Reading Report

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I. B-SPLINES & NURBS

Dear Deepesh. I have been checking the code of Hendrik S. and mine. Until I understood is like follow, when we want the ECT-space on $[x_{i-1}, x_i]$ and I use the linear bijection $\phi\colon [x_{i-1}, x_i] \to [0, 1]$ to compute the local Bernstein funcionts $B_{j,p}(\phi(x)), x \in [x_{i-1}, x_i]$, we have to change the parameter w by $w(x_i - x_{i-1})$. I compare the results of my code and Hendrik's code and now I get the same as him I show you some cases:

Given a basic interval $I := [x_1, x_2] \subset \mathbb{R}$, we denote with $\boldsymbol{\xi}$ and **open knot** vector of degree $p \in \mathbb{N}$ and length $n+p+1 \in \mathbb{N}$, i.e.,

$$\begin{aligned} \boldsymbol{\xi} &:= [\xi_1, \xi_2, \dots, \xi_{n+p+1}], \quad \xi_{i+1} \geq \xi_i, \\ \xi_1 &= \cdots = \xi_{p+1} = x_1 < \xi_{p+2}, \\ \xi_{n+1} &= \cdots = \xi_{n+p+1} = x_2 > \xi_n. \end{aligned} \tag{1}$$

The number of times m_i that ξ_i appears in $\boldsymbol{\xi}$ is called its **multiplicity**. And we assume that $1 \leq m_i \leq p-1$. The ***B*-splines** $\{b_{j,p} : j = 1, \dots, n\}$ are defined by recursively starting with the zeroth-order basis function ($p = 0$) as follows:

$$b_{j,0}(x) := \begin{cases} 1, & \text{if } x \in [\xi_j, \xi_{j+1}), \\ 0, & \text{otherwise;} \end{cases} \tag{2}$$

and for $p \geq 1$:

$$b_{j,p}(x) = \frac{x - \xi_j}{\xi_{j+p} - \xi_j} b_{j,p-1}(x) + \frac{\xi_{j+p+1} - x}{\xi_{j+p+1} - \xi_{j+1}} b_{j+1,p-1}(x), \tag{3}$$

the entire fraction will be defined like zero if the denominator is zero. We assume left continuity in the basis of the B-splines.

Example 1. A B-spline of degree 1 is also called a linear B-spline or a hat function. The recurrence relation 2 takes the form

$$b_{j,1}(x) = \frac{x - \xi_j}{\xi_{j+1} - \xi_j} b_{j,0}(x) + \frac{\xi_{j+2} - x}{\xi_{j+2} - \xi_{j+1}} b_{j+1,0}(x), \tag{4}$$

resulting in

$$b_{j,1}(x) := \begin{cases} \dfrac{x - \xi_j}{\xi_{j+1} - \xi_j}, & \text{if } x \in [\xi_j, \xi_{j+1}), \\ \dfrac{\xi_{j+2} - x}{\xi_{j+2} - \xi_{j+1}}, & \text{if } x \in [\xi_{j+1}, \xi_{j+2}), \\ 0, & \text{otherwise;} \end{cases} \tag{5}$$

Example 2. A B-spline of degree 2 is also called a quadratic B-spline. Using the relation 2, the three pieces of the quadratic B-spline $B_{j,2,\xi}$ are given by

$$b_{j,2}(x) := \begin{cases} \dfrac{(x - \xi_j)^2}{(\xi_{j+2} - \xi_j)(\xi_{j+1} - \xi_j)}, & \text{if } x \in [\xi_j, \xi_{j+1}), \\ \dfrac{(x - \xi_j)(\xi_{j+2} - x)}{(\xi_{j+2} - \xi_j)(\xi_{j+2} - \xi_{j+1})} \\ \quad + \dfrac{(x - \xi_{j+1})(\xi_{j+3} - x)}{(\xi_{j+2} - \xi_{j+1})(\xi_{j+3} - \xi_{j+1})}, & \text{if } x \in [\xi_{j+1}, \xi_{j+2}), \\ 0 & \text{otherwise;} \end{cases} \tag{6}$$

Example 3. Figures (1), (2) and (3) illustrate several sets of B-splines of degree $p = 1, 2, 3$. The same knot sequence is chosen for the different degrees, with only simple knots.

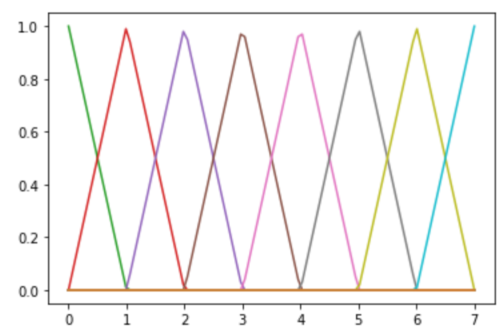

Fig. 1. B-spline of degree $p = 1$ and $\boldsymbol{\xi} = [0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 7, 7, 7]$

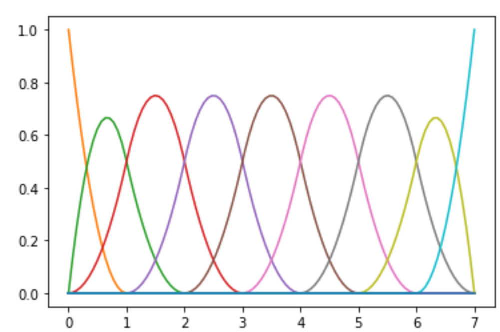

Fig. 2. B-spline of degree $p = 2$ and $\boldsymbol{\xi} = [0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 7, 7, 7]$

We deduce the the following properties of a B-spline.

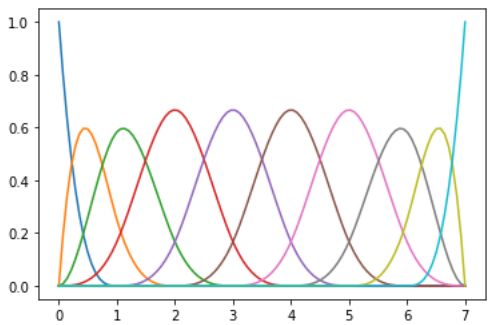

Fig. 3. B-spline of degree $p = 3$ and $\boldsymbol{\xi} = [0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 7, 7, 7]$

- **Local Support.** A B-spline is locally supported on the interval given by the extreme knots used in its definition.
$$b_{j,p}(x) = 0, \quad x \notin [\xi_j, \xi_{j+p+1}) \,. \tag{7}$$
- **Nonnegativity.** A B-spline is nonnegative everywhere, and positive inside its support, i.e.,
$$b_{j,p}(x) \geq 0, \quad x \in \mathbb{R} \text{ and } b_{j,p}(x) > 0 \quad x \in [\xi_j, \xi_{j+p+1}) \,. \tag{8}$$
- **Translation and Scaling Invariance.** A B-spline is invariant under a translation and/or scaling transformation of its knots sequence, i.e.,
$$\hat{b}_{j,p}(\alpha x + \beta) = b_{j,p}(x), \quad \alpha, \beta \in \mathbb{R}, \quad \alpha \neq 0, \tag{9}$$
where $\hat{b}_{j,p}$ is the B-spline obtained using the knot vector $\alpha\boldsymbol{\xi} + \beta := \{\alpha\xi_i + \beta\}$.

Given a $\boldsymbol{\xi}$, and let be $b_{j,p}$ their corresponding set of B-splines of degree p, there corresponds a polynomial $psi_{j,p,\boldsymbol{\xi}}$ of degree p with roots at the interior knots of the B-spline. We define $\psi_{j,0,\boldsymbol{\xi}} := 1$, and
$$\psi_{j,p,\boldsymbol{\xi}} := (y - \xi_{j+1}) \cdots (y - \xi_{j+p}), \quad y \in \mathbb{R}, \quad p \in \mathbb{N}. \tag{10}$$
This polynomial is called **dual polynomial**. Using the recurrence relation of the dual polynomial is possible to prove several properties of the B-splines.

Theorem 1. For $p \in \mathbb{N}$, $x, y \in \mathbb{R}$ and $\xi_{j+p} > \xi_j$, we have the **dual recurrence relation**
$$(x-y)\psi_{j,p-1,\boldsymbol{\xi}}(y) = \frac{x - \xi_j}{\xi_{j+p} - \xi_j}\psi_{j,p,\boldsymbol{\xi}}(y) + \frac{\xi_{j+p} - x}{\xi_{j+p} - \xi_j}\psi_{j-1,p,\boldsymbol{\xi}}(y), \tag{11}$$
and the **the dual difference formula**
$$\psi_{j,p-1,\boldsymbol{\xi}}(y) = \frac{\psi_{j-1,p,\boldsymbol{\xi}}(y)}{\xi_{j+p} - \xi_j} - \frac{\psi_{j,p,\boldsymbol{\xi}}(y)}{\xi_{j+p} - \xi_j} \tag{12}$$

Proof. With $y \in \mathbb{R}$ fixed, let be $l_y \colon \mathbb{R} \to \mathbb{R}$ given by $l_y := y - x$. Using linear interpolation, we have
$$l_y(x) = \frac{x - \xi_j}{\xi_{j+p} - \xi_j} l_y(\xi_{j+p}) + \frac{\xi_{j+p} - \xi}{\xi_{j+p} - \xi_j} l_y(\xi_j). \tag{13}$$
By multiplying both sides with $\psi_{j,p,\boldsymbol{\xi}}(y)$ we get (55). And (12) follows from (55) by differentiating with respect to x. $\square$

Theorem 2. (Local Marsden Identity) For $j \leq m \leq j + p$ and $\xi_m < \xi_{m+1}$, we have
$$(y-p)^p = \sum_{i=m-p}^{m} \psi_{i,p,\boldsymbol{\xi}}(y) b_{i,p}(x), \quad x \in [\xi_m, \xi_{m+1}) \,, \quad y \in \mathbb{R}. \tag{14}$$
If $b_{i,p}^{\{m\}}$ is the polynomial which is equal to $b_{i,p}(x)$ for $x \in [\xi_m, \xi_{m+1})$ then
$$(y - p)^p = \sum_{i=m-p}^{m} \psi_{i,p,\boldsymbol{\xi}}(y) b_{i,p}^{m}(x), \quad x, y \in \mathbb{R}. \tag{15}$$

