Week 2 – Conditional Probability and Independence

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Last class

- Sample spaces (finite)
- Probability functions on discrete spaces
- Uniform probability spaces
- Multiplicative counting principle
- Permutations/combinations

Repeated experiments/Product of sample spaces

E.g. Flip a coin twice

$\Omega_1 = \{H, T\}$

$\Omega = \{(H,H),(H,T),(T,H),(T,T)\} = \Omega_1 \times \Omega_1 = \Omega_1^2$

E.g. Flip a coin n times:

$\Omega = \underbrace{\Omega_1 \times \ldots \times \Omega_1}_{n \text{ times}} = \Omega_1^n$ all n-tuples with elements in $\{H, T\}$:

E.g. Flip a coin and then pick a month at random

$\Omega_1 = \{H, T\}, \Omega_2 = \{Jan, Feb, Mar, Apr, May, Jun, Jul, Aug, Sep, Oct, Nov, Dec\}$

$\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2), \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$

- Q: How many elements in Ω?

Repeated experiments/Product of sample spaces

If we have a probability function P_1 defined in Ω_1 and a probability P_1 defined in Ω_2 we can naturally defined a probability P in $\Omega^1 \times \Omega^2$ as:

$$P(\{\omega^1, \omega^2\}) = P_1(\omega^1)P_2(\omega^2)$$

E.g.

$P(\{H, Jul\}) = P(H) \times P(Jul) = \frac{1}{2} \times \frac{1}{12}$

This is how we model independence, which will cover laer today

Infinite (countable) sample spaces

E.g. number of coin flips until first heads appears:

What is the right probability space for this experiment?

$\Omega = \{1, 2, 3, \ldots\} = \mathbb{N}$

Sample space has to be infinite because no guarantee experiment will terminate in a finite number of steps!

If we assume that after k flips all $k-$tuples are equally likely what should the probability $P(k)$ be?

$\{k = 1\} = \{H\} \implies P(1) = \frac{1}{2}$

$\{k = 2\} = \{T, H\} \implies P(2) = \frac{1}{4}$

$\vdots$

$\{k\} = \{\underbrace{T, \ldots, T}_{\text{k-1 times}}, H\} \implies P(k) = \frac{1}{2^k}$

Does this result in a proper probability function?

Infinite (countable) sample spaces

For infinite sample spaces need to change additivity rule to countably additivity rule:

$P(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k)$ provided thy are disjoint ($A_k \cup A_l = \emptyset \; \forall k, l \in \mathbb{N}, k \neq l$)

$$P(\Omega) = P(\{1, 2, 3 \ldots\}) = P(1) + P(2) + P(3) + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

(Used that for a geometric series: $1 + r + r^2 + \ldots = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$)

By extension of the rule for finite sample spaces, the probability defined above is a proper probability function.

Conditional Probability

Example: You roll a fair 6-faced die. Let A be the event that the outcome is an odd number, $A = 1, 3, 5$ Let B be the even that the outcome is less than 4, $B = 1, 2, 3$. What is the probability of A? What is the probability of A given B?

$$P(A) = \frac{|A|}{|S|} = \frac{|\{1,2,3\}|}{|S|} = \frac{3}{6} = \frac{1}{2}$$

$$P(A \mid B) = \frac{|A \cap B|}{|B|} = \frac{2}{3}$$

We can write $P(A \mid B) = \frac{|A \cap B|}{|B|} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} = \frac{P(A \cap B)}{P(B)}$

Conditional Probability

If A and B are events, and $P(B) > 0$ the conditional probability of A given B is defined as:

$$P\left(A \mid B\right) = \frac{P(A \cap B)}{P(B)}$$

- Fraction of the probability B also in the event A
- Tells us how to update probability in the presence of new information
- Example:
  - What is the probability that two cards drawn at random from a deck of playing cards will both be aces? A: $\frac{\binom{4}{2}}{\binom{52}{2}} = \frac{4 \times 3}{52 \times 51}$
  - What is the probability that two cards drawn at random from a deck of playing cards will both be aces if after dealing the first card it is an Ace? A: $\frac{3}{51}$

Bayes Rule

From the definition we get the properties:

Multiplication rule:

$$P(A \cap B) = P(A \mid B) P(B)$$

Bayes rule:

$$P(B \mid A) = \frac{P(A \mid B) P(B)}{P(A)} \quad (\text{if } P(A) > 0)$$

Bayes Rule

Example: There are approximately 2.6 physicians per 1,000 people in the US (from world public health data by country)

Probability of choosing a physician if randomly choose a US inhabitant = $\frac{2.6}{1{,}000} = 0.0026$

$A = \{\text{Being a Physician in the US}\} = \{\text{Physician}\}$

$B = \{\text{Being a Woman in the US}\} = \{\text{Woman}\}$

$P(\text{Physician}) = \frac{2.6}{1{,}000}$

$P(\text{Woman}) = 0.508$ (From US Census)

$P(\text{Woman} \mid \text{Physician}) = 0.36$ (From Labor statistics)

$P(\text{Physician} \mid \text{Woman}) = \frac{P(\text{Woman} \mid \text{Physician})P(\text{Physician})}{P(\text{Woman})} = \frac{2.6}{1{,}000}\frac{0.36}{0.58} = \frac{1.6}{1{,}000}$

Some special cases

- A and B disjoint $\implies P(A \mid B) = P(B \mid A) = 0$
- $B \subset A \implies P(A \mid B) = 1$
- $A \subset B \implies P(A \mid B) = \frac{P(A)}{P(B)}$

