

Fibonacci Search

Assume $f : [a, b] \rightarrow \mathbb{R}$ is unimodal and continuous on interval $[a, b]$.

Without any evaluation of function f we can take the midpoint \hat{x} of $[a, b]$ as approximation of the minimum point x^* and in that case

$$|\hat{x} - x^*| \leq \frac{b - a}{2}.$$

If we evaluate f at one point, again the best we can do is to take the midpoint \hat{x} as an approximation of x^* .

Assume f is evaluated at two points A and B .

If $f(A) > f(B)$ then $x^* \in [A, b]$.

If $f(A) \leq f(B)$ then $x^* \in [a, B]$.

Let's take

$$A = \frac{b+a}{2} - 2\delta,$$

$$B = \frac{b+a}{2} + 2\delta.$$

Then we take the midpoint \hat{x} of appropriate subinterval $[A, b]$ or $[a, B]$ as approximation of x^* and

$$|\hat{x} - x^*| \leq \frac{b-a}{4} + \delta.$$

Assume f is evaluated at three points.

$$A = a + \frac{b - a}{3},$$

$$B = b - \frac{b - a}{3}.$$

If $f(A) > f(B)$ then $x^* \in [A, b]$.

If $f(A) \leq f(B)$ then $x^* \in [a, B]$.

Assume $x^* \in [A, b]$

Next evaluation is made at $B + \delta$.

If $f(B) > f(B + \delta)$ then $x^* \in [B, b]$. We take as approximation the midpoint of $[B, b]$ and

$$|\hat{x} - x^*| \leq \frac{b - a}{6}.$$

If $f(B) \leq f(B + \delta)$ then $x^* \in [a, B + \delta]$. We take as approximation the midpoint of $[a, B + \delta]$ and

$$|\hat{x} - x^*| \leq \frac{b - a}{6} + \frac{\delta}{2}.$$

Fibonacci Sequence

$$F_0 = 1, \quad F_1 = 1,$$

$$F_k = F_{k-1} + F_{k-2}, \quad k \geq 2.$$

By continuing the search pattern outlined, we find an estimate \hat{x} with only n evaluations of f and with an error

$$|\hat{x} - x^*| \leq \frac{b - a}{2F_n} + \frac{\delta}{2}.$$

Let's denote $[a_0, b_0] = [a, b]$.

$$\gamma_0 = \frac{F_{n-2}}{F_n}, \quad n \geq 3.$$

$$A_0 = a_0 + \gamma_0(b_0 - a_0)$$

$$B_0 = b_0 - \gamma_0(b_0 - a_0)$$

$$[a_1, b_1] = \begin{cases} [a_0, B_0], & \text{if } f(A_0) < f(B_0), \\ [A_0, b_0], & \text{if } f(A_0) \geq f(B_0). \end{cases}$$

$$x^* \in [a_1, b_1]$$

$$b_1 - a_1 = \frac{F_{n-1}}{F_n}(b - a)$$

$$\gamma_1 = \frac{F_{n-3}}{F_{n-1}}$$

$$A_1 = a_1 + \gamma_1(b_1 - a_1)$$

$$B_1 = b_1 - \gamma_1(b_1 - a_1)$$

If $[a_1, b_1] = [a_0, B_0]$, $B_1 = A_0$.

If $[a_1, b_1] = [A_0, b_0]$, $A_1 = B_0$.

$$\gamma_k = \frac{F_{n-2-k}}{F_{n-k}}, \quad k = 0, 1, \dots, n-3.$$

$$A_k = a_k + \gamma_k(b_k - a_k)$$

$$B_k = b_k - \gamma_k(b_k - a_k)$$

$$[a_{k+1}, b_{k+1}] = \begin{cases} [a_k, B_k], & \text{if } f(A_k) < f(B_k), \\ [A_k, b_k], & \text{if } f(A_k) \geq f(B_k). \end{cases}$$

$$x^* \in [a_{k+1}, b_{k+1}]$$

$$b_{k+1} - a_{k+1} = \frac{F_{n-1-k}}{F_n}(b - a)$$

At the end of step $k = n - 3$ we have an interval $[a_{n-2}, b_{n-2}]$ with a length of $b_{n-2} - a_{n-2} = \frac{2}{F_n}(b - a)$. At this step we made $n - 1$ evaluations.

