Problems with inequality constraints

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h_i(x) = 0, \quad i = 1, \dots m, \\ & g_j(x) \leq 0, \quad j = 1, \dots p \end{aligned}$$

where $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots m$, $m \leq n$ and $g_j : \mathbb{R}^n \to \mathbb{R}$ for $j = 1, \dots p$ are given functions.

Definition

Assume x^* is a feasible point. An inequality constraint $g_j(x) \leq 0$ is said to be active at x^* if $g_j(x^*) = 0$. It is inactive at x^* if $g_j(x^*) < 0$.

Definition

Let x^* be a feasible point, i.e. $h_i(x^*) = 0$, $i = 1, \dots m$ and $g_j(x^*) \leq 0$, $j = 1, \dots p$ and let $J(x^*)$ be the index set of active inequality constraints

$$J(x^*) = \left\{ j : g_j(x^*) = 0 \right\}.$$

We will say that x^* is a regular point if the vectors

$$\nabla h_i(x^*),\ i = 1, \dots m, \nabla g_j(x^*),\ j \in J(x^*)$$

are linearly independent.

Example

Consider the following constraints on $\mathbb{R}^2$

$$h(x_1, x_2) = x_1 - 2 = 0 \quad \text{and} \quad g(x_1, x_2) = (x_2 + 1)^3 \leq 0.$$

Find the set of feasible points. Are the feasible points regular?

Let's introduce the Lagrangian function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$, given by

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{p} \mu_j g_j(x).$$

Theorem (Karush-Kuhn-Tucker (KKT) Theorem)

Let x^* be a local minimizer of $f : \mathbb{R}^n \to \mathbb{R}$ subject to $h(x) = 0$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$, $g(x) \leq 0$, $g : \mathbb{R}^n \to \mathbb{R}^p$. Assume f, h_i, $i = 1, \dots m$ and g_j, $j = 1, \dots p$ are continuously differentiable functions and x^* is a regular point. Then, there exist $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ such that:

- $\mu_j^* \geq 0$, *for* $j = 1, \dots p$,
- $\frac{\partial \mathcal{L}}{\partial x_i}(x^*, \lambda^*, \mu^*) = 0$, *for* $i = 1, \dots n$,
- $\mu_j^* g_j(x^*) = 0$, *for* $j = 1, \dots p$.

Example

Consider the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \geq 5, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

Is it possible that the point $x^* = [2, 3]^T$ is a local minimizer of the formulated problem?

Theorem (Second Order Necessary Condition SONC)

Let x^* be a local minimizer of $f : \mathbb{R}^n \to \mathbb{R}$ subject to $h(x) = 0$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $m \le n$, $g(x) \le 0$, $g : \mathbb{R}^n \to \mathbb{R}^p$. Assume f, h_i, $i = 1, \ldots m$ and g_j, $j = 1, \ldots p$ are twice continuously differentiable functions and x^* is a regular point. Then, there exist $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ such that

- $\mu_j^* \ge 0$, *for* $j = 1, \ldots p$,
- $\frac{\partial \mathcal{L}}{\partial x_i}(x^*, \lambda^*, \mu^*) = 0$, *for* $i = 1, \ldots n$,
- $\mu_j^* g_j(x^*) = 0$, *for* $j = 1, \ldots p$,
- $\nabla^2 \mathcal{L}(x^*, \lambda^*, \mu^*)$ *is positive semidefinite on* $TS(x^*)$

  $$TS(x^*) = \left\{ y \in \mathbb{R}^n : y^T \nabla h_i(x^*) = 0, i = \overline{1, m}, y^T \nabla g_j(x^*) = 0, j \in J(x^*) \right\},$$

  i.e. for all $y \in TS(x^*)$, $y^T \nabla^2 \mathcal{L}(x^*, \lambda^*, \mu^*) y \ge 0$.

Example

Use SONC to find all possible local minimizers of the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \geq 5, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

$$J(x^*, \mu^*) = \{j : g_j(x^*) = 0, \mu_j^* > 0\}$$

$$TS(x^*, \mu^*) = \{y \in \mathbb{R}^n : y^T \nabla h_i(x^*) = 0, i = \overline{1, m}, y^T \nabla g_j(x^*) = 0, j \in J(x^*, \mu_j^*)\}$$

Theorem (Second Order Sufficient Condition SOSC)

Suppose f, h_i, $i = \overline{1, m}$, $g_j(x)$, $j = \overline{1, p}$ are twice continuously differentiable functions on $\mathbb{R}^n$ and there exist a point $x^* \in \mathbb{R}^n$ satisfying $h_i(x^*) = 0$, $i = \overline{1, m}$, $g_j(x^*) \leq 0$, $j = \overline{1, p}$, $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ such that:

- $\mu_j^* \geq 0$ *for* $j = 1, \ldots p$,
- $\frac{\partial \mathcal{L}}{\partial x_i}(x^*, \lambda^*, \mu^*) = 0$ *for* $i = 1, \ldots n$,
- $\mu_j^* g_j(x^*) = 0$ *for* $j = 1, \ldots p$,
- $\nabla^2 \mathcal{L}(x^*, \lambda^*, \mu^*)$ *is positive definite on* $TS(x^*, \mu^*)$, *i.e. for all* $y \in TS(x^*, \mu^*)$, $y \neq 0$, $y^T \nabla^2 \mathcal{L}(x^*, \lambda^*, \mu^*) y > 0$,

then x^* is a strict local minimizer of f subject to $h_i(x) = 0$, $i = 1, \ldots, m$ and $g_j(x) \leq 0$, $j = \overline{1, p}$.

Example

Consider the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & (x_1 - 1)^2 + x_2 + e^{x_3^2} \\ \text{subject to} \quad & x_2 - x_1 = 1, \\ & x_1 + x_2 \leq 2, \\ & x_3 \geq 0. \end{aligned}$$