# Proving existence/uniqueness of spatial models with Allen & Arkolakis

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# Setting goals and restricting scope

**Goal**: Understand how to prove existence and uniqueness of spatial models

**Scope**: A *spatial* model is a (GE) model in which some subset of goods or factors can move across locations

- **international trade**: labor doesn't move, usually static/stationary
- **economic geography**: labor can migrate/commute, usually static/stationary
- **international macro/finance**: labor doesn't move, capital does, nominal matters, usually dynamic

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Focus on **static GE of trade & geography models**, but results still useful for dynamic models (take "snapshots")

# Why bother proving existence/uniqueness?

## Existence

- proof by construction can suffice *ex post*, but super helpful as a researcher to have a guarantee **before** you code a solver
- data is in eqbm. $\implies$ SMM/GMM only searches over parameter space that yields eqbm.

## Uniqueness

- esp. in economic geography, **multiplicity is often expected** $\rightarrow$ we want to know *when*
- without it, counterfactual exercises are hard to interpret
  - equilibria are locally isolated (MWG), so can study small perturbations even with multiplicity
  - exact hat requires a selection rule (Ahlfeldt et al., 2015)

Most important: **You learn how your model really works!**

# What Allen, Arkolakis, and Takahashi (2020) do (and don't do)

**Do . . .**

- Define class of spatial models called *gravity models*
- List well-known examples of gravity models (Table 1)
- Show sufficient conditions for existence/interiority/uniqueness that depend only on gravity elasticities
- Show (local) counterfactual real price changes depend only on gravity elasticities and observed data

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**Don't . . .**

- make it easy to use their results (*not a cookbook!*)
  - no definitive checks if your model fits
  - no necessary conditions
- claim that all gravity models are isomorphic
  - same positive predictions given same estimated elasticities & data $\not\Rightarrow$ same normative predictions or optimal policy
  - different lenses on different data

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**AAT20 doesn't solve every problem, but it helps a lot!**

# How did folks prove existence/uniqueness before AAT20?

1. **Omit it.** Just trust your gut (Eaton and Kortum, 2002)
2. **Assume it.** But argue it'll work for sure if $N \to \infty$ (Costinot, 2009)
3. **Reduce it.** That is, use a two-location or symmetric location model s.t. equilibrium reduces to a scissors graph (Melitz, 2003; Krugman, 1991)
4. **Contort it (to fit MWG).** Find a fixed point of the excess demand function, which is unique if goods are gross substitutes (Alvarez and Lucas, 2007)
5. **Borrow it.** Cite Allen and Arkolakis (2014), the proto-AAT20 for just geography models (more on this later...)

# The setup

- Each location ($i \in S$) produces a **representative** good
- We'll tate **six conditions** about aggregate trade flows that reduce the equilibrium to **two equations** per location
- Definitions $\longrightarrow$

| | |
|---|---|
| output | $Q_i \geq 0$ |
| quantity traded | $Q_{ij} \geq 0$ |
| output price | $p_i \geq 0$ |
| bilateral price | $p_{ij} \geq 0$ |
| income | $Y_i \equiv p_i Q_i$ |
| trade flows | $X_{ij} \equiv p_{ij} Q_{ij}$ |
| expenditure | $E_i \equiv \sum_j X_{ji}$ |
| price index | $P_i \equiv \ldots$ |
| real expenditure | $W_i \equiv E_i / P_i$ |
| real output price | $p_i / P_i$ |

## The six conditions

**C.1 (Iceberg costs)** for some trade frictions $\{\tau_{ij}\}$, $p_{ij} = p_i \tau_{ij}$

**C.2 (CES aggregate demand)** $\exists$ exogenous (negative of the) demand elasticity $\phi \in \mathbb{R}$ s.t.

$$E_j = \left( \sum_i p_{ij}^{-\phi} \right)^{-1/\phi} W_j \equiv P_j W_j \implies X_{ij} = \frac{p_{ij}^{-\phi}}{\sum_i p_{ij}^{-\phi}} E_j$$

**C.3 (CES aggregate supply)** $\exists$ exogenous supply shifters $\{\bar{c}_i\}$, exogenous aggregate supply elasticity $\psi \in \mathbb{R}$, and endogenous scalar $\kappa > 0$ s.t.

