Linear Programming

A linear program is an optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where $c \in \mathbb{R}^n$, A is $m \times n$ matrix and $b \in \mathbb{R}^m$. The vector inequality $x \geq 0$ that each component of x is nonnegative.

Examples of Linear Programming Problems

Example (The Diet Problem)

How can we determine the most economical diet that satisfies the basic minimum nutritional requirements for good health? Such a problem might, for example, be faced by the dietitian of a large army. We assume that there are available at the market n different foods and that the j-th food sells at a price c_j per unit. In addition there are m basic nutritional ingredients and, to achieve a balanced diet, each individual must receive at least b_i units of the i-th nutrient per day. Finally, we assume that each unit of food j contains a_{ij} units of the i-th nutrient.

$$
\begin{aligned}
\text{minimize} \quad & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
\text{subject to} \quad & a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \geq b_i, \quad i = \overline{1, m} \\
& x_j \geq 0, \quad j = \overline{1, n}.
\end{aligned}
$$

Two-Dimensional Linear Programs

Example

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0, \end{aligned}$$

where $x = [x_1, x_2]^T$ and $c = [1, 5]^T$, $A = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}$, $b = [30, 12]^T$.

To solve the problem geometrically we start with drawing the feasible set

$$\text{FS} = \left\{(x_1, x_2)^T : 5x_1 + 6x_2 \leq 30,\ 3x_1 + 2x_2 \leq 12,\ x_1 \geq 0, x_2 \geq 0.\right\}$$

Then, we draw several level curves of objective function

$$f(x_1, x_2) = x_1 + 5x_2$$

to see how large we can make f while satisfying the constraints.

We see that the maximizer is $(0, 5)^T$.

If as objective function we take

- $f(x_1, x_2) = 5x_1 + 4x_2$ as maximizer we get $(1.5, 3.75)^T$;
- $f(x_1, x_2) = 5x_1 + 6x_2$ as maximizers we get $(0, 5)^T$ and $(1.5, 3.75)^T$ and all points on the line segment between thees two points;
- $f(x_1, x_2) = 2x_1 + x_2$ as maximizer we get $(6, 0)^T$.

Example

Solve the following Linear Programming Problem (LPP) graphically

$$\begin{aligned} \text{minimize} \quad & -x_1 - 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 5 \\ & 4x_2 - 3x_1 \geq -4 \\ & x_2 - 6x_1 \leq 6 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

As we see it is possible that LPP doesn't posses a solution. It is possible when the feasible set is unbounded.

Geometric View of Linear Programs

Recall that a set $\Omega \in \mathbb{R}^n$ is a convex set if $\forall x, y \in \Omega$ and $\forall \in [0,1]$ the point $\alpha x + (1-\alpha)y \in \Omega$.

Proposition. Let A be $m \times n$ matrix, $b \in \mathbb{R}^m$. The set of points $\Omega = \{x : Ax = b, x \geq 0\}$ is a convex set.

Definition

A point x in a convex set Ω is said to be an extreme point or corner point of Ω if there are no $x_1, x_2 \in \Omega$, $x_1 \neq x_2$ and $\alpha \in (0,1)$ such that $x = \alpha x_1 + (1-\alpha)x_2$.

Proposition. We need to look for the solution of LPP (if exists) among extreme points of the feasible set.

Proposition. If the feasible set is bounded, then LPP has a solution and, moreover, one of the extreme points will be a solution.

Duality

The dual problem is constructed from the cost function and constraints. The solution of the dual problem can be obtained from the solution of the primal problem and vice versa. Solving an LP problem via its dual can be simpler in some cases.

Suppose we are given a linear programming problem of the form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0. \end{aligned}$$

We refer to the above as *primal problem*. We define the corresponding *dual problem* as

$$\begin{aligned} \text{maximize} \quad & w^T b \\ \text{subject to} \quad & w^T A \leq c^T \\ & w \geq 0. \end{aligned}$$

The form of duality defined above is called the *symmetric form of duality*.

Example

Form the dual LPP of the following problem

$$\begin{aligned} \text{minimize} \quad & 3x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 \geq 2 \\ & x_1 + 2x_2 + 3x_3 \geq 5 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned}$$

To define the dual of an arbitrary linear programming problem we need to convert it into an equivalent problem of the primal form, then construct the corresponding dual problem.

Proposition. The dual of the dual problem is the primal problem.

Now let's consider an LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

The corresponding *dual problem* will be

$$\begin{aligned} \text{maximize} \quad & w^T b \\ \text{subject to} \quad & w^T A \leq c^T. \end{aligned}$$

The form of duality above is referred to as the *asymmetric form of duality*.

Example

Form the dual LPP of the following problem

$$
\begin{aligned}
\text{minimize} \quad & 2x_1 + x_2 + 4x_3 \\
\text{subject to} \quad & x_1 + x_2 + 2x_3 = 3 \\
& 2x_1 + x_2 + 3x_3 = 5 \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.
\end{aligned}
$$