ECON 164: Theory of Economic Growth

Week 3: The Neoclassical Growth Model

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Winter 2026

Recap

- Last time, we developed the **Solow model**, which has two building blocks:
  - the aggregate production function:

$$\boldsymbol{Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}}$$

  - **endogenous** capital accumulation...

$$\boldsymbol{\dot{K}_t = sY_t - \delta K_t}$$

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- But this picture is incomplete: ***what determines variation in s?***

- This week: Solow w/ endogenous saving → **Ramsey-Cass-Koopmans model**

(*in discrete time, h/t Adrien Bilal*)

What are we saving *for*?

- Endogenizing saving behavior requires taking a stance on what we're saving for
- Most of us save to **spend** more later!
- This is why we'll be analyzing **consumption** and **saving** ***jointly***

What should go into a model of consumption/saving?

- **Solow model**: consumption/saving only depend on **current income** Y_t

$$\boldsymbol{C_t} = (1 - s) \cdot \boldsymbol{Y_t}, \qquad \boldsymbol{I_t} = s \cdot \boldsymbol{Y_t}$$

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- Take *microfounded* approach: saving is **chosen** by a rational utility-maximizing rep. HH
- **This week**: in *discrete* time...
  - two periods (ECON 102)
  - ∞ periods
  - ***back to growth!***

Two-period consumption-saving

Utility function

Consider the behavior of a **representative household** over two periods ($t = 1, 2$)

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- $\boldsymbol{U'(C)} \equiv$ **marginal** utility
  - assume *positive* but *decreasing* ($\rightarrow U$ is strictly concave) [*interpretation?*]
  - e.g., if $U(C) = \ln C$ then $U'(C) = \frac{1}{C}$

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  - assume *positive* but *decreasing* ($\rightarrow U$ is strictly concave) [*interpretation?*]
  - e.g., if $U(C) = \ln C$ then $U'(C) = \frac{1}{C}$
- $\boldsymbol{U(C_1) + \beta U(C_2)} \equiv$ **lifetime utility**, where $\boldsymbol{\beta} \in (0, 1)$ is the discount factor

Budget constraint

Suppose the household receives income $\boldsymbol{Y_t}$ at date t for $t = 1, 2$

- saving $\boldsymbol{S}$ earns interest rate $\boldsymbol{r}$
- the **per-period (flow)** budget constraints:

$$C_1 + S = Y_1$$
$$C_2 = Y_2 + (1 + r)S$$

- How to turn these *two* constraints into a single one?

Present discounted value

The **present discounted value (PDV)** of a payment D at date t...

how much you need to save at date 0 to have D at date t

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- Examples:
  - To have \$1 at $t = 1$, you need to save PDV = $\frac{1}{1+r}$ at $t = 0$
  - To have \$1 at $t = 2$, you need to save PDV = $\frac{1}{(1+r)^2}$ at $t = 0$

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  - To have \$1 at $t = 2$, you need to save PDV $= \frac{1}{(1+r)^2}$ at $t = 0$
- What if you get \$1 each year from this year onward?

$$\text{PDV} = 1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots = \underbrace{\sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i}_{\text{geometric sum}} = \frac{1}{1 - \frac{1}{1+r}} = \frac{1+r}{r}$$

The lifetime budget constraint

Combine the per-period budget constraints into a single **lifetime budget constraint**:

1. Divide the second period constraint by $(1+r)$:

$$C_2 = Y_2 + (1+r)S \quad \rightarrow \quad \frac{C_2}{1+r} = \frac{Y_2}{1+r} + S$$

2. Substitute in $S = Y_1 - C_1$:

$$\underbrace{C_1 + \frac{C_2}{1+r}}_{\text{PDV of consumption}} = \underbrace{Y_1 + \frac{Y_2}{1+r}}_{\text{PDV of income}}$$

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Coming up: ***this holds more generally!***

Two formulations of the two-period consumption-saving problem

1. The **per-period** formulation:

$$\max_{C_1, C_2} \; U(C_1) + \beta U(C_2) \quad \text{s.t.} \quad \begin{aligned} C_1 + S &= Y_1 \\ C_2 &= Y_2 + (1+r)S \end{aligned}$$

2. The **lifetime** formulation:

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Two ways to solve

Substitution $\rightarrow$ *un*constrained optimization

using per-period budget constraints...

$$\max_S \; U(Y_1 - S) + \beta U(Y_2 + (1+r)S)$$

$$\rightarrow U'(Y_1 - S) = \beta(1+r)U'(Y_2 + (1+r)S)$$

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Lagrangian $=$ constrained optimization

using lifetime budget constraint...

$$\max_{C_1, C_2} \; U(C_1) + \beta U(C_2) + \boldsymbol{\lambda} \left(Y_1 + \frac{Y_2}{1+r} - C_1 - \frac{C_2}{1+r} \right)$$

$$\rightarrow U'(C_1) = \lambda, \quad \beta U'(C_2) = \frac{\lambda}{1+r}$$

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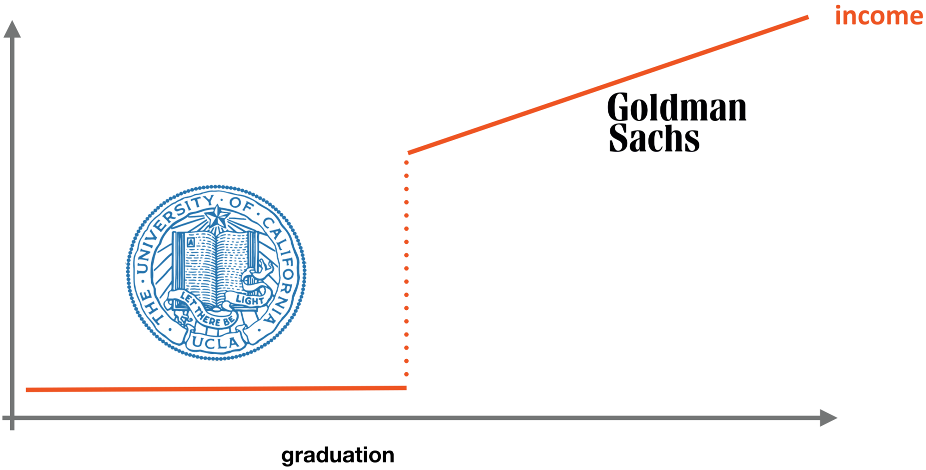
$$\max_{C_1, C_2} \; U(C_1) + \beta U(C_2)$$
$$+ \lambda \left(Y_1 + \frac{Y_2}{1+r} - C_1 - \frac{C_2}{1+r} \right)$$
$$\rightarrow U'(C_1) = \lambda, \quad \beta U'(C_2) = \frac{\lambda}{1+r}$$

Either way, end up with the same **Euler equation**:

$$\underbrace{\boldsymbol{U'(C_1)}}_{\text{MU if } \textit{consume} \text{ one more unit this period}} = \underbrace{\boldsymbol{\beta(1+r)U'(C_2)}}_{\substack{\text{MU if } \textit{save} \text{ one more unit this period} \\ \rightarrow \text{ consume } 1+r \text{ more next period}}}$$

With the budget constraint(s), uniquely pins down $\boldsymbol{C_1}$, $\boldsymbol{C_2}$, $\boldsymbol{S}$

Are *you* on your Euler equation?



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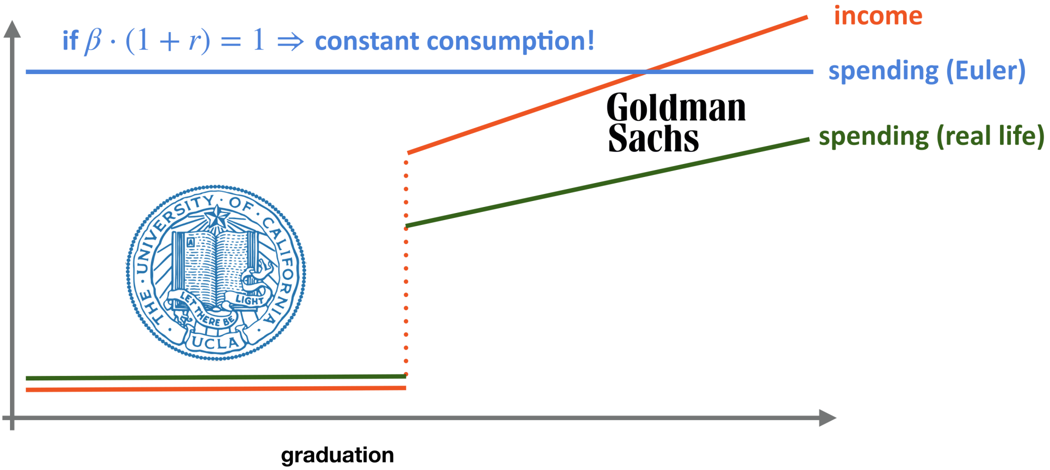

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2. If **log utility**:

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$$U'(C_1) = \underbrace{\beta(1+r)}_{=1} U'(C_2) \quad \rightarrow \quad \boldsymbol{C_1 = C_2}$$

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$$\frac{1}{C_1} = \beta(1+r)\frac{1}{C_2} \quad \rightarrow \quad \boldsymbol{C_2 = \beta(1+r)C_1}$$

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Infinite-horizon consumption-saving

Infinite-horizon utility function

- **Why ∞?** Clearly people do not live forever...
  - Yes, but each generation does care about the next one $\rightarrow$ **saves to leave bequests**
  - So could interpret each person as part of an "infinite dynasty"
  - Practically, the math for infinite lives is nicer than for finite lives
- Label periods by $t = 0, 1, 2, \ldots$ (start at date 0). Consumption at date $t \equiv C_t$.
- So now **lifetime utility** is

$$U(C_0) + \beta U(C_1) + \beta^2 U(C_2) + \ldots = \boldsymbol{\sum_{t=0}^{\infty} \beta^t U(C_t)}$$

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(*can we do this if* $\beta = 1$?)

