Red-Black Trees

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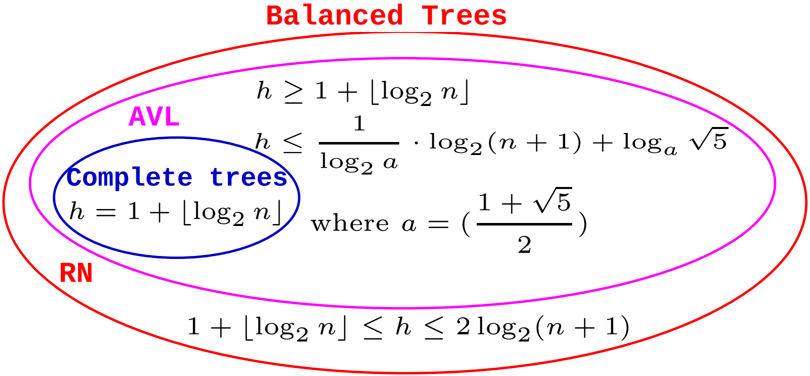
- **Height:** at most $2\lg(n+1)$ where n is the number of nodes;
- Insertion and deletion have time $O(\lg n)$;
- The path from the root to the farthest leaf is no more than twice as long as the path from the root to the nearest leaf.

Comparison of Balanced Trees



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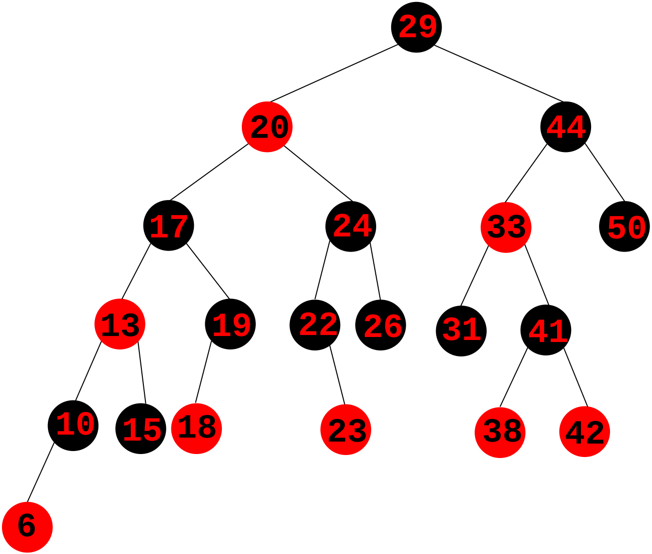
  2. For each node, all paths from a node to the leaves have the same number of black nodes;
  3. If a node is red, then both child nodes are black.

Example of Red-Black Tree



Red-Black Trees: Insertion

Node inserted q is red. Possibilities:

Case 1: v is black

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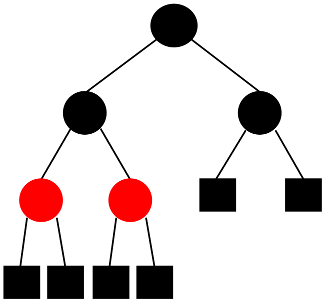

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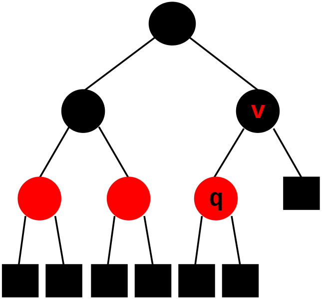

Red-Black Trees: Insertion

Node inserted q is red. Possibilities:

Case 2: v is red. Therefore, w (parent of v) is black.

Case 2.1: t is red;
We modify the color of v, t, w.

