Afternoon Exam

Duration: 4 hours

Mobile phones, tablets, computers, smart watches, and any electronic communication or storage devices, as well as any documents, are prohibited.

Calculators without memory (middle-school type) or calculators in exam mode are permitted.

The quality of the written presentation is an important factor in grading. Humility is appreciated throughout the reasoning. You may solve the problems in any order.

Problem 1 : *(Happy New Year!)*

2026 is a remarkable integer: it is a **beprisque number**, meaning a number lying between a perfect square and a prime number. Indeed,

$$2025 = 45^2 \quad \text{and} \quad 2027 \text{ is prime.}$$

At present, we conjecture that there exist infinitely many beprisque numbers, but we do not know how to prove it (if you have some time left, feel free to try).

Here is a modest number theory problem centered around 2025, 2026, and 2027 :)

1. *(Warm-up)* Write 2025 and 2026 as the sum of two squares of natural integers, then show that 2027 cannot be written as the sum of two squares of natural integers.

2. *(Things get trickier)* Let $a, b \in \mathbb{N}^*$ such that

$$\frac{a}{b} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1350} + \frac{1}{1351}.$$

   Show that a is a multiple of 2027.

3. Let $N \in \mathbb{N}$. We know there exists an integer $n \in \mathbb{N}$ and digits $a_0, \ldots, a_n \in \{0, \ldots, 9\}$ such that

$$N = \sum_{k=0}^{n} a_k 10^k \qquad \text{(base-10 representation of } N\text{)}.$$

   We define the function f by

$$f(N) = \sum_{k=0}^{n} a_k^2.$$

   We say that N is a **happy number** if the sequence $(u_k)_{k \in \mathbb{N}}$ defined by

$$\begin{cases} u_0 = N, \\ \forall k \in \mathbb{N},\ u_{k+1} = f(u_k) \end{cases}$$

   reaches the value 1, i.e. there exists $p \in \mathbb{N}$ such that

$$u_p = 1,$$

   otherwise, we say that N is an **unhappy number**.

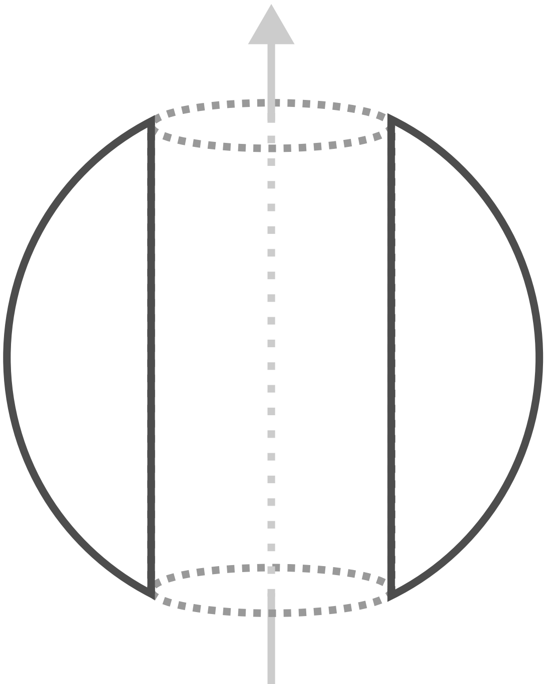
   a) Check that 2026 is a happy number, and that 2025 and 2027 are unhappy numbers :(
   b) Justify that there exist infinitely many happy numbers and infinitely many unhappy numbers.
   c) Show that one of the following two properties always holds:
      - N is a happy number;
      - there exists an index $r \in \mathbb{N}$ such that

$$u_r = 4,$$

        and the sequence $(u_k)_{k \in \mathbb{N}}$ associated with N is periodic from index r onward.

Problem 2 : ***(Axel's Present)***

Axel wants to craft a unique necklace for his girlfriend, so he buys a perfectly round solid gold sphere from a jeweler. To turn this sphere into an elegant golden bead (a "pearl"), he drills a straight cylindrical hole right through the exact center of the sphere.



He drills carefully, making sure to collect every bit of gold shavings, because the jeweler has agreed to buy back all unused gold at the same price per gram. This means Axel only pays for the amount of gold that actually remains in the final bead.

Once the hole is drilled, Axel places the newly shaped golden bead on the table. Its height (the distance from the bottom of the bead to the top) is exactly 1 cm.
Axel wants to know: What is the volume of the golden bead he created?

Axel checks the price of gold: ≈ 2500€ per cm^3. His girlfriend loves mathematics, so Axel decides to restrict himself to a neat budget of 100π€ for the golden bead.

He wonders: Which bead size(s) (i.e. which heights of the bead and which choices of original sphere radius / hole radius) exactly fit his budget of 100π €?
Explain all the size option(s) that pay exactly 100π €.

