Afternoon Exam

Duration: 4 hours

Mobile phones, tablets, computers, smart watches, and any electronic communication or storage devices, as well as any documents, are prohibited.

Calculators without memory (middle-school type) or calculators in exam mode are permitted.

The quality of the written presentation is an important factor in grading. Humility is appreciated throughout the reasoning. You may solve the problems in any order.

Problem 1 : ***(Happy New Year!)***

2026 is a remarkable integer: it is a **beprisque number**, meaning a number lying between a perfect square and a prime number. Indeed,

$$2025 = 45^2 \quad \text{and} \quad 2027 \text{ is prime.}$$

At present, we conjecture that there exist infinitely many beprisque numbers, but we do not know how to prove it (if you have some time left, feel free to try).

Here is a modest number theory problem centered around 2025, 2026, and 2027 :)

1. *(Warm-up)* Write 2025 and 2026 as the sum of two squares of natural integers, then show that 2027 cannot be written as the sum of two squares of natural integers.

   We have
$$2025 = 45^2 + 0^2,$$
   and also
$$2025 = 36^2 + 27^2,$$
   since $36^2 + 27^2 = 1296 + 729 = 2025$.

   Moreover,
$$2026 = 45^2 + 1^2,$$
   because $45^2 = 2025$.

   Impossibility of writing 2027 as a sum of two squares

   We begin by observing that
$$2027 \equiv 3 \pmod 4.$$
   Now, for any integer n, we have
$$n^2 \equiv 0 \text{ or } 1 \pmod 4.$$
   Thus, for all $a, b \in \mathbb{N}$,
$$a^2 + b^2 \equiv 0, 1 \text{ or } 2 \pmod 4,$$
   and therefore it can never be congruent to 3 modulo 4.

2. *(Things get trickier)* Let $a, b \in \mathbb{N}^*$ such that
$$\frac{a}{b} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1350} + \frac{1}{1351}.$$
   Show that a is a multiple of 2027.

   It is trivial that
$$\begin{aligned} a = {} & 263773158847080840725950396517784792107678908242703397147734944417 \\ & 821793255684499697860484377458864481654081356499417199551704093287626 \\ & 887802936317996020661790435042838458522805654712974941691541485083635 \\ & 835861067369957181168847471204180553844909667277452248866125286293291 \\ & 569683444233669152076963669512588600014114090592673413505656624904564 \\ & 288499812056357206949807497976678580884595001475761042204227808008124 \\ & 448402828830168191232124799892222094462977470529706918967441567697276 \\ & 809627145944647131427158446599326403040271549162542561769846367362446 \\ & 579988319600811441488829962364351 1, \end{aligned}$$
   which is obviously divisible by 2027.

   More seriously
$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1350} + \frac{1}{1351} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{1350} + \frac{1}{1351} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1350}\right) \\ &= \frac{1}{676} + \frac{1}{677} + \cdots + \frac{1}{1350} + \frac{1}{1351} \\ &= \left(\frac{1}{676} + \frac{1}{1351}\right) + \left(\frac{1}{677} + \frac{1}{1350}\right) + \dots \\ &= \frac{2027}{676 \times 1351} + \frac{2027}{677 \times 1350} + \dots \end{aligned}$$

The primality of 2027 allows us to conclude.

3. Let $N \in \mathbb{N}$. We know there exists an integer $n \in \mathbb{N}$ and digits $a_0, \ldots, a_n \in \{0, \ldots, 9\}$ such that

$$N = \sum_{k=0}^{n} a_k 10^k \qquad \text{(base-10 representation of } N\text{)}.$$

We define the function f by

$$f(N) = \sum_{k=0}^{n} a_k^2.$$

We say that N is a **happy number** if the sequence $(u_k)_{k\in\mathbb{N}}$ defined by

$$\begin{cases} u_0 = N, \\ \forall k \in \mathbb{N},\ u_{k+1} = f(u_k) \end{cases}$$

reaches the value 1, i.e. there exists $p \in \mathbb{N}$ such that

$$u_p = 1,$$

otherwise, we say that N is an **unhappy number**.

a) Check that 2026 is a happy number, and that 2025 and 2027 are unhappy numbers :(

b) Justify that there exist infinitely many happy numbers and infinitely many unhappy numbers.

