# MEE 3017 System Modelling and Analysis

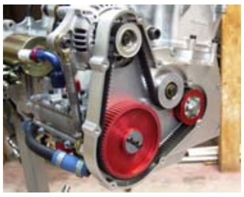
## L01 Mechanical Systems



### Mechanical Systems

Mechanical systems, machines and devices usually consist of:
- masses (point masses, rigid bodies);
- connecting elements (bars, beams, springs, belts, dampers);
- machine elements (bearings, gears, guidances, cylinders with pistons).





From system dynamics point of view all these components are classified in three groups:

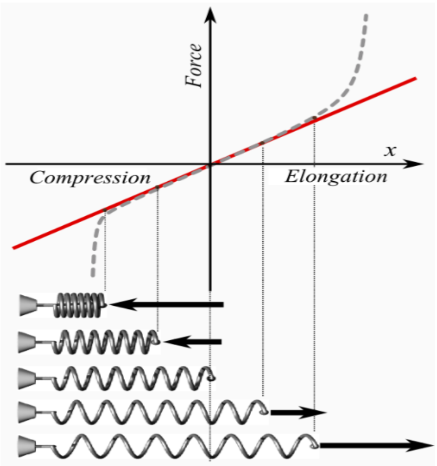
-Compliant elements – which can be specified as Springs
-Dissipitaitive elements –which can be specified as dampers
-Inertial elements – which can be specified as masses

## Translational Mechanical System Components

Mechanical translational systems are comprised of a group of mechanical elements that only translate. In one dimensional analysis mostly “x” is selected as motion axis.

### Springs

Any mechanical element that undergoes a change in shape (size) when subjected to a force can be characterized as a spring.
Ability to change in shape according to applied force is generally described as compliance and its reciprocal is called as stiffness.
The parameter that determines the ratio between applied force and change in size is the spring constant K.

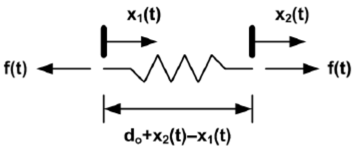







## Translational Mechanical System Components-Springs

If the force applied to an element is an algebraic function of the displacement across the element, the element is represented by a spring.




The constitutive law describing the force applied to the spring is

$$f(t) = K(x_2(t) - x_1(t)) = K\Delta x(t)$$

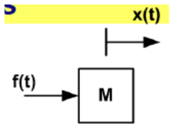
Spring elements are **mechanical potential energy** storage elements

$$U = E_p = \frac{1}{2}K(\Delta x)^2$$

K–spring stiffness, spring constant (N/m)
$d_o$–unstretched spring length (m)

Translational Mechanical System Components-Masses

Mass is a property of a physical system or body, observable and measurable as a resistance to being accelerated by a force.(therefore they are classified as inertial elements.)




The constitutive law describing the force applied to mass is

$$f(t) = M\frac{d^2x(t)}{dt^2}$$

Mass elements are **mechanical kinetic energy** storage elements

$$T = E_k = \frac{1}{2}Mv^2$$

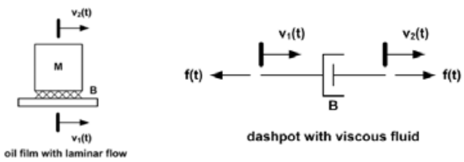
In many common situations, mass can be thought of as representing the quantity of matter in an object
M–mass (kg)





Translational Mechanical System Components-Frictional elements

Forces that are algebraic functions of velocity are frictional forces. They always acts against the direction of the motion. Due to the friction, some energy of the system is lost in frictional elements.Therefore they are called dissipative elements.