Proof. Suppose $x \in [\xi_m, \xi_{m+1})$. We prove (14) by induction, let $p = 0$.
$$\begin{aligned}(y - p)^0 &= 1 = \psi_{i,0,\xi}(y) b_{i,0}^{m}(x), \quad x \in [\xi_m, \xi_{m+1}) \\ &= (1) b_{i,0}^{m}(x), \quad x \in [\xi_m, \xi_{m+1}) \\ &= (1)(1) = 1.\end{aligned} \tag{16}$$
Now, we assume that it holds for degree $p - 1$. Then by (2) and (55) we obtain:
$$\begin{aligned}&(y - x)^p = (y - x)(x - x)^{p-1} = (y - x) \sum_{i=m-p}^{m} \psi_{i,p,\boldsymbol{\xi}}(y) b_{i,p}(x) \\ &= \sum_{i=m-p+1}^{m} \left(\frac{x - \xi_i}{\xi_{i+p} - \xi_i}\psi_{i,p,\boldsymbol{\xi}}(y) + \frac{\xi_{i+p} - x}{\xi_{i+p} - \xi_i}\psi_{i-1,p,\boldsymbol{\xi}}(y) \right) \\ &\times b_{i,p-1}(x) \\ = &\left(\frac{x - \xi_{m-p+1}}{\xi_{m+1} - \xi_{m-p+1}} \right) b_{m-p+1,p-1}(x)\psi_{m-p+1,p,\boldsymbol{\xi}}(y) \\ &+ \left(\frac{\xi_{m+1} - x}{\xi_{m+1} - \xi_{m-p+1}} \right) b_{m-p+1,p-1}(x)\psi_{m-p,p,\boldsymbol{\xi}}(y) \\ &+ \left(\frac{x - \xi_{m-p+2}}{\xi_{m+2} - \xi_{m-p+2}} \right) b_{m-p+2,p-1}(x)\psi_{m-p+2,p,\boldsymbol{\xi}}(y) \\ &+ \left(\frac{\xi_{m+2} - x}{\xi_{m+2} - \xi_{m-p+2}} \right) b_{m-p+2,p-1}(x)\psi_{m-p+1,p,\boldsymbol{\xi}}(y) + \cdots + \\ &+ \left(\frac{x - \xi_m}{\xi_{m+p} - \xi_m} \right) b_{m,p-1}(x)\psi_{m,p,\boldsymbol{\xi}}(y) \\ &+ \left(\frac{\xi_{m+p} - x}{\xi_{m+p} - \xi_m} \right) b_{m,p-1}(x)\psi_{m-1,p,\boldsymbol{\xi}}(y),\end{aligned} \tag{17}$$
using that $\frac{x-\xi_i}{\xi_{i+p}-\xi_i} b_{i,p-1}(x) = 0$ for $i = m - p, m + 1$, and associating the first and fourth elements of the previous sum and similarly rearranging all the other terms we have that (78) is equal to the following:
$$\begin{aligned}&= \sum_{i=m-p}^{m} \left(\frac{x - \xi_i}{\xi_{i+p} - \xi_i} b_{i,p-1}(x) + \frac{\xi_{i+p+1} - x}{\xi_{i+p+1} - \xi_{i+1}} b_{i+1,p-1}(x) \right) \\ &\times \psi_{i,p,\boldsymbol{\xi}}(y) \\ = &\sum_{i=m-p}^{m} \psi_{i,p,\boldsymbol{\xi}}(y) b_{i,p}(x).\end{aligned} \tag{18}$$
$\square$

If we now suppose that $\xi_m < \xi_{m+1}$ for some $j \leq m \leq j+p$, then we obtain the following.

Theorem 3. Local Representation of Monomials We have for $p \geq k$,

$$x^k = \sum_{i=m-p}^{m} \left((-1)^k \frac{k\,!}{p\,!} D^{p-k} \psi_{i,p,\boldsymbol{\xi}}(x) \right), \quad x \in [\xi_m, \xi_{m+1})\,. \tag{19}$$

Proof. Fix $x \in [\xi_m, \xi_{m+1})$. Differentiating $p-k$ times in (16) with respect to y in the left hand we get:

$$\frac{p\,!}{k\,!}(y-x)^k, \tag{20}$$

while in the left hand we have:

$$\sum_{i=m-p}^{m} D^{p-k} \psi_{i,p,\boldsymbol{\xi}}(y) b_{i,p}^m(x), \quad y \in \mathbb{R}, k = 0, \dots, p. \tag{21}$$

Joining (80) and (21):

$$\frac{(y-x)^k}{k\,!} = \sum_{i=m-p}^{m} \left(\frac{1}{p\,!} D^{p-k} \psi_{i,p,\boldsymbol{\xi}}(y) \right) b_{i,p}^m(x), \quad y \in \mathbb{R}, k = 0, \dots, p. \tag{22}$$

Setting $y = 0$ in both sides, we get (79). □

Theorem 4. Local Partition of Unity

$$\sum_{i=m-p}^{m} B_{i,p,\boldsymbol{\xi}}(x) = 1, \quad x \in [\xi_m, \xi_{m+1})\,. \tag{23}$$

Proof. Take $k = 0$ in (22) □

Theorem 5. Local Linear Independence The two sets $\{b_{i,p}\}_{i=m-p}^m$ and $\{\psi_{i,p,\boldsymbol{\xi}}\}_{i=m-p}^m$ form both a basis for the polynomial space $\mathbb{P}_p$ on any subset of $[\xi_m, \xi_{m+1})$ containing at least $p+1$ distinct points.

Proof. Let A be a subset of $[\psi_m, \psi_{m+1})$ containing at least $p+1$ distinct points. From eq (22) on A every polynomial of degree at most p can be written as linear combination of the $p+1$ polynomials $b_{j,p}^w$, $i = m-p, \dots, m$. Since the dimension of the space $\mathbb{P}_p$ on A is $p+1$, these polynomials must be linearly independent and a basis. □

Let us denote with $\boldsymbol{w}$ a **weight vector** of length n, i.e.,

$$\boldsymbol{w} := [w_1, w_2, \dots, w_n]\,, \quad w_i > 0. \tag{24}$$

We say that its corresponding set of NURBS $\left\{ b_{j,p}^w : 1, \dots, n \right\}$ is defined as follows:

$$b_{j,p}^w(x) := \frac{w_j b_{j,p}(x)}{\sum_{i=1}^n w_i b_{i,p}(x)}. \tag{25}$$

Each NURBS has the following properties:

- $b_{j,p}^w(x) \geq 0, \quad x \in I.$
- $b_{j,p}^w(x) = 0, \quad x \notin [\xi_j, \xi_{j+p+1}]\,.$

Moreover,

- The functions $b_{j,p}^w$ are linearly independent and form a partition of the unity, the proof follows from the Theorem (5).

They satisfy the following end-point conditions:

$$\begin{aligned} &b_{1,p}^w(x_1) = 1, \quad b_{j,p}^w(x_1) = 0, \quad j = 2, \dots, n, \\ &b_{n,p}^w(x_2) = 1, \quad b_{j,p}^w(x_2) = 0, \quad j = 1, \dots, n-1. \end{aligned} \tag{26}$$

According with the notation in [1], the NURBS space corresponding to ψ and $\boldsymbol{w}$ is denoted with $\mathcal{R}\left[\boldsymbol{\xi}, \boldsymbol{w}\right]$ and is defined as the span of $\left\{ b_{j,p}^{\psi} : j = 1, \dots, n-1 \right\}$. This is a space of piecewise-rational functions of degree p with smoothness C^{p-m_i} at knot ξ_i and its dimension is n. We identify a function $f \in \mathcal{R}\left[\boldsymbol{\xi}, \boldsymbol{w}\right]$ with the vector of its coefficients $[f_1, f_2, \dots, f_n]$.

II. Rational multi-degree B-splines

Now, contemplate m open knot vectors $\boldsymbol{\xi}^{(i)}$ of degree $p^{(i)}$, $i = 1, \dots, m$, defined like in eq (1). We denote the left and right end points of the interval $I^{(i)}$ associated to $\boldsymbol{\xi}^{(i)}$ with $x_1^{(i)}$ adn $x_2^{(i)}$, respectively. The collection $\boldsymbol{\Xi} := (\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(m)}$ is called an m-segment knot vector configuration. The multi-degree spline spaces will be constructed by considering spline spaces over the knot vector $\boldsymbol{xi}$, which are glued together with certain smoothness requirements at the end points $x_2^{(i)}$ and $x_1^{(i)}$ for $i \in \{1, 2\dots, m-1\}$. The equivalence class at the points is called the *ith segment join*. We define the mapping $\phi^{(i)}$ for each segment $i = 1, \dots, m$,

$$\phi^{(i)} := x - x_1^{(i)} + \tau^{(1)} + \sum_{l=1}^{i-1} (x_2^{(l)} - x_1^{(l)}), \tag{27}$$

with $\tau_1^{(1)} \in \mathbb{R}$ an arbitrarily chosen origin. Then $\Omega^{(i)} := \left[\tau_1^{(i)}, \tau_2^{(i)}\right] = \phi^{(i)} \left(\left[x_1^{(i)}, x_2^{(i)} \right] \right) \subset \mathbb{R}$. And the composed interval is defined like

$$\Omega := [t_1, t_2] := \Omega^{(1)} \cup \dots \cup \Omega^{(m)}. \tag{28}$$

Observe that $\tau_2^{(i)} = \tau_1^{(i+1)}$, $i \in \{1, \dots, m-1\}$ and $\tau_1^{(1)} = t_1$. In addition, let $\boldsymbol{W} := (\boldsymbol{w}^{(1)}, \dots, \boldsymbol{w}^{(m)})$ be a sequence of weights vectors defined in (24). The space of rational multi-degree splines is defined as

$$\mathcal{R}\left[\boldsymbol{\Xi}, \boldsymbol{W}\right] := \{ f \in C^1(\Omega) : f \circ \phi^{(i)} \in \mathcal{R}\left[\boldsymbol{\xi}^{(i)}, \boldsymbol{w}^{(i)} \right], \quad 1 \leq i \leq m \}, \tag{29}$$

and the periodic space of rational multi-degree splines as

$$\mathcal{R}^{\text{per}}\left[\boldsymbol{\Xi}, \boldsymbol{W}\right] := \{ f \in \mathcal{R}\left[\boldsymbol{\Xi}, \boldsymbol{W}\right] : f(t_1) = f(t_2), \quad \frac{df}{dt}(t_1) = \frac{df}{dt}(t_2) \}. \tag{30}$$

The elements of $\mathcal{R}\left[\boldsymbol{\Xi}, \boldsymbol{W}\right]$ and $\mathcal{R}^{\text{per}}\left[\boldsymbol{\Xi}, \boldsymbol{W}\right]$ are piecewise-NURBS functions such that the pieces meet with C^1 continuity at each segment join. We build basis for the spaces $\mathcal{R}\left[\boldsymbol{\Xi}, \boldsymbol{W}\right]$ and $\mathcal{R}^{\text{per}}\left[\boldsymbol{\Xi}, \boldsymbol{W}\right]$ as follows. On the ith knot vector $\boldsymbol{xi}^{(i)}$, we have $n^{(i)}$ NURBS $b_{j,p^{(i)}}^{w,(i)}$ of degree $p^(i)$ that span the spline space $\mathcal{R}\left[\boldsymbol{\xi}^{(i)}, \boldsymbol{w}^{(i)}\right]$. First, we map these basis functions from

