A GENERALIZATION OF GRÜNBAUM'S INEQUALITY

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Abstract. Grünbaum's inequality gives sharp bounds between the volume of a convex body and its part cut off by a hyperplane through the centroid of the body. We provide a generalization of this inequality for hyperplanes that do not necessarily contain the centroid. As an application, we obtain a sharp inequality that compares sections of a convex body to the maximal section parallel to it.

1. Introduction

A *convex body* K is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. The *centroid* (also called the *center of mass*, or *barycenter*) of K is the point

$$g(K) = \frac{1}{|K|}\int_K x\,dx.$$

Here and throughout the paper, $|\cdot|$ denotes either the n-dimensional Lebesgue measure (volume) of a convex body or the $(n-1)$-dimensional Lebesgue measure of its sections. It should be clear from the context what is meant in each particular case. An inequality of Grünbaum [5] states if $K \subset \mathbb{R}^n$ is a convex body with centroid at the origin then

$$\left(\frac{n}{n+1}\right)^n \le \frac{|K \cap \xi^+|}{|K|} \le 1 - \left(\frac{n}{n+1}\right)^n, \quad \text{for all } \xi \in S^{n-1}. \tag{1}$$

Here $\xi^+ = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \ge 0\}$ and $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product. We also write S^{n-1} for the unit sphere in $\mathbb{R}^n$. The bounds in (1) are sharp and equality occurs in the lower bound when, for example, K is the cone

$$K = \operatorname{conv}\left(\frac{-1}{n+1}\xi + B_2^{n-1}, \frac{n}{n+1}\xi\right), \tag{2}$$

where we denote by B_2^{n-1} the closed unit $(n-1)$-dimensional Euclidean ball in $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$. The upper bound in (1) is easily obtained from the lower bound, and equality occurs in the upper bound when, for example, K is the reflection about the origin of the cone in (2). For recent advancements in Grünbaum-type inequalities for sections and projections of convex bodies see [3], [8], [9], [12].

In light of (1), the goal of this paper is to establish a similar result with hyperplanes that do not necessarily contain the centroid. For $\alpha \in (-1, n)$ and $\xi \in S^{n-1}$

2010 *Mathematics Subject Classification.* Primary 52A20.

Key words and phrases. Convex body, centroid, sections.

The first author was supported by an NSERC USRA award. The second author was supported by an NSERC Discovery Grant.

consider the half-space

$$H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \alpha h_K(-\xi)\},$$

where h_K is the support function for K (see Section 2 for the precise definition). We ask the following question: Are there positive constants $C_1(\alpha, n)$ and $C_2(\alpha, n)$, depending only on α and n, such that

$$C_1(\alpha, n) \leq \frac{|K \cap H_\alpha^+|}{|K|} \leq C_2(\alpha, n), \tag{3}$$

for every convex body $K \subset \mathbb{R}^n$ with centroid at the origin? We give an affirmative answer to this question. Both bounds are sharp, and the values of $C_1(\alpha, n)$ and $C_2(\alpha, n)$ are presented in Theorem 4, which also discusses the equality cases. The case $n = 2$ for (3) was obtained earlier in [11], where it was used to prove a discrete version of Grünbaum's inequality. It is important to note that in (1) one bound automatically determines the other bound. On the other hand, the bounds in (3) need to be shown separately.

As an application of (3) we obtain a generalization of the following result of Makai and Martini [7]; see also [2]. Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin, then

$$\left|K \cap \xi^\perp\right| \geq \left(\frac{n}{n+1}\right)^{n-1} \max_{t\in\mathbb{R}} \left|K \cap \left(\xi^\perp + t\xi\right)\right|, \quad \text{for all } \xi \in S^{n-1}. \tag{4}$$

The bound is sharp, and equality holds again if, for example, K is a cone as in (2). In this paper, we establish an analogue of the inequality above for sections that do not necessarily pass through the centroid. Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin, $\alpha \in (-1, n)$, and $\xi \in S^{n-1}$. Consider the hyperplane

$$H_\alpha = \{x \in \mathbb{R}^n : \langle x, \xi\rangle = \alpha h_K(-\xi)\}.$$

Then

$$|K \cap H_\alpha| \geq D(\alpha, n) \max_{t\in\mathbb{R}} \left|K \cap \left(\xi^\perp + t\xi\right)\right|,$$

where $D(\alpha, n)$ is a constant depending only on α and n. The inequality is sharp, and the exact value of $D(\alpha, n)$ is discussed in Theorem 5, along with equality cases.

2. Preliminaries

The *support function* $h_K : \mathbb{R}^n \longrightarrow \mathbb{R}$ for a convex body $K \subset \mathbb{R}^n$ is

$$h_K(\xi) = \max\{\langle x, \xi\rangle : x \in K\}.$$

If $\xi \in S^{n-1}$ then $h_K(\xi)$ gives the signed distance from the origin to the supporting hyperplane for K in the direction ξ. A result of Minkowski and Randon [1, p. 58] states if $K \subset \mathbb{R}^n$ is a convex body with centroid at the origin and $\xi \in S^{n-1}$, then

$$\frac{1}{n} h_K(\xi) \leq h_K(-\xi) \leq n h_K(\xi). \tag{5}$$

Note that the choice of bounds for α in Theorems 4 and 5 is a result of (5).

Let $\xi \in S^{n-1}$. The *parallel section function* $A_{K,\xi} : \mathbb{R} \longrightarrow \mathbb{R}$ for a convex body K is

$$A_{K,\xi}(t) = \left|K \cap \left(\xi^\perp + t\xi\right)\right|.$$

Lemma 1. *Let $K \subset \mathbb{R}^n$ be a convex body. Then $A_{K,\xi}^{1/(n-1)}$ is concave on its support, for every $\xi \in S^{n-1}$.*

For the proof of Lemma 1, refer to [6, p. 18].

Let $\xi \in S^{n-1}$. The *volume cut-off function* $V_{K,\xi} : \mathbb{R} \longrightarrow \mathbb{R}$ for a convex body $K \subset \mathbb{R}^n$ is

$$V_{K,\xi}(t) = \int_t^{\infty} A_{K,\xi}(s)\, ds.$$

The following result is also well-known, but we include a proof for completeness.

Lemma 2. *Let $K \subset \mathbb{R}^n$ be a convex body. Then $V_{K,\xi}^{1/n}$ is concave on its support, for every $\xi \in S^{n-1}$.*

Proof. Let $\lambda \in [0,1]$ and $t_1, t_2 \in \operatorname{supp}(V_{K,\xi})$. Note that

$$\begin{aligned}\lambda\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_1\}\Big) + (1-\lambda)\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_2\}\Big)&\\ \subset \Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \lambda t_1 + (1-\lambda)t_2\}\Big)&.\end{aligned}$$

This, together with the Brunn-Minkowski inequality (see [4, p. 415] or [10, p. 369]), implies that

$$\begin{aligned}&\Big|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \lambda t_1 + (1-\lambda)t_2\}\Big|^{1/n}\\ &\quad\geq \Big|\lambda\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_1\}\Big) + (1-\lambda)\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_2\}\Big)\Big|^{1/n}\\ &\quad\geq \lambda\Big|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_1\}\Big|^{1/n} + (1-\lambda)\Big|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_2\}\Big|^{1/n},\end{aligned}$$

which proves the result. $\square$

Let $K \subset \mathbb{R}^n$ be a convex body and $\xi \in S^{n-1}$. The *Schwarz symmetral* of K with respect to ξ is the convex body $S_\xi K$ such that for all $t \in [-h_K(-\xi), h_K(\xi)]$, the set $S_\xi K \cap (\xi^\perp + t\xi)$ is an $(n-1)$-dimensional Euclidean ball centered at $t\xi$ and $A_{K,\xi}(t) = A_{(S_\xi K),\xi}(t)$. By construction we obtain

$$h_K(\pm\xi) = h_{S_\xi K}(\pm\xi) \quad \text{and} \quad V_{K,\xi}(t) = V_{(S_\xi K),\xi}(t), \tag{6}$$

for all $t \in \mathbb{R}$. Note that the centroid of $S_\xi K$ lies on the line $\ell = \{t\xi : t \in \mathbb{R}\}$ due to the rotational symmetry of $S_\xi K$ about ℓ. See [4, p. 62] for more information on Schwarz symmetrizations.