Problem 3 : ***(Santa's laser-propelled sleigh)***

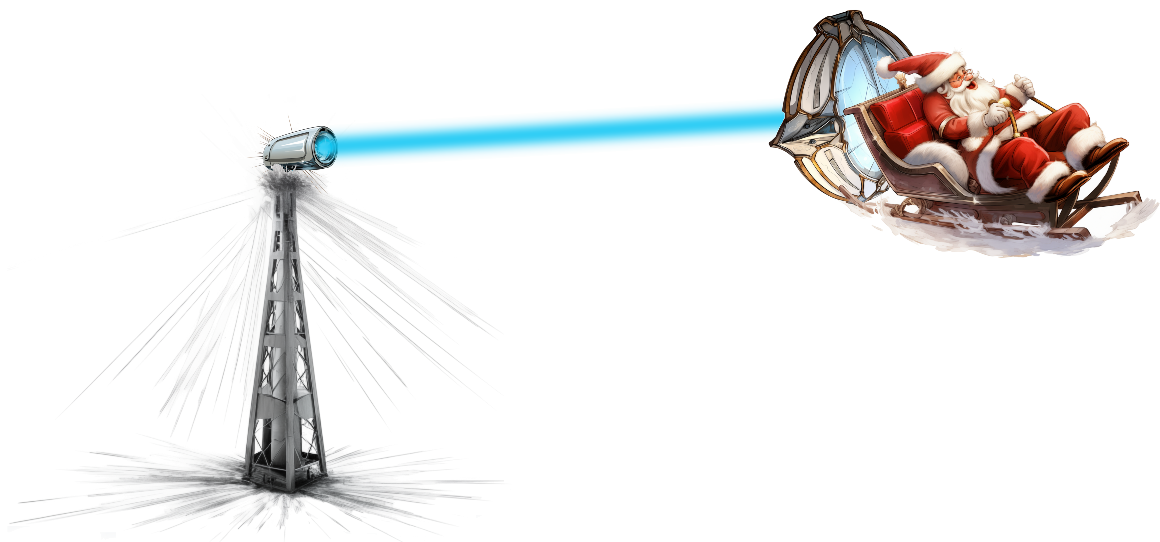
Albert school students spied on Christmas eve. They found Santa's secret: he has a very high tech sleigh! By making measurements, they managed to deduce the following about how it works:
The sleigh is powered by laser towers. The idea is simple:

- There are towers at regular intervals.
- Each tower shoots a laser that pushes the sleigh forward.
- The sleigh has a shield on the back to protect it from the laser.

When the sleigh flies past a tower, the distance to the tower is called D (in meters). The laser becomes weaker when the sleigh is further away; in fact, the speed of the sleigh is inversely proportional to the distance to the tower it is receiving the push from. The sleigh must stay at least 8 km away from every tower while receiving energy. When the sleigh passes the first tower (at time $t = 0$), it is already at the safe distance of 8 km and moving at 900 km/h (i.e. 250 m/s). The sleigh must never slow down below 40 m/s (which is 144 km/h). If it goes slower than that, it can't fly properly.

That is, how far apart can the towers be so that the sleigh never stops flying?

In the long run, what is the average speed of the sleigh?

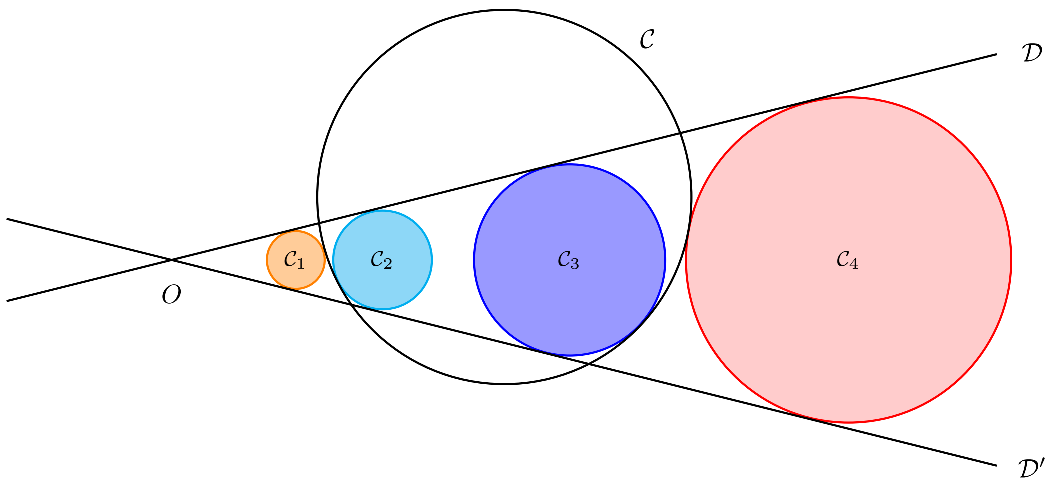


Problem 4 : ***(Sangaku!)***

Consider $\mathcal{D}$ and $\mathcal{D}'$ two lines intersecting at a point O, and $\mathcal{C}$ a circle intersecting $\mathcal{D}$ and $\mathcal{D}'$.

Consider additionally four circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ with respective radii R_1, R_2, R_3, R_4, all tangent to $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{C}$; circles $\mathcal{C}_2$ and $\mathcal{C}_3$ being internally tangent to $\mathcal{C}$ and circles $\mathcal{C}_1$ and $\mathcal{C}_4$ being externally tangent to $\mathcal{C}$.

As an illustration, here is an example of a configuration corresponding to the statement:



Prove Ohara's theorem, i.e.

$$R_1 R_4 = R_2 R_3.$$