MATH 420 Advanced Calculus: Assignment 6

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§9.4

(28) Prove that any group G of order p, p prime, is isomorphic to $\mathbb{Z}_p$.

Proof. By corollary 6.12, $|G|$ being a prime number means that G is cyclic. By theorem 9.8, G is isomorphic to $\mathbb{Z}_p$. □

(32) Prove $U(5) \cong \mathbb{Z}_4$. Can you generalize this result for $U(p)$, where p is prime?

Proof. $|U(5)| = |\{1, 2, 3, 4\}| = 4$. Similarly, $|\mathbb{Z}_4| = |\{0, 1, 2, 3\}| = 4$. By theorem 9.8, since $U(5)$ is cyclic, it is isomorphic to $\mathbb{Z}_4$. Since $U(p)$ for p prime is the set of natural numbers less than p which are coprime, it is the set $\{1, 2, \cdots, p-2, p-1\}$ for all p. This means that $|U(p)| = p-1$. Similarly, $|\mathbb{Z}_p - 1| = |\{0, 1, \cdots, p-3, p-2\}| = p-1$. So again by theorem 9.8, $U(p) \cong \mathbb{Z}_{p-1}$. □

(35) An ***automorphism*** of a group G is an isomorphism with itself. Prove that complex conjugation is an automorphism of the additive group of complex numbers; that is, show that the map $\phi(a+bi) = a - bi$ is an isomorphism from C to C.

Proof. Let $z_1 = a + bi$ and $z_2 = x + yi$. Then

$$\phi(z_1 + z_2) = \phi(a + bi + x + yi) = \phi((a + x) + (b + y)i) = (a + x) - (b + y)i.$$

Similarly,

$$\phi(z_1) + \phi(z_2) = a - bi + x - yi = (a + x) - (b + yi)$$

So ϕ is an isomorphism. □

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