1. **Section 1B-1C: Vector Spaces and Subspaces**
 - Definition of a Vector space V over $\mathbb{F}$.
 $-\vec{u}+\vec{v}=\vec{v}+\vec{u} \quad \forall\, \vec{u}, \vec{v} \in V$
 $-(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w}) \quad \forall\, \vec{u}, \vec{v}, \vec{w} \in V$
 $-\vec{u}+\vec{0}=\vec{u} \quad \forall\, \vec{u} \in V$
 $-\forall\, \vec{v} \in V, \exists\, \vec{u} \in V$ s.t. $\vec{v}+\vec{u}=\vec{0}$
 $-\lambda \vec{v} \in V \quad \forall\, \vec{v} \in V$, and $\lambda \in \mathbb{F}$
 $-1 \vec{v}=\vec{v} \quad \forall\, \vec{v} \in V$
 $-\lambda(\vec{v}+\vec{u})=\lambda \vec{v}+\lambda \vec{u} \quad \forall\, \vec{v}, \vec{u} \in V$, and $\lambda \in \mathbb{F}$
 $-(\lambda+\mu) \vec{v}=\lambda \vec{v}+\mu \vec{v} \quad \forall\, \vec{v} \in V$, and $\lambda, \mu \in \mathbb{F}$
 - Definition of a subspace U of V : A subset U of V which is itself a vector space
 - Criterion for a Subspace
 $-\vec{0} \in U$
 $-\vec{v}+\vec{u} \in U \quad \forall\, \vec{v}, \vec{u} \in U$
 $-\lambda \vec{v} \in U \quad \forall\, \vec{v} \in U, \lambda \in \mathbb{F}$
 - What a Sum of Subspaces is: for subspaces $V_1, \cdots, V_m$ of V

$$V_1+\cdots+V_m=\{\vec{v}_1+\cdots \vec{v}_m \mid \vec{v}_i \in V_i\}$$

 - What a direct sum of subspaces is: $V_1+\cdots+V_m$ is a direct sum is there is only one combination of $\vec{v}_i$'s that add up to each $\vec{v} \in V_1+\cdots+V_m$. A direct sum is denoted $V_1 \oplus \cdots \oplus V_m$.
 - Claim 1.45: Suppose $V_1, \cdots, V_m$ are subspaces of $V . V_1+\cdots+V_m$ is a direct sum if and only if $\sum \vec{v}_i=0$ implies that $\vec{v}_i=0$ for all $i \in[1, m]$.
 - Claim 1.46: Suppose U and W are subspaces of V. Then

$$U+W \text{ is a direct sum } \Longleftrightarrow U \cap W=\{\vec{0}\}.$$

2. **Section 2A-2C: Linear Independence, Bases, and Dimension**
 - Know what the span of a set of vectors is: span $\{\vec{v}_1, \cdots, \vec{v}_n\}=\{\sum a_i \vec{v}_i \mid i \in[1, n], a_i \in \mathbb{F}\}$
 - If span $\{\vec{v}_1, \cdots, \vec{v}_n\}=V$, then we say $\{\vec{v}_i \mid i \in[1, n]\}$ spans V.
 - A vector space is finite dimensional if it can be spanned by a finite number of vectors.
 - A set of vectors is linearly independent if no one vector is in the span of the rest.
 - Equivalently, **Definition 2.15:** $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent if $\sum a_i \vec{v}_i=\vec{0}$ implies that $a_i=0$ for all $i \in[1, n]$.
 - Definition of a basis for a vector space V : A linearly independent spanning set of vectors.
 - Claim 2.28: $\{\vec{v}_1, \cdots, \vec{v}_n\} \in V$ is a basis if and only if for every vector $\vec{v} \in V$ can be written as a unique linear combination of $\vec{v}_1, \cdots, \vec{v}_n$, or

$$\vec{v}=\sum a_i \vec{v}_i \text{ for unique } a_i \in \mathbb{F}$$

 - Claim 2.30: Every spanning set contains a basis.
 - Claim 2.30 is equivalent to saying any linearly dependent set can be reduced to a linearly independent one through deletion.
 - Claim 2.31: Every finite dimensional vector space has a basis.
 - Claim 2.33: Suppose V is finite dimensional and U is a subspace of V. Then there exists a subspace W such that $U \oplus W=V$.
- Definition of dimension: The dimension of a vector space is the number of basis vectors, denoted $\operatorname{dim} V$.
 - Claim 2.34: All bases of a finite dimensional vector space are the same length.
 - Claim 2.37: If V is a finite dimensional and U is a subspace of V, then $\operatorname{dim} U \leq \operatorname{dim} V$.
 - Every basis of a subspace can be extended to a basis of the entire vector space.
 - Claim 2.43: If V_1 and V_2 are subspaces of a finite dimensional vector space, then

$$\operatorname{dim}(V_1+V_2)=\operatorname{dim} V_1+\operatorname{dim} V_2-\operatorname{dim}(V_1 \cap V_2)$$

