Homework #6

1) Need to solve Diffusion equation

$$-D\frac{d}{dx}\frac{d\phi(x)}{dx} + \Sigma_a \phi(x) = S(x) \qquad \text{Bound cond: } \phi(\pm a) = 0$$

a). $S(x) = 0$ for $x \in [-a, a]$ $\quad \phi(\pm a) = 0 \qquad L = \sqrt{\frac{\Sigma_a}{D}}$

$$-\frac{1}{\Sigma_a}\frac{d}{dx}\frac{d\phi(x)}{dx} + L^2 \phi(x) = 0$$

$$\phi(x) = C_1 e^{x/L} + C_2 e^{-x/L}$$

$$0 = C_1 e^{a/L} + C_2 e^{-a/L}$$

$$C_1 e^{a/L} = -C_2 e^{-a/L}$$

$$C_1 = -C_2 e^{-2a/L}$$

$$\boxed{\phi(x) = 0}$$

$$0 = -C_2 e^{-2a/L} e^{a/L} + C_2 e^{-a/L}$$

so $C_2 = 0$, $C_1 = 0$

b). $S(x) = S_0$ for $x \in [-a, a]$ $\qquad$ with $L = \sqrt{\frac{\Sigma_a}{D}}$

$$-\frac{1}{L^2}\frac{d}{dx}\frac{d\phi(x)}{dx} + L^2 \phi(x) = S_0$$

$$\phi(x) = C_1 e^{x/L} + C_2 e^{-x/L} \qquad B_0 + B_1 x + B_2 x^2$$

$$-D(2B_2) + \Sigma_a (B_0 + B_1 x + B_2 x^2) = S_0$$

$B_2 = 0$, $B_1 = 0$ since no x and x^2 term exists.

$$B_0 \Sigma_a = S_0 \qquad B_0 = S_0/\Sigma_a$$

$$\phi(x) = C_1 e^{x/L} + C_2 e^{-x/L} + \frac{S_0}{\Sigma_a} \qquad \phi(x) = 0 \quad x = -a$$

$$\frac{d\phi}{dx} = 0 = \frac{C_1 e^{x/L}}{L} - \frac{C_2 e^{-x/L}}{L} \quad \text{so} \quad C_1 = C_2$$

$$\phi(a) = 0 = C e^{a/L} + C e^{-a/L} + \frac{S_0}{\Sigma_a} \qquad -\frac{S_0}{\Sigma_a} = C(e^{a/L} + e^{-a/L})$$

$$C = \frac{-S_0}{\Sigma_a(e^{a/L} + e^{-a/L})} \qquad \phi(x) = \frac{S_0}{\Sigma_a}\left(1 - \left(\frac{e^{x/L} + e^{-x/L}}{e^{a/L} + e^{-a/L}}\right)\right)$$

c). $S(x) = \cos(x)$ for $x \in [-a, a]$

$$-\frac{1}{L^2}\frac{d}{dx}\frac{d\phi(x)}{dx} + L^2\phi(x) = \cos(x)$$

$$L^2 = \frac{\Sigma_a}{D}$$

$$\phi_H(x) = C_1 \sinh\left(\frac{x}{L}\right) + C_2 \cosh\left(\frac{x}{L}\right)$$

prob trig part. $\phi_p(x) = A\sin(x) + B\cos(x)$

$$\frac{d^2\phi_p(x)}{dx^2} = -A\sin(x) - B\cos(x)$$

$$(A\sin(x) + B\cos(x)) + \frac{1}{L^2}(A\sin(x) + B\cos(x)) = \frac{\cos(x)}{D}$$

A must be 0 so B is

$$B\left(1+\frac{1}{L^2}\right) = \frac{1}{D} \qquad B = \frac{1}{D\left(1+\frac{1}{L^2}\right)}$$

$$\phi(x) = C_1 \sinh\left(\frac{x}{L}\right) + C_2 \cosh\left(\frac{x}{L}\right) + \frac{\cos x}{\Sigma_a(1+L^2)}$$

since $\frac{d\phi}{dx} = 0$ C_1 must be $\underline{0}$ to satisfy cosh in derivative being non zero.

now

$$\phi(a) = 0 = C_2 \cosh\left(\frac{a}{L}\right) + \frac{\cos(a)}{\Sigma_a(1+L^2)}$$

$$C_2 = \frac{-\cos(a)}{\cosh\left(\frac{a}{L}\right)\Sigma_a(1+L^2)}$$

so

$$\boxed{\phi(x) = \frac{\cos(x)}{\Sigma_a(1+L^2)} - \frac{\cos(a)}{\cos\left(\frac{a}{L}\right)\Sigma_a(1+L^2)}\cosh\left(\frac{x}{L}\right)}$$

since

$$\phi(x) = \phi_p + \phi_H$$