Infinite-horizon per-period budget constraints

- Now agents can save over **many periods**
  - $S_t \equiv$ stock of savings going into period t, earns interest rate r
  - $S_0 \equiv$ *initial wealth* at the beginning of period 0
- Get an **infinite sequence** of **per-period (flow)** budget constraints

$$
\begin{aligned}
C_0 + S_1 &= (1+r)S_0 + Y_0 \\
C_1 + S_2 &= (1+r)S_1 + Y_1 \\
C_2 + S_3 &= (1+r)S_2 + Y_2 \\
&\vdots
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$$\boldsymbol{C_t + S_{t+1} = (1+r)S_t + Y_t} \quad \text{for } t = 0, 1, 2, \ldots$$

Infinite-horizon lifetime budget constraint

- We can again combine all the per-period constraints into a lifetime constraint
- Solve out per-period consumption C_t and take **PDVs** from date 0

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\begin{aligned}
C_0 &= (1+r)S_0 + Y_0 - S_1 \quad \rightarrow & C_0 &= (1+r)S_0 + Y_0 - S_1 \\
C_1 &= (1+r)S_1 + Y_1 - S_2 \quad \rightarrow & \frac{C_1}{1+r} &= S_1 + \frac{Y_1}{1+r} - \frac{S_2}{1+r} \\
C_2 &= (1+r)S_2 + Y_2 - S_3 \quad \rightarrow & \frac{C_2}{(1+r)^2} &= \frac{S_2}{1+r} + \frac{Y_2}{(1+r)^2} - \frac{S_3}{(1+r)^2} \\
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$$
\sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} = (1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}
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Two formulations of the infinite-horizon consumption-saving problem

1. The **per-period** formulation:

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t U(C_t) \quad \text{s.t.} \quad C_t + S_{t+1} = (1+r)S_t + Y_t \quad \text{for } t = 0, 1, 2, \ldots$$

2. The **lifetime** formulation:

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2. The **lifetime** formulation: solve via **Lagrangian**

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t U(C_t) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} = (1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}$$

continuous time: solve via ***Hamiltonian***

(Barro and Sala-i Martin, 2004)

Solving via the Lagrangian

(Kurlat, 2020, § 6.3)

The Lagrangian for the lifetime formulation is

$$\mathcal{L}(\{C_t\}) = \sum_{t=0}^{\infty} \beta^t U(C_t) + \lambda \left[(1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t} - \sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} \right]$$

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The first-order conditions w.r.t. any particular C_t and C_{t+1} are

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial C_t} = \beta^t U'(C_t) - \lambda \frac{1}{(1+r)^t} \\ 0 &= \frac{\partial \mathcal{L}}{\partial C_{t+1}} = \beta^{t+1} U'(C_{t+1}) - \lambda \frac{1}{(1+r)^{t+1}} \end{aligned}$$

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The first-order conditions w.r.t. any particular C_t and C_{t+1} are

$$0 = \frac{\partial \mathcal{L}}{\partial C_t} = \beta^t U'(C_t) - \lambda \frac{1}{(1+r)^t}$$

$$0 = \frac{\partial \mathcal{L}}{\partial C_{t+1}} = \beta^{t+1} U'(C_{t+1}) - \lambda \frac{1}{(1+r)^{t+1}}$$

Solving for λ and rearranging recovers the **Euler equation**: for $t = 0, 1, 2, \ldots$

$$U'(C_t) = \beta(1+r)U'(C_{t+1})$$

Characterizing the solution

The **solution** is then the unique set of $C_0, C_1, C_2, \ldots$ that solve

$$U'(C_t) = \beta(1+r)U'(C_{t+1}) \;\; \forall\, t, \qquad \sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} = (1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}$$

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With **log utility**, the Euler equation simplifies to $\boldsymbol{C_{t+1} = \beta(1+r)C_t}$, so

$$\boldsymbol{C_t = \beta^t (1+r)^t C_0}$$

$$\begin{aligned}
\boldsymbol{C_0} &= \left((1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}\right) \Big/ \left(\sum_{t=0}^{\infty} \beta^t\right) \\
&= \left((1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}\right) \Big/ \left(\frac{1}{1-\beta}\right) \\
&= \boldsymbol{(1-\beta)\left((1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}\right)}
\end{aligned}$$

Taking stock

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So, turning back to **growth**...

opportunities vs. **necessities**

(modern macro inc. all these *and more!*)

Revisiting the Solow model

Consumption in the Solow model, revisited

Recall from **last week**...

$$y_t^{\text{BGP}} = A_t \left(\tilde{k}^{\text{ss}}\right)^{\alpha} = A_t \left(\frac{\boldsymbol{s}}{\delta + g_A + g_L}\right)^{\frac{\alpha}{1-\alpha}}$$

...**so should we just max out w/ $\boldsymbol{s = 1}$?**

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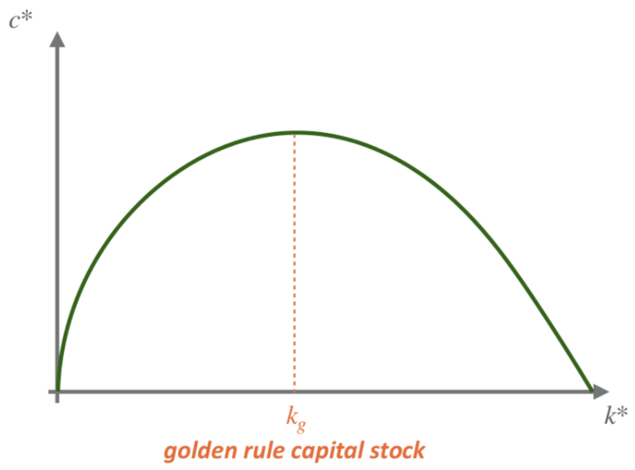
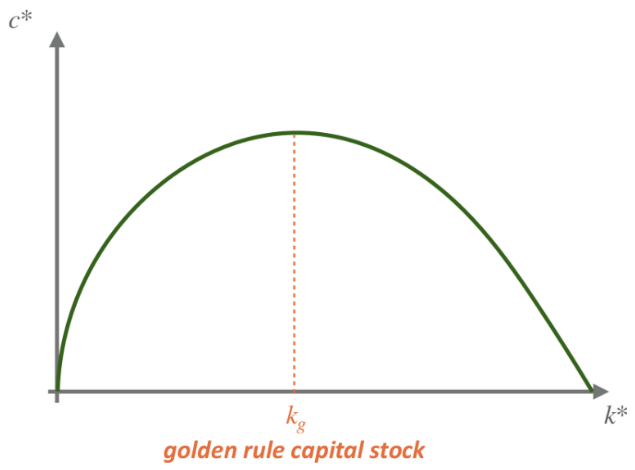
Not if what matters for utility is **consumption**:

$$c_t^{\text{BGP}} = (1 - \boldsymbol{s})y_t^{\text{BGP}} = (1 - \boldsymbol{s})A_t \left(\frac{\boldsymbol{s}}{\delta + g_A + g_L}\right)^{\frac{\alpha}{1-\alpha}}$$

