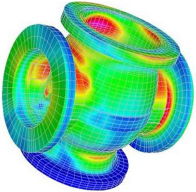
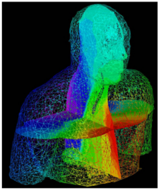
The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

Prof. Dr. Eleni Chatzi

Lecture 2 - 01 October, 2015





Linear Analysis Assumptions

- Infinitesimally small displacements
- Linearly elastic material
- No gaps or overlaps occurring during deformations - The displacement field is smooth
- Nature of Boundary Conditions remains unchanged

Steady State Assumption

- No dependence on time

What kind of problems are not steady state and linear?

- Material behaves Nonlinearly
- Geometric Nonlinearity (ex. p-Δ effects, follower force)
- Contact Problems (Hertzian stress)
- Loads vary fast compared to the eigenfrequencies of the structure
- Varying Boundary conditions (ex. freezing of water, welding)

General feature:

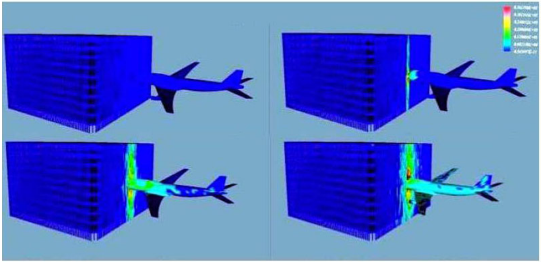
Response becomes load path dependent

What is the added value of being able to assess the Nonlinear non-steady state response of structures?

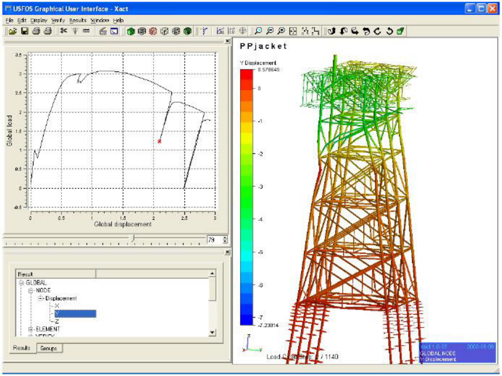
Assessing the:

- Structural response of structures to extreme events (rock-fall, earthquake, hurricanes)
- Performance (failures and deformations) of soils
- Verifying simple models

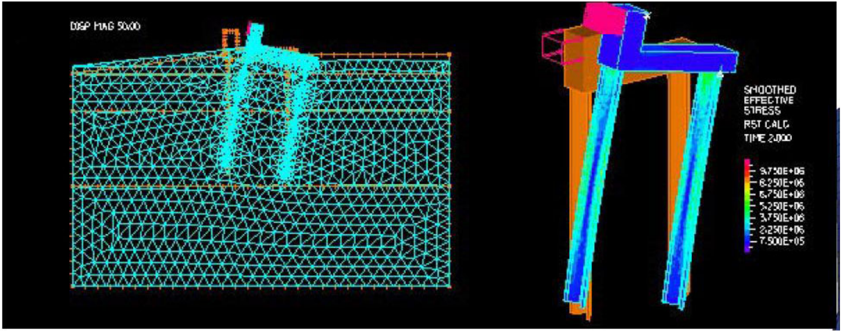
Collapse Analysis of the World Trade Center



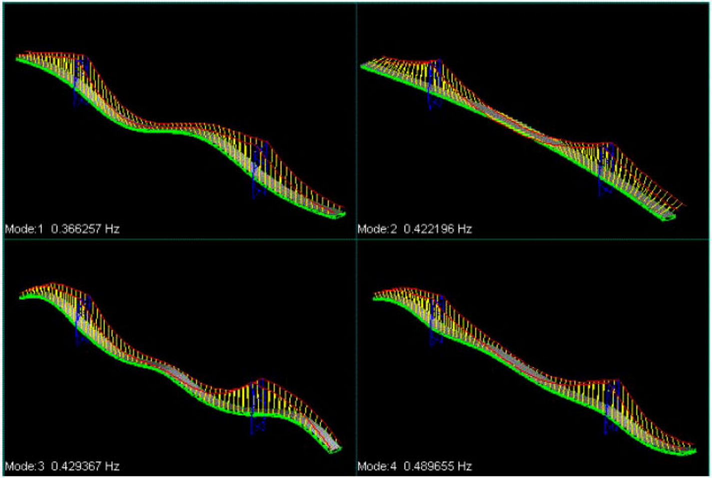
Ultimate collapse capacity of jacket structure



