The Convergence Rate of Numerical Sequence

Definition

A sequence x_n exhibits **linear** convergence to a limit x if there is a constant C in the interval $(0\ ,\ 1)$ and an integer N such that

$$|x_{n+1} - x| \leq C|x_n - x|, \quad \forall n \geq N.$$

Example

$$x_n = \frac{1}{2^n}.$$

Definition

A sequence x_n exhibits **superlinear** convergence to a limit x if there is a sequence β_n, which converges to 0, and an integer N such that

$$|x_{n+1} - x| \leq \beta_n |x_n - x|, \quad \forall n \geq N.$$

Example

$$x_n = \frac{n}{2^{n^2}} + 1.$$

Definition

We will say that $\alpha \geq 1$ is the rate of convergence of sequence x_n if α is the largest number for which there exist a constant $C > 0$ (if $\alpha = 1$ then $0 < C < 1$) and an integer N such that

$$|x_{n+1} - x| \leq C|x_n - x|^{\alpha}, \quad \forall n \geq N.$$

Example

Find the limit and the rate of convergence to that limit for the following sequences:

a. $x_n = \frac{1}{2^{2^n}}$;

b. $x_n = \frac{1}{2^{3^n} + n}$;