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Exact and Heuristic Algorithms for the Weapon-Target Assignment Problem

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The weapon-target assignment (WTA) problem is a fundamental problem arising in defense-related applications of operations research. This problem consists of optimally assigning n weapons to m targets so that the total expected survival value of the targets after all the engagements is minimal. The WTA problem can be formulated as a nonlinear integer programming problem and is known to be NP-complete. No exact methods exist for the WTA problem that can solve even small-size problems (for example, with 20 weapons and 20 targets). Although several heuristic methods have been proposed to solve the WTA problem, due to the absence of exact methods, no estimates are available on the quality of solutions produced by such heuristics. In this paper, we suggest integer programming and network flow-based lower-bounding methods that we obtain using a branch-and-bound algorithm for the WTA problem. We also propose a network flow-based construction heuristic and a very large-scale neighborhood (VLSN) search algorithm. We present computational results of our algorithms, which indicate that we can solve moderately large instances (up to 80 weapons and 80 targets) of the WTA problem optimally and obtain almost optimal solutions of fairly large instances (up to 200 weapons and 200 targets) within a few seconds.

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1. Introduction

The *weapon-target assignment* (*WTA*) *problem* arises in the modeling of combat operations where we wish to maximize the total expected damage caused to the enemy's target using a limited number of weapons. In the WTA problem, there are n targets, numbered $1, 2, \ldots, n$; and m weapon types, numbered $1, 2, \ldots, m$. Let V_j denote the value of the target j and W_i denote the number of weapons of type i available to be assigned to targets. Let p_{ij} denote the probability of destroying target j by a single weapon of type i. Hence, $q_{ij} = 1 - p_{ij}$ denotes the probability of survival of target j if a single weapon of type i is assigned to it. Observe that if we assign x_{ij} number of weapons of type i to target j, then the survival probability of target j is given by $q_{ij}^{x_{ij}}$. A target may be assigned weapons of different types. The WTA problem is to determine the number of weapons x_{ij} of type i to be assigned to target j to minimize the total expected survival value of all targets. This problem can be formulated as the following nonlinear integer programming problem:

$$\text{Minimize} \sum_{j=1}^{n} V_j \left(\prod_{i=1}^{m} q_{ij}^{x_{ij}} \right) \tag{1a}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leqslant W_i \quad \text{for all } i = 1, 2, \ldots, m, \tag{1b}$$

$$x_{ij} \geqslant 0 \text{ and integer}$$
$$\text{for all } i = 1, 2, \ldots, m \text{ and for all } j = 1, 2, \ldots, n. \tag{1c}$$

In the above formulation, we minimize the expected survival value of the targets while ensuring that the total number of weapons used is no more than those available. This formulation presents a simplified version of the WTA problem. In more practical versions, we may consider adding additional constraints, such as (i) lower and/or upper bounds on the number of weapons of type i assigned to a target j; (ii) lower and/or upper bounds on the total number of weapons assigned to target j; or (iii) a lower bound on the survival value of the target j. The algorithms proposed in this paper can be easily modified to handle these additional constraints.

The weapon-target assignment problem has been widely studied by researchers over three decades. There are two types of the WTA problem: *static* and *dynamic*. In the static WTA problem, all the inputs to the problem are fixed; that is, all targets are known, all weapons are known, and all weapons engage targets in a single stage. The dynamic WTA problem is a generalization of the static WTA problem. This problem is a multistage problem where some weapons are engaged at the targets at a stage, the outcome of this engagement is assessed, and strategy for the next stage is decided. The problem consists of a number of time stages, and each time stage allows the defense to fire a subset of its weapons and perfectly observe the outcomes of all of the engagements of its weapons. With the feedback of this information, the defense can make better use of its weapons because it will no longer engage targets that have already been destroyed. This is also called the *shoot-and-look strategy*. In real-life applications, solutions of the dynamic WTA problem are used in decision making. In this paper, we study the static WTA problem; however, our algorithms can be used as important subroutines to solve the dynamic WTA problem. Henceforth in this paper, the static WTA problem is simply referred to as the WTA problem.

Research on the WTA problem dates back to the 1950s and 1960s, where the modeling issues for the WTA problem were investigated (Manne 1958, Braford 1961, Day 1966). Lloyd and Witsenhausen (1986) established the NP-completeness of the WTA problem. Exact algorithms have been proposed to solve the WTA problem for the following special cases: (i) when all the weapons are identical (denBroeder et al. 1958, Katter 1986), or (ii) when targets can receive at most one weapon (Chang et al. 1987, Orlin 1987). Castanon et al. (1987) have formulated the WTA problem as a nonlinear network flow problem; they relax the integrality constraints and the fractional assignments are fixed during a postprocessing phase. Wacholder (1989) developed methods for solving the WTA problem using neural networks. The methods were tested on small-scale problems only. Green et al. (1997) applied a goal programming-based approach to the WTA problem to meet other realistic preferences in a combat scenario. Metler and Preston (1990) have studied a suite of algorithms for solving the WTA problem efficiently, which is critical for real-time applications of the WTA problem. Maltin (1970), Eckler and Burr (1972), and Murphey (1999) provide comprehensive reviews of the literature on the WTA problem. Research to date on the WTA problem either solves the WTA problem for special cases or develops heuristics for the WTA problem. Moreover, because no exact algorithm is available to solve WTA problems, it is not known how accurate the solutions obtained by these heuristic algorithms are.

