SDG 4 QUALITY EDUCATION

METHODS FOR SOLVING ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

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Lecture Outline

1. Zeros or Roots of An Equation
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   - Iterative Methods

2. Initial Approximation of An Iterative Procedure

3. Finding The Root of an Equation
   - Bisection Method
   - Method of False-Position
   - Newton-Raphson Method
   - Secant Method

Section 1

Zeros or Roots of An Equation

Introduction

A problem of great importance in science is determining the roots or zeros of a function.

Definition

A polynomial equation of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \quad (1)$$

is called an algebraic equation

Definition (Transcendental equation)

An equation which contains polynomial term, exponential term, logarithm term, and trigonometric term are called transcendental equations.

Some examples of transcendental equations are

1. $2xe^{2x-1} + 1 = 0$
2. $\cosh(x) + \cos(2x) + x^2 = 0$
3. $x^2 + \ln(2x) + \dfrac{1}{e^{x^2}} = 0$

Definition

A number α for which $f(x) = 0$ is called the roots/zero of the function f. Geometrically, the roots of an equation is the value of x where the graph crosses the $x-$axis.

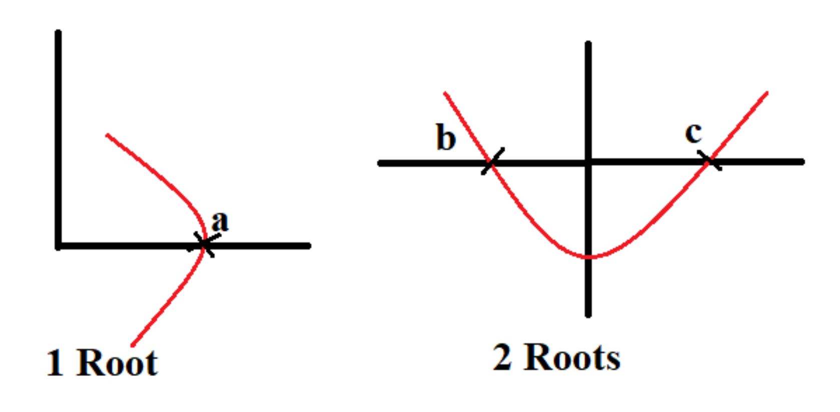


Figure 1: Roots of Equation

Roots

1. A polynomial equation of degree n has exactly n roots.
2. These roots can either be real numbers, complex numbers or combination of real and complex numbers.
3. Again, it can be a single root or multiple roots.

Example

$f(x) = 3x - 9$ has one root, and $f(x) = x^5 + x - 1$ will have five roots.

4. A transcendental equation may have one root, infinite number of roots, or no root.

The methods for finding the roots of an equation can be categorized as:

1. Direct methods
2. Iterative methods

Direct Method

This gives the exact values of all the roots in a finite number of steps.

Example

The roots of a quadratic equation: $ax^2 + bx + c = 0, \quad a \neq 0$ is

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Iteration Method and Stopping Criterion

This is based on the idea of successive approximation. It starts with one or two initial approximations to the root in order to obtain the other sequences. The initial value is sometimes guessed. A sequence x_k is said to converge to the exact root α if

$$\lim_{k\to\infty} x_k = \alpha \tag{2}$$

or

$$\lim_{k\to\infty} |x_k - \alpha| = 0 \tag{3}$$

Given an error tolerance ϵ, an iterative procedure is terminated when

$$|x_{k+1} - x_k| \leq \epsilon \tag{4}$$

That is, a current solution value (x_{k+1}) minus the previous solution value (x_k) should be less or equal to a given threshold value (ϵ).

This is often the stopping criterion for all iterative schemes.

Section 2

Initial Approximation of An Iterative Procedure

Theorem (Intermediate Value Theorem)

If $f(x)$ is continuous on the closed interval $[a,\ b]$ and

$$f(a) \times f(b) < 0 \tag{5}$$

then $f(x) = 0$ has at least one real root or an odd number of real roots in the open interval $(a,\ b)$.