The constitutive law describing the force applied to the damper is

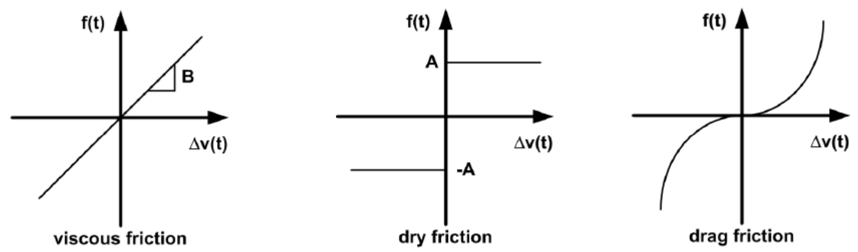
$$f(t) = B(v_2(t) - v_1(t)) = B\Delta V(t)$$

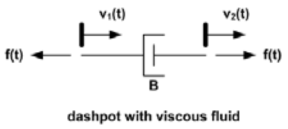
Virtual vork done by

$$\delta w = -Bv\delta x$$

C,B–viscous friction coefficient (N/(m/s))

## Translational Mechanical System Components-Frictional elements




$$f(t) = B(v_2(t) - v_1(t))$$

$$f(t) = A\,\mathrm{sgn}\left[\Delta v(t)\right] = \begin{cases} -A & \text{if} \quad \Delta v(t) < 0 \\ 0 & \text{if} \quad \Delta v(t) = 0 \\ A & \text{if} \quad \Delta v(t) > 0 \end{cases}$$

$$f(t) = D\left|\Delta v(t)\right|\Delta v(t)$$

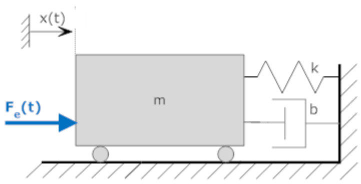
Drag friction (i.e., a body moving through a fluid) has the constitutive law





## Translational Mechanical System Analysis-Free Body diagrams

A schematic isolating an object (or part of an object) from its environment for the purpose of revealing all external forces and moments acting on the object. Free body diagrams are helpful in applying Newton’s 2nd Law of motion to objects.

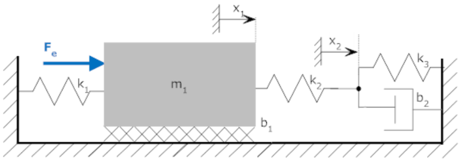




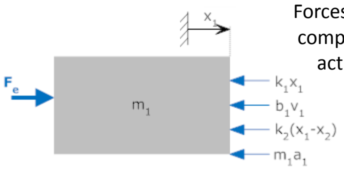
1.Isolate all the masses in the system.
2.İdentify the forces acting to each specific mass
3. Assume that the inertial force. (It is always opposite to the positive direction convention.)
4. Apply that sum of the forces are zero.

## Translational Mechanical System Analysis-FBD examples 2DoF

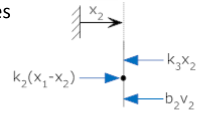
1.Isolate all the masses in the system.
2.Identify the forces acting to each specific mass
3. Assume that the inertial force. (It is always opposite to the positive direction convention.)
4. Apply that sum of the forces are zero.




$$m_1\ddot{x}_1 + b_1\dot{x}_1 + (k_2 + k_1)x_1 - k_2x_2 = F_e$$
$$b_2\dot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = 0$$




Forces transfered from one component to other using action reaction forces principle



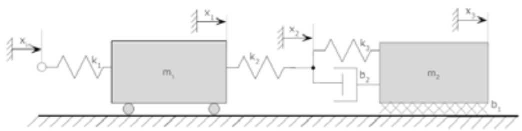

$$F_e - m_1a_1 - k_2(x_1 - x_2) - b_1v_1 - k_1x_1 = 0 \qquad -k_2(x_1 - x_2) + k_3x_2 + b_2v_2 = 0$$

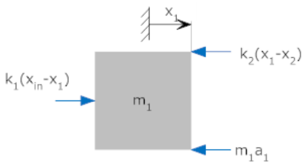


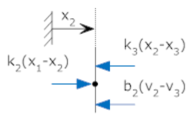


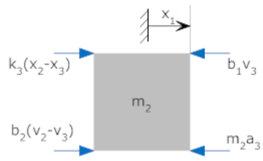
## Translational Mechanical System Analysis-FBD examples: Position input

1.Isolate all the masses in the system.
2.İdentify the forces acting to each specific mass.
3. Assume that the inertial force. (It is always opposite to the positive direction convention.)
4. Apply that sum of the forces are zero.





