Disjoint-set data structure: Union-find

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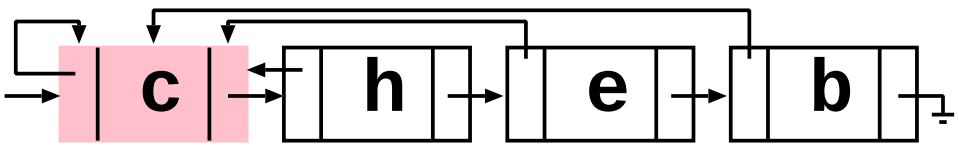

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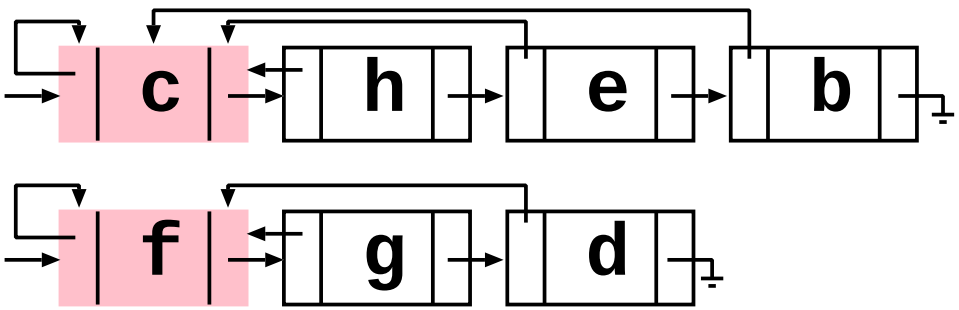

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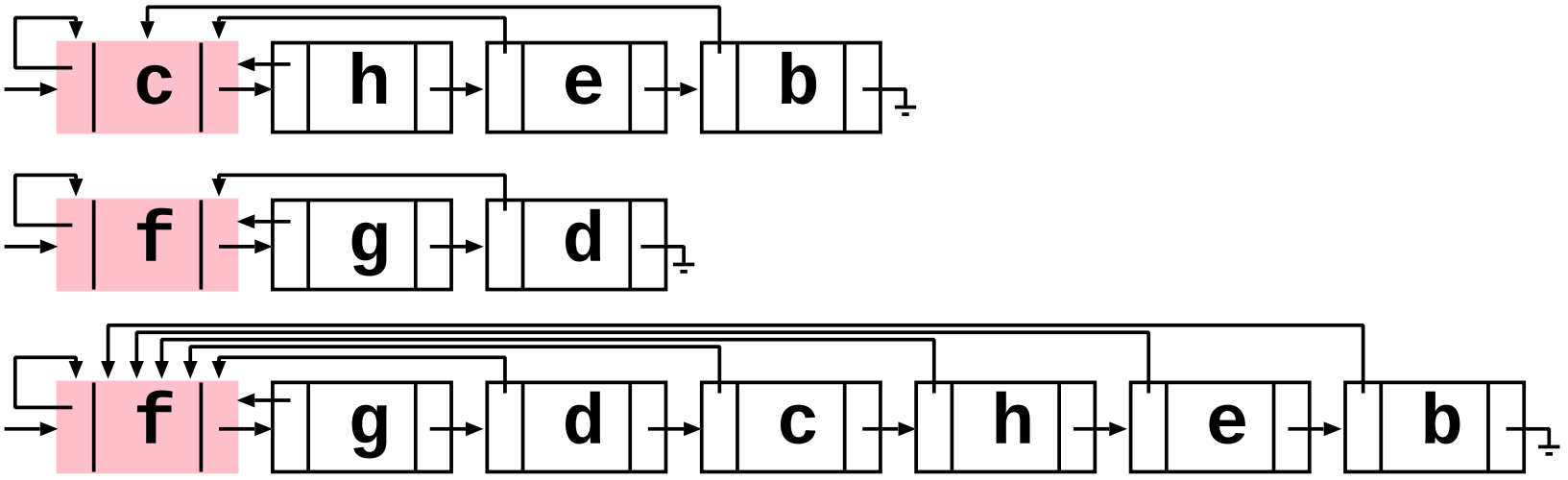

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  4. **representative member**: root of the tree;

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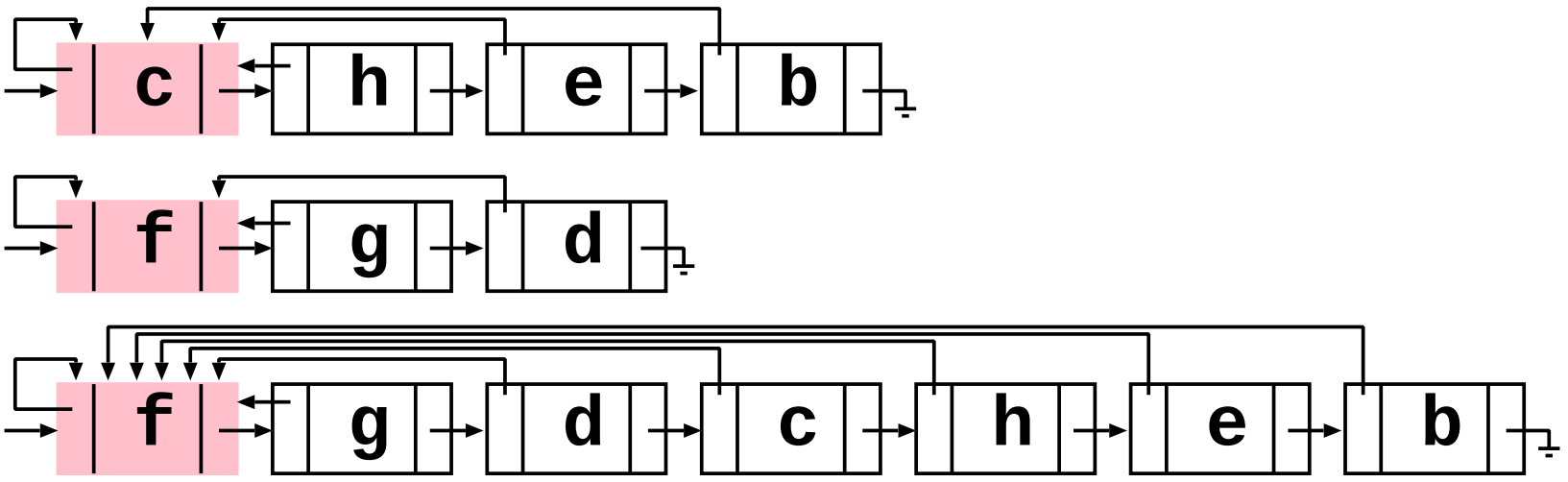