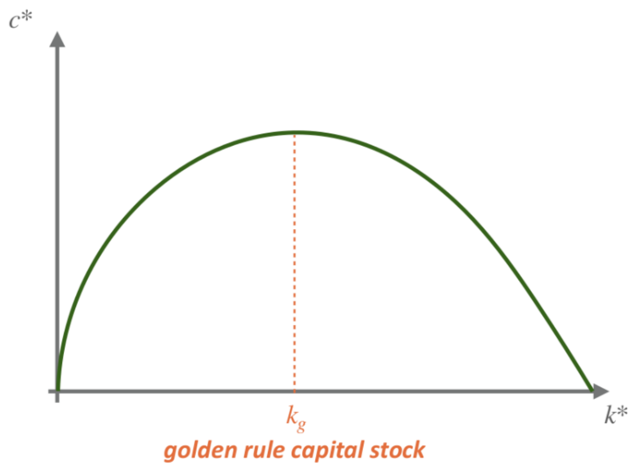
The "golden rule" to optimize steady-state consumption (Phelps, 1961)





Let's work w/ **detrended** consumption:

$$\tilde{c}^{\mathsf{ss}} \equiv \frac{c_t^{\mathsf{BGP}}}{A_t} = (1-s)\frac{y_t^{\mathsf{BGP}}}{A_t} \equiv (1-s)\tilde{y}^{\mathsf{ss}}$$



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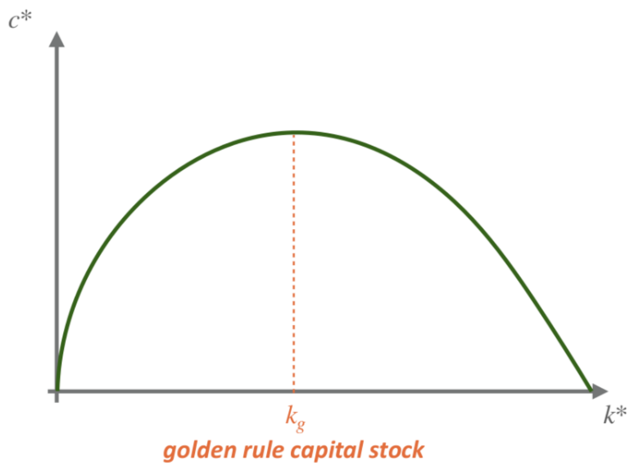

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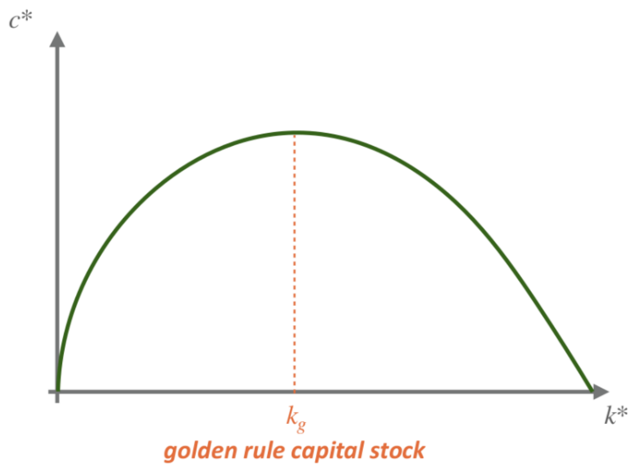
$\leftarrow$ ***What is that curve?***

$$\tilde{c}^{\text{ss}} = (1 - \boldsymbol{s}) \cdot \left[\tilde{k}^{\text{ss}}(\boldsymbol{s})\right]^{\alpha}$$

All values of $\tilde{c}$ *consistent with* $g_{\tilde{k}} = 0$



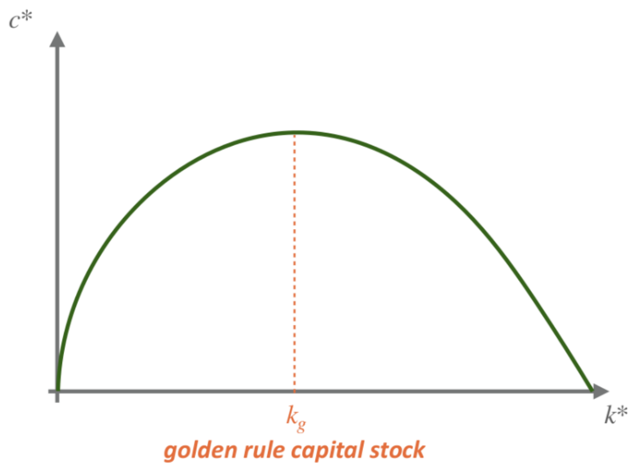
Solving $\max_s \tilde{c}^{ss}$:

$$\tilde{y}^{ss} = (1-s)\frac{\partial \tilde{y}^{ss}}{\partial s}$$

$$\frac{s}{1-s} = \frac{\partial \tilde{y}^{ss}}{\partial s}\frac{s}{\tilde{y}^{ss}}$$

$$\frac{s}{1-s} = \frac{\alpha}{1-\alpha}$$

$$\boldsymbol{s^{\mathsf{GR}} = \alpha}$$



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$$s^{GR} = \alpha$$

What's the intuition?

The Solow model in discrete time

As we switch into discrete time, let's simplify: $\boldsymbol{L_t = 1}$, $\boldsymbol{A_t = 1}$ $\forall t \rightarrow \boldsymbol{K_t = k_t = \tilde{k}_t}$

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$$\frac{k_{t+1} - k_t}{\Delta t} = sF(k_t, 1) - \delta k_t$$

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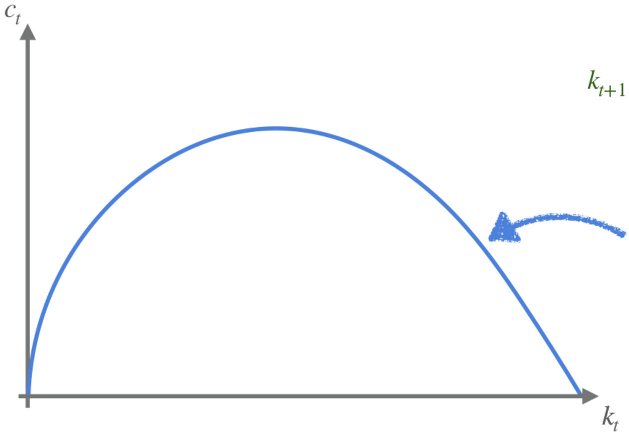
$$k_{t+1} = sF(k_t, 1) + (1 - \delta)k_t$$

but we can split that into...

$$k_{t+1} = [\boldsymbol{F(k_t, 1) - c_t}] + (1 - \delta)k_t, \qquad c_t = (\boldsymbol{1 - s})F(k_t, 1)$$

A **steady state** is a point where $\boldsymbol{\frac{k_{t+1} - k_t}{\Delta t} = 0 \rightarrow k_{t+1} = k_t \rightarrow c_{t+1} = c_t}$

(Re)introducing the c-k phase diagram



the law of motion:

$$k_{t+1} = F(k_t,1) - c_t + (1-\delta)\, k_t$$

Q: How can we find the points for which $k_{t+1} = k_t$?

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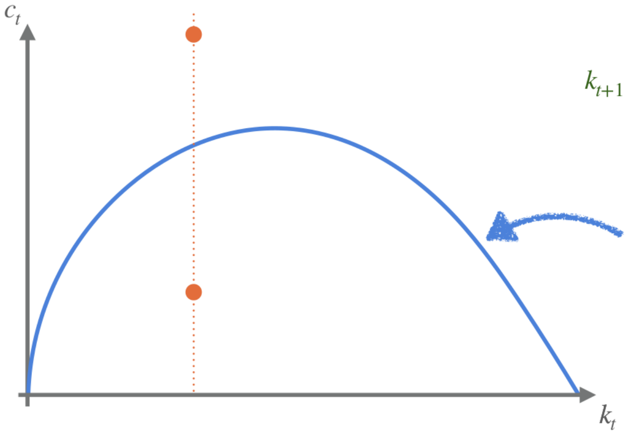

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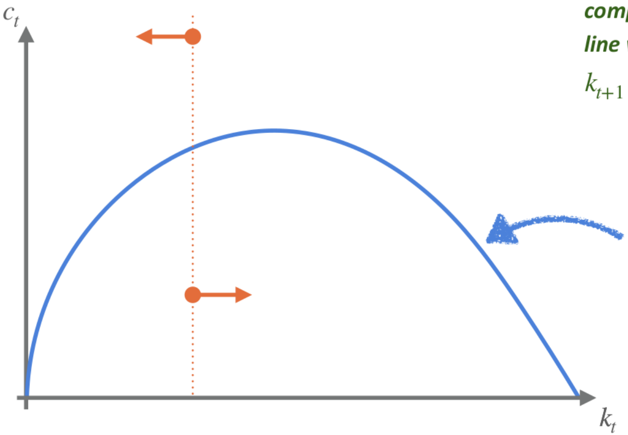
$\longrightarrow c_t = F(k_t, 1) - \delta k_t$ ***We also call this a locus***

Q: What happens if, for a given k_t we choose consumption above or below the blue line?