$I^{(i)}$ to $\Omega^{(i)}$ using $\phi^{(i)}$ in eq (27), and extend them on the entire on the entire interval Ω by defining them to be zero outside $\Omega^{(i)}$, i.e., defining the cumulative local dimension μ_i for $i = 0, \ldots, m$,

$$\mu_0 := 0, \quad \mu_i := \sum_{l=1}^{i} n^{(l)} = \mu_{i-1} + n^{(i)}, \quad i > 0, \tag{31}$$

we define for $i = 1, \ldots, m$ and $j = 1, \ldots, n^{(i)}$,

$$b_{\mu_{i-1}+j}(t) := \begin{cases} b^{w,(i)}_{j,p^{(i)}}(x), & \text{if } \tau_1^{(i)} \le t = \phi^{(i)}(x) < \tau_2^{(i)}, \\ b^{w,(i)}_{j,p^{(i)}}(x_2^{(i)}), & \text{if } i = m \text{ and } t = t_2, \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

We construct extraction matrices $\boldsymbol{H}$ and $\boldsymbol{H}^{\text{per}}$ such that the functions in $\{B_i : i = 1, \ldots, n\}$ and $\left\{B_i^{\text{per}} : i = 1, \ldots, n^{\text{per}}\right\}$, defined by:

$$\boldsymbol{B} := \boldsymbol{H}\boldsymbol{b}, \quad \boldsymbol{B}^{\text{per}} := \boldsymbol{H}^{\text{per}}\boldsymbol{b}, \tag{33}$$

span $\mathcal{R}[\boldsymbol{\Xi}, \boldsymbol{W}]$ and $\mathcal{R}^{\text{per}}[\boldsymbol{\Xi}, \boldsymbol{W}]$, respectively. To buil these matrices, we define counters η_i for $i = 0, \ldots, m$,

$$\eta_0 := 0, \quad \eta_i := \sum_{l=1}^{i} (n^{(l)} - 2) = \mu_i - 2i, \quad i > 0, \tag{34}$$

and parameters $\alpha^{(i)}$ and $\beta^{(i)}$ for $i = 1, \ldots, m-1$,

$$\begin{aligned} \alpha^{(i)} &= \frac{p^{(i)}}{x_2^{(i)} - \xi^{(i)}_{n^{(i)}}} \frac{w^{(i)}_{n^{(i)}-1}}{w^{(i)}_{n^{(i)}}} > 0, \\ \beta^{(i)} &= \frac{p^{(i+1)}}{\xi^{(i+1)}_{p^{(i+1)}+2} - x_1^{(i+1)}} \frac{w_2^{(i+1)}}{w_1^{(i+1)}} > 0, \end{aligned} \tag{35}$$

In the periodic setting, $\alpha^{(m)}$ and $\beta^{(m)}$ are computed using the above equations identifying the index $i+1$ with 1. Then we define a common sparse matrix $\boldsymbol{H}^c$ of size $\eta_m \times (\mu_m - 2)$, whose non-zero entries H^c_{ij} are identified as follows: for $i = 1, \ldots, m$ and $j = 1, \ldots, n^{(i)} - 2$,

$$H^c_{\eta_{i-1}+j, \mu_{i-1}+j} := 1, \tag{36}$$

and for $i = 1, \ldots, m-1$,

$$\begin{aligned} H^c_{\eta_i+1, \mu_i-1} &:= H^c_{\eta_i, \mu_i} := \frac{\alpha^{(i)}}{\alpha^{(i)} + \beta^{(i)}}, \\ H^c_{\eta_i+1, \mu_i-1} &:= H^c_{\eta_i+1, \mu_i} := \frac{\beta^{(i)}}{\alpha^{(i)} + \beta^{(i)}}, \end{aligned} \tag{37}$$

The desire extraction matrices in eq (33) are specified as follows:

$$\boldsymbol{H} := \begin{bmatrix} 1 & \boldsymbol{0} & 0 \\ \boldsymbol{0} & \boldsymbol{H}^c & \boldsymbol{0} \\ 0 & \boldsymbol{0} & 1 \end{bmatrix}, \tag{38}$$

$$\boldsymbol{H}^{\text{per}} := \begin{bmatrix} \frac{\beta^{(m)}}{\alpha^{(m)}+\beta^{(m)}} & & \frac{\beta^{(m)}}{\alpha^{(m)}+\beta^{(m)}} \\ \boldsymbol{0} & \boldsymbol{H}^c & \boldsymbol{0} \\ \frac{\alpha^{(m)}}{\alpha^{(m)}+\beta^{(m)}} & & \frac{\alpha^{(m)}}{\alpha^{(m)}+\beta^{(m)}} \end{bmatrix}, \tag{39}$$

the figure (5) is an illustration of the entire procedure mentioned above:

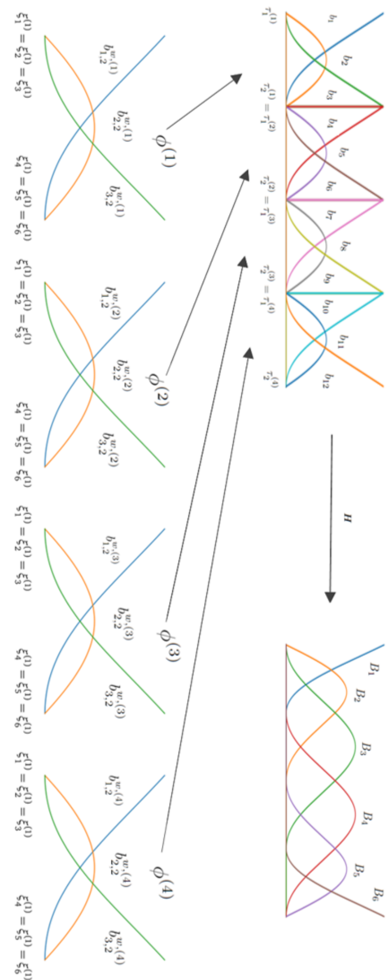

Fig. 4. Consider $\Omega = [0, 4]$, with $m = 4$, $\boldsymbol{\xi}^{(i)} = [0, 0, 0, 1, 1, 1]$ and $\boldsymbol{w}^{(i)} = \left[1, \frac{\sqrt{2}}{2}, 1\right]$, i = 1,…, 4.

III. Piecewise-rational polar surfaces

We first describe the initial setup a tensor-product spline space on a rectangular domain. We start from two univariate C^1 rational spline spaces $\mathcal{R}[\boldsymbol{\Xi}, \boldsymbol{W}]$ and $\mathcal{R}^{\text{per}}[\boldsymbol{\Xi}, \boldsymbol{W}]$ defined on the univariate domains $\Omega^s := [s_1, s_2]$ and $\Omega^t := [t_1, t_2]$, respectively; the superscripts of s and t are meant to indicate the symbols used for the respective coordinates. Using a Cartesian produc, we build the rectangular domain $\Omega := \Omega_s \otimes \Omega_t$, and on Ω we define the tensor-product spline space $\mathcal{R} := \mathcal{R}_s \otimes \mathcal{R}_t$. Without loss of generality, we assume that $s_1 = t_1 = 0$. This tensor-product spline space is spanned by tensor-product B-spline basis functions B_{ij}, $i = 1, \ldots, n^s$; $j = 1, \ldots, n^t$. Here, n^s and n^t denote the respective dimensions of the chosen univariate spline spaces; the basis functions spanning these spaces are denoted with B_i^s and B_j^t. Then, the tensor-product basis function B_{ij} is simply the product $B_i^s \otimes B_j^t$. The functions B_{ij} are assumed to be periodic in s and non-periodic in t.

Below is an example as in the article [1], choosing $\Omega^s = [0, 4]$ and $\Omega^t = [0, 2]$, and build the univariate rational spline space $\mathcal{R}^s$ (periodic) and $\mathcal{R}^t$ on them using the following sets of parameters:

$$\mathcal{R}^s, i=1,\dots,4: \begin{cases} \boldsymbol{\xi}^{(i)} = [0,0,0,1,1,1]\,, \\ \boldsymbol{w}^{(i)} = \left[1, \frac{\sqrt{2}}{2}, 1\right] \end{cases} \tag{40}$$

$$\mathcal{R}^t, i=1,2: \begin{cases} \boldsymbol{\xi}^{(i)} = [0,0,0,1,1,1]\,, \\ \boldsymbol{w}^{(i)} = \left[1, \frac{\sqrt{2}}{2}, 1\right] \end{cases} \tag{41}$$

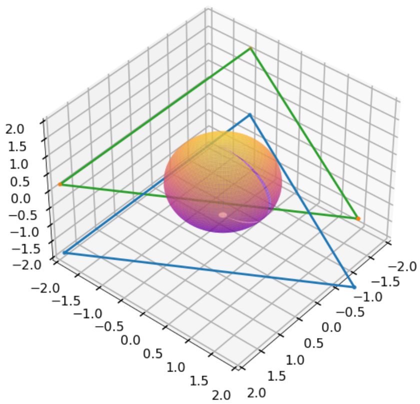

Fig. 5. C^1 description of degree $(2,2)$.

IV. The generalized Bernstein Trigonometric Space

In [2] they build the generalized Tchebycheffian splines (GT-splines) by the Tchebycheffian splines with pieces drawn from different extended Tchebycheff (ET-) spaces without the need for them to be of the same dimension, and their main contribution is an algorithm that allow the evaluation of the B-splines-like basis of the GT-splines.