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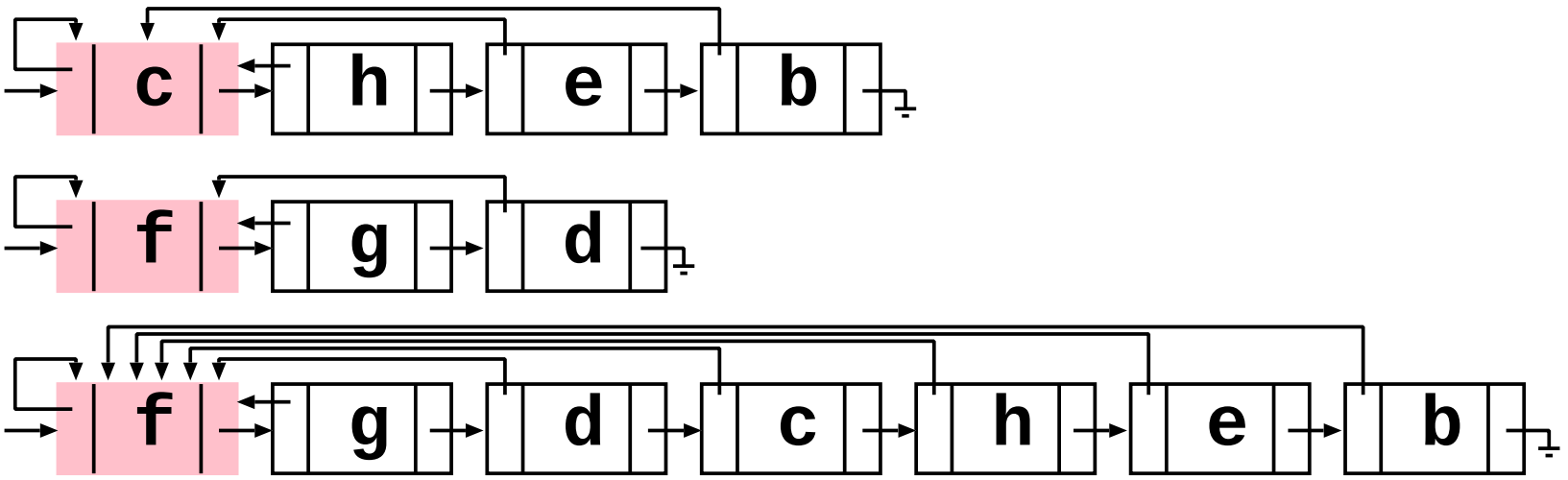

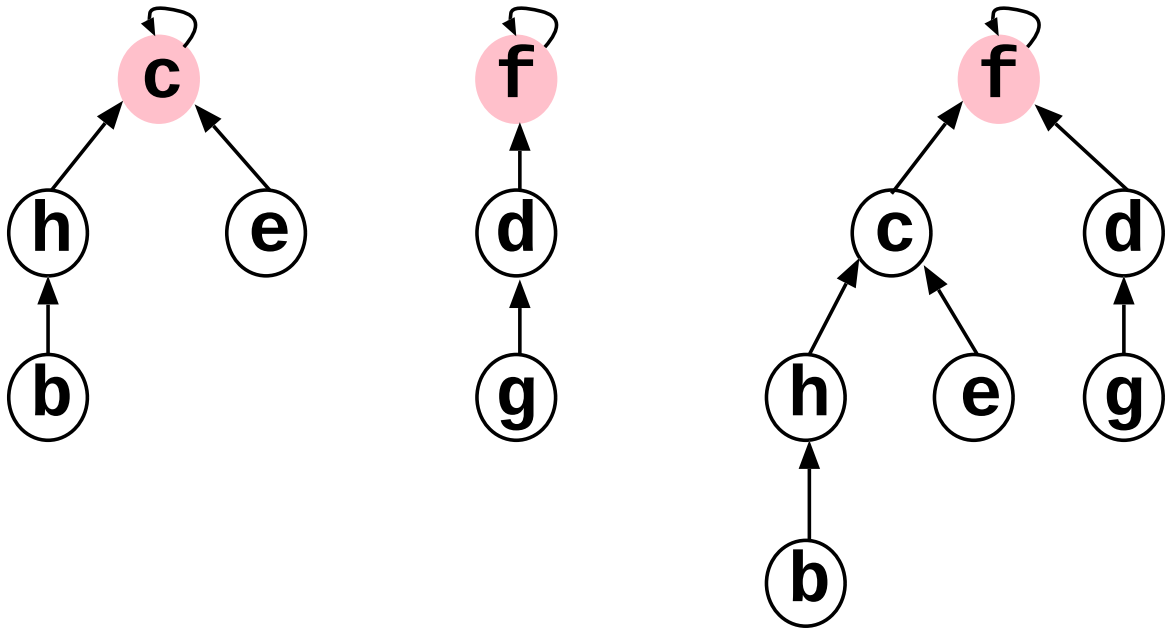

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  2. **path compression**: each node points directly to the root;

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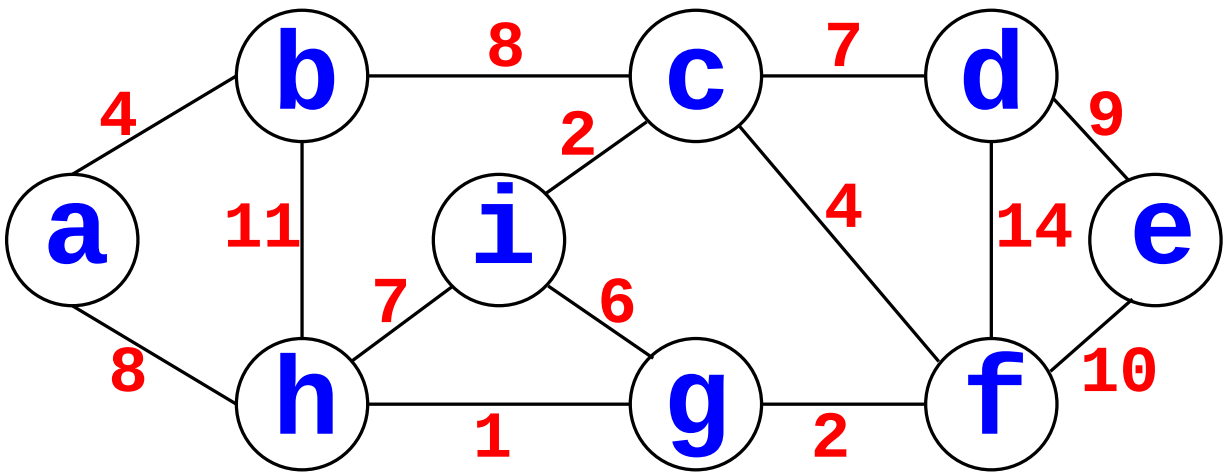

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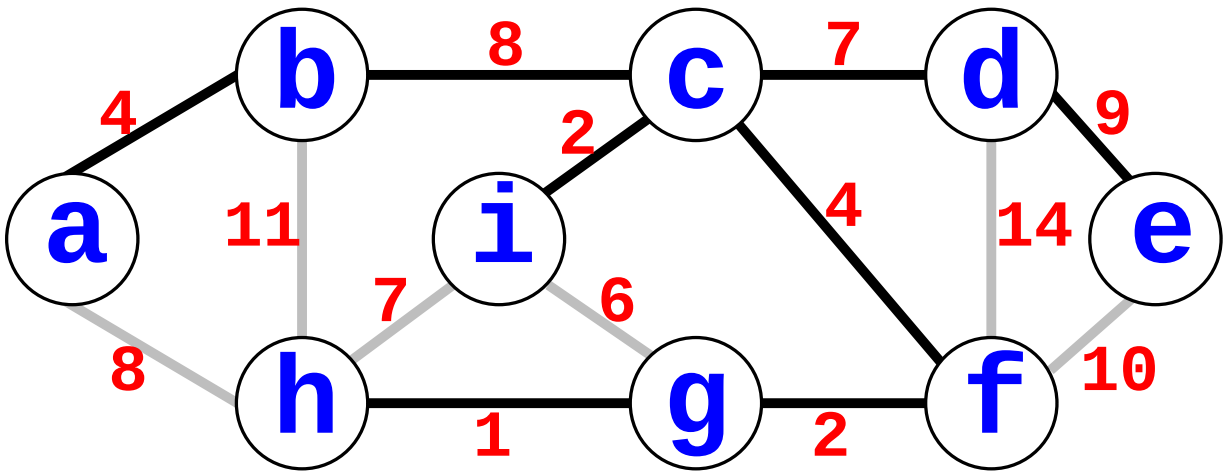