$$Q_i = \kappa \bar{c}_i \left( \frac{p_i}{P_i} \right)^{\psi}$$

**C.4 (Output market clearing)** $\forall i$, $Q_i = \sum_j \tau_{ij} Q_{ij}$ or, equivalently, $Y_i = \sum_j X_{ij}$

**C.5 (Trade balance)** $\forall i$, $E_i = p_i Q_i$ (they allow exogenous deficits, but not in Theorem 1. . . )

**C.6 (Normalization)** $\sum_i Y_i = 1$ (pins down *product* of $\kappa$ and price scale)

# The two equilibrium equations (per location)

**C.1.** $p_{ij} = p_i \tau_{ij}$

**C.2.** $X_{ij} = \frac{p_{ij}^{-\phi}}{\sum_i p_{ij}^{-\phi}} E_j$

**C.3.** $Q_i = \kappa \bar{c}_i \left(\frac{p_i}{P_i}\right)^{\psi}$

**C.4.** $Y_i = \sum_j X_{ij}$

**C.5.** $E_i = p_i Q_i$

**C.6.** $\sum_i Y_i = 1.$

An **equilibrium** is $\{Y_i, E_i, X_{ij}, p_i/P_i\}$ in levels and $\{Q_i, Q_{ij}, p_i, p_{ij}, P_i\}$ up to scale.

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Combine C.1 and C.2 to get

$$P_i^{-\phi} = \sum_j \tau_{ij}^{-\phi} p_j^{-\phi}, \quad \forall i \tag{7}$$

Combine C.1-5 with $Y_i \equiv p_i Q_i$ and rearrange to get

$$p_i^{1+\phi} \bar{c}_i \left(\frac{p_i}{P_i}\right)^{\psi} = \sum_j \tau_{ij}^{-\phi} P_j^{\phi} p_j \bar{c}_j \left(\frac{p_j}{P_j}\right)^{\psi}, \quad \forall i \tag{6}$$

**What really matters is if your model's equilibrium can be written like (6) and (7).**

# What's a gravity model and what's not? (We'll circle back . . . )

**Gravity models** (Table 1)

- Armington (1969); Anderson (1979); Anderson and van Wincoop (2003)
- Krugman (1980)
- Melitz (2003)
- Eaton and Kortum (2002)
- Caliendo and Parro (2015)
- Allen and Arkolakis (2014)
- Redding (2016)
- Redding and Sturm (2008)

**Not-gravity models**

- non-CES
  - Novy (2013) (translog gravity)
  - Fajgelbaum and Khandelwal (2016) (nonhomothetic demand)
  - Melitz and Ottaviano (2008) (outside good)
  - Head, Mayer, and Thoenig (2014) (lognormal productivity)
- non-constant factor intensities
- dynamic models with trade deficits
- models with tariffs

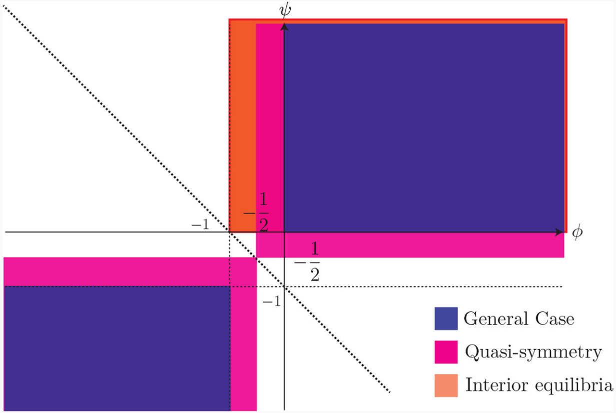
# Theorem 1

Consider any model contained within the universal gravity framework with

- balanced trade,
- $\tau_{ii} < \infty$ for all $i \in S$, and
- the graph of the matrix of trade frictions $\{\tau_{ij}\}$ is strongly connected

Then,

1. if $1 + \psi + \phi \neq 0$, $\exists$ interior eqbm.;
2. if $\phi \geq -1$ and $\psi \geq 0$, all equilibria are interior;
3. if $\{\phi \geq 0, \psi \geq 0\}$ or $\{\phi \leq -1, \psi \leq -1\}$, $\exists$ unique interior eqbm.



