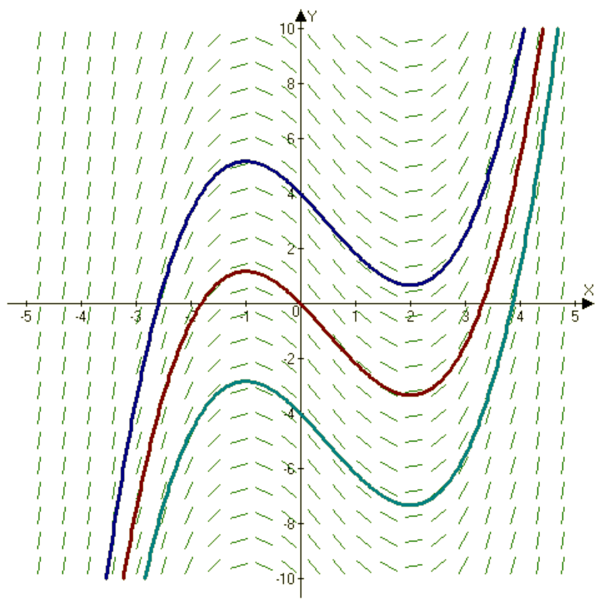
AP Calculus BC Test 3

H20 Slope fields and Euler's method

A slope field is a graphical representation of a differential equation that shows the slope of the solution curve at each point in the plane.

For example, the slope field for the differential equation $\frac{dy}{dx} = x^2 - x - 2$ can be drawn by calculating the slope at various points (x, y) and drawing small line segments with those slopes. The result is (from Wikipedia):



Euler's method is a technique used to approximate solutions to differential equations with a given initial value by using the slope of the function at a given point to estimate the value of the function at the next point.

Example: Use a table and Euler's method to approximate the value of y at $x = 1$ for the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $y(0) = 1$, using a step size of $h = 0.5$. Use $\Delta y = \frac{dy}{dx} \cdot \Delta x$ to find the change in y at each step.

Step 1: Make a table with the initial condition and known values.

x	y	Δx	$\frac{dy}{dx} = x + y$	Δy	$(x + \Delta x, y + \Delta y)$
0	1	0.5	-	-	$(0.5, y)$
0.5	-	0.5	-	-	$(1, y)$

Step 2: Calculate the slope $\frac{dy}{dx}$ at each point and use it to find Δy.

x	y	Δx	$\frac{dy}{dx} = x + y$	Δy	$(x + \Delta x, y + \Delta y)$
0	1	0.5	1	0.5	$(0.5, 1.5)$
0.5	1.5	0.5	2	1	$(1, 2.5)$

Example: Use tangent lines to do the same problem.

Step 1: Find the equation of the tangent line at the initial point.

Given:

$$\begin{gathered}\frac{dy}{dx} = x + y \\ y(0) = 1 \\ \Delta x = 0.5\end{gathered}$$

Finding the tangent line:

$$\begin{gathered}\frac{dy}{dx} = 0 + 1 = 1 \\ y - 1 = 1(x - 0) \\ y = x + 1\end{gathered}$$

Step 2: Use the tangent line to approximate y at $x = 0.5$.

At $x = 0.5$, $y = 0.5 + 1 = 1.5$.

Step 3: Use the new point to find the next tangent line.

$$\begin{gathered}\frac{dy}{dx} = 1.5 + 0.5 = 2 \\ y - 1.5 = 2(x - 0.5) \\ y = 2x + 0.5\end{gathered}$$

Step 4: Use the new tangent line to approximate y at $x = 1$.

At $x = 1$, $y = 2(1) + 0.5 = \boxed{2.5}$.

Both methods yield the same approximation of $y(1) \approx 2.5$.

H21 Separable differential equations

A separable differential equation is one that can be expressed in the form $\frac{dy}{dx} = f(x)g(y)$, allowing the variables to be separated on opposite sides of the equation for integration.

Example: Solve the separable differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$ with the initial condition $y(0) = 2$.

