Problem 8:

a) Trình bày các phương pháp giải gần đúng phương trình vi phân bậc nhất và áp dụng để giải phương trình

$$y' = 2xy, y(1) = 1, h = 1$$

Tính đến y_2.

b) Tính bằng lập trình đến y_{10} với h = 0,1.

c) Đánh giá sai số.

SOLUTION

Given Problem:

$$\frac{dy}{dt} = f(t,y) \ , \qquad y(a) = \alpha$$

- ***Euler's Method:***

Euler's Method constructs $w_i \approx y(t_i)$, for each $i = \overline{1, N-1}$:

$$\boldsymbol{w_0 = \alpha}$$

$$\boldsymbol{w_{i+1} = w_i + hf(t_i, w_i)} \textbf{ for each } \boldsymbol{i = \overline{1, N-1}}$$

Error Bound:

Suppose:

1. $\left|\frac{\partial f}{\partial y}(t,y)\right| \le L \ \forall \ (t,y) \in D = \{(t,y) | a \le t \le b \,, -\infty < y < \infty\}$
2. $|y''(t)| \le M \ \forall \ t \in [a,b]$

Then

$$\boldsymbol{|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right]}$$

LTE in Euler's Method: O(h^2)

GTE in Euler's Method: O(h)

- ***Second – Order Taylor's series Method:***

$w_i \approx y(t_i)$, for each $i = \overline{1, N-1}$:

$$\boldsymbol{w_0 = \alpha}$$

$$\boldsymbol{w_{i+1} = w_i + hT^{(2)}(t_i, w_i)} \textbf{ for each } \boldsymbol{i = \overline{1, N-1}}$$

where

$$\boldsymbol{T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)}$$

GTE in Second – Order Taylor's series Method: O(h^2)

- ***Midpoint Method:***

$w_i \approx y(t_i)$, for each $i = \overline{1, N-1}$:

$$\boldsymbol{w_0 = \alpha}$$

$$\boldsymbol{w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)} \textbf{ for each } \boldsymbol{i = \overline{1, N-1}}$$

LTE in Midpoint Method: O(h^3)

GTE in Midpoint Method: O(h^2)

- ***Heun's Predictor Corrector Method:***

Predictor: $$y_{i+1}^{0} = y_i + hf(t_i, y_i)$$

Corrector: $$y_{i+1}^{k+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f\left(t_{i+1}, y_{i+1}^{k}\right)\right]$$

LTE in Midpoint Method: O(h^3)

GTE in Midpoint Method: O(h^2)

- ***Runge – Kutta Method of Order 2:***

Select $\alpha \neq 0$:

$$K_1 = hf(t_i, y_i)$$

$$K_2 = hf(t_i + \alpha h, y_i + \alpha K_1)$$

$$y_{i+1} = y_i + \left(1 - \frac{1}{2\alpha}\right)K_1 + \frac{1}{2\alpha}K_2$$

For each $i = \overline{1, N-1}$

Choose $\alpha = \frac{1}{2}$: Second Order Runge – Kutta becomes the Midpoint Method.

Choose $\alpha = 1$: Second Order Runge – Kutta becomes Heun's Method with a Single Corrector.

LTE in Runge – Kutta Method of Order 2 Method: O(h^3)

GTE in Runge – Kutta Method of Order 2 Method: O(h^2)

- ***Runge – Kutta Method of Order 4:***

$$w_0 = \alpha$$

$$k_1 = hf(t_i, w_i)$$

$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right)$$

$$k_4 = hf(t_{i+1}, w_i + k_3)$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

For each $i = \overline{1, N-1}$

LTE in Runge – Kutta Method of Order 4 Method: O(h^5)

GTE in Runge – Kutta Method of Order 4 Method: O(h^4)

- ***4 – Step Adam – Molton Predictor – Corrector Method:***

Predictor: The 4th – order Adams – Bashforth Method:

$$w_0 = \alpha$$

$$w_1 = \alpha_1$$

$$w_2 = \alpha_2$$

$$w_3 = \alpha_3$$

$$w_{i+1}^{P} = w_i + \frac{1}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

Corrector: The 4th – order Adams – Moulton Method:

$$w_{i+1}^{C} = w_i + \frac{1}{24}\left[9f\left(t_{i+1}, w_{i+1}^{P}\right) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\right]$$

For each $i = \overline{1, N-1}$.

