AP Calculus BC Test 4

H1 Sequences

A **sequence** is a list of elements.

$$2, 4, 6, 8 \ldots \quad a_n = 2n \quad \text{arithmetic sequence}$$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots \quad a_n = \frac{1}{2^{n-1}} \quad \text{geometric sequence}$$

$$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120} \ldots \quad a_n = \frac{1}{n!}$$

H2 Series and convergence

A **series** is the sum of the elements of a sequence.

Vocabulary and formulas

$$\text{Infinite series:} \quad \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

$$\text{Geometric series:} \quad \sum_{n=0}^{\infty} ar^n$$

$$\text{Sum of geometric series:} \quad S = \frac{a}{1-r} \quad \text{if } |r| < 1$$

$$\text{Partial sum:} \quad S_n$$

$$\text{Sum of series:} \quad S$$

Proving divergence with the nth-term test

If $\displaystyle\sum_{n=1}^{\infty} a_n$ ***converges***, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

This is *not* saying that if the limit of the terms in the sequence goes to 0, then the series converges. If the limit goes to 0, further tests are needed to determine convergence.

If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ ***diverges.***

Example: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 2}{n}$ converges or diverges.

Determine the limit:

$$\lim_{n\to\infty} \frac{n^2 + 2}{n} = \infty$$

because the degree of the numerator is greater than the degree of the denominator.

Since the limit does not equal 0, the series **diverges** by the nth-term test.

Proving convergence of geometric series

All that is needed to prove convergence of a geometric series is to show that the common ratio r satisfies $|r| < 1$.

Example: Determine whether the series $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ converges or diverges. If it converges, find the sum.

Step 1: Find the common ratio and determine whether the series converges or diverges.

The common ratio is $r = \frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, the series **converges**.

Step 2: Find the sum.

$$S = \frac{a}{1 - r}$$
$$= \frac{1}{1 - \frac{3}{4}}$$
$$= \frac{1}{\frac{1}{4}}$$
$$= \boxed{\text{converges to } 4}$$

Telescoping series

A **telescoping series** is a series where many terms cancel out when writing the partial sums. These often involve partial fractions.

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges or diverges. If it converges, find the sum.

Step 1: Separate using partial fractions.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$
$$A(n+1) + B(n) = 1$$
$$An + A + Bn = 1$$
$$A = 1$$
$$B = -1$$
$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Step 2: Write out the sequence.

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} \cdots - \frac{1}{n+1}$$

Step 3: Take the limit of the series.

$$S = 1 - \frac{1}{n+1}$$
$$\lim_{n \to \infty} 1 - \frac{1}{n+1}$$
$$= \boxed{\text{converges to } 1}$$