Consider a partitioning, Δ, of the interval $[a,b]\subset\mathbb{R}$ into a sequence of breakpoints,

$$\Delta := \{a := x_0 < x_1 < \cdots < x_{m-1} < x_m := b\}\,. \tag{42}$$

In addition

$$J_i = \begin{cases} [x_{i-1}, x_i) & \text{if } i = 1,2,\dots,m-1, \\ [x_{m-1}, x_m] & \text{if } i = m. \end{cases} \tag{43}$$

We define an ECT-space of dimension p_i+1 on each closed interval $[x_{i+1}, x_i]$, $i=1,\dots,p_i$:

$$\mathbb{T}^{(i)}_{p_i} := \text{span}\left\{g^{(i)}_0,\dots,g^{(i)}_{p_i}\right\}, \quad g^{(i)}_j \in C^{p_i}\left([x_{i-1},x_i]\right) \\ j=0,\dots,p_i, \tag{44}$$

where $g^{(i)}_0,\dots,g^{(i)}_{p_i}$ are generalized powers defined in terms of positive weights $w^{(i)}_j \in C^{p_i}\left([x_{i-1},x_i]\right)$, $j=0,\dots,p_i$. Together, these functions span the following space:

$$\mathbb{S}^{\boldsymbol{p}}(\Delta) = \left\{s : [a,b]\to\mathbb{R} : s|_{J_i} \in \mathbb{T}^{(i)}_{p_i}, i=1,\dots,m\right\}. \tag{45}$$

In order to measure smoothness at the breakpoints, we define the following jump operator for a given $s\in\mathbb{S}^{\boldsymbol{p}}(\Delta)$:

$$\text{Jump}_{x_i,k}(s) := D^k_- s(x_i) - D^k_+ s(x_i). \tag{46}$$

Then we define the space of the generalized Tchebycheffian splines (GT-splines) as follows:

Given the sets of integers $\boldsymbol{p} = \{p_1,\dots,p_m\}$ and:

$$\boldsymbol{r} = \Big\{r_i\in\mathbb{Z} : -1\le r_i \le \min\{p_i,p_{i+1}\}, i=1,\dots,m-1, \\ r_0 = r_m = -1\Big\}$$

we define

$$\mathbb{S}^{\boldsymbol{p}}_{\boldsymbol{r}}(\Delta) = \Big\{s\in\mathbb{S}^{\boldsymbol{p}}(\Delta) : \text{Jump}_{x_i,k}(s) = 0, j=0,\dots,r_i \\ \text{and } i=1,\dots,m-1\Big\}$$

Given $p_i\ge 2$, let $u^{(i)}, v^{(i)} \in C^{p_i}([x_{i-1},x_i])$, and

$$U^{(i)} := D^{(p_i-1)}u^{(i)}, \quad V^{(i)} := D^{(p_i-1)}v^{(i)}, \tag{47}$$

such that $\mathbb{G}^{(i)} := \text{span}\left\{U^{(i)}, V^{(i)}\right\}$ is an ECT-space on $[x_{i-1},x_i]$. There exists a unique couple of functions $\tilde{U}^{(i)}, \tilde{V}^{(i)} \in \mathbb{G}^{(i)}$ such that

$$\tilde{U}^{(i)}(x_{i-1}) = 1, \quad \tilde{U}^{(i)}(x_i) = 0, \tag{48}$$

$$\tilde{V}^{(i)}(x_{i-1}) = 0, \quad \tilde{V}^{(i)}(x_i) = 1. \tag{49}$$

The generalized polynomial space of degree $p_i\ge 2$ on the closed interval $[x_{i-1},x_i]$ is defined by

$$\mathbb{G}^{(i)} := \text{span}\left\{1,x,x^2,\dots,x^{p_i-2},u^i(x),v^i(x)\right\}. \tag{50}$$

Where $u^i(x) = \sin(\omega x)$, $v^i(x) = \cos(\omega x)$, with $0<\omega(x_i-x_{i-1})<\pi$. For $q=1,\dots,p_i$ and $j=0,\dots,q$ the generalized Bernstein basis for the trigonometric space defined on $[x_{i-1},x_i]=[0,1]$ reads as follows, for degree $q=1$:

$$b_{0,1}(x) = \frac{\sin(\omega(1-x))}{\sin(\omega)}, \quad b_{1,1}(x) = \frac{\sin(\omega x)}{\sin(\omega)}, \tag{51}$$

and for degree $q=2$.

$$b_{0,2}(x) = \frac{1-\cos(\omega(1-x))}{1-\cos(\omega)}, \quad b_{2,2}(x) = \frac{1-\cos(\omega x)}{1-\cos(\omega)}, \\ b_{1,2}(x) = \frac{\cos(\omega(1-x))+\cos(\omega x)-\cos(\omega)-1}{1-\cos(\omega)}. \tag{52}$$

And the rest of the recurrence formula is given by:

$$b_{j,q}(x) := \begin{cases} 1-\int_{x_i-1}^{x}\frac{b_{0,q-1}(y)}{\delta_{0,q-1}}dy, & j=0 \\ \int_{x_i-1}^{x}\left[\frac{b_{j-1,q-1}(y)}{\delta_{j-1,q-1}}\frac{b_{j,q-1}(y)}{\delta_{j,q-1}}\right]dy, & 0<j<q, q>1 \\ \int_{x_i-1}^{x}\frac{b_{q-1,q-1}(y)}{\delta_{q-1,q-1}}dy, & j=q, \end{cases} \tag{53}$$

Example 1. Consider $\Delta = \{0,1,\frac{5}{2},5\}$, and $\omega=\frac{\pi}{2}$. And again, we glued the previous GTB-Trigonometric basis with the following function: Let $l := l(i,j) := \sum_{k=1}^{i-1}(p_k+1)+j$, and define

$$B_l(x) := \begin{cases} B^i_j(x), & x\in J_i \\ 0 & \text{otherwise}. \end{cases} \tag{54}$$

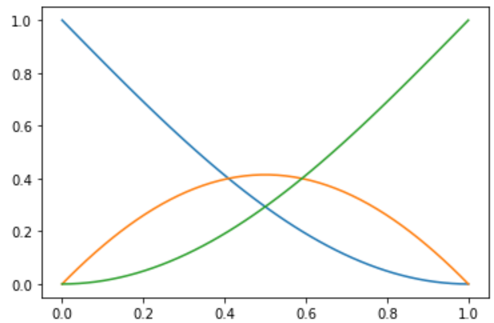

Fig. 6. GTB-Trigonometric basis with $p = 2$ in $x_0 = 0$ and $x_1 = 1$.

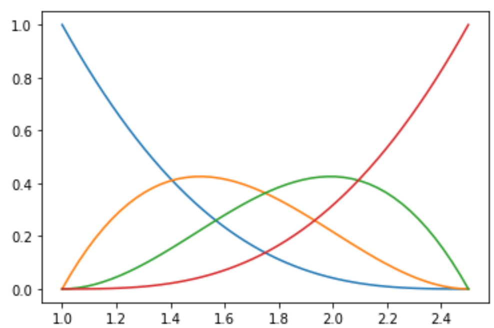

Fig. 7. GTB-Trigonometric basis with $p = 3$ in $x_1 = 1$ and $x_2 = \frac{5}{2}$.

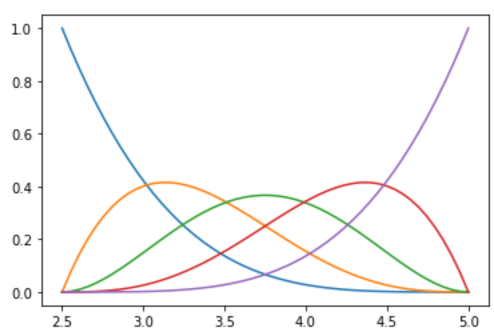

Fig. 8. GTB-Trigonometric basis with $p = 4$ in $x_2 = \frac{5}{2}$ and $x_3 = 5$.

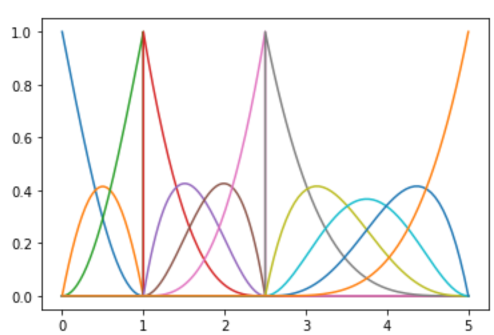

Fig. 9. GTB-Trigonometric basis glued together with the function in eq (62)

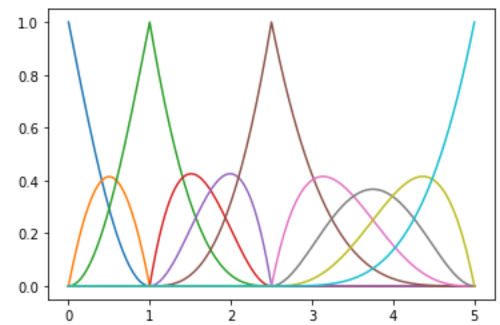

Fig. 10. GTB-Trigonometric basis composed with the extraction matrix C (1st iteration $r = [-1, 0, 0, -1]$, C^0 $[0, 5]$).

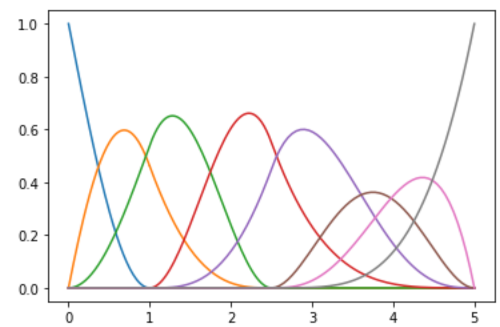

Fig. 11. GTB-Trigonometric basis composed with the extraction matrix C (2nd iteration $r = [-1, 1, 1, -1]$, C^1 $[0, 5]$).

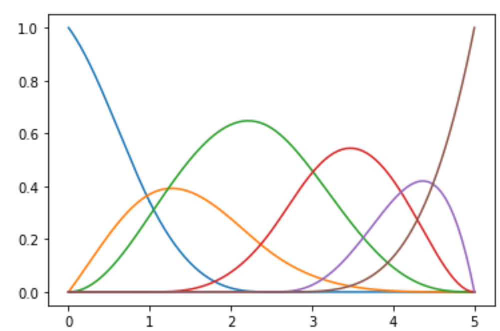

Fig. 12. GTB-Trigonometric basis composed with the extraction matrix C (3rd iteration $r = [-1, 2, 2, -1]$, C^2 $[0, 5]$).