# Proof of Theorem 1, Pt. 1: Solve nonlinear integral equations

Define $z$ as follows:

$$z \equiv \begin{pmatrix} (x_i)_i \\ (y_i)_i \end{pmatrix} \equiv \begin{pmatrix} \left(p_i^{1+\psi+\phi} P_i^{-\psi}\right)_i \\ \left(P_i^{-\phi}\right)_i \end{pmatrix}.$$

Then the system of equations (6) and (7) of the general equilibrium gravity model is rewritten in vector form:

$$\begin{pmatrix} (x_i)_i \\ (y_i)_i \end{pmatrix} = \begin{pmatrix} \sum_j K_{ij} \bar{c}_i^{-1} \bar{c}_j x_j^{a_{11}} y_j^{a_{12}} \\ \sum_j K_{ji} x_j^{a_{21}} y_j^{a_{22}} \end{pmatrix}, \qquad (23)$$

where $A = (a_{ij})_{i,j}$ is given by

$$A = \begin{pmatrix} \dfrac{1+\psi}{1+\psi+\phi} & -\dfrac{1+\phi}{1+\psi+\phi} \\ -\dfrac{\phi}{1+\psi+\phi} & \dfrac{\psi}{1+\psi+\phi} \end{pmatrix}.$$

- Nonlinear integral equations $\equiv$ solve for unknown functions $z$ under the (Lebesgue) integral
- $K_{ij} \equiv \tau_{ij}^{-\phi}$ is the "kernel" of the integral equation
- Domain of (23) is unbounded $\implies$ cannot use Brouwer's fixed point theorem straightaway

# Proof of Theorem 1, Pt. 1: Solve nonlinear integral equations

Therefore, consider the following "scaled" version of equation (23):

$$z = \begin{pmatrix} (x_i)_i \\ (y_i)_i \end{pmatrix} = \begin{pmatrix} \dfrac{\sum_j K_{ij}\bar{c}_i^{-1}\bar{c}_j x_j^{a_{11}} y_j^{a_{12}}}{\sum_{i,j} K_{ij}\bar{c}_i^{-1}\bar{c}_j x_j^{a_{11}} y_j^{a_{12}}} \\ \dfrac{\sum_j K_{ji} x_j^{a_{21}} y_j^{a_{22}}}{\sum_{i,j} K_{ji} x_j^{a_{21}} y_j^{a_{22}}} \end{pmatrix} \equiv F(z), \tag{24}$$

and $F$ is defined over the following compact set $C$:

$$C = \{x \in \Delta(R_+^N);\, x_i \in [\underline{x}, \bar{x}]\ \forall\ i\} \times \{y \in \Delta(R_+^N);\, y_i \in [\underline{y}, \bar{y}]\ \forall\ i\}, \tag{25}$$

where the bounds for $x$ and $y$ are given as follows:

$$\bar{x} \equiv \max_{i,j} \frac{K_{ij}\bar{c}_i^{-1}\bar{c}_j}{\sum_{i,j} K_{ij}\bar{c}_i^{-1}\bar{c}_j}, \quad \underline{x} \equiv \min_{i,j} \frac{K_{ij}\bar{c}_i^{-1}\bar{c}_j}{\sum_{i,j} K_{ij}\bar{c}_i^{-1}\bar{c}_j},$$
$$\bar{y} \equiv \max_{i,j} \frac{K_{ji}}{\sum_{i,j} K_{ji}}, \quad \underline{y} = \min_{i,j} \frac{K_{ji}}{\sum_{i,j} K_{ji}}.$$

It is trivial to show that $F$ maps from $C$ to $C$ and continuous over the compact set $C$, so that we can apply Brouwer's fixed point and there exists an fixed point $z^* \in C$.

There are two technical points to be proved: first, there exists a fixed point for the original (unscaled) system (eq. [23]); and second, the equilibrium $z^*$ is strictly positive. These two claims are proved in lemmas 1 and 2, respectively, in appendix B.4.