(Re)introducing the c-k phase diagram



For each point (k_t, c_t) in the diagram, we can compute next period's capital stock k_{t+1}, in line with the law of motion:

$$k_{t+1} = F(k_t,1) - c_t + (1-\delta)\,k_t$$

Q: How can we find the points for which $k_{t+1} = k_t$?

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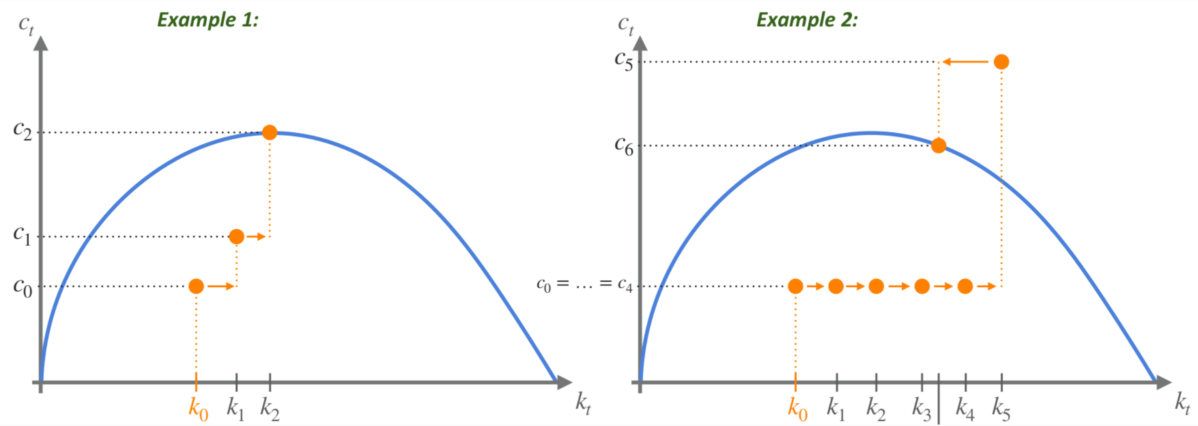
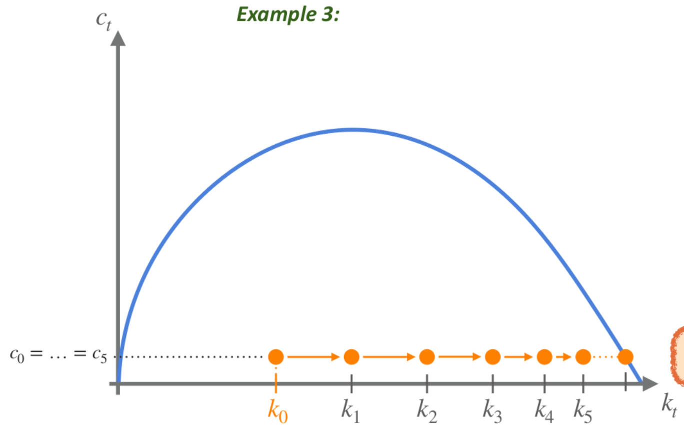
For any k_0, different consumption rules $\rightarrow$ different transition paths

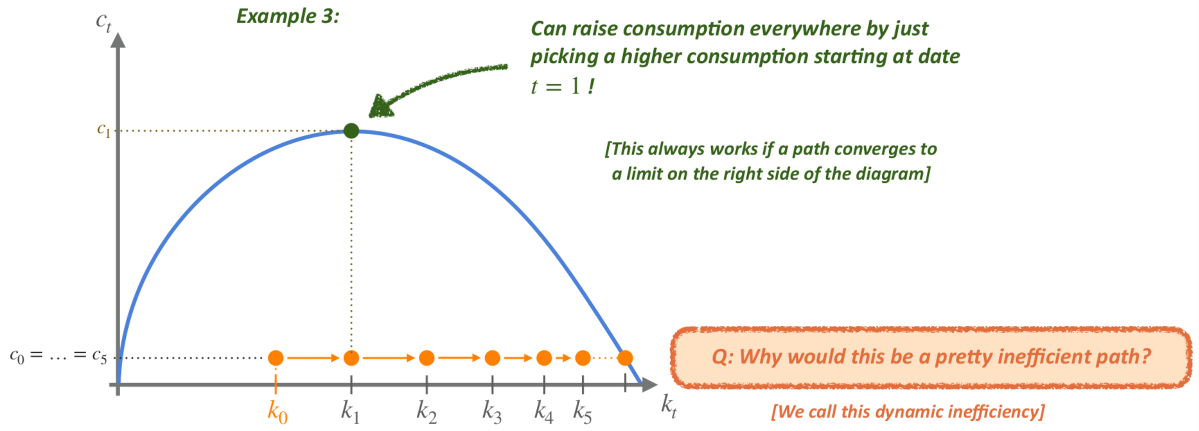


Some paths are clearly inferior to others. . .

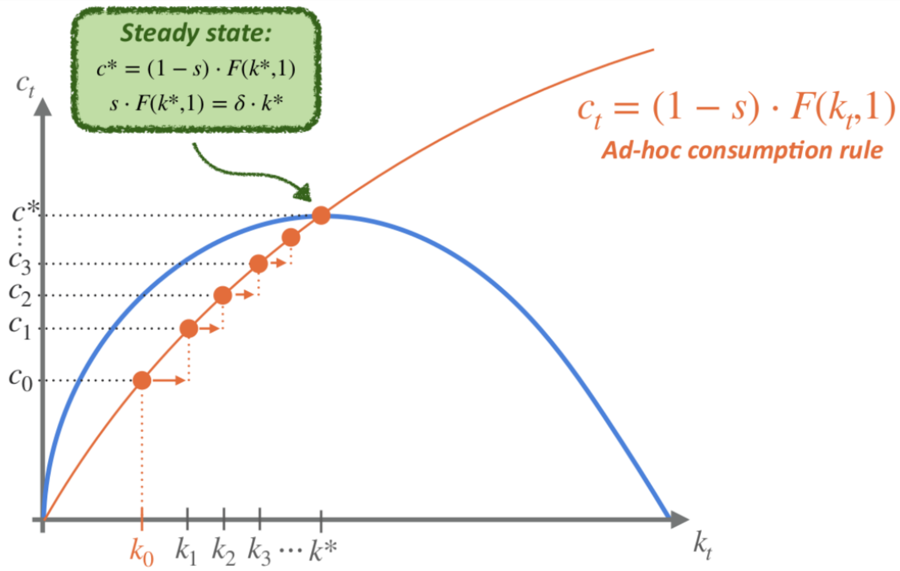


Q: Why would this be a pretty inefficient path?

Some paths are clearly inferior to others. . .



