ECE 131A Project Report

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1. Data Imputation

(a) we start by simplifying $\frac{d}{da_i}E_{MMSE}$:

$$\frac{d}{da_i}E_{MMSE} = \frac{d}{da_i}\left[\sum_{i \in K_{miss}} (X_i - a_i)^2\right]$$

$$\frac{d}{da_i}E_{MMSE} = E\left[\sum_{i \in K_{miss}} \frac{d}{da_i}(X_i - a_i)^2\right]$$

$$\frac{d}{da_i}E_{MMSE} = \sum_{i \in K_{miss}} E\left[2\left(a_i - x_i\right)\right]$$

$$\frac{d}{da_i}E_{MMSE} = 2\sum_{i \in K_{miss}} \left(E\left[a_i\right] - E\left[x_i\right]\right)$$

Because a_i is a constant, we have $E[a_i] = a_i$, $\frac{d}{da_i}E_{MMSE} = 0$ and $E[x_i] = \mu$, we have

$$2\sum_{i \in K_{miss}} (a_i - \mu) = 0$$

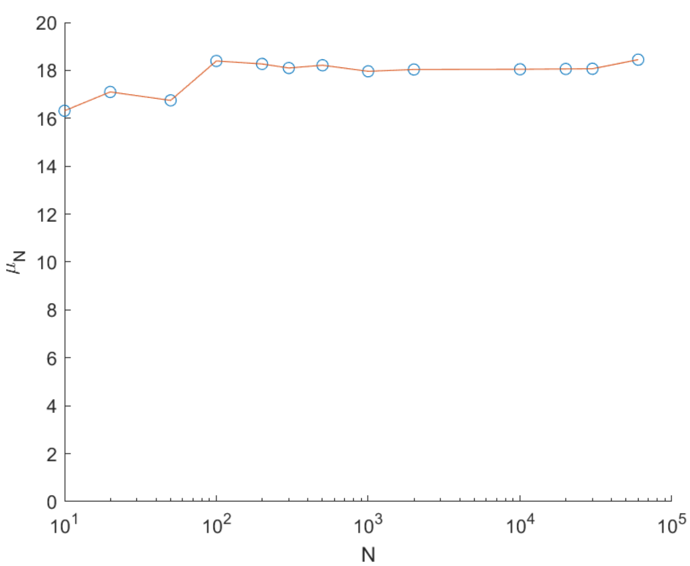
therefore,

$$a_i = \mu$$

(b)

N	10	20	50	100	200	300	500
$\hat{\mu}_N$	16.314	17.099	16.747	18.392	18.269	18.099	18.214

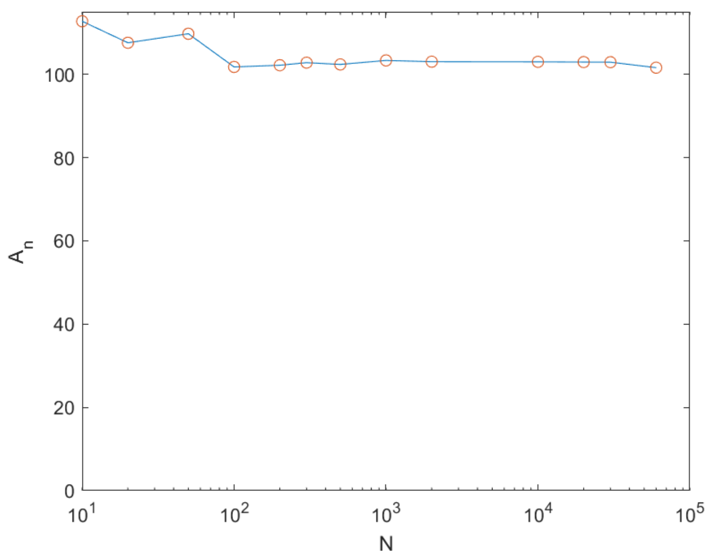
N	1000	2000	10000	20000	30000	60000	-
$\hat{\mu}_N$	17.961	18.037	18.046	18.060	18.067	18.447	-



The empirical estimate of $\hat{\mu}_N$ is consistent across all N's, as the average varies between 16-19. Therefore, we can estimate the average of all N to be around 18.