Example

Determine the maximum number of positive and negative roots and the interval of length one unit in which the real roots lies in the following

$$8x^3 - 12x^2 - 2x + 3 = 0, \quad x \in [-2,\ 3]$$

$$8x^3 - 12x^2 - 2x + 3 = 0, \quad x \in [-2,\ 3]$$

1. So we begin by finding the functional values of the given problem within the given interval.

$$f(-2) = 8 * (-2)^3 - 12 * (-2)^2 - 2(-2) + 3 = -105$$

$$f(-1) = 8 * (-1)^3 - 12 * (-1)^2 - 2(-1) + 3 = -15$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f(3) = 8 * 3^3 - 12 * 3^2 - 2(3) + 3 = 105$$

2. The other values are computed using this same procedure.

These values are presented in a tabular form as

x	-2	-1	0	1	2	3
$f(x)$	-105	-15	3	-3	15	105

Now let determine the intervals that satisfy the intermediate value theorem.

1. $f(-2) \times f(-1) = -105 \times -15 \implies > 0$, hence, there is no root in the interval $(-2, -1)$
2. $f(-1) \times f(0) = -15 \times 3 \implies < 0$, hence, there is a root in the interval $(-1, 0)$
3. $f(0) \times f(1) = 3 \times -3 \implies < 0$, hence, there is a root in the interval $(0, 1)$
4. $f(1) \times f(2) = -3 \times 15 \implies < 0$, hence, there is a root in the interval $(1, 2)$
5. $f(2) \times f(3) = 15 \times 105 \implies > 0$, hence, there is no root in the interval $(2, 3)$

Considering the obtained results $(-1,\ 0)$, $(0,\ 1)$, $(1,\ 2)$, the function $f(x)$ will have one negative root and two positive roots.

Section 3

Finding The Root of an Equation

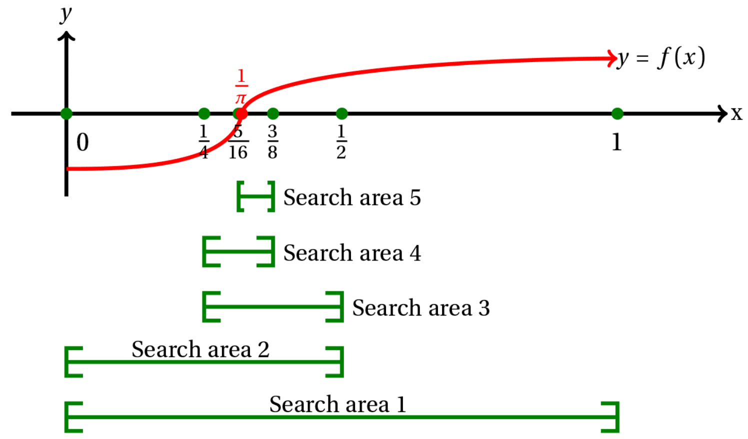
Finding The Root of an Equation

There are several numerical methods for finding the root of the equation $f(x) = 0$. However, the following four methods will be considered in this course. The methods are

1. Bisection or interval halving method
2. Method of false position or chord method
3. Newton-Raphson method
4. Secant method

Bisection Method

This is a simple and robust method for finding the root of an equation. It works by repeatedly dividing the interval in which the root is known to exist and narrowing down the interval until the desired level of accuracy is achieved.



Bisection Method

The method is applicable to functions of the form $f(x) = 0$, where the function f is continuous and defined on a closed interval $[a, b]$ and $f(a),\ f(b)$ have opposite signs. The function f must have one root in the open interval (a, b). Suppose we need to find the root of $f(x) = 0$ given the error tolerance ϵ, then the algorithm for the bisection method is as follows:

1. Find two numbers $a = x_0$ and $b = x_1$, for which f has different signs. That is, consider the interval $[a, b]$ or $[x_0, x_1]$.
2. Define c, such that $c = \dfrac{a+b}{2}$ or $c = \dfrac{x_0 + x_1}{2}$
3. If $b - c \leq \epsilon$, then accept c as the root of the equation and stop the iteration, otherwise continue
4. If $f(a) \times f(c) \leq 0$, then set c as the new b, otherwise set c as the new a.
   Return to step two

The procedure is continued until the interval is sufficiently small, ie $|x_{k+1} - x_k| \leq \epsilon$.

Example

Find the root of the equation

$$x^2 + 2x - 3 = 0$$

accurate to $\epsilon = 0.05$. Otherwise, stop after the 5th iteration. Assume that the root lies in the interval $[0,3]$.