- ***Solving detailed problem with estimating the errors:***

Denote $\boldsymbol{y(t_i) = y_i\,, i = \overline{0,2}}$

Euler's Method:

Ta có:

$$y_0 = w_0 = 1$$

$$y_1 = w_1 = w_0 + hf(t_0, w_0) = 1 + 1.f(1,1) = 1 + 1.2.1.1 = 3$$

$$\boldsymbol{y_2 = w_2 = w_1 + hf(t_1, w_1) = 3 + 1.f(2,3) = 3 + 1.2.2.3 = 15}$$

Second – Order Taylor's series Method:

$$f'(x, y) = 2y + 2y'x = 2y + 4x^2y$$

$$y_0 = w_0 = 1$$

$$y_1 = w_1 = w_0 + hf(t_0, w_0) + \frac{h^2}{2}f'(t_0, w_0) = 1 + 1.f(1,1) + \frac{1}{2}.f'(1,1) = 1 + 1.2.1.1 + \frac{1}{2}.2.1 = 6$$

$$\boldsymbol{y_2 = w_2 = w_1 + hf(t_1, w_1) + \frac{h^2}{2}f'(t_1, w_1) = 3 + 1.f(2,3) + \frac{1}{2}.f'(2,3) = 3 + 1.2.2.3 + \frac{1}{2}.2.3 = 84}$$

Midpoint Method:

$$y_0 = w_0 = 1$$

$$y_1 = w_1 = w_0 + hf\left(t_0 + \frac{h}{2}, w_0 + \frac{h}{2}f(t_0, w_0)\right) = 1 + 1.f\big(1.5, 1 + 0.5.f(1,1)\big) = 1 + 1.2.(1.5).(1 + 1) = 7$$

$$\boldsymbol{y_2 = w_2 = w_1 + hf\left(t_1 + \frac{h}{2}, w_1 + \frac{h}{2}f(t_1, w_1)\right) = 7 + 1.f\big(2.5, 7 + 0.5.f(2,7)\big) = 112}$$

Heun's Predictor Corrector Method:

Step 1: $\boldsymbol{y_1}$:

Predictor: $y_1^0 = y_0 + hf(t_0, y_0) = 3$

Corrector: $y_1^1 = y_0 + \frac{h}{2}[f(t_0, y_0) + f(t_1, y_1^0)] = 8$

Step 2: $\boldsymbol{y_2}$:

Predictor: $y_2^0 = y_1 + hf(t_1, y_1) = 40$

Corrector: $y_2^1 = y_1 + \frac{h}{2}[f(t_1, y_1) + f(t_2, y_2^0)] = 144$

Runge – Kutta Method of Order 2:

Choose $\alpha = \frac{1}{2}$: Second Order Runge – Kutta becomes the Midpoint Method.

Choose $\alpha = 1$: Second Order Runge – Kutta becomes Heun's Method with a Single Corrector:

Step 1: $\boldsymbol{y_1}$:

$$K_1 = hf(t_0, y_0) = 2$$

$$K_2 = hf(t_0 + h, y_0 + K_1) = 12$$

$$y_1 = y_0 + \left(1 - \frac{1}{2}\right)K_1 + \frac{1}{2}K_2 = 8$$

Step 2: $\boldsymbol{y_2}$:

$$K_1 = hf(t_1, y_1) = 32$$

$$K_2 = hf(t_1 + h, y_1 + K_1) = 240$$

$$y_1 = y_1 + \left(1 - \frac{1}{2}\right)K_1 + \frac{1}{2}K_2 = 144$$

Runge – Kutta Method of Order 4:

Step 1: $\boldsymbol{y_1}$:

$$w_0 = 1$$

$$k_1 = hf(t_0, w_0) = 2$$

$$k_2 = hf\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}k_1\right) = 6$$

$$k_3 = hf\left(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}k_2\right) = 12$$

$$k_4 = hf(t_1, w_0 + k_3) = 52$$

$$y_1 = w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 16$$

Step 2: $\boldsymbol{y_2}$:

$$k_1 = hf(t_1, w_1) = 64$$

$$k_2 = hf\left(t_1 + \frac{h}{2}, w_1 + \frac{1}{2}k_1\right) = 240$$

$$k_3 = hf\left(t_1 + \frac{h}{2}, w_1 + \frac{1}{2}k_2\right) = 680$$

$$k_4 = hf(t_2, w_1 + k_3) = 4176$$

$$y_1 = w_2 = w_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{3088}{3} \approx 1029{,}33$$

4 – Step Adam – Moulton Predictor – Corrector Method:

(Unavailable with y_2)

Error:

Euler's Method ($O(h)$) ≥ Second – Order Taylor's series Method ($O(h^2)$) ≥ Midpoint Method ($O(h^2)$) ≥ Heun's Predictor Corrector Method, RK2 ($O(h^2)$) ≥ RK4 ($O(h^4)$)