c) Show that one of the following two properties always holds:
   - N is a happy number;
   - there exists an index $r \in \mathbb{N}$ such that

$$u_r = 4,$$

   and the sequence $(u_k)_{k\in\mathbb{N}}$ associated with N is periodic from index r onward.

a) 2026 is a **happy** number because the recursive sequence defined by

$$\begin{cases} u_0 = 2025, \\ \forall k \in \mathbb{N},\ u_{k+1} = f(u_k) \end{cases}$$

satisfies $u_5 = 1$. Indeed,

$$\begin{aligned} u_1 &= 2^2 + 0^2 + 2^2 + 6^2 = 44 \\ u_2 &= 4^2 + 4^2 = 32 \\ u_3 &= 3^2 + 2^2 = 13 \\ u_4 &= 1^2 + 3^2 = 10 \\ u_5 &= 1^2 + 0^2 = 1. \end{aligned}$$

For the number 2025, the situation is different:

$$\begin{aligned} 2^2 + 0^2 + 2^2 + 5^2 &= 33 \\ 3^2 + 3^2 &= 18 \\ 1^2 + 8^3 &= 65 \\ 6^2 + 5^2 &= 61 \\ 6^2 + 1^2 &= 37 \\ 3^2 + 7^2 &= 58 \\ 5^2 + 8^2 &= 89 \\ 8^2 + 9^2 &= 145 \\ 1^2 + 4^2 + 5^2 &= 42 \end{aligned}$$

$$\begin{aligned}
4^2 + 2^2 &= 20 \\
2^2 + 0^2 &= 4 \\
4^2 &= 16 \\
1^2 + 6^2 &= 37.
\end{aligned}$$

We thus observe that the sequence associated with 2025 is periodic starting from index 5, and has the following period of length 8:

$$4 \longrightarrow 16 \longrightarrow 37 \longrightarrow 58 \longrightarrow 89 \longrightarrow 145 \longrightarrow 42 \longrightarrow 20.$$

In particular, there is no index such that the sequence associated with 2025 reaches the value 1: therefore 2025 is **unhappy**. Similarly,

$$\begin{aligned}
2^2 + 0^2 + 2^2 + 7^2 &= 57 \\
5^2 + 7^2 &= 74 \\
7^2 + 4^2 &= 65 \\
6^2 + 5^2 &= 61 \\
6^2 + 1^2 &= 37 \\
3^2 + 7^2 &= 58 \\
5^2 + 8^2 &= 89 \\
8^2 + 9^2 &= 145 \\
1^2 + 4^2 + 5^2 &= 42 \\
4^2 + 2^2 &= 20 \\
2^2 + 0^2 &= 4 \\
4^2 &= 16 \\
1^2 + 6^2 &= 37.
\end{aligned}$$

We thus observe that the sequence associated with 2027 is also periodic starting from index 5, and has the following period of length 8:

$$4 \longrightarrow 16 \longrightarrow 37 \longrightarrow 58 \longrightarrow 89 \longrightarrow 145 \longrightarrow 42 \longrightarrow 20.$$

In particular, there is no index such that the sequence associated with 2027 reaches the value 1: therefore 2027 is **unhappy**.

b) We have

$$\forall k \in \mathbb{N},\ 10^k \text{ is a happy number.}$$

Likewise,

$$\forall k \in \mathbb{N},\ 4 \times 10^k \text{ is an unhappy number.}$$

c) First, let us note that

$$\begin{aligned}
&1, 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 94, 97, 100, 103, 109, \\
&129, 130, 133, 139, 167, 176, 188, 190, 192, 193, 203, 208, 219, 226, 230, 236 \text{ and } 239 \text{ are happy,}
\end{aligned}$$

and all other numbers less than or equal to 243 are unhappy, and their associated sequence becomes periodic from some index onward, with period

$$4 \longrightarrow 16 \longrightarrow 37 \longrightarrow 58 \longrightarrow 89 \longrightarrow 145 \longrightarrow 42 \longrightarrow 20.$$

If N is a three-digit integer strictly greater than 244, then $f(N)$ is bounded above by

$$9^2 + 9^2 + 9^2 = 243.$$

Thus, we easily reduce the problem to the study carried out above.