3. Main Results

Before proving our main result, we will provide a simple remark that we will apply throughout the rest of the paper.

Remark 3. *Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin. Let $\alpha \in (-1, n)$ and $\xi \in S^{n-1}$. Denote $\overline{K} = K + h_K(-\xi)\xi$ and consider the two halfspaces $H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \alpha h_K(-\xi)\}$ and $\overline{H}_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq (\alpha+1)\langle g(\overline{K}), \xi\rangle\}$. Then*

$$\left|K \cap H_\alpha^+\right| = \left|\overline{K} \cap \overline{H}_\alpha^+\right|.$$

Proof. Since $\overline{K}$ is a translate of K, it is easy to see that the centroid of $\overline{K}$ is translated by the same vector, i.e., $g(\overline{K}) = h_K(-\xi)\xi$. Therefore,

$$\begin{aligned}\overline{H}_\alpha^+ &= \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq (\alpha+1)\left\langle g(\overline{K}), \xi\right\rangle\} \\ &= \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq (\alpha+1)h_K(-\xi)\} = H_\alpha^+ + h_K(-\xi)\xi,\end{aligned}$$

and the result follows. □

Statements analogous to Remark 3 also hold when $\geq$ is replaced with $\leq$ or $=$. We will now prove our main result.

Theorem 4. *Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin. Let $\alpha \in (-1, n)$ and $\xi \in S^{n-1}$. Consider the half-space*

$$H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \alpha h_K(-\xi)\}.$$

Then

$$C_1(\alpha, n) \leq \frac{|K \cap H_\alpha^+|}{|K|} \leq C_2(\alpha, n).$$

where

$$C_1(\alpha, n) = \begin{cases} \left(\frac{n-\alpha}{n+1}\right)^n, & \text{if } \alpha \in (-1, 0], \\ \left(\frac{n}{n+1}\right)^n (\alpha+1)^{n-1}(1-\alpha n), & \text{if } \alpha \in (0, 1/n), \\ 0, & \text{if } \alpha \in [1/n, n), \end{cases}$$

and

$$C_2(\alpha, n) = \begin{cases} 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n, & \text{if } \alpha \in (-1, 0], \\ c(\alpha, n), & \text{if } \alpha \in (0, n). \end{cases}$$

$c(\alpha, n)$ is a constant depending only on α and n. Determining the explicit value of $c(\alpha, n)$ involves finding the roots of a high-degree rational function. The lower bounds and upper bounds are sharp, and equality cases are discussed in the proof below.

Proof. Given K as written above, consider the Schwarz symmetral $S_\xi K$. Using the observations in (6) and Fubini's theorem we can conclude that the centroid of $S_\xi K$ is at the origin and that $|K \cap H_\alpha^+| = |(S_\xi K) \cap H_\alpha^+|$ for all $\alpha \in (-1, n)$. Therefore we will prove the result with $S_\xi K$, which we will denote by K for brevity. By Remark 3, it suffices to find bounds for $|\overline{K} \cap \overline{H}_\alpha^+|$, and after further abuse of notation, we will write K for $\overline{K}$ and H_α^+ for $\overline{H}_\alpha^+$. We will also write $H_\alpha = \{x \in \mathbb{R}^n : \langle x, \xi\rangle = (\alpha+1)\langle g(K), \xi\rangle\}$ and $H_\alpha^- = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \leq (\alpha+1)\langle g(K), \xi\rangle\}$. Let us remark that $\xi^\perp$ is now a supporting hyperplane of K and $0 \in \partial K$.

Let us first consider the case $\alpha \in (-1, 0]$. We will obtain the upper bound. Observe that

$$\left|K \cap H_\alpha^-\right| = |K| - \left|K \cap H_\alpha^+\right|.$$

Denote by $K/(\alpha+1)$ the dilation of K by a factor of $1/(\alpha+1) > 1$, and also write $H_\alpha^-/(\alpha+1) = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \leq \langle g(K), \xi\rangle\}$. Since $0 \in K$, we obtain $K \subset K/(\alpha+1)$

and thus

$$\begin{aligned}\left|K \cap H_\alpha^-\right| &= (\alpha+1)^n \left|\frac{1}{\alpha+1} K \cap \frac{1}{\alpha+1} H_\alpha^-\right| \\ &\geq (\alpha+1)^n \Big| K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \leq \langle g(K), \xi\rangle\}\Big| \\ &= (\alpha+1)^n \Big|\big(K - g(K)\big) \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \leq 0\}\Big| \geq (\alpha+1)^n \left(\frac{n}{n+1}\right)^n |K|,\end{aligned}$$

where we used Grünbaum's inequality (1). Therefore, for $\alpha \in (-1, 0]$, we have

$$\frac{|K \cap H_\alpha^+|}{|K|} \leq 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n,$$

as desired.

We will now obtain the lower bound for $\alpha \in (-1, 0]$. By Lemma 2, $V_{K,\xi}^{1/n}$ is concave on its support. Hence,

$$\begin{aligned}\left|K \cap H_\alpha^+\right|^{1/n} &= {V_{K,\xi}}^{1/n}\big((\alpha+1)\langle g(K), \xi\rangle\big) = {V_{K,\xi}}^{1/n}\big(-\alpha \cdot 0 + (\alpha+1)\langle g(K), \xi\rangle\big) \\ &\geq -\alpha {V_{K,\xi}}^{1/n}(0) + (\alpha+1) {V_{K,\xi}}^{1/n}\big(\langle g(K), \xi\rangle\big).\end{aligned}$$

Using Grünbaum's inequality and the observation that $V_{K,\xi}(0) = |K|$, we have

$$\left|K \cap H_\alpha^+\right|^{1/n} \geq -\alpha |K|^{1/n} + (\alpha+1)\left(\frac{n}{n+1}\right)|K|^{1/n},$$

which implies for $\alpha \in (-1, 0]$:

$$\left(\frac{n-\alpha}{n+1}\right)^n \leq \frac{|K \cap H_\alpha^+|}{|K|}.$$

Thus, we have shown the bounds for $\alpha \in (-1, 0]$.

