HW2-implementation

313511602
931190987

part 1

1. value-range:
   $\phi(x,y) = Acos(2\pi\omega_x x)\, sin(2\pi\omega_y y) \leq Acos(2\pi\omega_x x) \leq A.$
   $\phi(x,y) = Acos(2\pi\omega_x x)\, sin(2\pi\omega_y y) \geq -Acos(2\pi\omega_x x) \geq -A$
   Where we used the fact that $-1 \leq cos(x), sin(x) \leq 1$
   Now for the derivatives:
   $\frac{\partial\phi}{\partial x} = -2\pi\omega_x Asin(2\pi\omega_x x)\, sin(2\pi\omega_y y)$. So the energy E_x is:

$$E_x = \int_0^1\int_0^1 \left(\frac{\partial\phi}{\partial x}\right)^2 dxdxy = \int_0^1\int_0^1 4\pi^2\omega_x^2 A^2 sin^2(2\pi\omega_x x)\, sin^2(2\pi\omega_y y)\, dxdy =$$

$$4\pi^2\omega_x^2 A^2 \int_0^1 \left(sin^2(2\pi\omega_y y)\int_0^1 sin^2(2\pi\omega_x x)\, dx\right) dy =$$

$$4\pi^2\omega_x^2 A^2 \int_0^1 sin^2(2\pi\omega_y y)\, dy \int_0^1 sin^2(2\pi\omega_x x)\, dx$$

   Usign the trigonometric identety $sin^2(x) = \frac{1-cos(2x)}{2}$ We get:

$$\int_0^1 sin^2(2\pi\omega_x x)\, dx = \frac{1}{2} - \frac{sin(4\pi\omega_x)}{8\pi\omega_x}$$

   The same goes for y which means that:

$$E_x = 4\pi^2\omega_x^2 A^2 \left(\frac{1}{2} - \frac{sin(4\pi\omega_x)}{8\pi\omega_x}\right)\left(\frac{1}{2} - \frac{sin(4\pi\omega_y)}{8\pi\omega_y}\right) =$$

$$= 4\pi^2\omega_x^2 A^2 \left(\frac{1}{2} - \frac{sin(8\pi)}{8\pi\omega x}\right)\left(\frac{1}{2} - \frac{sin(49\pi)}{8\pi\omega_y}\right)$$

   Relaxing the term and plugging in the given values we get:

$$E_x = \omega_x^2\pi^2 A^2 = 4\pi^2 A^2$$

   $\frac{\partial\phi}{\partial y} = 2\pi\omega_y Acos(2\pi\omega_x x)\, cos(2\pi\omega_y y)$

$$E_y = \int_0^1\int_0^1 \left(\frac{\partial\phi}{\partial y}\right)^2 dxdy = \int_0^1\int_0^1 A^2 4\pi^2\omega_y^2 cos^2(2\pi\omega_x x)\, cos^2(2\pi\omega_y y)\, dxdy =$$

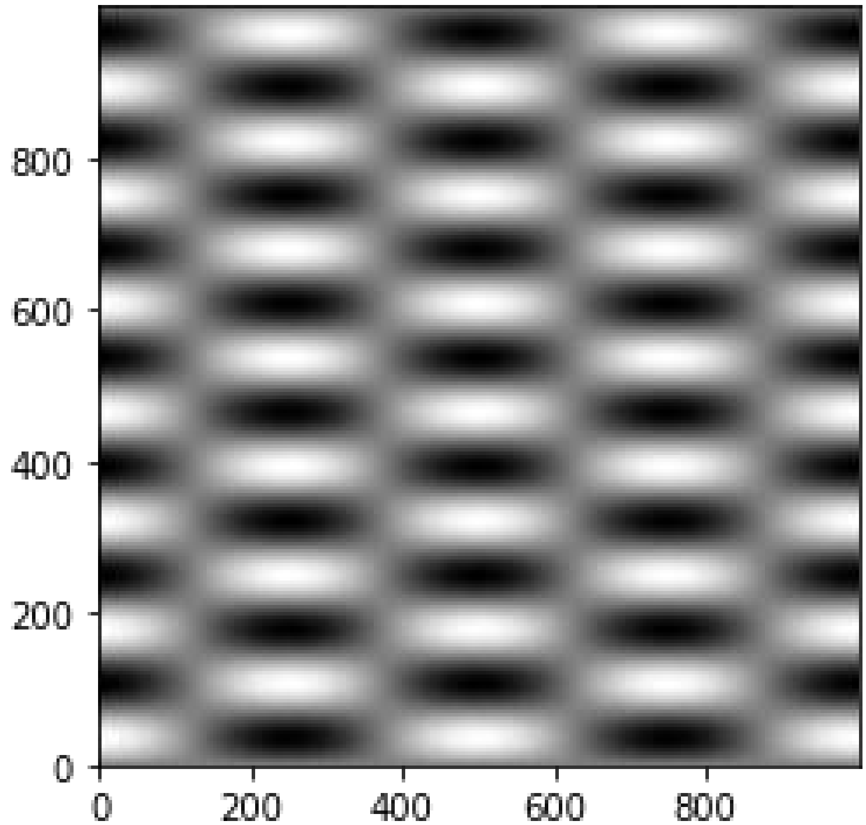
$$= A^2 4\pi^2 \omega_y^2 \int_0^1 cos^2 (2\pi\omega_x x)\, dx \int_0^1 cos^2 (2\pi\omega_y y)\, dy$$

$$\int_0^1 cos^2 (2\pi\omega_y y)\, dy = \int_0^1 \frac{1 + cos\,(4\pi\omega_y y)}{2} dy = \frac{1}{2}$$

Plugging this, we get:

$$E_y = \omega_y^2 \pi^2 A^2 = 49\pi^2 A^2$$

2. The image:



3. The numerical results are:

my signal max= 2499.9969095714937
my signal min= -2499.9969095714937
value range = 4999.993819142987
horizontal derative energey = 246246876.5472896
vertical derative energey = 3028614503.095629

That is opposed to the range that was $[-2500, 2500]$¸ The real horizontal energy $= \omega_x^2 \pi^2 A^2 = 246740110$ and the real vertical energy $= \omega_y^2 \pi^2 A^2$ =3022566348.
As we can see the numerical results are very close to the truth, and the higher the resolution the higher the approximation to the real values.

4. We want to minimize the $MSE = f\left(N_x, N_y, b\right)$ with the constraint $N_x N_y b - B = 0$.
Note: Actually the constraint is $N_x N_y b \leq B$, but we can safely say that if we do not utilize all our resources then we will not recieve the optimal result, thus we demand $N_x N_y b = b$.
To minimize this function with respect to the constraint we will use the notion of Lagrange Multipliers.
Denote $r =$ Range of the function

$$MSE = f\left(N_x, N_y, b\right) = \frac{E_x}{12N_x^2} + \frac{E_y}{12N_y^2} + \frac{r^2}{12 \cdot 4^b}$$

$$g\left(N_x, N_y, b\right) = N_x N_y b - B,\; g\left(N_x, N_y, b\right) = 0$$

$$f\left(N_x, N_y, b\right) + \lambda g\left(N_x, N_y, b\right) = u\left(N_x, N_y, b, \lambda\right), \nabla u\left(N_x, N_y, b, \lambda\right) = 0$$

With respect to N_x : $\frac{\partial u(N_x,N_y,b,\lambda)}{\partial N_x} = \lambda N_y b - \frac{E_x}{6N_x^3} = 0 \implies \frac{E_x}{6N_x^2} = \lambda B$
With respect to N_y : $\frac{\partial u(N_x,N_y,b,\lambda)}{\partial N_y} = \lambda N_x b - \frac{E_y}{6N_y^3} = 0 \implies \frac{E_y}{6N_y^2} = \lambda B$
With respect to b: $\frac{\partial u(N_x,N_y,b,\lambda)}{\partial b} = \lambda N_x N_y - \frac{ln(4) r^2 4^{-b}}{12} = 0 \implies \frac{ln(4) r^2 4^{-b}}{12 \cdot 4^b} = \lambda B$
From the above we deduct:

$$\frac{E_x}{6N_x^2} = \frac{E_y}{6N_y^2} \implies N_y = \frac{\sqrt{E_y} N_x}{\sqrt{E_x}},\; N_x = \frac{\sqrt{E_x} N_y}{\sqrt{E_y}}$$