If N is an integer with $n \geqslant 4$ digits, we may write

$$N = \sum_{k=0}^{n-1} a_k 10^k, \qquad \text{where } a_k \in \{0, \dots, 9\} \text{ and } \ a_{n-1} \geqslant 1.$$

But

$$\begin{aligned} f(N) &= \sum_{k=0}^{n-1} a_k^2 \\ &\leqslant 9^2 n = 81n. \end{aligned}$$

Now, for every $n \geqslant 4$, we have

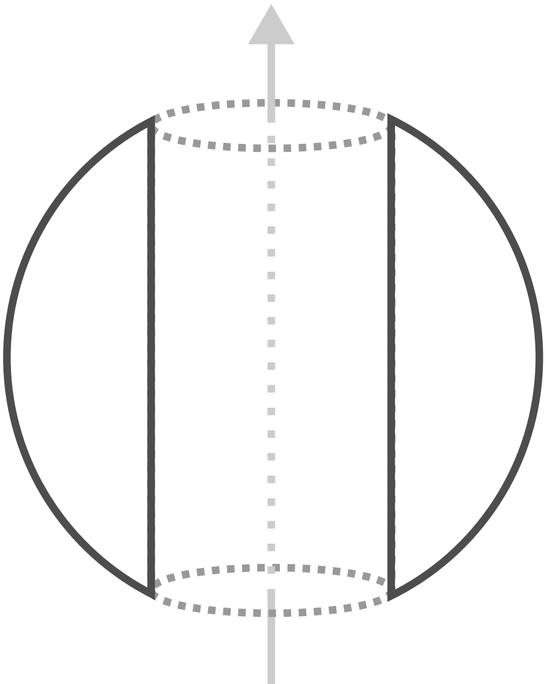
$$81n < 10^{n-1}.$$

One may draw the variation table of the function $x \longmapsto 10^{x-1} - 81x$ *to verify this: it is strictly increasing on* $[3, +\infty[$ *and vanishes at* $x_0 \approx 3.446$.

Thus, $f(N)$ has at most $n-1$ digits. By iterating f at most $n-3$ times, we eventually reduce to one of the two previous cases, which completes the proof.

<u>Problem 2</u> : *(Axel's Present)*

Axel wants to craft a unique necklace for his girlfriend, so he buys a perfectly round solid gold sphere from a jeweler. To turn this sphere into an elegant golden bead (a "pearl"), he drills a straight cylindrical hole right through the exact center of the sphere.



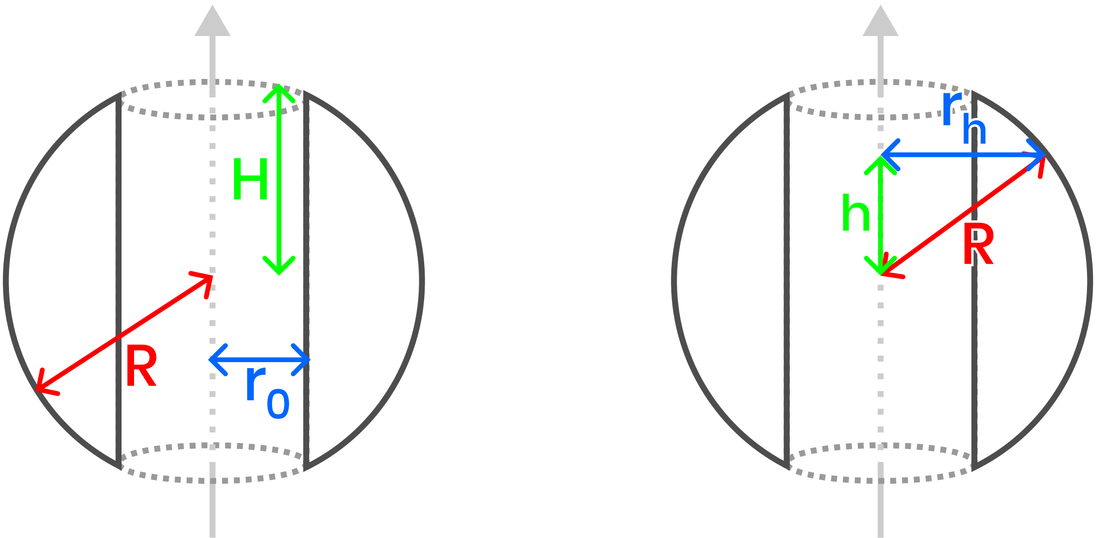
He drills carefully, making sure to collect every bit of gold shavings, because the jeweler has agreed to buy back all unused gold at the same price per gram. This means Axel only pays for the amount of gold that actually remains in the final bead.