In this paper, we propose exact and heuristic algorithms to solve the WTA problem using branch-and-bound techniques. Our branch-and-bound algorithms are the first implicit enumeration algorithms that can solve moderately sized instances of the WTA problem optimally. We also propose heuristic algorithms that generate near-optimal solutions within a few seconds of computational time. Our paper makes the following contributions:

- We formulate the WTA problem as an integer linear programming problem and as a generalized integer network flow problem on an appropriately defined network. The piecewise-linear approximation of the objective function of this formulation gives a lower bound on the optimal solution of the WTA problem. We describe this formulation in §2.1.
- We propose a minimum cost flow formulation that yields a different lower bound on the optimal solution of the WTA problem. This lower bound is, in general, not as tight as the bound obtained by the generalized integer network flow formulation described above, but it can be obtained in much less computational time. We describe this formulation in §2.2.
- We propose a third lower-bounding scheme in §2.3 that is based on simple combinatorial arguments and uses a greedy approach to obtain a lower bound.
- We develop branch-and-bound algorithms to solve the WTA problem employing each of the three bounds described above. These algorithms are described in §3.
- We propose a very large-scale neighborhood (VLSN) search algorithm to solve the WTA problem. The VLSN search algorithm is based on formulating the WTA problem as a partition problem. The VLSN search starts with a feasible solution of the WTA problem and performs a sequence of "cyclic and path exchanges" to improve the solution. In §4, we describe a heuristic method that obtains an excellent feasible solution of the WTA problem by solving a sequence of minimum cost flow problems, and then uses a VLSN search algorithm to iteratively improve this solution.
- We perform extensive computational investigations of our algorithms and report these results in §5. Our algorithms solve moderately large instances (up to 80 weapons and 80 targets) of the WTA problem optimally and obtain almost optimal solutions of fairly large instances (up to 200 weapons and 200 targets) within a few seconds.

Although the focus of this paper is to make contributions to a well-known and unsolved operations research problem arising in defense, the paper also makes some contributions that go beyond defense applications. We list some of these contributions next.

- Network cost flow problems are widely encountered in practice, but applications of network flow problems to solve nonlinear integer programming problems are rather infrequent. The paper illustrates an intuitive application of networks to this field. It shows that the generalized minimum cost flow and the (pure) minimum cost flow problems can be used to obtain lower bounds for nonlinear integer programming problems. It also describes a minimum cost flow-based algorithm to obtain a near-optimal solution. These novel applications are likely to motivate similar applications for other nonlinear programming problems.

• Most previous applications of VLSN search techniques reported in the literature are for solving integer linear programming problems. The paper shows that VLSN search algorithms can be applied to solve highly nonlinear integer programming problems as well. This work should lead to VLSN search algorithms for other integer nonlinear programming problems as well.

2. Lower-Bounding Schemes

In this section, we describe three lower-bounding schemes for the WTA problem, using linear programming, the integer programming, the minimum cost flow problem, and a combinatorial method. These three approaches produce lower bounds with different values and have different running times.

2.1. A Lower-Bounding Scheme Using an Integer Generalized Network Flow Formulation

In this section, we formulate the WTA problem as an integer programming problem with a convex objective function value. This formulation is based on a result reported by Manne (1958), who attributed it to Dantzig (personal communications).

In formulation (1), let $s_j = \prod_{i=1}^{m} q_{ij}^{x_{ij}}$. Taking logarithms on both sides, we obtain $\log(s_j) = \sum_{i=1}^{m} x_{ij} \log(q_{ij})$ or $-\log(s_j) = \sum_{i=1}^{m} x_{ij}(-\log(q_{ij}))$. Let $y_j = -\log(s_j)$ and $d_{ij} = -\log(q_{ij})$. Observe that because $0 \leqslant q_{ij} \leqslant 1$, we have $d_{ij} \geqslant 0$. Then, $y_j = \sum_{i=1}^{m} d_{ij} x_{ij}$. Also observe that $\prod_{i=1}^{m} q_{ij}^{x_{ij}} = 2^{-y_j}$. By introducing the terms d_{ij} and y_j in formulation (1), we get the following formulation:

$$\text{Minimize} \sum_{j=1}^{n} V_j 2^{-y_j} \tag{2a}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leqslant W_i \quad \text{for all } i = 1, 2, \ldots, m, \tag{2b}$$

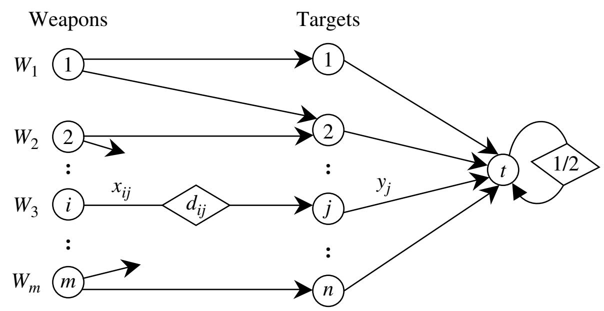
$$\sum_{i=1}^{m} d_{ij} x_{ij} = y_j \quad \text{for all } j = 1, 2, \ldots, n, \tag{2c}$$

$$x_{ij} \geqslant 0 \text{ and integer}$$
$$\text{for all } i = 1, \ldots, m \text{ and for all } j = 1, \ldots, n, \tag{2d}$$

$$y_j \geqslant 0 \quad \text{for all } j = 1, 2, \ldots, n. \tag{2e}$$

Observe that (2) is an integer programming problem with separable convex objective functions. This integer program can also be viewed as an integer generalized network flow problem with convex flow costs. Generalized network flow problems are flow problems where flow entering an arc may be different than the flow leaving the arc (see, for example, Ahuja et al. 1993). In a generalized network flow problem, each arc (i, j) has an associated multiplier γ_{ij} and the flow x_{ij} becomes $\gamma_{ij} x_{ij}$ as it travels from node i to node j. Formulation (2) is a generalized network flow problem on the network shown in the Figure 1. Next, we give some explanations of this formulation.

Figure 1. Formulating the WTA problem as an integer generalized network flow problem.



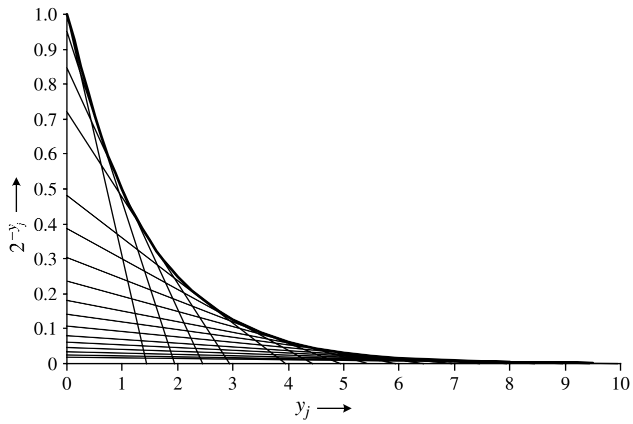
The network contains m weapon nodes, one node corresponding to each weapon type. The supply at node i is equal to the number of weapons available, W_i, for the weapon type i. The network contains n target nodes, one node corresponding to each target, and there is one sink node t. The supplies/demands of target nodes are zero. We now describe the arcs in the network. The network contains an arc connecting each weapon node to each target node. The flows on these arcs are given by x_{ij}, representing the number of weapons of type i assigned to the target j. The multipliers for these arcs are d_{ij}s. Because there is no cost coefficient for x_{ij}s in the objective function, the cost of flow on these arcs is zero. The network contains an arc from each of the target nodes to the sink node t. The flow on arc (j, t) is given by y_j and the cost of flow on this arc is $V_j 2^{-y_j}$. Finally, there is a loop arc (t, t) incident on node t with multiplier 1/2. The flow on this arc is equal to the double of the sum of the flow on incoming arcs from the target nodes. This constraint is required to satisfy the mass balance constraints at node t.