- RHS of (24) is positive and $\sum_i = 1 \implies$ upper bound puts all weight on the largest term
- Same for lower bound & smallest term
- **Note**: (24) is not a well-defined mapping unless entries of $A$ are finite $\implies$ unless $1 + \psi + \phi \neq 0$

# Proof of Theorem 1, Pt. 3: Uniqueness, by contradiction

It suffices to show that there exists a unique interior solution for equation (23). Suppose that there are two strictly positive solutions $(x_i, y_i)$ and $(\hat{x}_i, \hat{y}_i)$ such that there does not exist $t, s > 0$ satisfying

$$(x_i, y_i) = (t\hat{x}_i, s\hat{y}_i).$$

Namely, the two solutions are "linearly independent." First note that for any $i \in S$, we can evaluate the first row of equation (23).

$$\frac{x_i}{\hat{x}_i} = \frac{1}{\hat{x}_i}\sum_{j\in S} K_{ij}\bar{c}_i^{-1}\bar{c}_j\left(\frac{x_j}{\hat{x}_j}\right)^{\alpha_{11}}\left(\frac{y_j}{\hat{y}_j}\right)^{\alpha_{12}}(\hat{x}_j)^{\alpha_{11}}(\hat{y}_j)^{\alpha_{12}} \tag{26}$$

$$\leqslant \max_{j\in S}\left(\frac{x_j}{\hat{x}_j}\right)^{\alpha_{11}}\max_{j\in S}\left(\frac{y_j}{\hat{y}_j}\right)^{\alpha_{12}}. \tag{27}$$

Taking the maximum of the left-hand side,

$$\max_{i\in S}\frac{x_i}{\hat{x}_i} \leqslant \max_{j\in S}\left(\frac{x_j}{\hat{x}_j}\right)^{\alpha_{11}}\max_{j\in S}\left(\frac{y_j}{\hat{y}_j}\right)^{\alpha_{12}}. \tag{28}$$

Lemma 3, in appendix B.4, shows that the inequality is actually strict. Analogously, we obtain

$$\min_{i\in S}\frac{x_i}{\hat{x}_i} \geqslant \min_{j\in S}\left(\frac{x_j}{\hat{x}_j}\right)^{\alpha_{11}}\min_{j\in S}\left(\frac{y_j}{\hat{y}_j}\right)^{\alpha_{12}}. \tag{29}$$

- The key jump is from (26) to (27)

# Proof of Theorem 1, Pt. 3: Uniqueness, by contradiction

Dividing equation (28) by equation (29) shows that

$$1 \leqslant \mu_x \equiv \frac{\max_{i\in S}(x_i/\hat{x}_i)}{\min_{i\in S}(x_i/\hat{x}_i)} < \frac{\max_{j\in S}\left(x_j/\hat{x}_j\right)^{\alpha_{11}}}{\min_{j\in S}\left(x_j/\hat{x}_j\right)^{\alpha_{11}}} \times \frac{\max_{j\in S}\left(y_j/\hat{y}_j\right)^{\alpha_{12}}}{\min_{j\in S}\left(y_j/\hat{y}_j\right)^{\alpha_{12}}} = \mu_x^{|\alpha_{11}|} \times \mu_y^{|\alpha_{12}|},$$

where

$$\mu_y \equiv \frac{\max_{i\in S}(y_i/\hat{y}_i)}{\min_{i\in S}(y_i/\hat{y}_i)}.$$

The same argument is applied to the second row of equation (23) to obtain the following inequality:

$$1 \leqslant \mu_y \equiv \frac{\max_{i\in S}(y_i/\hat{y}_i)}{\min_{i\in S}(y_i/\hat{y}_i)} < \frac{\max_{j\in S}\left(x_j/\hat{x}_j\right)^{\alpha_{21}}}{\min_{j\in S}\left(x_j/\hat{x}_j\right)^{\alpha_{21}}} \times \frac{\max_{j\in S}\left(y_j/\hat{y}_j\right)^{\alpha_{22}}}{\min_{j\in S}\left(y_j/\hat{y}_j\right)^{\alpha_{22}}} = \mu_x^{|\alpha_{21}|} \times \mu_y^{|\alpha_{22}|}.$$

Taking logs in the two inequalities and exploiting the restriction, we can write

$$\begin{pmatrix} \ln \mu_x \\ \ln \mu_y \end{pmatrix} < \underbrace{\begin{pmatrix} |\alpha_{11}| & |\alpha_{12}| \\ |\alpha_{21}| & |\alpha_{22}| \end{pmatrix}}_{=|A|} \begin{pmatrix} \ln \mu_x \\ \ln \mu_y \end{pmatrix}, \tag{30}$$

which from the Collatz-Wielandt formula implies that the largest eigenvalue of $|A|$ is greater than one.

