t approaches 0, leading to a scenario where our body becomes a cone with its base positioned in the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 0\}$ and its vertex at e_1. As α increases from 0 towards $1/n$, $\|t\|$ ranges from 0 to 1. Thus, the base shifts from the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 0\}$ to the hyperplane $\{x \in \mathbb{R}^n \mid x_1 = 1\}$, while the vertices remain fixed at 0 and e_1. To observe this evolution of shapes, refer to Figure 4 which illustrates the changes as α varies from -1 to n. $\square$

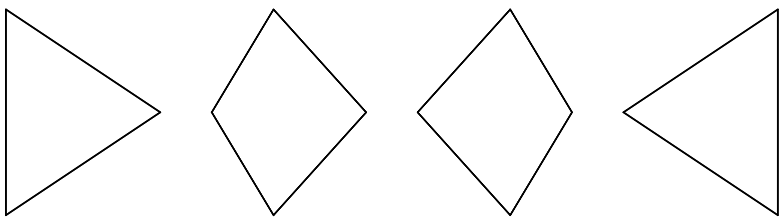

Figure 4: Extreme shapes for the lower bound

As an application of Theorem 3 we obtain a generalization of the following result of Makai and Martini [6]. Let K be a convex body in $\mathbb{R}^n$ with centroid at the origin. Then

$$\left|K \cap \xi^{\perp}\right| \geq \left(\frac{n}{n+1}\right)^{n-1} \sup_{t\in\mathbb{R}} \left|K \cap \left(\xi^{\perp} + t\xi\right)\right|.$$

We generalize this result to sections that do not necessarily contain the centroid of K.

Theorem 4. *Let $K \subseteq \mathbb{R}^n$ be a convex body with centroid at the origin. Let $\xi \in S^{n-1}$ and let $-1 < \alpha < n$. Consider the hyperplane*

$$H_\alpha = \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle = \alpha h_K(-\xi)\}.$$

Then

$$|K \cap H_\alpha| \geq c(\alpha, n) \sup_{t\in\mathbb{R}} \left|K \cap \left(\xi^{\perp} + t\xi\right)\right|,$$

where

$$c(\alpha, n) = \begin{cases} \left(\frac{n(\alpha+1)}{n+1}\right)^{n-1} & \alpha \in (-1, 0], \\ \left(\frac{n}{n+1}\right)^{n-1} (\alpha+1)^{\frac{(n-1)^2}{n}} (1-\alpha n)^{\frac{n-1}{n}} & \alpha \in (0, 1/n), \\ 0 & \alpha \in [1/n, n). \end{cases} \tag{12}$$

The bounds in (12) are sharp. The equality cases are discussed in the proof below.

Proof. To prove the theorem, we consider two cases:

1. $|K \cap H_\alpha| = \sup_{t\in\mathbb{R}} \left|K \cap \left(\xi^{\perp} + t\xi\right)\right|$,
2. $|K \cap H_\alpha| \neq \sup_{t\in\mathbb{R}} \left|K \cap \left(\xi^{\perp} + t\xi\right)\right|$.

The first case gives $c(\alpha, n) = 1$. In the second case, we can assume that

$$0 < |K \cap H_\alpha| < \sup_{t\in\mathbb{R}} \left|K \cap \left(\xi^{\perp} + t\xi\right)\right|,$$

since if $|K \cap H_\alpha| = 0$ then we deduce that $c(\alpha, n) = 0$ is the best possible. Without loss of generality, apply Schwartz symmetrization to K with respect to the line $\ell = \{x \in \mathbb{R}^n \mid x = t\xi,\ t \in \mathbb{R}\}$. Abusing notation, we will denote the new body again by K. Let $t_0 \in \mathbb{R}$ such that $K \cap \left(\xi^{\perp} + t_0\xi\right)$ is a maximal section for K. Since $0 < |K \cap H_\alpha| < \left|K \cap \left(\xi^{\perp} + t_0\xi\right)\right|$ we can find a cone with base equal to $K \cap \left(\xi^{\perp} + t_0\xi\right)$ and section equal to $K \cap H_\alpha$. Such a cone is uniquely determined by these two sections. Denote this cone by N_1. Consider $\gamma = h_{L_1}(\xi)\xi$. Due to the convexity of K, γ lies in $\mathrm{cl}(K^c)$. Define N_2 to be the cone with base equal to $K \cap H_\alpha$ and vertex γ. Finally, define H_α^* to be the halfspace given by the hyperplane H_α that contains N_2.