Iteration 1:

1. Step 1: Considering the given interval, it implies that $a = 0$ and $b = 3$.
2. Step 2: $c = \dfrac{a+b}{2} = \dfrac{0+3}{2} = 1.5$
3. Step 3: Check stopping criterion: $b - c = 3 - 1.5 = 1.5 \implies \nless \epsilon$.
4. Hence, we continue the iteration.
5. Step 4: $f(a) = f(0) = 0^2 + 2(0) - 3 = -3$
6. $f(c) = f(1.5) = 1.5^2 + 2(1.5) - 3 = 2.25$
7. $f(a) \times f(c) = -3(2.25) = -6.75 \implies < 0$, hence set c as new b
8. Therefore, the new interval is $[0, 1.5]$
9. Return to step 2.

Iteration 2:

1. Considering the new interval, $a = 0$ and $b = 1.5$.
2. Step 2: $c = \dfrac{a+b}{2} = \dfrac{0+1.5}{2} = 0.75$
3. Step 3: Check stopping criterion: $b - c = 1.5 - 0.75 = 0.75 \implies \not< \epsilon$
4. Hence, we continue the iteration.
5. Step 4: $f(a) = f(0) = 0^2 + 2(0) - 3 = -3$
6. $f(c) = f(0.75) = 0.75^2 + 2(0.75) - 3 = -0.9375$
7. $f(a) \times f(c) = -3(-0.9375) \implies > 0$, hence set c as new a
8. Therefore, the new interval is $[0.75, 1.5]$
9. Return to step 2.

Iteration 3:

1. Considering the new interval, $a = 0.75$ and $b = 1.5$.
2. Step 2: $c = \dfrac{a+b}{2} = \dfrac{0.75+1.5}{2} = 0.125$
3. Step 3: Check stopping criterion: $b - c = 1.5 - 0.125 = 0.375 \implies \nless \epsilon$
4. Hence, we continue the iteration.
5. Step 4: $f(a) = f(0.75) = 0.75^2 + 2(0.75) - 3 = -0.9375$
6. $f(c) = f(0.125) = 0.125^2 + 2(0.125) - 3 = 0.5156$
7. $f(a) \times f(c) = -0.936(0.5156) \implies < 0$, hence set c as new a
8. Therefore, the new interval is $[0.75, 1.125]$
9. Return to step 2.

Iteration 4:

1. Considering the new interval, $a = 0.75$ and $b = 1.125$.
2. Step 2: $c = \dfrac{a+b}{2} = \dfrac{0.75+1.125}{2} = 0.9375$
3. Step 3: Check stopping criterion: $b - c = 1.125 - 0.9375 = 0.1875 \implies \nless \epsilon$
4. Hence, we continue the iteration.
5. Step 4: $f(a) = f(0.75) = 0.75^2 + 2(0.75) - 3 = -0.937$
6. $f(c) = f(0.9375) = 0.9375^2 + 2(0.9375) - 3 = -0.2461$
7. $f(a) \times f(c) = -0.937(-0.2461) \implies > 0$, hence set c as new a
8. Therefore, the new interval is $[0.9375, 1.125]$
9. Return to step 2.

Iteration 5:

1. Considering the new interval, $a = 0.9375$ and $b = 1.125$.
2. Step 2: $c = \dfrac{a+b}{2} = \dfrac{0.9375+1.125}{2} = 1.0313$
3. Step 3: Check: $b - c = 1.125 - 1.0313 = 0.0938 \implies \nless \epsilon$
4. Though $b - c$ is not less than ϵ, the otherwise statement in the question implies that the computations could be halted at the 5th iteration.
5. Hence, $c = 1.0313$ is root of the equation that lies in the interval $[0,3]$

Note

The function $f(x) = x^2 + 2x - 3$ have two roots, but the iterative scheme could find only one root at a time.