In [2] they present an algorithm that computes the GTB-spline basis, whenever it exists, for the spline space using Bezier extraction (matrix C):

Example 2. Consider the same case as in Example 2

Let $\{b_{j,p_i}\}_{j=0}^{p_i}$ be the Bernstein basis corresponding to the ECT-space $\mathbb{T}_{p_i}^i$, $i = 1, \ldots, m$. We identify a function $f \in \mathbb{T}_{p_i}^i$ with the vector of its coefficients $[f_{i,1}, \ldots, f_{i,p_i}]$,

$$f = \sum_{j=0}^{p_i} f_{i,j} b_{j,p_i}, \tag{55}$$

then observe that the derivatives of the previous basis are as follows: $q=1$:

$$b'_{0,1}(x) = -\frac{\cos(\omega(1-x))\omega}{\sin(\omega)}, \quad b'_{1,1}(x) = \frac{\cos(\omega x)\omega}{\sin(\omega)}, \quad (56)$$

and for degree $q=2$:

$$\begin{aligned} b'_{0,2}(x) &= \frac{\sin(\omega(x-1))\omega}{1-\cos(\omega)}, \quad b'_{2,2}(x) = \frac{\sin(\omega x)\omega}{1-\cos(\omega)}, \\ b'_{1,2}(x) &= \frac{\sin(\omega(1-x))\omega - \sin(\omega x)\omega}{1-\cos(\omega)}. \end{aligned} \quad (57)$$

And the rest of the recurrence formula is given by:

$$b'_{j,q}(x) := \begin{cases} \frac{b_{0,q-1}(x)}{\delta_{0,q-1}}, & j=0 \\ \frac{b_{j-1,q-1}(x)}{\delta_{j-1,q-1}} - \frac{b_{j,q-1}(x)}{\delta_{j,q-1}} & 0<j<q, \quad q>1 \\ \frac{b_{q-1,q-1}(y)}{\delta_{q-1,q-1}}, & j=q, \end{cases} \quad (58)$$

Then from (55) we obtain:

$$\begin{aligned} f'(x) &= \sum_{j=0}^{p_i} f_{i,j} b'_{j,p_i} \\ &= -f_{i,0}\frac{b_{0,p_i-1}(x)}{\delta_{0,p_i-1}} + \sum_{j=1}^{p_i-1} f_{i,j}\Big[\frac{b_{j-1,p_i-1}(x)}{\delta_{j-1,p_i-1}} - \frac{b_{j,p_i-1}(x)}{\delta_{j,p_i-1}}\Big] \\ &+ f_{i,p_i}\frac{b_{p_i-1,p_i-1}(x)}{\delta_{p_i-1,p_i-1}} \\ &= \frac{b_{0,p_i-1}(x)}{\delta_{0,p_i-1}}(f_{i,0}-f_{i,1}) + \sum_{j=1}^{p_i-1}\frac{b_{j,p_i-1}(x)}{\delta_{j,p_i-1}}(f_{i,j+1}-f_{i,j}), \end{aligned} \quad (59)$$

in addition we have:

$$\begin{aligned} b_{1,p_i}(x_{i-1}) = 1, \quad b_{j,p_i}(x_{i-1}) = 0, \quad j = 2,\dots,p_i, \\ b_{p_i,p_i}(x_{i-1}) = 1, \quad b_{j,p_i}(x_{i-1}) = 0, \quad j = 1,\dots,p_i-1, \end{aligned} \quad (60)$$

then from (59) only the first (last) 2 basis functions contribute towards the 1st order derivative at the left (right) end points (x_{i-1}, x_i). Then we have:

$$\begin{aligned} f(x_{i-1}) = f_{i,1}, \quad \frac{df}{dx}(x_{i-1}) = \frac{(f_{i,0}-f_{i,1})}{\delta_{0,p_i-1}}, \\ f(x_i) = f_{i,p_i}, \quad \frac{df}{dx}(x_i) = \frac{(f_{i,p_i}+f_{i,p_i-1})}{\delta_{0,p_i-1}}. \end{aligned} \quad (61)$$

We glued the previous GTB-Trigonometric basis with the following function: Let $l := l(i,j) := \sum_{k=1}^{i-1}(p_k+1)+j$, and define

$$b_l(x) := \begin{cases} b_{j,p_i}(x), & x \in J_i \\ 0 & \text{otherwise.} \end{cases} \quad (62)$$

Let $\boldsymbol{b}$ the vector that collects the global Bernstein function $\{b_l\}_{l=1}^{\theta}$, where $\theta := \sum_{i=1}^{m}(p_i)+1$, since $\mathbb{S}_{\boldsymbol{r}}^{\boldsymbol{p}}(\Delta) \subseteq \mathbb{S}_r(\Delta)$, we can use the knot insertion procedure ([2], Algorithm 6.2) to convert from the global Bernstein basis $\{b_l\}_{l=1}^{\theta}$ to the smooth GTB-spline basis $\{B_k\}_{k=1}^{n}$, with $n=\phi+\theta$ where $\phi = \sum_{i}^{m-1}(r_i+1)$, i.e., through the Bézier extraction operator $C \in \mathbb{R}^{n\times\theta}$ we produce a GTB-spline basis $\{B_k\}_{k=1}^{n}$ for the spline space a $\mathbb{S}_{\boldsymbol{r}}^{\boldsymbol{p}}(\Delta)$ according to

$$\boldsymbol{B}(x) = \boldsymbol{H}\boldsymbol{b}(x). \quad (63)$$

Using the computed the GT-spline basis $\{B_k\}_{k=1}^{n}$ that span $\mathbb{S}_{\boldsymbol{r}}^{\boldsymbol{p}}(\Delta)$, we can define its derivative as:

$$\frac{d^2\boldsymbol{B}}{dx} = \boldsymbol{H}\frac{d\boldsymbol{b}}{dx}. \quad (64)$$

For non-periodic spaces, one of the properties of the building of $\boldsymbol{H}$ is that the 1th derivatives at the left (right) end of Δ will be completely defined by the derivatives of the first (last) 2 basis functions on the first (last) segment, under the assumption $2 \le p_1$ ($2 \le p_m$). A consequence of the partition of unity property of the basis function is that at the left end point a:

$$\sum_{j=1}^{2}\frac{d^2B_j}{dx}(a) = 0 \quad (2 \le p_1). \quad (65)$$

Now consider a function f in $\mathbb{S}_{\boldsymbol{r}}^{\boldsymbol{p}}(\Delta)$ using a vector of its coefficients $[f_1,\dots,f_n]$:

$$f(x) = \sum_{l=1}^{n} f_l B_l(x). \quad (66)$$

Then using (65) and (66) we observe that the first derivative at the left end involves only the first two basis functions and can be expressed as follows:

$$\frac{df}{dx}(a) = f_1\frac{dB_1}{dx}(a) + f_2\frac{dB_2}{dx}(a) = (f_2-f_1)\frac{dB_2}{dx}(a). \quad (67)$$

Similarly for the right end point b:

$$\frac{df}{dx}(b) = f_{n-1}\frac{dB_{n-1}}{dx}(b) + f_n\frac{dB_n}{dx}(b) = (f_n-f_{n-1})\frac{dB_n}{dx}(b). \quad (68)$$

Now consider a surface that can be obtained by starting from a bivariate tensor product (GTB-spline) and collapsing one of its edges as illustrated in Fig13, the result of that collapsing creates a polar point and can be achieved by coalescing the control points related to basis functions with non-zero values on the edge. However, the following is one way to build C^1 smooth representations for polar surfaces containing one single polar point, because we want to avoid the case when the control point coalescing introduce kinks at the poles and the surface representation that are not smooth.

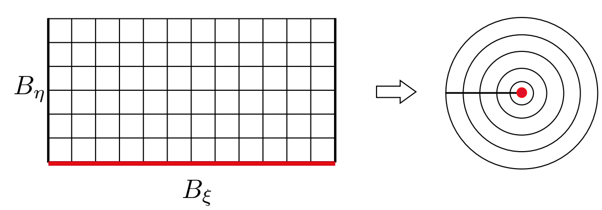

Fig. 13. Singular polar singularity

Consider two univariate C^1 GTB-splines spaces $\mathbb{S}_{\boldsymbol{r}_\xi}^{\boldsymbol{p}_\xi}(\Delta)^\xi$, $\mathbb{S}_{\boldsymbol{r}}^{\boldsymbol{p}_\eta}(\Delta)^\eta$ defined Δ^ξ and Δ^η the partitionings of the intervals $[\xi_1,\xi_2]$ and $[\eta_1,\eta_2]$ respectively. We built the rectangular partition $\Omega = [\xi_1,\xi_2]\times[\eta_1,\eta_2]$, and on Ω we define the tensor product GTB-spline space $\mathbb{S}_{\boldsymbol{r}}^{\boldsymbol{p}}(\Delta) := \mathbb{S}_{\boldsymbol{r}_\xi}^{\boldsymbol{p}_\xi}(\Delta)^\xi \otimes \mathbb{S}_{\boldsymbol{r}_\eta}^{\boldsymbol{p}_\eta}(\Delta)^\eta$, we assume that $\xi_1 = \eta_1 = 0$. The tensor product is spanned by tensor-product GTB-splines basis functions $B_{ij} = B_i^\xi \otimes B_j^\eta$,

$i = 1, \ldots, n^{\xi}$; $j = 1, \ldots, n^{\eta}$, where $n_{\xi} = \phi^{\xi} + \theta_{\xi}$ and $n_{\eta} = \phi^{\eta} + \theta_{\eta}$ denote the respective dimensions of the chosen univariate GTB-splines, and we assume that B_{ij} is periodic in ξ and non-periodic in η.

Consider a planar disk-like domain Ω^{pol} called the polar parametric domain, via a suitable polar map $\boldsymbol{F}$. Assign the control points $\boldsymbol{F}_{ij} := (\rho_j \cos\theta_i, \rho_j \sin\theta_i) \in \mathbb{R}^2$ to the basis function B_{ij}, with:

$$\rho_j := \frac{j-1}{n^{\eta}-1} \in [0,1]\,, \tag{69}$$

and

$$\theta_i := 2\pi + \frac{(1-2i)\pi}{n^{\xi}} \in [0, 2\pi]\,. \tag{70}$$

Through these control points, we construct the disk-like domain Ω^{pol} with the aid of the map $\boldsymbol{F}$ from Ω to Ω^{pol}:

$$\boldsymbol{F} : \underset{(\xi,\eta)}{\Omega} \underset{\mapsto}{\rightarrow} \underset{\boldsymbol{F}(\xi,\eta)=(u,v)}{\Omega^{\text{pol}}} \tag{71}$$

defined as:

$$\boldsymbol{F}(\xi,\eta) := (F_u(\xi,\eta), F_v(\xi,\eta)) := \sum_{i=1}^{n^{\xi}} \sum_{j=1}^{n^{\eta}} \boldsymbol{F}_{ij} B_{ij}(\xi,\eta). \tag{72}$$

As can be deduced from Fig. 13, we have that for all $\xi \in \Omega^{\xi}$:

$$\boldsymbol{F}(\xi, 0) = (0,0), \tag{73}$$

with $(0,0)$ the polar point, which implies that:

$$\left.\frac{\partial F_u}{\partial \xi}\right|_{\eta=0} \equiv 0 \equiv \left.\frac{\partial F_v}{\partial \xi}\right|_{\eta=0}. \tag{74}$$

Using (71) we define B_{ij}^{pol} like the be the image B_{ij} in composition with the map $\boldsymbol{F} : \Omega \rightarrow \Omega^{\text{pol}}$, i.e.:

$$B_{ij}(u,v) = B_{ij}(\boldsymbol{F}(\xi,\eta)) = B_{ij}^{\text{pol}}(\xi,\eta). \tag{75}$$

We define a polar trigonometric spline function over Ω^{pol}as:

$$f^{\text{pol}}(u,v) = \sum_{i=1}^{n^{\xi}} \sum_{j=1}^{n^{\eta}} f_{ij} B_{ij}^{\text{pol}}(u,v), \tag{76}$$

with f_{ij} coefficients given. Pulling f^{pol} back to Ω we get:

$$\begin{aligned} f(\xi,\eta) := f^{\text{pol}}(\boldsymbol{F}(\xi,\eta)) &= \sum_{i=1}^{n^{\xi}} \sum_{j=1}^{n^{\eta}} f_{ij} B_{ij}^{\text{pol}}(\boldsymbol{F}(\xi,\eta)) \\ &= \sum_{i=1}^{n^{\xi}} \sum_{j=1}^{n^{\eta}} f_{ij} B_{ij}(\xi,\eta). \end{aligned} \tag{77}$$