Plugging inot g and equalizing to 0:

$$N_x^2 = \frac{\sqrt{E_x}}{b\sqrt{E_y}} B,\; N_y^2 = \frac{\sqrt{E_y}}{bE_x} B$$

Plugging into f and recieving:

$$MSE = \frac{b\sqrt{E_y E_x}}{6B} + \frac{r}{12 \cdot 4^b}$$

Now plugging in $g\left(N_x, N_y, b\right) = 0$ into third constraint and deriving we get:

$$\frac{\sqrt{E_x E_y}}{6B} - \frac{ln\left(4\right) r^2}{12 \cdot 4^b} = 0 \implies \frac{\sqrt{E_x E_y}}{6B} = \frac{ln\left(4\right) r^2}{12 \cdot 4^b} \implies \frac{ln\left(4\right) r^2 B}{2\sqrt{E_y E_x}} = 4^b \implies$$

$$\implies \frac{ln\left(\frac{ln(4) r^2 B}{2\sqrt{E_y E_x}}\right)}{ln\left(4\right)} = b$$

Now that we have b we can easily calculate N_x and N_y by plugging it into:

$$N_x = \sqrt{\frac{\sqrt{E_x}}{b\sqrt{E_y}} B}$$

and

$$N_y = \sqrt{\frac{\sqrt{E_y}}{b\sqrt{E_x}} B}$$

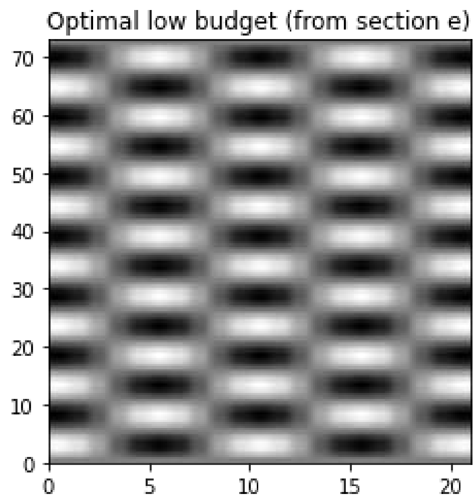
5. The results from numerically calculating the best allocation are:

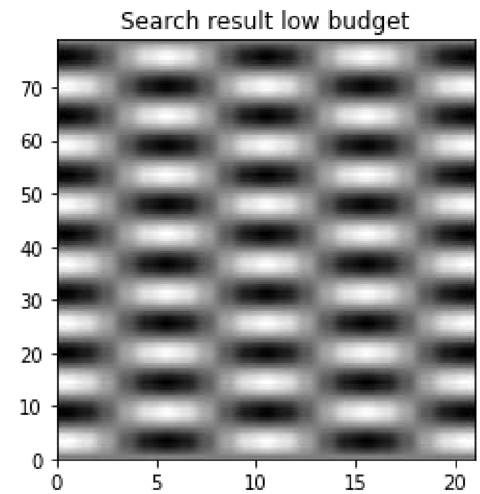
Optimal low budget Nx = 21 Ny = 73 b = 3
Optimal high budget Nx = 53 Ny = 188 b = 5

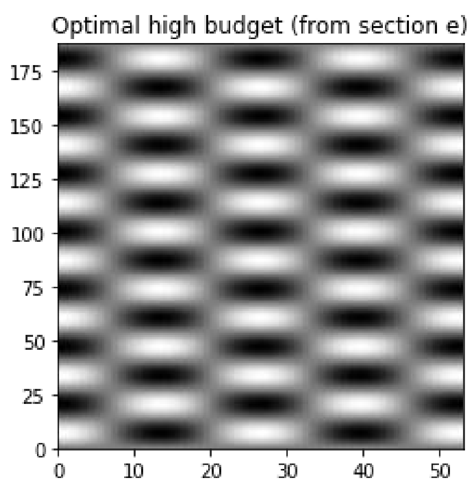
7. The result of the search are:

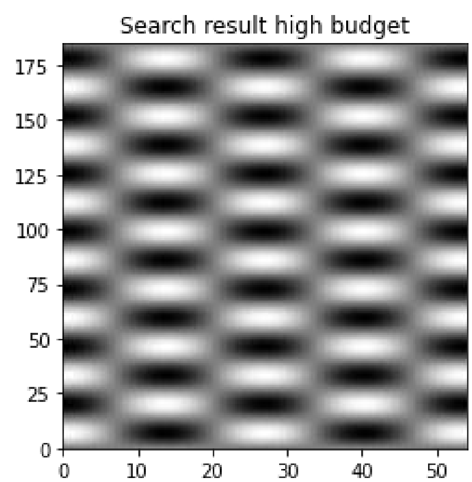
Search result low budget: Nx = 21, Ny = 79, b = 3
Search result high budget: Nx = 54, Ny = 185, b = 5

As we can see the solution found using the searching technique is similiar (floored v alues) t o the numerical values we found earlier.









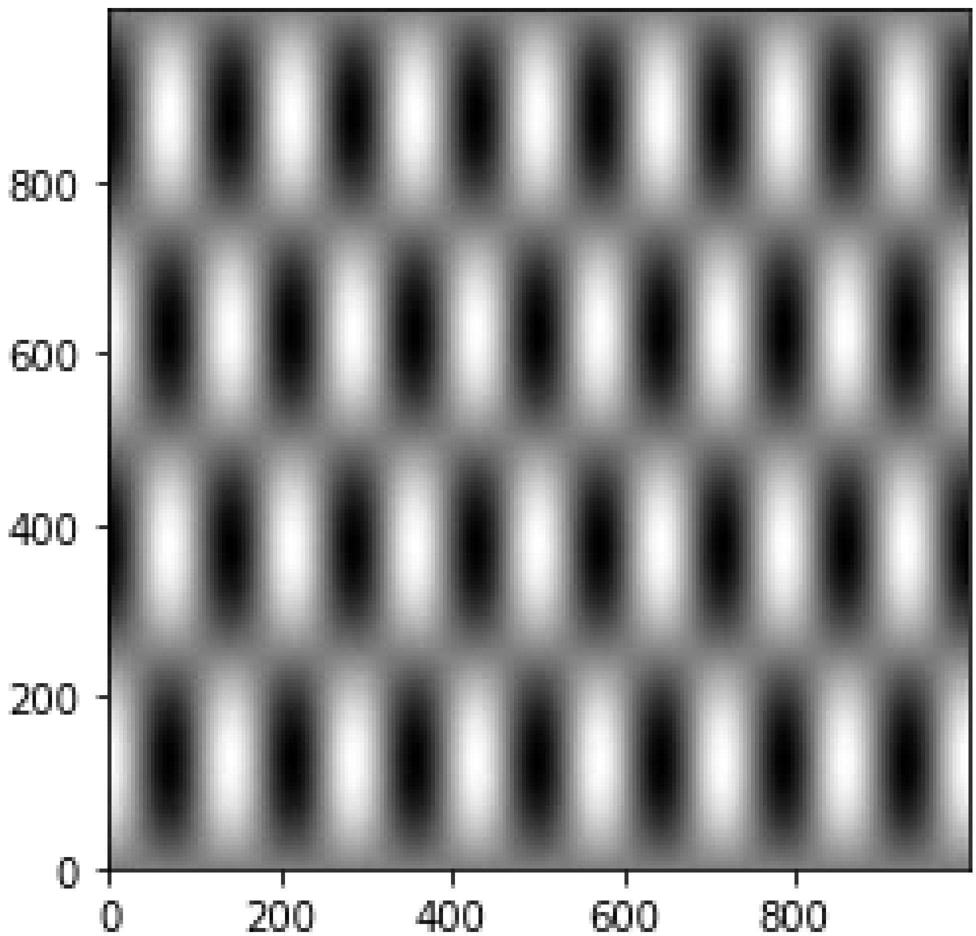
8. First of all we will notice that switching the ω_x and ω_y will switch between the partial derivatives values but will not change the range. This means that for the new function:

$$E_x^{new} = \left(\omega_x^{new}\right)^2 \pi^2 A^2 = 49\pi^2 A^2$$

$$E_y^{new} = \left(\omega_y^{new}\right)^2 \pi^2 A^2 = 4\pi^2 A^2$$

This is deduted simply by the calculations we made in point 1.

