

AP Calculus BC Test 2

H10 Linearization

Linearization is a method used to approximate the value of a function near a given point using the tangent line at that point.

Example: Approximate the cube root of 82 using linearization.

Step 1: Identify the function and the point of tangency.

$$f(x) = \sqrt[3]{x}, \quad a = 81$$

Step 2: Find the derivative of the function.

$$f'(x) = \frac{1}{3x^{2/3}}$$

Step 3: Evaluate the function and its derivative at the point of tangency.

$$f(81) = 3, \quad f'(81) = \frac{1}{27}$$

Step 4: Write the equation of the tangent line.

H11 Optimization

Optimization involves finding the maximum or minimum values of a function subject to certain constraints.

Example: A farmer wants to build a rectangular pen with a fixed amount of fencing (100 meters). What dimensions will maximize the area of the pen?

Step 1: Define the variables and equations.

Let x be the length and y be the width of the pen. The area A is given by

$$A = xy$$

and the constraint from the fencing is

$$2x + 2y = 100$$

or

$$x + y = 50$$

Step 2: Express one variable in terms of the other using the constraint.

$$y = 50 - x$$

Step 3: Substitute into the area equation.

$$\begin{aligned} A(x) &= x(50 - x) \\ &= 50x - x^2 \end{aligned}$$

Step 4: Find the critical points by taking the derivative and setting it to zero.

$$\begin{aligned} A'(x) &= 50 - 2x \\ 0 &= 50 - 2x - 2x = -50x = 25 \end{aligned}$$

Step 5: Find the corresponding value of y .

$$y = 50 - 25 = 25$$

Step 6: Conclude with a justification:

The dimensions that maximize the area of the pen are 25 meters by 25 meters because the sign of the first derivative changes from positive to negative at $x = 25$, indicating a maximum.

H12 Estimating with Riemann sums

RRAM Right Rectangular Approximation Method - heights of rects = heights of right endpoints

LRAM Left Rectangular Approximation Method - heights of rects = heights of left endpoints

MRAM Midpoint Rectangular Approximation Method - heights of rects = heights of midpoints

Trapezoidal approximation Average of RRAM and LRAM

Example: Estimate with MRAM the value of $\int_0^4 (x^2 + 1) dx$ using 4 subintervals.

Step 1: Determine Δx .

$$\Delta x = \frac{4 - 0}{4} = 1$$

Step 2: Identify the midpoints of each subinterval.

- Subinterval $[0, 1]$: midpoint $= 0.5$
- Subinterval $[1, 2]$: midpoint $= 1.5$
- Subinterval $[2, 3]$: midpoint $= 2.5$
- Subinterval $[3, 4]$: midpoint $= 3.5$

Step 3: Evaluate the function at each midpoint.

$$\begin{aligned}f(0.5) &= (0.5)^2 + 1 = 1.25 \\f(1.5) &= (1.5)^2 + 1 = 3.25 \\f(2.5) &= (2.5)^2 + 1 = 7.25 \\f(3.5) &= (3.5)^2 + 1 = 13.25\end{aligned}$$

Step 4: Calculate the Riemann sum.

$$\begin{aligned}\text{MRAM} &= \Delta x[f(0.5) + f(1.5) + f(2.5) + f(3.5)] \\&= 1[1.25 + 3.25 + 7.25 + 13.25] \\&= \boxed{25}\end{aligned}$$

H13 Writing and interpreting Riemann sums

When converting a Riemann sum of the form

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b}{n} f\left(a + \frac{bi}{n}\right)$$

to a definite integral, follow these steps:

1. Identify Δx (the width of each subinterval) as $\frac{b}{n}$.
2. Identify the function $f(x)$ inside the sum.
3. Determine the limits of integration:
 - The lower limit is a (the starting point of the interval).
 - The upper limit is $a + b$ (the endpoint of the interval).
4. Write the definite integral as

$$\int_a^{a+b} f(x) \, dx$$

Similarly, when writing a Riemann sum for a definite integral of the form

$$\int_a^{a+b} f(x) dx$$

follow these steps:

1. Identify the interval $[a, a + b]$ and the function $f(x)$.
2. Determine Δx as $\frac{b}{n}$ (subtract upper limit of integration from lower) and plug into the equation for x as $\frac{bi}{n}$
3. Write the Riemann sum as

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b}{n} f\left(a + \frac{bi}{n}\right)$$

Example: Write the Riemann sum for the definite integral $\int_2^5 (x^2 + 1) dx$.

Step 1: Identify the interval and function.

The interval is $[2, 5]$ and the function is $f(x) = x^2 + 1$.

Step 2: Determine Δx and express x_i .

$$\Delta x = \frac{5 - 2}{n} = \frac{3}{n}$$
$$x_i = 2 + \frac{3i}{n}$$

Step 3: Write the Riemann sum.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left[\left(2 + \frac{3i}{n} \right)^2 + 1 \right]$$

(H14 intentionally skipped.)

H15 Antiderivatives and indefinite integration

Antiderivative rules are the same as derivative rules, but in reverse. Always +C to the end of indefinite integrals.