(c)

N	10	20	50	100	200	300	500
$\hat{A}_N$	112.721	107.561	109.72	101.753	102.164	102.779	102.356
N	1000	2000	10000	20000	30000	60000	-
$\hat{A}_N$	103.318	103.016	102.980	102.928	102.898	101.580	-



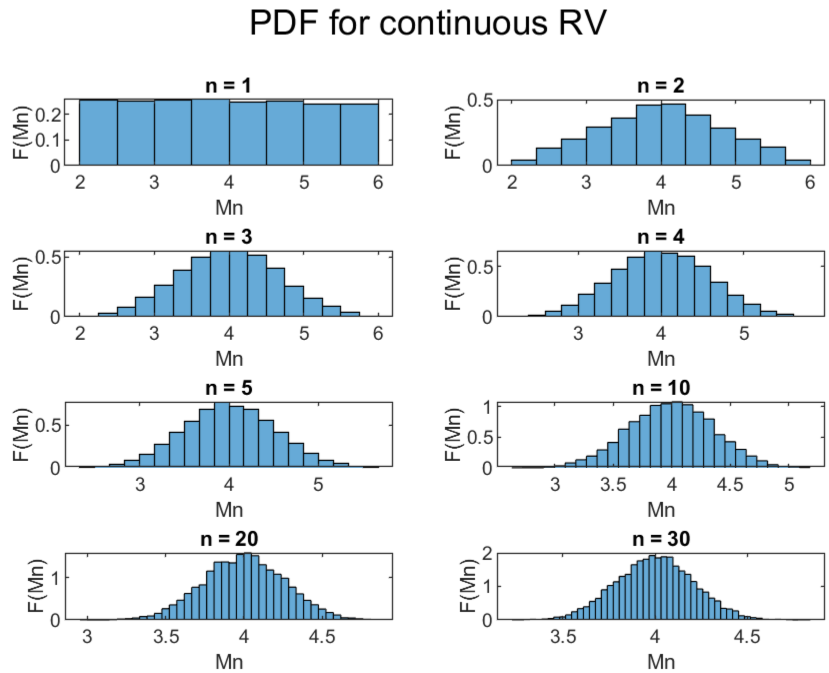
(d) As N increases, $\hat{A}_N$ approaches around 100. $\hat{A}_N$ does not approaches because as N becomes large, $\hat{A}_N = \sum_{i \in K_{avail}} (x_i - \hat{\mu}_N)^2 / |K_{avail}|$ approaches the variance of x_i which is not necessary 0.

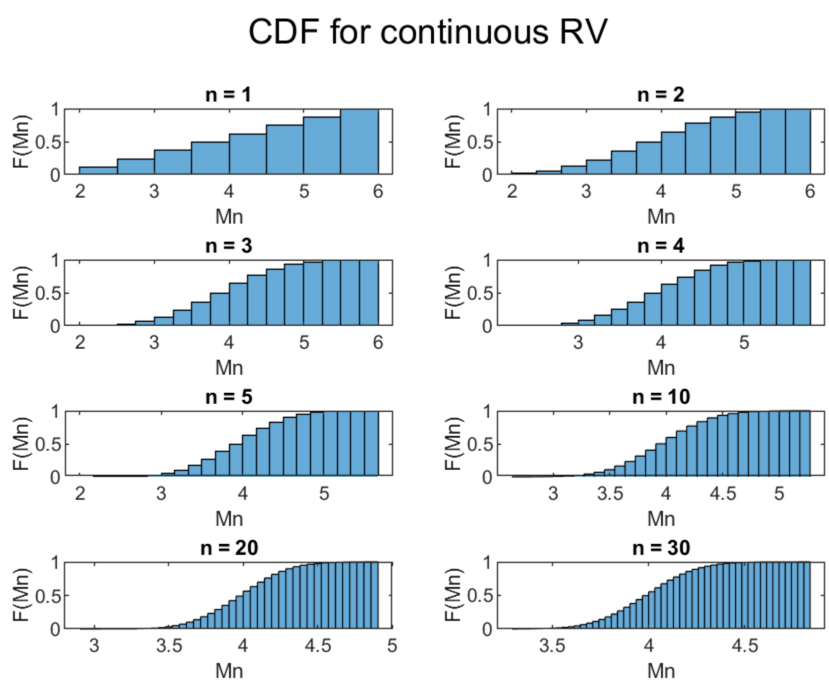
(e) As N becomes large, $\hat{A}_N \to 100$. Therefore, we can estimate $\sigma^2 \approx 100$.