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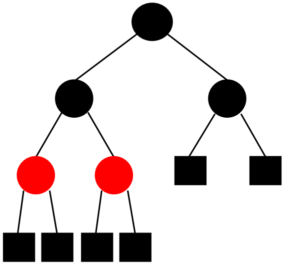

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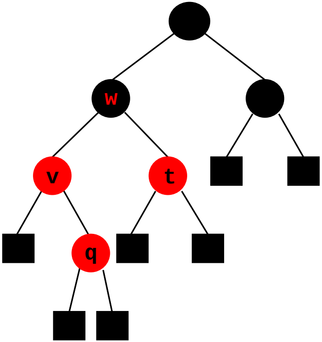

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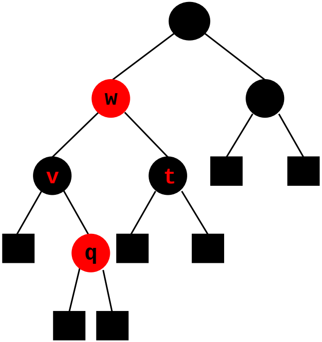
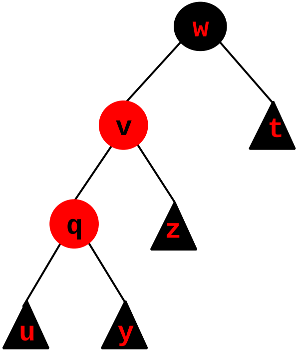


Rotations – Case 2.2: t is black

Case 2.2.1: q is the left child of v, and v is the left child of w.
Change the color of v and w.



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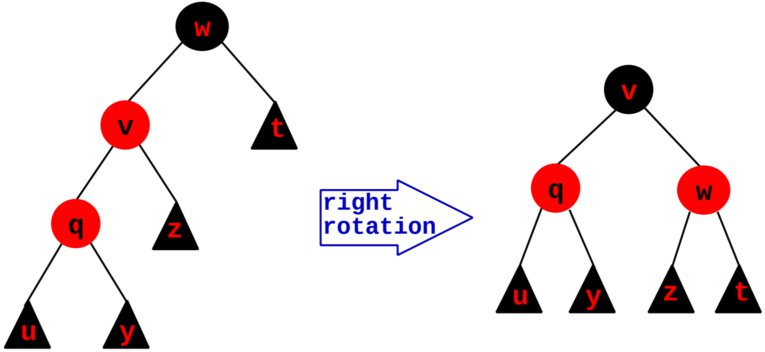

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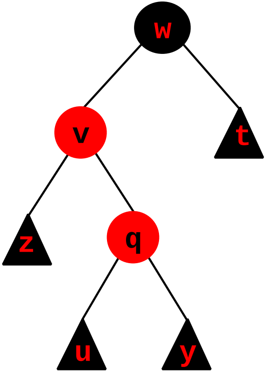
Case 2.2.2: q is the right child of v, and v is the left child of w. Change color of q and w.



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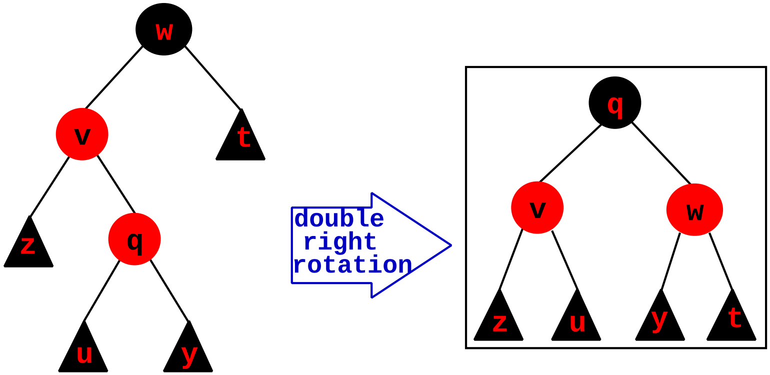

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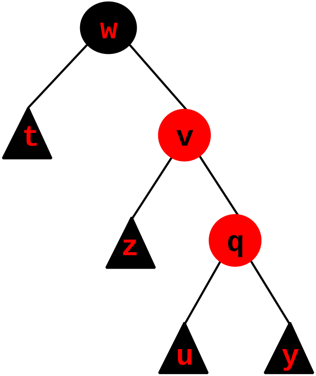
Case 2.2.3: *q* is the right child of *v*, and *v* is the right child of *w*. Change color of *v* and *w*.