We will now investigate the case $\alpha \in (0, n)$. We will prove the upper bound first. We can assume that $K \cap H_\alpha$ is non-empty, otherwise the bound is trivial. Let B_2^{n-1} be the unit $(n-1)$-dimensional Euclidean ball in $\xi^\perp$. By continuity we can find $r_1 \geq 0$ such that

$$K \cap \xi^\perp \subset r_1 B_2^{n-1} \quad \text{and} \quad \left|\operatorname{conv}(r_1 B_2^{n-1}, K \cap H_\alpha)\right| = \left|K \cap H_\alpha^-\right|.$$

Denote $L^- = \operatorname{conv}(r_1 B_2^{n-1}, K \cap H_\alpha)$. Then again by continuity, there are $r_2 \geq 0$ and μ with $(\alpha+1)\langle g(K), \xi\rangle < \mu < h_K(\xi)$ such that

$$\left|\operatorname{conv}(K \cap H_\alpha, r_2 B_2^{n-1} + \mu\xi)\right| = \left|K \cap H_\alpha^+\right|$$

and

$$L^- \cup \operatorname{conv}(K \cap H_\alpha, r_2 B_2^{n-1} + \mu\xi) = \operatorname{conv}(r_1 B_2^{n-1}, r_2 B_2^{n-1} + \mu\xi).$$

Denote $L^+ = \operatorname{conv}(K \cap H_\alpha, r_2 B_2^{n-1} + \mu\xi)$. Then $L = L^- \cup L^+$ is a truncated cone whose sections parallel to $\xi^\perp$ are Euclidean balls; see Figure 1. Note that $\langle g(L^-), \xi\rangle \leq \langle g(K \cap H_\alpha^-), \xi\rangle$ and $\langle g(L^+), \xi\rangle \leq \langle g(K \cap H_\alpha^+), \xi\rangle$, and thus

$$\langle g(L), \xi\rangle \leq \langle g(K), \xi\rangle.$$

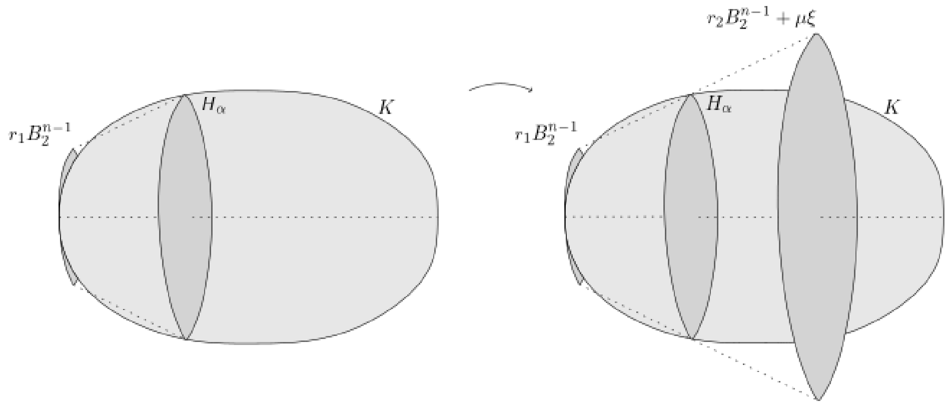


FIGURE 1. Constructing $r_1 B_2^{n-1}$ and $r_2 B_2^{n-1} + \mu\xi$.

By construction, we have $|K| = |L|$ and

$$\begin{aligned}|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq (\alpha+1)\langle g(K), \xi\rangle\}| \\ &= |L \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq (\alpha+1)\langle g(K), \xi\rangle\}| \\ &\leq |L \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq (\alpha+1)\langle g(L), \xi\rangle\}| \,.\end{aligned}$$

Hence, it suffices to work with L instead of K. After rescaling, we may assume that $h_L(\xi) = 1$ and then for $0 \leq t \leq 1$ we can assume that

$$A_{L,\xi}(t) = (mt+b)^{n-1}, \tag{7}$$

where either $m = 0$ and $b > 0$, or $b \geq 0$ and either (1) $m > 0$ or (2) $m < 0$ and $m + b \geq 0$. We will focus on the case $m \neq 0$ first, and address the case $m = 0$ later. By Fubini's theorem and (7) we obtain

$$|L| = \int_0^1 A_{L,\xi}(t)\, dt = \frac{(b+m)^n - b^n}{mn},$$

and similarly we find

$$\langle g(L), \xi\rangle = \frac{1}{|L|}\int_0^1 tA_{L,\xi}(t)\, dt = \frac{b^{n+1} + (mn-b)(b+m)^n}{m(n+1)\big((b+m)^n - b^n\big)}.$$

Denote $G_L = (\alpha+1)\,\langle g(L), \xi\rangle$. Now we can compute

$$\frac{|L \cap H_\alpha^+|}{|L|} = \frac{1}{|L|}\int_{G_L}^1 A_{L,\xi}(t)\, dt = \frac{(b+m)^n - (b+mG_L)^n}{(b+m)^n - b^n}.$$

Denote by φ the above equation of m and b for when $m \neq 0$. If $b > 0$ then we have $\varphi(m,b) \xrightarrow{m\to 0} (1-\alpha)/2$, which is readily verified to agree with the case $m = 0$. Making the change of variables $z = b/m$ allows us to write φ as a function of z. That is, we obtain

$$\varphi(z) = \frac{(z+1)^n - (z+G_L)^n}{(z+1)^n - z^n},$$

where

$$G_L = (\alpha+1)\frac{z^{n+1} + (n-z)(z+1)^n}{(n+1)\big((z+1)^n - z^n\big)},$$

for $z \in (-\infty, -1] \cup [0, \infty)$. For $\alpha \in (0, n)$, the rational function φ is not monotonic, so determining $c(\alpha, n)$ becomes an unfeasible task, as this involves finding roots of high-degree rational functions. When $n = 2$ one can explicitly solve for $c(\alpha, n)$ (see [11] for the derivation):

$$c(\alpha, 2) = \begin{cases} \frac{5-3\alpha}{9(\alpha+1)}, & \text{if } \alpha \in (0, 1), \\ \frac{1}{9}(2-\alpha)^2, & \text{if } \alpha \in [1, 2). \end{cases}$$

When $n \geq 3$ and $\alpha \in (0, n)$ all we can write is

$$\frac{|K \cap H_\alpha^+|}{|K|} \leq c(\alpha, n) = \sup\{\varphi(z) : z \in (-\infty, -1] \cup [0, \infty)\}.$$

We will now obtain the lower bound for $\alpha \in (0, n)$. Note for $\alpha \in [1/n, n)$ if K is the cone

$$K = \operatorname{conv}\left(-\frac{n}{n+1}\xi, \frac{1}{n+1}\xi + B_2^{n-1}\right),$$

then $|K \cap H_\alpha^+| = 0$. Therefore we cannot do better than $C_1(\alpha, n) = 0$.

Now assume that $\alpha \in (0, 1/n)$. By continuity, there is $v \geq h_K(\xi)$ such that

$$\left|\operatorname{conv}(K \cap H_\alpha, v\xi)\right| = \left|K \cap H_\alpha^+\right|.$$

Denote $M^+ = \operatorname{conv}(K \cap H_\alpha, v\xi)$. Then again by continuity there are r and β with $r > 0$ and $0 \leq \beta \leq (\alpha+1)\langle g(K), \xi\rangle$ such that

$$\operatorname{conv}(rB_2^{n-1} + \beta\xi, M^+) = \operatorname{conv}(rB_2^{n-1} + \beta\xi, v\xi),$$

and

$$\left|\operatorname{conv}(0, rB_2^{n-1} + \beta\xi, K \cap H_\alpha)\right| = \left|K \cap H_\alpha^-\right|.$$

Denote $M^- = \operatorname{conv}(0, rB_2^{n-1} + \beta\xi, K \cap H_\alpha)$. If $\beta > 0$ then $M = M^- \cup M^+$ is a convex body formed by the union of two cones with a common base in $\xi^\perp + \beta\xi$; see Figure 2. Such a body will be called a double cone. If $\beta = 0$ then M is a cone.