In formulation (2), the cost of the flow in the network equals the objective function (2a), the mass balance constraints of weapon nodes are equivalent to constraint (2b), and mass balance constraints of target nodes are equivalent to constraint (2c). It follows that an optimal solution of the above generalized network flow problem will be an optimal solution of the WTA problem.

The generalized network flow formulation (2) is substantially more difficult than the standard generalized network flow problem (see Ahuja et al. 1993) because the flow values x_{ij}s are required to be integer numbers (instead of real numbers) and the costs of flows on some arcs is a convex function (instead of a linear function). We will approximate each convex function by a piecewise-linear convex function so that the optimal solution of the modified formulation gives a lower bound on the optimal solution of the generalized formulation (2).

We consider the cost function $V_j 2^{-y_j}$ at values y_j that are integer multiples of a parameter $p > 0$, and draw tangents of $V_j 2^{-y_j}$ at these values. Let $F_j(p, y_j)$ denote the upper envelope of these tangents. It is easy to see that the function

Figure 2. Approximating a convex function by a lower envelope of linear segments.



$F_j(p, y_j)$ approximates $V_j 2^{-y_j}$ from below, and for every value of y_j provides a lower bound on $V_j 2^{-y_j}$. Figure 2 shows an illustration of this approximation.

Thus, in formulation (2), if we replace the objective function (2a) by the following objective function,

$$\sum_{j=1}^{n} F_j(p, y_j), \tag{2a$'$}$$

we obtain a lower bound on the optimal objective function of (2a). Using this modified formulation, we can derive lower bounds in two ways:

LP-Based Lower-Bounding Scheme. Observe that the preceding formulation is still an integer programming problem because arc flows x_{ij}s are required to be integer valued. By relaxing the integrality of the x_{ij}s, we obtain a mathematical programming problem with linear constraints and piecewise-linear convex objective functions. It is well known (see Murty 1976) that linear programs with piecewise-linear convex functions can be transformed to linear programs by introducing a variable for every linear segment. We can solve this linear programming problem to obtain a lower bound for the WTA problem. Our computational results indicate that the lower bounds generated by this scheme are not very tight, and therefore we do not include this scheme in our discussions.

MIP-Based Lower-Bounding Scheme. In this scheme, we do not relax the integrality of the x_{ij}s, which keeps the formulation as an integer programming formulation. However, we transform the piecewise-linear convex functions to linear cost functions by introducing a variable for every linear segment. We then use cutting plane methods to obtain a lower bound on the optimal objective function value. We have used the built-in routines in the software CPLEX 8.0 to generate Gomory cuts and mixed-integer rounding cuts to generate fairly tight lower bounds for the WTA problem. It can be observed that for a sufficiently high number of segments and when given enough time to converge, this method will converge to an (near) optimal solution of the original problem.

We summarize the discussion in this section as follows:

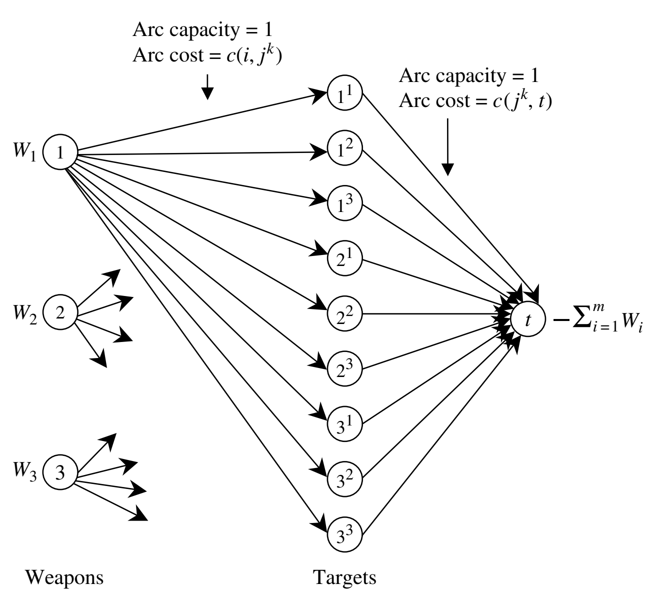
Theorem 1. *The generalized network flow formulation gives a lower bound on the optimal objective function value for the WTA problem.*

2.2. A Minimum Cost Flow-Based Lower-Bounding Scheme

The objective function of the WTA problem can also be interpreted as maximizing the expected damage to the targets. In this section, we develop an upper bound on the expected damage to the targets. Subtracting this upper bound on the expected damage from the total value of the targets ($\sum_{j=1}^{n} V_j$) will give us a lower bound on the minimum survival value. We formulate the problem of maximizing the damage to targets as a maximum cost flow problem. The underlying network G for the maximum cost flow formulation is shown in Figure 3.

This network has three layers of nodes. The first layer contains a supply node i for every weapon type i with supply equal to W_i. We denote these supply nodes by the set N_1. The second layer of nodes, denoted by the set N_2, contains nodes corresponding to targets, but each target j is represented by several nodes $j^1, j^2, \ldots, j^k$, where k is the maximum number of weapons that can be assigned to target j. A node j^p represents the pth weapon striking the target j, i.e., $p-1$ number of weapons has already been assigned to the target j. For example, the node labeled 3^1 represents the event of the first weapon being assigned to target 3, the node labeled 3^2 represents the event of the second weapon being assigned to target 3, and so on. All nodes in the second layer have zero supplies/demands. Finally, the third layer contains a singleton node t with demand equal to $\sum_{i=1}^{m} W_i$.

Figure 3. A network flow formulation of the WTA problem.



