AP Calculus BC Test 2

H10 Linearization

Linearization is a method used to approximate the value of a function near a given point using the tangent line at that point.

Example: Approximate the cube root of 82 using linearization.

Step 1: Identify the function and the point of tangency.

$$f(x) = \sqrt[3]{x}, \quad a = 81$$

Step 2: Find the derivative of the function.

$$f'(x) = \frac{1}{3x^{2/3}}$$

Step 3: Evaluate the function and its derivative at the point of tangency.

$$f(81) = 3, \quad f'(81) = \frac{1}{27}$$

Step 4: Write the equation of the tangent line.

H11 Optimization

Optimization involves finding the maximum or minimum values of a function subject to certain constraints.

Example: A farmer wants to build a rectangular pen with a fixed amount of fencing (100 meters). What dimensions will maximize the area of the pen?

Step 1: Define the variables and equations.

Let x be the length and y be the width of the pen. The area A is given by

$$A = xy$$

and the constraint from the fencing is

$$2x + 2y = 100$$

or

$$x + y = 50$$

Step 2: Express one variable in terms of the other using the constraint.

$$y = 50 - x$$

Step 3: Substitute into the area equation.

$$\begin{aligned} A(x) &= x(50 - x) \\ &= 50x - x^2 \end{aligned}$$

Step 4: Find the critical points by taking the derivative and setting it to zero.

$$A'(x) = 50 - 2x$$
$$0 = 50 - 2x - 2x = -50x = 25$$

Step 5: Find the corresponding value of y.

$$y = 50 - 25 = 25$$

Step 6: Conclude with a justification:

The dimensions that maximize the area of the pen are 25 meters by 25 meters because the sign of the first derivative changes from positive to negative at $x = 25$, indicating a maximum.

H12 Estimating with Riemann sums

RRAM Right Rectangular Approximation Method - heights of rects = heights of right endpoints

LRAM Left Rectangular Approximation Method - heights of rects = heights of left endpoints

MRAM Midpoint Rectangular Approximation Method - heights of rects = heights of midpoints

Trapezoidal approximation Average of RRAM and LRAM

Example: Estimate with MRAM the value of $\int_0^4 (x^2 + 1)\, dx$ using 4 subintervals.

Step 1: Determine Δx.

$$\Delta x = \frac{4 - 0}{4} = 1$$

Step 2: Identify the midpoints of each subinterval.

- Subinterval $[0, 1]$: midpoint $= 0.5$
- Subinterval $[1, 2]$: midpoint $= 1.5$
- Subinterval $[2, 3]$: midpoint $= 2.5$
- Subinterval $[3, 4]$: midpoint $= 3.5$

Step 3: Evaluate the function at each midpoint.

$$\begin{aligned} f(0.5) &= (0.5)^2 + 1 = 1.25 \\ f(1.5) &= (1.5)^2 + 1 = 3.25 \\ f(2.5) &= (2.5)^2 + 1 = 7.25 \\ f(3.5) &= (3.5)^2 + 1 = 13.25 \end{aligned}$$

Step 4: Calculate the Riemann sum.

$$\begin{aligned} \text{MRAM} &= \Delta x[f(0.5) + f(1.5) + f(2.5) + f(3.5)] \\ &= 1[1.25 + 3.25 + 7.25 + 13.25] \\ &= \boxed{25} \end{aligned}$$

H13 Writing and interpreting Riemann sums

When converting a Riemann sum of the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{b}{n} f\left(a + \frac{bi}{n}\right)$$

to a definite integral, follow these steps:

1. Identify Δx (the width of each subinterval) as $\frac{b}{n}$.
2. Identify the function $f(x)$ inside the sum.
3. Determine the limits of integration:
   - The lower limit is a (the starting point of the interval).
   - The upper limit is $a + b$ (the endpoint of the interval).
4. Write the definite integral as

$$\int_a^{a+b} f(x)\, dx$$

Similarly, when writing a Riemann sum for a definite integral of the form

$$\int_{a}^{a+b} f(x)\,dx$$

follow these steps:

1. Identify the interval $[a, a+b]$ and the function $f(x)$.
2. Determine Δx as $\frac{b}{n}$ (subtract upper limit of integration from lower) and plug into the equation for x as $\frac{bi}{n}$
3. Write the Riemann sum as

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{b}{n} f\left(a + \frac{bi}{n}\right)$$

Example: Write the Riemann sum for the definite integral $\int_{2}^{5} (x^2 + 1)\,dx$.

