Modern Alegbra
Homework 6

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5.4 (12) Let $\sigma \in S_n$ have order n. Show that for all integers i and j, $\sigma^i = \sigma^j$ if and only if $i \equiv j \mod n$.

Proof.

$\Rightarrow$ Suppose that $\sigma^i = \sigma^j$. Then $\sigma^i = \sigma^j \sigma^n$, since $\sigma^n = \text{id}$, so $\sigma^n = \sigma^k n$ for $k \in \mathbb{N}_0$, so $\sigma^i = \sigma^j \sigma^{kn}$. Thus $\sigma^j \sigma^{kn} = \sigma^{j+kn}$. This implies that $i = j + kn$, so $i \equiv j \mod n$.

$\Leftarrow$ Next, suppose that $i \equiv j \mod n$, so $i = j + kn$ for $k \in \mathbb{N}_0$. If $k = 0$, then $i = j$, so $\sigma^i = \sigma^j$. If $k > 0$, then $\sigma^i = \sigma^{j+kn} = \sigma^j \sigma^{kn}$ for $k \in N$. But because σ has order n, $\sigma^{kn} = \sigma^n \sigma^{n(k-1)} = \text{id}\sigma^{n(k-1)}\sigma^{n(k-2)} = \cdots = \text{id}\sigma^{1n} = \text{id}^2 = \text{id}$, so $\sigma^i = \sigma^{j+kn} = \sigma^j \sigma^{kn} = \text{id}\sigma^j = \sigma^j$ and therefore $\sigma^i = \sigma^j$. □

(13) Let $\sigma = \sigma_1 \cdots \sigma_m \in S_n$ be the product of disjoint cycles. Prove that the order of σ is the least common multiple of the lengths of the cycles $\sigma_1, \cdots, \sigma_m$.

Proof. Let the order of σ be n. Since the cycles are disjoint, they commute and so $(\sigma_1 \cdots \sigma_m)^n = \sigma_1^n \cdots \sigma_m^n$. Let the length of each σ_i^n be k_i. σ_i is the identity permutation if and only if $k_i | n$. Therefore, the order of σ is the smallest integer n that is divisible be the lengths of all the cycles $\sigma_1, \cdots, \sigma_m$. That is the definition of the least common multiple, and so the order of σ is the least common multiple. □