2. Central Limit Theorem

In this Problem, the blue histogram represents data from generated RVs and orange histogram represents fitted gaussian RV.

(a)





(b) Because X_i's are i.i.d uniform RVs, we have

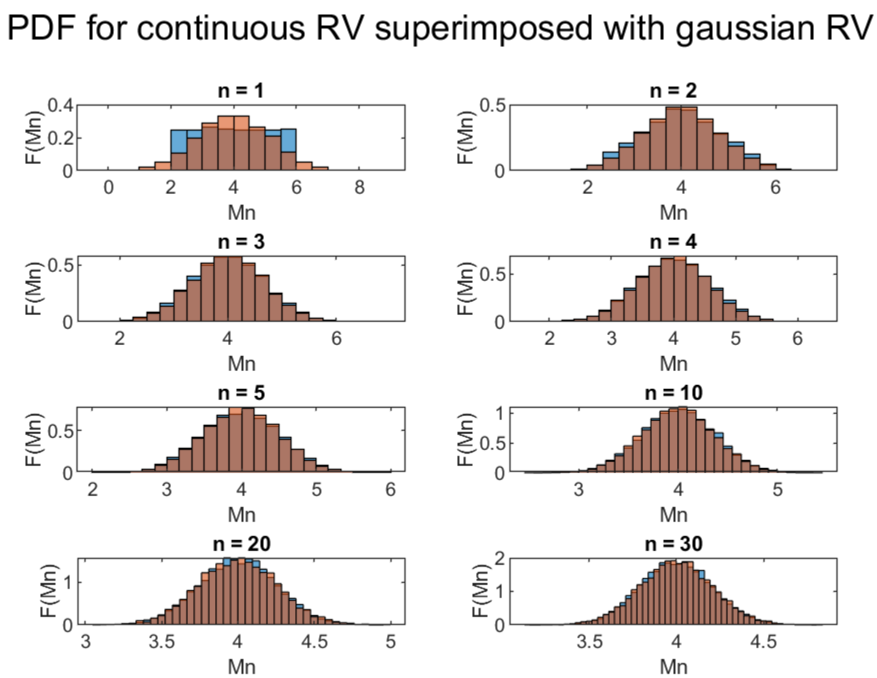
$$\mu = E\left[X_i\right] = \frac{2+6}{2} = 4$$