Step 1: Separate the variables.

$$\frac{dy}{dx} = \frac{x}{y}$$
$$y\,dy = x\,dx$$

Step 2: Integrate both sides to find the general solution.

$$\int y\,dy = \int x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$
$$y^2 = x^2 + C$$

Step 3: Use the initial condition to find the particular solution.

$$y^2 = x^2 + C$$
$$y(0) = 2$$
$$2^2 = 0^2 + C$$
$$C = 4$$
$$y = \pm\sqrt{x^2 + 4}$$
$$2 = \pm\sqrt{0^2 + 4}$$
$$2 = \sqrt{4}$$
$$\boxed{y = \sqrt{x^2 + 4}}$$

H22 Logistic equations

A logistic equation is a type of differential equation that models population growth with a carrying capacity.

Typically, the logistic equation is expressed as

$$\frac{dP}{dt} = kP(M - P)$$

where P is the population size, k is the growth rate, and M is the carrying capacity.

Important things to note:

- The population grows fastest at $P = \dfrac{M}{2}$, or in the middle of the curve at the point of inflection.
- As P approaches M, the growth rate slows down and the population stabilizes. The rate of change approaches zero as the population approaches its carrying capacity.
- If the equation is not in the standard form, it may need to be manipulated algebraically to identify k and M.

H23 (review of integrals, skipped)

H24 Integration by parts, solving for the integral, and tabular integration

Integration by parts

Integration by parts is a technique used to integrate products of functions following the formula[1]:

$$\int u\, dv = uv - \int v\, du$$

where u and dv are parts of the original integral.

To identify u, use the initialism **LIATE**, which stands for **L**ogarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, and **E**xponential functions. The function that appears first in this list should be chosen as u.

Example: Use integration by parts to evaluate the integral $\int xe^x\, dx$.

Step 1: Identify u and dv with du and v.

Let $u = x$ (algebraic) and $dv = e^x\, dx$. Thus, $du = dx$ and $v = e^x$.[2]

Step 2: Follow the formula.

[1] This formula can be memorized with the mnemonic device "ultraviolet voodoo", origin unknown :)

[2] While $u = x$ is not a valid choice for u-substitution since it does not simplify the integral, it does work well for integration by parts.

$$
\begin{aligned}
&uv - \int v \, du \\
&= xe^x - \int e^x \, dx
\end{aligned}
$$

Step 3: Integrate.

$$
\begin{aligned}
&xe^x - \int e^x \, dx \\
&= \boxed{xe^x - e^x + C}
\end{aligned}
$$

Solving for the integral

Sometimes, applying integration by parts results in an equation that contains the original integral. In such cases, solve for the integral algebraically.

Consider the following integral:

$$\int e^x \cos x \, dx$$

This integral looks solvable, but applying integration by parts twice will lead back to the original integral. To test this, let $u = e^x$ and $dv = \cos x \, dx$, and therefore $du = e^x \, dx$ and $v = \sin x$. Following the formula

$$\int u \, dv = uv - \int v \, du$$

yields:

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Now, apply integration by parts again to the remaining integral $\int e^x \sin x \, dx$. Let $u = e^x$ and $dv = \sin x \, dx$, so $du = e^x \, dx$ and $v = -\cos x$. Applying the formula again gives:

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

Substituting this back into the previous equation results in:

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx \right)$$

This simplifies to:

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

Now, the original integral is on both sides of the equation. To solve for it, add $\int e^x \cos x \, dx$ to both sides:

$$2 \int e^x \cos x \, dx = e^x (\sin x + \cos x)$$

Finally, divide both sides by 2 to isolate the integral:

$$\int e^x \cos x \, dx = \boxed{\frac{e^x (\sin x + \cos x)}{2} + C}$$

Tabular integration

Tabular integration is a method used to simplify the process of integration by parts when one function can be differentiated repeatedly until it becomes zero, and the other function can be integrated repeatedly.

Example: Use tabular integration to evaluate the integral $\int x^3 e^x \, dx$.

Step 1: Set up the table and differentiate u until the left side reaches 0. Integrate dv the same number of times.

u	dv
x^3	e^x
$3x^2$	e^x
$6x$	e^x
6	e^x
0	e^x

Step 2: Multiply diagonally, alternating signs, and sum the results.

$$
\begin{gathered}
+x^3e^x \\
-3x^2e^x \\
+6xe^x \\
-6e^x
\end{gathered}
$$

Step 3: Write the final answer.

$$\int x^3 e^x \, dx = \boxed{e^x(x^3 - 3x^2 + 6x - 6) + C}$$