Since we want to ensure that f^{pol} is C^1 at $(0,0)$, then the we have that:

$$\begin{aligned} &\lim_{(u,v)\rightarrow(0,0)} f^{\text{pol}}(u,v) = \alpha_1 \\ &\lim_{(u,v)\rightarrow(0,0)} \frac{\partial f^{\text{pol}}}{\partial u}(u,v) = \alpha_2 \\ &\lim_{(u,v)\rightarrow(0,0)} \frac{\partial f^{\text{pol}}}{\partial v}(u,v) = \alpha_3, \quad \{\alpha_1, \alpha_2, \alpha_3\} \in \mathbb{R}, \end{aligned} \tag{78}$$

i.e. the previous limits exist. From (74):,

$$f(\xi, 0) = \alpha_1 \quad \xi \in \Omega^{\xi}, \tag{79}$$

and similarly using the chain rule in (77) we have:

$$\frac{\partial f}{\partial \xi}(\xi,0) = \alpha_2 \frac{\partial F_u}{\partial \eta}(\xi,0) + \alpha_3 \frac{\partial F_v}{\partial \eta}(\xi,0) \quad \xi \in \Omega^{\xi}, \tag{80}$$

Only B_{ij} $j \leq 2$, have non-zero values and derivatives when $\eta = 0$ (see Fig 14), we get the following requirements for all $\xi \in [\xi_1, \xi_2]$:

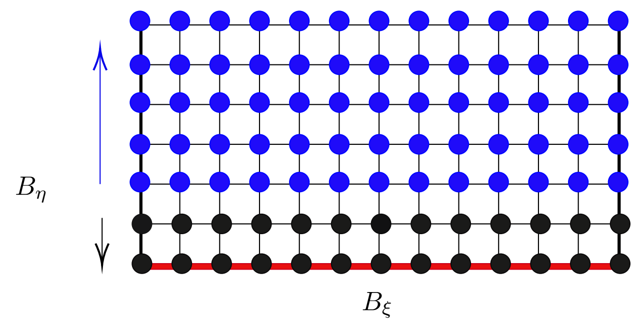

Fig. 14. $B_{ij}, j \leq 2$ have non-zero (black points) and derivatives when $\eta = 0$.

$$\begin{aligned} &\alpha_1 = f(\xi,0) = \sum_{i=1}^{n^{\xi}} \sum_{j=1}^{n^{\eta}} f_{ij} B_{ij}^{\text{pol}}(\boldsymbol{F}(\xi,0)) = \sum_{i=1}^{n^{\xi}} B_{i1}(\xi,0) \\ &\sum_{i=1}^{n^{\xi}} \sum_{j=1}^{2} \left[f_{ij} - \boldsymbol{F}_{ij} \cdot (\alpha_2, \alpha_3)\right] \frac{\partial B_{ij}}{\partial \eta}(\xi,0) = 0. \end{aligned} \tag{81}$$

Let be $\boldsymbol{B}$ the vector in which is arranged the set of basis functions $\left\{B_{ij}^{\text{pol}} : i = 1, \ldots, n^{\xi}; j = 1, \ldots, n^{\eta}\right\}$, where the $i + (j-1)n^{\xi}$th entry is occupied by B_{ij}. And let $\boldsymbol{N}$ the vector in which is arranged the spline basis functions $\{N_l\}_{l=1}^{\mathbf{n}}$ that are C^1 ar $(0,0)$ and satisfies:

$$\boldsymbol{N} := \boldsymbol{E}\boldsymbol{B}, \tag{82}$$

with $\boldsymbol{E}$ an extraction operator that we need to build as follow. For fixed j, the set $\left\{B_{ij} : i = 1, \ldots, n^{\xi}\right\}$ is called the $(j-1)$th polar ring of basis functions. When $j \leq 2$, all basis function in the $(j-1)$th ring already satisfy the C^1 continuity conditions at the polar point because their derivatives are identically zero there. We need to substitute the others three by smooth polar basis functions. Then E will be a matrix with $\mathbf{n} := n^{\xi}(n^{\eta} - 2) + 3$ rows and $n^{\xi} n^{\eta}$ columns taking the following sparse block-diagonal form:

$$\boldsymbol{E} := \begin{bmatrix} \bar{\boldsymbol{E}} & \\ & \boldsymbol{I} \end{bmatrix}. \tag{83}$$

with $\boldsymbol{I}$ is the matrix of size $n^{\xi}(n^{\eta}-2) \times n^{\xi}(n^{\eta}-2)$ and $\bar{\boldsymbol{E}}$ is a matrix of size $3 \times 2n^{\xi}$. The entry of $\bar{\boldsymbol{E}}$ corresponding to its

lth row and $(i+(j-1)n^{\xi})$th column is denoted with $\bar{E}_{l,(ij)}$. We rewrite (82) as follows for $l=1,2,3$,

$$N_l^{\text{pol}}(u,v) = \sum_{i=1}^{n^{\xi}}\sum_{j=1}^{2} \bar{E}_{l,(ij)} B_{ij}^{\text{pol}}(u,v). \tag{84}$$

Using (77) we pull the previous back to Ω and get the equivalent representation for $l=1,2,3$,

$$N_l(\xi,\eta) = \sum_{i=1}^{n^{\xi}}\sum_{j=1}^{2} \bar{E}_{l,(ij)} B_{ij}(\xi,\eta). \tag{85}$$

We enforce C^1 continuity at the polar point by requiring the basis function N_l^{pol} to satisfy a linearly independent Hermite data set at polar point thinking in (78). For that we use three source basis functions $\{T_l : l=1,2,3\}$, these functions provide us the appropriate Hermite data. Let $\triangle$ with vertices $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$, let $(\lambda_1,\lambda_2,\lambda_3)$ be the unique barycentric coordinates of point (u,v) with respect to $\triangle$ such that:

$$\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \lambda_3\boldsymbol{v}_3 = (u,v), \quad \lambda_1+\lambda_2+\lambda_3 = 1. \tag{86}$$

Then we define

$$T_l(u,v) := \lambda_l, \quad l=1,2,3. \tag{87}$$

These functions can be view as triangular Bernstein polynomials of degree 1. They are non-negative on the domain triangle $\triangle$. Moreover, they are linearly independent, form a partition of unity, and span the space of bivariate polynomials of total degree less than or equal to 1. So, we need that N_l in (85) is a spline function that it satisfies the continuity constraints in (79) with:

$$\alpha = T_l(0,0) \quad \beta = \frac{\partial T_l}{\partial u}(0,0), \quad \gamma = \frac{\partial T_l}{\partial v}(0,0), \tag{88}$$

for $l=1,2,3$. We choose the triangle $\triangle$ as equilateral with vertices:

$$\boldsymbol{v}_1 = (2\rho_2, 0), \quad \boldsymbol{v}_2 = (\rho_2, \sqrt{3}\rho_2), \quad \boldsymbol{v}_2 = (-\rho_2, -\sqrt{3}\rho_2); \tag{89}$$

Note: Bernstein basis function and derivatives on the triangle with vertices as in (89). For $l=1,2,3$.

$$\begin{aligned} &T_l(0,0) = \frac{1}{3} \\ &\frac{\partial T_l}{\partial u}(0,0) = \frac{2i_1 - i_2 - i_3}{2\rho_2}, \quad \frac{\partial T_l}{\partial v}(0,0) = \frac{i_2 - i_3}{2\sqrt{3}\rho_2}, \end{aligned} \tag{90}$$

where $(i_1,i_2,i_3) \in (\mathbb{N}\cup\{0\})^3$, with $i_1+i_2+i_3 = 1$.

For (79) we require that N_l^{pol} to be such that for $l=1,2,3$:

$$\lim_{(u,v)\to(0,0)} N_l^{\text{pol}}(u,v) = T_l(0,0) = \frac{1}{3}. \tag{91}$$

$$\begin{aligned} \lim_{(u,v)\to(0,0)} \frac{\partial N_l^{\text{pol}}}{\partial u}(u,v) &= \frac{\partial T_l}{\partial u}(0,0) \\ \lim_{(u,v)\to(0,0)} \frac{\partial N_l^{\text{pol}}}{\partial v}(u,v) &= \frac{\partial T_l}{\partial v}(0,0). \end{aligned} \tag{92}$$

Pulling back the spline basis function and derivatives to Ω^{pol}, we need that for all ξ_0:

$$\lim_{(\xi,\eta)\to(\xi_0,0)} N_l(\xi,\eta) = \frac{1}{3}, \tag{93}$$

$$\begin{aligned} \lim_{(\xi,\eta)\to(\xi_0,0)} \frac{\partial N_l}{\partial \xi}(\xi,\eta) &= \lim_{(\xi,\eta)\to(\xi_0,0)} \frac{\partial F_u}{\partial \xi}(\xi,\eta)\frac{\partial T_l}{\partial u}(0,0) \\ &\quad + \frac{\partial F_v}{\partial \xi}(\xi,\eta)\frac{\partial T_l}{\partial v}(0,0) \\ \lim_{(\xi,\eta)\to(\xi_0,0)} \frac{\partial N_l}{\partial \eta}(\xi,\eta) &= \lim_{(\xi,\eta)\to(\xi_0,0)} \frac{\partial F_u}{\partial \eta}(\xi,\eta)\frac{\partial T_l}{\partial u}(0,0) \\ &\quad + \frac{\partial F_v}{\partial \eta}(\xi,\eta)\frac{\partial T_l}{\partial v}(0,0). \end{aligned} \tag{94}$$

Since

$$N_l(\xi_0,0) = \sum_{i=1}^{n^{\xi}} \bar{E}_{l,(i1)} B_{i1}(\xi_0,0) = \sum_{i=1}^{n^{\xi}} \bar{E}_{l,(i1)} B_i^{\xi}(\xi_0,0), \tag{95}$$

and $\{N_k\}_{k=1}^{\boldsymbol{n}}$ form a partition of unity, then $\forall i$:

$$E_{l,(i1)} = \frac{1}{3}. \tag{96}$$

The previous implies that $\frac{\partial N_l}{\partial \eta}(\xi_0,0) = 0$ for all ξ_0. Then, the only constraint that still needs to be satisfied is:

$$\frac{\partial N_l}{\partial \eta}(\xi_0,0) = \frac{\partial F_u}{\partial \eta}(\xi_0,0)\frac{\partial T_l}{\partial v}(0,0) + \frac{\partial F_v}{\partial \eta}(\xi_0,0)\frac{\partial T_l}{\partial v}(0,0) \tag{97}$$

