Precalculus Honors Spring Final Study Guide

Test date: June 2, 2025

Notation

1. Always keep writing the full limit (with "lim") until there are no more variables in the equation. For example:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3}$$

   not:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{x-3}$$

2. When using LHR, show your justification in some way and invoke LHR before solving the problem. For example:

$$\lim_{x\to 0} \sin x = 0 = \lim_{x\to 0} x$$

   LHR

   do NOT write this:

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0}$$

   because $\frac{0}{0}$ does not exist.

3. Justifications should be complete and concise. For example, "A local maximum occurs at (a, b) because the first derivative changes sign from positive to negative at $x = a$." Don't write more than necessary!

4. Don't simplify using algebraic methods unless the question specifies otherwise to avoid losing points for algebra mistakes. Once the calculus is over, the problem is done.

Limits and continuity

Definition and applications of the limit

- A **limit** is a y-value where you expect to be at a certain x-value. It does not have to equal the actual value of the function at that point.

- If the limit from the right $\lim_{x \to a^+} f(x)$ equals the limit from the left $\lim_{x \to a^-} f(x)$, the limit exists.
- If the limit from the right does not equal the limit from the left, or both limits go toward negative or positive infinity, the limit does not exist (DNE).

Methods to evaluate limits

Plugging in directly

Example: Evaluate $\lim_{x \to 5} \frac{x^2+2}{x-2}$.

$$\begin{aligned} \lim_{x \to 5} \frac{x^2+2}{x-2} &= \frac{5^2+2}{5-2} \\ &= \frac{25+2}{5-2} \\ &= \boxed{\frac{27}{3}} \end{aligned}$$

Algebraic methods

Example: Evaluate $\lim_{x \to 3} \frac{x^2-9}{x-3}$.

Factor:

$$\begin{gathered} \lim_{x \to 3} \frac{x^2-9}{x-3} \\ = \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} \\ = \lim_{x \to 3} x+3 \\ = \boxed{6} \end{gathered}$$

Multiplying by the conjugate

Example: Evaluate $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$.

Multiply the numerator and denominator by the conjugate of the numerator:

$$\begin{aligned} \lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} &= \lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} \\ &= \lim_{x \to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} \end{aligned}$$

$$= \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)}$$
$$= \lim_{x\to 4} \frac{1}{\sqrt{x}+2}$$
$$= \frac{1}{\sqrt{4}+2}$$
$$= \frac{1}{2+2}$$
$$= \boxed{\frac{1}{4}}$$

L'Hôpital's rule for indeterminate forms

(when the limit $= \frac{0}{0}$ or $\frac{\infty}{\infty}$)

Example: Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

This is an indeterminate form ($\frac{0}{0}$), requiring the application of L'Hôpital's rule (LHR). **Before anything, invoke LHR and justify. This step is very important and should not be skipped over.**

$$\lim_{x\to 0} \sin x = 0 = \lim_{x\to 0} x$$
$$\text{LHR}$$

Apply L'Hôpital's rule, taking the derivative of the numerator over the derivative of the denominator.[1]

$$\lim_{x\to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)}$$
$$= \lim_{x\to 0} \frac{\cos x}{1}$$
$$= \cos 0$$
$$= \boxed{1}$$

The derivative and its applications

Definitions of the derivative

Conceptual

The **derivative** is the slope of a function at a given point or over the entire curve (instantaneous rate of change, IROC).

[1]This can be applied as many times as necessary.

Limit definition

$$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Alternate limit definition

for finding the limit at a given point a:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Basic derivatives

c denotes a constant. Differentiation is with respect to x.

Power rule

$$\frac{d}{dx} x^c = c x^{c-1}$$

Product rule

$$\frac{d}{dx} f(x) \times g(x) = f'(x) g(x) + f(x) g'(x)$$

Quotient rule

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x) f'(x) - f(x) g'(x)}{(g(x))^2}$$

Chain rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Logs and exponentials

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx} a^x = a^x \ln a$$

$$\frac{d}{dx} e^x = e^x$$

Trigonometric functions

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\csc x = -\csc x \cot x$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Composite and inverse differentiation

Chain rule

"Differentiate the mother, keep the baby inside, differentiate the baby."

Example: Differentiate $(5x+3)^3$ with respect to x.

$$\begin{aligned}\frac{d}{dx}(5x+3)^3 &= 3(5x+3)^2 \cdot 5 \\ &= \boxed{15(5x+3)^2}\end{aligned}$$

Common derivatives of functions

u denotes a function.

$$\frac{d}{dx}\ln u = \frac{u'}{u}$$
$$\frac{d}{dx}\log_a u = \frac{u'}{u \ln a}$$
$$\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arccos u = -\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}a^u = u'a^u \ln a$$

$$\frac{d}{dx}e^u = u'e^u$$

Inverse differentiation

A function $f(x)$ has the inverse $g(x)$. At (a, b), the derivative $f'(a)$ is equal to $\frac{1}{g'(b)}$.