Step 1: Identify the interval and function.

The interval is $[2, 5]$ and the function is $f(x) = x^2 + 1$.

Step 2: Determine Δx and express x_i.

$$\Delta x = \frac{5-2}{n} = \frac{3}{n}$$
$$x_i = 2 + \frac{3i}{n}$$

Step 3: Write the Riemann sum.

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \left[\left(2 + \frac{3i}{n}\right)^2 + 1\right]}$$

H15 Definite Integration

Properties of integration

$$\text{Flipping the limits:}\quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\text{Sum/Difference:}\quad \int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\text{Constant multiple:}\quad \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\text{Additivity over intervals:}\quad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\text{Even function:}\quad \int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

$$\text{Odd function:}\quad \int_{-a}^a f(x)\,dx = 0$$

$$\text{Integral of a constant:}\quad \int_a^b c\,dx = c(b-a)$$

Example: $\int_0^\pi \sin x\,dx = 2$. Given this integral, find:

a) $\int_\pi^{2\pi} \sin x\,dx$

b) $\int_0^{2\pi} 3\sin x\,dx$

c) $\int_0^{\frac{\pi}{2}} 3\sin x\,dx$

d) $\int_0^{\pi} 3\sin x\,dx$

e) $\int_{-\pi}^{\pi} \sin x\,dx$

Answers:

a) -2 (flipped limits)

b) 6 (constant multiple)

c) 3 (additivity over intervals)

d) 6 (constant multiple)

e) 0 (odd function)

H15 Antiderivatives and indefinite integration

Antiderivative rules are the same as derivative rules, but in reverse. Always +C to the end of indefinite integrals.

Example: Find the antiderivative of $f(x) = 3x^2 - 4x + 5$.

Apply the power rule to each term:

$$\begin{aligned} &\int (3x^2 - 4x + 5)\, dx \\ &= \int (\frac{3x^{2+1}}{2+1} - \frac{4x^{1+1}}{1+1} + \frac{5x^{0+1}}{0+1})\, dx \\ &= \boxed{x^3 - 2x^2 + 5x + C} \end{aligned}$$

H16 The Fundamental Theorem of Calculus

First Fundamental Theorem

If f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Second Fundamental Theorem

The derivative of the integral of a function is the original function:

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

Example: Evaluate $\frac{d}{dx} \int_1^{3x} (2t + 3)\, dt$.

Step 1: Use the Second FTC and plug in the upper limit $(3x)$ into the function.

$$\begin{aligned} &2(3x) + 3 \\ &= 6x + 3 \end{aligned}$$

Step 2: Multiply by the derivative of the upper limit.

$$
\begin{aligned}
&(6x+3)\cdot \frac{d}{dx}(3x) \\
&= (6x+3)\cdot 3 \\
&= \boxed{18x+9}
\end{aligned}
$$

Average value

The **average value** of a function f on the interval $[a, b]$ is given by

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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H17 Integration by u-substitution and change of variable

Use u-substitution when the integral contains a function and its derivative.

Example: Evaluate $\int 2x\sqrt{x^2+1}\,dx$.

Step 1: Choose u to be the inner function.

$$
\begin{aligned}
u &= x^2+1 \\
du &= 2x\,dx
\end{aligned}
$$

Step 2: Rewrite the integral in terms of u.

$$\int \sqrt{u}\,du$$

Step 3: Integrate with respect to u.

$$
\begin{aligned}
\int \sqrt{u}\,du &= \int u^{1/2}\,du \\
&= \frac{u^{3/2}}{(3/2)} + C \\
&= \frac{2}{3}u^{3/2} + C
\end{aligned}
$$

Step 4: Substitute back to x.

$$\boxed{\frac{2}{3}(x^2+1)^{3/2} + C}$$

Sometimes, u-substitution requires a **change of variable** when the integral does not directly contain the derivative of the inner function.

Example: Evaluate $\int x\sqrt{x-1}\,dx$.

Step 1: Choose u to be the inner function.

$$u = x - 1$$
$$du = dx$$

This u-substitution is not sufficient because there is still an x in the integral. To fix this, apply a change of variable.

