POLYNOMIAL APPROXIMATION AND INTERPOLATION

(APPROXIMATION WITH UNEVENLY SPACED POINTS)

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Lecture Outline

Introduction

We now discuss the problem of approximating a given function by polynomials.

Why approximating polynomials

1. To reconstruct the function $f(x)$ when it is not given explicitly and only values of $f(x)$ and/or its certain order derivatives are given at a set of distinct points called nodes or tabular points.
2. to perform the required operations which were intended for $f(x)$, like determination of roots, differentiation and integration etc. can be carried out using the approximating polynomial $P(x)$.
3. The approximating polynomial $P(x)$ can be used to predict the value of $f(x)$ at a non-tabular point.

The deviation of $P(x)$ from $f(x)$, that is $f(x) - P(x)$, is called the error of approximation.

Introduction

1. Let $f(x)$ be a continuous function defined on some interval $[a, b]$, and be prescribed at $n+1$ distinct tabular points $x_0, x_1, \cdots, x_n$ such that
$$a = x_0 < x_1 < x_2 < \cdots < x_n = b \tag{1}$$
2. The distinct tabular points $x_0, x_1, \cdots, x_n$ may be non-equispaced or equispaced, that is
$$x_{k+1} - x_k = h; \quad k = 0, 1, 2, \cdots, n-1 \tag{2}$$
3. The problem of polynomial approximation is to find a polynomial $P_n(x)$, of degree $\leq n$, which fits the given data exactly, that is,
$$P_n(x_i) = f(x_i); \quad i = 0, 1, 2, \cdots, n \tag{3}$$

The polynomial $P_n(x)$ is called the interpolating polynomial. The conditions given in eq. (3) are called the interpolating conditions.

Introduction

Remark

1. Through two distinct points, we can construct a unique polynomial of degree 1 (straight line).
2. Through three distinct points, we can construct a unique polynomial of degree 2 (parabola) or a unique polynomial of degree1 (straight line).
3. That is, through three distinct points, we can construct a unique polynomial of degree ≤ 2.
4. In general, through $n+1$ distinct points, we can construct a unique polynomial of degree $\leq n$.

Lagrange Interpolation

x	x_0	x_1	x_2	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_n)$

Let the data above be given at distinct unevenly spaced points or non-uniform points $x_0, x_1, \cdots, x_n$. This data may also be given at evenly spaced points. We can fit a unique polynomial of degree $\leq n$ given as

$$P_n(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + \cdots + l_n(x)f(x_n) \tag{4}$$

$$= l_0(x)f_0 + l_1(x)f_1 + \cdots + l_n(x)f_n \tag{5}$$

where $l_i(x);\ i = 0, 1, 2, \cdots, n$ are polynomials of degree n defined as

$$l_i(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)} \tag{6}$$

Equation (5) is called the Lagrange interpolating polynomial and eq. (6) are called the Lagrange fundamental polynomials. Note that the denominator is obtained by setting $x = x_i$ in the numerator.

Linear interpolation

1. For $n = 1$, we have the data

x	x_0	x_1
$f(x)$	f_0	f_1

2. The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x - x_1)}{(x_0 - x_1)}, \qquad l_1(x) = \frac{(x - x_0)}{(x_1 - x_0)} \tag{7}$$

3. The Lagrange linear interpolation polynomial is given by

$$P_1(x) = l_0(x) f_0 + l_1(x) f_1 \tag{8}$$

Quadratic interpolation

1. For $n = 2$, we have the data

x	x_0	x_1	x_2
$f(x)$	f_0	f_1	f_2

2. The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \quad l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}, \text{ and}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \tag{9}$$

3. The Lagrange linear interpolation polynomial is given by

$$P_2(x) = l_0(x)f_0 + l_1(x)f_1 + l_2(x)f_2 \tag{10}$$

Two Data Values

Example

Using the data $\sin(0.1) = 0.09983$ and $\sin(0.2) = 0.19867$, find an approximate value of $\sin(0.15)$ by Lagrange interpolation.

$$x = 0.15, \quad x_0 = 0.1, \quad x_1 = 0.2, \quad f_0 = 0.09983, \quad f_1 = 0.19867$$

$$l_0(0.15) = \frac{(x - x_1)}{(x_0 - x_1)} = \frac{0.15 - 0.2}{0.1 - 0.2} = 0.5 \tag{11}$$

$$l_1(0.15) = \frac{(x - x_0)}{(x_1 - x_0)} = \frac{0.15 - 0.1}{0.2 - 0.1} = 0.5 \tag{12}$$

Then

$$P_1(x) = l_0(x)f_0 + l_1(x)f_1 = 0.5f_0 + 0.5f_1 \tag{13}$$

$$P_1(0.15) = (0.5)(0.09983) + (0.5)(0.19867) = 0.14925. \tag{14}$$

Example

Given that $f(0)=1,\ f(1)=3,\ f(3)=55$, find the unique polynomial of degree 2 or less, which fits the given data.