Analysis of soil performance



Analysis of bridge response



Steady state problems (Linear/Nonlinear):

The response of the system does not change over time

$$\mathbf{K}\mathbf{U} = \mathbf{R}$$

Propagation problems (Linear/Nonlinear):

The response of the system changes over time

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{R}(t)$$

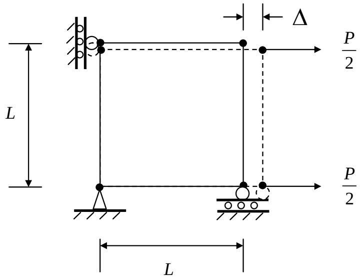
Eigenvalue problems:

No unique solution to the response of the system

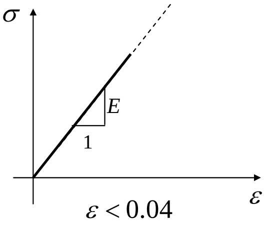
Classification of Nonlinear analyses

Type of analysis	**Description**	**Typical formulation used**	**Stress and strain measures used**
Materially-nonlinear only	Infinitesimal displacements and strains; stress train relation is non-linear	Materially-nonlinear-only (MNO)	Engineering strain and stress
Large displacements, large rotations but small strains	Displacements and rotations of fibers are large; but fiber extensions and angle changes between fibers are small; stress strain relationship may be linear or non-linear	Total Lagrange (TL) Updated Lagrange (UL)	Second Piola-Kirchoff stress, Green-Lagrange strain Cauchy stress, Almansi strain
Large displacements, large rotations and large strains	Displacements and rotations of fibers are large; fiber extensions and angle changes between fibers may also be large; stress strain relationship may be linear or non-linear	Total Lagrange (TL) Updated Lagrange (UL)	Second Piola-Kirchoff stress, Green-Lagrange strain Cauchy stress, Logarithmic strain

Linear Elastic

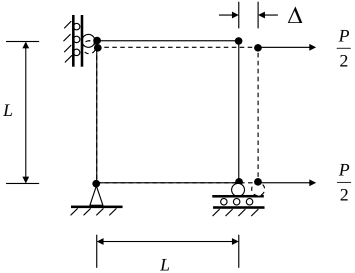


$$\boxed{\begin{aligned}\sigma &= P/A\\ \varepsilon &= \sigma/E\\ \Delta &= \varepsilon L\end{aligned}}$$



Infinitesimal Displacements

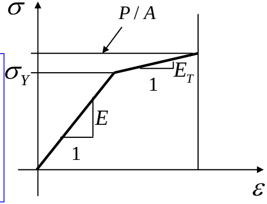
Material Nonlinearity only



$$\sigma = P / A$$

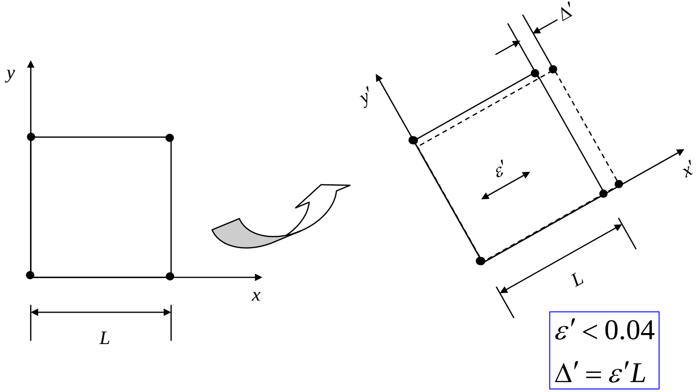
$$\varepsilon = \frac{\sigma_Y}{E} + \frac{\sigma - \sigma_Y}{E_T}$$