A method exhibits linear convergence, that is the error in the approximation decreases by a constant factor in each iteration. Mathematically, if e_n is the error at iteration n, a method has linear convergence if there exists a constant $0 < c < 1$ such that:

$$e_{n+1} \leq c \cdot e_n$$

Advantage

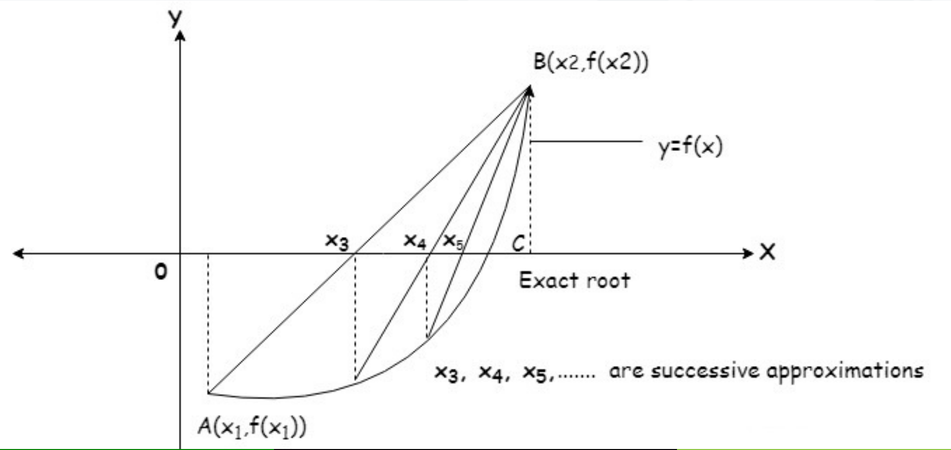
1. Simplicity: The bisection method is straightforward to implement and understand.
2. Robustness: The bisection method is very robust and reliable. It guarantees convergence to a root as long as there is an initial interval containing a single root and the function is continuous on that interval.

Disadvantages

1. Limited applicability: The bisection method is only applicable when there is a change of sign in the function within the initial interval.
2. It cannot detect multiple roots.

Method of False-Position (Regula Falsi)

This method is an improvement over the bisection method and uses linear interpolation to find the root. It uses a weighted average of the function values at the endpoints to determine the next interval.



Method of False-Position

1. This method also requires the interval in which the root is expected to lie. The iterative formula is defined as

$$x_{k+1} = \frac{[x_{k-1} \times f(x_k)] - [x_k \times f(x_{k-1})]}{f(x_k) - f(x_{k-1})} \tag{6}$$

2. The iterative procedure begins to find x_2 (that is $k = 1$), given that x_0 and x_1 are given. That is, starting with the initial interval $[x_0, x_1]$ is which the root lies, then x_2 is computed as

$$x_2 = \frac{[x_0 \times f(x_1)] - [x_1 \times f(x_0)]}{f(x_1) - f(x_0)} \tag{7}$$

3. If $f(x_0) \times f(x_2) < 0$, then the root lies in the interval (x_0, x_2), otherwise the root lies in the interval (x_2, x_1).

Method of False-Position

1. To simplify the subsequent computations and iterations, we let $x_2 = x_1$ when the chosen interval is (x_0, x_2) , likewise, we set $x_2 = x_0$ when the chosen interval is (x_2, x_1).
2. That could help us use the iterative formula (7) repeatedly without any complications.
3. The iteration is continued until the required accuracy criterion is satisfied. That is when

$$|x_{k+1} - x_k| \leq \epsilon$$

ϵ is a given tolerance level.

Example

Find the positive root of the function

$$f(x) = x^2 + 2x - 3$$

accurate to $\epsilon = 0.05$ using the method of false-position. Otherwise, stop after the 5th iteration. Take initial interval [0, 2]

Iteration 1

1. Step 1: We start with the initial interval $[x_0, x_1] = [0, 2]$. Then

$$x_2 = \frac{x_0 \times f(x_1) - x_1 \times f(x_0)}{f(x_1) - f(x_0)} \tag{8}$$

2. Since $x_0 = 0$ and $x_1 = 2$ are known, let compute their corresponding functional values. Hence

$$f(x_0) = f(0) = -3, \qquad f(x_1) = f(2) = 5$$

1. Thus
$$x_2 = \frac{x_0 \times f(x_1) - x_1 \times f(x_0)}{f(x_1) - f(x_0)} = \frac{0(5) - 2(-3)}{5 - (-3)} = \frac{6}{8} = 0.75$$
2. Step 2: Check stopping criterion: $|x_{k+1} - x_k| \leq \epsilon$.
3. Since this is the first iteration, we will skip this step. There is no previous iterative value to make such comparison. Hence, we continue the iteration.
4. Step 3: Check the new interval: $f(x_2) = f(0.75) = 0.75^2 + 2(0.75) - 3 = -0.9375$
   Note that $f(x_0) = -3$
5. Therefore, $f(x_0) \times f(x_2) = -0.9375(-3) = 2.8125 \implies > 0$.
   Hence, the root will lie in the interval (x_2, x_1).
6. With proper substitution, the new interval is $[0.75, 2]$

