Homework 2

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10. Prove that matrices of the form $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ is a group under multiplication.

Proof. For 3 matrices A, B, C in the Heisenburg group s.t. $A = \begin{bmatrix} 1 & a_A & b_A \\ 0 & 1 & c_A \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & a_B & b_B \\ 0 & 1 & c_B \\ 0 & 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & a_C & b_C \\ 0 & 1 & c_C \\ 0 & 0 & 1 \end{bmatrix}$ with $a_A, a_B, a_C, b_A, b_B, b_B, c_A, c_B, c_C \in R$, by the definition given in the book,

$$\begin{aligned}
(AB)C &= \left(\begin{bmatrix} 1 & a_A + a_B & b_A + b_B + a_A \cdot c_B \\ 0 & 1 & c_A + c_B \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & a_C & b_C \\ 0 & 1 & c_C \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & a_A + a_B + a_C & a_A \cdot c_B + c_C \cdot a_A + c_C \cdot a_B + b_A + b_B + b_C \\ 0 & 1 & c_A + c_B + c_C \\ 0 & 0 & 1 \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
A(BC) &= \begin{bmatrix} 1 & a_A & b_A \\ 0 & 1 & c_A \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & a_B + a_C & b_B + b_C + a_B \cdot c_C \\ 0 & 1 & c_B + c_C \\ 0 & 0 & 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 & a_A + a_B + a_C & a_A \cdot c_B + a_A \cdot c_C + a_B \cdot c_C + b_A + b_B + b_C \\ 0 & 1 & c_A + c_B + c_C \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

so the group is associative.

There is a multiplicitive identity I s.t for $a, b, c \in R$, $AI = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

Similarly, $IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$. As given in example 12.1 in the book, a matrix is invertable if and only if its determinant is nonzero. For any matrix A of the form $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ with $a, b, c \in R$, by Leibniz formula $\det A = 1 \cdot 1 \cdot 1 + a \cdot c \cdot 0 + b \cdot 0 \cdot 0 - b \cdot 1 \cdot 0 - a \cdot 0 \cdot 1 - 1 \cdot c \cdot 0 = 1$, so A is invertable. $\square$

13. Prove that $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is a group under multiplication.

Proof. Multiplication in $\mathbb{R}^*$ is associative because multiplication in $\mathbb{R}$ is. The identity element is 1, that is, for $a \in \mathbb{R}^*, 1a = a1 = a$. The inverse element $a^{-1} = \frac{1}{a}$. $\square$

14. Show that $G = \mathbb{R}^* \times \mathbb{Z}$ is a group under $(a, m) \circ (b, n) = (ab, m + n)$.

Proof.

$$\begin{aligned} ((a_1, n_1) \circ (a_2, n_2)) \circ (a_3, n_3) &= (a_1 a_2, n_1 + n_2) \circ (a_3, n_3) \\ &= ((a_1 a_2) a_3, (n_1 + n_2) + n_3) \\ &= (a_1 a_2 a_3, n_1 + n_2 + n_3) \end{aligned}$$

. Similarly,

$$\begin{aligned} (a_1, n_1) \circ ((a_2, n_2) \circ (a_3, n_3)) &= (a_1, n_1) \circ (a_2 a_3, n_2 + n_3) \\ &= (a_1 (a_2 a_3), n_1 + (n_2 + n_3)) \\ &= (a_1 a_2 a_3, n_1 + n_2 + n_3). \end{aligned}$$

The identity element is (1, 0), that is, $(a, n) \circ (1, 0) = (1a, n + 0) = (a, n)$. The inverse is $\left(\frac{1}{a}, -n\right)$, that is, $(a, n) \circ \left(\frac{1}{a}, -n\right) = \left(a\frac{1}{a}, n - n\right) = (1, 0)$. $\square$