Single-source shortest path with positive weights

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- Given a map of cities and their distances between each other, what is the shortest path between any two cities A and B?
- This problem can be modeled as a graph:
  - **City:** vertices;
  - **Roads between cities:** weighted edges representing the distance between cities.
  - **Weighted Edges:** can indicate time, cost, penalties, losses, etc;
- This problem is known as **Shortest Path Problem.**

Approach

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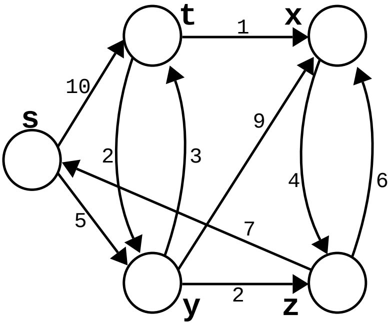
- We can use a similar idea to the **Breadth-first search**;
- Important to consider:
  - the problem has **optimal substructure**: which means that a shortest path between two vertices can be constructed from the shortest paths of subproblems.

Single-source shortest path: Dijkstra's algorithm



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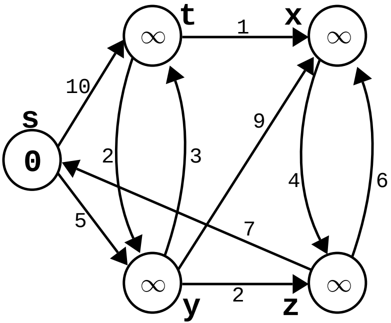

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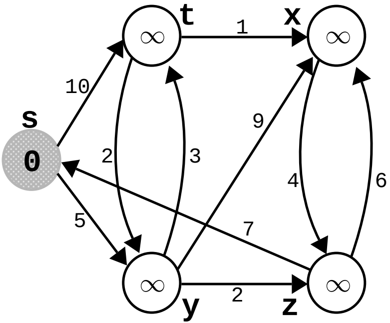

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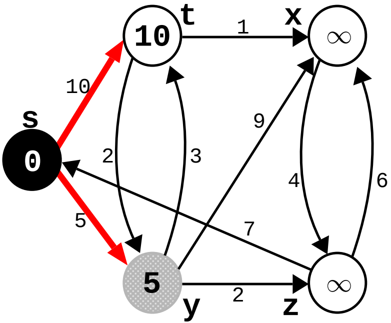

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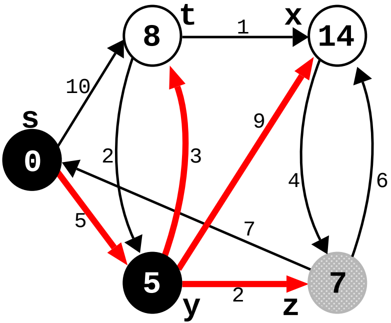

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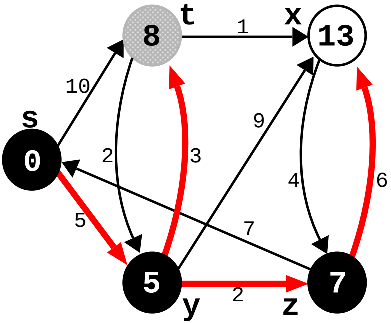

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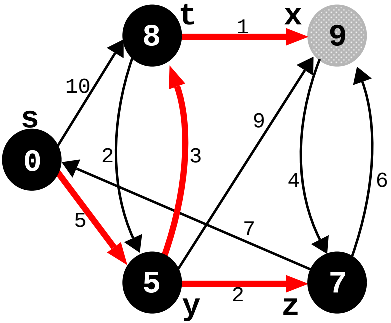

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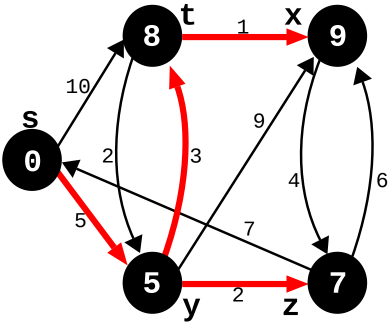

Single-source shortest path: Dijkstra's algorithm

```
relax(u, v):
  if v.d > u.d + p(u, v) then
    v.d = u.d + p(u, v)
    v.p = u
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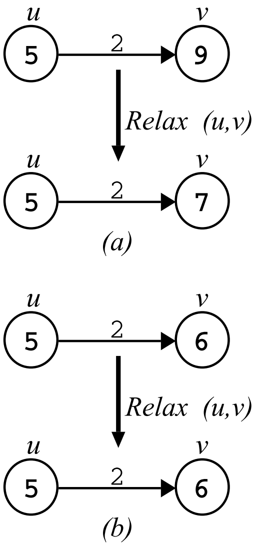

Single-source shortest path: Dijkstra's algorithm

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dijkstra(G, s):
   for u in V(G) do
       u.d = ∞
       u.p = None
   s.d = 0
   s.p = s
   A = Heap(V(G)) \\ using value of d
   S = [ ]
   while size(A) > 1 do
       u = remove_min(A)
       S = S + u
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- Total complexity: $O((n + m) \log n)$.

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Shortest Path Problems

- **Single-source shortest path:** Dijkstra's algorithm;
- **Single-destination shortest path:** invert direction of edges and then apply Dijstra's algorithm;
- **Shortest path between any vertices u and v:** Dijkstra's algorithm;
- **Shortest path from all vertices to all vertices:** Floyd-Warshall's algorithm in time $O(n^3)$.

Limitations of Dijkstra's algorithm

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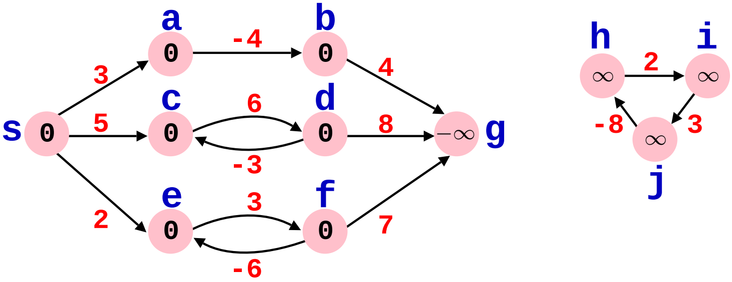
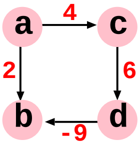
- **Option:** Bellman-Ford's algorithm with complexity $O(n \cdot m)$;



Shortest Path in Graphs with Negative Weights

- What if we add a constant to the weight of each edge in order to make all weights positive?



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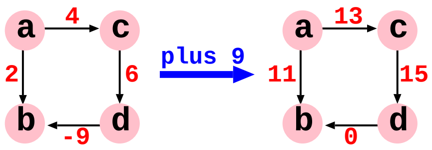

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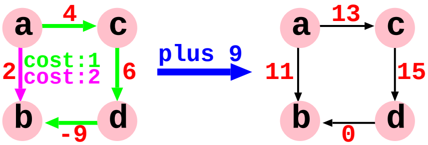

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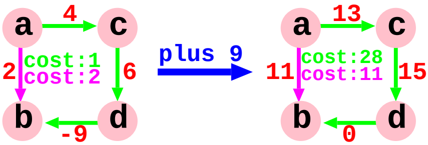

Bibliography

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