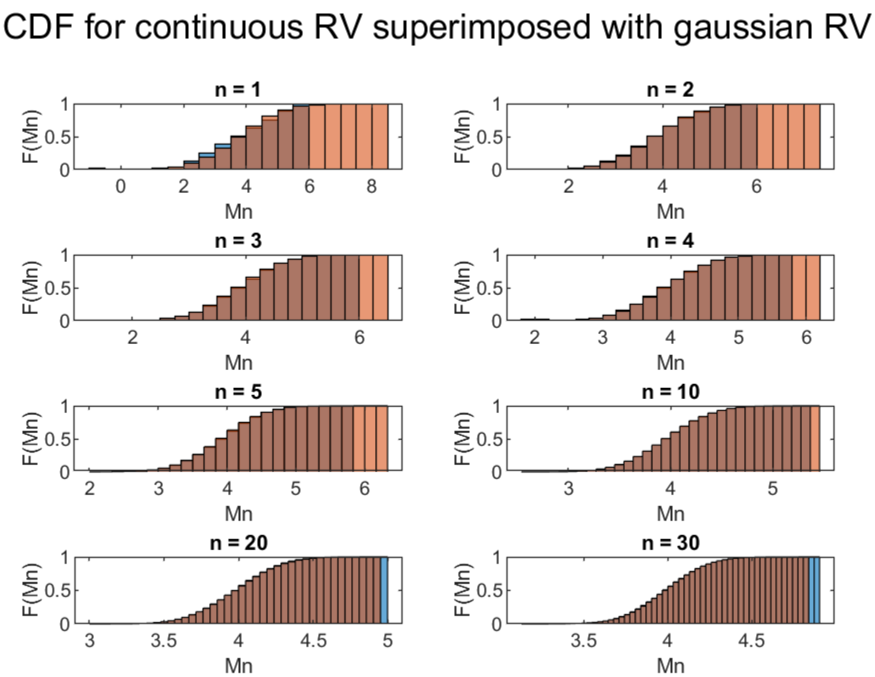
$$\sigma^2 = VAR(X_i) = \frac{(6-2)^2}{12} = \frac{4}{3}$$

$$E\left[M_n\right] = \frac{nE\left[X_i\right]}{n} = 4$$

$$VAR(M_n) = VAR\left(\frac{X_1 + X_2 + ... + X_n}{n}\right) = \frac{n\sigma^2}{n^2} = \frac{4}{3n}$$

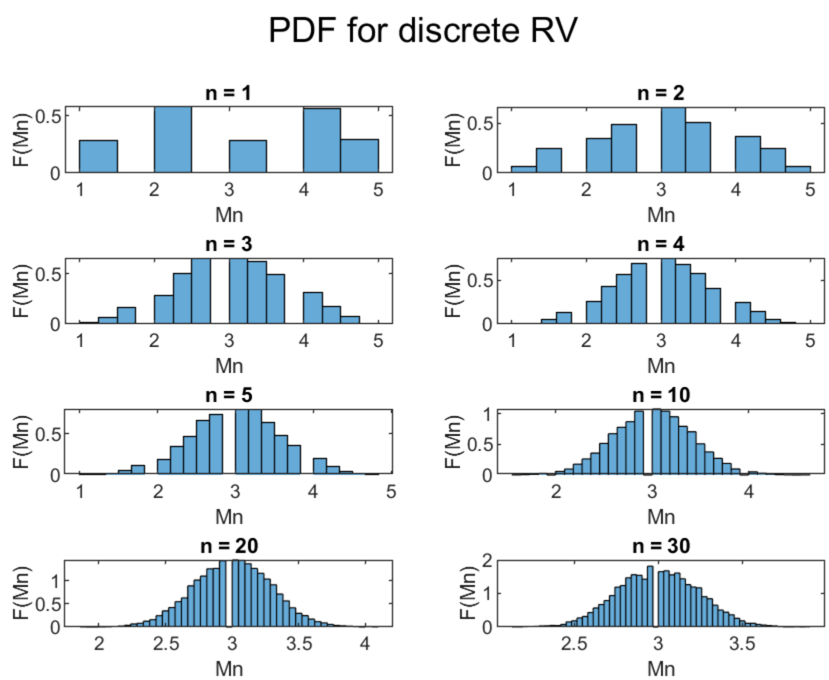
(c)

