AP Calculus BC Test 4

H1 Sequences

A **sequence** is a list of elements.

$$\begin{aligned} 2, 4, 6, 8 \dots \quad & a_n = 2n \quad \text{arithmetic sequence} \\ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \dots \quad & a_n = \frac{1}{2^{n-1}} \quad \text{geometric sequence} \\ 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120} \dots \quad & a_n = \frac{1}{n!} \end{aligned}$$

H2 Series and convergence

A **series** is the sum of the elements of a sequence.

Vocabulary and formulas

$$\begin{aligned} \textbf{Infinite series:} \quad & \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots \\ \textbf{Geometric series:} \quad & \sum_{n=0}^{\infty} ar^n \\ \textbf{Sum of geometric series:} \quad & S = \frac{a}{1-r} \quad \text{if } |r| < 1 \\ \textbf{Partial sum:} \quad & S_n \\ \textbf{Sum of series:} \quad & S \end{aligned}$$

nth-term test

If $\sum_{n=1}^{\infty} a_n$ *converges,* then $\lim_{n\to\infty} a_n = 0$.

This is *not* saying that if the limit of the terms in the sequence goes to 0, then the series converges. If the limit goes to 0, further tests are needed to determine convergence.

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ *diverges.*

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2+2}{n}$ converges or diverges.

Determine the limit:

$$\lim_{n\to\infty} \frac{n^2+2}{n} = \infty$$

because the degree of the numerator is greater than the degree of the denominator.

Since the limit does not equal 0, the series **diverges** by the nth-term test.

Geometric series test

All that is needed to prove convergence of a geometric series is to show that the common ratio r satisfies $|r| < 1$.

Example: Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ converges or diverges. If it converges, find the sum.

Step 1: Find the common ratio and determine whether the series converges or diverges.

The common ratio is $r = \dfrac{3}{4}$. Since $|r| = \dfrac{3}{4} < 1$, the series **converges**.

Step 2: Find the sum.

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{1}{1-\frac{3}{4}} \\ &= \frac{1}{\frac{1}{4}} \\ &= \boxed{\text{converges to } 4} \end{aligned}$$

Telescoping series test

A **telescoping series** is a series where many terms cancel out when writing the partial sums. These often involve partial fractions.

Example: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges or diverges. If it converges, find the sum.

Step 1: Separate using partial fractions.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$
$$A(n+1) + B(n) = 1$$
$$An + A + Bn = 1$$
$$A = 1$$
$$B = -1$$
$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Step 2: Write out the sequence.

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} \cdots - \frac{1}{n+1}$$

Step 3: Take the limit of the series.

$$S = 1 - \frac{1}{n+1}$$
$$\lim_{n\to\infty} 1 - \frac{1}{n+1}$$
$$= \boxed{\text{converges to } 1}$$

H3 Integral test and p-series

Integral test

If $f(x)$ is **positive, continuous, and decreasing** for $x \geq 1$ and $f(n) = a_n$, then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

p-series

A p-**series** is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is a positive constant.

If $p > 1$, the series *converges.* Otherwise, the series *diverges.*

A special case occurs at $p = 1$, which is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. This series *diverges.*

H4 Comparison tests

Direct comparison test

The direct comparison test involves comparing the given series to a known series.

Given a_n and b_n are **positive terms** for all n:

If $\sum b_n$ *converges* and $a_n \leq b_n$ for all n, then $\sum a_n$ *converges.*

If $\sum a_n$ *diverges* and $a_n \leq b_n$ for all n, then $\sum b_n$ *diverges.*

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$ converges or diverges.

Step 1: Choose a series to compare to and set up the inequality.

$$\frac{n^2+2}{n^4+5} < \frac{n^2+2}{n^4}$$

Step 2: Simplify the inequality.

$$\frac{n^2+2}{n^4} = \frac{1}{n^2} + \frac{2}{n^4}$$

Step 3: Determine whether the comparison series converges or diverges.

Because both $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{2}{n^4}$ are p-series with $p > 1$, both series **converge.** Therefore, by the direct comparison test, the original series **converges.**

Limit comparison test

Given a_n and b_n are positive terms for all n, and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Example: Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{3^n - n}$ converges or diverges given that the series has positive terms.

Step 1: Choose a series to compare to.

Because the dominant term on the denominator is 3^n, we can compare to the geometric series $\sum_{n=0}^{\infty} \frac{1}{3^n}$.

Step 2: Find the limit of the ratio of the two series.

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\frac{1}{3^n-n}}{\frac{1}{3^n}} \\
&= \lim_{n\to\infty} \frac{3^n}{3^n - n} \\
&= \lim_{n\to\infty} \frac{1}{1 - \frac{n}{3^n}} \\
&= 1
\end{aligned}$$

The limit is a finite, positive value, so we move on with the limit comparison test.