Solow consumption rule → *just one particular path*



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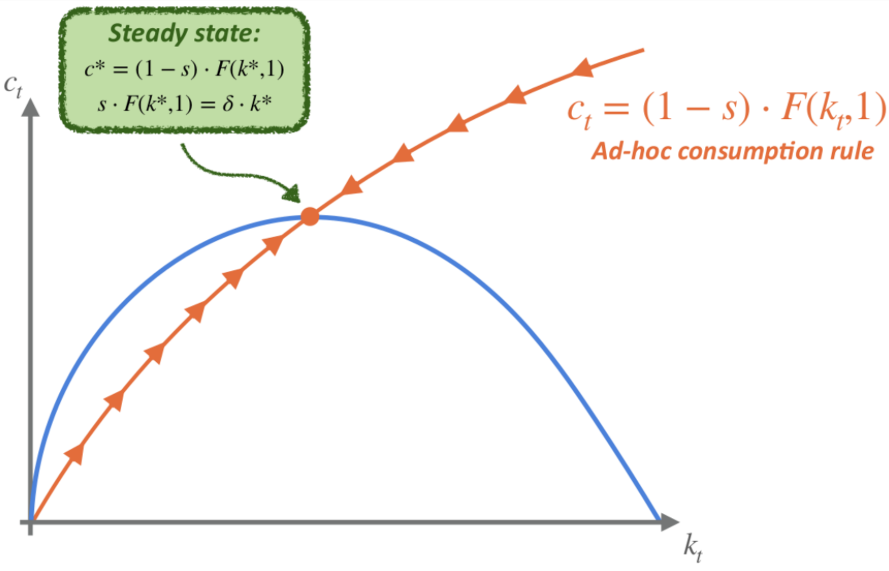
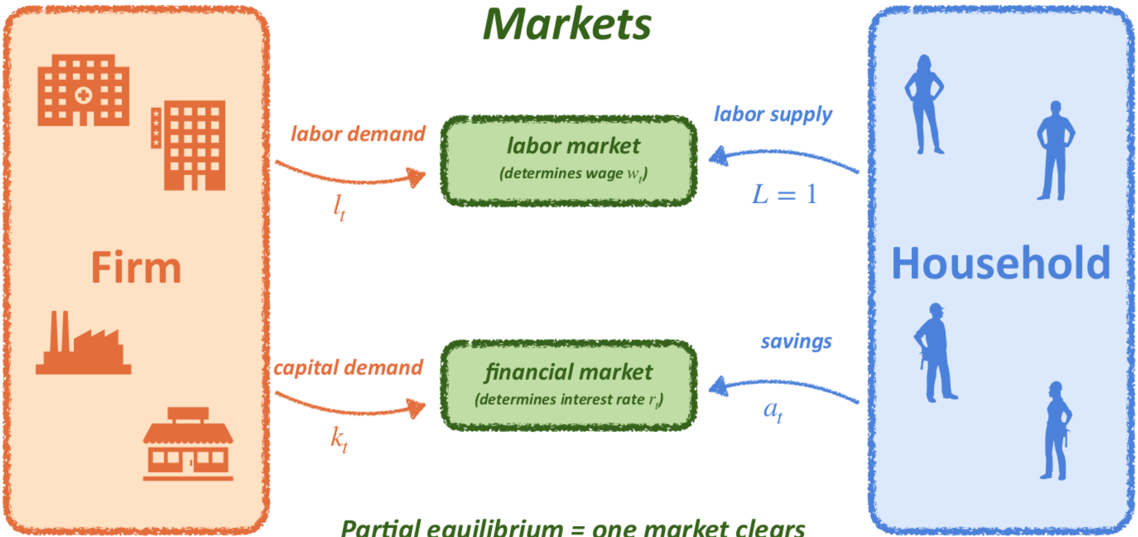


The Ramsey-Cass-Koopmans model

Overview

- The Ramsey-Cass-Koopmans model will be our first **fully modern** macro model
- **Main idea**: Want to replace the ad-hoc consumption rule of the Solow model...
    - ...with consumption behavior from a utility-maximizing ∞-horizon household
- Two approaches to set up this model
    - Describe as **general equilibrium**
        - Representative household & firm, interact thru markets [*will focus on this one!*]
    - Describe as **planning problem**
        - A "benevolent planner" maximizes household utility s.t. resource constraints
- *Note*: Both yield the same, so we say that the model is **first-best efficient**

General equilibrium



Partial equilibrium = one market clears

General equilibrium = all markets clear simultaneously

Household optimization problem

- Household earns **wage $\boldsymbol{w_t}$** and accumulates a stock of **assets $\boldsymbol{a_t}$** entering period t
  - *last class*: income was denoted by Y_t and savings by S_t
  - in general equilibrium, using w_t and a_t is somewhat more common

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$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \quad \text{s.t.} \quad c_t + a_{t+1} = (1 + r_t)a_t + w_t \quad \text{for } t = 0, 1, 2, \ldots$$

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which is characterized by the **Euler equation**

$$U'(c_t) = \beta(1 + r_{t+1})U'(c_{t+1}) \quad \text{for } t = 0, 1, 2, \ldots$$

and **budget feasibility** (*recall from last class!*)

Firm optimization problem

- In a general equilibrium model, we also need to describe:
  - **what** the household saves in...
  - ...and **who** pays the wage
- Suppose there is a representative firm that, **each period**:
  - Rents capital k_t from the household at rate $R_t = r_t + \delta$
    - compensate household for *opportunity cost* (r_t) and *depreciation* (δ)
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$$\max_{k_t, \ell_t} \underbrace{F(k_t, \ell_t)}_{\text{revenue}} \overbrace{-(r_t + \delta)k_t}^{\substack{\text{capital}\\ \text{expenditures}}} \underbrace{-w_t \ell_t}_{\substack{\text{wage}\\ \text{payments}}}$$

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$$\frac{\partial F(k_t,\ell_t)}{\partial k_t} \equiv \boldsymbol{F_k(k_t,\ell_t) = r_t + \delta}$$

$$\frac{\partial F(k_t,\ell_t)}{\partial \ell_t} \equiv \boldsymbol{F_\ell(k_t,\ell_t) = w_t}$$

Defining equilibrium

A **competitive equilibrium** consists of quantities $\{c_t, a_t, k_t, \ell_t\}$ and prices $\{r_t, w_t\}$ s.t.

1. The household optimally picks $\{c_t, a_t\}$ given $\{r_t, w_t\}$

$$U'(c_t) = \beta(1 + r_{t+1})U'(c_{t+1}), \quad c_t + a_{t+1} = (1 + r_t)a_t + w_t \quad \text{for } t = 0, 1, 2, \ldots$$

2. The firm optimally picks $\{k_t, \ell_t\}$ given $\{r_t, w_t\}$

$$F_k(k_t, \ell_t) = r_t + \delta, \quad F_\ell(k_t, \ell_t) = w_t$$

3. The asset market clears: $k_t = a_t$
4. The labor market clears: $\ell_t = 1$
5. The resource constraint holds: (*check*: already implied by #1–4)

$$k_{t+1} = F(k_t, 1) - c_t + (1 - \delta)k_t$$

Ramsey-Cass-Koopmans vs. Solow

Both have **same** resource constraint. . .

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. . . but **different** consumption rules:

$$c_t = (1 - s)F(k_t, 1)$$

vs.

$$U'(c_t) = \beta[1 + r_{t+1}]U'(c_{t+1})$$

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How can we **solve** this system? **Just like Solow!**

1. Characterize the **steady state**
2. Analyze transitions using c-k **phase diagram**

Steady state

A **steady state** is a pair $\{\boldsymbol{c}^{\mathsf{ss}}, \boldsymbol{k}^{\mathsf{ss}}\}$ at which **optimally**...

$$c_t = c_{t+1} = c^{\mathsf{ss}} \quad \rightarrow \quad U'(c^{\mathsf{ss}}) = \beta\left[1 + F_k\left(k^{\mathsf{ss}}, 1\right) - \delta\right] U'(c^{\mathsf{ss}})$$

$$\boldsymbol{F_k\left(k^{\mathsf{ss}}, 1\right) = \frac{1}{\beta} - 1 + \delta}$$

$$k_t = k_{t+1} = k^{\mathsf{ss}} \quad \rightarrow \quad k^{\mathsf{ss}} = F(k^{\mathsf{ss}}, 1) - c^{\mathsf{ss}} + (1-\delta)k^{\mathsf{ss}}$$

$$\boldsymbol{c^{\mathsf{ss}} = F(k^{\mathsf{ss}}, 1) - \delta k^{\mathsf{ss}}}$$

Can be shown: optimal $\{c_t, k_t\}$ **always converges** to $\{c^{\mathsf{ss}}, k^{\mathsf{ss}}\}$

(*Check*: What is the relationship between β and r^{ss}?)

The "*modified* golden rule" to optimize the consumption *path*

Recall the **golden rule** ($\boldsymbol{s^{\mathsf{GR}} = \alpha}$): (Cobb-Douglas, $L_t = 1$, $A_t = 1$)

$$k^{\mathsf{ss}}_{\mathsf{Solow}} = \left(\frac{s}{\delta + g_A + g_L}\right)^{\frac{1}{1-\alpha}} \quad \rightarrow \quad k^{\mathsf{GR}} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}$$

But, from the Kaldor facts: $s \approx \frac{1}{4} < \frac{1}{3} \approx \alpha \rightarrow$ **should we be saving more?**

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The **modified golden rule** says *not necessarily*:

$$F_k\left(k^{\text{ss}}, 1\right) = \frac{1}{\beta} - 1 + \delta \quad \rightarrow \quad k^{\text{MGR}} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta}\right)^{\frac{1}{1-\alpha}} < k^{\text{GR}}$$

The golden rule **didn't account for impatience**!

What happens away from steady state?