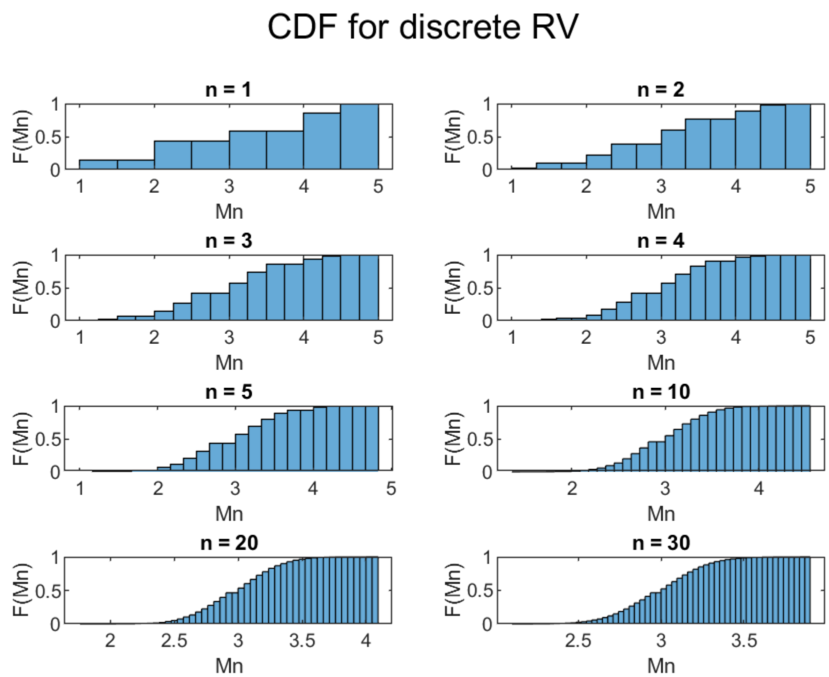


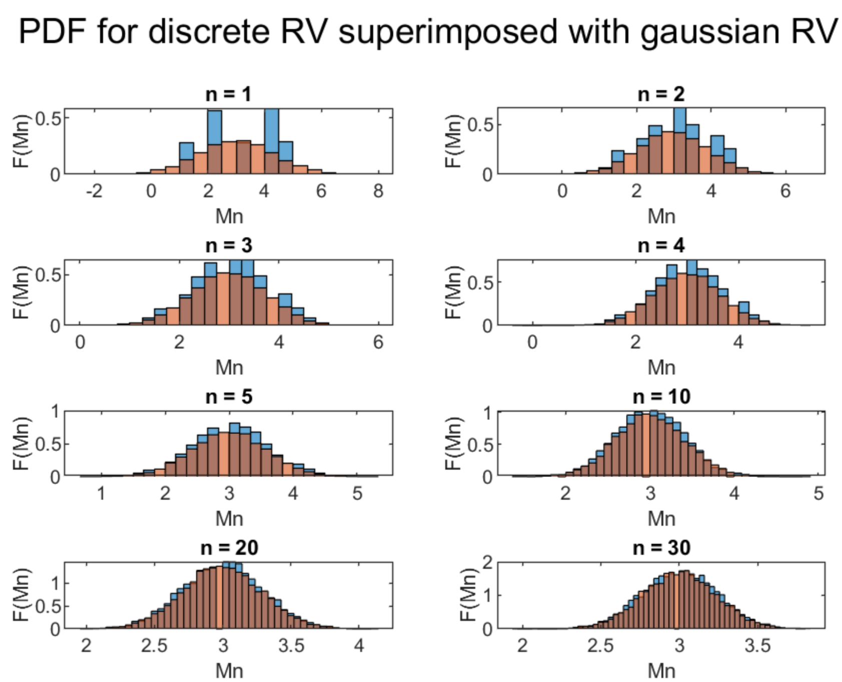


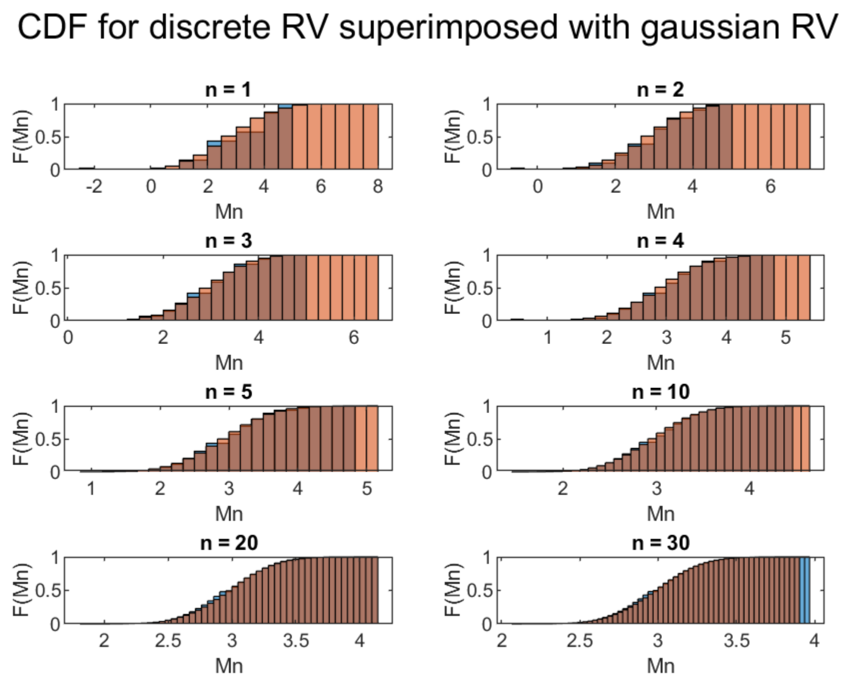
(d)

$$E\left[X_i\right] = 3$$
$$VAR\left(X_i\right) = \frac{12}{7n}$$




CDF for discrete RV
n = 1
n = 2
n = 3
n = 4
n = 5
n = 10
n = 20
n = 30
F(Mn)
Mn


PDF for discrete RV superimposed with gaussian RV
n = 1
n = 2
n = 3
n = 4
n = 5
n = 10
n = 20
n = 30
F(Mn)
Mn


CDF for discrete RV superimposed with gaussian RV
n = 1
n = 2
n = 3
n = 4
n = 5
n = 10
n = 20
n = 30
F(Mn)
Mn

3. GDA: Gaussian Discriminant Analysis

(a) Using Bayes' Rule, we have

$$P(y=0|\vec{x}) = \frac{P(\vec{x}|y=0)P(y=0)}{P(\vec{x})} = \frac{P(\vec{x}|y=0)}{2P(\vec{x})}$$