Once the hole is drilled, Axel places the newly shaped golden bead on the table. Its height (the distance from the bottom of the bead to the top) is exactly 1 cm.
Axel wants to know: What is the volume of the golden bead he created?

Axel checks the price of gold: $\approx$ 2500€ per cm^3. His girlfriend loves mathematics, so Axel decides to restrict himself to a neat budget of 100π€ for the golden bead.

He wonders: Which bead size(s) (i.e. which heights of the bead and which choices of original sphere radius / hole radius) exactly fit his budget of 100π €?
Explain all the size option(s) that pay exactly 100π €.

Let's annotate the figure, setting the origin at the center of the sphere:



From this, we get via Pythagoras:

$$H^2 + r_0^2 = R^2 \implies r_0 = \sqrt{R^2 - H^2}$$

and

$$h^2 + r_h^2 = R^2 \implies r_h = \sqrt{R^2 - h^2}$$

Let's express the volume we are looking for as an integral:

$$\begin{aligned}
\mathcal{V} &= \int_{-H}^{H} \left(\int_{r_0}^{r_h} 2\pi r \, dr \right) \, dh \\
&= \pi \int_{-H}^{H} \left(\int_{\sqrt{R^2-H^2}}^{\sqrt{R^2-h^2}} 2r \, dr \right) \, dh \\
&= \pi \int_{-H}^{H} \left[r^2 \right]_{\sqrt{R^2-H^2}}^{\sqrt{R^2-h^2}} \, dh \\
&= \pi \int_{-H}^{H} (R^2 - h^2) - (R^2 - H^2) \, dh \\
&= \pi \int_{-H}^{H} H^2 - h^2 \, dh \\
&= \pi \left[H^2 h - \frac{1}{3} h^3 \right]_{-H}^{H} \\
&= \pi \left(\left(H^3 - \frac{1}{3} H^3 \right) - \left(-H^3 + \frac{1}{3} H^3 \right) \right) \\
&= \frac{4\pi}{3} H^3
\end{aligned}$$

This is a famous result: the volume of a sphere with a hole drilled through it depends only on the height of the resulting solid, not on the original sphere radius.
Replacing with value $H = 0.5$cm:

$$\mathcal{V} = \frac{4\pi}{3} \left(\frac{1}{2} \right)^3 = \frac{\pi}{6} \text{ cm}^3$$

Axel can use a maximum volume

$$\mathcal{V} = \frac{100\pi}{2500} = \frac{\pi}{25} \text{ cm}^3$$

of gold. So we have

$$\frac{4\pi}{3} H^3 = \frac{\pi}{25} \implies H^3 = \frac{3}{100} \implies H = \sqrt[3]{0.03} \approx 0.31 \text{ cm.}$$

Hence, any $R > H = \sqrt[3]{0.03} \approx 0.31$ cm. can work, provided the drilled cylinder has radius $r_0 = \sqrt{R^2 - H^2}$ (with H defined above).