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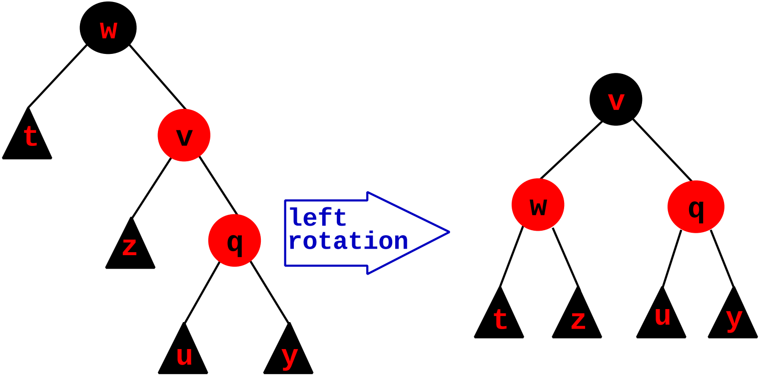

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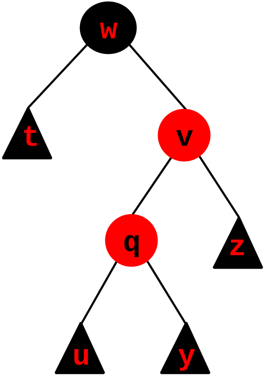
Case 2.2.4: q is the left child of v, and v is the right child of w. Change color of q and w.



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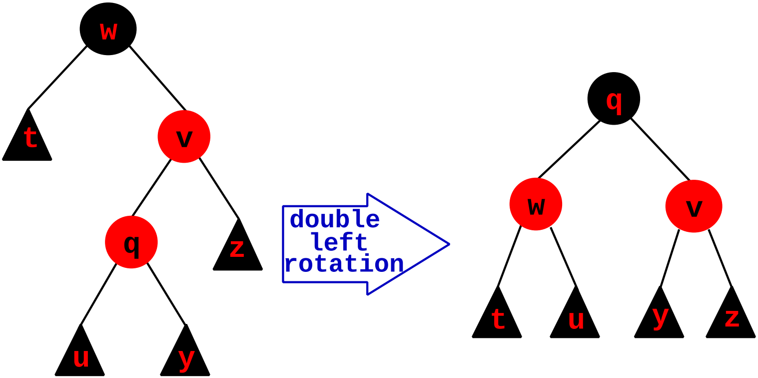
Case 2.2.4: q is the left child of v, and v is the right child of w. Change color of q and w.



Red-Black Trees: insertion algorithm

```
InsertionRB(x, ptv, ptw, ptr, a):
    if ptv = external then
        new(ptv)
        ptv ↑ .left ← ptv ↑ .right ← external
        ptv ↑ .key ← x; ptv ↑ .color ← R
        if ptroot = external then
            ptv ↑ .color ← B; ptroot ← ptv
        else if x < ptw ↑ .key then
            ptw ↑ .left ← ptv
        else ptw ↑ .right ← ptv
    else if x ≠ ptv ↑ .key then
        if x < ptv ↑ .key then ptq ← ptv ↑ .left
        else ptq ← ptv ↑ .right
            InsertionRB(x, ptq, ptv, ptw, a)
            if a = 1 then route(ptq, ptv, ptw, ptr, a)
            else if a = 0 then a = 1
            else "Invalid Insertion"
```

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Comparison: AVL trees are more strictly balanced than Red-Black trees, making insertion and deletion slower but retrieval (search) faster;

Exercises

1. Prove or give a counterexample:
    1.1 Every complete tree is AVL.
    1.2 Every AVL tree is complete.
    1.3 Every AVL tree is red-black.
    1.4 Every red-black tree is AVL.
    1.5 Every complete tree is red-black.
    1.6 Every red-black tree is complete.

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4. Write the procedure of deletion for Red-Black trees.
5. Prove or give a counterexample: given a Red-Black tree with a red root, if we change its color for black, the resulting tree is still a Red-Black tree.

Bibliography

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