$$P(y=0|\vec{x}) = \frac{P(\vec{x}|y=1)P(y=0)}{P(\vec{x})} = \frac{P(\vec{x}|y=1)}{2P(\vec{x})}$$

Therefore, we can simplify the inequality as

$$P(\vec{x}|y=0) \geq P(\vec{x}|y=1) \tag{3.1}$$

Because

$$f_x = \frac{1}{(2\pi)|\Sigma|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right] \tag{3.2}$$

we get

$$\frac{1}{(2\pi)|\Sigma|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(x-\mu_0)^\top \Sigma^{-1}(x-\mu_0)\right] \geq \frac{1}{(2\pi)|\Sigma|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(x-\mu_1)^\top \Sigma^{-1}(x-\mu_1)\right]$$

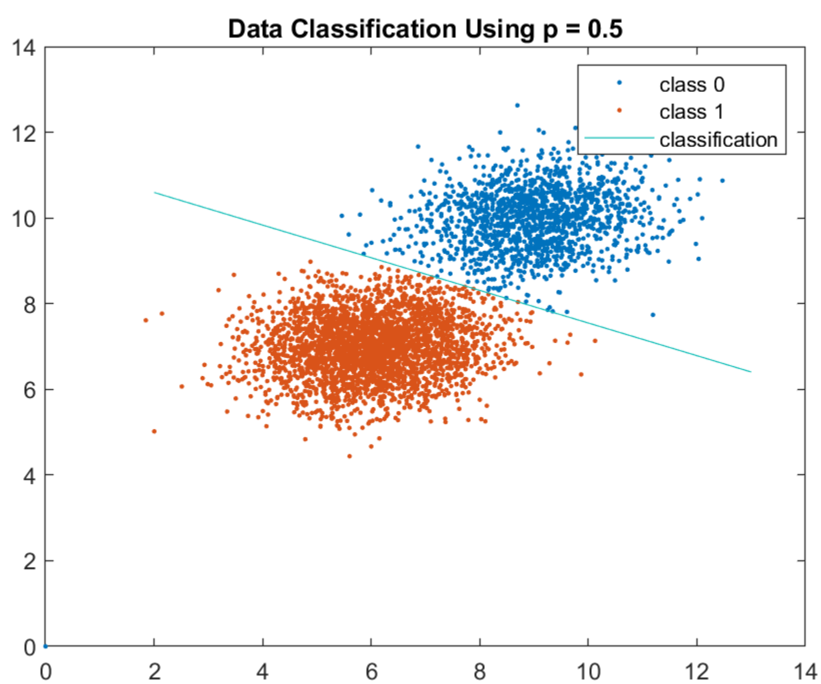
$$-\frac{1}{2}(x-\mu_0)^\top \Sigma^{-1}(x-\mu_0) \geq -\frac{1}{2}(x-\mu_1)^\top \Sigma^{-1}(x-\mu_1)$$