$$\varepsilon < 0.04$$



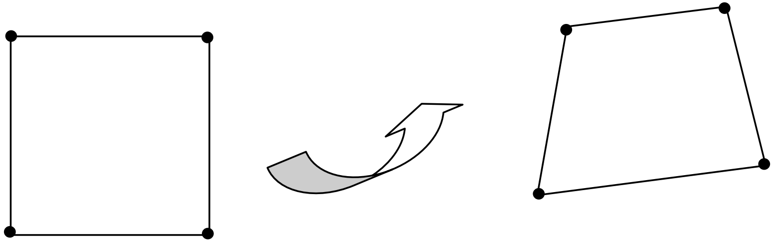
Infinitesimal Displacements, but Nonlinear Stress Strain relation

Large displacements, small strains



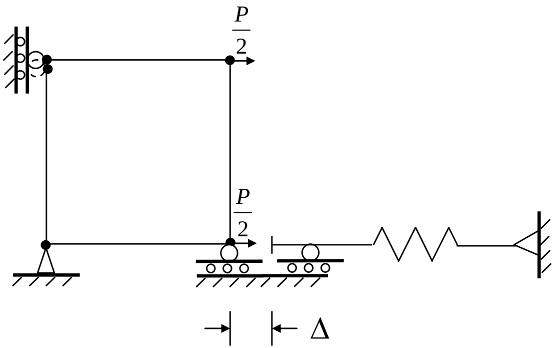
Linear or Nonlinear material behavior

Large displacements, large strains



Linear or Nonlinear material behavior

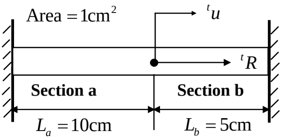
Change in BC for displacement

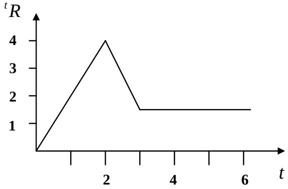


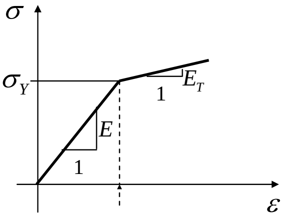
Example: Simple Bar Structure

Material Nonlinearity only

Assumptions: Small displacements, strains, load is applied slowly.







$$\varepsilon_Y = 0.002$$

$E = 10^7 \, \mathrm{N/cm^2}$

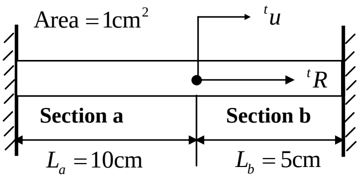
$E_T = 10^5 \, \mathrm{N/cm^2}$

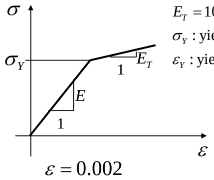
σ_Y : yield stress

ε_Y : yield strain

$\Rightarrow$ Calculate the displacement at the point of load application.

Example: Simple Bar Structure



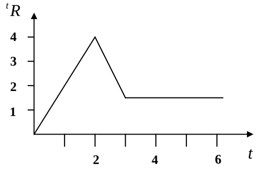


$E = 10^7\,\mathrm{N/cm^2}$

$E_T = 10^5\,\mathrm{N/cm^2}$

σ_Y : yield stress

ε_Y : yield strain



$$^t\varepsilon_a = \frac{^tu}{L_a},\ ^t\varepsilon_b = -\frac{^tu}{L_b}$$

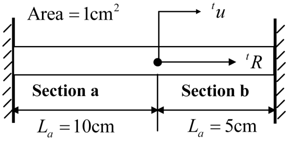
$$^tR + {^t\sigma_b} A = {^t\sigma_a} A$$