3. **Section 3A-3D: Linear Maps and Matrices**
- Definition of a Linear Map: A function $T: V \rightarrow W$ which respects addition and scaling.

$$\begin{gathered}
T(\vec{v}+\vec{u})=T(\vec{v})+T(\vec{u}) \quad \forall\, \vec{u}, \vec{v} \in V \\
T(\lambda \vec{v})=\lambda T(\vec{v}) \quad \forall\, \vec{v} \in V, \lambda \in \mathbb{F}
\end{gathered}$$

 - $\mathcal{L}(V, W)$ are the linear maps from V to W, and $\mathcal{L}(V)$ are linear maps of V to itself.
 - Claim 3.4: Suppose $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis of V and $\vec{w}_1, \cdots, \vec{w}_n \in W$. Then there exists a unique linear map $T \in \mathcal{L}(V, W)$ such that

$$T(\vec{v}_i)=\vec{w}_i$$

 - $\mathcal{L}(V, W)$ is itself a vector space with an added product: composition.
 - Claim 3.10: For any linear map $T, T(\vec{0})=\vec{0}$.
- Definition of null space of T : null $T=\{\vec{v} \in V \mid T(\vec{v})=\vec{0}\}$
 - Claim 3.13: Suppose $T \in \mathcal{L}(V, W)$. Then nullT is a subspace of V.
- Definition of injective: $T(\vec{u})=T(\vec{v})$ implies that $\vec{u}=\vec{v}$.
 - Claim 3.15: $T \in \mathcal{L}(V, W)$ is injective if and only if null $T=\{\vec{0}\}$.
- Definition of the range of T : range $T=\{\vec{w} \in W \mid \exists\, \vec{v} \in V$ s.t. $T(\vec{v})=\vec{w}\}$.
 - Claim 3.18: Suppose $T \in \mathcal{L}(V, W)$. Then rangeT is a subspace of W.
- Definition of surjective: range $T=W$.
 - Fundamental Theorem of Linear Maps: For finite dimensional V and $T \in \mathcal{L}(V, W)$, range T is finite dimensional and

$$\operatorname{dim} V=\operatorname{dim} \operatorname{null} T+\operatorname{dim} \operatorname{range} T$$

 - Claim 3.22: Suppose V and W are finite dimensional with $\operatorname{dim} V>\operatorname{dim} W$. Then there is no injective linear map from V to W.
 - Claim 3.24: Suppose V and W are finite dimensional with $\operatorname{dim} V<\operatorname{dim} W$. Then there is no surjective linear map from V to W.
 - Each linear map has a matrix representation $\mathcal{M}(T)=A$ such that $T(\vec{v})=A \vec{v}$.
 - For $T \in \mathcal{L}(V, W)$ and bases $\{\vec{v}_i\} \in V$ and $\{\vec{w}_i\} \in W, T(\vec{v}_i)=A_{1, i} \vec{w}_1+\cdots+A_{m, i} \vec{w}_m=$ the i^{th} column of A.
 - Know how to add, subtract, multiply, and transpose matrices.
 - Know how to multiply a vector by a matrix.
 - $T \in \mathcal{L}(V, W)$ is invertible if there exists $S \in \mathcal{L}(W, V)$ such that $S T=I_v$ and $T S=I_W$. We denote $S=T^{-1}$. (This is identical to the definition of having an invertible matrix)
 - Claim 3.63: T is invertible if and only if it is injective and surjective. - Claim 3.65: If $\operatorname{dim} V=\operatorname{dim} W<\infty$ with $T \in \mathcal{L}(V, W)$

$$T \text{ invertible } \Longleftrightarrow T \text{ injective } \Longleftrightarrow T \text{ surjective}$$