$$(x-\mu_0)^\top \Sigma^{-1}(x-\mu_0) \leq (x-\mu_1)^\top \Sigma^{-1}(x-\mu_1)$$

Because $\Sigma_0 = \Sigma_1 = \Sigma$, $\vec{x}^\top \Sigma_0^{-1}\vec{x} = \vec{x}^\top \Sigma_1^{-1}\vec{x}$. We get

$$\left(\left(\vec{\mu}_0 - \vec{\mu}_1\right)\Sigma^{-1}\right) \cdot \vec{x} + \frac{1}{2}\left(\vec{\mu}_1^\top \Sigma^{-1}\vec{\mu}_1 - \vec{\mu}_0^\top \Sigma^{-1}\vec{\mu}_0\right) \geq 0 \tag{3.3}$$

(b) The precentage of samples identified to be from class 0 is 49.83%.



(c) Using Bayes' Rule, we have

$$P(y=0|\vec{x}) = \frac{P(\vec{x}|y=0)P(y=0)}{P(\vec{x})} = \frac{P(\vec{x}|y=0)(1-p)}{P(\vec{x})}$$

$$P(y=0|\vec{x}) = \frac{P(\vec{x}|y=1)P(y=0)}{P(\vec{x})} = \frac{P(\vec{x}|y=1)\cdot p}{P(\vec{x})}$$

Therefore, the inequality simplifies to

$$P(\vec{x}|y=0)(1-p) \geq P(\vec{x}|y=1)\cdot p \tag{3.4}$$

use Eqn. 3.2, we have

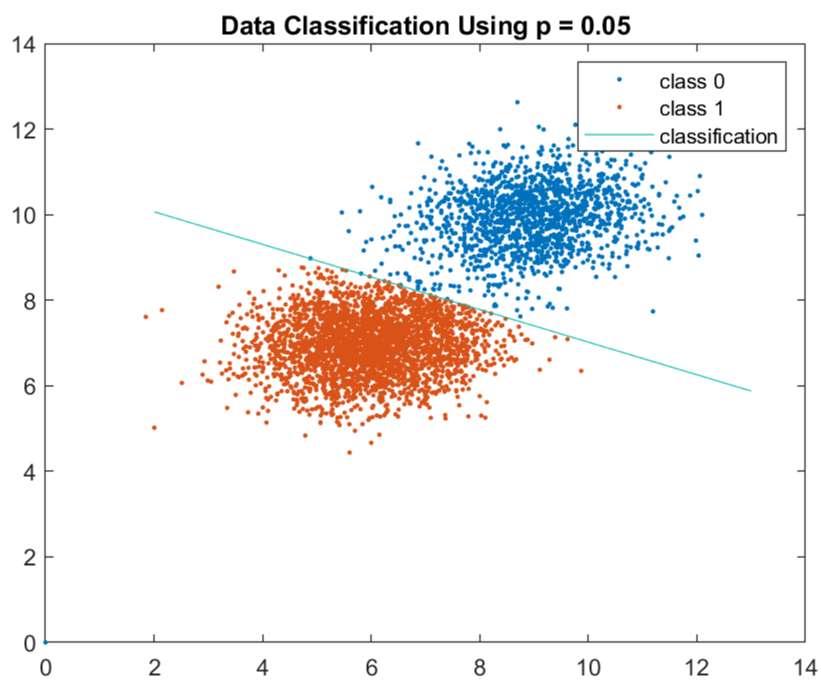
$$\frac{1}{(2\pi)|\Sigma|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}(x-\mu_0)^\top\Sigma^{-1}(x-\mu_0)\right]\cdot p \geq \frac{1}{(2\pi)|\Sigma|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}(x-\mu_1)^\top\Sigma^{-1}(x-\mu_1)\right](1-p)$$