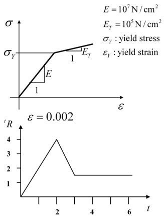
$$^t\varepsilon = \frac{^t\sigma}{E} \text{ (elastic region)}$$

$$^t\varepsilon = \varepsilon_Y + \frac{^t\sigma - \sigma_Y}{E_T} \text{ (plastic region)}$$

$$\Delta\varepsilon = \frac{\Delta\sigma}{E} \text{(unloading)}$$

In the beginning both Sections are elastic





$$
{}^t\varepsilon_a = \frac{{}^tu}{L_a},\ {}^t\varepsilon_b = -\frac{{}^tu}{L_b}
$$

$$
{}^tR + {}^t\sigma_b A = {}^t\sigma_a A
$$

$$
{}^t\varepsilon = \frac{{}^t\sigma}{E} \text{ (elastic region)}
$$

$$
{}^t\varepsilon = \varepsilon_Y + \frac{{}^t\sigma - \sigma_Y}{E_T} \text{ (plastic region)}
$$

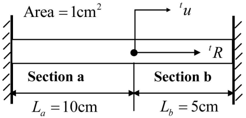
$$
\Delta\varepsilon = \frac{\Delta\sigma}{E} \text{(unloading)}
$$

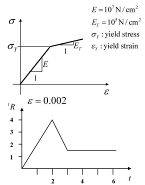
$$
{}^tR = EA\,{}^tu\left(\frac{1}{L_a} + \frac{1}{L_b}\right) \Rightarrow {}^tu = \frac{{}^tR}{3\cdot 10^6}
$$

$$
\sigma_a = \frac{{}^tR}{3A},\ \sigma_b = -\frac{2}{3}\frac{{}^tR}{A}
$$

Section A is elastic while Section B is plastic

Since the stress on Section B is higher, it will yield first at time t^*:





$$ {}^t\varepsilon_a = \frac{{}^t u}{L_a},\ {}^t\varepsilon_b = -\frac{{}^t u}{L_b} $$

$$ {}^t R + {}^t\sigma_b A = {}^t\sigma_a A $$

$$ {}^t\varepsilon = \frac{{}^t\sigma}{E} \text{ (elastic region)} $$

$$ {}^t\varepsilon = \varepsilon_Y + \frac{{}^t\sigma - \sigma_Y}{E_T} \text{ (plastic region)} $$

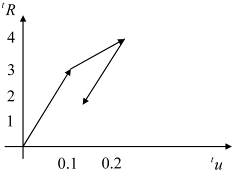
$$ \Delta\varepsilon = \frac{\Delta\sigma}{E} \text{(unloading)} $$

section b will be plastic when ${}^{t^*}R = \frac{3}{2}\sigma_Y A$

$$ \sigma_a = E\frac{{}^t u}{L_a},\ \sigma_b = -E_T\left(\frac{{}^t u}{L_b} - \varepsilon_Y\right) - \sigma_Y $$

$$ {}^t R = \frac{EA\,{}^t u}{L_a} + \frac{E_T A\,{}^t u}{L_b} - E_T\varepsilon_Y A + \sigma_Y A \Rightarrow $$

$$ {}^t u = \frac{{}^t R / A + E_T\varepsilon_Y - \sigma_Y}{E/L_a + E/L_b} = \frac{{}^t R}{1.02\cdot 10^6} - 1.9412\cdot 10^{-2} $$



Unloading occurs before Section A yields.

Conclusion from the previous example:

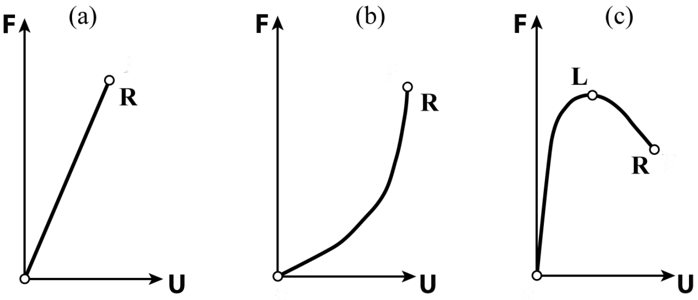
The basic problem in general Nonlinear analysis is to find a state of equilibrium between externally applied loads and element nodal forces

