Definition

Assume $f : \Omega \to \mathbb{R}$, $\Omega \to \mathbb{R}^n$ and $x_0 \in \Omega$. If all second order partial derivatives of f exist at x_0, then the following matrix

$$\nabla^2 f(x_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x_0) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x_0) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_0) & \frac{\partial^2 f}{\partial x_2^2}(x_0) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x_0) \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x_0) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x_0) & \dots & \frac{\partial^2 f}{\partial x_n^2}(x_0) \end{pmatrix}.$$

is called the Hessian matrix of f at x_0.

If second order partial derivatives of f are all continuous at x_0 then Hessian matrix is symmetric.

Example

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x_1, x_2, x_3) = x_1^3 + 4x_2^2 x_1 + \sin(x_3)$. Compute the Hessian matrix $\nabla^2 f(x_0)$ at $x_0 = (1, 1, 0)^T$.

Definition

Assume $A = \left[a_{ij}\right]_{i,j=1}^{n}$ is a $n \times n$ symmetric matrix, i.e. $a_{ij} = a_{ji}$. A function $QF_A : \mathbb{R}^n \to \mathbb{R}$ is called a **quadratic form** associated to the matrix A if

$$QF_A(h) = h^T A h = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} h_i h_j.$$

Example

Construct the quadratic form associated to the matrix A if

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 5 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Definition

We will say that the symmetric matrix A or the quadratic form QF_A is

- **positive definite** if $QF_A(h) > 0$, $\forall h \in \mathbb{R}^n$ and $h \neq 0$;
- **positive semidefinite** if $QF_A(h) \geq 0$, $\forall h \in \mathbb{R}^n$;
- **negative definite** if $QF_A(h) < 0$, $\forall h \in \mathbb{R}^n$ and $h \neq 0$;
- **negative semidefinite** if $QF_A(h) \leq 0$, $\forall h \in \mathbb{R}^n$;
- **indefinite** if there exist $h_1, h_2 \in \mathbb{R}^n$ such that $QF_A(h_1) > 0$ and $QF_A(h_2) < 0$.

Example

Determine whether the matrix A is positive definite (semidefinite), negative definite (semidefinite) or indefinite if

a.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix};$$

b.

$$A = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix};$$

c.

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}.$$

Definition

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ be $n \times n$ matrix. The leading principal minors are $\det A$ and the minors obtained by successively removing the last row and the last column. That is, the leading principal minors are

$$\Delta_1 = a_{11},\ \Delta_2 = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ \Delta_3 = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ldots,$$

$\Delta_n = \det A$.

Theorem

Let $A = [a_{ij}]_{i,j=1}^{n}$ *be* $n \times n$ *symmetric matrix. The following three statements are equivalent*

- *A is positive definite;*
- *All eigenvalues of A are positive;*
- *All leading principal minors of A are positive (Sylvester's criterion).*

Definition

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ be $n \times n$ matrix. The principal minors are $\det A$ itself and the determinants of matrices obtained by successively removing an i-th row and and i-th column. That is, the principal minors are

$$\det \begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \dots & a_{i_1 i_p} \\ a_{i_2 i_1} & a_{i_2 i_2} & \dots & a_{i_2 i_p} \\ \vdots & & & \\ a_{i_p i_1} & a_{i_p i_2} & \dots & a_{i_p i_p} \end{pmatrix}, 1 \leq i_1 < i_2 < \cdots < i_p \leq n, p = 1, \dots, n.$$

Theorem

Let $A = [a_{ij}]_{i,j=1}^{n}$ be $n \times n$ symmetric matrix. The following three statements are equivalent

- *A is positive semidefinite;*
- *All eigenvalues of A are nonnegative;*
- *All principal minors of A are nonnegative.*

Example

Determine whether the matrix A is positive definite (semidefinite) if

a.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix};$$

b.

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Theorem

Let $A = [a_{ij}]_{i,j=1}^{n}$ be $n \times n$ symmetric matrix. The following three statements are equivalent

- *A is negative definite;*
- *All eigenvalues of A are negative;*
- *All leading principal minors of even order are positive and of odd order negative (Sylvester's criterion).*

Theorem

Let $A = [a_{ij}]_{i,j=1}^{n}$ *be* $n \times n$ *symmetric matrix. The following three statements are equivalent*

- *A is negative semidefinite;*
- *All eigenvalues of A are nonpositive;*
- *All principal minors of even order are nonnegative and of odd order nonpositive.*

Example

Determine whether the matrix A is positive definite (semidefinite), negative definite (semidefinite) or indefinite if

a.

$$A = \begin{pmatrix} -1 & -2 & 0 \\ -2 & -5 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

b.

$$A = \begin{pmatrix} -1 & -3 & 0 \\ -3 & -9 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

c.

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open, convex set and $f \in \mathbb{C}^2(\Omega)$, then f is convex if and only if

$$\nabla^2 f(x) \succeq 0, \quad \forall x \in \Omega.$$

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open, convex set and $f \in \mathbb{C}^2(\Omega)$ such that $\nabla^2 f(x) \succ 0$, $\forall x \in \Omega$, then f is strictly convex.

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open, convex set and $f \in \mathbb{C}^2(\Omega)$, then f is concave if and only if

$$\nabla^2 f(x) \preceq 0, \quad \forall x \in \Omega.$$

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open, convex set and $f \in \mathbb{C}^2(\Omega)$ such that $\nabla^2 f(x) \prec 0$, $\forall x \in \Omega$, then f is strictly concave.

Example

Check whether f is convex (strictly convex), concave (strictly concave) on Ω if

a. $f(x_1, x_2, x_3) = x_1^2 + 3x_1x_2 + 4x_2^2 + x_3^2 - x_1x_3$, $\Omega = \mathbb{R}^3$;

b. $f(x_1, x_2) = x_1^2 - 2x_1x_2 - 2x_2^2 + 4x_2$, $\Omega = \mathbb{R}^2$;

c. $f(x_1, x_2) = -x_1^4 + 2x_1x_2 - x_2^4 - x_1^2 - 2x_2^2$, $\Omega = \mathbb{R}^2$.