# Introduction to Data Processing and Representation
# (236201)
## Winter 2022-2023

# Homework 4

- **Published date: 11/01/2023**
- **Deadline date: 26/01/2023**

Guidelines:

- Submission is in pairs only.
- Submit your entire solution (including the theoretical part, the Python part and the Python code) electronically via the course website. The file should be a zip file containing the your PDF submission and code.
- The submission should be in English and in a clear printed form (recommended) or a clear hand-writing.
- Please follow the FAQ at course website, as clarifications and corrections may be published only there.
- Rigorous mathematical proofs and reasoning are required for theoretical questions. Vague answers and unjustified claims will not be accepted.

# I Theory

## 1. Inverting the Second Derivative Operator

In this exercise we consider a sensor that computes the second derivative of its incoming periodic signal of known period. For simplicity, we consider both the input and output signals to be finite $M$-dimensional vectors. The sensor only provides the output of its computation and the input signal is lost. However, we would like to retrieve the input signal. We model our problem as follows. Our $M$-dimensional signals $\boldsymbol{\varphi} \in \mathbb{R}^M$ are distorted following the model:

$$\boldsymbol{\varphi}_{data} = H\boldsymbol{\varphi}$$

where $\boldsymbol{\varphi}_{data}$ is the output of the sensor, a given, and $H$ is a linear operator of size $M \times M$ representing the second derivative operator. Manufacturers of the sensor have tested $H$ and found that it is well approximated by the centered finite difference model of order 4, namely that, by writing the indices modulo $M$ and starting indexing at 0:

$$\boldsymbol{\varphi}_{data,j} = -\frac{1}{12}\boldsymbol{\varphi}_{j-2[M]} + \frac{4}{3}\boldsymbol{\varphi}_{j-1[M]} - \frac{5}{2}\boldsymbol{\varphi}_{j[M]} + \frac{4}{3}\boldsymbol{\varphi}_{j+1[M]} - \frac{1}{12}\boldsymbol{\varphi}_{j+2[M]}$$

where we define $k[M]$ the rest of the euclidean division of $k$ by $M$, i.e. the result of the modulo operation of $k$ by $M$.

The goal is now to recover the original signal $\boldsymbol{\varphi}$ based solely on the available distorted signal $\boldsymbol{\varphi}_{data}$ and our prior knowledge on our sensor $H$.

a. Provide the explicit formulation of the degradation matrix $H$ and its components;

b. Give the pseudo-inverse filter for our task. Note that in the mathematical expression you provide you should explicitly give the singular or eigendecomposition of $H$.

c. Without any prior knowledge on the clean input signal $\boldsymbol{\varphi}$, could any $\boldsymbol{\varphi}$ be perfectly recovered using the previously provided inverse filter? If so prove it. If not provide the set of input signals that would be and prove your answer.

## 2. Let's Randomise

In this exercise, signals are column vectors $\boldsymbol{\varphi} = [\varphi_1, ..., \varphi_N]^T$ of dimension $N$, where $N$ is even, of the form

$$\boldsymbol{\varphi} = [M, ..., M, M+L, M, ..., M]^T,$$

where the $M+L$ term is in the $K$-th element. Namely, all vector components have the value $M$ except for the $K$-th element, which has the value $M+L$. We have that $M$ and $K$

are independent random variables, where $K$ is a uniform random variable over the integers $\{1, \ldots, N\}$, and the random variable $M$ satisfies that $\mathbb{E}(M) = 0$ and $\mathbb{E}(M^2) = c$, where $c$ is a constant having real value in $(0, 1)$. The random variable $L$ satisfies

$$L = \begin{cases} L_1, & \text{if } K \leq \frac{N}{2} \\ L_2, & \text{otherwise} \end{cases},$$

where $L_1$ and $L_2$ are two independent random variables satisfying $\mathbb{E}(L_1) = \mathbb{E}(L_2) = 0$, $\mathbb{E}(L_1^2) = Na$ and $\mathbb{E}(L_2^2) = Nb$, where $a$ and $b$ are constants having real value in $(0, 1)$. We further assume that $M$, $(L_1, L_2)$, and $K$ are independent random variables.