$$
\begin{aligned}
&{}^{t}\mathbf{R} - {}^{t}\mathbf{F} = 0 \\
&{}^{t}\mathbf{R} = {}^{t}\mathbf{R}_B + {}^{t}\mathbf{R}_S + {}^{t}\mathbf{R}_C \\
&{}^{t}\mathbf{F} = \sum_m \int_{{}^{t}V_m} {}^{t}\mathbf{B}^{(m)T} \; {}^{t}\boldsymbol{\tau}^{(m)} \; {}^{t}d\mathbf{V}^{(m)}
\end{aligned}
$$

where $\mathbf{R}_B$: body forces, $\mathbf{R}_S$: surface forces, $\mathbf{R}_C$: nodal forces

- We must achieve equilibrium for all time steps when incrementing the loading
- Very general approach
- Includes implicitly also dynamic analysis!

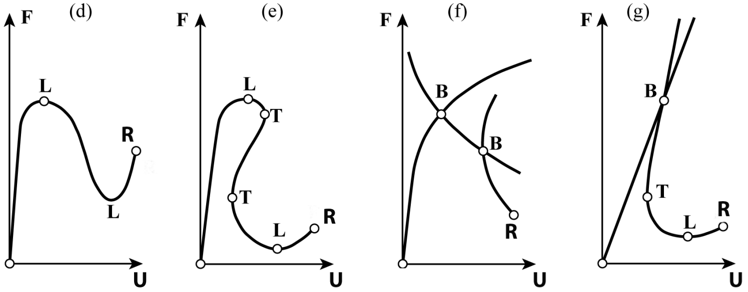
Basic Types



Basic flavors of nonlinear response: (a) Linear until brittle failure, (b) Stiffening or hardening, (c) Softening.

(a) elastic (brittle behavior materials); (b) increase of stiffness as load increases (pneumatic structures); (c) commonly observed behavior (concrete or steel)

Complex Types

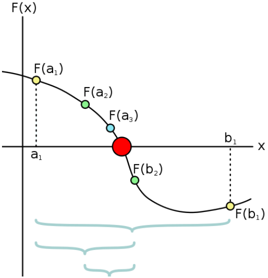


More complex response patterns: (d) snap-through, (e) snap-back, (f) bifurcation, (g) bifurcation combined with limit points and snap-back.

(d) typical for curved structures; (e) the response turns back on itself (trussed dome); (f),(g) Buckling & Stability problems

Root finding for single variable NL problems $f(x) = 0$

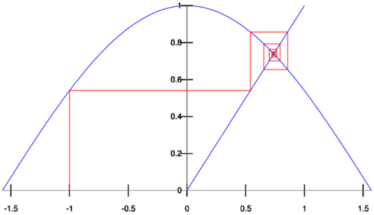
- Bisection Method



Assumption: $f[a,\ b] \in \Re$ and continuous

If $f(a) > 0, f(b) < 0 \Rightarrow a \leq \bar{x} \leq b \Rightarrow f(\bar{x}) = 0$

- Fixed Point Iteration

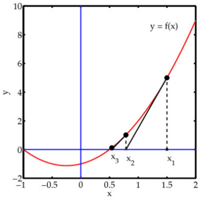


Write $f(x) = 0$ in the form $f(x) = x - q(x)$, the solution $\bar{x}$ satisfies $\bar{x} = q(\bar{x})$
Recurrence relation: $x_{k+1} = g(x_k)$

Convergence: If $g'(x)$ is defined over $[a,\ b]$ and a positive constant K exists with $|\dot{g}(x)| \leq K$, $\forall x \in [a,\ b]$ then $g(x)$ has a unique fixed point $\bar{x} \in [a,\ b]$.