We now describe the arcs in this network. The network contains an arc (i, j^k) for each node $i \in N_1$ and each node $j^k \in N_2$; this arc represents the assignment of a weapon of type i to target j as the kth weapon. This arc has a unit capacity. The network also contains an arc (j^k, t) with unit capacity for each node $j^k \in N_2$.

We call a flow x in this network a *contiguous flow* if it satisfies the property that if $x(i, j^k) = 1$, then $x(i, j^l) = 1$ for all $l = 1, 2, \ldots, k-1$. In other words, the contiguous flow implies that weapon i is assigned to target j as the kth weapon provided that $(k-1)$ weapons have already been assigned to it. The following observation directly follows from the manner we have constructed the network G:

Observation 1. There is a one-to-one correspondence between feasible solutions of the WTA problem and contiguous flows in G.

Although there is a one-to-one correspondence between feasible solutions, it is not a cost-preserving correspondence if we require costs to be linear. We instead provide linear costs that will overestimate the true nonlinear costs. We define our approximate costs next.

The arc (i, j^k) represents the assignment of a weapon of type i to target j as the kth weapon. If $k = 1$, then the cost of this arc is the expected damage caused to the target,

$$c(i, j^1) = V_j(1 - q_{ij}), \tag{3}$$

which is the difference between the survival value of the target before strike (V_j) and the survival value of the target after strike $(V_j q_{ij})$. Next, consider the cost $c(i, j^2)$ of the arc (i, j^2), which denotes the change in the survival value of target j when weapon i is assigned to it as the second weapon. To determine this, we need to know the survival value of target j before weapon i is assigned to it. However, this cost depends upon which weapon was assigned to it as the first weapon. The first weapon striking target j can be of any weapon type $1, 2, \ldots, m$, and we do not know its type a priori. Therefore, we cannot determine the cost of the arc (i, j^2). However, we can determine an upper bound on the cost of the arc (i, j^2). Next, we will derive the expression for the cost of the arc (i, j^k), which as a special case includes (i, j^2).

Suppose that the first $(k-1)$ weapons assigned to target j are of weapon types $i_1, i_2, \ldots, i_{k-1}$, and suppose that the type of the kth assigned weapon is of type i. Then, the survival value of target j after the first $(k-1)$ weapons is $V_j q_{i_1 j} q_{i_2 j} \ldots q_{i_{k-1} j}$ and the survival value of the target j after k weapons is $V_j q_{i_1 j} q_{i_2 j} \ldots q_{i_{k-1} j} q_{ij}$. Hence, the cost of the arc (i, j^k) is the difference between the two terms, which is

$$c(i, j^k) = V_j q_{i_1 j} q_{i_2 j} \ldots q_{i_{k-1} j}(1 - q_{ij}). \tag{4}$$

Let $q_j^{\max} = \max\{q_{ij}\colon 1, 2, \ldots, m\}$. Then, we can obtain an upper bound on $c(i, j^k)$ by replacing each q_{ij} by $q_j^{\max}$. Hence, if we set

$$c(i, j^k) = V_j (q_j^{\max})^{k-1}(1 - q_{ij}), \tag{5}$$

we get an upper bound on the total destruction on assigning weapons to targets. It directly follows from (5) that

$$c(i, j^1) > c(i, j^2) > \cdots > c(i, j^{k-1}) > c(i, j^k), \tag{6}$$

which implies that the optimal maximum cost flow in the network G will be a contiguous flow. It should be noted here that because this is a maximization problem, we solve it by first multiplying all arc costs by -1 and then using any minimum cost flow algorithm. Let z^* represent the upper bound on the destruction caused to targets after all the assignments obtained by solving this maximum cost flow problem. Then, the lower bound on the objective function of formulation (1) is $\sum_{j=1}^{n} V_j - z^*$.

We can summarize the preceding discussion as follows:

Theorem 2. *If z^* is the optimal objective function value of the maximum cost flow problem in the network G, then $\sum_{j=1}^{n} V_j - z^*$ is a lower bound for the WTA problem.*

2.3. Maximum Marginal Return-Based Lower-Bounding Scheme

In this section, we describe a different relaxation that provides a valid lower bound for the WTA problem. This approach is based on the underestimation of the survival of a target when hit by a weapon because we assume that every target is hit by the best weapons.

Let $q_j^{\min}$ be the survival probability for target j when hit by the weapon with the smallest survival probability, i.e., $q_j^{\min} = \min\{q_{ij}\colon i = 1, 2, \ldots, m\}$. Replacing the term q_{ij} in formulation (1) by $q_j^{\min}$, we can formulate the WTA problem as follows:

$$\text{Minimize} \sum_{j=1}^{n} V_j \prod_{i=1}^{m} (q_j^{\min})^{x_{ij}} \tag{7a}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leqslant W_i \quad \text{for all } i = 1, 2, \ldots, m, \tag{7b}$$

$$x_{ij} \geqslant 0 \text{ and integer}$$
$$\text{for all } i = 1, 2, \ldots, m \text{ and for all } j = 1, 2, \ldots, n. \tag{7c}$$

Let $x_j = \sum_{i=1}^{m} x_{ij}$, and if we let $g_j(x_j) = V_j (q_j^{\min})^{x_j}$, then we can rewrite (7) as

$$\text{Minimize} \sum_{j=1}^{n} g_j(x_j) \tag{8a}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leqslant W_i \quad \text{for all } i = 1, 2, \ldots, m, \tag{8b}$$

$$\sum_{i=1}^{n} x_{ij} = x_j \quad \text{for all } i = 1, 2, \ldots, m, \tag{8c}$$

$$x_{ij} \geqslant 0 \text{ and integer}$$
$$\text{for all } i = 1, 2, \ldots, m \text{ and for all } j = 1, 2, \ldots, n. \tag{8d}$$

It is also possible to eliminate the variables x_{ij} entirely. If we let $W = \sum_{i=1}^{m} W_i$, formulation (8) can be rewritten as an equivalent integer program (9):

$$\text{Minimize} \sum_{j=1}^{n} g_j(x_j) \tag{9a}$$

subject to

$$\sum_{j=1}^{n} x_j \leqslant W, \tag{9b}$$

$$x_j \geqslant 0 \text{ and integer} \quad \text{for all } j = 1, 2, \ldots, n. \tag{9c}$$

It is straightforward to transform a solution for (9) into one for (8) because all weapon types are identical in formulation (8).