Substitute the **resource constraint** into the **Euler equation**

$$U'(c_t) = \beta \left[1 + F_k(\underbrace{F(k_t, 1) - c_t + (1-\delta)k_t}_{k_{t+1}:\ \text{resource constraint}}, 1) - \delta \right] U'(c_{t+1})$$

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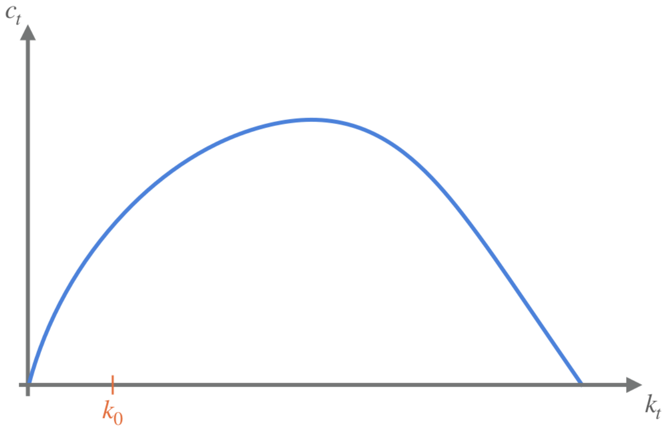
With **Cobb-Douglas** production and **log utility**...

$$c_{t+1} = \beta \left[1 + \alpha(k_t^\alpha - c_t + (1-\delta)k_t)^{\alpha-1} - \delta \right] c_t$$

From any given k_0, how do we transition to steady state?



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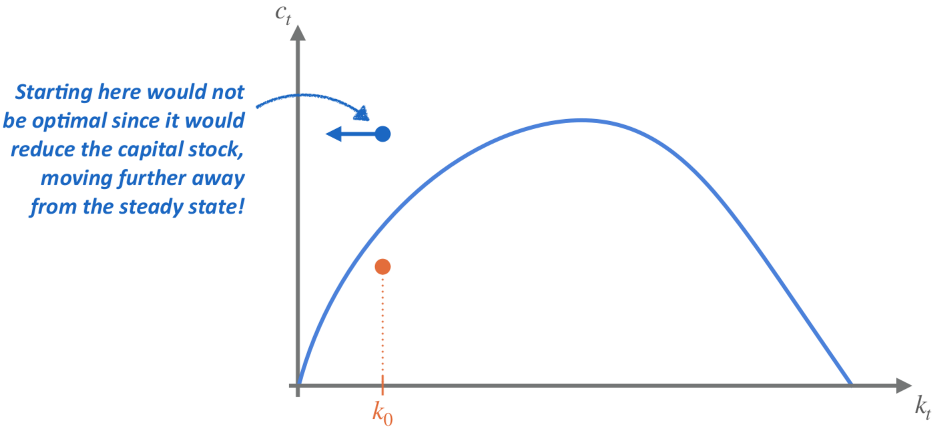

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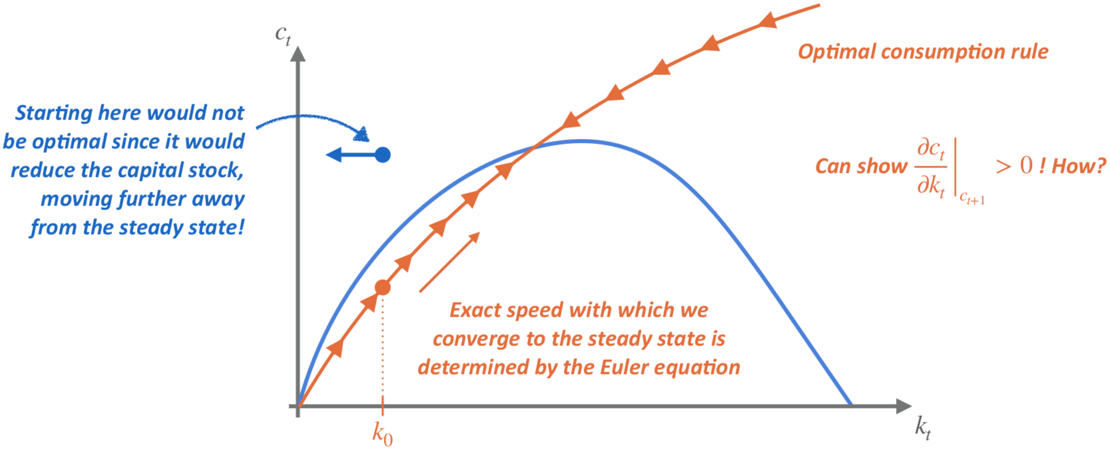

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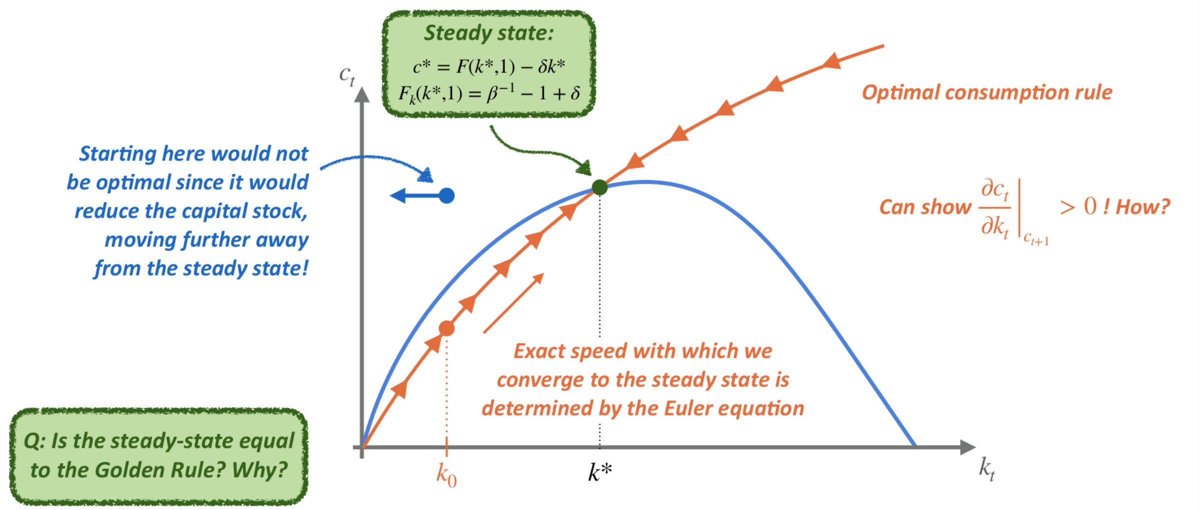


The Brock-Mirman special case (Brock and Mirman, 1972)

Here's a special case we can solve in **closed-form**: (*Problem Set #3*)

- Cobb-Douglas production function: $F(k_t, 1) = k_t^\alpha$
- Log utility: $U(c_t) = \ln(c_t)$
- Full depreciation: $\delta = 1$ (*pretty unrealistics unless we use long time periods*)

Then the **optimal consumption rule** is given by...

$$c_t = (1 - \alpha\beta)k_t^\alpha$$

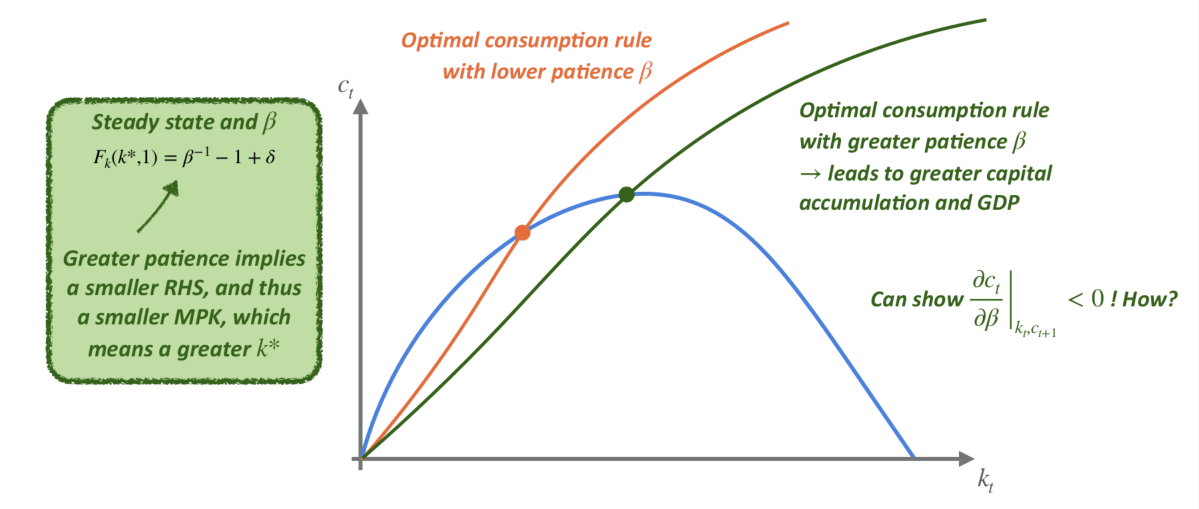
...which looks exactly like Solow just with endogenous savings rate $s = \alpha\beta$

Applications

Let's study **two applications** of the Ramsey-Cass-Koopmans model:

1. Compare two economies with different **discount factors**
    - *idea*: Protestant culture should lead to greater patience
    - *idea*: Aging economies should look like more patient economies
2. Introduce **capital taxes**
    - *idea*: Economies with extractive institutions should end up saving less