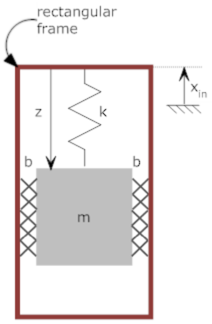

$$\sum_{\text{at } x_1} F = 0$$
$$k_1(x_{in} - x_1) - k_2(x_1 - x_2) - m_1a_1 = 0$$
$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = k_1x_{in}$$

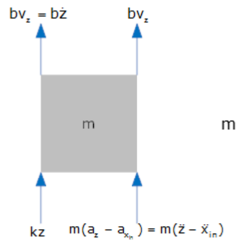
$$\sum_{\text{at } x_2} F = 0$$
$$k_2(x_1 - x_2) - k_3(x_2 - x_3) - b_2(v_2 - v_3) = 0$$
$$b_2\dot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 - b_2\dot{x}_3 - k_3x_3 = 0$$

$$\sum_{\text{at } x_3} F = 0$$
$$k_3(x_2 - x_3) + b_2(v_2 - v_3) - m_2a_3 - b_1v_3 = 0$$
$$m_2\ddot{x}_3 + (b_1 + b_2)\dot{x}_3 + k_3x_3 - b_2v_2 - k_3x_2 = 0$$

## Translational Mechanical System Analysis-FBD examples: Relative coordinates

1.Isolate all the masses in the system.
2.İdentify the forces acting to each specific mass.
3. Assume that the inertial force. (It is always opposite to the positive direction convention.)
4. Apply that sum of the forces are zero.




$$\sum_{\text{at } z} F = 0$$

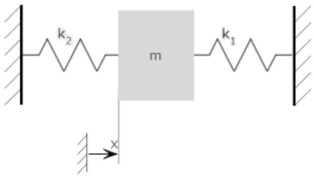
$$m \cdot (\ddot{z} - \ddot{x}_{in}) + 2b \cdot \dot{z} + k \cdot z = 0$$

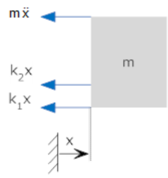
$$m \cdot \ddot{z} + 2b \cdot \dot{z} + k \cdot z = m \cdot \ddot{x}_{in}$$





## Translational Mechanical System Analysis-FBD examples: Parallel Springs




$$\sum_{\text{at } x} F = 0$$

$$k_1 \cdot x + k_2 \cdot x = f$$

$$(k_1 + k_2) \cdot x = f$$

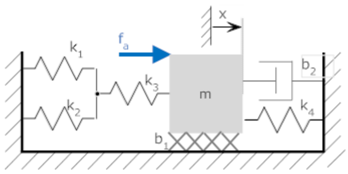
$$k_{eq} \cdot x = f$$

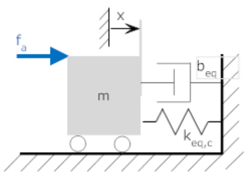
When two (or more) spring or friction elements are in parallel (i.e., such that both ends of the elements are connected), their values add

$$k_{eq} = k_1 + k_2 + \ldots \qquad b_{eq} = b_1 + b_2 + \ldots$$

When two (or more) spring or friction elements are in series (such that they have one end in common, with nothing else attached at that node), the inverse of the values add.

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots \qquad \frac{1}{b_{eq}} = \frac{1}{b_1} + \frac{1}{b_2} + \ldots$$



## Rotational Mechanical System Components

The quantities used in rotating systems are very similar to those in translating systems. The table below lists the analogous quantities for rotating and translating systems.