Observe that formulations (1) and (7) have the same constraints; hence, they have the same set of solutions. However, in formulation (7), we have replaced each q_{ij} by $q_j^{\min} = \min\{q_{ij}\colon i = 1, 2, \ldots, m\}$. Noting that the optimal solution value for problems (7), (8), and (9) are all identical, we get the following result:

Theorem 3. *An optimal solution value for* (9) *is a lower bound for the WTA problem.*

Integer program (9) is a special case of the knapsack problem in which the separable costs are monotone decreasing and concave. As such, it can be solved using the greedy algorithm as presented in Figure 6. In the following, assigned (j) is the number of weapons assigned to target j, and value(i, j) is the incremental cost of assigning the next weapon to target j.

This lower-bounding scheme is in fact a variant of a popular algorithm to solve the WTA problem, which is known as the *maximum marginal return algorithm*. In this algorithm, we always assign a weapon with maximum improvement in the objective function value. This algorithm is a heuristic algorithm to solve the WTA problem, but is known to give an optimal solution if all weapons are identical.

We now analyze the running time of our lower-bounding algorithm. In this algorithm, once the value (value(j)) of each target is initialized, we assign W weapons iteratively, and in each iteration we: (i) identify the target with minimum value(j) and (ii) decrease the value of the identified target. If we maintain a Fibonacci heap of targets with their values as their keys, then the running time of our algorithm in each iteration is O(1) for identifying the minimum value target plus O(1) in updating the heap when the value of the identified target is decreased (Fredman and Tarjan 1984). As the algorithm performs W iterations, the running time of the Fibonacci heap implementation of our maximum marginal return algorithm is $O(W)$. In our implementation, we used binary heap in place of Fibonacci heap, which runs in $O(W \log n)$ time and is still fast enough for the problem sizes that we considered.

Figure 4. Combinatorial lower-bounding algorithm.

```
algorithm combinatorial-lower-bounding;
begin
  for j := 1 to n do
  begin
    assigned(j) := 0;
    value(j) := g_j(assigned(j) + 1) − g_j(assigned(j));
  end
  for i = 1 to m do
  begin
    find j corresponding to the minimum value(j);
    assigned(j) := assigned(j) + 1;
    value(j) := g_j (assigned(j) + 1) − g_j(assigned(j));
  end;
end.
```

3. A Branch-and-Bound Algorithm

We developed and implemented a branch-and-bound algorithm and experimented with it for three lower-bounding strategies. A branch-and-bound algorithm is characterized by the branching, lower bounding, and search strategies. We now describe these strategies for our approaches.

Branching strategy. For each node of the branch-and-bound tree, we find the weapon-target combination that gives the best improvement and we set the corresponding variable as the one to be branched on next. Ties are broken arbitrarily. To keep the memory requirement low, the only information we store at any node is which variable we branch on at that node, and the lower and upper bounds at the node. The upper bound at a node is the sum of remaining survival value of all the targets. To recover the partial solution associated with a node of the branch-and-bound tree, we trace back to the root of the tree.

Lower-bounding strategy. We used the three lower-bounding strategies: (i) generalized network flow (§2.1), (ii) minimum cost flow (§2.2), and (iii) maximum marginal return (§2.3). We provide a comparative analysis of these bounding schemes in §5.

Search strategy: We implemented both the breadth-first and depth-first search strategies. We found that for smaller-size problems (i.e., up to 10 weapons and 10 targets), the breadth-first search strategy gave overall better results, but for larger problems, the depth-first search strategy had a superior performance. We report the results for the depth-first search in §5.

4. A Very Large-Scale Neighborhood Search Algorithm

In the previous section, we described a branch-and-bound algorithm for the WTA problem. This algorithm is the first exact algorithm that can solve moderate-size instances of the WTA problem in reasonable time. Nevertheless, there is still a need for heuristic algorithms that can solve large-scale instances of the WTA problems. In this section, we describe a neighborhood search algorithm for the WTA problem that has exhibited excellent computational results. A neighborhood search algorithm starts with a feasible solution of the optimization problem and successively improves it by replacing the solution by an improved neighbor until it converges to a locally optimal solution. The quality of the locally optimal solution depends both upon the quality of the starting feasible solution and the structure of the neighborhood, that is, how we define the neighborhood of a given solution. We next describe the method we used to construct the starting feasible solution followed by our neighborhood structure. This algorithm is an application of a VLSN search to the WTA problem. A VLSN search algorithm is a neighborhood search algorithm where the size of the neighborhood is very large and we use some implicit enumeration algorithm to identify an improved neighbor.

4.1. A Minimum Cost Flow Formulation-Based Construction Heuristic

We developed a construction heuristic that solves a sequence of minimum cost flow problems to obtain an excellent solution of the WTA problem. This heuristic uses the minimum cost flow formulation shown in Figure 3, which we used to determine a lower bound on the optimal solution of the WTA problem. Recall that in this formulation, we define the arc costs $(i, j^1), (i, j^2), \ldots, (i, j^k)$, which, respectively, denote the cost of assigning the first, second, and kth weapon of type i to target j. Also recall that only the cost of the arc (i, j^1) was computed correctly, and for the other arcs, we used a lower bound on the cost. We call the arcs whose costs are computed correctly *exact-cost arcs*, and call the rest of the arcs *approximate-cost arcs*.

This heuristic works as follows. We first solve the minimum cost flow problem with respect to the arc costs, as defined earlier. In the optimal solution of this problem, exact-cost arcs as well as approximate-cost arcs may carry positive flow. We next fix the part of the WTA corresponding to the flow on the exact-cost arcs and remove those arcs from the network. In other words, we construct a partial solution for the WTA by assigning weapons only for exact-cost arcs. After fixing this partial assignment, we again compute the cost of each arc. Some previous approximate-cost arcs will now become exact-cost arcs. For example, if we set the flow on arc (i, j^1) equal to one, we know that that weapon i is the first weapon striking target j, and hence we need to update the costs of the arcs (l, j^k) for all $l = 1, 2, \ldots, m$ and for all $k \geqslant 2$. Also observe that the arcs (l, j^2) for all $l = 1, 2, \ldots, m$ now become exact-cost arcs. We next solve another minimum cost flow problem and again fix the flow on the exact-cost arcs. We recompute arc costs, make some additional arcs exact cost, and solve another minimum flow problem. We repeat this process until all weapons are assigned to the targets.