Problem 3 : ***(Santa's laser-propelled sleigh)***

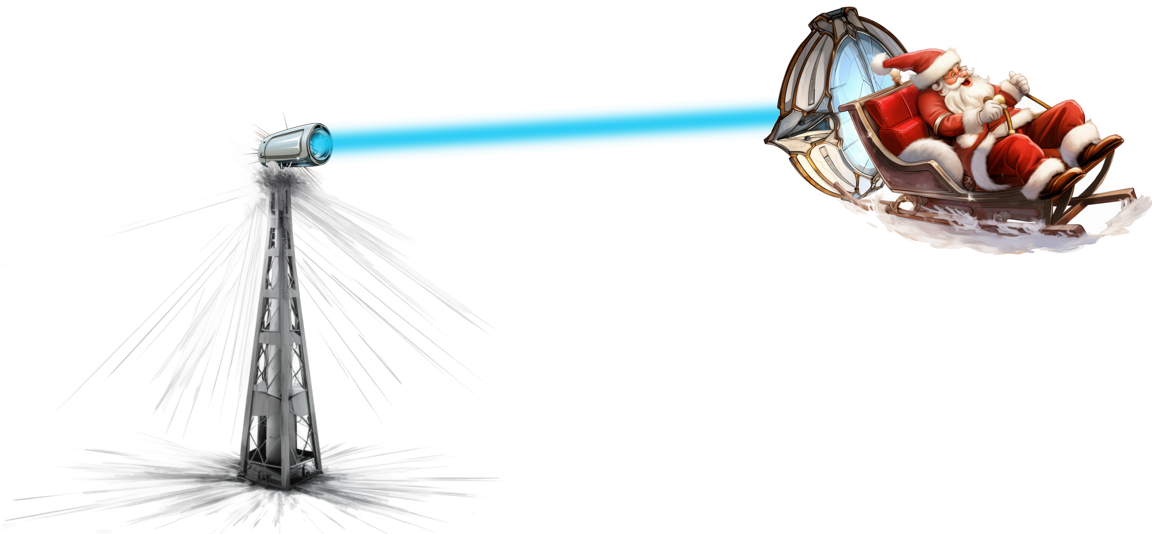
Albert school students spied on Christmas eve. They found Santa's secret: he has a very high tech sleigh! By making measurements, they managed to deduce the following about how it works:
The sleigh is powered by laser towers. The idea is simple:

- There are towers at regular intervals.
- Each tower shoots a laser that pushes the sleigh forward.
- The sleigh has a shield on the back to protect it from the laser.

When the sleigh flies past a tower, the distance to the tower is called D (in meters). The laser becomes weaker when the sleigh is further away; in fact, the speed of the sleigh is inversely proportional to the distance to the tower it is receiving the push from. The sleigh must stay at least 8 km away from every tower while receiving energy. When the sleigh passes the first tower (at time $t = 0$), it is already at the safe distance of 8 km and moving at 900 km/h (i.e. 250 m/s). The sleigh must never slow down below 40 m/s (which is 144 km/h). If it goes slower than that, it can't fly properly.

That is, how far apart can the towers be so that the sleigh never stops flying?

In the long run, what is the average speed of the sleigh?



Problem: The sleigh moves past laser towers that push it forward. The speed of the sleigh depends on its distance D to the tower:

$$V = \frac{\beta}{D}, \quad \beta = 2{,}000{,}000 \text{ s}^{-1}.$$

We are asked to find the maximum possible spacing between two towers such that the sleigh never goes below $V_{\min} = 40\,\text{m/s}$.

Step 1: Determine the minimum distance to a tower corresponding to the minimum speed. The sleigh must satisfy

$$V \geq V_{\min} \implies \frac{\beta}{D} \geq V_{\min} \implies D \leq \frac{\beta}{V_{\min}}.$$

Substitute $\beta = 2{,}000{,}000\,\text{s}^{-1}$ and $V_{\min} = 40\,\text{m/s}$:

$$D_{\max} = \frac{2{,}000{,}000}{40} = 50{,}000 \text{ m}.$$

Step 2: Compute the distance the sleigh can travel between two towers. The sleigh starts at $D_0 = 8{,}000\,\text{m}$ (closest safe distance to the first tower) and moves forward until it reaches $D_{\max} = 50{,}000\,\text{m}$ from that tower. The distance traveled along the trajectory corresponds to the difference between these distances, assuming the towers are aligned along a straight line and the distance D is measured along that line:

$$\Delta x = D_{\max} - D_0 = 50{,}000 - 8{,}000 = 42{,}000 \text{ m}.$$

Step 3: Conclusion Hence, the maximum spacing $L_{\max}$ between two consecutive towers so that the sleigh never drops below $V_{\min}$ is

$$\boxed{L_{\max} = 42{,}000 \text{ m} = 42 \text{ km}}.$$

The sleigh moves past each tower with speed

$$V(D) = \frac{\beta}{D}, \quad \beta = 2{,}000{,}000 \text{ s}^{-1}.$$

The distance to each tower varies between the closest safe distance $D_{\min} = 8{,}000$ m and the maximum distance $D_{\max} = 50{,}000$ m, which corresponds to the minimum speed $V_{\min} = 40$ m/s.