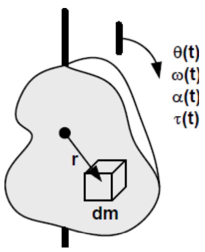
| Rotating Systems | | | Translating Systems | |
|---|---|---|---|---|
| Quantity | Unit | | Quantity | Unit |
| Moment of Inertia - J | kg-m² | | Mass - m | kg |
| Torque - $\tau$ | N-m | | Force - f | N |
| Angle - $\theta$ | rad | | Length - l | m |
| Angular velocity - $\dot{\theta} = \omega$ | rad/sec | | Velocity - $\dot{x} = v$ | m/s |
| Angular acceleration - $\ddot{\theta} = \alpha$ | rad/sec² | | Acceleration - $\ddot{x} = a$ | m/s² |
| Spring Constant - $K_r$ | N-m/rad | | Spring Constant - k | N/m |
| Friction Coefficient - $B_r$ | N-m-s/rad | | Friction Coefficient - g | N-s/m |





## Rotational Mechanical System Components-Inertia Element

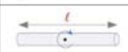
In rotating mechanical systems, the inertia elements are masses that rotate and are characterized by moment of inertia.
Applying Newton's Second Law to differential mass dm and integrating over the entire body, the time rate of change of the angular momentum is equal to the summation of the applied torques




$$\frac{d}{dt}(Jw(t)) = \sum T(t)$$

If the inertia is constant

$$J\frac{d^2\theta}{dt^2} = \sum T(t)$$

| Shape | Image | Moment of Inertia, J |
|---|---|---|
| Cylinder, radius=r, mass=m Rotating about center axis |  | $\frac{1}{2}mr^2$ |
| Solid Sphere, radius=r, mass=m Rotating about center |  | $\frac{2}{5}mr^2$ |
| Uniform Rod, length=ℓ, mass=m Rotating about end |  | $\frac{1}{3}m\ell^2$ |
| Uniform Rod, length=ℓ, mass=m Rotating about center |  | $\frac{1}{12}m\ell^2$ |
| Mass at end of massless rod, length=ℓ, mass=m Rotating about end |  | $m\ell^2$ |

Inertial elements are energy storage elements

$$T = E_K = \frac{1}{2}J\omega^2$$

J – moment of inertia (kg·m²)
$\sum T$ – summation of applied torques (N·m)

## Rotational Mechanical System Components-Inertia Element

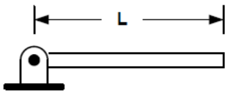
Calculation of moment of inertia

For rotations not about the center of mass, use the Parallel Axis Theorem

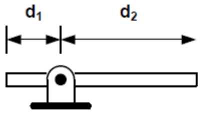
$$J = J_o + ma^2$$

a – distance between axis of rotation and axis through center of mass (m)

$J_o$ – moment of inertia about axis through center of mass (kg·m²)




$$J = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$



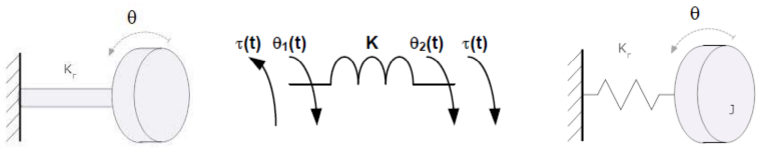

$$J = \frac{1}{3}M\frac{d_1^3 + d_2^3}{d_1 + d_2}$$



L02 Mechanical Systems

## Rotational Mechanical System Components-Spring Element

A rotational spring is an element that is deformed (wound or unwound) in direct proportion to the amount of torque applied.Therefore the torque applied to an element is an algebraic function of the angular displacement across the element




Constitutive law describing the torque applied to the spring is

$$\tau(t) = K\left[\theta_2(t) - \theta_1(t)\right] = K\Delta\theta(t)$$

Spring elements are energy storage elements

$$U = E_p = \frac{1}{2}K(\Delta\theta)^2$$

K – spring stiffness, spring constant (N·m/rad)

## Rotational Mechanical System Components-Dissipitative Element

Torques that are algebraic functions of rotational velocity are frictional torques.