a. Show that the random vector $\boldsymbol{\varphi}$ has a zero mean.

b. Compute the autocorrelation matrix of $\boldsymbol{\varphi}$, which we denote as $\boldsymbol{R}_{\boldsymbol{\varphi}}$.

c. What are the most general conditions of $a$, $b$ and $c$ such that $DFT^*$ matrix is the PCA matrix corresponding to the autocorrelation matrix $\boldsymbol{R}_{\boldsymbol{\varphi}}$?

# 3. Let's Randomise Again!

In this exercise, signals are column vectors $\boldsymbol{\varphi} = [\varphi_1, ..., \varphi_N]^T$ of dimension $N$, where $N$ is even, of the form

$$\boldsymbol{\varphi} = [M, ..., M, M + L, M, ..., M, M + L, M, ..., M]^T,$$

where the $M + L$ terms appear in the $K$-th and in the $(K + \frac{N}{2})$-th elements. Namely, all the vector components take the value $M$ except for the $K$-th and $(K + \frac{N}{2})$-th elements whose value are $M + L$. The random variables $K$, $M$ and $L$ are independent random variables, where $K$ is a uniform random variable over the integers $\{1, \ldots, \frac{N}{2}\}$, $M$ satisfies $\mathbb{E}(M) = 0$ and $\mathbb{E}(M^2) = c$, and $L$ satisfies $\mathbb{E}(L) = 0$ and $\mathbb{E}(L^2) = \frac{N}{2}(1 - c)$, where $c$ is a constant having real value in $(0, 1)$.

a. Compute the autocorrelation matrix of $\boldsymbol{\varphi}$, which we denote as $\boldsymbol{R}_{\boldsymbol{\varphi}}$. Is it circulant?

b. Explain how to compute the eigenvalues of $\boldsymbol{R}_{\boldsymbol{\varphi}}$ in a simpler way than explicit diagonalization/eigendecomposition procedure.

# 4. Wiener Filter

Consider a random vector $\boldsymbol{\varphi}$, that is distorted by a linear degradation operator $\mathcal{H}$ and an additive independent noise vector $\boldsymbol{n}$ having zero mean, resulting in the degraded signal

$$\boldsymbol{\varphi}^* = \mathcal{H}\boldsymbol{\varphi} + \boldsymbol{n},$$

We denote the autocorrelation matrix of $\boldsymbol{\varphi}$ by $\boldsymbol{R}_{\boldsymbol{\varphi}}$ and the autocorrelation matrix of $\boldsymbol{n}$ by $\boldsymbol{R}_{\boldsymbol{n}}$.

a. What is the autocorrelation matrix of $\boldsymbol{\varphi}^*$?

b. Consider $\boldsymbol{R}_{\boldsymbol{n}} = \sigma_n^2 \boldsymbol{I}$, where $\sigma_n > 0$. Formulate the Wiener filter appropriate to the above problem.

c. Prove that if a $N \times N$ matrix $\boldsymbol{A}$ is diagonalized by the $DFT^*$ matrix of size $N \times N$, then $\boldsymbol{A}$ is a circulant matrix.

d. For any arbitrary $\boldsymbol{R}_{\boldsymbol{\varphi}}$, $\mathcal{H}$, and $\sigma_n$, is the Wiener filter a shift-invariant system?

e. In the case that the Wiener filter is not shift-invariant in general, describe the general conditions on $\boldsymbol{R}_{\boldsymbol{\varphi}}$, $\mathcal{H}$, and $\sigma_n$ in order that the Wiener filter will be shift-invariant.

# II Implementation

Instruction:

- Figures should be titled in a appropriate font size.
- Submit the code and a report describing the results and your understanding of the exercise.

