Numerical Methods for Unconstrained Optimization

One Dimensional Search Methods

Here we consider the minimization of univariate function $f : [a, b] \to \mathbb{R}$.

In an iterative algorithm we start with an initial candidate solution x_0 and generate sequence of points x_1, x_2, ... Each x_{k+1} iteration depends on previous points $x_0, x_1, \ldots, x_k$. The algorithm may also use the value of f or f' or even f'' at some points:

- Golden section method (uses only f);
- Fibonacci method (uses only f);
- Bisection method (uses only f');
- Gradient descent (uses only f');
- Newton's method (uses f' and f'').

Definition

The function $f : [a, b] \to \mathbb{R}$ is called a unimodal function on interval $[a, b]$, if f has only one local minimizer on $[a, b]$.

Example

Check if the function f is unimodal on the given interval, if

a. $f(x) = \sin(x)$, $x \in [\pi/2, 2\pi]$;

b. $f(x) = \sin(x)$, $x \in [0, 2\pi]$;

c. $f(x) = \frac{x^5}{5} - x^3$, $x \in [-5, 2]$.

If $f : [a, b] \to \mathbb{R}$ is continuous and unimodal on $[a, b]$, then f is strictly decreasing up to the minimum point x^* and increasing thereafter.

Golden Section Search

Assume $f : [a, b] \to \mathbb{R}$ is unimodal and continuous on interval $[a, b]$. Let x^* be the minimum point of f over $[a, b]$.

Let's denote

$$[a_0, b_0] = [a, b].$$

$$A = a_0 + \gamma(b_0 - a_0),$$

$$B = b_0 - \gamma(b_0 - a_0),$$

where $\gamma \in \left(0, \frac{1}{2}\right)$.

We define the new interval in the following way

$$[a_1, b_1] = \begin{cases} [a_0, B], & \text{if } f(A) < f(B), \\ [A, b_0], & \text{if } f(A) \geq f(B). \end{cases}$$

$$x^* \in [a_1, b_1]$$

$$b_1 - a_1 = (1 - \gamma)(b_0 - a_0).$$

The first approximation will be

$$x_1 = \frac{a_1 + b_1}{2}.$$

n-th step

$$[a_n, b_n] = \begin{cases} [a_{n-1}, B], & \text{if } f(A) < f(B), \\ [A, b_{n-1}], & \text{if } f(A) \geq f(B). \end{cases}$$

$$x^* \in [a_n, b_n]$$

$$b_n - a_n = (1 - \gamma)(b_{n-1} - a_{n-1})$$

The *n*-th approximation will be

$$x_n = \frac{a_n + b_n}{2}.$$

Theorem

If f is a unimodal and continuous function on $[a, b]$ and $\gamma \in \left(0, \frac{1}{2}\right)$, then the golden section search approximation x_n converges to x^* and we have

$$|x_n - x^*| \leq 0.5(1-\gamma)^n(b-a).$$

Example

Calculate the second approximation x_2 of the golden section search method with $\gamma = \frac{1}{3}$ for function $f(x) = \frac{x^3}{3} - 2x$ on the interval $[-3, 0]$.

In general case at each step we calculate the value of f at two points: A and B.

It is possible to find $\gamma \in \left(0, \frac{1}{2}\right)$ such that we evaluate f at two points at first step but at each next step we evaluate f at one point.

Let's assume that at the step n $f(A) < f(B)$, which means that our new interval $[a_n, b_n]$ is going to be $[a_{n-1}, B]$.

$$\begin{aligned} a_{n-1} + \gamma(b_{n-1} - a_{n-1}) &= b_n - \gamma(b_n - a_n) \\ &= b_{n-1} - \gamma(2-\gamma)(b_{n-1} - a_{n-1}) \end{aligned}$$

This implies

$$\gamma^2 - 3\gamma + 1 = 0$$

and

$$\gamma = \frac{3-\sqrt{5}}{2} \approx 0.382.$$

$$|x_n - x^*| \leq 0.5 \left(\frac{\sqrt{5}-1}{2} \right)^n (b-a).$$