Solution to Homework 10: Game Theory

Problem 1

Winston (2004), Page 192, Section 4.1-3, Prob. 1

A soldier can hide in one of five foxholes (1, 2, 3, 4, or 5) (see Figure 1). A gunner has a single shot and may fire at any of the four spots A, B, C, or D. A shot will kill a soldier if the soldier is in a foxhole adjacent to the spot where the shot was fired. For example, a shot fired at spot B will kill the soldier if he is in foxhole 2 or 3, while a shot fired at spot D will kill the soldier if he is in foxhole 4 or 5. Suppose the gunner receives a reward of 1 if the soldier is killed and a reward of 0 if the soldier survives the shot.

Figure 1: Foxholes and spots

a) Assuming this to be a zero-sum game, construct the reward matrix.

b) Find and eliminate all dominated strategies.

c) We are given that an optimal strategy for the soldier is to hide 1/3 of the time in foxholes 1, 3, and 5. We are also told that for the gunner, an optimal strategy is to shoot 1/3 of the time at A, 1/3 of the time at D, and 1/3 of the time at B or C. Determine the value of the game to the gunner.

d) Suppose the solider chooses the following nonoptimal strategy: 1/2 of the time, hide in foxhole 1; 1/4 of the time, hide in foxhole 3; and 1/4 of the time, hide in foxhole 5. Find a strategy for the gunner that ensures that his expected reward will exceed the value of the game.

e) Write down each player's LP and verify that the strategies given in part (c) are optimal strategies.

Solution:

a)

The reward/payoff matrix is as follows.

Gunner	Soldier 1	2	3	4	5
A	1	1	0	0	0
B	0	1	1	0	0
C	0	0	1	1	0
D	0	0	0	1	1

b)

Column 2 is dominated by Column 1, Column 4 is dominated by Column 5. After elimination of the columns, we can get

Gunner	Soldier 1	3	5
A	1	0	0
B	0	1	0
C	0	1	0
D	0	0	1

c)

For the gunner:

- $p_1 = P(\text{shoot at A}) = \frac{1}{3}$, $E_1 = \frac{1}{3}$ is the expected payoff when shooting at A
- $p_2 = P(\text{shoot at D}) = \frac{1}{3}$, $E_2 = \frac{1}{3}$ is the expected payoff when shooting at D
- $p_3 = P(\text{shoot at B or C}) = \frac{1}{3}$, $E_3 = \frac{1}{3}$ is the expected payoff when shooting at B or C

The value of the game to the gunner:

$$\begin{aligned} E_1p_1 + E_2p_2 + E_3p_3 &= \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

d)

Since there is a greater chance that the soldier is hiding in foxhole 1, the shooter will be more likely to receive a higher expected reward if he shoots at A. If the gunner only shoots at A, his expected reward is $1 \times \frac{1}{2} = \frac{1}{2} > \frac{1}{3}$.

e)

- $x_1/x_2/x_3/x_4$: probability that the gunner will shoot at A/B/C/D.
- $y_1/y_2/y_3$: probability that the soldier will hide in foxholes 1/3/5.

For the gunner:

$$
\begin{aligned}
\max \quad & v \\
\text{s.t.} \quad & v - x_1 \leq 0, \\
& v - x_2 - x_3 \leq 0, \\
& v - x_4 \leq 0, \\
& x_1 + x_2 + x_3 + x_4 = 1, \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
$$

For the soldier:

$$
\begin{aligned}
\min \quad & w \\
\text{s.t.} \quad & w - y_1 \geq 0, \\
& w - y_2 \geq 0, \\
& w - y_3 \geq 0, \\
& y_1 + y_2 + y_3 = 1, \\
& y_1, y_2, y_3 \geq 0.
\end{aligned}
$$

Using Gurobi to solve the two LPs, we can get the optimal solutions: $(x_1, x_2, x_3, x_4) = (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})$, $(y_1, y_2, y_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, optimal objective functions: $v = w = \frac{1}{3}$.

The strategies given in (c) are $x_1 = \frac{1}{3}, x_2 + x_3 = \frac{1}{3}, x_4 = \frac{1}{3}$ and $y_1 = y_2 = y_3 = \frac{1}{3}$. Substituting these two solutions into the two corresponding LP models described above, all constraints are satisfied, so they are both feasible solutions, and the corresponding objective function values are both $\frac{1}{3}$, so the policies given in (c) are optimal.

Problem 2

Winston (2004), Page 197, Section 4.4, Prob. 3

The New York City Council is ready to vote on two bills that authorize the construction of new roads in Manhattan and Brooklyn. If the two boroughs join forces, they can pass both bills, but neither borough by itself has enough power to pass a bill. If a bill is passed, it will cost the taxpayers of each borough \$1 million, but if roads are built in a borough, the benefits to the borough are estimated to be \$10 million. The council votes on both bills simultaneously, and each councilperson must vote on the bills without knowing how anybody else will vote. Assuming that each borough supports its own bill, determine whether this game has any equilibrium points. Is this game analogous to the Prisoner's Dilemma? Explain why or why not.

Solution:

The reward/payoff matrix is as follows.

Manhattan	Brooklyn	
	Don't Vote for Manhattan	Vote for Manhattan
Don't vote for Brooklyn	(0,0)	(9,-1)
Vote for Brooklyn	(-1,9)	(8,8)

In the prisoner's dilemma, we have the following:

- NC Noncooperative action: don't vote for the other borough
- C Cooperative action: vote for the other borough
- P Punishment for not cooperating: \$0
- S Payoff to person who is double-crossed: $-$\$1 billion
- R Reward for cooperating if both players cooperate: \$8 billion
- T Temptation for double-crossing opponent: \$9 billion

In this game, $T > R > P > S$, so it has an equilibrium point $(0, 0)$, which is analogous to a prisoner's dilemma.