Iteration 2: $x_0 = 0.75$ and $x_1 = 2$

1. The corresponding functional values are

$$f(0.75) = -0.9375, \qquad f(2) = 5$$

2. Then

$$x_2 = \frac{x_0 \times f(x_1) - x_1 \times f(x_0)}{f(x_1) - f(x_0)} = \frac{0.75(5) - 2(-0.9375)}{5 - (-0.9375)} = 0.9474$$

3. Step 2: Check stopping criterion: $|x_{k+1} - x_k| = |0.9474 - 0.75| = 0.1974 \implies \not< \epsilon$
   Hence we continue the iteration.
4. Step 3: Check the new interval: $f(x_2) = f(0.9474) = 0.9474^2 + 2(0.9474) - 3 = -0.2076$.
   Note that $f(x_0) = -0.9375$
5. Therefore, $f(x_0) \times f(x_2) > 0$.
   Hence, the root will lie in the interval (x_2, x_1).
6. With proper substitution the new interval is $[0.9474, 2]$

Iteration 3: $x_0 = 0.9474$ and $x_1 = 2$

1. The corresponding functional values are

$$f(0.9474) = -0.2076, \qquad f(2) = 5$$

2. Then

$$x_2 = \frac{x_0 \times f(x_1) - x_1 \times f(x_0)}{f(x_1) - f(x_0)} = \frac{0.9474(5) - 2(-0.2076)}{5 - (-0.2076)} = 0.9894$$

3. Step 2: Check stopping criterion: $|x_{k+1} - x_k| = |0.9894 - 0.9474| = 0.04 \implies < \epsilon$
4. Since the stopping criterion is satisfied, we halt the iteration process here.
5. Hence the root of the equation is 0.9894.

Advantages and Disadvantages of the Method False-Position

Advantages

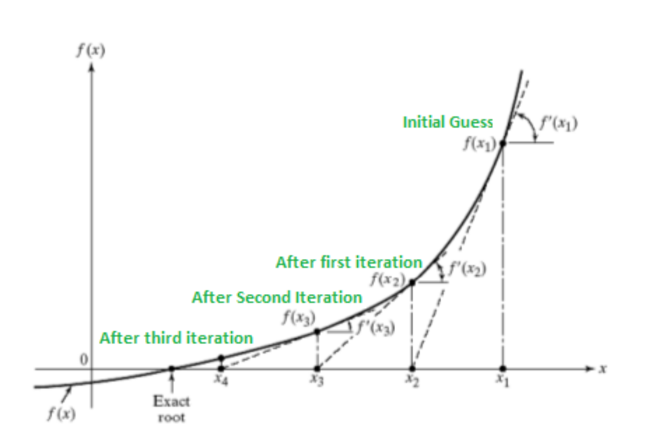
1. The method is guaranteed to converge.
2. The method converges faster than the bisection method.

Disadvantage

1. It cannot detect multiple roots.

Newton-Raphson Method/ Newton's method

This method uses the derivative of the function to approximate the root. Starting with an initial guess, the method iteratively improves the estimate by using the tangent line of the function at each point.



The iterative procedure is explained as follows

Given x_0, then $x_1, x_2, \cdots, x_{k+1}$ are obtained as:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \qquad \text{where } f'(x_0) \neq 0 \tag{9}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \qquad \text{where } f'(x_1) \neq 0 \tag{10}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}, \qquad \text{where } f'(x_2) \neq 0 \tag{11}$$

$$\vdots \qquad \vdots$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad \text{where } f'(x_k) \neq 0 \tag{12}$$

The process is continued until the required accuracy criterion is obtained. That is when

$$|x_{k+1} - x_k| < \epsilon$$

Example

Use the Newton's method to find the root of the function

$$f(x) = x^2 + 2x - 3$$

that lies in the interval [0, 2]. Take $\epsilon = 0.05$ and $x_0 = 0$.