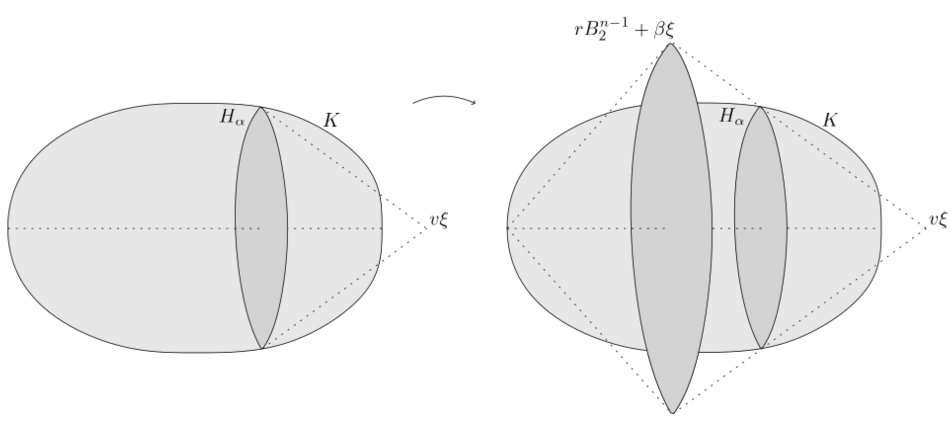


FIGURE 2. Constructing $rB_2^{n-1} + \beta\xi$ and $v\xi$.

Note that $\langle g(M^-),\xi\rangle \geq \langle g(K\cap H_\alpha^-),\xi\rangle$ and $\langle g(M^+),\xi\rangle \geq \langle g(K\cap H_\alpha^+),\xi\rangle$, and thus

$$\langle g(M),\xi\rangle \geq \langle g(K),\xi\rangle .$$

As a result we have constructed a convex body M where $|K|=|M|$ and

$$\begin{aligned}|K\cap\{x\in\mathbb{R}^n : x_1\geq(\alpha+1)\langle g(K),\xi\rangle\}|\\ &= |M\cap\{x\in\mathbb{R}^n : x_1\geq(\alpha+1)\langle g(K),\xi\rangle\}|\\ &\geq |M\cap\{x\in\mathbb{R}^n : x_1\geq(\alpha+1)\langle g(M),\xi\rangle\}| .\end{aligned}$$

Hence, it suffices to work with M instead of K. After rescaling, we may assume that $h_M(\xi)=1$ and $\left|rB_2^{n-1}\right|=n$. Define

$$M_1 = M\cap\{x\in\mathbb{R}^n \mid \langle x,\xi\rangle\leq\beta\} \quad\text{and}\quad M_2 = M\cap\{x\in\mathbb{R}^n \mid \langle x,\xi\rangle\geq\beta\},$$

to be the cones forming M. M_1 is degenerate if $\beta=0$. Since the heights of M_1 and M_2 are β and $1-\beta$ respectively, and $\left|rB_2^{n-1}\right|=n$, we get $|M_1|=\beta$ and $|M_2|=1-\beta$. It is a well-known fact that the centroid of a cone in $\mathbb{R}^n$ divides its height in the ratio $[1:n]$. Hence, we obtain $\langle g(M_1),\xi\rangle=(\beta n)/(n+1)$ and $\langle g(M_2),\xi\rangle=(\beta n+1)/(n+1)$, and thus it follows that

$$\begin{aligned}\langle g(M),\xi\rangle &= |M_1|\,\langle g(M_1),\xi\rangle + |M_2|\,\langle g(M_2),\xi\rangle\\ &= \beta\frac{\beta n}{n+1}+(1-\beta)\frac{\beta n+1}{n+1} = \frac{\beta(n-1)+1}{n+1}.\end{aligned}$$

Denote $G_M=(\alpha+1)\langle g(M),\xi\rangle$. We are interested in computing the volume of the intersection of M with the halfspace $H_\alpha^+=\{x\in\mathbb{R}^n : \langle x,\xi\rangle\geq G_M\}$. We will consider two cases, first when $0\leq\beta\leq G_M$, and then when $G_M\leq\beta<1$. These cases are equivalent to $0\leq\beta\leq\frac{\alpha+1}{2-(n-1)\alpha}$ and $\frac{\alpha+1}{2-(n-1)\alpha}\leq\beta<1$, respectively. In the first case, note that $M\cap H_\alpha^+$ is a cone homothetic to M_2 with the homothety coefficient equal to $(1-G_M)/(1-\beta)$. Therefore,

$$\left|M\cap H_\alpha^+\right| = \left(\frac{1-G_M}{1-\beta}\right)^n(1-\beta) = \frac{(1-G_M)^n}{(1-\beta)^{n-1}}.$$

In the second case, $M\cap H_\alpha^-$ is a cone homothetic to M_1 with the homothety coefficient equal to G_M/β. Thus,

$$\left|M\cap H_\alpha^+\right| = 1-\left|M\cap H_\alpha^-\right| = 1-\left(\frac{G_M}{\beta}\right)^n\beta = 1-\frac{G_M^n}{\beta^{n-1}}.$$

Summarizing, $|M\cap H_\alpha^+|$ is equal to the following piecewise function

$$\psi(\beta)=\begin{cases}\frac{(1-G_M)^n}{(1-\beta)^{n-1}}, & \text{if } 0\leq\beta\leq\frac{\alpha+1}{2-(n-1)\alpha},\\ 1-\frac{G_M^n}{\beta^{n-1}}, & \text{if } \frac{\alpha+1}{2-(n-1)\alpha}\leq\beta<1.\end{cases}$$

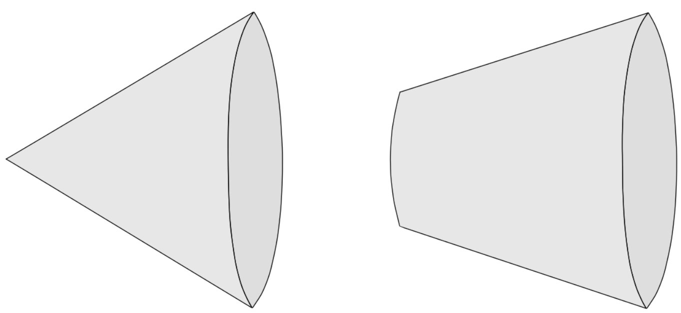
Our goal is to find the infimum of ψ on $[0,1)$ when $\alpha\in(0,1/n)$. Calculations show that the derivative of ψ vanishes at $\beta_0=\big((n+1)\alpha\big)/(\alpha+1)\in(0,\frac{\alpha+1}{2-(n-1)\alpha})$. Furthermore, ψ is decreasing on $[0,\beta_0)$ and increasing on $(\beta_0,1)$. Thus, the minimum of ψ is

$$\psi(\beta_0)=\left(\frac{n}{n+1}\right)^n(\alpha+1)^{n-1}(1-\alpha n),$$

which is the value of $C_1(\alpha,n)$ when $\alpha\in(0,1/n)$.

We will now discuss the equality cases. Recall that in both the upper and lower bound constructions, we performed the Schwarz symmetrization to transform the sections of K in the direction of ξ into $(n-1)$-dimensional Euclidean balls. We also performed dilations and translations. If we have an equality body K for either bound under these operations, then we can undo these operations to produce a new body whose sections are no longer $(n-1)$-dimensional Euclidean balls but instead $(n-1)$-dimensional convex bodies homothetic to each other.