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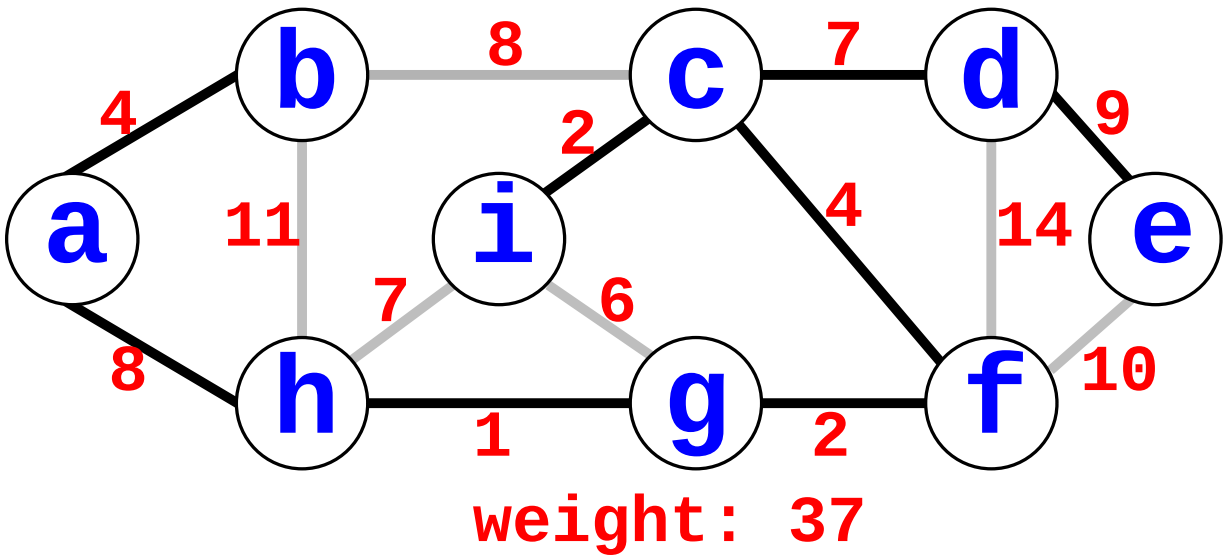

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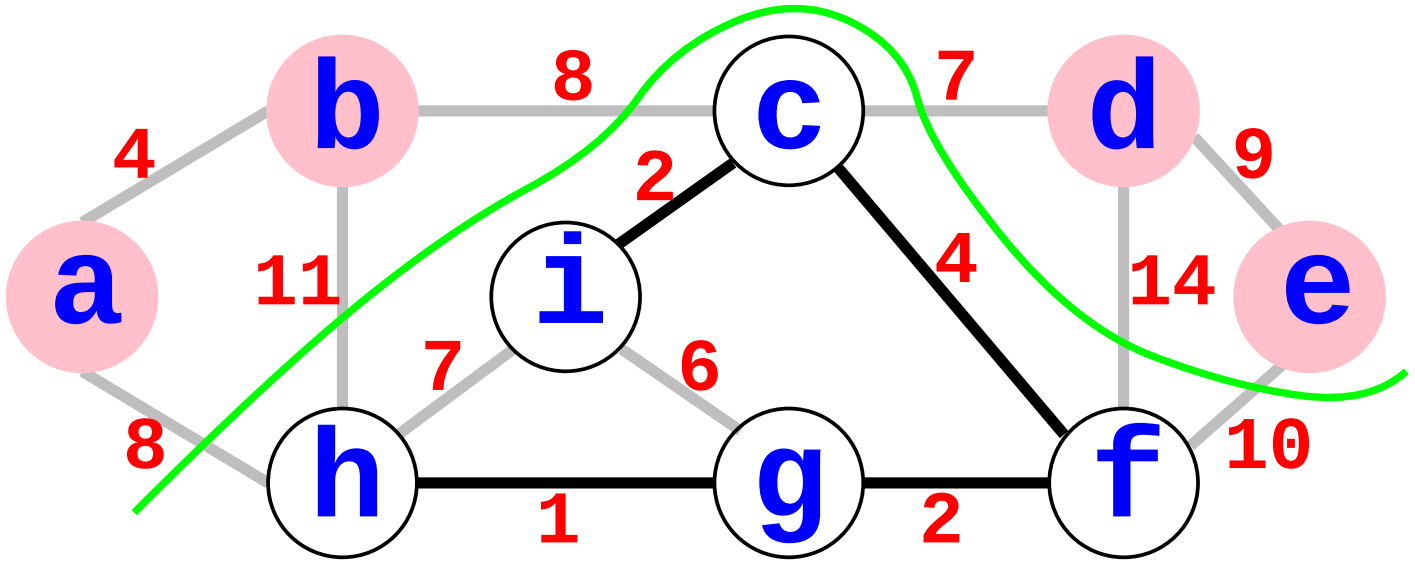
Safe edge: does not create cycles in A

Minimum spanning tree: general algorithm

Light edge: edge with the minimum weight crossing the cut;



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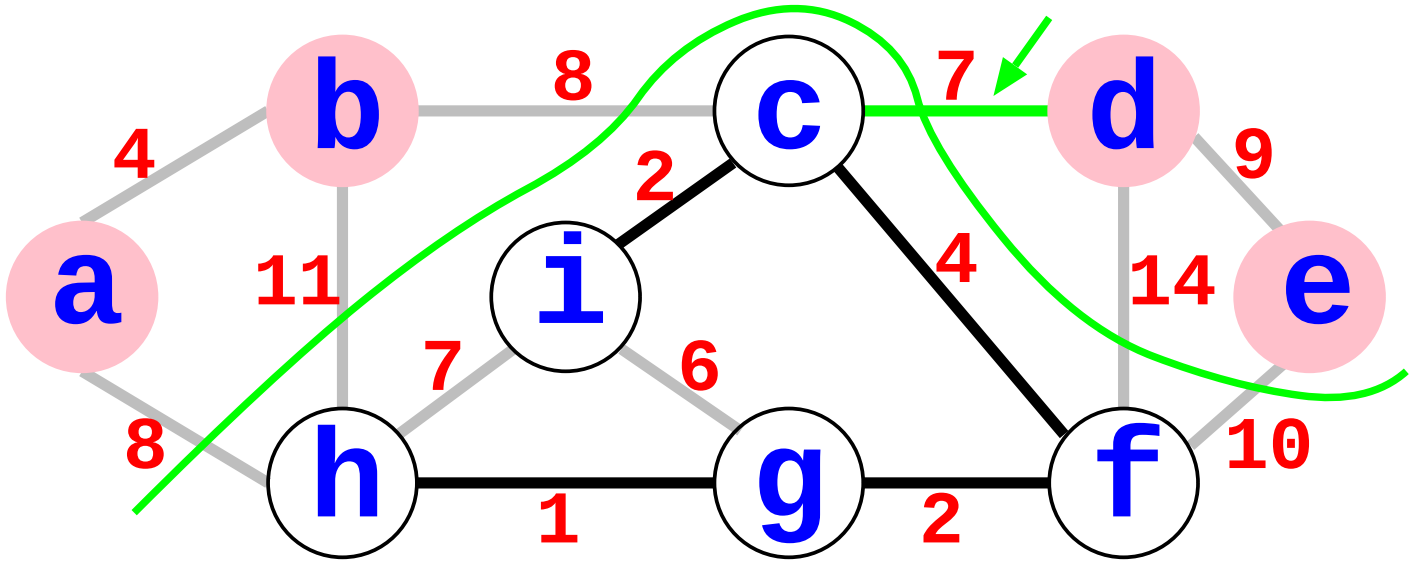

