Homework 3

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Proposition 3.21.

Let G be a group and a and b be any two elements in G. Then the equations $ax = b$ and $xa = b$ have unique solutions in G.

Proof. As provided in the textbook, suppose that $ax = b$. We must show that such an x exists. We can multiply both sides of $ax = b$ by a^{-1} to find $x = ex = a^{-1}ax = a^{-1}b$.

To show uniqueness, suppose that x_1 and x_2 are both solutions of $ax = b$; then $ax_1 = b = ax_2$. So $x_1 = a^{-1}ax_1 = a^{-1}a_2 = x_2$. The proof for the existence and uniqueness of the solution xa=b is similar. □

Proposition 3.22.

If G is a group and $a, b, c \in G$, then $ba = ca \implies b = c$ and $ab = ac \implies b = c$.

Proof. $ba = ca \implies (ba)a^{-1} = (ca)a^{-1} \implies baa^{-1} = caa^{-1} \implies be = ce \implies b = c$. The proof is similar for $ab = ac$. □

Theorem 3.23

In a group, the usual laws of exponents hold; that is, for all $g, h \in G$,
1. $g^m g^n = g^{m+n}$
2. $(g^m)^n = g^{mn}$
3. $(gh)^n = (h^{-1}g^{-1})^{-n}$
4. If G is abelian, then $(gh)^n = g^n h^n$.

Proof. 1. $g^m g^n = \underbrace{g \cdot g \cdots g \cdot g}_{n \text{ times}} \cdot \underbrace{g \cdot g \cdots g \cdot g}_{m \text{ times}} = \underbrace{g \cdot g \cdots g \cdot g}_{m+n \text{ times}} = g^{m+n}$.

2. $(g^m)^n = (\underbrace{g \cdot g \cdots g \cdot g}_{m \text{ times}})^n = \underbrace{g \cdot g \cdots g \cdot g}_{mn \text{ times}} = g^{mn}$.

3. $(gh)^n = (gh)^{(-1)(-1)n} = ((gh)^{-1})^{-n}$. By proposition 3.19, $((gh)^{-1})^{-n} = (g^{-1}h^{-1})^{-n}$.

4. Because G is abelian, we can say $(gh)^n = \underbrace{gh \cdot gh \cdots gh \cdot gh}_{n \text{ times}}$

$= \underbrace{g \cdot g \cdots g \cdot g}_{n \text{ times}} \cdot \underbrace{h \cdot h \cdots h \cdot h}_{n \text{ times}} = g^n h^n$. □