The new image we get is:



And the result we get:

my signal max= 2499.9969095714937
my signal min= -2499.9969095714937

value range = 4999.993819142987
horizontal derative energey = 3016524237.7042975
vertical derative energey = 247233836.98739842

Optimal low budget: Nx = 72, Ny = 21, b = 3
Optimal high budget: Nx = 187, Ny = 53, b = 4

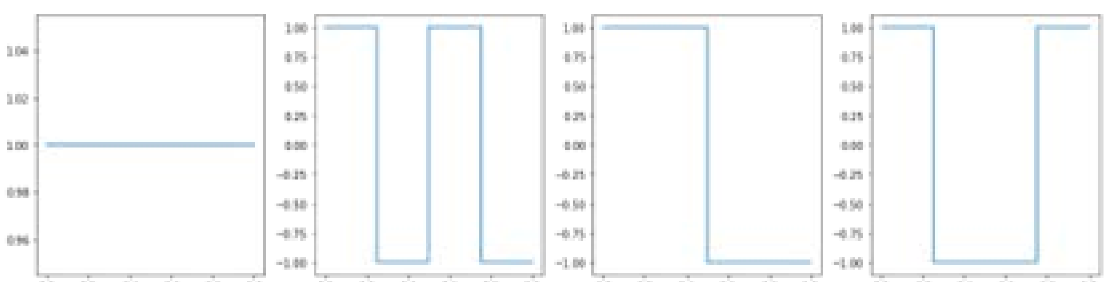
Search result low budget: Nx = 79, Ny = 21, b = 3
Search result high budget: Nx = 185, Ny = 54, b = 5

Where we see that the energies switched, N_x, N_y Switched as they are dependent on the enery (The more the energy the more samples we need), the bit count remained the same.

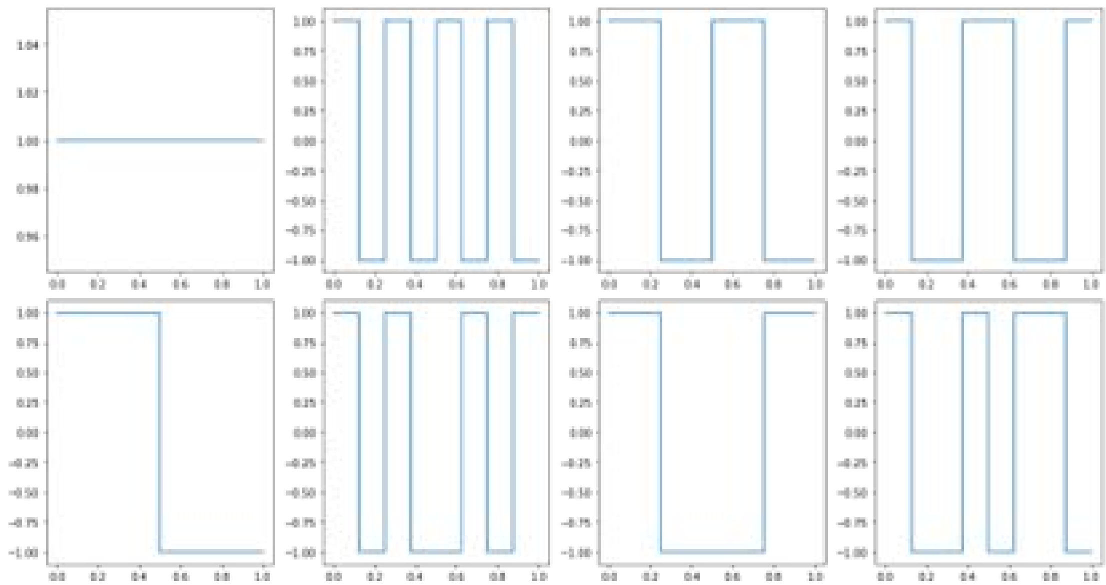
part 2

2. The basis is:

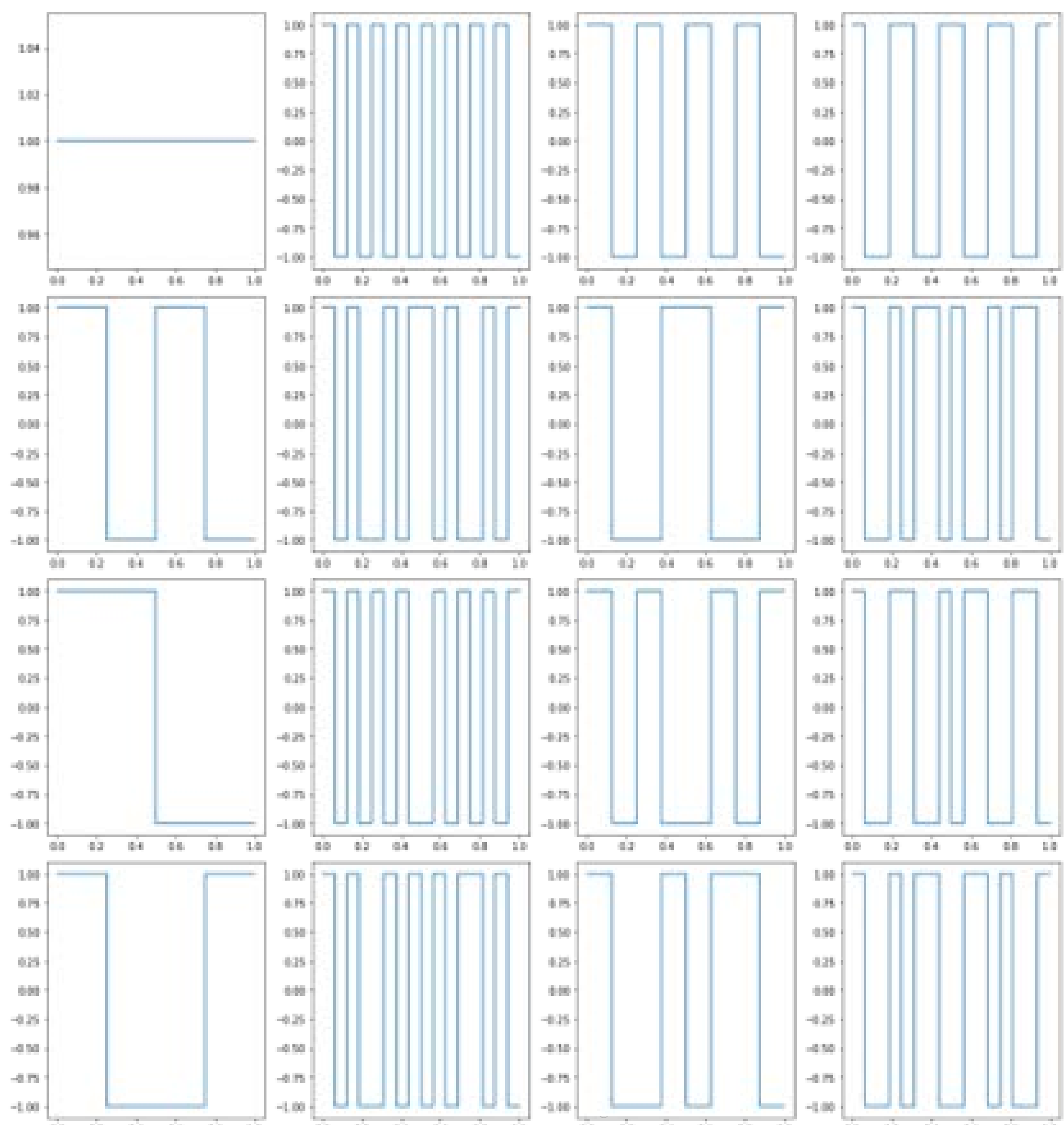
for n=2:



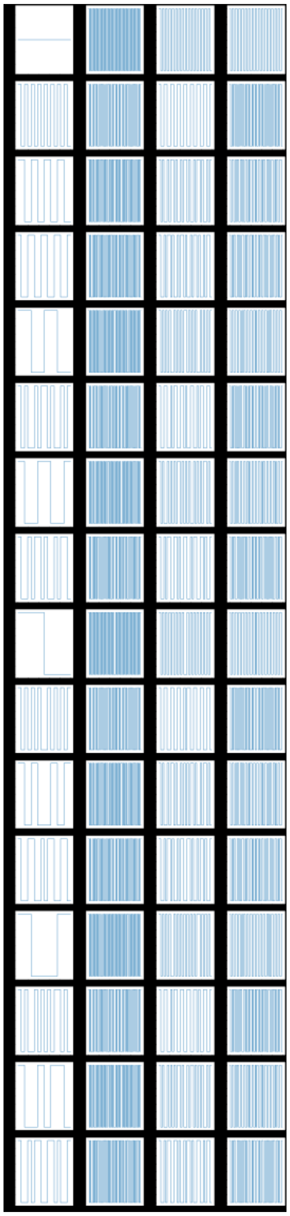
for n=3:



for n=4:



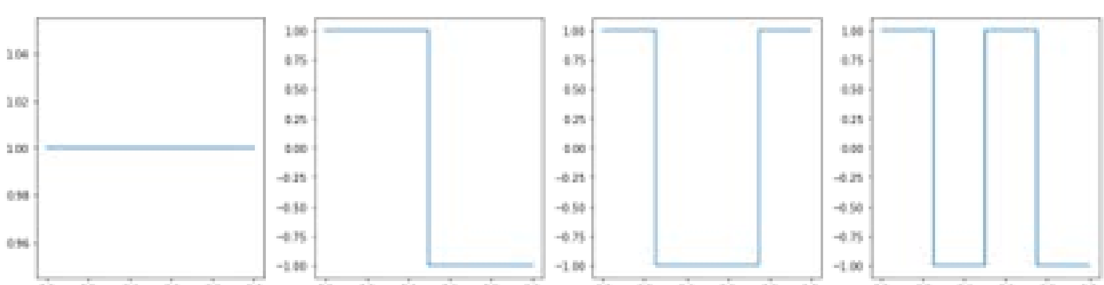
for n=6:



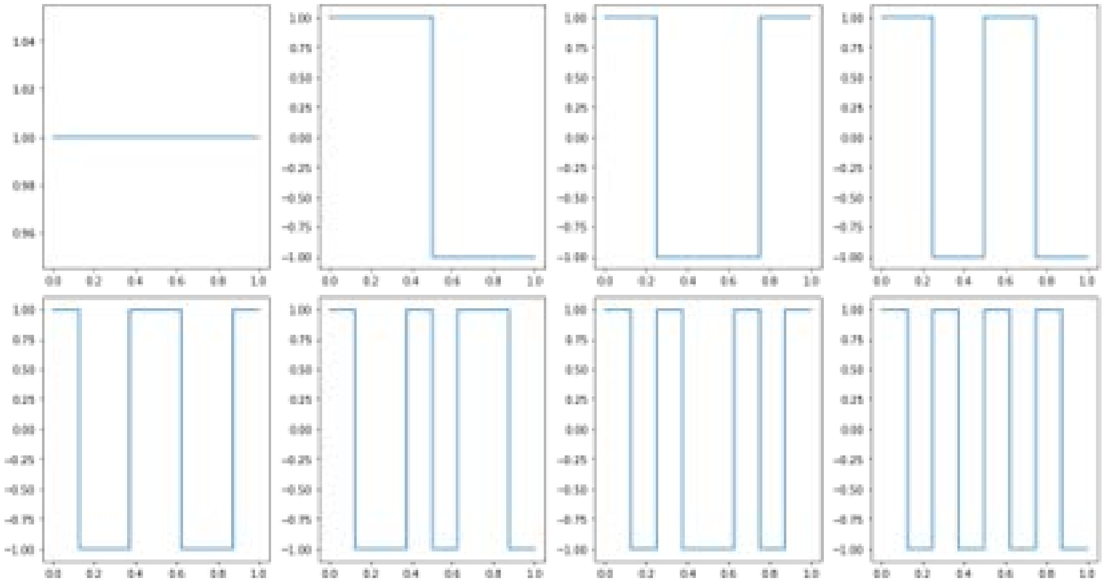
for n=5:

4. The Walsh Hadamard basis:

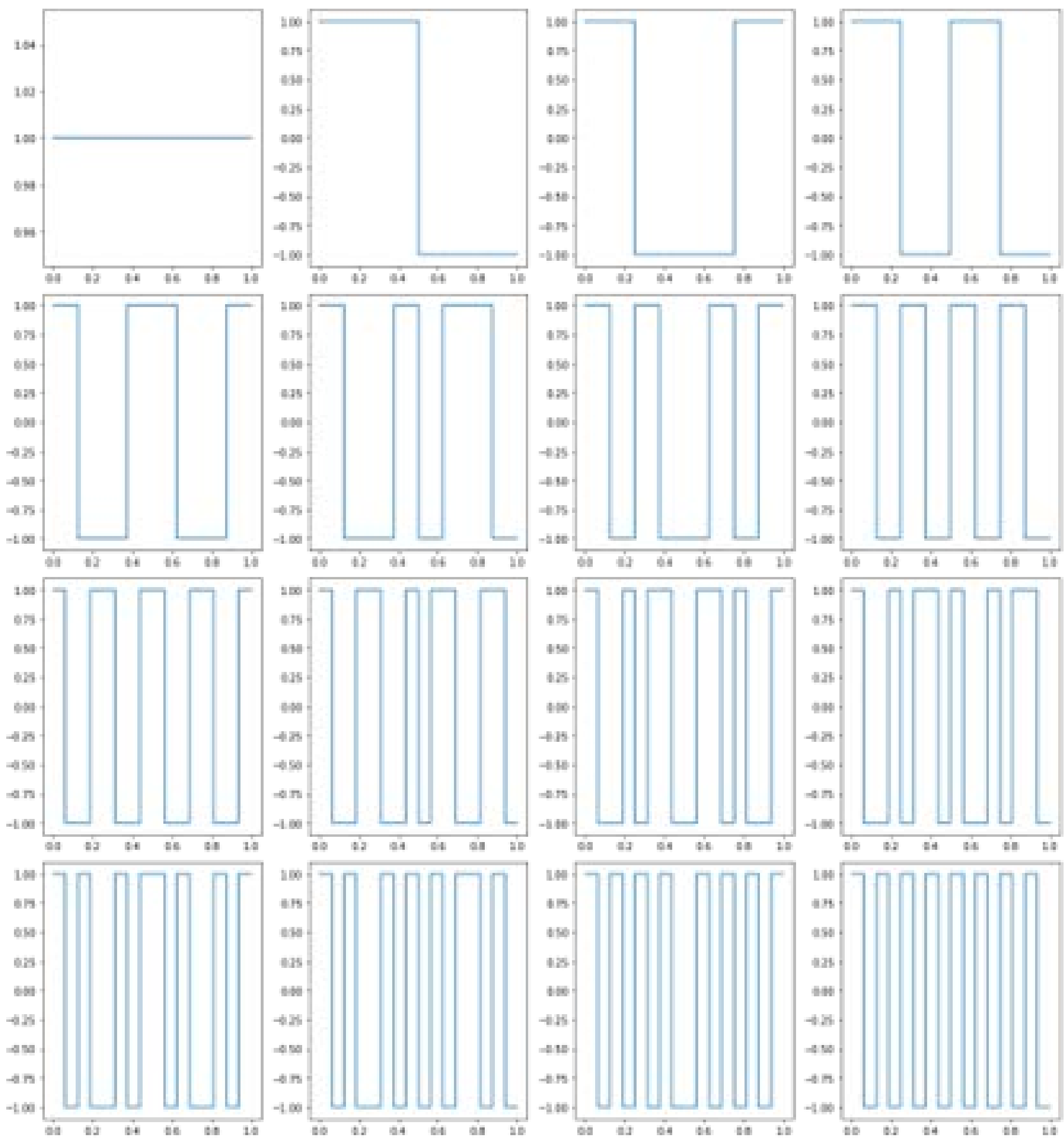
for n=2:



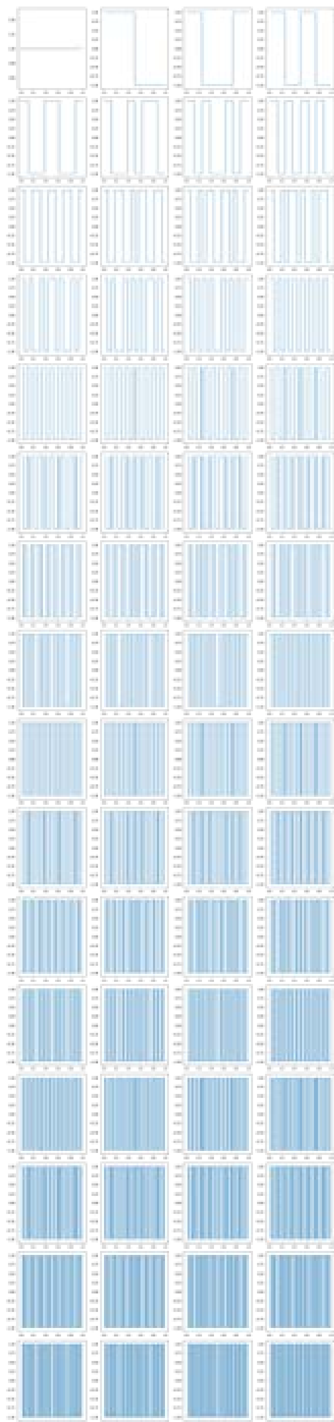
for n=3:



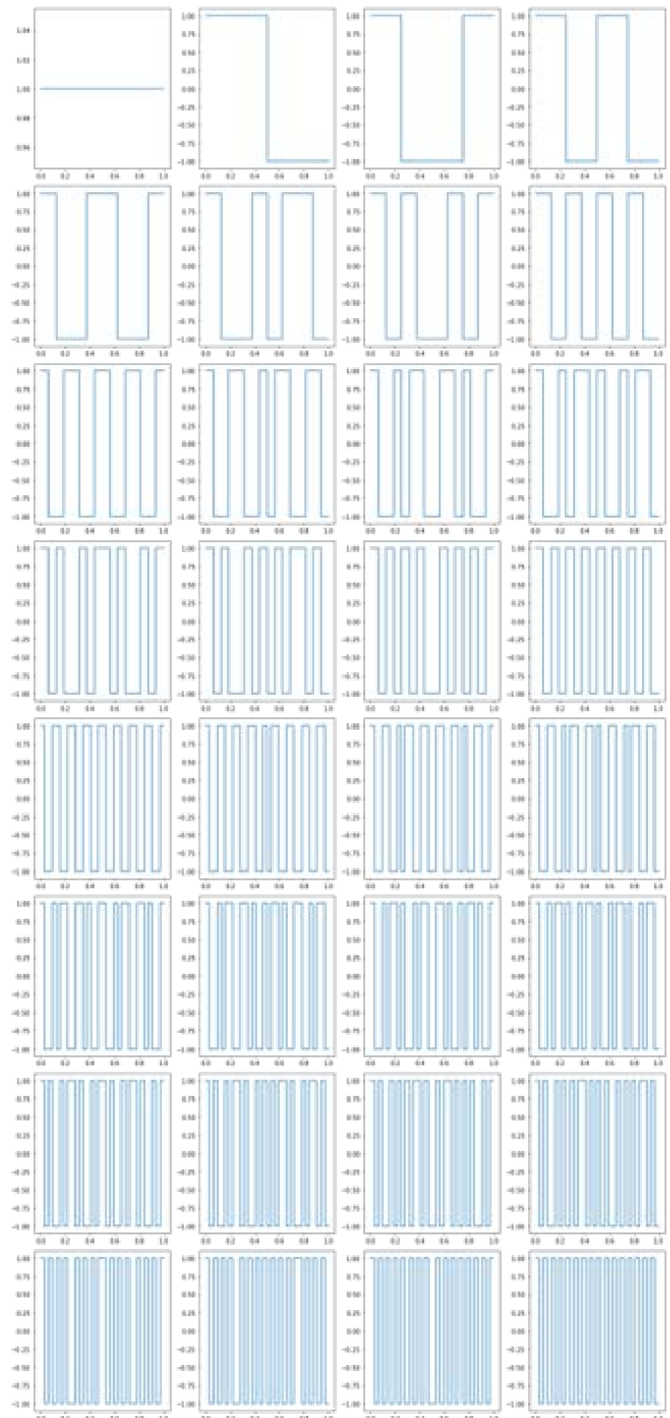
for n=4:



for n=6:

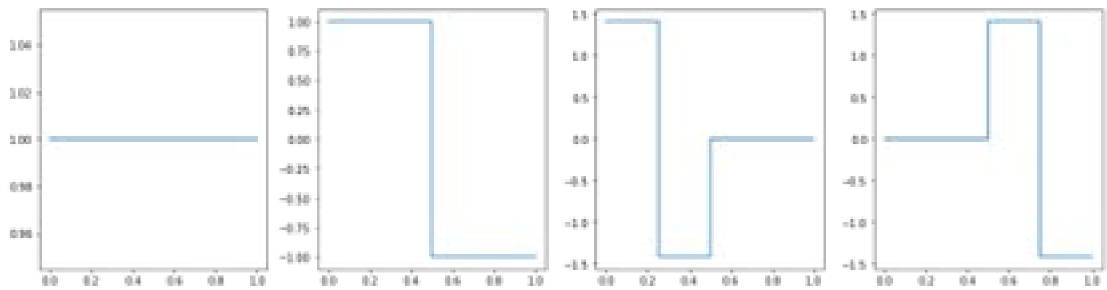


for n=5:

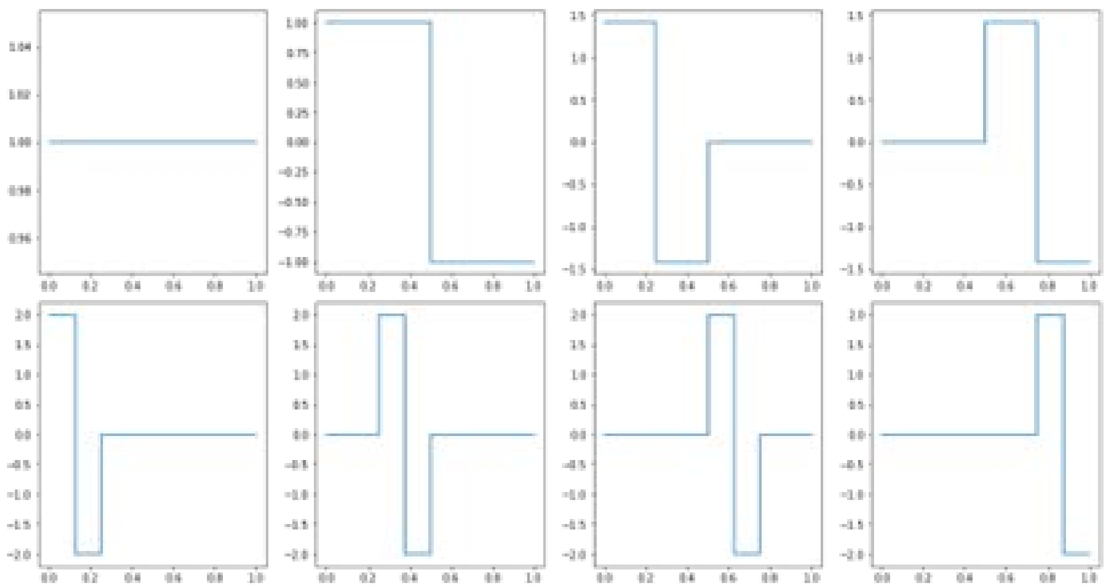


6. The Haar basis is:

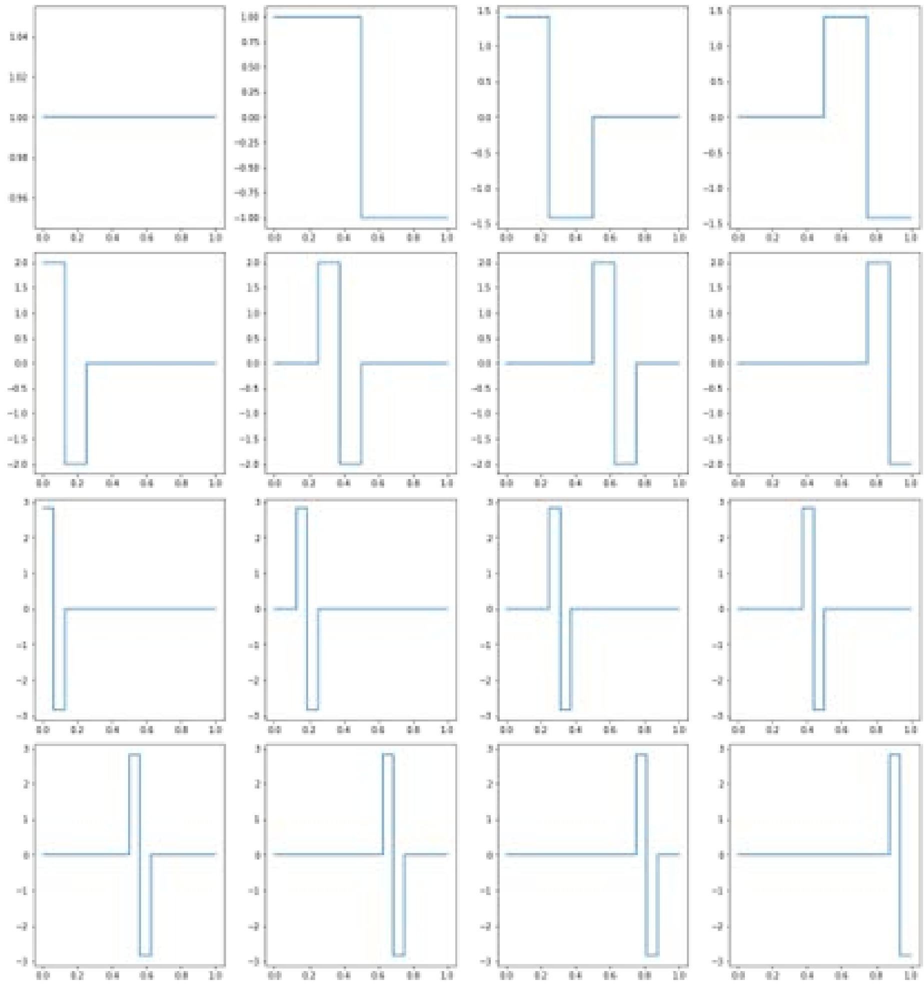
for n=2:



for n=3:

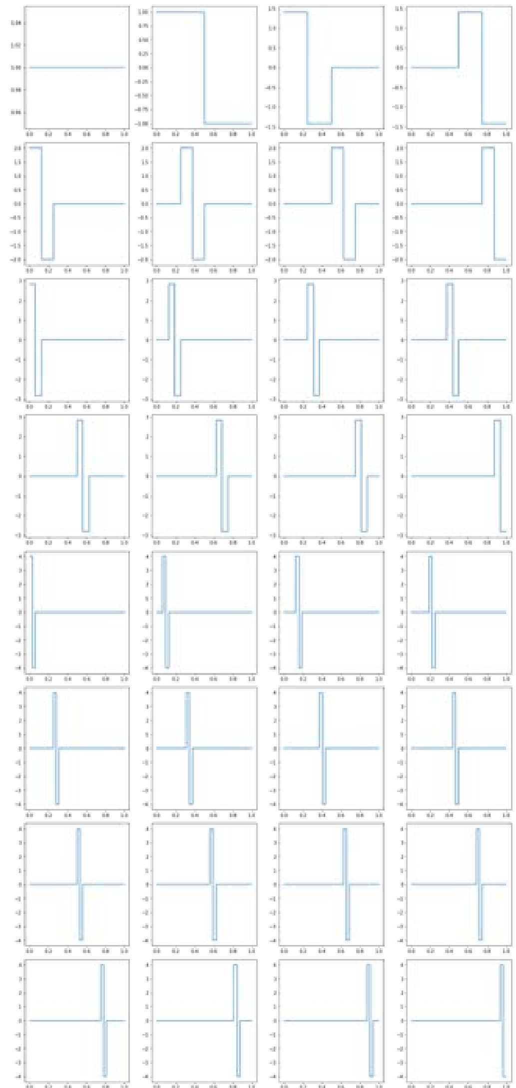


for n=4:



for n=6:

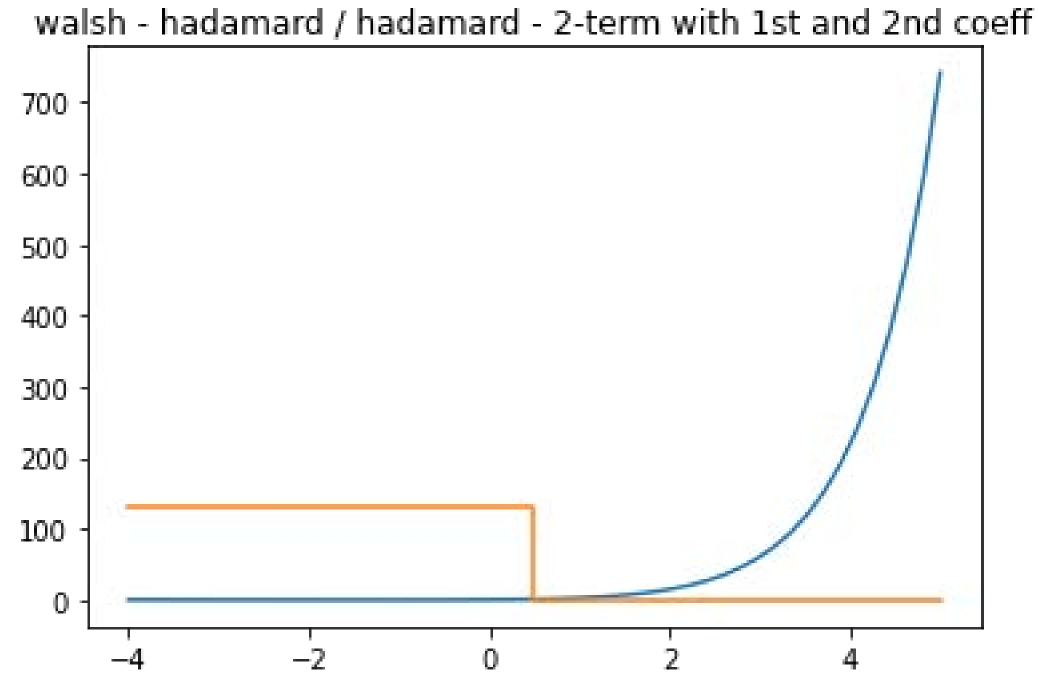
for n=5:

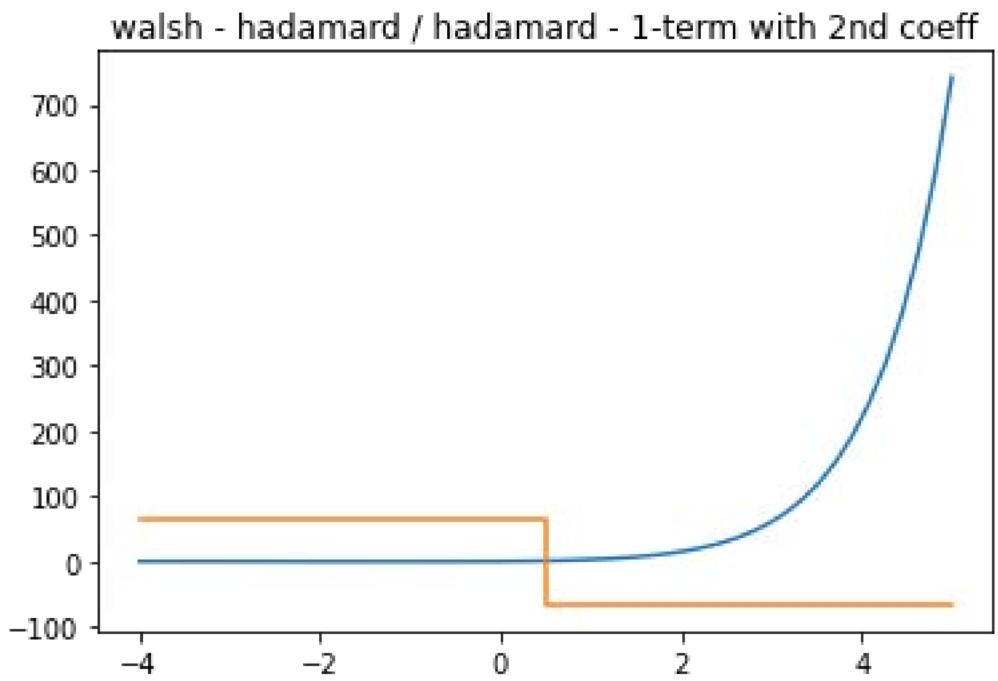


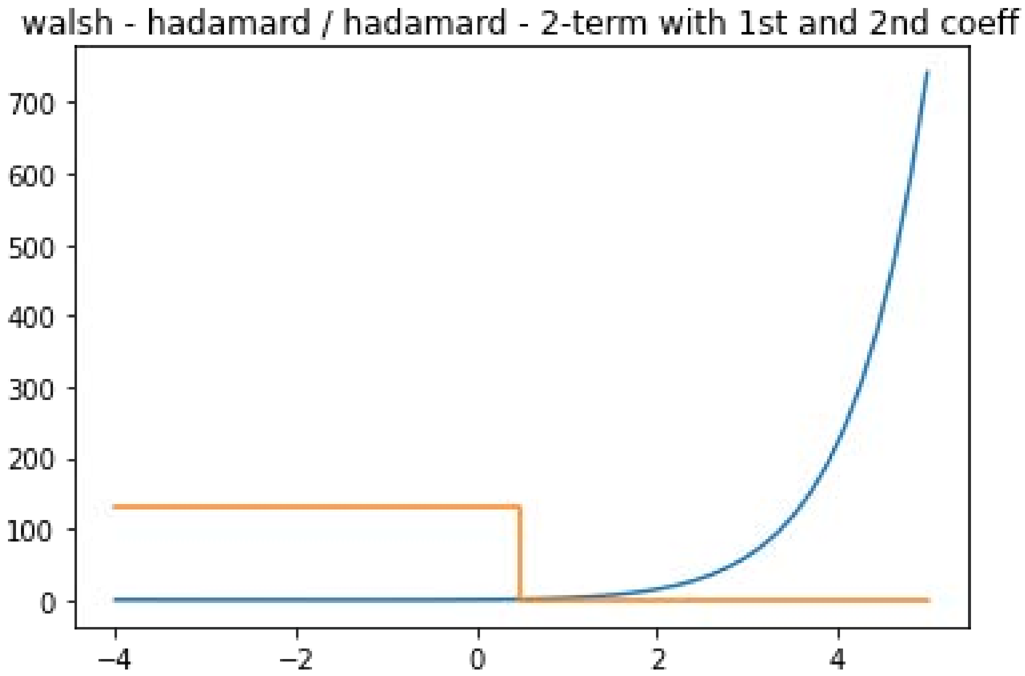
7. For this question we implemented the function k-term-aproximation which finds the highest coefficiants and with respect to k chooses the biggest coefficiants(in absolute value) and returns the function that is the projection of $\phi(t)$ on to the basis. That is:

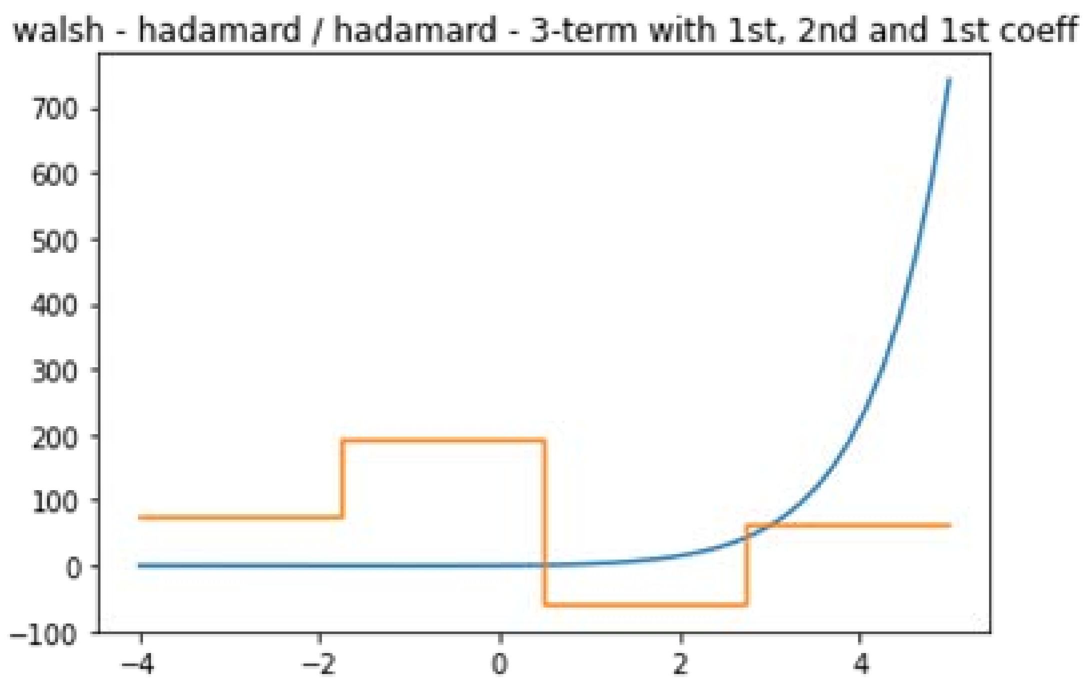
$$\sum_{i=1}^{k} \langle \beta_i, \phi \rangle \beta_i$$

For Hadamard basis:


walsh - hadamard / hadamard - 2-term with 1st and 2nd coeff
700
600
500
400
300
200
100
0
−4
−2
0
2
4


walsh - hadamard / hadamard - 1-term with 2nd coeff
700
600
500
400
300
200
100
0
−100
−4
−2
0
2
4

