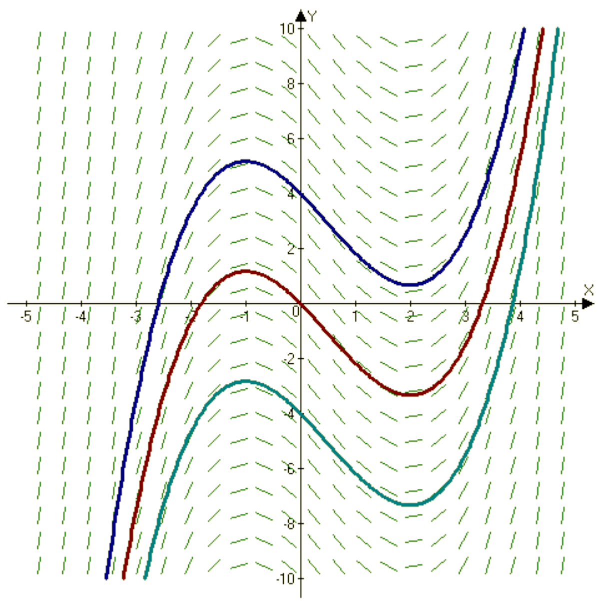
AP Calculus BC Test 3

H20 Slope fields and Euler's method

A slope field is a graphical representation of a differential equation that shows the slope of the solution curve at each point in the plane.

For example, the slope field for the differential equation $\frac{dy}{dx} = x^2 - x - 2$ can be drawn by calculating the slope at various points (x, y) and drawing small line segments with those slopes. The result is (from Wikipedia):



Euler's method is a technique used to approximate solutions to differential equations with a given initial value by using the slope of the function at a given point to estimate the value of the function at the next point.

Example: Use a table and Euler's method to approximate the value of y at $x = 1$ for the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $y(0) = 1$, using a step size of $h = 0.5$. Use $\Delta y = \frac{dy}{dx} \cdot \Delta x$ to find the change in y at each step.

Step 1: Make a table with the initial condition and known values.

x	y	Δx	$\frac{dy}{dx} = x + y$	Δy	$(x + \Delta x, y + \Delta y)$
0	1	0.5	-	-	$(0.5, y)$
0.5	-	0.5	-	-	$(1, y)$

Step 2: Calculate the slope $\frac{dy}{dx}$ at each point and use it to find Δy.

x	y	Δx	$\frac{dy}{dx} = x + y$	Δy	$(x + \Delta x, y + \Delta y)$
0	1	0.5	1	0.5	$(0.5, 1.5)$
0.5	1.5	0.5	2	1	$(1, 2.5)$

Example: Use tangent lines to do the same problem.

Step 1: Find the equation of the tangent line at the initial point.

Given:

$$\frac{dy}{dx} = x + y$$
$$y(0) = 1$$
$$\Delta x = 0.5$$

Finding the tangent line:

$$\frac{dy}{dx} = 0 + 1 = 1$$
$$y - 1 = 1(x - 0)$$
$$y = x + 1$$

Step 2: Use the tangent line to approximate y at $x = 0.5$.

At $x = 0.5$, $y = 0.5 + 1 = 1.5$.

Step 3: Use the new point to find the next tangent line.

$$\frac{dy}{dx} = 1.5 + 0.5 = 2$$
$$y - 1.5 = 2(x - 0.5)$$
$$y = 2x + 0.5$$

Step 4: Use the new tangent line to approximate y at $x = 1$.

At $x = 1$, $y = 2(1) + 0.5 = \boxed{2.5}$.

Both methods yield the same approximation of $y(1) \approx 2.5$.

H21 Separable differential equations

A separable differential equation is one that can be expressed in the form $\frac{dy}{dx} = f(x)g(y)$, allowing the variables to be separated on opposite sides of the equation for integration.

Example: Solve the separable differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$ with the initial condition $y(0) = 2$.

Step 1: Separate the variables.

$$\frac{dy}{dx} = \frac{x}{y}$$
$$y\,dy = x\,dx$$

Step 2: Integrate both sides to find the general solution.

$$\int y\,dy = \int x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$
$$y^2 = x^2 + C$$

Step 3: Use the initial condition to find the particular solution.

$$y^2 = x^2 + C$$
$$y(0) = 2$$
$$2^2 = 0^2 + C$$
$$C = 4$$
$$y = \pm\sqrt{x^2 + 4}$$
$$2 = \pm\sqrt{0^2 + 4}$$
$$2 = \sqrt{4}$$
$$\boxed{y = \sqrt{x^2 + 4}}$$

H22 Logistic equations

A logistic equation is a type of differential equation that models population growth with a carrying capacity.

Typically, the logistic equation is expressed as

$$\frac{dP}{dt} = kP(M - P)$$

where P is the population size, k is the growth rate, and M is the carrying capacity.

Important things to note:

- The population grows fastest at $P = \frac{M}{2}$, or in the middle of the curve at the point of inflection.
- As P approaches M, the growth rate slows down and the population stabilizes. The rate of change approaches zero as the population approaches its carrying capacity.
- If the equation is not in the standard form, it may need to be manipulated algebraically to identify k and M.