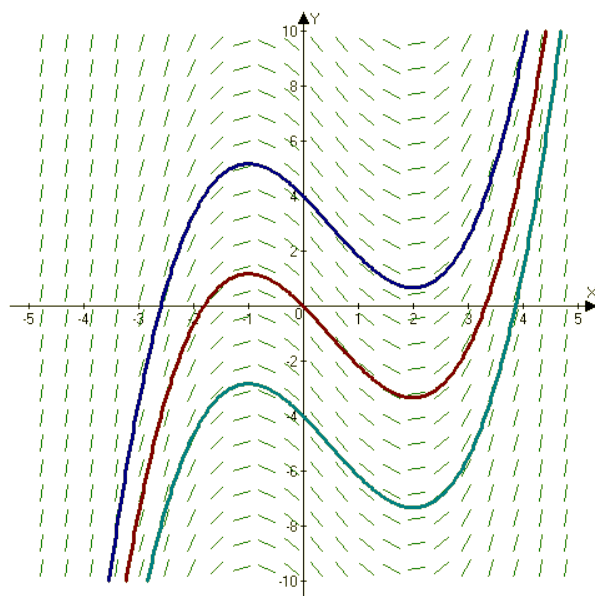# AP Calculus BC Test 3

H20 Slope fields and Euler's method

A slope field is a graphical representation of a differential equation that shows the slope of the solution curve at each point in the plane.

For example, the slope field for the differential equation $\frac{dy}{dx} = x^2 - x - 2$ can be drawn by calculating the slope at various points (x, y) and drawing small line segments with those slopes. The result is (from Wikipedia):

Euler's method is a technique used to approximate solutions to differential equations with a given initial value by using the slope of the function at a given point to estimate the value of the function at the next point.

Example: Use a table and Euler's method to approximate the value of y at $x = 1$ for the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $y(0) = 1$, using a step size of $h = 0.5$. Use $\Delta y = \frac{dy}{dx} \cdot \Delta x$ to find the change in y at each step.

Step 1: Make a table with the initial condition and known values.

x	y	Δx	$\frac{dy}{dx} = x + y$	Δy	$(x + \Delta x, y + \Delta y)$
0	1	0.5	-	-	$(0.5, y)$
0.5	-	0.5	-	-	$(1, y)$

Step 2: Calculate the slope $\frac{dy}{dx}$ at each point and use it to find Δy.

x	y	Δx	$\frac{dy}{dx} = x + y$	Δy	$(x + \Delta x, y + \Delta y)$
0	1	0.5	1	0.5	$(0.5, 1.5)$
0.5	1.5	0.5	2	1	$(1, 2.5)$

Example: Use tangent lines to do the same problem.

Step 1: Find the equation of the tangent line at the initial point.

Given:

$$\frac{dy}{dx} = x + y$$
$$y(0) = 1$$
$$\Delta x = 0.5$$

Finding the tangent line:

$$\frac{dy}{dx} = 0 + 1 = 1$$
$$y - 1 = 1(x - 0)$$
$$y = x + 1$$

Step 2: Use the tangent line to approximate y at $x = 0.5$.

At $x = 0.5$, $y = 0.5 + 1 = 1.5$.

Step 3: Use the new point to find the next tangent line.

$$\frac{dy}{dx} = 1.5 + 0.5 = 2$$
$$y - 1.5 = 2(x - 0.5)$$
$$y = 2x + 0.5$$

Step 4: Use the new tangent line to approximate y at $x = 1$.

At $x = 1$, $y = 2(1) + 0.5 = \boxed{2.5}$.

Both methods yield the same approximation of $y(1) \approx 2.5$.

H21 Separable differential equations

A separable differential equation is one that can be expressed in the form $\frac{dy}{dx} = f(x)g(y)$, allowing the variables to be separated on opposite sides of the equation for integration.

Example: Solve the separable differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$ with the initial condition $y(0) = 2$.

Step 1: Separate the variables.

$$\frac{dy}{dx} = \frac{x}{y}$$
$$y \, dy = x \, dx$$

Step 2: Integrate both sides to find the general solution.

$$\int y \, dy = \int x \, dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$
$$y^2 = x^2 + C$$

Step 3: Use the initial condition to find the particular solution.

$$y^2 = x^2 + C$$
$$y(0) = 2$$
$$2^2 = 0^2 + C$$
$$C = 4$$
$$y = \pm\sqrt{x^2 + 4}$$
$$2 = \pm\sqrt{0^2 + 4}$$
$$2 = \sqrt{4}$$
$$\boxed{y = \sqrt{x^2 + 4}}$$

H22 Logistic equations

A logistic equation is a type of differential equation that models population growth with a carrying capacity.

Typically, the logistic equation is expressed as

$$\frac{dP}{dt} = rP(M - P)$$

where P is the population size, $k = r \cdot M$ is the growth rate, and M is the carrying capacity.