Linear approximation

Use a derivative to approximate a function at a given point.

Example: Estimate $f(1.5)$ for $f(x) = \cos x$. ($\frac{\pi}{2} \approx 1.57$ for this problem.)

Find the first derivative:

$$f'(x) \cos x = -\sin x$$

Approximate at the first derivative:

$$f'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$

This is the slope of the tangent line. Find the value of the function at $\frac{\pi}{2}$:

$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

Write the line in slope-intercept form:

$$y - 0 = -\left(x - \frac{\pi}{2}\right)$$

Approximate $f(1.5)$ using this line, $f'(1.57)$:

$$y = -(1.5 - 1.57)$$

$$y = -1.5 + 1.57$$

$$y = 0.07$$

$$\boxed{f(1.5) \approx 0.07}$$

IVT and MVT

Intermediate Value Theorem

If f is continuous on $[a, b]$ and k is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number c in the interval $[a, b]$ such that $f(c) = k$.

In other words, if a function is continuous over an interval, it has a y-value for every x-value in that interval.

Example: Let $f(x) = x^3 - 4x + 1$. Show that there is at least one root ($f(x) = 0$) in the interval $[1, 2]$.

MVT applies because this is a polynomial function and $f(x)$ is continuous over the entire interval.

Mean Value Theorem

If $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b), then c exists such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

.

If $f(x)$ is continuous on the closed interval and differentiable on the open interval, there is a point where AROC (average rate of change, the secant line) = IROC (instantaneous rate of change, the tangent line).

Graph analysis

Extrema

Local (relative) extrema can only occur within the given interval. **Absolute extrema** (the single lowest/highest point on the interval) can occur at endpoints or within the interval.

The first derivative

The first derivative (y', $f'(x)$, $\frac{dy}{dx}$) determines whether a graph is **increasing or decreasing**. When the first derivative >0, the graph is increasing. When the first derivative <0, the graph is decreasing. At $\frac{dy}{dx} = 0$, there is a minimum or a maximum.

The second derivative

The second derivative (y'', $f''(x)$, $\frac{d^2y}{dx^2}$) determines the **concavity** of a graph (concave up, concave down). When the second derivative >0, the graph is concave up. When the second derivative <0, the graph is concave down. At $\frac{d^2y}{dx^2} = 0$, there is a point of inflection where the graph changes concavity.

Example: Analyze the function $f(x) = x^3 - 3x^2 + 4$. Find the interval(s) on which the function is increasing/decreasing, local extrema, interval(s) on which the function is concave up/down, and point(s) of inflection. Justify.

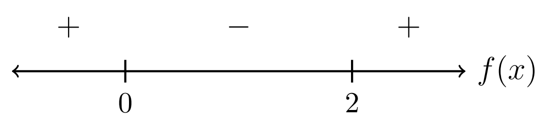
First derivative:

$$3x^2 - 6x$$

Set to 0:

$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$
$$x = 0, x = 2$$

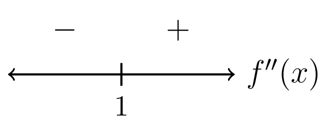
0 and 2 are the critical numbers. Test numbers in the following intervals by plugging them into the first derivative.



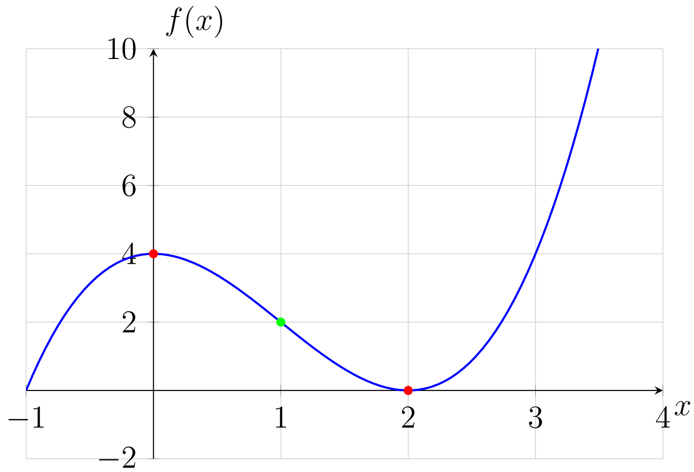
Second derivative:

$$6x - 6 = 0$$
$$x = 1$$

Do the same with $x < 1$ and $x > 1$:



Graph:



increasing: $(-\infty, 0)$ and $(2, \infty)$
decreasing: $(0, 2)$
concave up: $(1, \infty)$
concave down: $(-\infty, 1)$
maximum: $x = 0$ because the sign of the first derivative changes from positive to negative at $x = 0$
minimum: $x = 2$ because the sign of the first derivative changes from negative to positive at $x = 2$
point of inflection: $x = 1$ because the sign of the second derivative changes from negative to positive at $x = 1$