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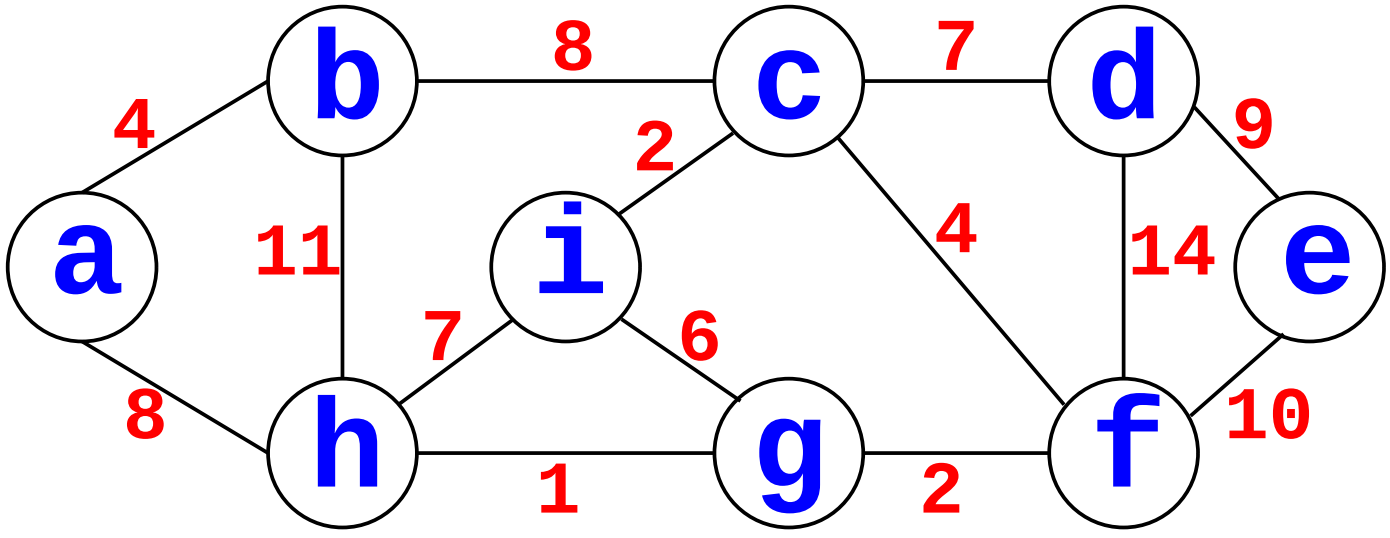

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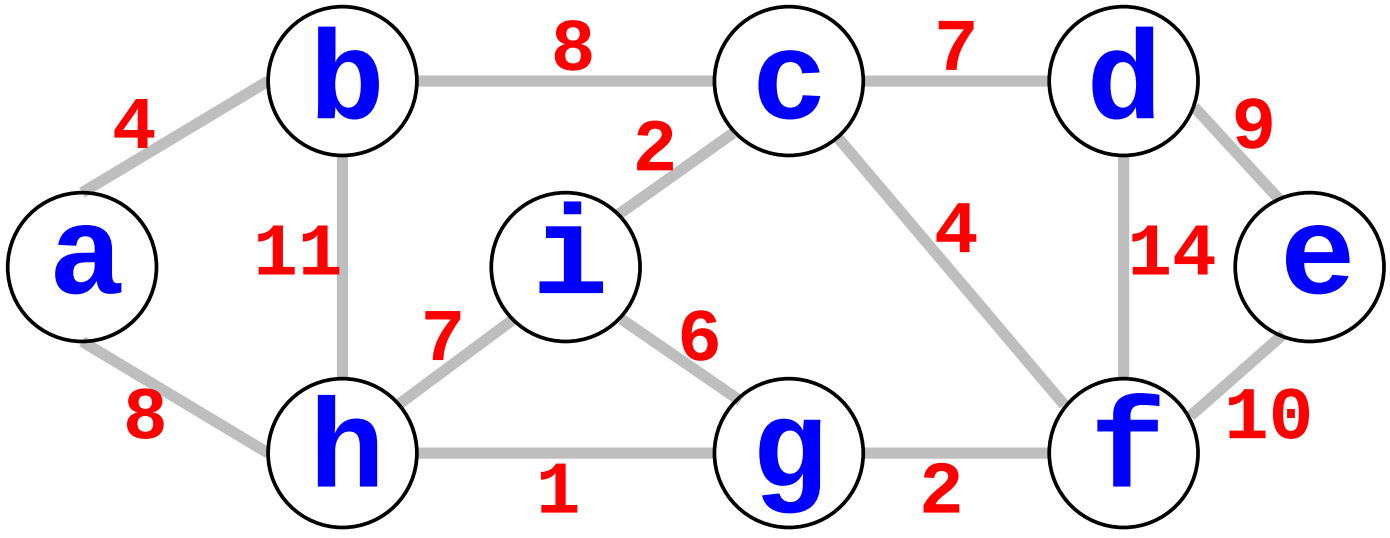

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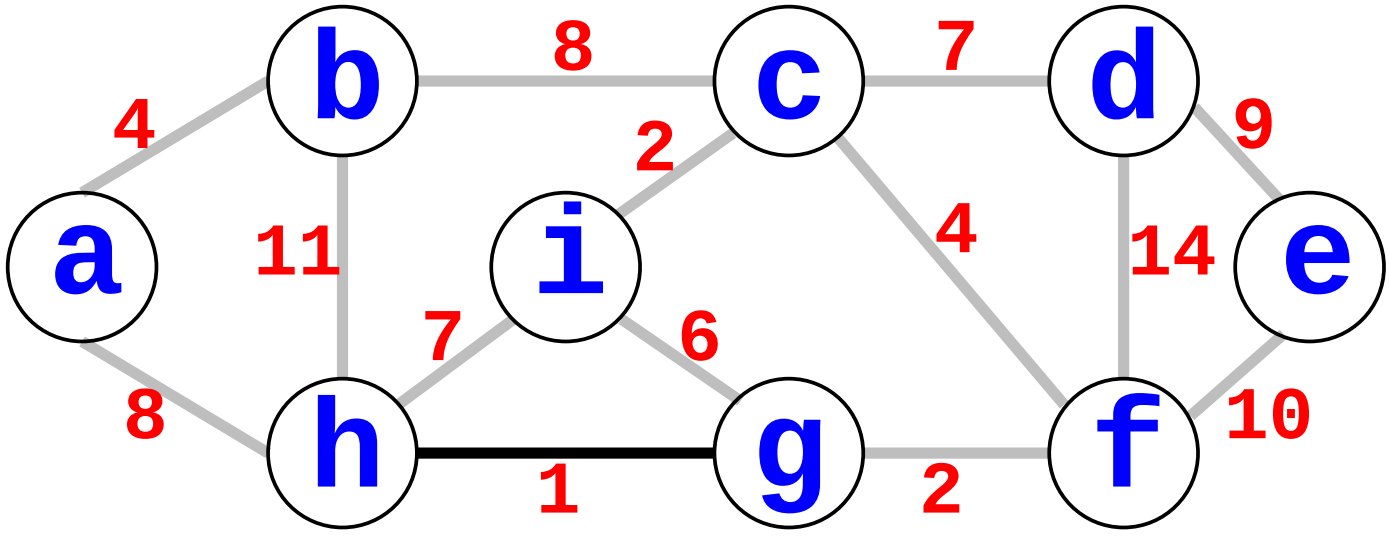