Then we need to evaluate f at a point δ away from the midpoint of $[a_{n-2}, b_{n-2}]$.

As approximation we take the midpoint of final interval.

Bisection Method

Assume $f : [a, b] \rightarrow \mathbb{R}$ is a unimodal and continuously differentiable function on $[a, b]$ and $f'(a)f'(b) < 0$.

Let's denote

$$[a_0, b_0] = [a, b]$$

and

$$x_0 = \frac{a_0 + b_0}{2}.$$

If $f'(x_0) = 0$, then we stop here.

$$[a_1, b_1] = \begin{cases} [a_0, x_0], & \text{if } f'(x_0) > 0, \\ [x_0, b_0], & \text{if } f'(x_0) < 0. \end{cases}$$

n -th step

$$[a_n, b_n] = \begin{cases} [a_{n-1}, x_{n-1}], & \text{if } f'(x_{n-1}) > 0, \\ [x_{n-1}, b_{n-1}], & \text{if } f'(x_{n-1}) < 0. \end{cases}$$

$$x_n = \frac{a_n + b_n}{2}.$$

If at n -th step $f'(x_n) = 0$, then we terminate our search.

Stopping conditions for the Bisection Method

- $|x_n - x_{n-1}| < \varepsilon$ or $\frac{|x_n - x_{n-1}|}{|x_{n-1}|} < \varepsilon$, if $x_{n-1} \neq 0$
- $|f'(x_n)| < \varepsilon$
- $|f(x_n) - f(x_{n-1})| < \varepsilon$ or $\frac{|f(x_n) - f(x_{n-1})|}{|f(x_{n-1})|} < \varepsilon$, if $f(x_{n-1}) \neq 0$

Example

Calculate the second approximation x_2 of the bisection method for the function $f(x) = -\frac{x^3}{3} + 2x$ on the interval $[-4, 0]$.

Gradient Descent

Assume $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function on \mathbb{R} .

To define the gradient descent algorithm we need to choose a parameter $h > 0$ called step-size. After choosing a starting point $x_0 \in \mathbb{R}$, we generate a sequence using the rule

$$x_{n+1} = x_n - hf'(x_n), \quad n = 0, 1, \dots$$

Theorem

Let x^* be a local minimizer of f . Assume that for some $\varepsilon > 0$,

- the initial point x_0 lies in the interval $[x^* - \varepsilon, x^* + \varepsilon]$,
- x^* is the unique local minimizer of f in $[x^* - \varepsilon, x^* + \varepsilon]$,
- $f'(x)$ is Lipschitz in $[x^* - \varepsilon, x^* + \varepsilon]$ with constant M , that is $|f'(x) - f'(y)| \leq M(x - y)$, $\forall x, y \in [x^* - \varepsilon, x^* + \varepsilon]$,
- f is m -strongly convex in $[x^* - \varepsilon, x^* + \varepsilon]$, that is $f(x) \geq f(y) + f'(y)(x - y) + \frac{m}{2}(x - y)^2$, $\forall x, y \in [x^* - \varepsilon, x^* + \varepsilon]$,

then for the step-size $h \leq \frac{1}{M}$, all the iterates x_k of the gradient descent algorithm lie in the interval $[x^* - \varepsilon, x^* + \varepsilon]$ and

$$|x_k - x^*|^2 \leq \frac{2(f(x_0) - f(x^*))}{m}(1 - hm)^k.$$

Stopping conditions

- $|x_n - x_{n-1}| < \varepsilon$ or $\frac{|x_n - x_{n-1}|}{|x_{n-1}|} < \varepsilon$, if $x_{n-1} \neq 0$
- $|f'(x_n)| < \varepsilon$
- $|f(x_n) - f(x_{n-1})| < \varepsilon$ or $\frac{|f(x_n) - f(x_{n-1})|}{|f(x_{n-1})|} < \varepsilon$, if $f(x_{n-1}) \neq 0$