We will start classifying equality cases for the upper bound. For $\alpha \in (-1, 0]$, we have equality from the equality conditions of Grünbaum's theorem, in other words $L = \text{conv}(B, v)$ is a cone with its base B being an $(n-1)$-dimensional convex body lying parallel to $\xi^{\perp}$ in ξ^{+} and vertex v lying in $\xi^{-} = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq 0\}$. For $\alpha \in (0, n)$, L is the convex hull of an $(n-1)$-dimensional convex body B and a homothetic copy of B. The coefficient of homothety is not explicit, but it can be found numerically for each n and α. In particular, up to translation and dilation, L is the convex hull of an $(n-1)$-dimensional convex body B in $\xi^{\perp}$ and its homothetic copy λB in $\xi^{\perp} + \xi$, where $\lambda = 1 + \frac{1}{z_0}$ and z_0 is the point where the maximum of φ is attained.



$-1 < \alpha \leq 0$ $\qquad$ $0 < \alpha < n$

FIGURE 3. Extremizing shapes for the upper bound.

We will now classify equality cases for the lower bound. For $\alpha \in (-1, 0]$, we have equality from the equality conditions of Grünbaum's theorem, in other words $M = \text{conv}(B, v)$ is a cone with its base B being an $(n-1)$-dimensional convex body lying parallel to $\xi^{\perp}$ in ξ^{-} and vertex v lying in ξ^{+}. For $\alpha \in (0, 1/n)$, the extremizing body is the union of two cones which share the same base, and whose heights are proportional to each other with the coefficient $\beta_0/(1-\beta_0)$. Recall that β_0 is the unique point of minimum for the function ψ. As α increases from 0 towards $1/n$, β_0 increases from 0 to 1, so B shifts in the direction of ξ. When $\alpha = 1/n$ we have the equality $|K \cap H_{\alpha}^{+}| = 0$ only in the case when $h_K(\xi) = \frac{1}{n} h_K(-\xi)$ (assuming the centroid of K is at the origin). The latter is possible only when K is a cone, which follows from the equality case in (5). For $\alpha \in (1/n, n)$, we have many bodies centered at the origin with property $h_K(\xi) < \alpha h_K(-\xi)$. All of them satisfy $|K \cap H_{\alpha}^{+}| = 0$.

□

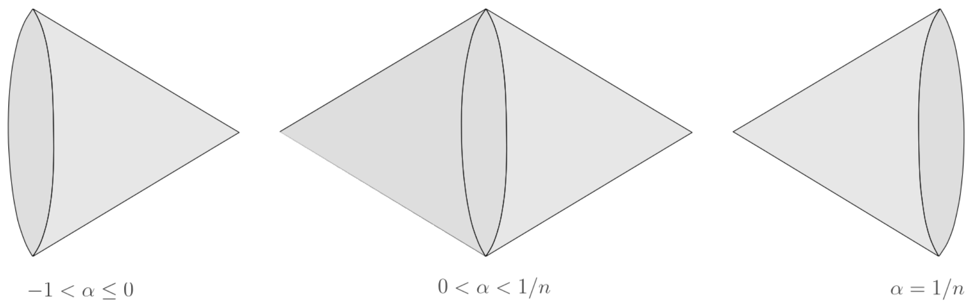


FIGURE 4. Extremizing shapes for the lower bound.

As an application of Theorem 4 we obtain a generalization of the result of Makai and Martini [7] stated in the introduction.

Theorem 5. *Let K be a convex body with centroid at the origin. Let $\xi \in S^{n-1}$ and $\alpha \in (-1, n)$. Consider the hyperplane*

$$H_\alpha = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = \alpha h_K(-\xi)\}.$$

Then

$$|K \cap H_\alpha| \geq D(\alpha, n) \sup_{t \in \mathbb{R}} \left| K \cap \left(\xi^\perp + t\xi\right) \right|,$$

where

$$D(\alpha, n) = \begin{cases} \left(\frac{n(\alpha+1)}{n+1}\right)^{n-1}, & \text{if } \alpha \in (-1, 0], \\ \left(\frac{n-\alpha}{n+1}\right)^{n-1}, & \text{if } \alpha \in (0, 1/n], \\ 0, & \text{if } \alpha \in (1/n, n). \end{cases}$$

The bound is sharp and equality cases are discussed in the proof below.

Proof. Note for $\alpha \in (1/n, n)$, if K is the cone

$$K = \operatorname{conv}\left(\frac{-n}{n+1}\xi, \frac{1}{n+1}\xi + B_2^{n-1}\right), \tag{8}$$

then it follows that $|K \cap H_\alpha| = 0$. Therefore for such α we cannot do better than $D(\alpha, n) = 0$.

We will now consider $\alpha \in (-1, 0]$. We can assume that

$$|K \cap H_\alpha| < \sup_{t \in \mathbb{R}} \left| K \cap \left(\xi^\perp + t\xi\right) \right|,$$

otherwise, the theorem follows immediately. We will apply the Schwarz symmetrization S_ξ to K. Abusing notation, we will denote the new body again by K. We will write

$$t_0 = \min\{t \in \mathbb{R} : A_{K,\xi}(t) = \max_{t \in \mathbb{R}} A_{K,\xi}(t)\},$$

so that $K \cap (\xi^\perp + t_0\xi)$ is a section of K orthogonal to ξ of maximal volume. Since $0 < |K \cap H_\alpha| < \left|K \cap \left(\xi^\perp + t_0\xi\right)\right|$ we can find a cone with base equal to $K \cap \left(\xi^\perp + t_0\xi\right)$ and section equal to $K \cap H_\alpha$. Such a cone is uniquely determined by these two sections. Denote this cone by N_1. Let $\gamma\xi$ be the vertex of N_1, for some number γ (either positive or negative). Due to the convexity of K, $\gamma\xi$ lies outside of K or on the boundary of K. Define N_2 to be the cone with base equal to $K \cap H_\alpha$ and vertex $\gamma\xi$; see Figure 5. Finally, we will let H_α^* be the halfspace bounded by the hyperplane H_α that contains N_2. We will consider two cases: $H_\alpha^* = H_\alpha^+ = \{x \in$

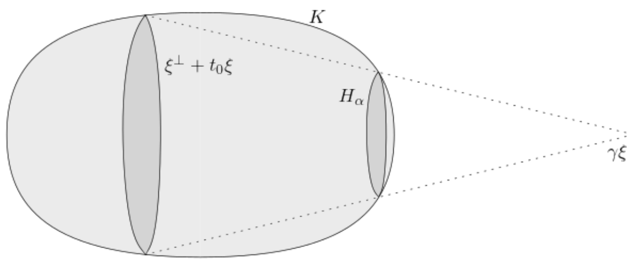


FIGURE 5. Constructing N_1 and N_2.

$\mathbb{R}^n : \langle x, \xi\rangle \geq \alpha h_K(-\xi)\}$ and $H_\alpha^* = H_\alpha^- = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \leq \alpha h_K(-\xi)\}$. Denote $h = \alpha h_K(-\xi)$. When $H_\alpha^* = H_\alpha^+$ the following inequality holds:

$$|K \cap H_\alpha| = \frac{|N_2|\, n}{|\gamma - h|} \geq \frac{|K \cap H_\alpha^+|\, n}{|\gamma - h|}.$$

Then by Theorem 4 and using that

$$|K| = \left|K \cap H_\alpha^+\right| + \left|K \cap H_\alpha^-\right| \geq C_1(\alpha, n)\, |K| + \left|K \cap H_\alpha^-\right|$$

we note that $\big(1 - C_1(\alpha, n)\big)|K| \geq |K \cap H_\alpha^-|$. We arrive at the following estimates

$$\begin{aligned} |K \cap H_\alpha| &\geq \frac{|K \cap H_\alpha^+|\, n}{|\gamma - h|} \geq C_1(\alpha, n)\frac{|K|\, n}{|\gamma - h|} \\ &\geq \frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\frac{|K \cap H_\alpha^-|\, n}{|\gamma - h|} \geq \frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\frac{|N_1 \setminus N_2|\, n}{|\gamma - h|}. \end{aligned}$$