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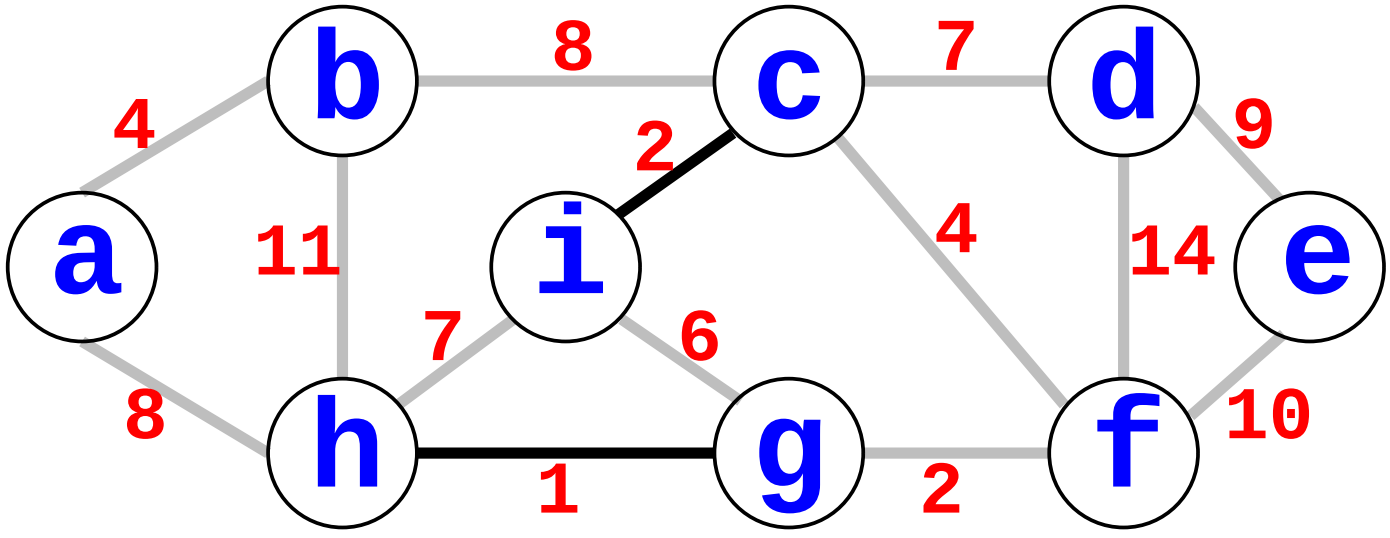

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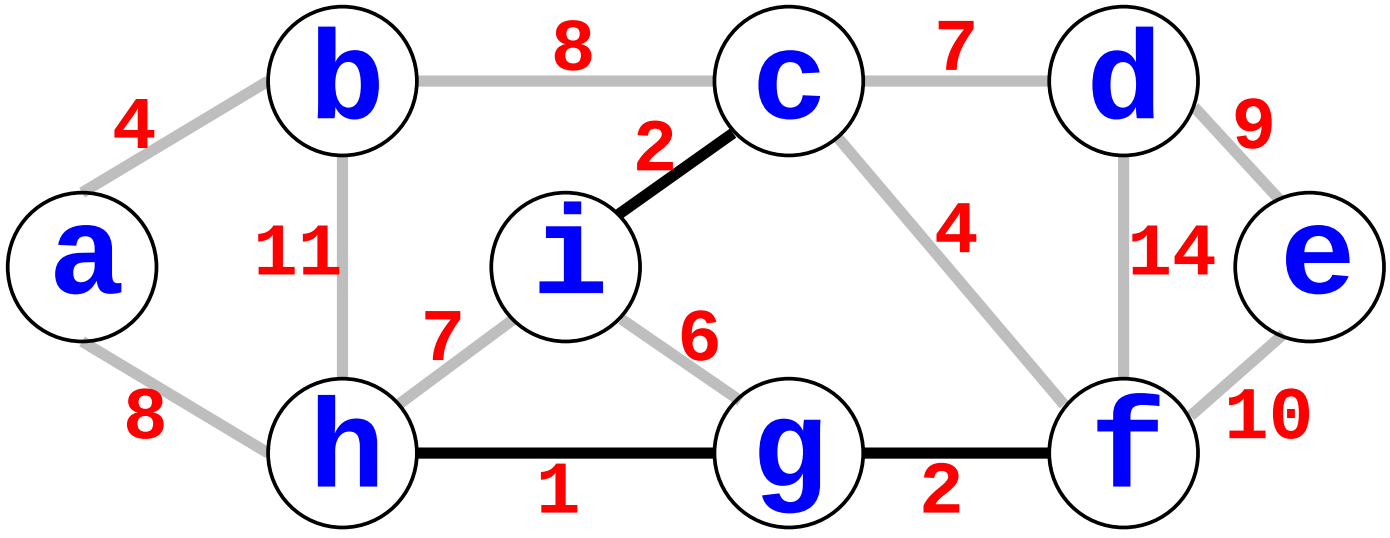

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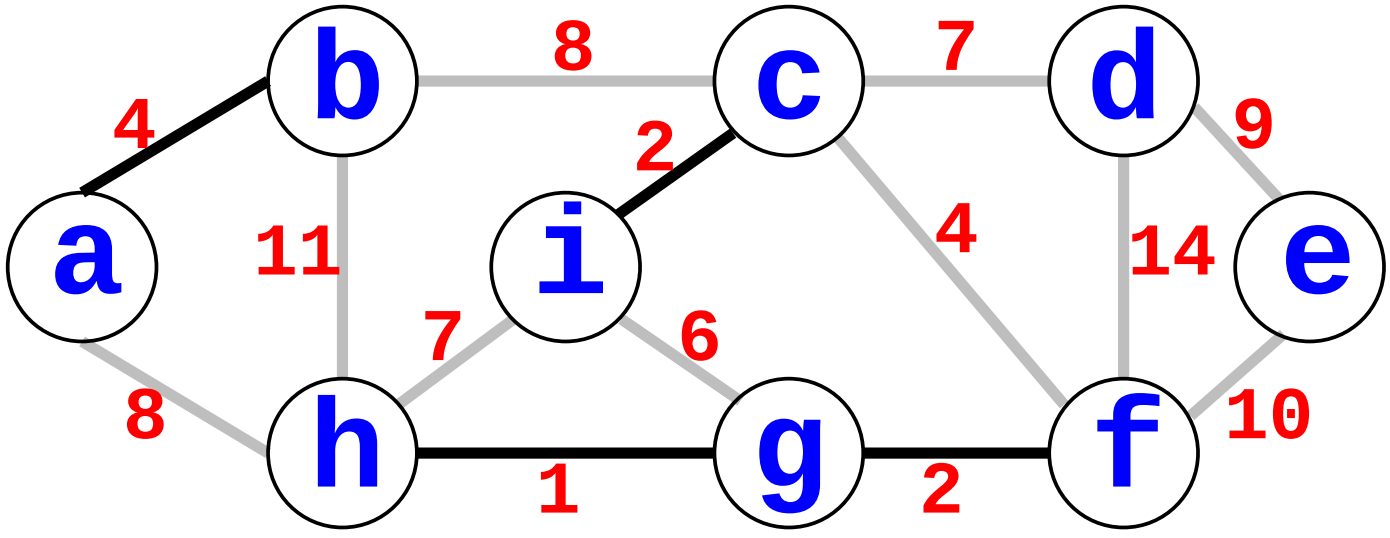