We tried another modification in the minimum cost flow formulation that gave better computational results. The formulation we described determines the costs of approximate-cost arcs assuming that the worst weapons (with the largest survival probabilities) are assigned to targets. However, we observed that in any near-optimal solution, the best weapons are assigned to the targets. Keeping this observation in mind, we determine the costs of valid arcs assuming that the best weapons (with the smallest survival probabilities) are assigned to targets. Hence, the cost of the arc (i, j^k), which is $c(i, j^k) = V_j q_{i_1 j} q_{i_2 j} \ldots q_{i_{k-1} j} \cdot (1 - q_{ij})$ is approximated by $c(i, j^k) = V_j [q_{\min}(j)]^{k-1} \cdot (1 - q_{ij})$. Our experimental investigation shows that this formulation generates better solutions compared to the previous formulation. We present computational results of this formulation in §5.

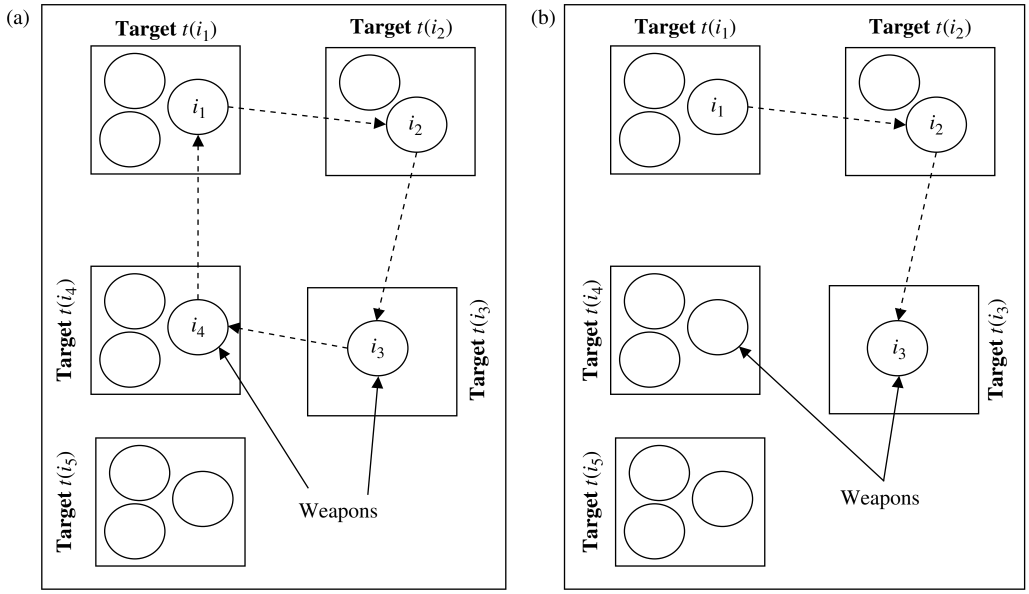
4.2. The VLSN Neighborhood Structure

The VLSN search heuristic uses *multiexchange neighborhood structures* that can be regarded as generalizations of the two-exchange neighborhood structures. This neighborhood structure is based on the multiexchange neighborhood structure developed by Thompson and Psaraftis (1993) and Thompson and Orlin (1989). Glover (1996) refers to this type of exchange as *ejection chains* (see also Rego and Roucairol 1996, and Kelley and Xu 1996). Thompson and Psaraftis (1993) and Gendreau et al. (1998) have used this neighborhood structure to solve vehicle-routing problems and obtained impressive results. We refer the reader to the paper by Ahuja et al. (2002) for an overview of VLSN search algorithms. In this paper, we use multiexchange neighborhood structures to solve the WTA problem.

The WTA problem can be conceived of as a partition problem defined as follows. Let $\mathbf{S} = \{a_1, a_2, a_3, \ldots, a_n\}$ be a set of n elements. The partition problem is to partition the set $\mathbf{S}$ into the subsets $S_1, S_2, S_3, \ldots, S_K$ such that the cost of the partition is minimal, where the cost of the partition is the sum of the cost of each part. The WTA problem is a special case of the partition problem where the set of all weapons is partitioned into n subsets $S_1, S_2, \ldots, S_n$, and subset j is assigned to target j, $1 \leqslant j \leqslant n$. Thompson and Orlin (1989) and Thompson and Psaraftis (1993) proposed a VLSN search approach for partitioning problems that proceeds by performing *cyclic exchanges*. Ahuja et al. (2001, 2003) proposed further refinements of this approach and applied it to the capacitated minimum spanning tree problem. We will present a brief overview of this approach when applied to the WTA problem.

Let $\mathbf{S} = (S_1, S_2, \ldots, S_n)$ denote a feasible solution of the WTA problem where the subset S_j, $1 \leqslant j \leqslant n$, denotes the set of weapons assigned to target j. Our neighborhood search algorithm defines neighbors of the solution $\mathbf{S}$ as those solutions that can be obtained from $\mathbf{S}$ by performing multiexchanges. A *cyclic multiexchange* is defined by a sequence of weapons $i_1 - i_2 - i_3 - \cdots - i_r - i_1$, where the weapons $i_1, i_2, i_3, \ldots, i_r$ belong to different subsets S_js. Let $t(i_1), t(i_2), t(i_3), \ldots, t(i_r)$, respectively, denote the targets to which weapons $i_1, i_2, i_3, \ldots, i_r$, are assigned. The cyclic multiexchange $i_1 - i_2 - i_3 - \cdots - i_r - i_1$ represents that weapon i_1 is reassigned from target $t(i_1)$ to target $t(i_2)$,

Figure 5. (a) An example of a cyclic multiexchange; (b) an example of a path multiexchange.



weapon i_2 is reassigned from target $t(i_2)$ to target $t(i_3)$, and so on, and finally weapon t_r is reassigned from target $t(i_r)$ to target $t(i_1)$. In Figure 5(a), we show a cyclic multiexchange involving four weapons (i_1, i_2, i_3, and i_4). We can similarly define a *path multiexchange* by a sequence of weapons $i_1 - i_2 - i_3 - \cdots - i_r$, which differs from the cyclic multiexchange in the sense that the last weapon i_r is not reassigned and remains assigned to target $t(i_r)$. For example, in Figure 5(b), weapon i_1 leaves target $t(i_1)$ and gets assigned to target $t(i_2)$, weapon i_2 leaves target $t(i_2)$ and gets assigned to target $t(i_3)$, and no weapon leaves target $t(i_3)$ as a result of path multiexchange.