Expressing the volumes of N_1 and N_2 in terms of their bases, we see

$$\begin{aligned} |K \cap H_\alpha| &\geq \frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\frac{(|N_1| - |N_2|)n}{|\gamma - h|} \\ &= \frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\frac{\left|K \cap \left(\xi^\perp + t_0\xi\right)\right| |\gamma - t_0|}{|\gamma - h|} - \frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\, |K \cap H_\alpha|\,. \end{aligned}$$

And so,

$$\begin{aligned} |K \cap H_\alpha| &\geq \frac{\frac{C_1(\alpha,n)}{1 - C_1(\alpha,n)}}{1 + \frac{C_1(\alpha,n)}{1 - C_1(\alpha,n)}}\frac{|K \cap (\xi^\perp + t_0\xi)||\gamma - t_0|}{|\gamma - h|} \\ &= C_1(\alpha, n)\frac{|K \cap (\xi^\perp + t_0\xi)||\gamma - t_0|}{|\gamma - h|}. \end{aligned}$$

Because N_1 is a homothetic copy of N_2, we can write

$$\frac{|\gamma - t_0|}{|\gamma - h|} = \frac{\left|K \cap \left(\xi^\perp + t_0\xi\right)\right|^{1/(n-1)}}{|K \cap H_\alpha|^{1/(n-1)}}.$$

Thus,

$$|K\cap H_\alpha| \geq C_1(\alpha,n)\left|K\cap\left(\xi^\perp+t_0\xi\right)\right| \frac{\left|K\cap\left(\xi^\perp+t_0\xi\right)\right|^{1/(n-1)}}{|K\cap H_\alpha|^{1/(n-1)}},$$

which implies

$$|K\cap H_\alpha| \geq C_1(\alpha,n)^{\frac{n-1}{n}}\left|K\cap\left(\xi^\perp+t_0\xi\right)\right|. \tag{9}$$

Now suppose $H_\alpha^* = H_\alpha^-$. Then the following inequality holds

$$|K\cap H_\alpha| = \frac{|N_2|\,n}{|\gamma-h|} \geq \frac{|K\cap H_\alpha^-|\,n}{|\gamma-h|}.$$

By Theorem 4 we have $\big((1-C_2(\alpha,n)\big)|K| \leq |K\cap H_\alpha^-|$ and so the following inequalities hold

$$\begin{aligned}|K\cap H_\alpha| &\geq \frac{|K\cap H_\alpha^-|\,n}{|\gamma-h|} \geq \big(1-C_2(\alpha,n)\big)\frac{|K|\,n}{|\gamma-h|}\\ &\geq \frac{1-C_2(\alpha,n)}{C_2(\alpha,n)}\frac{|K\cap H_\alpha^+|\,n}{|\gamma-h|} \geq \frac{1-C_2(\alpha,n)}{C_2(\alpha,n)}\frac{|N_1\setminus N_2|\,n}{|\gamma-h|}.\end{aligned}$$

Expressing the volumes of N_1 and N_2 in terms of their bases, we get

$$\begin{aligned}|K\cap H_\alpha| &\geq \frac{1-C_2(\alpha,n)}{C_2(\alpha,n)}\frac{(|N_1|-|N_2|)n}{|\gamma-h|}\\ &= \frac{1-C_2(\alpha,n)}{C_2(\alpha,n)}\frac{\left|K\cap\left(\xi^\perp+t_0\xi\right)\right||\gamma-t_0|}{|\gamma-h|} - \frac{1-C_2(\alpha,n)}{C_2(\alpha,n)}|K\cap H_\alpha|.\end{aligned}$$

So,

$$\begin{aligned}|K\cap H_\alpha| &\geq \frac{\frac{1-C_2(\alpha,n)}{C_2(\alpha,n)}}{1+\frac{1-C_2(\alpha,n)}{C_2(\alpha,n)}}\frac{|K\cap(\xi^\perp+t_0\xi)||\gamma-t_0|}{|\gamma-h|}\\ &= (1-C_2(\alpha,n))\frac{|K\cap(\xi^\perp+t_0\xi)||\gamma-t_0|}{|\gamma-h|}.\end{aligned}$$

Again using the homothety of N_1 and N_2, we arrive at

$$|K\cap H_\alpha| \geq \big(1-C_2(\alpha,n)\big)\left|K\cap\left(\xi^\perp+t_0\xi\right)\right| \frac{\left|K\cap\left(\xi^\perp+t_0\xi\right)\right|^{1/(n-1)}}{|K\cap H_\alpha|^{1/(n-1)}},$$

which implies

$$|K\cap H_\alpha| \geq \big(1-C_2(\alpha,n)\big)^{\frac{n-1}{n}}\left|K\cap\left(\xi^\perp+t_0\xi\right)\right|. \tag{10}$$

Now to determine $D(\alpha,n)$ we need to find the minimum of the two constants in equations (9) and (10) for fixed α. Note that $n\alpha \leq -\alpha$ for $\alpha\in(-1,0]$. Then it follows that

$$\big(1-C_2(\alpha,n)\big)^{\frac{n-1}{n}} = \left(\frac{n(\alpha+1)}{n+1}\right)^{n-1} \leq \left(\frac{n-\alpha}{n+1}\right)^{n-1} = C_1(\alpha,n)^{\frac{n-1}{n}},$$

for all $\alpha\in(-1,0]$, and thus we have our desired constant.

We will now consider $\alpha\in(0,1/n]$. We claim that it is enough to solve the problem for the class of double cones and truncated cones as it was done in Theorem 4. Our plan of attack to prove this claim is to show that when we construct such bodies following the procedure from Theorem 4, we can only decrease $|K\cap H_\alpha|$

and only increase the volume of the maximal section of K. It suffices to show this for the Schwarz symmetral $S_\xi K$ of K, which after abuse of notation we will denote by K. We will also employ Remark 3 and prove the result for $\overline{K}$ and $\overline{H}_\alpha = \{x \in \mathbb{R}^n : \langle x, \xi\rangle = (\alpha+1)\left\langle g(\overline{K}), \xi\right\rangle\}$. Again after abuse of notation, we will write K for $\overline{K}$ and H_α for $\overline{H}_\alpha$. Let

$$t_0 = \min\{t \in \mathbb{R} : A_{K,\xi}(t) = \max_{t\in\mathbb{R}} A_{K,\xi}(t)\},$$

so that $K \cap (\xi^\perp + t_0\xi)$ is a section of K orthogonal to ξ of maximal volume. For brevity, we will denote $G_K = (\alpha+1)\langle g(K), \xi\rangle$. We will split the analysis into two parts, according to whether $G_K < t_0 \le h_K(\xi)$ or $0 \le t_0 < G_K$. The case $t_0 = G_K$ is trivial.