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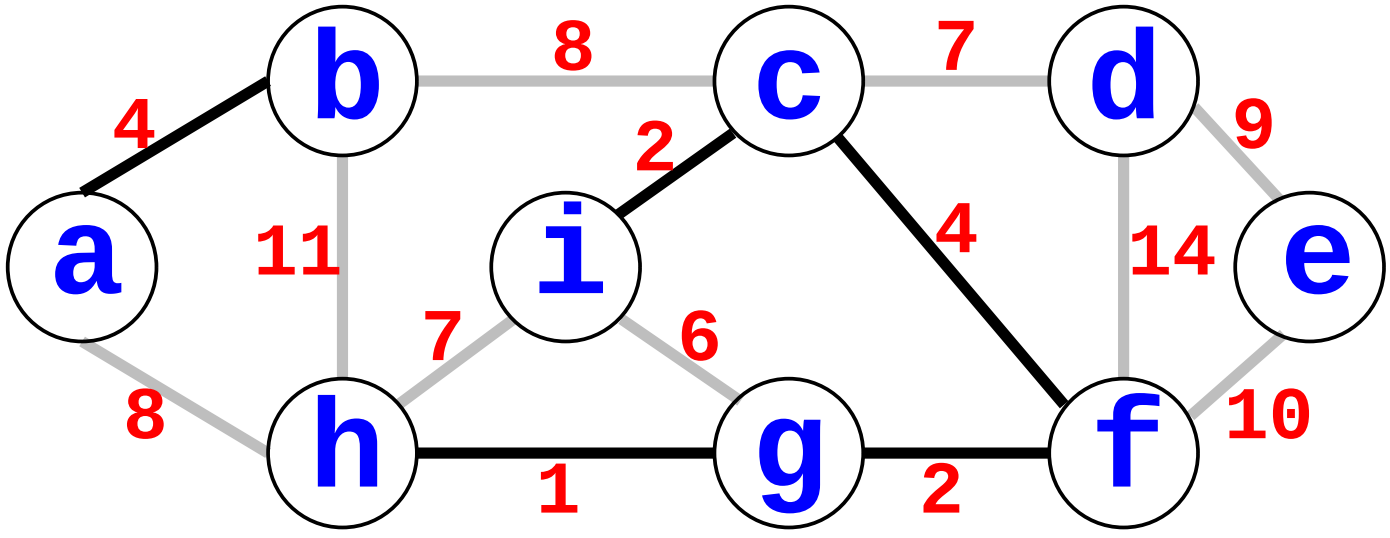

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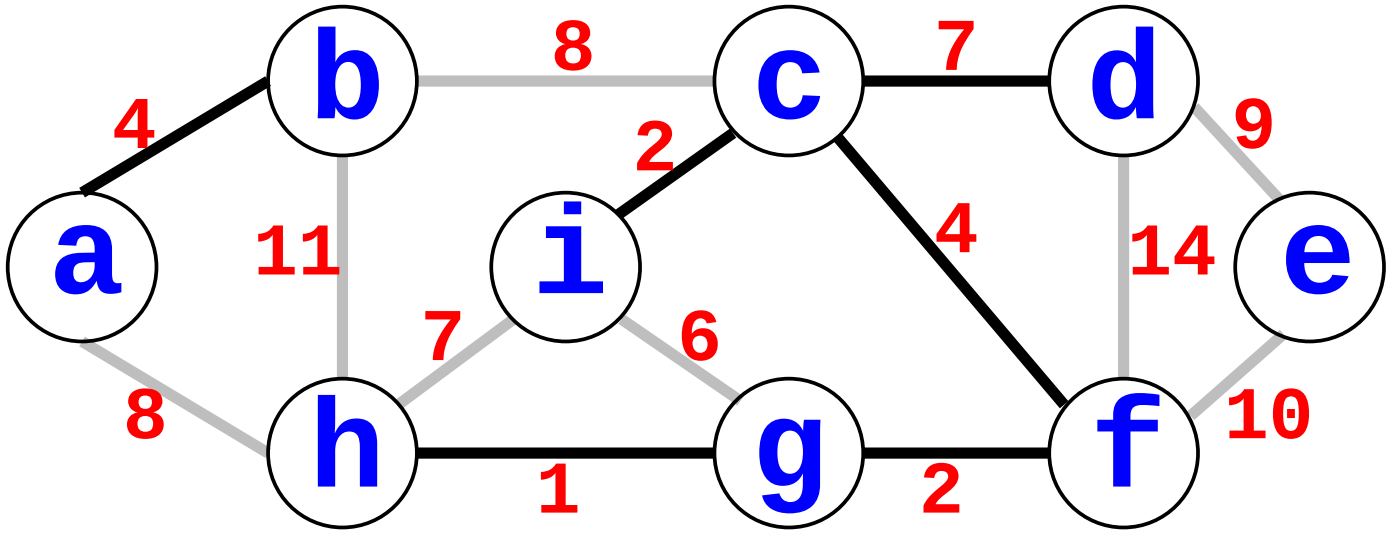

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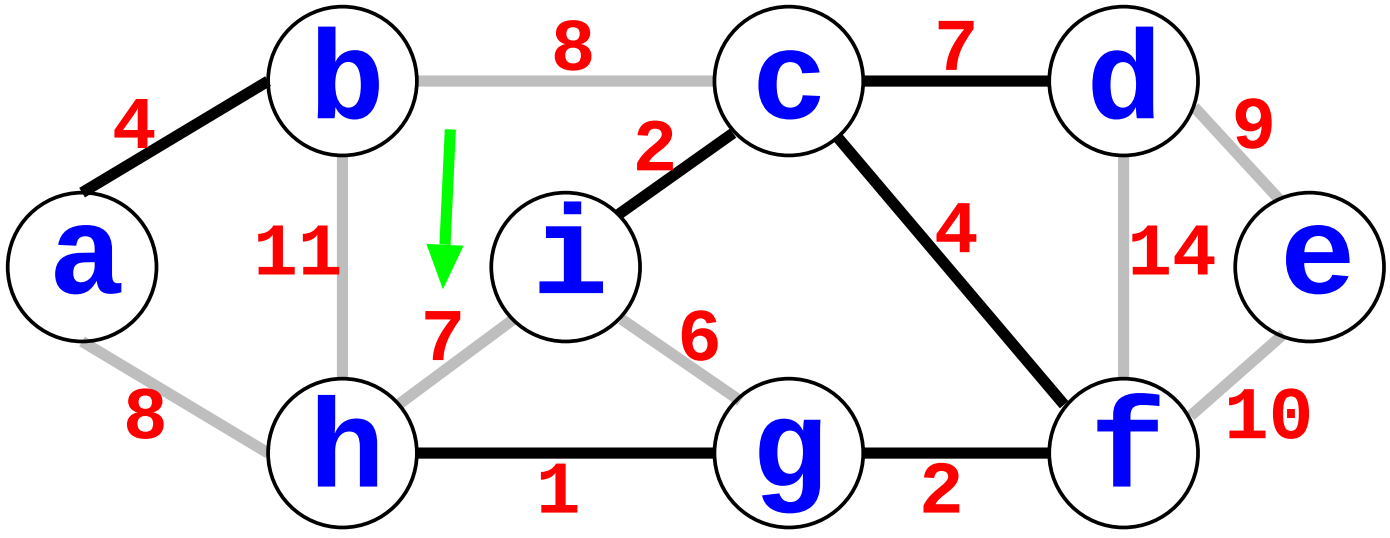

