LECTURE 11

§53. CONDITIONAL DISTRIBUTION AND CONDITIONAL EXPECTATION

One of the most useful concepts in probability theory is that of conditional distribution and conditional expectation. The reason is twofold. First, in practice, we are often interested in calculating probabilities and expectations when some partial information is available. Secondly, in calculating a desired probability or expectation it is often extremely useful to first "condition" on some appropriate random variable.

§53-1. The discrete case.

Recall that for any two events A and B, the conditional probability of A given B is defined, as long as $P(B) > 0$, by

$$P(A/B) = \frac{P(A \cap B)}{P(B)}.$$

Hence, if η_1 and η_2 are discrete random variables, then it is natural to define the conditional probability mass function of η_1 given that $\eta_2 = y$, by

$$p_{\eta_1/\eta_2}(x/y) = P(\eta_1 = x/\eta_2 = y) = \frac{P(\eta_1 = x \cap \eta_2 = y)}{P(\eta_2 = y)} = \frac{p(x,y)}{p_{\eta_2}(y)}$$

for all values of y such that $P(\eta_2 = y) > 0$. Here $p(x, y)$ is the joint probability mass function of (η_1, η_2).

Similarly, the conditional distribution function of η_1 given that $\eta_2 = y$ is defined, for all y such that $P(\eta_2 = y) > 0$, by

$$F_{\eta_1/\eta_2}(x/y) = P(\eta_1 \le x/\eta_2 = y) = \sum_{a \le x} p_{\eta_1/\eta_2}(a/y).$$

Finally, the conditional expectation of η_1 given that $\eta_2 = y$ is defined by

$$E[\eta_1/\eta_2 = y] = \sum_x x\,P(\eta_1 = x/\eta_2 = y) = \sum_x x\,p_{\eta_1/\eta_2}(x/y).$$

In other words, the definitions are exactly as before with the exception that everything is now conditional on the event that $\eta_2 = y$. If η_1 is independent of η_2, then the conditional mass function, distribution, and expectation are the same as the unconditional ones. This follows, since if η_1 is independent of η_2, then

$$p_{\eta_1/\eta_2}(x/y) = P(\eta_1 = x/\eta_2 = y) = \frac{P(\eta_1 = x \cap \eta_2 = y)}{P(\eta_2 = y)} = \frac{P(\eta_1 = x)P(\eta_2 = y)}{P(\eta_2 = y)} = P(\eta_1 = x).$$

Example 89. Suppose that $p(x, y)$, the joint probability mass function of η_1 and η_2, is given by

$$p(0,0) = 0.4, \quad p(0,1) = 0.2, \quad p(1,0) = 0.1, \quad p(1,1) = 0.3.$$

Calculate the conditional probability mass function of η_1 given that $\eta_2 = 1$.

SOLUTION: We first note that

$$p_{\eta_2}(1) = \sum_x p(x,1) = p(0,1) + p(1,1) = 0.5.$$

Hence

$$p_{\eta_1/\eta_2}(0/1) = P(\eta_1 = 0/\eta_2 = 1) = \frac{P(\eta_1 = 0 \cap \eta_2 = 1)}{P(\eta_2 = 1)} = \frac{2}{5}.$$

and

$$p_{\eta_1/\eta_2}(1/1) = P(\eta_1 = 1/\eta_2 = 1) = \frac{P(\eta_1 = 1 \cap \eta_2 = 1)}{P(\eta_2 = 1)} = \frac{3}{5}.$$

§**53-2. The continuous case.**

If η_1 and η_2 have a joint density function $f(x, y)$, then the conditional density function of η_1, given that $\eta_2 = y$, is defined for all values of y such that $f_{\eta_2}(y) > 0$, by

$$f_{\eta_1/\eta_2}(x_1|x_2) = \frac{f(x_1, x_2)}{f_{\eta_2}(x_2)}$$

(see Definition 19).

The conditional expectation of η_1 , given that $\eta_2 = y$, is defined for all values of y such that $f_{\eta_2}(y) > 0$, by

$$E(\eta_1/\eta_2 = y) = \int_{-\infty}^{+\infty} x\, f_{\eta_1/\eta_2}(x|y)\, dx.$$

Example 90. The joint density function of η_1 and η_2 is given by

$$f(x,y) = \begin{cases} \frac{15}{2}x(2-x-y) & \text{if} \quad 0 < x < 1 \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the conditional density of η_1, given that $\eta_2 = y$, where $0 < y < 1$.

SOLUTION: For $0 < x < 1 \quad 0 < y < 1$, we have

$$f_{\eta_1/\eta_2}(x|y) = \frac{f(x,y)}{f_{\eta_2}(y)} = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dx} = \frac{x(2-x-y)}{\int_0^1 x(2-x-y)\,dx} = \frac{x(2-x-y)}{\dfrac{2}{3}-\dfrac{y}{2}} = \frac{6x(2-x-y)}{4-3y}.$$

§54. REGRESSION AND CORRELATION ANALYSIS.

The conditional mean of Y

$$E(Y/X=x) = \int_{-\infty}^{+\infty} yf(y/x)dy$$

is called the regression equation of Y on X.

Example 91. Let X and Y be two random variables with the joint probability density function

$$f(x,y) = \begin{cases} x\,e^{-x(1+y)}, & \text{if} \quad x>0,\ y>0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the regression equation of Y on X.

SOLUTION: The marginal density of X is given by

$$g(x) = \int_0^{\infty} x\,e^{-x(1+y)}\,dy = \int_0^{\infty} x\,e^{-x}\,e^{-x\,y}\,dy = x\,e^{-x}\int_0^{\infty} e^{-x\,y}\,dy = x\,e^{-x}\left[-\frac{1}{x}e^{-x\,y}\right]_0^{\infty} = e^{-x}.$$

The conditional density of Y given $X = x$ is

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{x\,e^{-x(1+y)}}{e^{-x}} = x\,e^{-x\,y}, \quad y>0.$$

The mean of Y given $X = x$ is given by

$$E(Y|x) = \int_{-\infty}^{\infty} y\,f(y|x)\,dy = \int_0^{\infty} y\,x\,e^{-x\,y}\,dy = \frac{1}{x}.$$

Thus the regression equation of Y on X is

$$E(Y|x) = \frac{1}{x}, \quad x>0.$$