Suppose that $G_K < t_0 \le h_K(\xi)$. Then following the upper bound construction in Theorem 4, we can construct a convex body

$$L = \operatorname{conv}(r_1 B_2^{n-1}, r_2 B_2^{n-1} + \mu\xi),$$

for some $r_1 \ge 0$ and $r_2 \ge 0$ such that $r_1 + r_2 > 0$, and μ such that $G_K < \mu \le h_K(\xi)$. Here, as before, B_2^{n-1} stands for the unit Euclidean ball in $\xi^\perp$. Write

$$r_{K,\xi}(t) = \omega^{-1/(n-1)} A_{K,\xi}^{1/(n-1)}(t) \quad \text{and} \quad r_{L,\xi}(t) = \omega^{-1/(n-1)} A_{L,\xi}^{1/(n-1)}(t),$$

where $\omega = |B_2^{n-1}|$. Observe that $r_{L,\xi}$ is affine on its support, and $r_{K,\xi}$ is concave on its support by Lemma 1. In fact we can write:

$$r_{L,\xi}(t) = \frac{r_2 - r_1}{\mu} t + r_1.$$

We claim that $r_2 > r_1$, i.e., $r_{L,\xi}$ is increasing. To reach a contradiction, assume $r_{L,\xi}$ is non-increasing. Since the graphs of $r_{L,\xi}$ and $r_{K,\xi}$ intersect at $t = G_K$ and at some other point $t < G_K$, the concavity of $r_{K,\xi}$ implies that $r_{K,\xi}(t) \le r_{L,\xi}(t)$ for $t \ge G_K$. Since we are assuming that $G_K < t_0$, we get $r_{K,\xi}(t_0) \le r_{L,\xi}(t_0) \le r_{L,\xi}(G_K) = r_{K,\xi}(G_K)$. This means that $t_0 \le G_K$, which is a contradiction.

Let us denote $G_L = (\alpha+1)\langle g(L), \xi\rangle$. Since $r_{L,\xi}$ is increasing, $\langle g(L), \xi\rangle \le \langle g(K), \xi\rangle$ (as in the proof of Theorem 4), and $A_{L,\xi}(G_K) = A_{K,\xi}(G_K)$, we obtain

$$\begin{aligned} |L \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle = G_L\}| &\le |L \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle = G_K\}| \\ &= |K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle = G_K\}|. \end{aligned}$$

We now want to show that the volume of the maximal section of L is no smaller than the volume of the maximal section of K. Suppose the opposite, that is

$$A_{K,\xi}(t_0) > A_{L,\xi}(\mu). \tag{11}$$

Then it follows by the construction of L and concavity of $r_{K,\xi}$ on its support that $\mu < t_0$. (Otherwise we would have $A_{K,\xi}(t_0) \le A_{L,\xi}(t_0) \le A_{L,\xi}(\mu)$). Raising both sides of (11) to the power $1/(n-1)$, we see

$$r_{K,\xi}(t_0) > r_{L,\xi}(\mu) = r_2.$$

Observe that the linear functions $\frac{r_2 - r_1}{\mu} t + r_1$ and $\frac{r_2 - r_{K,\xi}(G_K)}{\mu - G_K}(t - G_K) + r_{K,\xi}(G_K)$ coincide. Indeed, at $t = \mu$ they are both equal to r_2, and at $t = G_K$ they are equal to $\frac{r_2-r_1}{\mu}G_K + r_1 = r_{L,\xi}(G_K) = r_{K,\xi}(G_K)$. Thus we obtain

$$\begin{aligned}
|L \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq G_K\}| &= \omega \int_{G_K}^{\mu} \left(\frac{r_2 - r_1}{\mu} t + r_1\right)^{n-1} dt \\
&= \omega \int_{G_K}^{\mu} \left(\frac{r_2 - r_{K,\xi}(G_K)}{\mu - G_K}(t - G_K) + r_{K,\xi}(G_K)\right)^{n-1} dt \\
&= \frac{\omega}{n} \frac{\mu - G_K}{r_2 - r_{K,\xi}(G_K)} (r_2^n - r_{K,\xi}^n(G_K)) \\
&< \frac{\omega}{n} \frac{t_0 - G_K}{r_2 - r_{K,\xi}(G_K)} (r_2^n - r_{K,\xi}^n(G_K)) \\
&= \omega \int_{G_K}^{t_0} \left(\frac{r_2 - r_{K,\xi}(G_K)}{t_0 - G_K}(t - G_K) + r_{K,\xi}(G_K)\right)^{n-1} dt,
\end{aligned}$$

where we used $\mu < t_0$ for the above inequality. Denote

$$\zeta(t) = \frac{r_2 - r_{K,\xi}(G_K)}{t_0 - G_K}(t - G_K) + r_{K,\xi}(G_K).$$

Note that $\zeta(G_K) = r_{K,\xi}(G_K)$ and $\zeta(t_0) = r_2$. Since by assumption $r_2 < r_{K,\xi}(t_0)$, it follows from concavity that $\zeta(t) < r_{K,\xi}(t)$ for all $t \in (G_K, t_0]$, and thus we have

$$\omega \int_{G_K}^{t_0} \zeta^{n-1}(t)\, dt < \int_{G_K}^{t_0} A_{K,\xi}(t)\, dt \leq |K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq G_K\}|\,.$$

Combining all of the above inequalities, we obtain

$$|L \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq G_K\}| < |K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq G_K\}|\,,$$

reaching a contradiction to our construction in Theorem 4. Therefore, we must have

$$A_{K,\xi}(t_0) \leq A_{L,\xi}(\mu),$$

as desired.

From now on we can work with the body L instead of K. We are interested in minimizing $\frac{A_{L,\xi}(G_L)}{A_{L,\xi}(\mu)}$. As in Theorem 4, computing this minimum explicitly is not feasible, but we will work around this fact. For now, it is enough to note that $A_{L,\xi}(t)$ is increasing in t on its support, so it follows that $A_{L,\xi}(\langle g(L), \xi\rangle) \leq A_{L,\xi}(G_L)$. Hence we obtain for $\alpha \in (0, 1/n]$ the following inequalities

$$\left(\frac{n}{n+1}\right)^{n-1} \leq \frac{A_{L,\xi}(\langle g(L), \xi\rangle)}{A_{L,\xi}(\mu)} \leq \frac{A_{L,\xi}(G_L)}{A_{L,\xi}(\mu)}, \tag{12}$$

where we used the result of Makai and Martini (4).

Now suppose that $0 \leq t_0 < G_K$. Then following the lower bound construction in Theorem 4, we can construct a convex body $M = \mathrm{conv}(0, rB_2^{n-1} + \beta\xi, v\xi)$ for some $v \geq h_K(\xi)$, $r > 0$, and β such that $0 \leq \beta \leq G_K$. Note that $M \cap (\xi^\perp + \beta\xi)$ is the maximal section of M in the direction ξ. To be precise, in Theorem 4 the construction that produced a double cone was performed for $\alpha \in (0, 1/n)$. However the same construction works also for $\alpha = 1/n$, unless $H_{1/n}$ is a supporting hyperplane to K. In the latter case we can see that K is a body that can be reduced to the cone (8) and therefore is not an extremizer. Thus below we will exclude such bodies and work with all $\alpha \in (0, 1/n]$. In particular, we will have $\beta < v$.