Using (67) we obtain:

$$\frac{\partial N_l}{\partial \eta}(\xi_0,0) = \frac{dB_2^{\xi}}{d\eta}(\xi_0)\sum_{i=1}^{n^{\xi}}\Big(\bar{E}_{l,i2} - \bar{E}_{l,i1}\Big)B_i^{\xi}(\xi_0) \tag{98}$$

Then from (97):

$$\begin{aligned} &\frac{dB_2^{\xi}}{d\eta}(\xi_0)\sum_{i=1}^{n^{\xi}}\Big(\bar{E}_{l,i2} - \bar{E}_{l,i1}\Big)B_i^{\xi}(\xi_0) = \\ &\frac{dB_2^{\xi}}{d\eta}(\xi_0)(\rho_2-\rho_1)\frac{\partial T_l}{\partial u}(\xi_0,0)\sum_{i=1}^{n^{\xi}}\cos\theta_i B_i^{\xi}(\xi_0) + \\ &\frac{dB_2^{\xi}}{d\eta}(\xi_0)(\rho_2-\rho_1)\frac{\partial T_l}{\partial v}(\xi_0,0)\sum_{i=1}^{n^{\xi}}\sin\theta_i B_i^{\xi}(\xi_0). \end{aligned} \tag{99}$$

We know that $\rho_1 = 0$, then the following equality holds for $i=1,\dots,n^{\xi}$.

$$\Big(\bar{E}_{l,i2} - \bar{E}_{l,i1}\Big) - \rho_2\cos\theta_i\frac{\partial T_l}{\partial u}(0,0) - \rho_2\sin\theta_i\frac{\partial T_l}{\partial u}(0,0) = 0. \tag{100}$$

Or equivalently:

$$\bar{E}_{l,i2} = \frac{1}{\sqrt{3}}\Big(\cos\theta_i\frac{2i_1-i_2-i_3}{2\sqrt{3}} + \sin\theta_i\frac{i_2-i_3}{2\sqrt{3}} + \frac{1}{\sqrt{3}}\Big), \tag{101}$$

For $(i_1, i_2, i_3) = ((1,0,0),(0,1,0),(0,0,1))$ respectively we obtain that the following equality must hold for $i = 1, \dots, n^\xi$, then the final form of the $(n^\xi + 1)$th column of $\bar{\boldsymbol{E}}$ is:

$$\begin{bmatrix} \bar{E}_{1,(i2)} \\ \bar{E}_{2,(i2)} \\ \bar{E}_{3,(i2)} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{2\sqrt{3}} & \frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{2\sqrt{3}} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \\ 1 \end{bmatrix} \tag{102}$$

The above relation says that $\left(\bar{E}_{1,(i2)}, \bar{E}_{2,(i2)}, \bar{E}_{3,(i2)}\right)$ are simply the barycentric coordinates of the control point $(\rho_2\cos\theta_i, \rho_2\sin\theta_i)$ with $\triangle$. We have that $\left(\bar{E}_{1,(i2)}, \bar{E}_{2,(i2)}, \bar{E}_{3,(i2)}\right)$ is non-negative, since $\triangle$ encloses the circle centered at $(0,0)$.

Then

$$\bar{\boldsymbol{E}} = \begin{bmatrix} \frac{1}{3} & \cdots & \frac{1}{3} & \bar{E}_{1,(12)} & \cdots \bar{E}_{1,(i2)} & \cdots \bar{E}_{1,(n^\xi 2)} \\ \frac{1}{3} & \cdots & \frac{1}{3} & \bar{E}_{2,(12)} & \cdots \bar{E}_{2,(i2)} & \cdots \bar{E}_{2,(n^\xi 2)} \\ \frac{1}{3} & \cdots & \frac{1}{3} & \bar{E}_{3,(12)} & \cdots \bar{E}_{3,(i2)} & \cdots \bar{E}_{3,(n^\xi 2)} \end{bmatrix} \tag{103}$$

As has been proved in [1], in the case of collapsing a pair of two opposite edges, the extraction matrix is for $n^\xi \geq 4$:

$$\boldsymbol{E}^{(2)} = \begin{bmatrix} \hat{\boldsymbol{E}} & & \\ & \boldsymbol{I} & \\ & & \tilde{\boldsymbol{E}} \end{bmatrix} \tag{104}$$

with $\hat{\boldsymbol{E}}_{3\times 2n^\xi} = \hat{E}$ the previous extraction matrix, $\boldsymbol{I}_{n^\xi(n^\eta-4)\times n^\xi(n^\eta-4)}$ and $\tilde{\boldsymbol{E}}_{3\times 2n^\xi} := \boldsymbol{R}_{3\times 3}\boldsymbol{E}^{(2)}\boldsymbol{R}_{2n^\xi\times 2n^\xi}$ and $\boldsymbol{J}_{r\times r}$ is the anti-diagonal matrix of size r.

V. Polar Trigonometric Splines Examples

Consider the following GT-spline spaces $\mathbb{S}^{\boldsymbol{p}_\xi}_{\boldsymbol{r}_\xi}(\Delta_\xi)$ (periodic) and $\mathbb{S}^{\boldsymbol{p}_\eta}_{\boldsymbol{r}_\eta}(\Delta_\xi)$ (non-periodic) defined by:

$$\Delta_\xi = \{0,1,2,3,4\}\,, \quad \boldsymbol{p}_\xi = \{2,2,2,2\}\,, \quad \boldsymbol{r}_\xi = \{1,1,1,1,1\}\,.$$

$$\Delta_\eta = \{0,1,2\}\,, \quad \boldsymbol{p}_\eta = \{2,2\}\,, \quad \boldsymbol{r}_\eta = \{-1,1,-1\}\,.$$

Let $\boldsymbol{B}_\xi$ and $\boldsymbol{B}_\eta$ the vectors that collect the global Bernstein basis for $\mathbb{S}^{\boldsymbol{p}_\xi}(\Delta_\xi)$ and $\mathbb{S}^{\boldsymbol{p}_\eta}(\Delta_\eta)$ respectively (see Fig. 15).

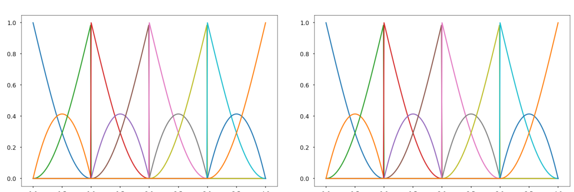

Fig. 15. $\boldsymbol{B}_\xi$ (on the left) and $\boldsymbol{B}_\eta$ (on the right)

Let $\boldsymbol{C}_\xi$ and $\boldsymbol{C}_\eta$ the Bézier extraction matrices that map functions from $\mathbb{S}^{\boldsymbol{p}_\xi}(\Delta_\xi)$ to $\mathbb{S}^{\boldsymbol{p}_\xi}_{\boldsymbol{r}_\xi}(\Delta_\xi)$ and from $\mathbb{S}^{\boldsymbol{p}_\eta}(\Delta_\eta)$ to $\mathbb{S}^{\boldsymbol{p}_\eta}_{\boldsymbol{r}_\eta}(\Delta_\eta)$ respectively (see Fig. 16).

We use 8 trigonometric smooth splines of bi-degree $(2,2)$ and will define six C^1 polar trigonometric splines and associated control points as below to built Fig. 17.

$$\begin{aligned} &\boldsymbol{f}_1 = (0, 2\sqrt{2}a_y, a_z), \quad \boldsymbol{f}_2 = (-\sqrt{6}a_x, -2\sqrt{2}a_y, a_z), \\ &\boldsymbol{f}_3 = (\sqrt{6}a_x, -\sqrt{2}a_y, a_z), \quad \boldsymbol{f}_4 = (-\sqrt{6}a_x, -\sqrt{2}a_y, -a_z), \\ &\boldsymbol{f}_5 = (\sqrt{6}a_x, -\sqrt{2}a_y, -a_z), \quad \boldsymbol{f}_6 = (0, 2\sqrt{2}a_y, -a_z). \end{aligned} \tag{105}$$

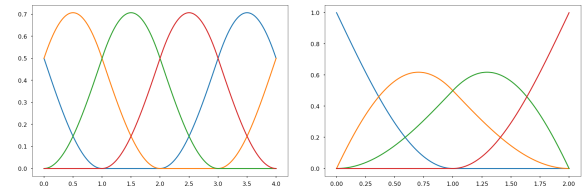

Fig. 16. Vector that collect the GTB-spline basis $\boldsymbol{N}_\xi$ (on the left) and $\boldsymbol{N}_\eta$ (on the right) of $\mathbb{S}^{\boldsymbol{p}_\xi}_{\boldsymbol{r}_\xi}(\Delta_\xi)$ and $\mathbb{S}^{\boldsymbol{p}_\eta}_{\boldsymbol{r}_\eta}(\Delta_\eta)$ respectively.

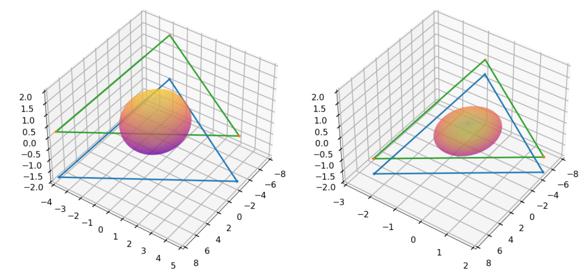

Fig. 17. C^1 smooth description of an ellipsoid using the control points in eq (105) with $a_x = a_y = a_z = 1$ on the left and $a_x = 2a_y = 3a_z = 1$ on the right.

Now, we consider the following GT-spline spaces $\mathbb{S}^{\boldsymbol{p}_\xi}_{\boldsymbol{r}_\xi}(\Delta_\xi)$ (periodic) and $\mathbb{S}^{\boldsymbol{p}_\eta}_{\boldsymbol{r}_\eta}(\Delta_\xi)$ (non-periodic) defined by:

$$\Delta_\xi = \{0,1,2,3,4\}\,, \quad \boldsymbol{p}_\xi = \{2,2,2,2\}\,, \quad \boldsymbol{r}_\xi = \{1,1,1,1,1\}\,.$$

$$\Delta_\eta = \{0,1\}\,, \quad \boldsymbol{p}_\eta = \{3\}\,, \quad \boldsymbol{r}_\eta = \{-1,-1\}\,.$$

Let $\boldsymbol{B}_\xi$ and $\boldsymbol{B}_\eta$ the vectors that collect the global Bernstein basis for $\mathbb{S}^{\boldsymbol{p}_\xi}(\Delta_\xi)$ and $\mathbb{S}^{\boldsymbol{p}_\eta}(\Delta_\eta)$ respectively (see Fig. 19).