$$\ln p + \left[-\frac{1}{2}(x-\mu_0)^\top{\Sigma_0}^{-1}(x-\mu_0)\right] \geq \ln(1-p) + \left[-\frac{1}{2}(x-\mu_1)^\top{\Sigma_1}^{-1}(x-\mu_1)\right]$$

Using $\vec{x}^\top\Sigma_0^{-1}\vec{x} = \vec{x}^\top\Sigma_1^{-1}\vec{x}$,

$$\left((\vec{\mu}_0-\vec{\mu}_1)\,\Sigma^{-1}\right)\cdot\vec{x} + \frac{1}{2}\left(\vec{\mu}_1^\top\Sigma^{-1}\vec{\mu}_1 - \vec{\mu}_0^\top\Sigma^{-1}\vec{\mu}_0\right) + \ln\left(\frac{1-p}{p}\right) \geq 0 \tag{3.5}$$

(d) The precentage of samples identified to be from class 0 is 51.38%.



The changed p moved the classfination line slightly down in the graph.

(e) From Eqn. 3.2 and 3.4, we get

$$\frac{1}{(2\pi)|\Sigma_0|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(x-\mu_0)^\top {\Sigma_0}^{-1}(x-\mu_0)\right] \cdot (1-p) \geq \frac{1}{(2\pi)|\Sigma_1|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(x-\mu_1)^\top {\Sigma_1}^{-1}(x-\mu_1)\right] p$$

$$\ln\left(\frac{1-p}{|\Sigma_0|^{\frac{1}{2}}}\right) + \left[-\frac{1}{2}(x-\mu_0)^\top {\Sigma_0}^{-1}(x-\mu_0)\right] \geq \ln\left(\frac{p}{|\Sigma_1|^{\frac{1}{2}}}\right) + \left[-\frac{1}{2}(x-\mu_1)^\top {\Sigma_1}^{-1}(x-\mu_1)\right]$$

Expand the equation, we get

$$\begin{aligned} &-\frac{1}{2}\ln\left(\frac{|\Sigma_0|^{\frac{1}{2}}}{1-p}\right) - \frac{1}{2}x^\top \Sigma_0^{-1} x - \frac{1}{2}\mu_0^\top \Sigma_0^{-1}\mu_0 + \mu_0^\top \Sigma_0^{-1} x + \ln(1-p) \\ =&-\frac{1}{2}\ln\left(\frac{|\Sigma_1|^{\frac{1}{2}}}{p}\right) - \frac{1}{2}x^\top \Sigma_1^{-1} x - \frac{1}{2}\mu_1^\top \Sigma_1^{-1}\mu_1 + \mu_1^\top \Sigma_1^{-1} x + \ln(p) \end{aligned}$$

which simplifies to

$$\vec{x}^\top \left(\frac{1}{2}\left(\Sigma_0^{-1} - \Sigma_1^{-1}\right)\right)\vec{x} + \left(\mu_1^\top \Sigma_1^{-1} - \mu_0^\top \Sigma_0^{-1}\right)\cdot\vec{x} + \frac{1}{2}\left(\mu_0^\top \Sigma_0^{-1}\mu_0 - \mu_1^\top \Sigma_1^{-1}\mu_1\right) + \ln\left(\frac{(1-p)|\Sigma_1|^{\frac{1}{2}}}{p\,|\Sigma_0|^{\frac{1}{2}}}\right) \geq 0 \quad (3.6)$$

(f) The precentage of samples identified to be from class 0 is 50.41%