Step 3: Determine whether the comparison series converges or diverges.

$$\frac{1}{3^n} = \left(\frac{1}{3}\right)^n$$

Because the common ratio r of the geometric series is $\frac{1}{3}$ ($|r| < 1$), the series **converges.** Therefore, $\boxed{\text{by the limit comparison test, the original series } \textbf{converges.}}$

H5 Alternating series test

The **alternating series test** can be used on any alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ or $\sum_{n=1}^{\infty}(-1)^n a_n$, where $a_n > 0$ to test if it **converges.**

The series converges if both of the following conditions are met:

(a) $a_{n+1} \leq a_n$ for all n (the terms are decreasing)

(b) $\lim_{n\to\infty} a_n = 0$

If either condition is not met, the test is **inconclusive.**

Example: Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges or diverges.

Step 1: Check if the limit of the terms goes to 0.

Because the degree of the denominator is greater than the degree of the numerator,

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Step 2: Check if the terms decrease.

Because $\frac{1}{n+1} < \frac{1}{n}$ for all n, the terms decrease.

Step 3: Conclude and justify.

$\boxed{\text{By the alternating series test, the series converges.}}$

Alternating Series Remainder

The **remainder** R_N is the absolute difference between the sum of the series S and the Nth partial sum S_N.

From the textbook:

"If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}.$$

"

In simpler terms, for a convergent alternating series that is decreasing, the error made by stopping at the Nth term is less than or equal to the absolute value of the next term.

For example, for the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, if we approximate the sum by stopping at the 4th term (which is S_4), the error, $|S - S_4|$, is less than or equal to $\frac{1}{5}$ (which is a_5).

$$|S - S_4| \leq \frac{1}{5}$$

Absolute and conditional convergence

A series $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges.

If a series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ diverges, then the series is said to be **conditionally convergent**.

H7 Ratio and root tests

Ratio test

Given a series $\sum_{n=1}^{\infty} a_n$, let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- If $L < 1$, the series **converges absolutely.**
- If $L > 1$, the series **diverges.**
- If $L = 1$, the test is **inconclusive.**

Root test

Given a series $\sum_{n=1}^{\infty} a_n$, let

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

- If $L < 1$, the series **converges absolutely.**
- If $L > 1$, the series **diverges.**
- If $L = 1$, the test is **inconclusive.**

Review of tests

Test	**Conditions**	**Procedure**	**C or D?**
nth-term	none	Find $\lim_{n\to\infty} a_n$. If the limit is 0, the series diverges.	Divergence
geometric series	none	Find the common ratio r. If $\lvert r\rvert < 1$, converges. Otherwise, diverges.	Both
p-series	none	If $p > 1$, converges. Otherwise, diverges.	Both
telescoping series	none	Cancel the middle terms and take the limit as n goes to infinity of the first and last term. If the limit exists and is finite, the series converges.	Both
integral test	positive, continuous, and decreasing	Evaluate $\int_1^\infty f(x)\,dx$. If the integral converges, the series converges. Otherwise, it diverges.	Both
direct comparison	positive	Compare original series to a known series. Designate the smaller series as a_n and the greater as b_n ($a_n < b_n$). **If a_n converges, the original series converges. If b_n diverges, the original series diverges.**	Both
limit comparison	positive	Compare original series a_n to a known series b_n. Find $\lim_{n\to\infty} \frac{a_n}{b_n} = c$. If c is finite and $c > 0$, both series either converge or diverge.	Both
alternating series	positive, decreasing	Check that $\lim_{n\to\infty} a_n = 0$ to conclude the series converges	Convergence
ratio	none	Find $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L$. If $L < 1$, converges. If $L > 1$, diverges. If $L = 1$, inconclusive.	Both
root	none	Find $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L$. If $L < 1$, converges. If $L > 1$, diverges. If $L = 1$, inconclusive.	Both

Taylor polynomials

The Taylor series of a function $f(x)$ centered at $x = a$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

An nth-degree Taylor polynomial is given by:

$$P_n(x) = a_0 + a_1(x-a) + \frac{a_2}{2!}(x-a)^2 + \frac{a_3}{3!}(x-a)^3 + \cdots + \frac{a_n}{n!}(x-a)^n$$

where

Similar to linearization, this series approximates the function $f(x)$ near the point $x = a$ with a polynomial function.

Example: Find the 5th-order (5 derivatives) Taylor polynomial for $f(x) = e^{1-x}$ centered at $x = 0$.

Step 1: Find the derivatives of $f(x)$ and evaluate them at $x = 0$.

$$\begin{aligned} f(0) &= e \\ f'(0) &= -e \\ f''(0) &= e \\ f'''(0) &= -e \\ f^{(4)}(0) &= e \\ f^{(5)}(0) &= -e \end{aligned}$$

Step 2: Substitute into the Taylor polynomial formula.

$$P_5(x) = e - ex + \frac{e}{2!}x^2 - \frac{e}{3!}x^3 + \frac{e}{4!}x^4 - \frac{e}{5!}x^5$$