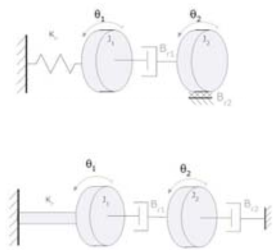

Constitutive law describing the torque applied to the spring is

$$\tau(t)=B\left[\omega_2(t)-\omega_1(t)\right]=B\Delta\omega(t)$$

Dissipitative elements do virtual work on the system.

$$\delta w = -Bw\delta\theta$$

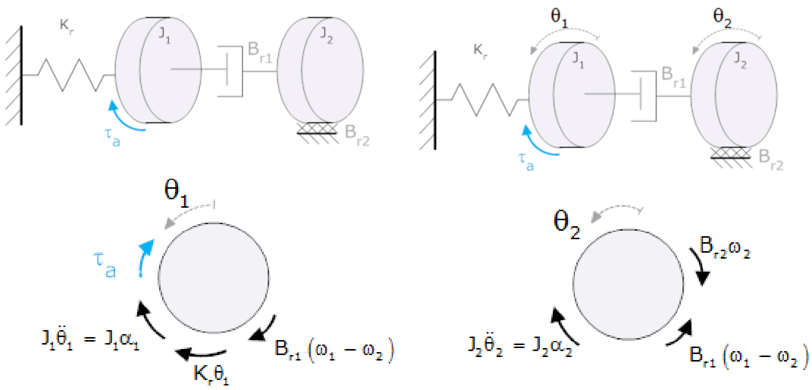
B – viscous friction coefficient (N·m/(rad/s)) )







## Rotational Mechanical System Components- Example Free Body Diagram



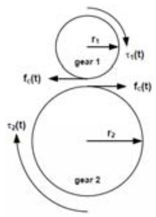

$$\tau_a + J_1\alpha_1 + B_{r1}(\omega_1-\omega_2) + K_r\theta_1 = 0$$

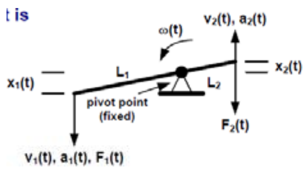
$$J_1\ddot{\theta}_1 + B_{r1}\dot{\theta}_1 + K_r\theta_1 - B_{r1}\dot{\theta}_2 = -\tau_a$$

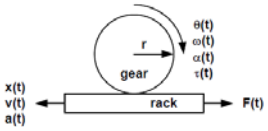
$$J_2\alpha_2 + B_{r2}\omega_2 - B_{r1}(\omega_1-\omega_2) = 0$$

$$J_2\ddot{\theta}_2 + (B_{r2}+B_{r1})\dot{\theta}_2 - B_{r1}\dot{\theta}_1 = 0$$

## Rotational Mechanical System Components- Transmissions




$$\frac{\theta_1}{\theta_2} = \frac{\alpha_1}{\alpha_2} = N$$

$$\frac{\tau_2}{\tau_1} = \frac{r_2}{r_1} = N$$

$$\frac{x_1}{x_2} = \frac{a_1}{a_2} = N_L$$

$$\frac{v_1}{v_2} = \frac{L_1}{L_2}$$

$$\frac{F_2}{F_1} = N_L$$

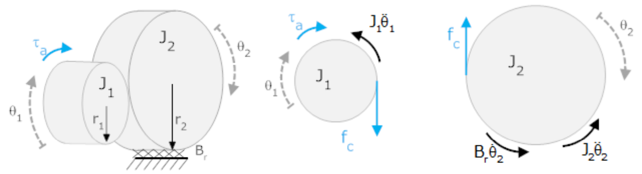
$$v = r\omega \;\rightarrow\; \frac{v}{\omega} = r = N_r$$

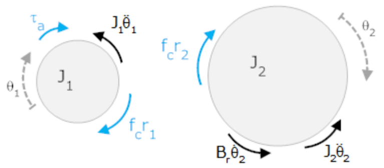
$$f_c = F = \frac{\tau}{r} \;\rightarrow\; \frac{\tau}{F} = N_r$$





## Rotational Mechanical System Components- Example Gears




$$\tau_a + f_c r_1 - J_1\ddot{\theta}_1 = 0$$
$$f_c r_2 - J_2\ddot{\theta}_2 - B_r\dot{\theta}_2 = 0$$

