AP Calculus BC Test 4

H1 Sequences

A **sequence** is a list of elements.

$$
\begin{aligned}
2, 4, 6, 8 \dots \quad & a_n = 2n \quad \text{arithmetic sequence} \\
1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \dots \quad & a_n = \frac{1}{2^{n-1}} \quad \text{geometric sequence} \\
1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120} \dots \quad & a_n = \frac{1}{n!}
\end{aligned}
$$

H2 Series and convergence

A **series** is the sum of the elements of a sequence.

Vocabulary and formulas

$$
\begin{aligned}
\textbf{Infinite series:} \quad & \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots \\
\textbf{Geometric series:} \quad & \sum_{n=0}^{\infty} ar^n \\
\textbf{Sum of geometric series:} \quad & S = \frac{a}{1-r} \quad \text{if } |r| < 1 \\
\textbf{Partial sum:} \quad & S_n \\
\textbf{Sum of series:} \quad & S
\end{aligned}
$$

nth-term test

If $\sum_{n=1}^{\infty} a_n$ ***converges,*** **then** $\lim_{n \to \infty} a_n = 0$.

This is *not* saying that if the limit of the terms in the sequence goes to 0, then the series converges. If the limit goes to 0, further tests are needed to determine convergence.

If $\lim_{n \to \infty} a_n \neq 0$, **then** $\sum_{n=1}^{\infty} a_n$ ***diverges.***

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2+2}{n}$ converges or diverges.

Determine the limit:

$$\lim_{n\to\infty} \frac{n^2+2}{n} = \infty$$

because the degree of the numerator is greater than the degree of the denominator.

Since the limit does not equal 0, the series **diverges** by the nth-term test.

Geometric series test

All that is needed to prove convergence of a geometric series is to show that the common ratio r satisfies $|r| < 1$.

Example: Determine whether the series $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ converges or diverges. If it converges, find the sum.

Step 1: Find the common ratio and determine whether the series converges or diverges.

The common ratio is $r = \frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, the series **converges**.

Step 2: Find the sum.

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{1}{1-\frac{3}{4}} \\ &= \frac{1}{\frac{1}{4}} \\ &= \boxed{\text{converges to } 4} \end{aligned}$$

Telescoping series test

A **telescoping series** is a series where many terms cancel out when writing the partial sums. These often involve partial fractions.

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges or diverges. If it converges, find the sum.

Step 1: Separate using partial fractions.

$$
\begin{aligned}
\frac{1}{n(n+1)} &= \frac{A}{n} + \frac{B}{n+1} \\
A(n+1) + B(n) &= 1 \\
An + A + Bn &= 1 \\
A &= 1 \\
B &= -1 \\
\frac{1}{n(n+1)} &= \frac{1}{n} - \frac{1}{n+1}
\end{aligned}
$$

Step 2: Write out the sequence.

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} \cdots - \frac{1}{n+1}$$

Step 3: Take the limit of the series.

$$
\begin{aligned}
S &= 1 - \frac{1}{n+1} \\
&\lim_{n\to\infty} 1 - \frac{1}{n+1} \\
&= \boxed{\text{converges to } 1}
\end{aligned}
$$

H3 Integral test and p-series

Integral test

If $f(x)$ is **positive, continuous, and decreasing** for $x \geq 1$ and $f(n) = a_n$, then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

p-series

A p**-series** is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is a positive constant.

If $p > 1$, the series *converges.* Otherwise, the series *diverges.*

A special case occurs at $p = 1$, which is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. This series *diverges.*

H4 Comparison tests

Direct comparison test

The direct comparison test involves comparing the given series to a known series.

Given a_n and b_n are **positive terms** for all n:

If $\sum b_n$ *converges* and $a_n \leq b_n$ for all n, then $\sum a_n$ *converges.*

If $\sum a_n$ *diverges* and $a_n \leq b_n$ for all n, then $\sum b_n$ *diverges.*

Example: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$ converges or diverges.

Step 1: Choose a series to compare to and set up the inequality.

$$\frac{n^2+2}{n^4+5} < \frac{n^2+2}{n^4}$$

Step 2: Simplify the inequality.

$$\frac{n^2+2}{n^4} = \frac{1}{n^2} + \frac{2}{n^4}$$

Step 3: Determine whether the comparison series converges or diverges.

Because both $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^4}$ are p-series with $p > 1$, both series **converge.** Therefore, by the direct comparison test, the original series **converges.**

Limit comparison test

Given a_n and b_n are positive terms for all n, and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Example: Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n - n}$ converges or diverges.

H5 Alternating series test

The **alternating series test** can be used on any alternating series of the form $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ or $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$, where $a_n > 0$ to test if it **converges.**

The series converges if both of the following conditions are met:

(a) $a_{n+1} \leq a_n$ for all n (the terms are decreasing)

(b) $\lim_{n\to\infty} a_n = 0$

If either condition is not met, the test is **inconclusive.**

Example: Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges or diverges.

Step 1: Check if the limit of the terms goes to 0.

Because the degree of the denominator is greater than the degree of the numerator,

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Step 2: Check if the terms decrease.

Because $\frac{1}{n+1} < \frac{1}{n}$ for all n, the terms decrease.

Step 3: Conclude and justify.

$\boxed{\text{By the alternating series test, the series converges.}}$