- Just cranking through to (30)
- Collatz-Wielandt formula: $\rho(\mathbf{A}) = \max_{\mathbf{x}}\{\min_i [\mathbf{A}\mathbf{x}]_i / x_i\}$
- why is $\rho(|\mathbf{A}|) > 1$ a problem?

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- **Lemma 4** (Appx B.4). If $\phi, \psi \geq 0$ or $\phi, \psi \leq -1$, then the eigenvalues for $|\mathbf{A}|$ are

$$\lambda_1 = 1$$

$$\lambda_2 = \frac{\phi - \psi}{1 + \phi + \psi}$$

with $|\lambda_2| < 1$, hence $\rho(|\mathbf{A}|) \leq 1$.

# Comparison to older results

1. Alvarez and Lucas (2007): show **excess demand function $\mathbf{z}$** satisfies
    - $\mathbf{z}$ is continuous;
    - $\mathbf{z}$ is homogeneous of degree zero in $\mathbf{p}$;
    - $\mathbf{p} \cdot \mathbf{z} = 0$ for all strictly positive price vectors (Walras' law);
    - there is a $\underline{z} > 0$ such that $z_\ell(\mathbf{p}) > -\underline{z}$ for every commodity $\ell$ and all $p$;
    - if $\mathbf{p}^n \to \mathbf{p}$, where $\mathbf{p} \neq 0$ but $p_\ell = 0$ for some $\ell$, then
    $$\max\{z_1(\mathbf{p}^n), \ldots, z_{IK+1}(\mathbf{p}^n)\} \to \infty;$$
    - $\frac{\partial z_\ell(\mathbf{p})}{\partial p_{\ell'}} > 0$ for all $\ell, \ell'$ with $\ell \neq \ell'$ and all $\mathbf{p} > \mathbf{0}$. ["gross substitutes"]

    But gross substitutes **fails** for $\psi > \phi \geq 0$ and $\psi < \phi \leq 1$, where AAT20 **still** unique
2. Allen and Arkolakis (2014): AAT20 **generalizes** their Theorem 2 in three ways
    - allows for asymmetric trade frictions
    - allows for infinite trade frictions between non-$ii$ pairs
    - applies to larger class of models (inc. $\psi = 0$)

# Comparison to their newer results: Allen, Arkolakis, and Li (2020)

- Economies where $N$ heterogeneous agents engage in $H$ types of interactions with equilibria characterized by

$$x_{ih} = \sum_{j=1}^{N} f_{ijh}(x_{j1}, \ldots, x_{jH})$$

- Existence and uniqueness (up to scale) if

$$\rho(\mathbf{A}) \leq 1, \quad \mathbf{A} \equiv \left[ \frac{\partial \ln f_{ijh}(x_j)}{\partial \ln x_{jh'}} \right]_{hh'}$$

  by multi-dimensional extension of the contraction mapping theorem
- constant elasticity ("gravity") representation

$$\prod_{h'} x_{ih'}^{\gamma_{hh'}} = \lambda_k \sum_j K_{ijh} \prod_{h'} x_{ih}^{\kappa_{hh'}} x_{jh'}^{\beta_{hh'}}$$

- Generalize AAT20 by allowing for...
  - general (non-constant elasticity) functional forms
  - more than two types of economic interactions

# So I have my spatial model…now what?

**Ask yourself:**

1. can I easily map my model to C.1-5?
2. can I derive equilibrium conditions that look like (6) and (7)?
3. can I point out an obvious violation of C.1-5?

**Decision tree:**

- If "yes" to 1 or 2, you can almost surely use AAT20 (or AAL20).
- Else if "yes" to 3, throw your hands up OR figure out an extension, then email Treb & Costas to coauthor AA[your initial here].
- If "no" to all three, circle back to 2 and keep trying with AAL20.

# A (not so) random example

Consider a spatial model with ...

- Armington varieties (at country level), iceberg costs
- a **quasilinear** homogeneous outside good, freely traded
- discrete choice over production of each Armington variety & outside good at **sub-country** level

Obvious violations of C.1-5:

- Demand side
  - if all together: fail C.2 because the outside good is not CES
  - if just Armington block: fail C.5 because expenditure on outside good $\cong$ endogenous deficit
- Supply side: no mapping to a country-level *representative* good (fail C.3)

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