Example

Show that the linear function $f(x) = a^T x + b$, where $a, x \in \mathbb{R}^n$ and $b \in \mathbb{R}$, is a convex function.

Example

Let $f(x) = x_1^2 + x_2^2 + ... + x_n^2$, $x \in \mathbb{R}^n$. Show that f is strictly convex function.

Theorem

Let $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$. If f is a convex function and x_0 is a local minimum point of f over Ω, then x_0 is a global minimum point of f in Ω, i.e. $x_0 = arg\ min_{x \in \Omega} f(x)$.

Theorem

Let $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$. If f is a strictly convex function and x_0 is a local minimum point of f over Ω, then x_0 is the unique global minimum point of f in Ω, i.e. $x_0 = arg\ min_{x \in \Omega} f(x)$.

Convexity by using the first order derivative

Let's denote by $\nabla f(x)$ the gradient of f at x i.e.

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right]^T.$$

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open convex set and $f : \Omega \to \mathbb{R}$ is a differentiable function, then f is convex if and only if

$$f(x) \geq f(x_0) + \nabla f(x_0)^T (x - x_0), \quad \forall x, x_0 \in \Omega.$$

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open convex set and $f : \Omega \to \mathbb{R}$ is a differentiable function, then f is strictly convex if and only if

$$f(x) > f(x_0) + \nabla f(x_0)^T (x - x_0), \quad \forall x, x_0 \in \Omega, x \neq x_0.$$

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open convex set and $f : \Omega \to \mathbb{R}$ is a differentiable function, then f is convex if and only if

$$(\nabla f(x) - \nabla f(y))^T (x - y) \geq 0, \quad \forall x, y \in \Omega.$$

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open convex set and $f : \Omega \to \mathbb{R}$ is a differentiable function, then f is strictly convex if and only if

$$(\nabla f(x) - \nabla f(y))^T (x - y) > 0, \quad \forall x, y \in \Omega, x \neq y.$$

Convexity by using the second order derivative

Theorem

Let $f : (a, b) \to \mathbb{R}$. If f is a twice differentiable function on (a, b) then f is convex if and only if $f''(x) \geq 0$, for all $x \in (a, b)$.

Theorem

Let $f : (a, b) \to \mathbb{R}$. If f is a twice differentiable function on (a, b) and $f''(x) > 0$, for all $x \in (a, b)$, then f is strictly convex.

Example

Check if the following functions are convex (strictly convex), concave (strictly concave), if

a. $\frac{1}{1+x^2}, \quad x \in \mathbb{R};$

b. $-\cos x, \quad x \in \left(0, \frac{\pi}{2}\right);$

c. $-(x-1)^4 + 1, \quad x \in \mathbb{R}.$