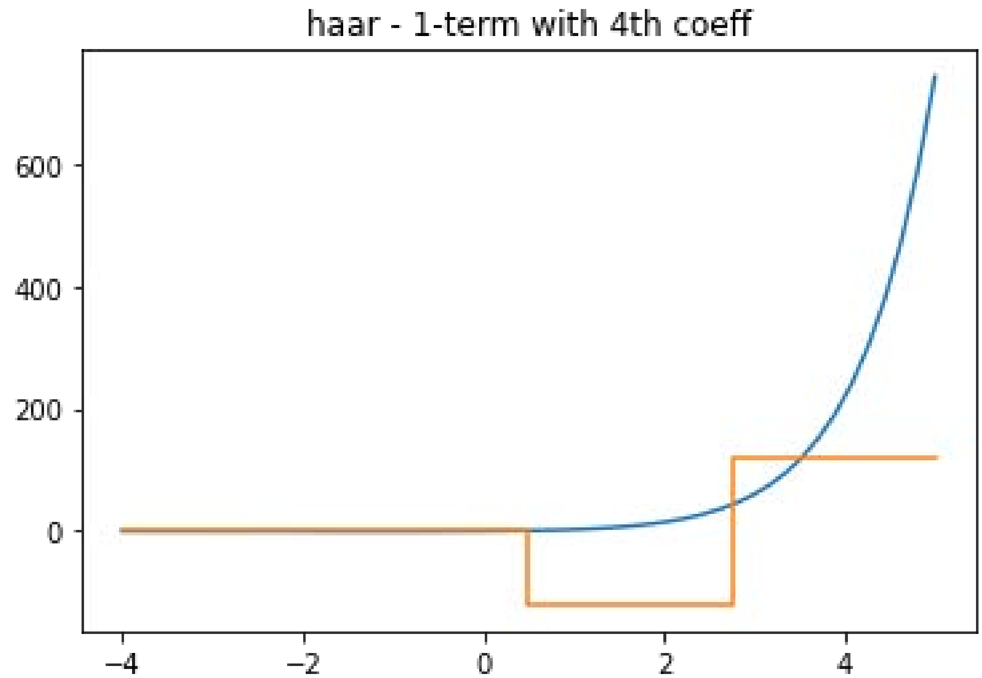


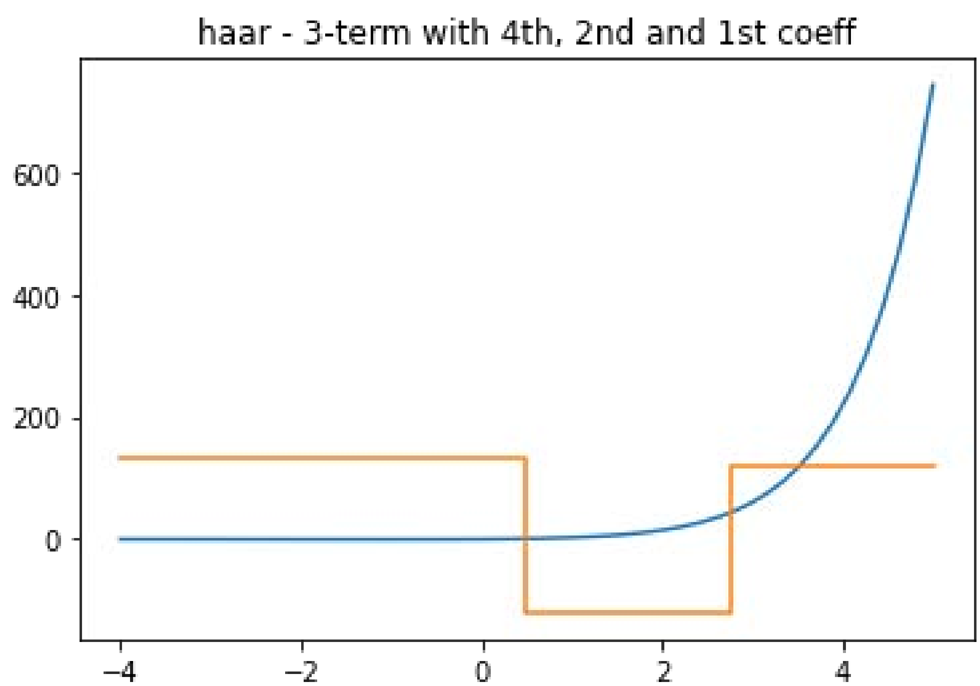


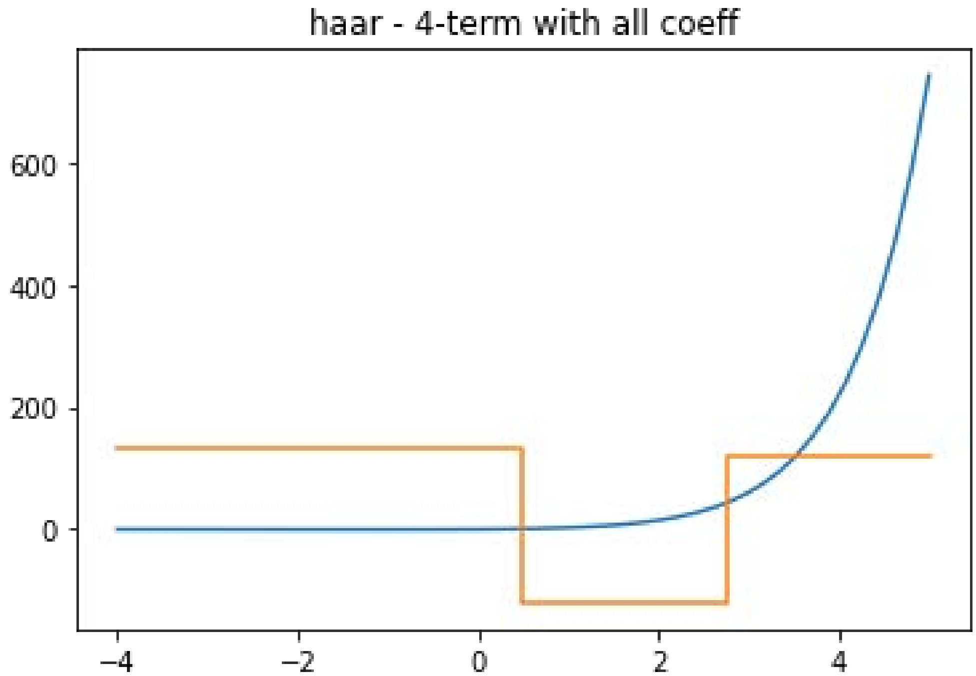
For the Walsh Hadamard basis: The only change is that now the rows are reorderd, but the k approximation remains the same.
That means that the plots and the errors will stay the same.
The MSE Errors are:

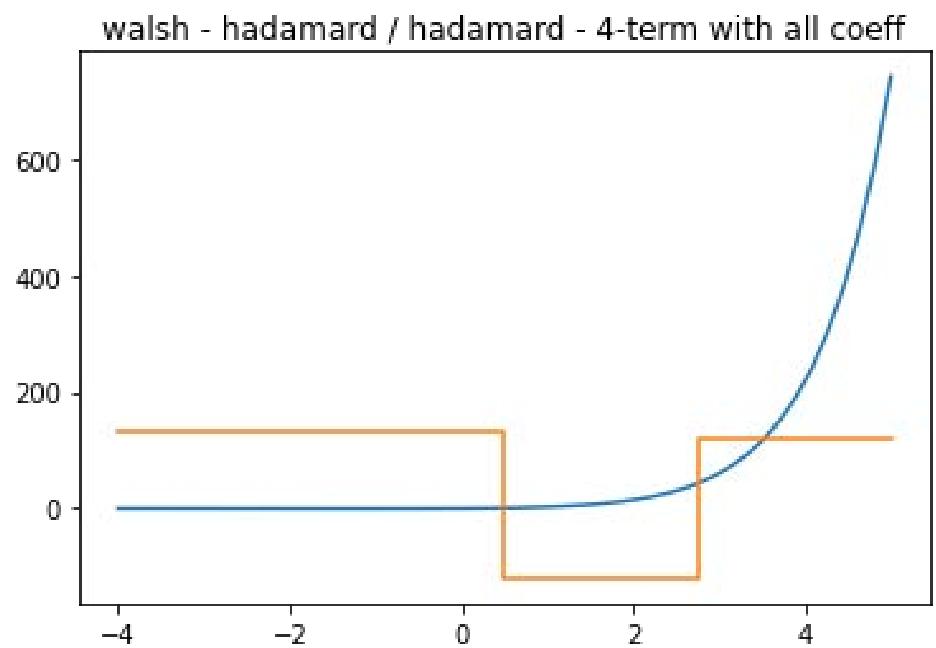
[1.97914738e+02 1.98403260e+02 -1.87703188e-02 -2.53652903e+02]

As for the Haar basis:


haar - 1-term with 4th coeff
600
400
200
0
−4
−2
0
2
4


haar - 3-term with 4th, 2nd and 1st coeff
600
400
200
0
−4
−2
0
2
4





As for the errors:

[1.97914738e+02 1.98403260e+02 -1.87703188e-02 -2.53652903e+02]

We can see that for 1,2 the approximation by Haar is better, for 3 the approximation by hadamard is better, but when using the whole basis
we can see that the error is the same simply because the hadamard, walsh-hadamard and haar span the same space of functions.