Iteration 1: k = 0

Given the formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad f'(x_0) \neq 0$$

Since we know $x_0 = 0$, we let $k = 0$, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \qquad f'(x_0) \neq 0 \tag{13}$$

We need to find $f(x_0)$ and $f'(x_0)$. Note that f' is the derivative of the given function.

1. Thus,

$$f'(x) = 2x + 2 \tag{14}$$
$$f'(x_0) = f'(0) = 2(0) + 2 = 2 \tag{15}$$
$$f(x_0) = f(0) = 0^2 + 2(x) - 3 = -3 \tag{16}$$

2. Substituting these values into equation (13).

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-3)}{2} = 1.5 \tag{17}$$

3. Checking the stopping criterion: Since this is the first iteration, we will skip this step. There is no previous iterative value to make such comparison.
4. Hence, we continue to the next iteration and find x_2.

Iteration 2: With $k = 1$, the formula reduces to

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

1. From the above, we know that $x_1 = 1.5$. Therefore, the functional values are:

$$f'(x_1) = f'(1.5) = 2(1.5) + 2 = 5$$

$$f(x_1) = f(1.5) = 1.5^2 + 2(1.5) - 3 = 2.25$$

2. Substituting these values into the formula

$$x_2 = 1.5 - \frac{2.25}{5} = 1.05$$

3. Checking the stopping criterion: $|x_2 - x_1| = |1.05 - 1.5| = 0.45 \implies \not< \epsilon$. Hence, compute x_3.

Iteration 3: With $k = 2$, the formula reduces to

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

1. From the above, we know that $x_2 = 1.05$. Therefore, the functional values are:

$$f'(x_2) = f'(1.05) = 2(1.05) + 2 = 4.1$$

$$f(x_2) = f(1.05) = 1.05^2 + 2(1.05) - 3 = 0.2025$$

2. Substituting these values into the formula

$$x_3 = 1.05 - \frac{0.2025}{4.10} = 1.0006$$

3. Checking the stopping criterion: $|x_3 - x_2| = |1.0006 - 1.05| = 0.049 \implies < \epsilon$. Hence, stop the iteration.

Therefore the root of the equation is 1.0006

Advantage and Disadvantage of the Newton-Raphson Method

This method has a quadratic convergence rate, that is the error in the approximation squares at each iteration. Mathematically, if e_n is the error at iteration n, a method has quadratic convergence if there exists a constant $c > 0$ such that:

$$e_{n+1} \leq c \cdot e_n^2$$

This means that the number of correct digits approximately doubles with each iteration.

Advantage

1. The method converges faster than the bisection and false position methods.

Disadvantage

1. The method diverge if the initial approximation is far from the root.
2. May fail if $f'(x) = 0$ is zero or if the function is not differentiable.

Secant Method

1. Similar to the Newton-Raphson method, the secant method estimates the derivative using a finite difference approximation by considering two points on the curve.
2. Assuming we need to find the root of the equation $f(x) = 0$. We choose two point x_0 and x_1 for which the root may lie or not.
3. Then the two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ form a straight line called the **secant line** which is viewed as the approximation to the graph of $f(x)$.
4. The point where this secant line crosses the $x-$axis is considered as the root of the equation.
5. The iteration is continued until the interval in which the root lies become significantly small. That is when

$$|x_{k+1} - x_k| < \epsilon$$

Secant Method

This method is similar to the method false position, but with different iterative procedure. The iterative procedure for the secant method is given by:

$$x_2 = x_1 - f(x_1)\left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] \tag{18}$$

$$x_3 = x_2 - f(x_2)\left[\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right] \tag{19}$$

$$\vdots \qquad \vdots$$

$$x_{k+1} = x_k - f(x_k)\left[\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right] \tag{20}$$

Example

Use the Secant method to find the root of the function

$$f(x) = x^2 + 2x - 3$$

that lies in the interval [0, 2]. Take $\epsilon = 0.06$.

Iteration 1: k=1

Given the general formula

$$x_{k+1} = x_k - f(x_k)\left[\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right] \tag{21}$$

For $k = 1$, we have

$$x_2 = x_1 - f(x_1)\left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] \tag{22}$$

From the given interval $x_0 = 0$ and $x_1 = 2$.