Root finding for single variable NL problems $f(x) = 0$

- Newton (Raphson) Method

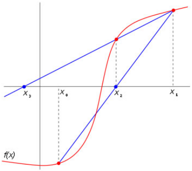


Defined by the **recurrence relation**

$$\mathbf{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

terminate when $|x_{k+1} - x_k| \leq \epsilon$, $\epsilon <<$
Convergence: quadratic
$\rightarrow |\bar{x} - x_{k+1}| \leq C|\bar{x} - x_{k+1}|^2$

- Secant Method



Defined by the **recurrence relation**

$$\mathbf{x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}}$$

Convergence: superlinear order
$\rightarrow \alpha = \frac{1+\sqrt{5}}{2}$ (golden ratio)

The basic approach in incremental analysis is:
Find a state of equilibrium between externally applied loads and element nodal forces

$$ {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F} = 0 $$

Assuming that ${}^{t+\Delta t}\mathbf{R}$ is independent of the deformations we have

$$ {}^{t+\Delta t}\mathbf{R} = {}^{t}\mathbf{F} + \mathbf{F} $$

We know the solution ${}^{t}\mathbf{F}$ at time t and F is the increment in the nodal point forces corresponding to an increment in the displacements and stresses from time t to time $t + \Delta t$. This we can approximate by

$$ \mathbf{F} = {}^{t}\mathbf{K}\mathbf{U} $$

Newton-Raphson Method

Assume the **tangent stiffness matrix**:

$$ {}^{t}\mathbf{K} = \frac{\partial^{t}\mathbf{F}}{\partial^{t}\mathbf{U}} $$

We may now substitute the tangent stiffness matrix into the equilibrium relation

$$ {}^{t}\mathbf{K}\mathbf{U} = {}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F} $$

which gives us a scheme for the calculation of the displacements:

$$ {}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \mathbf{U} $$

The exact displacements at time $t + \Delta t$ correspond to the applied loads at $t + \Delta t$, however we only determined these approximately as we used a tangent stiffness matrix thus we may have to **iterate** to find the solution.

Incremental Analysis

We may use the Newton-Raphson iteration scheme to find the equilibrium within each load increment

$$^{t+\Delta t}\mathbf{K}^{(i-1)}\Delta\mathbf{U}^{(i)} =^{t+\Delta t} \mathbf{R} -^{t\Delta t} \mathbf{F}^{(i-1)}$$

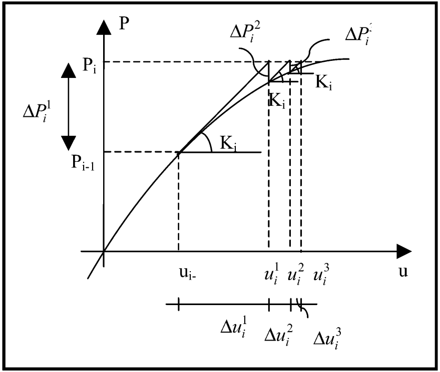
(**out of balance load vector**)

$$^{t+\Delta t}\mathbf{U}^{(i)} =^{t+\Delta t} \mathbf{U}^{(i-1)} + \Delta\mathbf{U}^{(i)}$$

with Initial Conditions

$$^{t+\Delta t}\mathbf{U}^{(0)} =^{t} \mathbf{U};\ ^{t+\Delta t}\mathbf{K}^{(0)} =^{t} \mathbf{K};\ ^{t+\Delta t}\mathbf{F}^{(0)} =^{t} \mathbf{F}$$

It may be expensive to calculate the tangent stiffness matrix. In the **Modified Newton-Raphson** iteration scheme it is only calculated in the beginning of each new load step



In the **quasi-Newton** iteration schemes the secant stiffness matrix is used instead of the tangent matrix

Simple Bar Example - Revisited

$$({}^{t}K_a + {}^{t}K_b)\Delta u^{(i)} = {}^{t+\Delta t}R - ({}^{t+\Delta t}F_a^{(i-1)} - {}^{t+\Delta t}F_b^{(i-1)})$$

$${}^{t+\Delta t}u^{(i)} = {}^{t+\Delta t}u^{(i-1)} + \Delta u^{(i)}$$

with initial conditions

$${}^{t+\Delta t}u^{(0)} = {}^{t}u; \quad {}^{t+\Delta t}F_a^{(0)} = {}^{t}F_a \qquad {}^{t+\Delta t}F_b^{(0)} = {}^{t}F_b$$

$${}^{t}K_a = \frac{{}^{t}CA}{L_a}; \quad {}^{t}K_b = \frac{{}^{t}CA}{L_b}$$

$${}^{t}C \begin{cases} = E & \text{if section is elastic} \\ = E_T & \text{if section is plastic} \end{cases}$$