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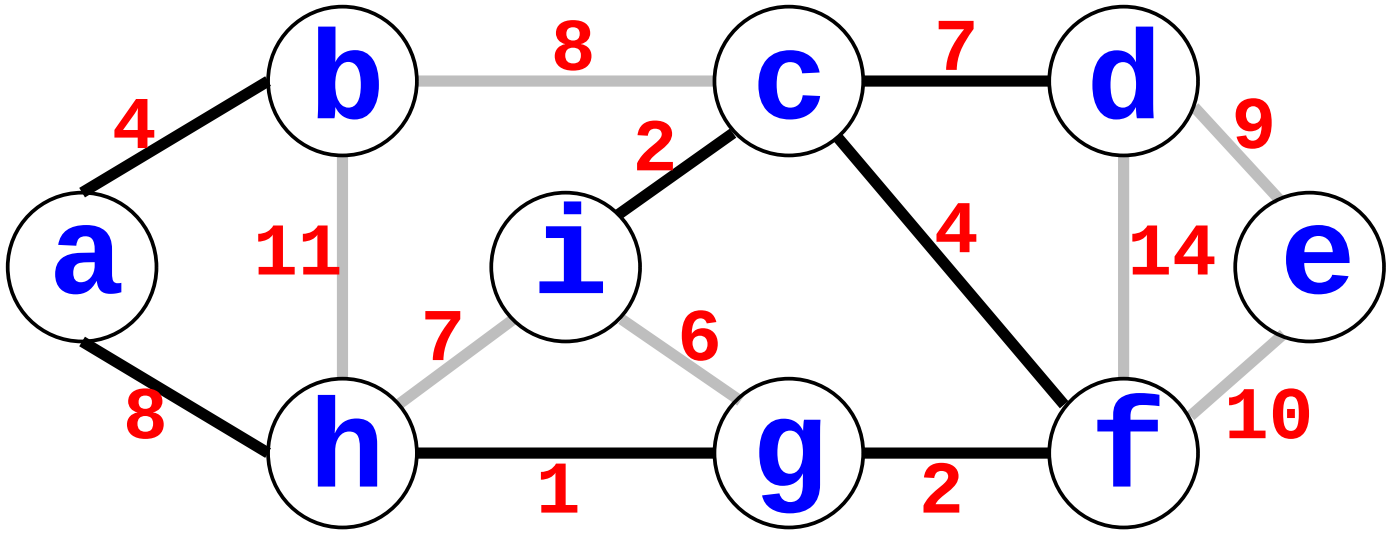

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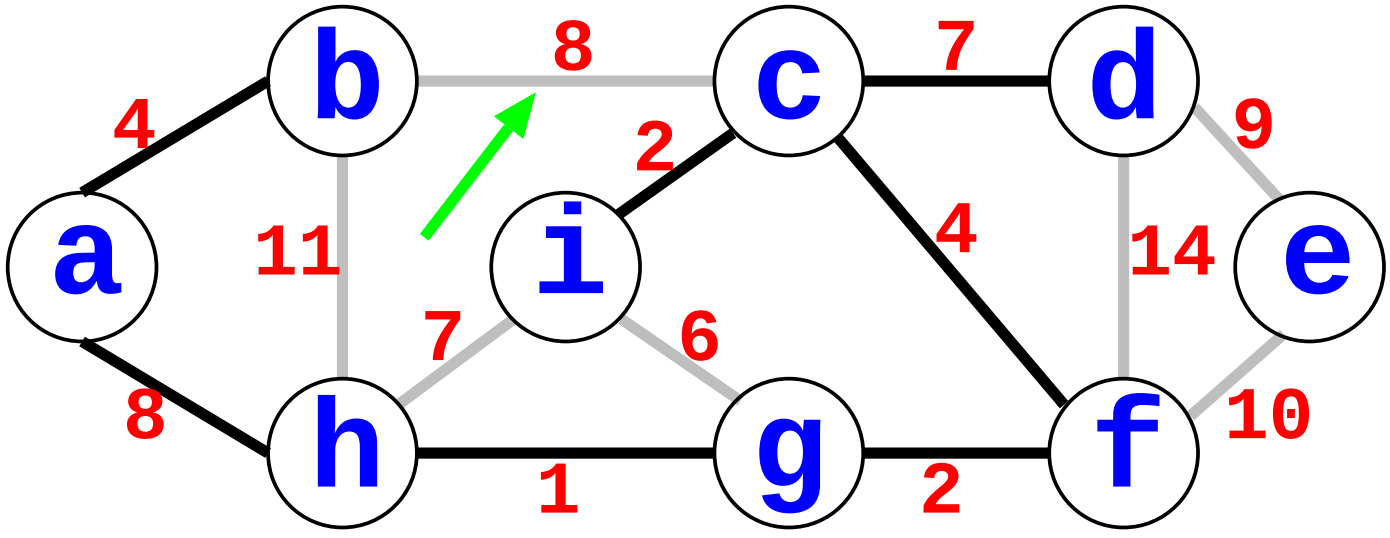

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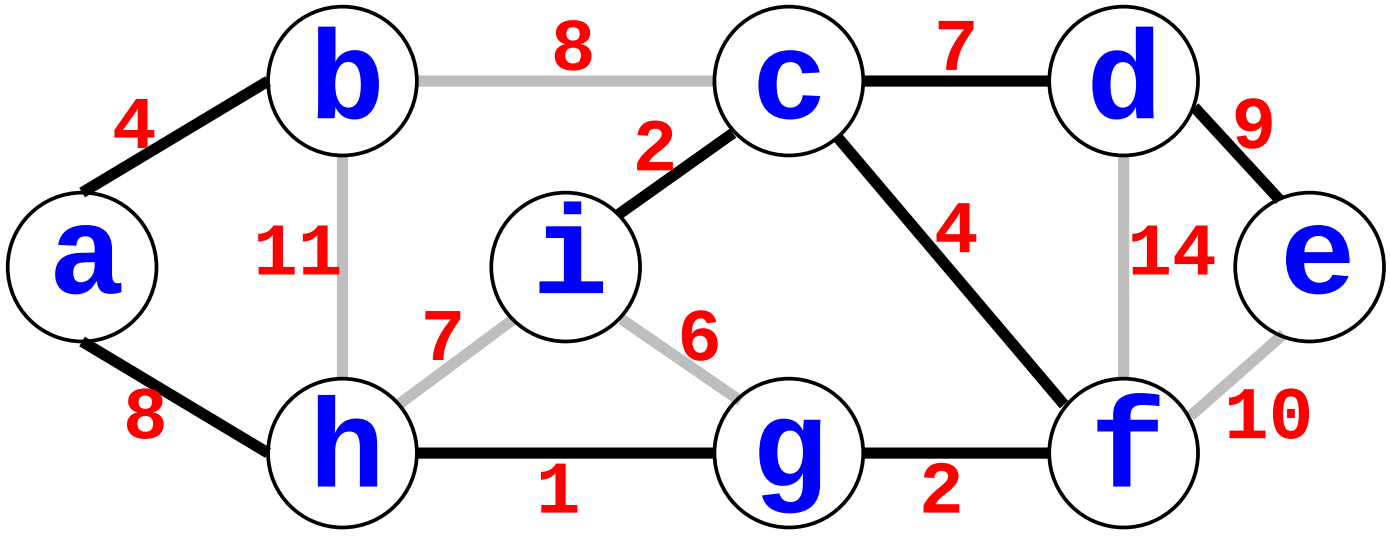