Implicit differentiation

Example 1: Find $\dfrac{dy}{dx}$ for $x^2 + 3xy + y = 5$.

$$\begin{gathered}
\frac{dy}{dx}(x^2 + 3xy + y) = \frac{dy}{dx}5 \\
2x + 3(xy' + y) + y' = 0 \\
2x + 3xy' + 3y + y' = 0 \\
3xy' + y' = -3y - 2x \\
y'(3x + 1) = -3y - 2x \\
y' = \frac{-3y - 2x}{3x + 1}
\end{gathered}$$

Position, acceleration, and velocity

position where the object is

distance total length of path travelled

displacement change in position from starting point to end point

velocity the rate at which the position is changing

speed the absolute value of velocity

acceleration the rate at which the velocity is changing

Original function	**First derivative**	**Second derivative**
position $s(t)$	velocity $v(t)$	acceleration $a(t)$
velocity $v(t)$	acceleration $a(t)$	-
acceleration $a(t)$	-	-

Example: A particle moves along a straight line with velocity given by

$$v(t) = 3t^2 - 12t + 9$$

and position given by

$$s(t) = t^3 - 6t^2 + 9t$$

for $t \in [0, 4]$. What is the total distance traveled and the displacement?

Find the critical numbers where the velocity is 0:

$$\begin{gathered} 3t^2 - 12t + 9 = 0 \\ 3(t^2 - 4t + 3) = 0 \\ (t-3)(t-1) = 0 \\ t = 3, 1 \end{gathered}$$

The number line is separated into three intervals: from 0 to 1, 1 to 3, and 3 to 4. Test values for each interval and decide whether the object moves forward or backward depending on the sign.

When t is on the interval $[0, 1]$, the object moves forward. On the interval $[1, 3]$, the object moves backward, and forward again on $[3, 4]$.

Evaluate position at both endpoints and key values:

$$\begin{gathered} s(0) = 0 \\ s(1) = 4 \\ s(3) = 0 \\ s(4) = 4 \end{gathered}$$

Calculate distance between position changes.

From 0 to 1, the object moves right 4 units.

From 1 to 3, the object moves left 4 units.

From 3 to 4, the object moves right 4 units.

$$4 + 4 + 4 = 12$$

Distance: $\boxed{\text{12 units}}$
Displacement $s(4) - s(0)$: $\boxed{\text{4 units}}$

Related rates

For related rates problems, write the given, what you're trying to find, and a unifying equation.

Example: A balloon is being inflated so that it remains a perfect sphere. If the radius of the balloon is decreasing at a rate of 2 cm/s, how fast is the volume of the balloon decreasing when the radius is 5 cm?

Equation: $V = \dfrac{4}{3}\pi r^3$
Given: $\dfrac{dr}{dt} = -2$, $r = 5$
Find: $\dfrac{dV}{dt}$

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} \\ \frac{dV}{dt} &= 4\pi(5)^2(-2) \\ \frac{dV}{dt} &= -200\pi \end{aligned}$$

The volume is $\boxed{\text{decreasing at a rate of } 200\pi \text{ cm}^3/\text{s}}$ when $r = 5$.

Optimization

Example: A company wants to build a rectangular fenced-in area next to a straight river, and they do not need to fence the side along the river. They have 600 meters of fencing available. What is the maximum area that can be enclosed?

Let the side across from the river be x, and the sides adjacent to the river be y. Then the perimeter of the area is

$$600 = x + 2y$$

This is the restriction. The area is

$$A = xy$$

First, solve for x:

$$x = 600 - 2y$$

plug it into the area formula:

$$A = (600 - 2y)(y)$$
$$A = 600y - 2y^2$$

To find the maximum of this equation, find the first derivative:

$$A' = 600 - 4y$$

set to 0 to find critical numbers:

$$0 = 600 - 4y$$
$$-4y = -600$$
$$y = 150$$

going back to the original condition:

$$600 = x + 2y$$
$$600 = x + 300$$
$$x = 300$$

The dimensions are 300m $\times$ 150m, so the area is $\boxed{45000\text{m}^2}$.

Integration

Taking an integral (or antidifferentiation) is the opposite of taking a derivative.

Basic integration

Example: Integrate $\int (3x + 5)\, dx$.

$$\begin{aligned}\int (3x + 5)\, dx &= \frac{3x^{1+1}}{2} + 5x + C \\ &= \boxed{\frac{3}{2}x^2 + 5x + C}\end{aligned}$$

Always remember the $+C$ (constant of integration) for indefinite integrals!

Important integrals

Just the derivatives but flipped with $+C$.

Example: Integrate $\int \frac{2x}{x^2 + 1}\, dx$.

$$\int \frac{2x}{x^2 + 1}\, dx = \boxed{\ln(x^2 + 1) + C}$$