Step 1: Express time to travel a small distance dD. The sleigh moves away from the tower along the line connecting the tower and the sleigh. At distance D, the instantaneous speed is

$$V(D) = \frac{dD}{dt} = \frac{\beta}{D} \implies dt = \frac{D}{\beta}\,dD.$$

Step 2: Compute the total time to go from $D_{\min}$ to $D_{\max}$.

$$T = \int_{D_{\min}}^{D_{\max}} \frac{D}{\beta}\,dD = \frac{1}{\beta}\left[\frac{D^2}{2}\right]_{D_{\min}}^{D_{\max}} = \frac{D_{\max}^2 - D_{\min}^2}{2\beta}.$$

Step 3: Compute the total distance traveled along that interval. The distance along the line between the towers is

$$L = D_{\max} - D_{\min}.$$

Step 4: Compute the average speed.
The average speed is

$$\overline{V} = \frac{\text{distance}}{\text{time}} = \frac{L}{T} = \frac{D_{\max} - D_{\min}}{\frac{D_{\max}^2 - D_{\min}^2}{2\beta}} = \frac{2\beta(D_{\max} - D_{\min})}{D_{\max}^2 - D_{\min}^2}.$$

Factor $D_{\max} - D_{\min}$ in the denominator:

$$D_{\max}^2 - D_{\min}^2 = (D_{\max} - D_{\min})(D_{\max} + D_{\min}),$$

so

$$\overline{V} = \frac{2\beta}{D_{\max} + D_{\min}}.$$

Step 5: Substitute numbers.

$$\overline{V} = \frac{2 \cdot 2{,}000{,}000}{50{,}000 + 8{,}000} = \frac{4{,}000{,}000}{58{,}000} \approx 68.97 \text{ m/s} \approx 248 \text{ km/h}.$$

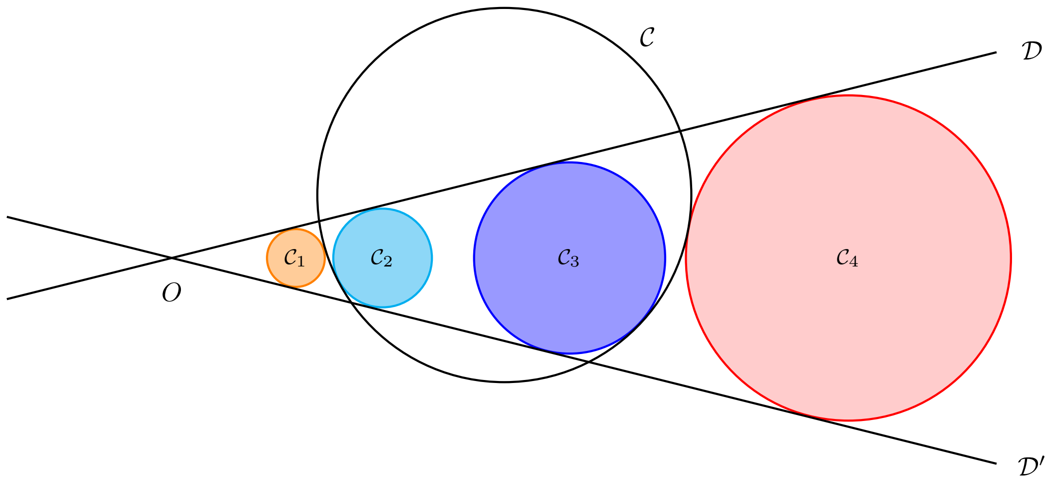
Hence: $\boxed{\overline{V} \approx 69 \text{ m/s (or 248 km/h)}}$

Problem 4 : ***(Sangaku!)***

Consider $\mathcal{D}$ and $\mathcal{D}'$ two lines intersecting at a point O, and $\mathcal{C}$ a circle intersecting $\mathcal{D}$ and $\mathcal{D}'$.