As was done above above, we may write:

$$r_{M,\xi}(t) = \omega^{-1/(n-1)} A_{M,\xi}^{1/(n-1)}(t) = \begin{cases} \frac{r}{\beta} t, & \text{if } t \in [0,\beta], \\ \frac{r}{\beta - v}(t-\beta) + r, & \text{if } t \in (\beta, v]. \end{cases}$$

Recall from Theorem 4 that $\langle g(M), \xi\rangle \geq \langle g(K), \xi\rangle$. Denoting $G_M = (\alpha+1)\langle g(M), \xi\rangle$, we see that $\beta \leq G_K \leq G_M$, and hence since $A_{M,\xi}$ is decreasing on $[\beta, v]$, it follows that

$$\begin{aligned} |M \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle = G_M\}| &\leq |M \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle = G_K\}| \\ &= |K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle = G_K\}|. \end{aligned}$$

Now we want to show that the volume of the maximal section of M is no smaller than the volume of the maximal section of K. Again, suppose the opposite, that is

$$A_{K,\xi}(t_0) > A_{M,\xi}(\beta). \tag{13}$$

Then it follows by the construction of M and concavity of $r_{M,\xi}$ on its support that $\beta > t_0$. (Otherwise, using that $A_{K,\xi}(t) \leq A_{M,\xi}(t)$ for $t \in [\beta, G_K]$, $A_{M,\xi}(t)$ is decreasing for $t \geq \beta$, and $\beta \leq t_0 \leq G_K$, we would get $A_{K,\xi}(t_0) \leq A_{M,\xi}(t_0) \leq A_{M,\xi}(\beta)$). Raising both sides of (13) to the power $1/(n-1)$, we again obtain

$$r_{K,\xi}(t_0) > r_{M,\xi}(\beta) = r.$$

Since $r_{M,\xi}(G_K) = r_{K,\xi}(G_K)$, we have $\frac{r}{\beta - v} = \frac{r - r_{K,\xi}(G_K)}{\beta - G_K}$, and therefore

$$\begin{aligned} &\frac{1}{\omega}|M \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \leq G_K\}| \\ &\quad = \int_0^\beta \left(\frac{r}{\beta}\right)^{n-1} t^{n-1} dt + \int_\beta^{G_K} \left(\frac{r}{\beta - v}(t-\beta) + r\right)^{n-1} dt \\ &\quad = \int_0^\beta \left(\frac{r}{\beta}\right)^{n-1} t^{n-1} dt + \int_\beta^{G_K} \left(\frac{r - r_{K,\xi}(G_K)}{\beta - G_K}(t-\beta) + r\right)^{n-1} dt \\ &\quad = \frac{1}{n}\left(\beta r^{n-1} + \frac{\beta - G_K}{r - r_{K,\xi}(G_K)}\left(r_{K,\xi}^n(G_K) - r^n\right)\right). \end{aligned}$$

Observe that the latter linear function of β is decreasing, since its slope is negative:

$$r^{n-1} + \frac{r_{K,\xi}^n(G_K) - r^n}{r - r_{K,\xi}(G_K)} = \frac{r_{K,\xi}(G_K)\left(r_{K,\xi}^{n-1}(G_K) - r^{n-1}\right)}{r - r_{K,\xi}(G_K)} < 0.$$

Therefore, using $\beta > t_0$, we obtain

$$\begin{aligned} &\frac{1}{\omega}|M \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \leq G_K\}| \\ &\quad < \frac{1}{n}\left(t_0 r^{n-1} + \frac{t_0 - G_K}{r - r_{K,\xi}(G_K)}\left(r_{K,\xi}^n(G_K) - r^n\right)\right) \\ &\quad = \int_0^{t_0} \left(\frac{r}{t_0}\right)^{n-1} t^{n-1} dt + \int_{t_0}^{G_K} \left(\frac{r - r_{K,\xi}(G_K)}{t_0 - G_K}(t-t_0) + r\right)^{n-1} dt, \end{aligned}$$

Let us write

$$\zeta(t) = \begin{cases} \frac{r}{t_0} t, & \text{if } t \in [0, t_0], \\ \frac{r - r_{K,\xi}(G_K)}{t_0 - G_K}(t - t_0) + r, & \text{if } t \in (t_0, G_K]. \end{cases}$$

Note that $\zeta(G_K) = r_{K,\xi}(G_K)$. Since by assumption $r < r_{K,\xi}(t_0)$, it follows from concavity that $\zeta(t) < r_{K,\xi}(t)$ for all $t \in [0, G_K)$, and thus we have

$$\omega \int_0^{G_K} \zeta^{n-1}(t)\, dt < \int_0^{G_K} A_{K,\xi}(t)\, dt = \left|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \le G_K\}\right|.$$

Combining all of the above inequalities, we obtain

$$\left|M \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \le G_K\}\right| < \left|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \le G_K\}\right|,$$

reaching a contradiction to our construction in Theorem 4. Therefore, we must have

$$A_{K,\xi}(t_0) \le A_{M,\xi}(\beta),$$

and hence, it suffices to work with M instead of K. After rescaling, we may assume that $h_M(\xi) = 1$ and $\left|rB_2^{n-1}\right| = n$. As before, we will define

$$M_2 = M \cap \{x \in \mathbb{R}^n \mid \langle x, \xi\rangle \ge \beta\}.$$

Taking our computations from Theorem 4, we have $|M_2| = 1 - \beta$ and

$$\left|M \cap H_\alpha^+\right| = \frac{(1 - G_M)^n}{(1 - \beta)^{n-1}},$$

where

$$G_M = (\alpha + 1)\frac{\beta(n-1) + 1}{n+1}.$$

Expressing the volumes of the sections we are interested in terms of the volumes of the cones M_2 and $M \cap H_\alpha^+$, we can write

$$\frac{|M \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle = G_M\}|}{\left|M \cap (\xi^\perp + \beta\xi)\right|} = \frac{|M \cap H_\alpha^+|}{|M_2|}\frac{1-\beta}{1 - G_M} = \left(\frac{1 - G_M}{1-\beta}\right)^{n-1}.$$

Denote by ψ the above function of β. Our goal is to find the minimum of ψ when $0 \le \beta \le G_M$, or equivalently when $\beta \in [0, \frac{\alpha+1}{2-(n-1)\alpha}]$ (as taken from Theorem 4). One can check that ψ is increasing in β and therefore the minimum of ψ is attained at $\beta = 0$,

$$\psi(\beta) \ge \psi(0) = \left(\frac{n - \alpha}{n+1}\right)^{n-1}. \tag{14}$$

Additionally, since $\beta = 0$ the extremizing body is a cone.

Now to determine the value of $D(\alpha, n)$ for $\alpha \in (0, 1/n]$, we need to find the lower of the two constants in (12) and (14). It is enough to note for $\alpha \in (0, 1/n]$ that

$$\left(\frac{n-\alpha}{n+1}\right)^{n-1} \le \left(\frac{n}{n+1}\right)^{n-1},$$

and the result follows.

Discussing equality cases, we see for $\alpha \in (-1, 0]$ that equality follows from the equality cases for the upper bound in Theorem 3 (which comes from Grünbaum's original theorem), and thus the equality bodies are, up to translation, cones of the form $\mathrm{conv}(B, v)$ with B an $(n-1)$-dimensional convex body lying parallel to $\xi^\perp$ in ξ^+ and vertex v lying in ξ^-. For $\alpha \in (0, 1/n]$, the equality bodies are cones of the form $\mathrm{conv}(B, v)$ with B an $(n-1)$-dimensional convex body lying parallel to $\xi^\perp$ in ξ^- and vertex v lying in ξ^+; see Figure 6. For $\alpha \in (1/n, n)$, there are many bodies for which $K \cap H_\alpha = 0$. Therefore we omit this case from Figure 6. $\square$

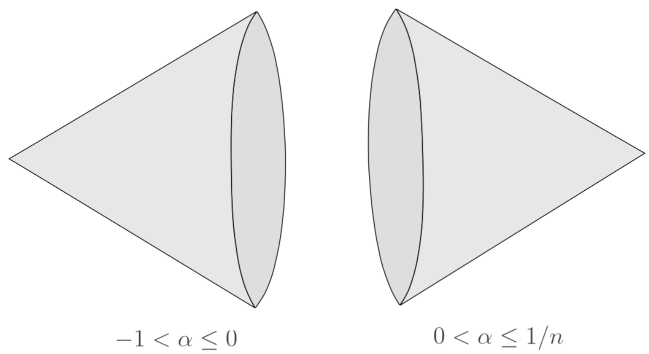


FIGURE 6. Extremizing shapes for Theorem 5.

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