- **Main Textbooks:**

ChZ Edwin K. P. Chong, Stanislaw H. Zak, An Introduction to Optimization, 4th Ed, *Volume 76 of Wiley Series in Discrete Mathematics and Optimization*, John Wiley & Sons, 2013

HL Frederick S. Hillier, Gerald J. Lieberman, *Introduction to Operations Research*, 10th Ed, McGraw-Hill Education, 2015

BJSh Mokhtar S. Bazaraa, John J. Jarvis, Hanif D. Sherali, *Linear Programming and Network Flows*, John Wiley & Sons, 2010

BV Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*, Cambridge University Press, available on-line at http://stanford.edu/ boyd/cvxbook/

- **Additional Textbooks:**

LY David G. Luenberger, Yinyu Ye, *Linear and Nonlinear Programming*, Springer International Publishing, 2016

NW Jorge Nocedal, Stephen J. Wright , *Numerical Optimization*, Springer, 2006

V Robert Vanderbei, *Linear Programming Foundations and Extensions*, 4th Ed, Springer Science & Business Media, 2013

F Roger Fletcher, *Practical Methods of Optimization*, Second Edition, John Wiley & Sons, 2013

Introduction

We are going to consider the following problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, with $n \geq 1$.

The function f that we wish to minimize is called the **objective function** or **cost function**.
The vector x is an $n-$vector of independent variables: $x = [x_1, x_2, ..., x_n]^T \in \mathbb{R}^n$. The variables $x_1, x_2, ..., x_n$ are often referred to as **decision variables**.
The set Ω is called the **constraint set** or **feasible set**.
If Ω is a proper subset of $\mathbb{R}^n$ then we have **constrained optimization problem**.
If $\Omega = \mathbb{R}^n$ then we have **unconstrained optimization problem**.

Classification of Optimization Problems

- *Discrete Optimization Problem*
  Knapsack Problem. Given a set of items, each with a weight and a value, determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit and the total value is as large as possible.
  The Mathematical Model. Let the weight of $i-$th item be w_i and the value v_i . W is the maximum weight capacity of knapsack.

$$\begin{array}{rl} \text{maximize} & \sum_{i=1}^{n} x_i v_i \\ \text{subject to} & \sum_{i=1}^{n} x_i w_i \leq W, \\ & x_i \in \mathbb{Z}_+, \quad i = 1, ..., n. \end{array}$$

The defining feature of *Discrete Optimization Problem* is that the unknown x is drawn from a finite (but often very large) set.

- *Continuous Optimization Problem*
  Transportation Problem. A chemical company has 2 factories F_1 and F_2 and a dozen retail outlets R_1, R_2,..., R_{12}. Each factory can produce certain amount of chemical product each week which is called the capacity of the plant. Each retail outlet R_j has a known weekly demand. The problem is to determine how much of the product to ship from each factory to each outlet so as to satisfy all the requirements and minimize cost
  The Mathematical Model. Let the capacity of $i-$th factory be a_i and the demand of $j-$th retail office b_j. c_{ij} is the cost of shipping one ton of the product from factory F_i to retail outlet R_j. x_{ij} is the number of tons of the product shipped from factory F_i to retail outlet R_j

$$
\begin{array}{rl}
\text{minimize} & \sum_{i=1}^{2} \sum_{j=1}^{12} x_{ij} c_{ij} \\
\text{subject to} & \sum_{j=1}^{12} x_{ij} \leq a_i, \quad i = 1, 2 \\
& \sum_{i=1}^{2} x_{ij} \geq b_j, \quad j = 1, 2, ..., 12, \\
& x_{ij} \geq 0 \quad i = 1, 2, \quad j = 1, 2, ..., 12.
\end{array}
$$

In continuous optimization, the variables in the model are nominally allowed to take on a continuous range of values, usually real numbers. The feasible set for continuous optimization problems is usually uncountably infinite.

Topics

- One Dimensional Optimization
- Unconstrained Finite-Dimensional Optimization
- One-Dimensional Search Methods
- Unconstrained Finite-Dimensional Optimization: Numerical Solution
- Nonlinear Programming
- Linear Programming
- Integer and Combinatorial Programming

One Dimensional Optimization

Assume $f : (a, b) \to \mathbb{R}$. Our aim is to solve the following problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in (a, b). \end{aligned} \tag{1}$$

To solve the problem (1) means to find the global minimum point of $f(x)$ in (a, b).

During this process we face following questions:

1. Existence of solution.
2. Uniqueness of solution.
3. If exists, how to find that solution analytically?
4. If it is not possible to find the solution analytically how to find it numerically?
5. Is the solution stable? Stability won't be discussed in this course.

1. Existence

Weierstrass Extreme Value Theorem

If $f \in \mathbb{C}[a, b]$, then f attains a minimum at least once.

Note. In EVT we have $[a, b]$ (closed, bounded interval) not (a, b).

2. Uniqueness

Convexity helps

If f is strictly convex on (a, b) and has a global minimum, then the global minimum is unique.

If f is strictly convex on (a, b) and has a local minimizer, then the local minimum is the unique global minimum.