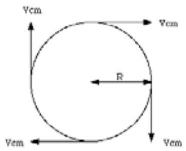
$f_c r_2 - J_2\ddot{\theta}_2 - B_r\dot{\theta}_2 = 0$ and $\theta_2 = -\frac{r_1}{r_2}\theta_1$, so

$$f_c r_2 + J_2\frac{r_1}{r_2}\ddot{\theta}_1 + B_r\frac{r_1}{r_2}\dot{\theta}_1 = 0$$

$$f_c = -J_2\frac{r_1}{r_2^2}\ddot{\theta}_1 - B_r\frac{r_1}{r_2^2}\dot{\theta}_1$$

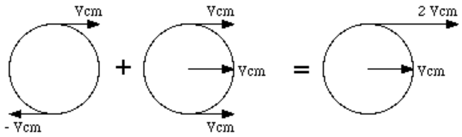
## The rolling motion

Rotational Motion of Wheel




$$V_{cm} = \omega R$$

Motion of wheel is sum of rotational and translational motion.




When the wheel is in contact with the ground, its bottom part is at rest with respect to the ground. This implies that besides a rotational motion the wheel experiences a linear motion with a velocity equal to $+ v_{cm}$ .

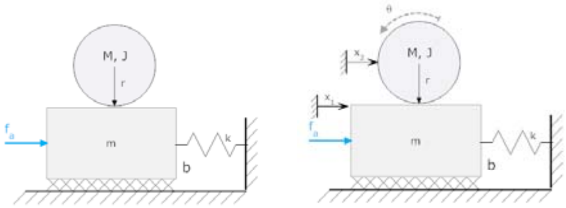
It is clear that the top of the wheel moves twice as fast as the center and the bottom of the wheel does not move at all.

$$T = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

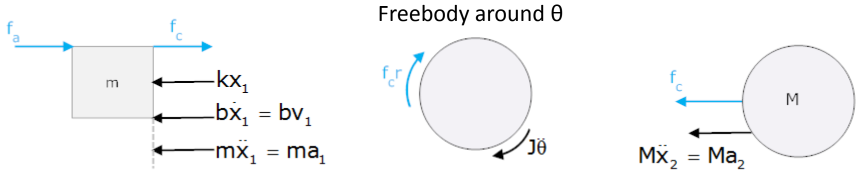




## Rotational Mechanical System Components- Translating and rotating systems




Freebody around θ




$$m\ddot{x}_1 + b\dot{x}_1 + kx_1 - f_c - f_a = 0 \qquad J\ddot{\theta} + f_c R = 0 \qquad f_c + M\ddot{x}_2 = 0$$

If $x_1$ changes while $x_2$ remains constant, then $r \cdot \theta = x_1$

If $x_2$ changes while $x_1$ remains constant, then $r \cdot \theta = -x_2$

Therefore, by superposition

$$\mathbf{r \cdot \theta = x_1 - x_2}$$

## Mechanical System Analysis

Translational

| Fundamental Quantities | SI unit |
|---|---|
| Time - t | second (s) |
| Mass - m | kilogram (kg) |
| Length - l | meter (m) |
| Force - f | Newton (N) |
| Energy - w | Joule (J) [W-s, N-m] |
| Power - p | Watt (W) [J/s] |
| | |
| Spring Constant - k | (N/m) |
| Friction Coefficient - b | (N-s/m) |

Rotational

| Fundamental Quantities | SI unit |
|---|---|
| Time - t | second (s) |
| Moment of Inertia - J | kilogram (kg-m□) |
| Angle - θ | radians (rad) |
| Torque - τ | Newton (N-m) |
| Energy - w | Joule (J) [W-s, N-m] |
| Power - p | Watt (W) [J/s] |
| | |
| Spring Constant - $K_r$ | (N-m/rad) |
| Friction Coefficient - $B_r$ | (N-m-s/rad) |





## References

These slides are heavily adapted from the Prof Erik Cheever's excelent website http://lpsa.swarthmore.edu/ and from the lecture notes of Dr. Robert G. Landers http://web.mst.edu/~landersr/