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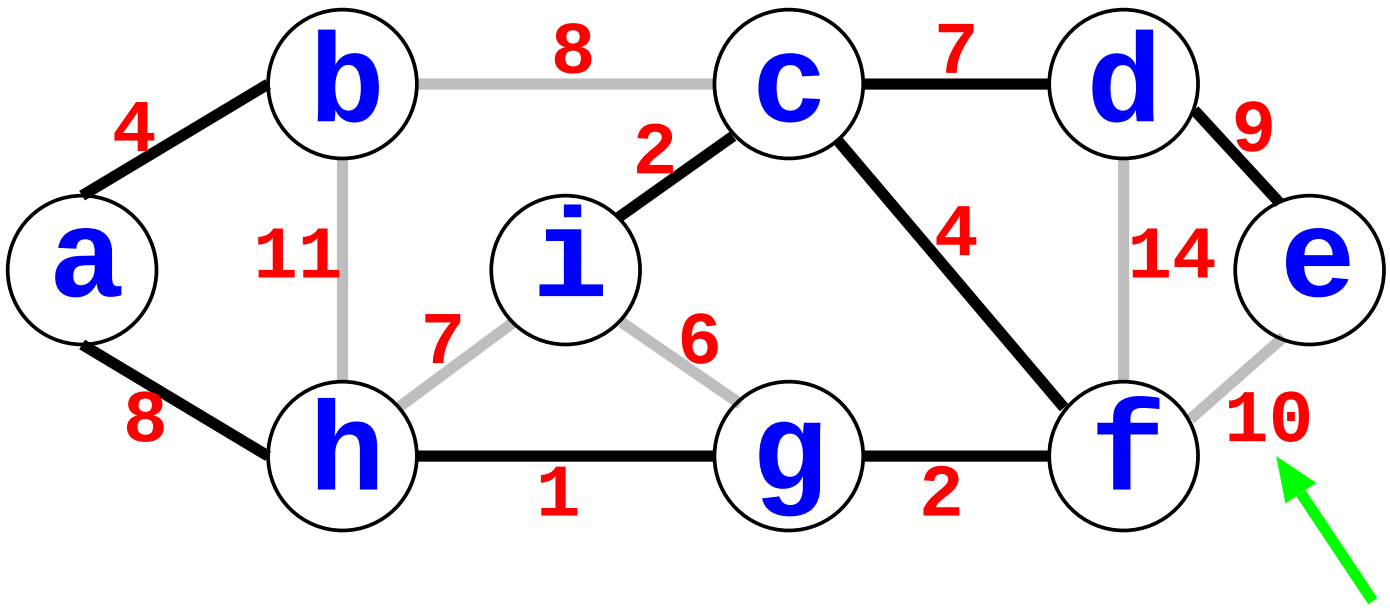

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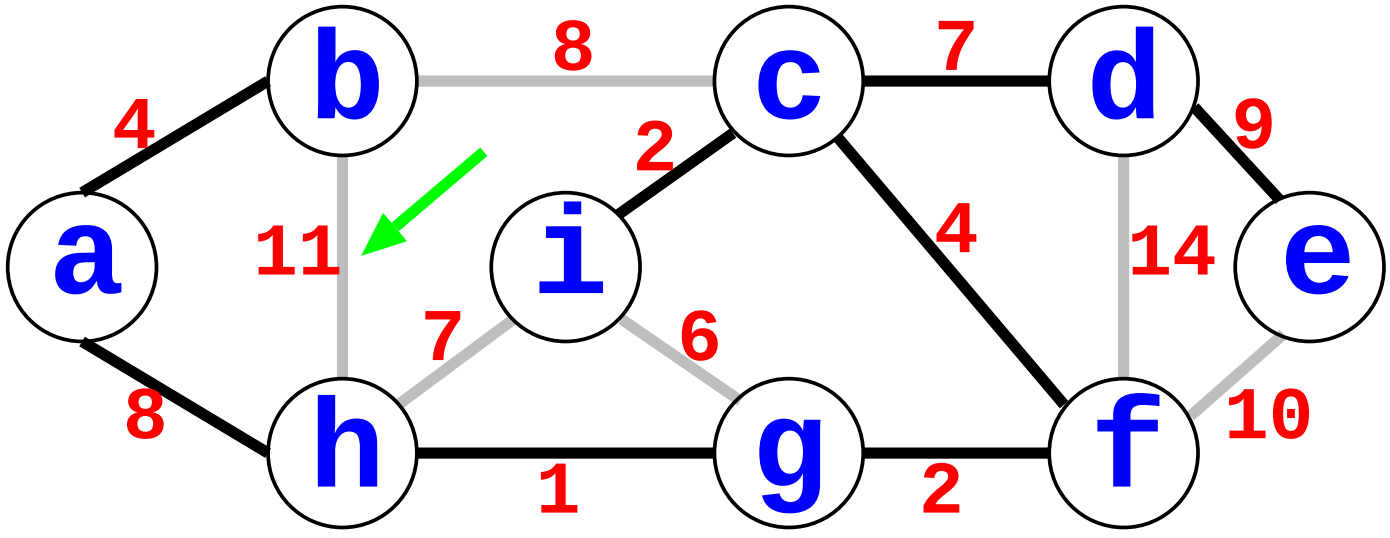

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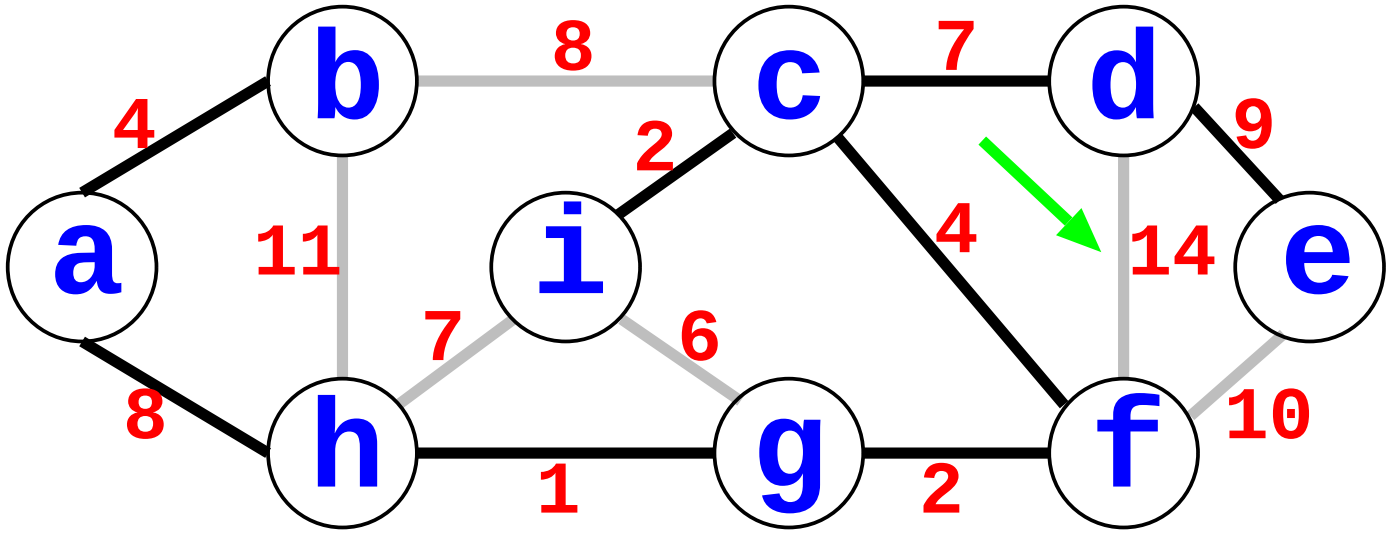

Minimum spanning tree: Kruskal's algorithm

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Implementation using union-find:

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1  kruskal(G):
2      A ← ∅
3      for each vertex v ∈ V(G) do
4          makeSet(v)
5      sort edges of E by their weight w in ascending order
6      for each edge (u, v) ∈ E(G) in ascending order do
7          if findSet(u) ≠ findSet(v) then
8              union(u, v)
9      return A
```

Exercises

1. Write a pseudocode for **makeSet**(**x**), **union**(**x**, **y**) and **findSet**(**x**) using the representation of linked lists and the weighted-union heuristic. Suppose each object x has an attribute *rep* pointing to the representative member of the set which contains x, and each set S has the attributes *begin*, *end* and *size*, where *size* is the length of the list.
2. The tree returned by Kruskal's algorithm is unique, i.e., the algorithm always returns the same minimum spanning tree? Prove or give a counterexample.

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