Application 1: Differences in discount factors



Application 2: Introduce capital taxes

- Think of capital taxes as **reducing the return on saving** in the Euler equation

$$U'(c_t) = \beta[1 + r_{t+1}\boldsymbol{(1-\tau)}]U'(c_{t+1}) \;\; \rightarrow \;\; \boldsymbol{(1-\tau)}\left[F_k\left(k^{\mathsf{ss}}, 1\right) - \delta\right] = \frac{1}{\beta} - 1$$

- capital taxes ($\boldsymbol{\tau} \uparrow$) have the **same effect** as less patience ($\boldsymbol{\beta} \downarrow$)
- reduce capital accumulation and steady-state GDP per capita

The neoclassical growth model (NGM)

$$F(K_t, A_t L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}, \qquad K_{t+1} = \boldsymbol{F(K_t, A_t L_t) - C_t} + (1-\delta)K_t$$

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... with **exogenous** savings (the **Solow** model)

- ad-hoc consumption rule $\longrightarrow$ savings depend on current output $\boldsymbol{y_t}$ only

... with **endogenous** savings (the **Ramsey-Cass-Koopmans** model)

- optimal consumption path $\longrightarrow$ savings depend on $\boldsymbol{\{w_t, r_t, \tau_t\}, \beta, k_0, \ldots}$
- can reincorporate population $(\boldsymbol{g_L})$ and productivity $(\boldsymbol{g_A})$ growth

How does the NGM answer our three organizing questions?

1. **Why are we so rich and they so poor?**

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Next week: How do these fit the data?

Bonus: A better welfare metric?

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- *begs the question*: can we come up w/ a better welfare metric than just y_{it}? say, one that incorporates...
  - optimal consumption-savings decisions?
  - optimal labor-leisure decisions?
  - mortality?
  - inequality?

Consumption-equivalent welfare (Jones and Klenow, 2016)

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Being a random person in country i (say, France) = Being a random person in the US with your consumption scaled by $\boldsymbol{\lambda_i}$.

A concrete example

Suppose everyone in the world has the **same preferences**:

$$U(c, \ell) = \bar{u} + \ln(c) + v(\ell)$$

with, for each country i, no discounting ($\beta = 1$), no consumption growth ($g_c = 0$), constant leisure (ℓ_i), known life expectancy (e_i), and consumption distributed LogNormal(c_i, σ_i^c) across people in i.

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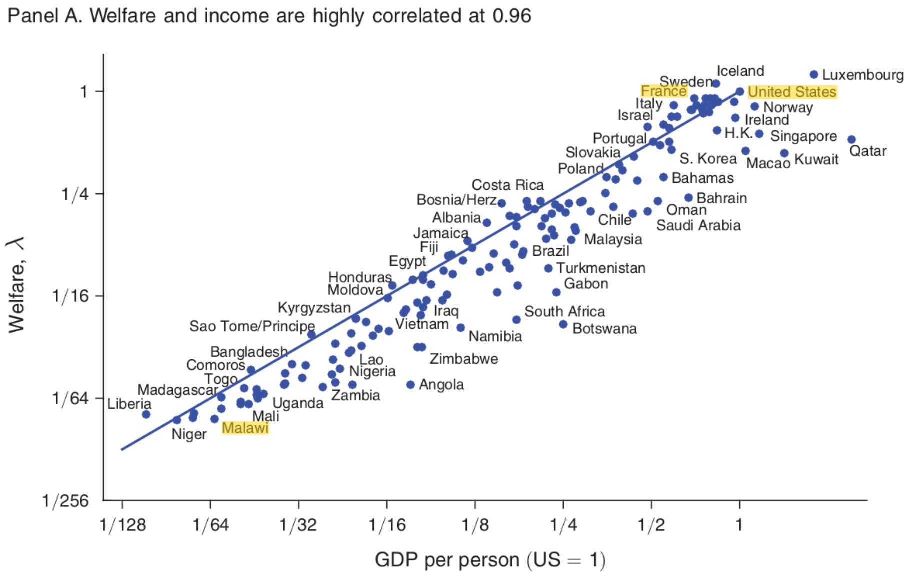
...and the (log) **consumption-equivalent welfare** in country i *relative to* the US is

$$\begin{aligned} \ln \boldsymbol{\lambda_i} = {} & \frac{e_i - e_{\text{US}}}{e_{\text{US}}} \left[\bar{u} + \ln(c_i) + v(\ell_i) - 0.5\sigma_i^c\right] && \text{(Life expectancy)} \\ & + \ln c_i - \ln c_{\text{US}} && \text{(Consumption)} \\ & + v(\ell_i) - v(\ell_{\text{US}}) && \text{(Leisure)} \\ & - 0.5(\sigma_i^c - \sigma_{\text{US}}^c) && \text{(Inequality)} \end{aligned}$$

GDP per capita is *highly* informative. . . (Jones and Klenow, 2016)



. . . but still masks important variation

(Jones and Klenow, 2016)

Panel B. But this masks substantial variation in the ratio of λ to GDP per capita. The mean absolute deviation from unity is about 27%

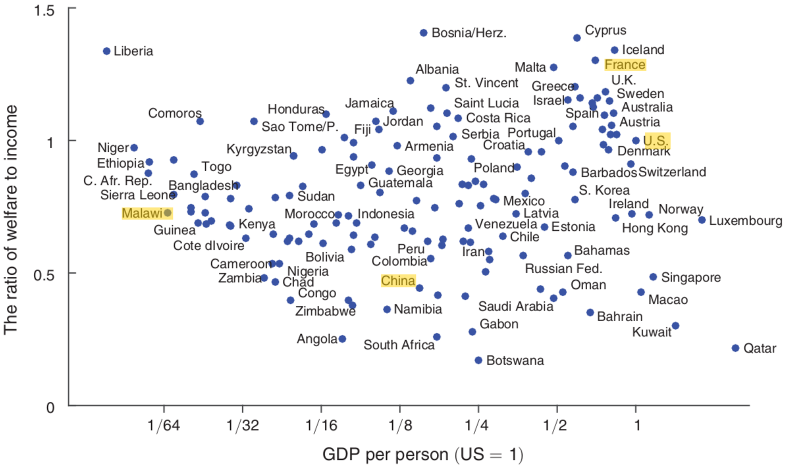


FIGURE 7. WELFARE USING MACRO DATA, 2007

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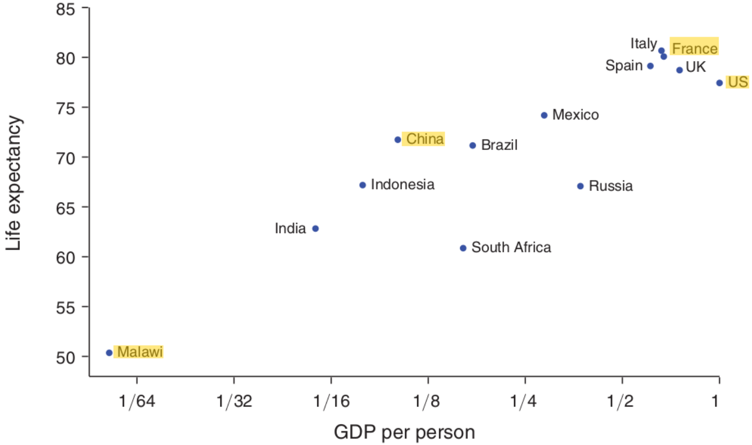

Figure 4. Life Expectancy

Note: Life expectancy at birth in each country is measured as the sum over all ages of the probability of surviving to each age, using life tables from the World Health Organization.

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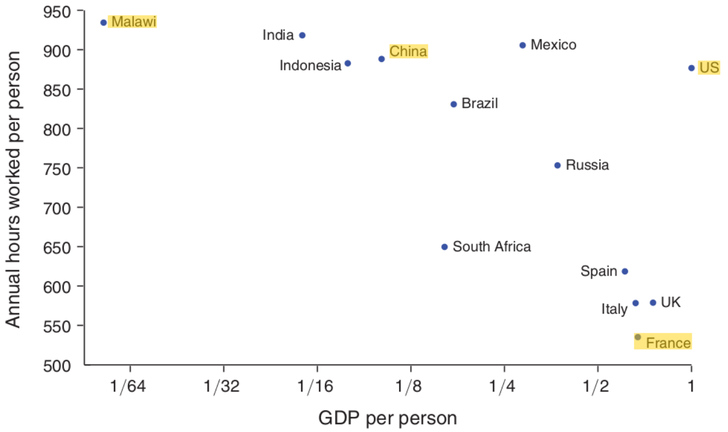

FIGURE 2. ANNUAL HOURS WORKED ACROSS COUNTRIES

Notes: The measure shown here of annual hours worked per capita is computed from the household surveys noted in Table 1, using survey-specific sampling weights and US survival rates across ages as in equation (16), with no time discounting.