Example: Find the antiderivative of $f(x) = 3x^2 - 4x + 5$.

Apply the power rule to each term:

$$\begin{aligned}
& \int (3x^2 - 4x + 5) \, dx \\
&= \int \left(\frac{3x^{2+1}}{2+1} - \frac{4x^{1+1}}{1+1} + \frac{5x^{0+1}}{0+1} \right) dx \\
&= \boxed{x^3 - 2x^2 + 5x + C}
\end{aligned}$$

H16 The Fundamental Theorem of Calculus

First Fundamental Theorem

If f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Second Fundamental Theorem

The derivative of the integral of a function is the original function:

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$$

Example: Evaluate $\frac{d}{dx} \int_1^{3x} (2t + 3) \, dt$.

Step 1: Use the Second FTC and plug in the upper limit ($3x$) into the function.

$$\begin{aligned}
& 2(3x) + 3 \\
&= 6x + 3
\end{aligned}$$

Step 2: Multiply by the derivative of the upper limit.

$$\begin{aligned}
& (6x + 3) \cdot \frac{d}{dx}(3x) \\
&= (6x + 3) \cdot 3 \\
&= \boxed{18x + 9}
\end{aligned}$$

H17 Integration by u -substitution and change of variable

Use u -substitution when the integral contains a function and its derivative.

Example: Evaluate $\int 2x\sqrt{x^2+1} \, dx$.

Step 1: Choose u to be the inner function.

$$\begin{aligned}u &= x^2 + 1 \\du &= 2x \, dx\end{aligned}$$

Step 2: Rewrite the integral in terms of u .

$$\int \sqrt{u} \, du$$

Step 3: Integrate with respect to u .

$$\begin{aligned}\int \sqrt{u} \, du &= \int u^{1/2} \, du \\&= \frac{u^{3/2}}{(3/2)} + C \\&= \frac{2}{3}u^{3/2} + C\end{aligned}$$

Step 4: Substitute back to x .

$$\boxed{\frac{2}{3}(x^2+1)^{3/2} + C}$$

H18 Inverse trig integration

$$\begin{aligned}\int \frac{1}{\sqrt{a^2-x^2}} \, dx &= \arcsin\left(\frac{x}{a}\right) + C \\ \int \frac{1}{a^2+x^2} \, dx &= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \\ \int \frac{1}{x\sqrt{x^2-a^2}} \, dx &= \frac{1}{a} \operatorname{arcsec}\left(\frac{|x|}{a}\right) + C\end{aligned}$$

(no need to memorize formulas - memorize forms, use u -substitution instead)

Example: Evaluate $\int \frac{1}{\sqrt{16-9x^2}} \, dx$.

Step 1: Reduce the constant to 1.

$$\begin{aligned} & \int \frac{1}{\sqrt{16 - 9x^2}} dx \\ &= \int \frac{1}{16\sqrt{1 - \frac{9}{16}x^2}} dx \\ &= \frac{1}{16} \int \frac{1}{\sqrt{1 - \left(\frac{3}{4}x\right)^2}} dx \end{aligned}$$

Step 2: Use u -substitution with $u = \frac{3}{4}x$.

$$\begin{aligned} u &= \frac{3}{4}x \\ du &= \frac{3}{4}dx \\ \frac{4}{3}du &= dx \\ \frac{1}{16} \cdot \frac{4}{3} \int \frac{1}{\sqrt{1 - u^2}} du \\ &= \frac{1}{12} \int \frac{1}{\sqrt{1 - u^2}} du \end{aligned}$$

Step 3: Use the inverse trig formula.

$$\begin{aligned} \frac{1}{12} \int \frac{1}{\sqrt{1 - u^2}} du &= \frac{1}{12} \arcsin(u) + C \\ &= \boxed{\frac{1}{12} \arcsin\left(\frac{3}{4}x\right) + C} \end{aligned}$$

H19 Integration by division

Use long division or synthetic division when the degree of the numerator is greater than or equal to the degree of the denominator.

Example: Evaluate $\int \frac{x^2 + 3x + 5}{x + 1} dx$.

Step 1: Use long division to divide the polynomials.

$$x^2 + 3x + 5 = x + 2 + \frac{3}{x + 1}$$

Step 2: Rewrite the integral and solve.

$$\begin{aligned} & \int \frac{x^2 + 3x + 5}{x + 1} dx \\ &= \int \left(x + 2 + \frac{3}{x + 1}\right) dx \\ &= \boxed{\frac{x^2}{2} + 2x + 3 \ln |x + 1| + C} \end{aligned}$$

How to know which integration method to use

- **Basic antiderivatives:** Check if the integral matches a basic antiderivative formula.
- **u -substitution:** If the integral contains a function and its derivative, consider u -substitution.
- **Change of variable:** Use when the
- **Long or synthetic division:** Use when the integrand is a rational function where the degree of the numerator is greater than or equal to the degree of the denominator.
- **u -substitution with trigonometry:** Use when the integrand contains inverse trig derivatives.