1. Therefore, the functional values are

$$f(x_0) = f(0) = 0^2 + 2(0) - 3 = -3$$

$$f(x_1) = f(1) = 2^2 + 2(2) - 3 = 5$$

2. Substituting these value into the iterative formula (22), we obtain

$$x_2 = 2 - 5\left[\frac{2-0}{5-(-3)}\right] = 2 - \frac{10}{8} = 0.75$$

3. Checking the stopping criterion: Since this is the first iteration, we will skip this step. There is no previous iterative value to make such comparison.
4. Hence, we continue the iteration.

Iteration 2: For $k = 2$, we have

$$x_3 = x_2 - f(x_2)\left[\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right] \tag{23}$$

1. The functional values are

$$f(x_2) = f(0.75) = 0.75^2 + 2(0.75) - 3 = -0.9375$$
$$f(x_1) = f(2) = 2^2 + 2(2) - 3 = 5$$

2. Substituting these value into the iterative formula (23), we obtain

$$x_3 = 0.75 - (-0.9375)\left[\frac{0.75 - 2}{-0.9375 - 5}\right] = 0.75 + 0.197 = 0.947$$

3. Checking the stopping criterion: $|x_3 - x_2| = |0.947 - 0.75| = 0.197 \implies \nless \epsilon$.
4. Hence continue to find x_4.

Iteration 3: For $k = 3$, we have

$$x_4 = x_3 - f(x_3)\left[\frac{x_3 - x_2}{f(x_3) - f(x_2)}\right] \tag{24}$$

1. The functional values are

$$f(x_2) = f(0.75) = 0.75^2 + 2(0.75) - 3 = -0.9375$$

$$f(x_3) = f(0.947) = 0.947^2 + 2(0.947) - 3 = -0.21$$

2. Substituting these value into the iterative formula (23), we obtain

$$x_4 = 0.947 - (-0.21)\left[\frac{0.947 - 0.75}{-0.21 - (-0.9375)}\right] = 0.947 + 0.0568 = 1.0038$$

3. Checking the stopping criterion: $|x_4 - x_3| = |1.0038 - 0.947| = 0.0568 \implies < \epsilon$. Hence, stop the iteration.

Therefore the root of the equation is 1.0038

Advantages and Disadvantage of the Secant Method

This method has superlinear convergence rate, that is the error decreases faster than linearly, but not necessarily as fast as quadratic.
The convergence rate is approximately 1.618 (the golden ratio).

Advantages

1. The method is faster after few initial iterations.
2. Does not require $f(x_0)$ and $f(x_1)$ to have opposite signs, making it more flexible.
3. Compared to the Newton's method, this does not require differentiation.

Disadvantage

1. It is slow compared to the Newton-Raphson method.
2. May not always converge if the initial guesses are not good or if the function has discontinuities or multiple roots close to each other.

Comparison of some root-finding methods

Method	Speed of Convergence	Robustness	Bracketing	Derivative Needed	Initial Guess Sensitivity	Comments
Method of False Position	Linear	High	Yes	No	Low	Guarantees root is within interval, slow convergence
Secant Method	Superlinear	Moderate	No	No	Moderate	Faster than Regula Falsi, may not always converge
Newton-Raphson	Quadratic	Low	No	Yes	High	Very fast near root, may fail with bad initial guess
Bisection Method	Linear	Very High	Yes	No	Low	Always converges if conditions met, slow

Exercise

1. Determine the maximum number of positive and negative roots and the interval of length one unit in which the real roots lies in the following
   1. $3x^3 - 2x^2 - x + 3 = 0, \quad x \in [-3,\ 3]$
   2. $xe^x - \cos(x) = 0, \quad x \in [-3,\ 3]$
2. Find a positive root of the following equations by choosing an appropriate initial interval or initial starting point accurate to $\epsilon = 0.01$:

$$f(x) = \tanh\left(x^2\right) - 2x, \qquad f(x) = \frac{\log(7x)}{\exp(7x)} + 1, \qquad f(x) = \sec\left(x^2 + 4x\right) + 1$$

   1. the interval halving or bisection method
   2. the method of false position
   3. the Newton-Raphson method
   4. the secant method,

Stop the iteration using their respective stopping criterion, or at 6th iteration otherwise.

END OF LECTURE