Step 2: Solve for x in terms of u.

$$x = u + 1$$

Step 3: Substitute.

$$\begin{aligned} &\int x\sqrt{x-1}\,dx \\ &= \int (u+1)\sqrt{u}\,du \\ &= \int (u+1)u^{1/2}\,du \end{aligned}$$

Step 4: Distribute.

$$\begin{aligned} &\int (u+1)u^{1/2}\,du \\ &= \int (u^{3/2} + u^{1/2})\,du \end{aligned}$$

Step 5: Integrate with respect to u.

$$\begin{aligned} &\int (u^{3/2} + u^{1/2})\,du \\ &= \boxed{\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C} \end{aligned}$$

For definite integrals, either substitute back to x or change the limits of integration to u.

Example: Evaluate the same integral as a definite integral: $\int_1^2 x\sqrt{x-1}\,dx$.

Step 1: As defined previously, $u = x - 1$. Change the limits of integration to u.

When $x = 1$, $u = 1 - 1 = 0$.

When $x = 2$, $u = 2 - 1 = 1$.

Step 2: Evaluate at the new limits of integration.

$$
\begin{aligned}
&\left[\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_{-1}^{0} \\
&= \left(\frac{2}{5} + \frac{2}{3}\right) - (0 + 0) \\
&= \boxed{\frac{16}{15}}
\end{aligned}
$$

H18 Inverse trig integration

$$
\begin{aligned}
\int \frac{1}{\sqrt{a^2 - x^2}}\,dx &= \arcsin\left(\frac{x}{a}\right) + C \\
\int \frac{1}{a^2 + x^2}\,dx &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \\
\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx &= \frac{1}{a}\operatorname{arcsec}\left(\frac{|x|}{a}\right) + C
\end{aligned}
$$

(no need to memorize formulas - memorize forms, use u-substitution instead)

Example: Evaluate $\int \frac{1}{\sqrt{16 - 9x^2}}\,dx$.

Step 1: Reduce the constant to 1.

$$
\begin{aligned}
&\int \frac{1}{\sqrt{16 - 9x^2}}\,dx \\
&= \int \frac{1}{16\sqrt{1 - \frac{9}{16}x^2}}\,dx \\
&= \frac{1}{16}\int \frac{1}{\sqrt{1 - \left(\frac{3}{4}x\right)^2}}\,dx
\end{aligned}
$$

Step 2: Use u-substitution with $u = \frac{3}{4}x$.

$$\begin{aligned} u &= \frac{3}{4}x \\ du &= \frac{3}{4}dx \\ \frac{4}{3}du &= dx \\ \frac{1}{16} \cdot \frac{4}{3} \int \frac{1}{\sqrt{1-u^2}}\, du \\ = \frac{1}{12} \int \frac{1}{\sqrt{1-u^2}}\, du \end{aligned}$$

Step 3: Use the inverse trig formula.

$$\begin{aligned} \frac{1}{12} \int \frac{1}{\sqrt{1-u^2}}\, du &= \frac{1}{12} \arcsin(u) + C \\ &= \boxed{\frac{1}{12} \arcsin\left(\frac{3}{4}x\right) + C} \end{aligned}$$

H19 Integration by division

Use long division or synthetic division when the degree of the numerator is greater than or equal to the degree of the denominator.

Example: Evaluate $\int \frac{x^2+3x+5}{x+1}\, dx$.

Step 1: Use long division to divide the polynomials.

$$x^2 + 3x + 5 = x + 2 + \frac{3}{x+1}$$

Step 2: Rewrite the integral and solve.

$$\begin{aligned} &\int \frac{x^2+3x+5}{x+1}\, dx \\ &= \int (x + 2 + \frac{3}{x+1})\, dx \\ &= \boxed{\frac{x^2}{2} + 2x + 3\ln|x+1| + C} \end{aligned}$$

How to know which integration method to use

- **Basic antiderivatives:** Check if the integral matches a basic antiderivative formula.
- **u-substitution:** If the integral contains a function and its derivative, consider u-substitution.
- **Change of variable:** Use when u-substitution is not sufficient and a more complex substitution is needed.
- **Long or synthetic division:** Use when the integrand is a rational function where the degree of the numerator is greater than or equal to the degree of the denominator.
- **u-substitution with trigonometry:** Use when the integrand contains inverse trig derivatives.