3. Determination of solution

Fermat's Theorem

If $x_0 \in (a, b)$ is a local minimum point and f is differentiable at x_0, then $f'(x_0) = 0$.
According to Fermat's Theorem we need to seek the solution among the following points

a. critical points,
b. endpoints.

This is a necessary condition but we need sufficient conditions.

Assume f is differentiable in some δ neighborhood of x_0 and x_0 is a critical point, then

a. If $f'(x) < 0$ on $(x_0 - \delta, x_0)$ and $f'(x) > 0$ on $(x_0, x_0 + \delta)$, then x_0 is a local minimum point.

b. If $f'(x) > 0$ on $(x_0 - \delta, x_0)$ and $f'(x) < 0$ on $(x_0, x_0 + \delta)$, then x_0 is a local maximum point.

c. If $f'(x)$ has the same sign on $(x_0 - \delta, x_0)$ and $(x_0, x_0 + \delta)$, then x_0 is not an extreme point.

Assume f is twice differentiable in some neighborhood of x_0 and x_0 is a critical point, then

a. If $f''(x_0) > 0$, then x_0 is a local minimum point.

b. If $f''(x_0) < 0$, then x_0 is a local maximum point.

c. If $f''(x_0) = 0$, then nothing can be said about x_0.

4. Numerical solution

   We can use some methods e.g. Fibonacci, Golden Ratio Search methods.

Finite-Dimensional Optimization

We are going to consider the following problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega, \end{aligned} \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, with $n \geq 1$.

Definition

A point $x^* \in \Omega$ is a **local minimizer** of f over Ω if there exists $\varepsilon > 0$ such that $f(x) \geq f(x^*)$ for all $x \in \Omega \backslash \{x^*\}$ and $||x - x^*|| < \varepsilon$. A point $x^* \in \Omega$ is a **global minimizer** of f over Ω if $f(x) \geq f(x^*)$ for all $x \in \Omega \backslash \{x^*\}$.

If in the definitions above we replace "$\geq$" with "$>$" then we have a **strict local minimizer** and a **strict global minimizer**, respectively.

If x^* is a global minimizer of f over Ω, we write $f(x^*) = \min_{x \in \Omega} f(x)$and $x^* = \arg\min_{x \in \Omega} f(x)$. If the minimization is unconstrained, we simply write $x^* = \arg\min_x f(x)$ or $x^* = \arg\min f(x)$

Existence of solution

Weierstrass Extreme Value Theorem

If $f \in \mathbb{C}(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is compact, then the problem (2) has a solution.

Definition

A point $x \in \mathbb{R}^n$ is said to be a **limit point** of $\Omega \subset \mathbb{R}^n$, if each neighborhood of x contains a point of Ω other than x.

Example

Let $\Omega = [0, 3)$. Is x a limit point of Ω?

a. $x = 0$

b. $x = 3$

c. $x = 2$

Example

Let $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1 \quad \text{and} \quad x_1 > 0\}$. Is x a limit point of Ω, if

a. $x = [0, 0]^T$;

b. $x = [1, 0]^T$.

Definition

A set $\Omega \subset \mathbb{R}^n$ is said to be **closed set** if it contains all its limit points.

Example

Check if the set Ω is a closed set, if

a. $\Omega = [0, 3);$

b. $\Omega = [0, 3];$

c. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\};$

d. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1 \quad \text{and} \quad x_1 > 0\}.$

Definition

A set $\Omega \subset \mathbb{R}^n$ is said to be **bounded** if there exists $M \in \mathbb{R}$ such that $||x|| \leq M$, for all $x \in \Omega$.

Example

Check if the set Ω is bounded, if

a. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\};$

b. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \geq 1 \quad \text{and} \quad x_1 > 0\}.$

Definition

A set $\Omega \subset \mathbb{R}^n$ is said to be **compact** if Ω is closed and bounded.

Example

Check if the set Ω is compact, if

a. $\Omega = [0, 3)$;

b. $\Omega = [0, 3]$;

c. $\Omega = \{x = [x_1, x_2, x_3]^T : x_1^2 + x_2^2 + x_3^2 \leq 1\}$;

d. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \geq 1 \quad \text{and} \quad x_1 > 0\}$.

Uniqueness of solution

Definition

A set $\Omega \subset \mathbb{R}^n$ is a **convex set** if $\alpha x + (1-\alpha)y \in \Omega$, $\forall x, y \in \Omega$ and $\forall \alpha \in [0, 1]$.

Example

Check if the set Ω is convex, if

a. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\}$;

b. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1 \quad \text{and} \quad x_1 x_2 \geq 0\}$.

Definition

A function $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ is a **convex function** if Ω is a convex set and $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$, $\forall x, y \in \Omega$, $x \neq y$, $\forall \alpha \in (0, 1)$.

If in the definition above we replace "$\leq$" with "$<$", then we have the definition of **strict convex function.**

Definition

A function f is a **concave function** if $-f$ is convex.

Definition

A function f is a **strictly concave function** if $-f$ is strictly convex.