Simple Bar Example - Revisited

Load step 1: $t = 1$:

$$({}^{0}K_a + {}^{0}K_b)\Delta u^{(1)} = {}^{1}R - {}^{1}F_a^{(0)} - {}^{1}F_b^{(0)}$$

$$\Downarrow$$

$$\Delta u^{(1)} = \frac{2\times 10^4}{10^7(\frac{1}{10}+\frac{1}{5})} = 6.6667\times 10^{-3}$$

Iteration 1: $(i = 1)$

$${}^{1}u^{(1)} = {}^{1}u^{(0)} + \Delta u^{(1)} = 6.6667\times 10^{-3}$$

$${}^{1}\varepsilon_a^{(1)} = \frac{{}^{1}u^{(1)}}{L_a} = 6.6667\times 10^{-4} < \varepsilon_Y \text{ (elastic section!)}$$

$${}^{1}\varepsilon_b^{(1)} = \frac{{}^{1}u^{(1)}}{L_b} = 1.3333\times 10^{-3} < \varepsilon_Y \text{ (elastic section!)}$$

$${}^{1}F_a^{(1)} = 6.6667\times 10^{3}; \qquad {}^{1}F_b^{(1)} = 1.3333\times 10^{4}$$

$$({}^{0}K_a + {}^{0}K_b)\Delta u^{(2)} = {}^{1}R - {}^{1}F_a^{(1)} - {}^{1}F_b^{(1)} = 0$$

Convergence in one iteration!

$${}^{1}u = 6.6667\times 10^{-3}$$

Simple Bar Example - Revisited

Load step 2: $t = 2$:

$$({}^1K_a + {}^1K_b)\Delta u^{(1)} = {}^2R - {}^2F_a^{(0)} - {}^2F_b^{(0)}$$

$$\Downarrow$$

$$\Delta u^{(1)} = \frac{(4\times10^4) - (6.6667\times10^3) - (1.333\times10^4)}{10^7(\frac{1}{10} + \frac{1}{5})} = 6.6667\times10^{-3}$$

Iteration 1: $(i = 1)$

$${}^2u^{(1)} = {}^2u^{(0)} + \Delta u^{(1)} = 1.3333\times10^{-2}$$

$${}^2\varepsilon_a^{(1)} = 1.3333\times10^{-3} < \varepsilon_Y \text{ (elastic section!)}$$

$${}^2\varepsilon_b^{(1)} = 2.6667\times10^{-3} > \varepsilon_Y \text{ (plastic section!)}$$

$${}^1F_a^{(1)} = 1.3333\times10^4; \qquad {}^1F_b^{(1)} = (E^T({}^2\varepsilon_b^{(1)} - \varepsilon_Y) + \sigma_Y)A = 2.0067\times10^4$$

$$({}^1K_a + {}^1K_b)\Delta u^{(2)} = {}^2R - {}^2F_a^{(1)} - {}^2F_b^{(1)} \Rightarrow \Delta u^{(2)} = 2.2\times10^{-3}$$

Simple Bar Example - Revisited

The procedure is repeated and the results of successive iterations are tabulated in the accompanying table.

i	$\Delta u^{(i)}$	${}^{2}u^{(i)}$
2		
3	1.45E-03	1.70E-02
4	9.58E-04	1.79E-02
5	6.32E-04	1.86E-02
6	4.17E-04	1.90E-02
7	2.76E-04	1.93E-02

The decision on where to stop, depends on **engineering judgment**, i.e., you may specify a tolerance level when the calculated displacement increment ΔU is small enough to terminate the iteration process and subsequently continue with the next load step at time $t = 3$.