We have $x_0=0,\ f_0=1,\ x_1=1,\ f_1=3,\ x_2=3,\ f_2=55$. Then

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x^2-4x+3) \tag{15}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{x(x-3)}{(1-0)(0-2)} = \frac{1}{2}(3x-x^2) \tag{16}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{x(x-1)}{(3-0)(2-0)} = \frac{1}{6}(x^2-x) \tag{17}$$

The Lagrange linear interpolation polynomial is given by

$$P_2(x) = l_0(x)f_0 + l_1(x)f_1 + l_2(x)f_2 \tag{18}$$

$$= \frac{1}{3}(x^2-4x+3)(1) + \frac{1}{2}(3x-x^2)(3) + \frac{1}{6}(x^2-x)(55) \tag{19}$$

$$= 8x^2-6x+1 \tag{20}$$

Example

Construct the Lagrange interpolation polynomial for the data below and hence, interpolate at $x = 5$.

x	-1	1	4	7
$f(x)$	-2	0	63	342

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-4)(x-7)}{(-1-1)(-1-4)(-1-7)} = \frac{-1}{80}(x^3-12x^2+39x-28) \quad (21)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x+1)(x-4)(x-7)}{(1+1)(1-4)(1-7)} = \frac{1}{36}(x^3-10x^2+17x+28) \quad (22)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x+1)(x-1)(x-7)}{(4+1)(4-1)(4-7)} = -\frac{1}{45}(x^3-7x^2-x+7) \quad (23)$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x+1)(x-1)(x-4)}{(7+1)(7-1)(7-4)} = \frac{1}{144}(x^3-4x^2-x+4) \quad (24)$$

$$
\begin{aligned}
P_3(x) &= l_0(x)f_0 + l_1(x)f_1 + l_2(x)f_2 + l_3(x)f_3 && (25)\\
&= \frac{2}{80}(x^3 - 12x^2 + 39x - 28) + 0 - \frac{63}{45}(x^3 - 7x^2 - x + 7) + \frac{342}{144}(x^3 - 4x^2 - x + 4) && (26)\\
&= \left(\frac{1}{40} - \frac{7}{5} + \frac{171}{72}\right)x^3 + \left(-\frac{3}{10} + \frac{49}{5} - \frac{171}{18}\right)^2 + \left(\frac{39}{40} + \frac{7}{5} - \frac{171}{72}\right)x + \left(-\frac{7}{5} - \frac{49}{5} + \frac{171}{8}\right) && (27)\\
&= x^3 - 1 && (28)
\end{aligned}
$$

Hence,

$$f(5) = P_3(5) = 53 - 1 = 124 \qquad (29)$$

Permanence Property

Remarks

1. Suppose that the data $(x_i, f(x_i))$, $i = 0, 1, 2, \cdots, n$, is given.
2. Assume that a new value $(x_{n+1}, f(x_{n+1}))$ at the distinct point x_{n+1} is added at the end of the table.
3. The data, $(x_i, f(x_i)), i = 0, 1, 2, \cdots, n+1$, represents a polynomial of degree $\leq (n+1)$.
4. If this polynomial of degree $(n+1)$ can be obtained by adding an extra term to the previously obtained n_{th} degree interpolating polynomial, then the interpolating polynomial is said to have the permanence property.

Note

The Lagrange interpolating polynomial does not have the permanence property.

Divided differences

Let the data, $(x_i, f(x_i))$, $i = 0, 1, 2, \cdots, n$, be given. We define the divided differences as follows.

First divided difference

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{f(x_{i+1})}{x_{i+1} - x_i} + \frac{f(x_i)}{x_i - x_{i+1}}; \quad i = 0, 1, 2, \cdots, n-1 \tag{30}$$

Example

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \; f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \; \text{and } f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} \tag{31}$$

Second divided difference

Consider any three consecutive data values $(x_i, f(x_i)),\ (x_{i+1}, f(x_{i+1})),\ (x_{i+2}, f(x_{i+2}))$, then

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}; \quad i = 0, 1, 2, \cdots, n-2 \tag{32}$$

$$= \frac{f_i}{(x_i - x_{i+1})(x_i - x_{i+2})} + \frac{f_{i+1}}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} + \frac{f_{i+2}}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} \tag{33}$$

Example

A 2_{nd} divided difference can be defined as:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}, \quad \text{and} \quad f[x_6, x_7, x_8] = \frac{f[x_7, x_8] - f[x_6, x_7]}{x_8 - x_6} \tag{34}$$

n_{th} Divided Difference

The n_{th} divided difference using all the data values in the table, is defined as

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0} \tag{35}$$

Table of Divided differences

x	$f(x)$	First	Second	Third
x_0	f_0			
		$f[x_0, x_1]$		
x_1	f_1		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	f_2		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	f_3			