Conditional probability is a probability function

For fixed C with $P(C) > 0$ the conditional probability $P_C(\cdot) = P(\cdot \mid C)$ is a probability function:

1. $P_C(A) = P(A \mid C) \geq 0$
2. $P_C(\Omega) = P(\Omega \mid C) = 1$
3. $P_C(A \cup B) = P_C(A) + P_C(B)$ if $A \cap B = \emptyset$

Law of total probability

If the sample space can be partition as $\Omega = \cup_{i=1}^{n} A_i$, with $A_1, \ldots, A_n$ disjoint, then:

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i)$$

(holds even for a countable partition)

In particular, for any event A, the sample space is can be partitioned as $\Omega = A \cup A^c$:

$$P(B) = P(B \mid A)P(A) + P(B \mid A^c)P(A^c)$$

Law of total probability example

The probability of infection from a certain virus upon exposure is 10% for children age < 13, 5% for ages $13 - 60$, and 15% for ages $60+$. What is the probability that a random individual is infected upon exposure in a population where $P(\text{Age} < 13) = 0.2$, $P(13 \leq \text{Age} \leq 60) = 0.6$, $P(\text{Age} > 60) = 0.2$?

Let I denote the event of infection:

$$
\begin{aligned}
P(I) = {} & P(I \mid \text{Age} < 13)P(\text{Age} < 13) + P(I \mid 13 \leq \text{Age} \leq 60)P(13 \leq \text{Age} \leq 60) + \\
& + P(I \mid \text{Age} > 60)P(\text{Age} > 60) = \\
& = 0.1 \times 0.2 + 0.05 \times 0.6 + 0.15 \times 0.2 = 0.08
\end{aligned}
$$

Medical testing example

- A diagnostic test has 99% sensitivity and 98% specificity
- If the population prevalence of the disease is 3%, what is the probability that an individual who tests positive is affected with the disease?

Testing example

- $\text{Sensitivity} = P(\text{Test+} \mid \text{Affected}) = 0.99$
- $\text{Specificity} = P(\text{Test-} \mid \text{Not Affected}) = 1 - P(\text{Test+} \mid \text{Not Affected})$
- $(P(\text{Test-} \mid \text{Not Affected}) = 1 - \text{Specificity} = 0.02)$
- $\text{Prevalence} = P(\text{Affected}) = 0.03$

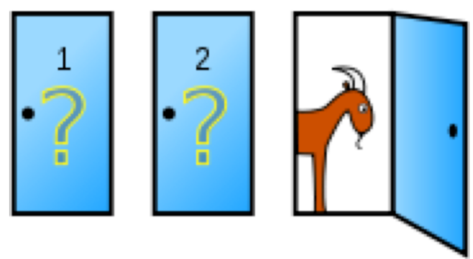
$$P(\text{Affected} \mid \text{Test+}) = ?$$

$$\begin{aligned} P(\text{Test+}) &= P(\text{Test+} \mid \text{Affected})P(\text{Affected}) + P(\text{Test+} \mid \text{Not Affected})P(\text{Not Affected}) \\ &= 0.99 \times 0.03 + 0.02 \times 0.97 = 0.0297 + 0.0194 = 0.0491 \end{aligned}$$

$$P(\text{Affected} \mid \text{Test+}) = \frac{P(\text{Test+} \mid \text{Affected})P(\text{Affected})}{P(\text{Test+})} = \frac{0.0297}{0.0491} = 0.605$$

Monty Hall Problem

- You’re on a game show, and you’re given the choice of three doors:
  - Behind one door is a car
  - behind the others, goats
- You pick a door, say No. 1, and the host, who knows what’s behind the doors, opens another door, say No. 3, which has a goat.
- He then says to you, “Do you want to switch to door No. 2 or keep prize behind door No. 1?”
- **Should you switch?**



- By Cepheus - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=1234194

Independence

Two events A and B are independent iff (if and only if) $P(A \cap B) = P(A)P(B)$

E.g. fair coin tossed twice:

$P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$

$P(\{H \text{ in first toss}\}) = P(HH) + P(HT) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(\{H \text{ in second toss}\}) = P(HH) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(\{H \text{ in first toss}\}) \cap \{H \text{ in second toss}\}) = P(\{H \text{ in first toss}\})P(\{H \text{ in second toss}\})$

The events $\{H \text{ in first toss}\}$ and $\{H \text{ in second toss}\}$ are independent

(in fact any first toss outcome is independent of any second toss outcome)

Independence

Independence between A and B is equivalent to:

1. $P(A \mid B) = P(A)$
2. $P(B \mid A) = P(B)$
3. A and B^c are independent (or A^c and B are independent, or A^c and B^c are independent)

Independence of multiple events

$A_1, \ldots A_n$ are independent iff

$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2) \ldots P(A_n)$

And also the equation above also holds replacing any number of the A_is by their complements (2^n equations!)

Pairwise independence does not imply independence!!

Example: Two tosses of a fair coin

$A = \{H \text{ in first toss}\}$

$B = \{H \text{ in second toss}\}$

$C = \{ \text{ two tosses are equal}\}$

Next week

- Read Ch 3 from 'Introduction to Probability, Statistics, and Random Processes'
- Try to solve the example problems before looking at the solutions to check your understanding
- Do the end-of-chapter exercises to practice
- Homework 1 due