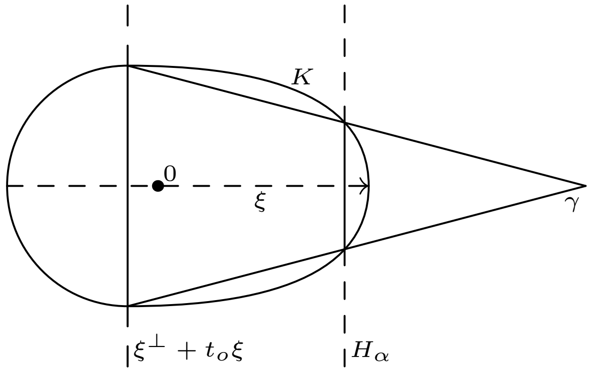

Figure 5: Constructing γ

We will now use N_1 and N_2 along with H_α^* to derive (12). Denote $h = \alpha h_K(-\xi)\xi$. We will consider two cases when $H_\alpha^* = H_\alpha^+$ and when $H_\alpha^* = H_\alpha^-$. When $H_\alpha^* = H_\alpha^+$ the following inequalities hold

$$|K \cap H_\alpha| = \frac{|N_2|\, n}{\|\gamma - h\|} \geq \frac{|K \cap H_\alpha^+|\, n}{\|\gamma - h\|}.$$

Then by Theorem 3 and using that $|K \cap H_\alpha^+| + |K \cap H_\alpha^-| = |K|$ we arrive at the following inequalities

$$\begin{aligned}
&\frac{|K \cap H_\alpha^+|\, n}{\|\gamma - h\|} \geq C_1(\alpha, n) \frac{|K|\, n}{\|\gamma - h\|} \\
&\geq \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)} \right) \frac{|K \cap H_\alpha^-|\, n}{\|\gamma - h\|} \geq \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)} \right) \frac{|N_1 \setminus N_2|\, n}{\|\gamma - h\|}.
\end{aligned}$$

Now expressing N_1 and N_2 in terms of their $(n-1)$-dimensional sections

$$\begin{aligned}
&\left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)} \right) \frac{|N_1 \setminus N_2|\, n}{\|\gamma - h\|} = \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)} \right) \frac{(|N_1| - |N_2|) n}{\|\gamma - h\|} \\
&= \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)} \right) \frac{\left|K \cap \left(\xi^{\perp} + t_o \xi\right)\right| \|\gamma - t_o \xi\|}{\|\gamma - h\|} - \left(\frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)} \right) |K \cap H_\alpha|.
\end{aligned}$$

Because N_1 is a homothetic copy of N_2 we can express $\|\gamma - t_o\xi\| / \|\gamma - h\|$ as $\left|K \cap \left(\xi^{\perp} + t_o \xi\right)\right|^{\frac{1}{n-1}} / \left|K \cap H_\alpha\right|^{\frac{1}{n-1}}$. Thus

$$\begin{aligned}
|K \cap H_\alpha| &\geq \left(\frac{\left(\frac{C_1(\alpha,n)}{1 - C_1(\alpha,n)} \right)}{1 + \left(\frac{C_1(\alpha,n)}{1 - C_1(\alpha,n)} \right)} \right) \left|K \cap \left(\xi^{\perp} + t_o \xi\right)\right| \frac{\left|K \cap \left(\xi^{\perp} + t_o \xi\right)\right|^{\frac{1}{n-1}}}{|K \cap H_\alpha|^{\frac{1}{n-1}}} \\
&= C_1(\alpha, n) \left|K \cap \left(\xi^{\perp} + t_o \xi\right)\right| \frac{\left|K \cap \left(\xi^{\perp} + t_o \xi\right)\right|^{\frac{1}{n-1}}}{|K \cap H_\alpha|^{\frac{1}{n-1}}},
\end{aligned}$$

which ultimately implies

$$|K \cap H_\alpha| \geq C_1(\alpha, n)^{\frac{n-1}{n}} \left|K \cap \left(\xi^{\perp} + t_o \xi\right)\right|. \tag{13}$$

Now suppose $H_\alpha^* = H_\alpha^-$. Then the following inequalities hold

$$|K \cap H_\alpha| = \frac{|N_2|\, n}{\|\gamma - h\|} \geq \frac{|K \cap H_\alpha^-|\, n}{\|\gamma - h\|}.$$

Then by Theorem 3 and using that $|K \cap H_\alpha^+| + |K \cap H_\alpha^-| = |K|$ we arrive at the following inequalities

$$\begin{aligned}
&\frac{|K \cap H_\alpha^-|\, n}{\|\gamma - h\|} \geq \big(1 - C_2(\alpha,n)\big) \frac{|K|\, n}{\|\gamma - h\|} \\
&\geq \left(\frac{1 - C_2(\alpha,n)}{C_2(\alpha,n)}\right) \frac{|K \cap H_\alpha^+|\, n}{\|\gamma - h\|} \geq \left(\frac{1 - C_2(\alpha,n)}{C_2(\alpha,n)}\right) \frac{|N_1 \setminus N_2|\, n}{\|\gamma - h\|},
\end{aligned}$$