Consider additionally four circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ with respective radii R_1, R_2, R_3, R_4, all tangent to $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{C}$; circles $\mathcal{C}_2$ and $\mathcal{C}_3$ being internally tangent to $\mathcal{C}$ and circles $\mathcal{C}_1$ and $\mathcal{C}_4$ being externally tangent to $\mathcal{C}$.

As an illustration, here is an example of a configuration corresponding to the statement:



Prove Ohara's theorem, i.e.

$$R_1 R_4 = R_2 R_3.$$

Let us first observe that since the lines $\mathcal{D}$ and $\mathcal{D}'$ intersect at O, any circle tangent to both $\mathcal{D}$ and $\mathcal{D}'$ must necessarily have its center lying on the bisector of the angle $\widehat{\mathcal{D}O\mathcal{D}'}$.
Without loss of generality, we work in an orthonormal coordinate system such that:

- $O = (0, 0)$,
- the angle bisector is the x-axis,
- the angle between each of the lines $\mathcal{D}, \mathcal{D}'$ and the x-axis is $\theta \in \left]0, \frac{\pi}{2}\right[$.

In this configuration, any circle $\mathcal{C}'$ with center $(x, 0)$ tangent to $\mathcal{D}$ and $\mathcal{D}'$ has radius

$$r = x \sin\theta.$$

We will use the following characterization:

$$\begin{aligned} &\text{Two circles of radii } R \text{ and } r \text{ are externally tangent} \\ &\iff \text{the distance between their centers equals } R + r, \\ &\text{Two circles of radii } R \text{ and } r \text{ are internally tangent} \\ &\iff \text{the distance between their centers equals } |R - r|. \end{aligned} \tag{$\star$}$$

Let $\mathcal{C}$ be a circle with center (α, β) and radius R, intersecting the lines $\mathcal{D}$ and $\mathcal{D}'$. *Note that β is not necessarily zero, which makes Ohara's theorem truly remarkable.*

Consider an arbitrary circle $\mathcal{C}'$ tangent to $\mathcal{D}$, $\mathcal{D}'$, and to $\mathcal{C}$, with center $(x, 0)$ and radius

$$r = x \sin\theta.$$

D'après $(\star)$, the condition of external tangency ensures that

$$\sqrt{(x - \alpha)^2 + \beta^2} = R + r.$$

Squaring both sides, we obtain

$$(x - \alpha)^2 + \beta^2 - (R + r)^2 = 0.$$

Since $x = \dfrac{r}{\sin\theta}$, we obtain

$$\left(\frac{r}{\sin\theta} - \alpha\right)^2 + \beta^2 - (R+r)^2 = 0. \tag{E_+}$$

Let us note that the solutions of (E_+) (as a quadratic equation in r, thus having at most two solutions) are R_1 and R_4.

Similarly, the condition of internal tangency ensures that

$$\sqrt{(x-\alpha)^2 + \beta^2} = R - r,$$

Squaring both sides, we obtain

$$(x-\alpha)^2 + \beta^2 - (R-r)^2 = 0.$$

Since $x = \dfrac{r}{\sin\theta}$, we obtain

$$\left(\frac{r}{\sin\theta} - \alpha\right)^2 + \beta^2 - (R-r)^2 = 0. \tag{E_-}$$

Let us note that the solutions of (E_-) (as a quadratic equation in r, thus having at most two solutions) are R_2 and R_3.

Thus, the equations (E_+) and (E_-) are two quadratic equations in r having:

- the same leading coefficient $\dfrac{1}{\sin^2\theta} - 1$,
- the same constant term $\alpha^2 + \beta^2 - R^2$.

By the relationship between coefficients and roots, the product of the roots must be identical for both equations, hence

$$R_1R_4 = R_2R_3 = \frac{\alpha^2 + \beta^2 - R^2}{\dfrac{1}{\sin^2\theta} - 1}.$$

This proves Ohara's theorem.

Recall that for a quadratic equation $ar^2 + br + c = 0$ with $a \neq 0$, the product of the roots is c/a. Here, the fact that both equations share the same leading coefficient and the same constant term guarantees that the products of their roots coincide. A grand finale.