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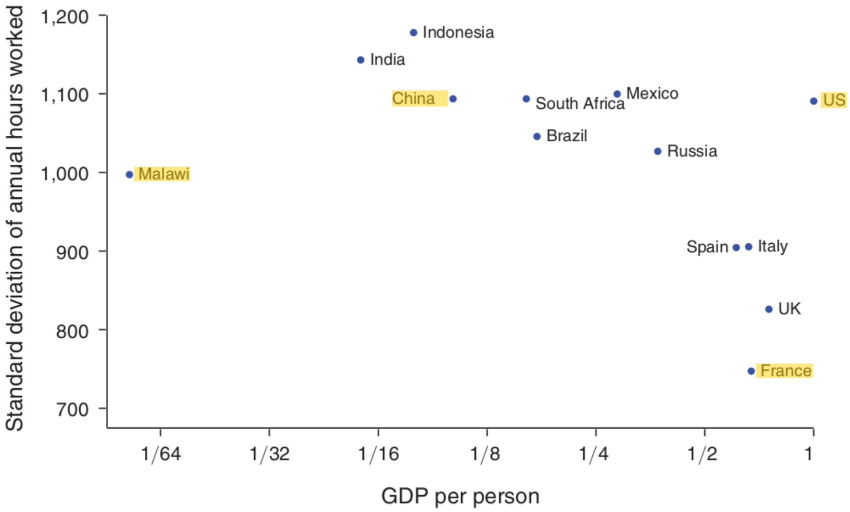

FIGURE 3. INEQUALITY IN ANNUAL HOURS WORKED

. . . but still masks important variation (Jones and Klenow, 2016)

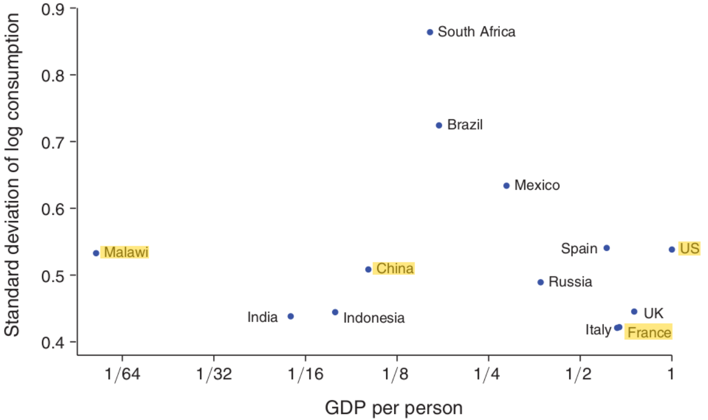

FIGURE 1. WITHIN-COUNTRY INEQUALITY

Notes: The standard deviation of log consumption within each economy is measured from the household surveys listed in Table 1. We use survey-specific sampling weights and US survival rates across ages using an analog of equation (17), with no discounting or growth.

That was for *levels*, but it's likewise for *growth* (Jones and Klenow, 2016)

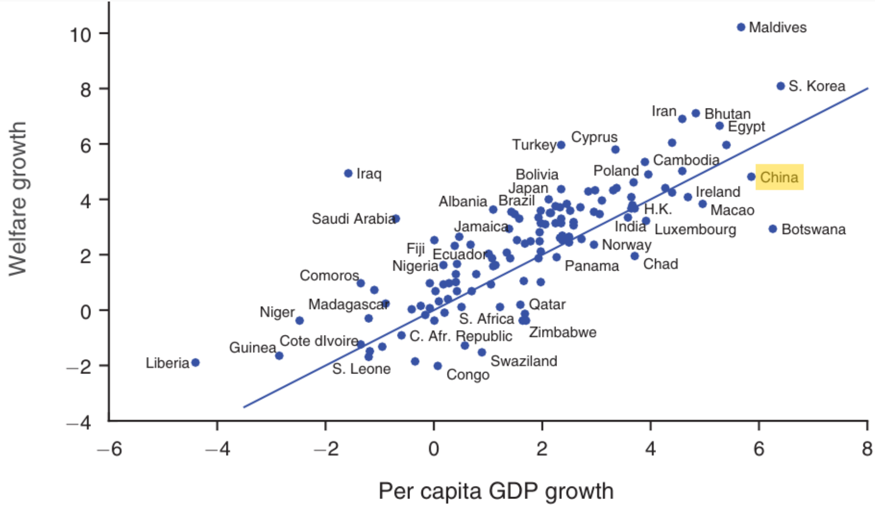

FIGURE 8. WELFARE AND INCOME GROWTH, 1980–2007 (*Percent*)

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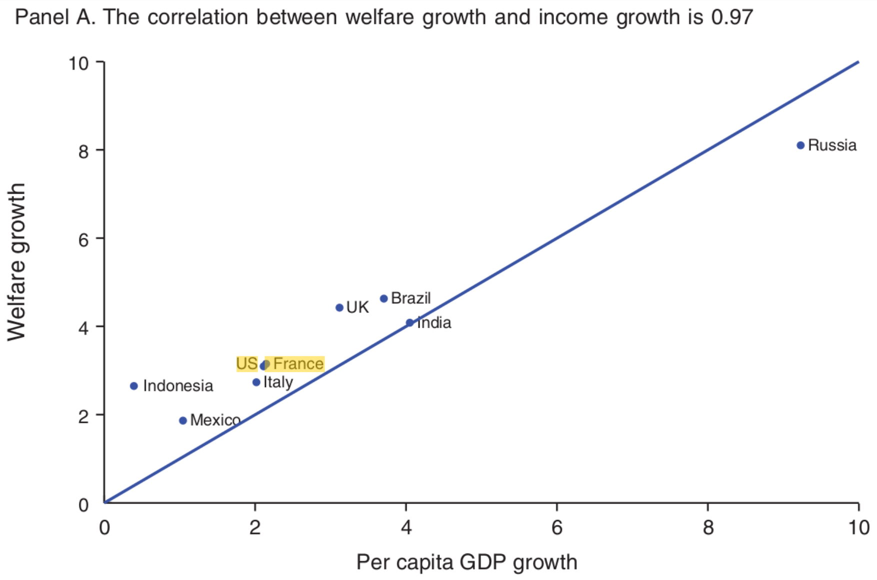

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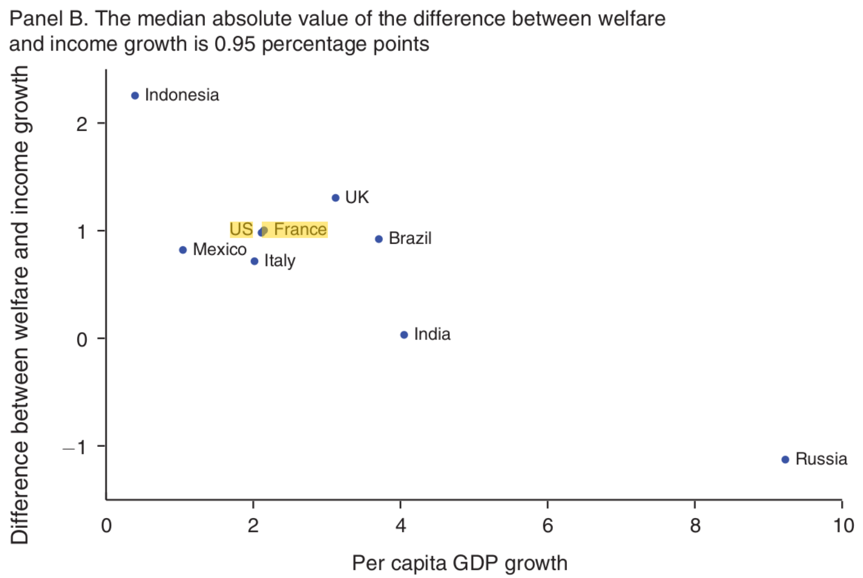

References

Barro, Robert J. and Xavier Sala-i Martin. 2004. *Economic growth*. Cambridge, MA: The MIT Press, second ed.

Brock, W. and L. Mirman. 1972. "Optimal growth under uncertainty: The discounted case." *Journal of Economic Theory* 4:479–513.

Jones, Charles I. and Peter J. Klenow. 2016. "Beyond GDP? Welfare across countries and time." *American Economic Review* 106 (9):2426–2457.

Kurlat, Pablo. 2020. *A course in modern macroeconomics*.

Phelps, Edmund. 1961. "The golden rule of accumulation: A fable for growthmen." *American Economic Review* 51 (4):638–643.