Conditions for local minimizers

Theorem (First-Order Necessary Conditions (FONC))

If x^* is a local minimizer of f and f is continuously differentiable in an open neighborhood of x^*, then

$$\nabla f(x^*) = 0.$$

Definition

We call x^* a **stationary point** if $\nabla f(x^*) = 0$.

Theorem (Second-Order Necessary Conditions (SONC))

If x^* is a local minimizer (maximizer) of f and f is twice continuously differentiable in an open neighborhood of x^*, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive (negative) semidefinite.

Definition

A **saddle point** is a stationary point which is not a local extremum.

Example

Show that $x^* = (0,0)^T$ is a saddle point for the function $f(x_1, x_2) = x_1^2 + 8x_1x_2 + x_2^2$.

Theorem (Second-Order Sufficient Conditions (SOSC))

Assume f is twice continuously differentiable in an open neighborhood of x^* such that $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive (negative) definite, then x^* is a strict local minimizer (maximizer) of f.

Example

Find all stationary points of f and check if these points are local maximum, minimum or saddle points for that function if

a. $f(x_1, x_2) = 4x_1^4 + x_2^4 + 4x_1x_2$;

b. $f(x_1, x_2) = \frac{1}{3}x_1^3 - 4x_1 + \frac{1}{3}x_2^3 - 16x_2$;

c. $f(x_1, x_2, x_3) = 3x_1^3 - 9x_1 + x_2^3 + x_3^3 - 6x_3^2 - 10$.

Theorem

When f is convex, any local minimizer x^* is a global minimizer of f. If in addition f is differentiable, then any stationary point x^* is a global minimizer of f.

Example

Find the global minimizer of f on Ω if

a. $f(x_1, x_2) = x_1^4 + x_2^4 + x_1^2 x_2^2$, $\Omega = \mathbb{R}^2$;

b. $f(x) = x^T A x$, where A is symmetric and positive definite matrix and $\Omega = \mathbb{R}^n$.