Now expressing N_1 and N_2 in terms of their $(n-1)$-dimensional sections

$$\begin{aligned}
&\left(\frac{1 - C_2(\alpha,n)}{C_2(\alpha,n)}\right) \frac{|N_1 \setminus N_2|\, n}{\|\gamma - h\|} = \left(\frac{1 - C_2(\alpha,n)}{C_2(\alpha,n)}\right) \frac{(|N_1| - |N_2|)n}{\|\gamma - h\|} \\
&= \left(\frac{1 - C_2(\alpha,n)}{C_2(\alpha,n)}\right) \frac{\left|K \cap \left(\xi^\perp + t_o\xi\right)\right| \|\gamma - t_o\xi\|}{\|\gamma - h\|} - \left(\frac{1 - C_2(\alpha,n)}{C_2(\alpha,n)}\right) |K \cap H_\alpha|\,.
\end{aligned}$$

Again using that N_1 and N_2 are homothetic we arrive at

$$\begin{aligned}
&|K \cap H_\alpha| \geq \left(\frac{\left(\frac{1 - C_2(\alpha,n)}{C_2(\alpha,n)}\right)}{1 + \left(\frac{1 - C_2(\alpha,n)}{C_2(\alpha,n)}\right)}\right) \left|K \cap \left(\xi^\perp + t_o\xi\right)\right| \frac{\left|K \cap \left(\xi^\perp + t_o\xi\right)\right|^{\frac{1}{n-1}}}{|K \cap H_\alpha|^{\frac{1}{n-1}}} \\
&= \big(1 - C_2(\alpha,n)\big) \left|K \cap \left(\xi^\perp + t_o\xi\right)\right| \frac{\left|K \cap \left(\xi^\perp + t_o\xi\right)\right|^{\frac{1}{n-1}}}{|K \cap H_\alpha|^{\frac{1}{n-1}}},
\end{aligned}$$

which ultimately implies

$$|K \cap H_\alpha| \geq \big(1 - C_2(\alpha,n)\big)^{\frac{n-1}{n}} \left|K \cap \left(\xi^\perp + t_o\xi\right)\right|. \tag{14}$$

Now to determine $c(\alpha,n)$ we need to find the minimum of the two constants in equations (13) and (14) for fixed α. Note that $n\alpha \leq -\alpha$ for $\alpha \in (-1,0]$. Then it follows that

$$\big(1 - C_2(\alpha,n)\big)^{\frac{n-1}{n}} = \left(\frac{n(\alpha+1)}{n+1}\right)^{n-1} \leq C_1(\alpha,n)^{\frac{n-1}{n}} = \left(\frac{n-\alpha}{n+1}\right)^{n-1},$$

for all $\alpha \in (-1,0]$. A direct comparison of our constants is not possible for $\alpha \in (0,n)$. However given that $C_2(\alpha) \leq 1$ for all $\alpha \in (-1,n)$, we can deduce for $\alpha \in [1/n, n)$ that

$$C_1(\alpha,n)^{\frac{n-1}{n}} = 0 \leq \big(1 - C_2(\alpha,n)\big)^{\frac{n-1}{n}}.$$

Observe that $V_{K,\xi}$ is monotone decreasing on $\mathbb{R}$. Let A be an equality case for the upper bound in Theorem 3. Then

$$C_2(\alpha,n)|A| = |A \cap H_\alpha^+| \leq |A \cap H_0^+| \leq C_2(0,n)|A|.$$

Thus, for $\alpha \in (0, 1/n)$ it follows that

$$C_2(\alpha, n) \leq 1 - \left(\frac{n}{n+1}\right)^n = C_2(0, n).$$

Observe that the function $f(\alpha) = (\alpha+1)^{n-1}(1-\alpha n)$ is positive and monotone decreasing on $(0, 1/n)$. Thus $(\alpha+1)^{n-1}(1-\alpha n) \leq 1$ so it follows that

$$\begin{aligned} C_1(\alpha, n)^{\frac{n-1}{n}} &= \left(\frac{n}{n+1}\right)^{n-1} (\alpha+1)^{\frac{(n-1)^2}{n}} (1-\alpha n)^{\frac{n-1}{n}} \\ &\leq \left(\frac{n}{n+1}\right)^{n-1} \leq \big(1 - C_2(\alpha, n)\big)^{\frac{n-1}{n}}. \end{aligned}$$

Thus, we have now shown the values for $c(\alpha, n)$ as seen in (12). For $\alpha \in (-1, n)$, equality is achieved when K is a convex body with $\xi^\perp + t_0\xi$ and H_α are two sections of a cone whose boundary coincides with the boundary of K. See Figure 6 for an example. □

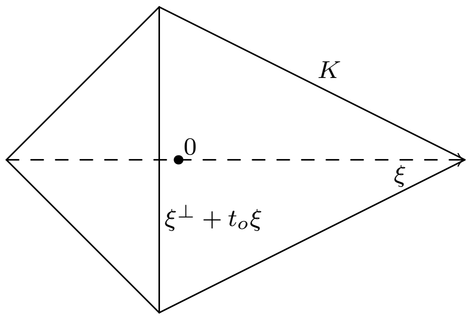

Figure 6: An example of the equality case in Theorem 4

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