Important things to note:

- The population grows fastest at $P = \dfrac{M}{2}$, or in the middle of the curve at the point of inflection.

- As P approaches M, the growth rate slows down and the population stabilizes. The rate of change approaches zero as the population approaches its carrying capacity.

- If the equation is not in the standard form, it may need to be manipulated algebraically to identify k and M.

The equation is alternatively expressed as

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where k is the growth rate and M is the carrying capacity.

H23 (review of integrals, skipped)

H24 Integration by parts, solving for the integral, and tabular integration

Integration by parts

Integration by parts is a technique used to integrate products of functions following the formula[1]:

$$\int u\, dv = uv - \int v\, du$$

where u and dv are parts of the original integral.

To identify u, use the initialism **LIATE**, which stands for **L**ogarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, and **E**xponential functions. The function that appears first in this list should be chosen as u.

Example: Use integration by parts to evaluate the integral $\displaystyle\int xe^x\, dx$.

Step 1: Identify u and dv with du and v.

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[1]This formula can be memorized with the mnemonic device "ultraviolet voodoo", origin unknown :)

Let $u = x$ (algebraic) and $dv = e^x \, dx$. Thus, $du = dx$ and $v = e^x$.[2]

$$uv - \int v \, du$$

$$= xe^x - \int e^x \, dx$$

$$xe^x - \int e^x \, dx$$

$$= \boxed{xe^x - e^x + C}$$

Solving for the integral

Sometimes, applying integration by parts results in an equation that contains the original integral. In such cases, solve for the integral algebraically.

Consider the following integral:

$$\int e^x \cos x \, dx$$

This integral looks solvable, but applying integration by parts twice will lead back to the original integral. To test this, let $u = e^x$ and $dv = \cos x \, dx$, and therefore $du = e^x \, dx$ and $v = \sin x$. Following the formula

$$\int u \, dv = uv - \int v \, du$$

yields:

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Now, apply integration by parts again to the remaining integral $\int e^x \sin x \, dx$. Let $u = e^x$ and $dv = \sin x \, dx$, so $du = e^x \, dx$ and $v = -\cos x$. Applying the formula again gives:

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[2] While $u = x$ is not a valid choice for u-substitution since it does not simplify the integral, it does work well for integration by parts.

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

Substituting this back into the previous equation results in:

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx \right)$$

This simplifies to:

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

Now, the original integral is on both sides of the equation. To solve for it, add $\int e^x \cos x \, dx$ to both sides:

$$2 \int e^x \cos x \, dx = e^x (\sin x + \cos x)$$

Finally, divide both sides by 2 to isolate the integral:

$$\int e^x \cos x \, dx = \boxed{\frac{e^x (\sin x + \cos x)}{2} + C}$$

Tabular integration

Tabular integration is a method used to simplify the process of integration by parts when one function can be differentiated repeatedly until it becomes zero, and the other function can be integrated repeatedly.

Example: Use tabular integration to evaluate the integral $\int x^3 e^x \, dx$.

Step 1: Set up the table and differentiate u until the left side reaches 0. Integrate dv the same number of times.

u	dv
x^3	e^x
$3x^2$	e^x
$6x$	e^x
6	e^x
0	e^x

Step 2: Multiply diagonally, alternating signs, and sum the results.

$$+x^3 e^x$$
$$-3x^2 e^x$$
$$+6x e^x$$
$$-6e^x$$

Step 3: Write the final answer.

$$\int x^3 e^x \, dx = \boxed{e^x(x^3 - 3x^2 + 6x - 6) + C}$$

H25 Partial fractions

Partial fraction decomposition is a technique used to break down a complex rational function into simpler fractions that are easier to integrate.

For each fraction in the partial fraction decomposition, the numerator should always be **one degree more** than the denominator.

Example: Evaluate $\int \dfrac{2x + 3}{(x - 1)(x + 2)} \, dx$.

Step 1: Set up the decomposition. The denominator is always one degree more than the numerator.

$$\frac{2x + 3}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}$$

Step 2: Multiply both sides by the denominator to eliminate the fractions.

$$2x + 3 = A(x + 2) + B(x - 1)$$
$$2x + 3 = Ax + 2A + Bx - B$$

Step 3: Solve for A and B.

$$A + B = 2$$
$$2A - B = 3$$
$$3A = 5$$
$$A = \frac{5}{3} \quad B = \frac{1}{3}$$

Step 4: Substitute and integrate.

$$\int \frac{2x+3}{(x-1)(x+2)}\,dx = \int \frac{5/3}{x-1} + \frac{1/3}{x+2}\,dx$$
$$= \boxed{\frac{5}{3}\ln|x-1| + \frac{1}{3}\ln|x+2| + C}$$

Area between curves

The area between two curves can be found by integrating the difference between the two functions from the leftmost intersection point to the rightmost intersection point. For sideways curves, integrate from the bottommost intersection point to the topmost intersection point (as a function of y).