 - Definition of isomorphism: An invertible linear map.
 - Definition of isomorphic: Two vector spaces are isomorphic if there exists an isomorphism between them.
 - Claim 3.78: If V and W finite dimensional, $\operatorname{dim} \operatorname{range} T=\operatorname{rank} \mathcal{M}(T)$.
 - We can include which bases we are using in our denotation of $\mathcal{M}(T): \mathcal{M}(T,\{\vec{v}_i\},\{\vec{w}_i\})$ where $\{\vec{v}_i\}$ is the basis of V and $\{\vec{w}_i\}$ is the basis of W.
 - Change of Basis Formula: Take $T \in \mathcal{L}(V)$ and two bases of $V,\{\vec{v}_i\}$ and $\{\vec{u}_i\}$. Let $A=\mathcal{M}(T,\{\vec{v}_i\}), B=\mathcal{M}(T,\{\vec{u}_i\})$, and $C=\mathcal{M}(I,\{\vec{v}_i\},\{\vec{u}_i\})$. Then

$$A=C^{-1} B C .$$

4. **Section 3E: Product and Quotient Spaces**
 - Definition of Catresian product: for vector spaces over $\mathbb{F}, V_1, \cdots, V_m$,

$$V_1 \times \cdots \times V_m=\{(\vec{v}_1, \cdots, \vec{v}_m) \mid \vec{v}_i \in V_i\}$$

 - The Cartesian product is still a vector space.
 - Claim 3.92: Suppose $V_1, \cdots, V_m$ are finite dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite dimensional and

$$\operatorname{dim}(V_1 \times \cdots \times V_m)=\sum \operatorname{dim} V_i$$

 - Definition of a translate: Shifting an entire subset U of V by a vector in V

$$\vec{v}+U=\{\vec{v}+\vec{u} \mid \vec{u} \in U\}$$

 - Definition of a quotient space: The collection of translates of a subspace U

$$V / U=\{\vec{v}+U \mid \vec{v} \in V\}$$

 - Claim 3.105: Suppose V is finite dimensional and U is a subspace of V. Then $\operatorname{dim} V / U=\operatorname{dim} V-\operatorname{dim} U$.
5. **Section 5A and 5C: Eigenvalues, Eigenvectors, and Upper Triangular Matrices**
 - Definition of linear operator: $T \in \mathcal{L}(V)$
 - Definition of invariant under an operator T : A subspace U is invariant under T is for all $\vec{u} \in U, T(\vec{u}) \in U$.
 - Definition of an eigenvalue: $\lambda \in \mathbb{F}$ such that there exists a vector $\vec{v} \in V$ where $T(\vec{v})=\lambda \vec{v}$.
 - Definition of an eigenvector: For any eigenvalue λ, any vector $\vec{v}$ such that $T(\vec{v})=\lambda \vec{v}$ is an eigenvector.
 - Claim 5.7: For finite dimensional V and $T \in \mathcal{L}(V)$, T.F.A.E.
 $-\lambda$ is an eigenvalue of T
 $-T-\lambda I$ is not injective
 $-T-\lambda I$ is not surjective
 $-T-\lambda I$ is not invertible
 - Know how to use exponents on linear operators
 - Know how to plug a linear operator into a polynomial
 - Claim 5.18: For $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$, the null $p(T)$ and range $p(T)$ are invariant under T.
 - Definition of Upper Triangular Matrices
 - Claim 5.39: For $T \in \mathcal{L}(V)$ and basis $\{\vec{v}_1, \cdots, \vec{v}_n\}$ of V T.F.A.E: - $\mathcal{M}(T,\{\vec{v}_i\})$ is upper triangular
 $-$ span $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is invariant under T for all $k \in[1, n]$
 $-T(\vec{v}_k) \in \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_k\}$ for all $k \in[1, n]$
 - Claim 5.40: If $\mathcal{M}(T,\{\vec{v}_i\})$ is upper triangluar with diagonal entries $\lambda_1, \cdots, \lambda_n$, then

$$(T-\lambda_1 I) \cdots(T-\lambda_n I)=0$$

 - Claim 5.41: If $\mathcal{M}(T,\{\vec{v}_i\})$ is upper triangluar with diagonal entries $\lambda_1, \cdots, \lambda_n$, then $\lambda_1, \cdots, \lambda_n$ are eigenvalues of T.