Line Search in Multidimensional Optimization

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function that we wish to minimize.

Iterative algorithms for finding a minimizer of f are of the form

$$x^{(k+1)} = x^{(k)} + \alpha_k d^{(k)}, \quad k = 0, 1, \ldots,$$

where x_0 is the initial approximation and $\alpha_k \geq 0$ is called step-size and $d^{(k)} \in \mathbb{R}^n$ is called search direction.

At each iteration we face two problems:

- first, we need to choose the search direction
- second, we need to choose the step size α_k when $d^{(k)}$ is fixed

There are two methods for choosing α_k:

- exact line search i.e. find the minimum point of $\Phi_k(\alpha) = f\left(x^{(k)} + \alpha_k d^{(k)}\right)$, $\alpha \geq 0$.
- we need to choose α_k to ensure that $f(x^{(k+1)}) < f(x^{(k)})$ but α_k shouldn't be too short or too long.

Choosing the step size

Assume we use a descent direction $d^{(k)}$ i.e. $\nabla f\left(x^{(k)}\right)^T d^{(k)} < 0$.

Let $\varepsilon \in (0,1)$, $\gamma > 1$ and $\eta \in (\varepsilon, 1)$.

One usually takes $\varepsilon = 10^{-4}$.

The *Armijo condition* ensures that α_k is not too large by requiring that

$$\Phi_k(\alpha_k) \leq \Phi_k(0) + \varepsilon \alpha_k \Phi_k'(0)$$

or

$$f\left(x^{(k)} + \alpha_k d^{(k)}\right) \leq f\left(x^{(k)}\right) + \varepsilon \alpha_k \nabla f\left(x^{(k)}\right)^T d^{(k)}.$$

It also ensures that α_k is not too short by requiring that

$$\Phi_k(\gamma \alpha_k) \geq \Phi_k(0) + \varepsilon \gamma \alpha_k \Phi_k'(0)$$

or

$$f\left(x^{(k)} + \gamma \alpha_k d^{(k)}\right) \geq f\left(x^{(k)}\right) + \varepsilon \gamma \alpha_k \nabla f\left(x^{(k)}\right)^T d^{(k)}.$$

The *Goldstein condition* (Armijo-Goldstein):

$$\Phi_k(\alpha_k) \leq \Phi_k(0) + \varepsilon \alpha_k \Phi_k'(0),$$

$$\Phi_k(\alpha_k) \geq \Phi_k(0) + \eta \alpha_k \Phi_k'(0).$$

The *Wolfe condition*:

$$\Phi_k(\alpha_k) \leq \Phi_k(0) + \varepsilon \alpha_k \Phi_k'(0),$$

$$\Phi_k'(\alpha_k) \geq \eta \Phi_k'(0).$$

The *strong Wolfe condition*:

$$\Phi_k(\alpha_k) \leq \Phi_k(0) + \varepsilon \alpha_k \Phi_k'(0),$$

$$\left|\Phi_k'(\alpha_k)\right| \leq \eta \left|\Phi_k'(0)\right|.$$

Armijo backtracking algorithm to chose the step size α_k

- Step 1: We start with some candidate value $\alpha_k^{(0)}$ for the step size α_k. Take a constant factor $\tau \in (0, 1)$ (typically $\tau = 0.5$) and $\ell = 0$.
- Step 2: If $\alpha_k^{(\ell)}$ satisfies a prespecified termination condition (usually the first Armijo inequality) then return $\alpha_k^{(\ell)}$ for α_k. If the condition is not satisfied, then take

$$\alpha_k^{(\ell+1)} = \tau \alpha_k^{(\ell)},$$

$$\ell \longmapsto \ell + 1$$

and do the Step 2.

Example

Assume we want to find the minimizer of

$$f(x_1, x_2) = 2x_1^2 + x_2^2,$$

using the line search method. We start with $\left(x_1^{(0)}, x_2^{(0)}\right) = (1, 1)$ and as search direction we take $d^{(0)} = -\nabla f\left(x^{(0)}\right)$. In order to calculate the next approximation $\left(x_1^{(1)}, x_2^{(1)}\right)$ we need a step size α_0 which we are going to find by using Armijo backtracking algorithm. In Armijo backtracking algorithm let's take $\alpha_0^{(0)} = 2$, $\tau = 0.5$ and $\varepsilon = 0.1$. Then we calculate $\left(x_1^{(1)}, x_2^{(1)}\right)$.