Reconsider the third exercise in the Theory part. We now also consider the following setting: 1) the length of the signal is $N = 64$ samples, and, 2) the random variables $M$ and $L$ follow a Gaussian distribution which satisfies the second-order statistics of the class of a parameter $c = 0.6$.

a. Produce a large (and sufficient) amount of realizations of the class defined above (note that each signal realization relies on realizations of $K$, $M$ and $L$). Compute the empirical approximation of the mean signal and the autocorrelation matrix of the class. Plot the empirical mean of the class, and show the empirically estimated autocorrelation matrix (using the function *imshow* in Pyplot). How well do your empirical results approximate the analytical results obtained in the third question in the Theory part? What is the number of realizations needed for obtaining a good empirical approximation of the second-order statistics?

b. Consider a degraded signal from the above signal $\boldsymbol{\varphi}$ by adding a white Gaussian noise having zero mean

$$\boldsymbol{\varphi}^* = \boldsymbol{\varphi} + \boldsymbol{n},$$

where the autocorrelation matrix of $\boldsymbol{n}$ is defined by $\boldsymbol{R_n} = \sigma_n^2 \boldsymbol{I}$. Consider $\sigma_n = 1$, construct numerically the Wiener filter for denoising the above defined noisy signals. Show the filter matrix using the *imshow* function in Pyplot and explain the filter structure. Produce a large (and sufficient) amount of realizations of the class defined above, each should be degraded by a (different) realization of the noise vector defined above. Note that the clean signal should be kept for the purpose of computing the MSE of the denoised signal. However, the denoising task should be applied with respect to the noisy signal only. Use the Wiener filter constructed above to denoise the realizations of the noisy signals by plotting several examples of denoised signals with respect to their clean and noisy versions. For each denoised realization, compute the MSE with respect to the clean version of the same realization. Average the MSE for all the realizations and write this number. This is the empirical approximation of the expected MSE of denoising using the Wiener filter.

c. Consider now that our random signals $\boldsymbol{\varphi}$, as defined in exercise 3, are not only degraded with an additive random white noise but also undergo a linear degradation operator as defined in exercise 1. That is:

$$\boldsymbol{\varphi}^* = H\boldsymbol{\varphi} + \boldsymbol{n}$$

where the $j$-th component of $H\boldsymbol{\varphi}$ is:

$$-\frac{1}{12}\boldsymbol{\varphi}_{j-2[M]} + \frac{4}{3}\boldsymbol{\varphi}_{j-1[M]} - \frac{5}{2}\boldsymbol{\varphi}_{j[M]} + \frac{4}{3}\boldsymbol{\varphi}_{j+1[M]} - \frac{1}{12}\boldsymbol{\varphi}_{j+2[M]}$$

(indices are written with the modulo operation), and $\boldsymbol{n}$ is defined as in the previous question. Do the same as in the previous question with this new problem.

d. Repeat the above questions by considering the new noise variance $\sigma_n^2 = 5$. What are the differences in the obtained results?

e. Compute $H^\dagger$ the pseudo-inverse filter of $H$. Plot the results of the products $H^\dagger H$ and $HH^\dagger$ using *imshow* (Pyplot). Comment on the results. Can you find two hand-crafted signals $\phi_1$ and $\phi_2$ such that $\|\phi_1 - \phi_2\|_2 \geq 256$ satisfying $H^\dagger\phi_1 = H^\dagger\phi_2$? If yes, explain how you chose these two signals. In the report, provide $\|\phi_1 - \phi_2\|_2$, $\|H^\dagger\phi_1 - H^\dagger\phi_2\|_2$, plot $\phi_1$, $\phi_2$, $\phi_1 - \phi_2$, $H^\dagger\phi_1$, $H^\dagger\phi_2$, and $H^\dagger\phi_1 - H^\dagger\phi_2$. If no, explain why.