The number of neighbors in the multiexchange neighborhood is too large to be enumerated explicitly. However, using the concept of an *improvement graph*, a profitable multi exchange can be identified using network algorithms. The improvement graph $G(\mathbf{S})$ for a given feasible solution $\mathbf{S}$ of the WTA problem contains a node r corresponding to each weapon r and contains an arc (r, l) between every pair of nodes r and l with $t(r) \neq t(l)$. The arc (r, l) signifies the fact that weapon r is reassigned to the target (say j) to which weapon l is currently assigned and weapon l is unassigned from its current target j. The cost of this arc (say c_{rl}), is set equal to the change in the survival value of the target j due to reassignments. Let V_j' denote the survival value of the target j in the current solution. Then, the cost of the arc (r, l) is $c_{rl} = V_j'((q_{rj}/q_{lj}) - 1)$. The terms V_j'/q_{lj} and $V_j' q_{rj}$ in a previous expression, respectively, denote that weapon l leaves the target j and weapon r gets assigned to it. We say that a directed cycle $W = i_1 - i_2 - i_3 - \cdots - i_k - i_1$ in $G(\mathbf{S})$ is *subset disjoint* if each of the weapons $i_1, i_2, i_3, \ldots, i_k$ is assigned to a different target. Thompson and Orlin (1989) showed the following result:

Lemma 1. *There is a one-to-one correspondence between multiexchanges with respect to* $\mathbf{S}$ *and directed subset-disjoint cycles in* $G(\mathbf{S})$*, and both have the same cost.*

This lemma allows us to solve the WTA problem using the neighborhood search algorithm presented in Figure 6.

We obtain the starting feasible solution $\mathbf{S}$ by using the minimum cost flow-based heuristic described in §4.1. The improvement graph $G(\mathbf{S})$ contains W nodes and $O(W^2)$ arcs, and the cost of all arcs can be computed in $O(W^2)$ time. We use a dynamic programming-based algorithm (Ahuja et al. 2003) to obtain subset-disjoint cycles. This algorithm first looks for profitable two-exchanges involving two targets only; if no profitable two-exchange is found, it looks for profitable three-exchanges involving three targets, and so on. In each iteration of a neighborhood search, the algorithm either finds a profitable multiexchange or terminates when it is unable to find a multiexchange involving k targets (we set $k = 5$). In the former case, we improve the current solution, and in the latter case, we declare the current solution to be locally optimal, and stop. The

Figure 6. The VLSN search algorithm for the WTA problem.

```
algorithm WTA-VLSN search;
begin
   obtain a feasible solution S of the WTA problem;
   construct the improvement graph G(S);
   while G(S) contains a negative cost subset-disjoint cycle do
   begin
      obtain a negative cost subset-disjoint cycle W in G(S);
      perform the multiexchange corresponding to W;
      update S and G(S);
   end;
end;
```

running time of the dynamic programming algorithm is $O(W^2 2^k)$ per iteration, and is typically much faster because most cyclic exchanges found by the algorithm are two-exchanges, i.e., $k = 2$.

5. Computational Results

We implemented each of the algorithms described in the previous section and tested them extensively. We tested our algorithms on randomly generated instances because data for the real-life instances were classified. We generated the data in the following manner. We generated the target survival values V_js as uniformly distributed random numbers in the range 25–100. We generated the kill probabilities for weapons-target combinations as uniformly distributed random numbers in the range 0.60–0.90. We performed all our tests on a 2.8 GHz Pentium 4 Processor computer with 1 GB RAM PC. In this section, we present the results of these investigations.

5.1. Comparison of the Lower-Bounding Schemes

In our first investigation, we compared the tightness of the lower bounds generated by the lower-bounding algorithms developed by us. We have proposed three lower-bounding schemes described in §2: (i) the generalized network flow-based lower-bounding scheme; (ii) the minimum cost-based lower-bounding scheme; and (iii) the maximum marginal return-based lower-bounding scheme. The computational results of these three lower-bounding schemes are given in Table 1. For each of these schemes, the first column gives the % gap from the optimal objective function value and the second column gives the time taken to obtain the bound. For the generalized network flow scheme, we used a piecewise approximation of the objective function and then solved the problem using CPLEX 8.0 MIP solver. For the piecewise approximation of the objective function, we did some sensitivity analysis for the number of segments used. We found that with a lesser number of segments, the lower bounds achieved are not very good, and the algorithm enumerates too many candidate problems resulting in high running times. However, if we use more segments, then lower bounds are better, but the running time at each node increases, thereby again increasing the total running time of the algorithm. Thus, one needs to balance these two conflicting considerations to find the overall best number of segments. We experimented with 5, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, and 200 segments. By empirical testing, we found that 50 segments gives the best overall performance. Thus, for all our experimentation, we used 50 segments.

We can derive the following observations from Table 1:

(i) The generalized network flow-based lower-bounding scheme gives the tightest lower bounds, but also takes the maximum computational time.

(ii) The minimum cost flow-based lower-bounding scheme gives fairly tight lower bounds when the number of weapons (W) is less than or equal to the number of targets (n). It can be recalled that in the minimum cost flow formulation, the cost of arcs assigning the first weapon to a target are exact. We also observed that when $W < n$, then the number of weapons assigned to most targets is at most one in the optimal solution. Therefore, this scheme gives very tight lower bounds when $W < n$.

(iii) The maximum marginal return-based lower-bounding scheme is very efficient and running times are quite low. However, the high optimality gap for maximum marginal return demonstrates that a simple greedy method is not very efficacious for the WTA problem.

5.2. Comparison of Three Lower-Bounding Schemes in the Branch-and-Bound Algorithm

We developed a branch-and-bound algorithm using the three lower-bounding schemes and one hybrid algorithm. The hybrid algorithm computes lower bounds using both the minimum cost flow-based and the maximum marginal return-based lower-bounding schemes and uses the better of these two bounds. Table 2 gives the results of these algorithms. The cells containing "—" mean that our branch-and-bound algorithm could not find an optimal solution

Table 1. Comparison of the three lower-bounding schemes.