Example

Obtain the divided difference table for the data

x	-1	0	2	3
$f(x)$	-8	3	1	12

x	$f(x)$	First	Second	Third
-1	-8			
		$\frac{3+8}{0+1} = 11$		
0	3		$\frac{-1-11}{2+1} = -4$	
		$\frac{1-3}{2-0} = -1$		$\frac{4+4}{3+1} = 2$
2	1		$\frac{11+1}{3-0} = 4$	
		$\frac{12-1}{3-2} = 11$		
3	12			

Newton's Divided Difference Interpolation

The Newton's divided difference interpolating polynomial is defined as

$$\begin{aligned} f(x) =& P_n(x) \\ =& f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ &+ \cdots + (x - x_0)(x - x_1), \cdots, (x - x_{n-1}) f[x_0, x_1, \cdots, x_n] \end{aligned} \tag{36}$$

Note

Newton's divided difference interpolating polynomial possesses the permanence property.

Example

Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

x	$f(x)$	First	Second	Third	Fourth
-4	1245				
		$\frac{33-1245}{-1+4} = -404$			
-1	33		$\frac{-28+404}{4} = 94$		
		$\frac{5-33}{0+1} = -28$		$\frac{10-94}{2+4} = -14$	
0	5		$\frac{2+28}{2+1} = 10$		$\frac{13+14}{5+4} = 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5+1} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

$$
\begin{aligned}
f(x) = f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] + \\
(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3] + \\
(x-x_0)(x-x_1)(x-x_2)(x-x_3)f[x_0,x_1,x_2,x_3,x_4]
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
f(x) = 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)x(-14) + \\
(x+4)(x+1)x(x-2)(3)
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
f(x) = 1245 - 404x - 1616 + (x^2+5x+4)(94) + (x^3+5x^2+4x)(-14) + \\
(x^4+3x^3-6x^2-8x)(3)
\end{aligned} \tag{39}
$$

$$
f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5 \tag{40}
$$

Example

Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula

x	1	3	4	5	7	10
$f(x)$	3	31	69	131	351	1011

Hence,

1. Interpolate at $x = 3.5$ and $x = 8.0$
2. Find, $f'(3)$ and $f''(1.5)$

x	$f(x)$	First	Second	Third	Fourth
1	3				
		14			
3	31		8		
		38		1	
4	69		12		0
		62		1	
5	131		16		0
		110		1	
7	351		22		
		220			
10	1011				

Since, the fourth order differences are zeros, the data represents a third degree polynomial.

Newton's divided difference formula gives the polynomial as

$$\begin{aligned} f(x) = f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] + \\ (x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3] \end{aligned} \tag{41}$$

$$\begin{aligned} f(x) &= 3 + (x-1)(14) + (x-1)(x-3)(8) + (x-1)(x-3)(x-4)(1) && (42) \\ &= 3 + 14x - 14 + 8x^2 - 32x + 24 + x^3 - 8x^2 + 19x - 12 && (43) \\ &= x^3 + x + 1 && (44) \end{aligned}$$

$$f(3.5) \approx P_3(3.5) = (3.5)^3 + 3.5 + 1 = 47.375 \tag{45}$$

$$f(8.0) \approx P_3(8.0) = (8.0)^3 + 8.0 + 1 = 521.0 \tag{46}$$

$$P_3'(x) = 3x^2 + 1 \tag{47}$$

$$P_3''(x) = 6x \tag{48}$$

$$f'(3) \approx P'(3) = 3(9) + 1 = 28 \tag{49}$$

$$f''(1.5) \approx P''(1.5) = 6(1.5) = 9 \tag{50}$$

Inverse Interpolation

1. Suppose that a data $(x_i, f(x_i))$; $i = 0, 1, 2, \cdots, n$, is given.
2. In interpolation, we predict the value of the ordinate $f(x_k)$ at a non-tabular point $x = x_k$.
3. In many applications, we require the value of x_k for a given value of $f(x_k)$.
4. For other problem, we consider the given data as $(f(x_i), x_i)$; $i = 0, 1, 2, \cdots, n$ and construct the interpolation polynomial.
5. That is, we consider $f(x)$ as the independent variable and x as the dependent variable. This procedure is called inverse interpolation

Exercise

1. Using Lagrange interpolation, find the unique polynomial $P(x)$ of degree 2 or less such that
$$P(1) = 1, \qquad P(3) = 27, \qquad P(4) = 64$$
2. A third degree polynomial passes through the points $(0,-1),(1,1),(2,1)$, and $(3,2)$. Determine this polynomial using Lagrange's interpolation. Hence, find the value at 1.5.
3. Using Lagrange interpolation, find y(10) given that
$$y(5) = 12,\ y(6) = 13,\ y(9) = 14,\ y(11) = 16.$$
4. Using Newton's divided difference method, find $f(1.5)$ using the data $f(1.0) = 0.7651977$, $f(1.3) = 0.6200860$, $f(1.6) = 0.4554022$, $f(1.9) = 0.2818186$, and $f(2.2) = 0.1103623$.

END OF LECTURE

THANK YOU