Let $\boldsymbol{C}_\xi$ and $\boldsymbol{C}_\eta$ the Bézier extraction matrices that map functions from $\mathbb{S}^{\boldsymbol{p}_\xi}(\Delta_\xi)$ to $\mathbb{S}^{\boldsymbol{p}_\xi}_{\boldsymbol{r}_\xi}(\Delta_\xi)$ and from $\mathbb{S}^{\boldsymbol{p}_\eta}(\Delta_\eta)$ to $\mathbb{S}^{\boldsymbol{p}_\eta}_{\boldsymbol{r}_\eta}(\Delta_\eta)$ respectively (see Fig. 20).

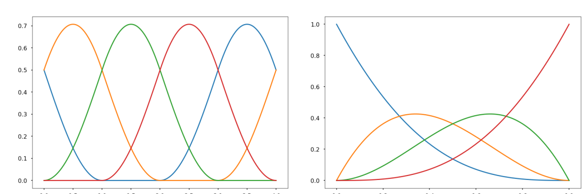

Fig. 20. Vector that collect the GTB-spline basis $\boldsymbol{N}_\xi$ (on the left) and $\boldsymbol{N}_\eta$ (on the right) of $\mathbb{S}^{\boldsymbol{p}_\xi}_{\boldsymbol{r}_\xi}(\Delta_\xi)$ and $\mathbb{S}^{\boldsymbol{p}_\eta}_{\boldsymbol{r}_\eta}(\Delta_\eta)$ respectively.

We use 4 trigonometric smooth splines of bi-degree $(2,3)$ and will define six C^1 polar trigonometric splines and associated control points as below to built Fig. 21:

$$\begin{aligned} &\boldsymbol{f}_1 = (0, 4\sqrt{2}a_y, a_z), \quad \boldsymbol{f}_2 = (-2\sqrt{6}a_x, -2\sqrt{2}a_y, a_z), \\ &\boldsymbol{f}_3 = (2\sqrt{6}a_x, -2\sqrt{2}a_y, a_z), \quad \boldsymbol{f}_4 = (-2\sqrt{6}a_x, -2\sqrt{2}a_y, -a_z), \\ &\boldsymbol{f}_5 = (2\sqrt{6}a_x, -2\sqrt{2}a_y, -a_z), \quad \boldsymbol{f}_6 = (0, 4\sqrt{2}a_y, -a_z). \end{aligned} \tag{106}$$

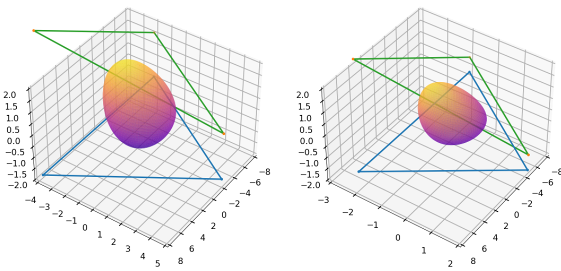

Fig. 18. C^1 smooth description of an ellipsoid like Fig. 17 but replacing the third control point f_3 by $f_3 + (0, 0, 4a_x)$.

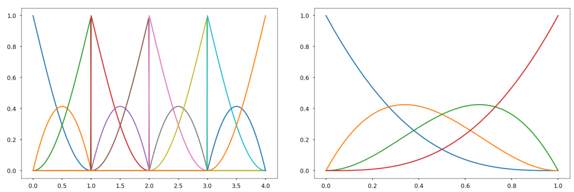

Fig. 19. $\boldsymbol{B}_\xi$ (on the left) and $\boldsymbol{B}_\eta$ (on the right)

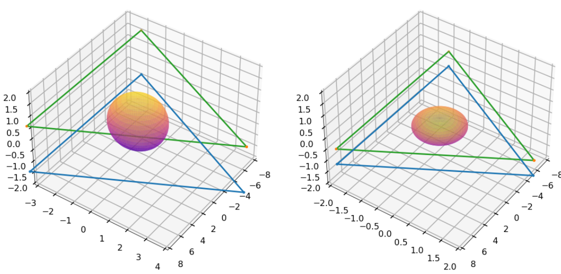

Fig. 21. C^1 smooth description of an ellipsoid using the control points in eq (106) with $a_x = a_y = a_z = 1$ on the left and $a_x = 2a_y = 3a_z = 1$ on the right.

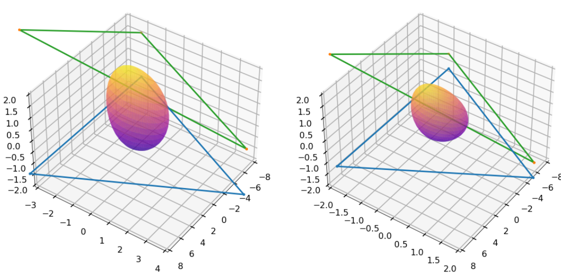

Fig. 22. C^1 smooth description of an ellipsoid like Fig. 21 but replacing the third control point f_3 by $f_3 + (0, 0, 4a_x)$.

Now, we consider the following GT-spline spaces $\mathbb{S}^{\boldsymbol{p}_\xi}_{\boldsymbol{r}_\xi}(\Delta_\xi)$ (periodic) and $\mathbb{S}^{\boldsymbol{p}_\eta}_{\boldsymbol{r}_\eta}(\Delta_\xi)$ (non-periodic) defined by:

$$\Delta_\xi = \{0, 1, 2\}, \quad \boldsymbol{p}_\xi = \{3, 3\}, \quad \boldsymbol{r}_\xi = \{1, 1, 1\}.$$

$$\Delta_\eta = \{0, 1\}, \quad \boldsymbol{p}_\eta = \{3\}, \quad \boldsymbol{r}_\eta = \{-1, -1\}.$$

Let $\boldsymbol{B}_\xi$ and $\boldsymbol{B}_\eta$ the vectors that collect the global Bernstein basis for $\mathbb{S}^{\boldsymbol{p}_\xi}(\Delta_\xi)$ and $\mathbb{S}^{\boldsymbol{p}_\eta}(\Delta_\eta)$ respectively (see Fig. 23).

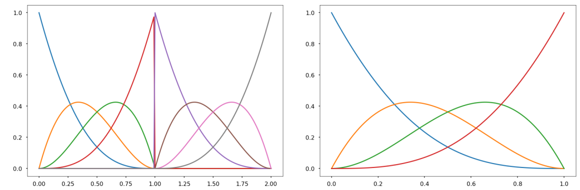

Fig. 23. $\boldsymbol{B}_\xi$ (on the left) and $\boldsymbol{B}_\eta$ (on the right)

Let $\boldsymbol{C}_\xi$ and $\boldsymbol{C}_\eta$ the Bézier extraction matrices that map functions from $\mathbb{S}^{\boldsymbol{p}_\xi}(\Delta_\xi)$ to $\mathbb{S}^{\boldsymbol{p}_\xi}_{\boldsymbol{r}_\xi}(\Delta_\xi)$ and from $\mathbb{S}^{\boldsymbol{p}_\eta}(\Delta_\eta)$ to $\mathbb{S}^{\boldsymbol{p}_\eta}_{\boldsymbol{r}_\eta}(\Delta_\eta)$ respectively (see Fig. 24).

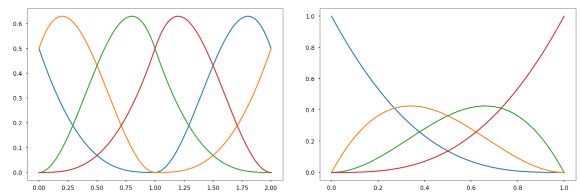

Fig. 24. Vector that collect the GTB-spline basis $\boldsymbol{N}_\xi$ (on the left) and $\boldsymbol{N}_\eta$ (on the right) of $\mathbb{S}^{\boldsymbol{p}_\xi}_{\boldsymbol{r}_\xi}(\Delta_\xi)$ and $\mathbb{S}^{\boldsymbol{p}_\eta}_{\boldsymbol{r}_\eta}(\Delta_\eta)$ respectively.

We use 2 trigonometric smooth splines of bi-degree $(3, 3)$ and will define six C^1 polar trigonometric splines and associated control points as below to built Fig. 25:

$$\begin{aligned}
&\boldsymbol{f}_1 = (0, 4\sqrt{2}a_y, a_z), \quad \boldsymbol{f}_2 = (-4\sqrt{6}a_x, -2\sqrt{2}a_y, a_z), \\
&\boldsymbol{f}_3 = (4\sqrt{6}a_x, -2\sqrt{2}a_y, a_z), \quad \boldsymbol{f}_4 = (-4\sqrt{6}a_x, -2\sqrt{2}a_y, -a_z), \\
&\boldsymbol{f}_5 = (4\sqrt{6}a_x, -2\sqrt{2}a_y, -a_z), \quad \boldsymbol{f}_6 = (0, 4\sqrt{2}a_y, -a_z).
\end{aligned} \tag{107}$$

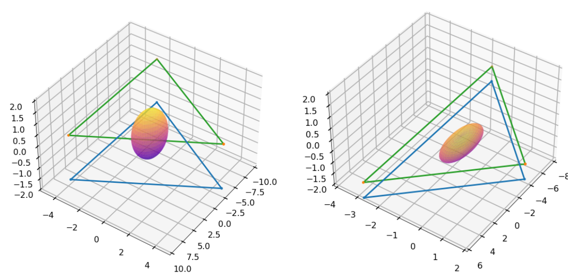

Fig. 25. C^1 smooth description of an ellipsoid using the control points in eq (107) with $a_x = a_y = a_z = 1$ on the left and $a_x = 2a_y = 3a_z = 1$ on the right.

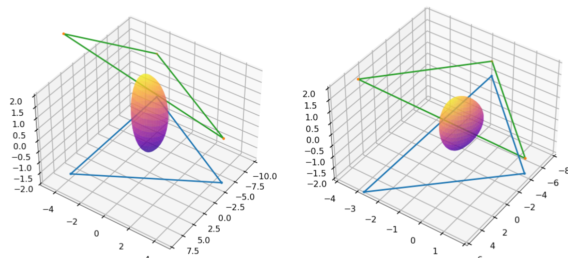

Fig. 26. C^1 smooth description of an ellipsoid like Fig. 25 but replacing the third control point f_3 by $f_3 + (0, 0, 4a_x)$.

References

[1] H. Speleers and D. Toshniwal, "A general class of c1 smooth rational splines: Application to construction of exact ellipses and ellipsoids," *Computer-Aided Design*, vol. 132, p. 102982, 2021.

[2] R. R. Hiemstra, T. J. Hughes, C. Manni, H. Speleers, and D. Toshniwal, "A tchebycheffian extension of multidegree b-splines: algorithmic computation and properties," *SIAM Journal on Numerical Analysis*, vol. 58, no. 2, pp. 1138–1163, 2020.