		Generalized network flow scheme		Min. cost flow scheme		Maximum marginal return scheme	
Number of weapons	Number of targets	% Gap	Time (in secs.)	% Gap	Time (in secs.)	% Gap	Time (in secs.)
5	5	0.21	0.016	1.66	<0.001	10.61	<0.001
10	10	0.12	0.031	0.00	<0.001	11.01	<0.001
10	20	0.04	0.062	0.00	<0.001	1.45	<0.001
20	10	0.53	0.156	21.32	<0.001	19.00	<0.001
20	20	0.25	0.109	1.32	<0.001	6.40	<0.001
20	40	0.04	0.296	0.00	<0.001	1.57	<0.001
40	10	2.12	0.609	42.41	<0.001	46.89	<0.001
40	20	0.45	0.359	25.52	0.015	13.53	<0.001
40	40	0.11	0.703	1.63	0.015	3.05	<0.001
40	80	0.03	1.812	0.00	0.046	0.88	<0.001

Table 2. Comparison of variations in lower-bounding schemes in the branch-and-bound algorithm.

		Generalized network flow-based B&B algorithm		Min. cost flow-based B&B algorithm		Maximum marginal return-based B&B algorithm		Hybrid algorithm	
Number of weapons	Number of targets	Nodes visited	Time (in secs.)	Nodes visited	Time (in secs.)	Nodes visited	Time (in secs.)	Nodes visited	Time (in secs.)
5	5	15	0.14	11	<0.001	23	<0.001	11	<0.001
10	10	29	0.56	1	<0.001	181	<0.001	1	<0.001
10	20	23	0.83	1	<0.001	83	<0.001	1	0.015
20	10	101	7.27	—	—	28,611	1.34	20,251	2.52
20	20	109	6.56	2,383	4.39	15,936	0.94	1,705	2.50
20	40	105	16.58	1	<0.001	111,603	10.14	1	0.015
40	10	1,285	327.27	—	—	—	—	—	—
40	20	205	35.19	—	—	$\sim 10^8$	13,651.9	$\sim 10^7$	25,868.9
40	40	211	50.96	$\sim 10^6$	10,583.62	$\sim 10^6$	943.03	383,275	1,891.83
40	80	385	235.41	1	0.031	—	—	1	0.031
80	40	117,227	43,079.55	—	—	—	—	—	—
80	80	44,905	58,477.31	—	—	—	—	—	—
80	160	1,055	3,670.49	1	0.062	—	—	1	0.062

even after running for more than 48 hours. We observe that the branch-and-bound algorithm using the MIP-based lower bounding gives the most consistent results and is able to solve the largest-size problems (containing 80 weapons and 80 targets). We also find that the hybrid algorithm also gives excellent results for those instances where the number of weapons is less than or equal to the number of targets.

5.3. Performance of the VLSN Search Algorithm

We now present computational results of the minimum cost flow-based construction heuristic and the VLSN search algorithm. In Table 3, we give the objective function values of the solutions obtained by the construction heuristic and the improved values when the VLSN search algorithm is applied to these solutions. For small instances, the optimal values were determined using the branch-and-bound method. For larger instances, the branch-and-bound algorithm could not be executed until optimality. However, observe that the minimum of the lower bounds of the active (node not pruned yet) nodes gives an overall lower bound on the objective function. We used this value to compute the optimality gap.

We believe that the success of the VLSN search on the WTA problem can be attributed to three factors. First, the WTA problem is unconstrained in nature and VLSN approaches are very efficacious in solving unconstrained partition problems. Second, we search the solution space for up to five-exchanges. This implies that we consider neighborhoods of size = (no-of-weapons)^5. For example, in the case of 80 weapons, the VLSN algorithm enumerates neighborhood of size $80^5 \approx 3$ billion, which is very large. Third, our construction heuristic generates an excellent starting solution for neighborhood search, which leads to a higher-quality local optimal solution. The net effect of these three factors is that we get solutions that are very close to the optimal solution.

We observe that the construction heuristic obtained optimal solutions for over half of the instances, and for the remaining instances, the VLSN search algorithm converted them into optimal or almost optimal solutions. The computational times taken by these algorithms are also very low and even fairly large instances are solved within three seconds. It appears that the VLSN search algorithm is ideally suited to solve the WTA problem.

Table 3. Results of the construction heuristic and the VLSN search algorithm.

		Construction heuristic		VLSN algorithm	
Number of weapons	Number of targets	Optimality gap (%)	Time (in secs.)	Optimality gap (%)	Time (in secs.)
10	5	0	<0.001	0	<0.001
10	10	0	<0.001	0	<0.001
10	20	0	<0.001	0	<0.001
20	10	0	<0.001	0	<0.001
20	20	0	<0.001	0	<0.001
20	40	0	0.015	0	0.015
20	80	0	0.015	0	0.031
40	10	1.79	0.015	0	0.031
40	20	0.33	0.015	0	0.015
40	40	0	0.015	0	0.015
40	80	0	0.031	0	0.078
40	120	0	0.062	0	0.109
80	20	2.33	0.109	0	0.156
80	40	0.10	0.062	0	0.109
80	80	0.0003	0.093	0.0003	0.156
80	160	0	0.172	0	0.219
80	320	0	0.390	0	0.625
100	50	0.79	0.120	0.0015	0.437
100	100	0.001	0.187	0.0009	0.250
100	200	0	0.375	0	0.609
200	100	0.01	0.656	0.0059	0.828
200	200	0.001	0.921	0.0008	1.109
200	400	0	1.953	0	2.516

6. Conclusions

In this paper, we consider the WTA problem, which is considered to be one of the classical operations research problems that has been extensively studied in the literature, but still has remained unsolved. Indeed, this problem is considered to be the holy grail of defense-related operations research. Although the WTA problem is a nonlinear integer programming problem, we use its special structure to develop LP, MIP, network flow, and combinatorial lower-bounding schemes. Using these lower-bounding schemes in branch-and-bound algorithms gives us effective exact algorithms to solve the WTA problem. Our VLSN search algorithm also gives highly impressive results and gives either optimal or almost optimal solutions for all instances to which it is applied. To summarize, we can now state that the WTA problem is a well-solved problem and its large-scale instances can also be solved in real time.

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