Introduction to Data Processing and representation

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HW3

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Theory

1. Circulant Matrices

a. Considering the matrix $J = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}$, we want to compute J^k,

for $k \in \mathbb{N}$:
Let's check the behavior of J^k for $k = 2$

$$J^2 = J^1 \cdot J = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & \cdots & 1 & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \end{pmatrix}$$

As can be seen from the outcome above, we got matrix J with one cyclic shift of the rows downward or $n-1$ cyclic shifts of the rows upward. Thus for $k \in \mathbb{N}$ we get $J^k = j^{k-1}J$, meaning the result is matrix J with $k-1$ cyclic shifts of the rows downward. For $k = n$ we get $J^n = j^{n-1}J$, meaning the result is matrix J with $n-1$ cyclic shifts of the rows downward or one cyclic shift of the rows upwards. The first row of J would be the n^{th} row of the outcome matrix now, after shifting all the rows we get

$$J^n = \begin{pmatrix} 1 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} = I_{n\times n}$$

This means, for $k = n$ the result is the identity matrix of order n.
J is called permutation matrix which means a square binary matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere.

b. We want to compute the eigenvalues of J
Using the definition, if there is a vector $V \in \mathbb{R}^n \neq 0$ such that

$$JV = \lambda V$$

For some scalar λ, then λ is called the eigenvalues of J with corresponding eigenvector V.
This is equivalent to $(J-\lambda I)V=0$, and nontrivial solution is available iff the determinant vanishes, so the solutions of $(J-\lambda I)V=0$ are given by

$$\det(J-\lambda I)=0$$

By applying the determinant of upper or lower triangular matrix is the product of all the diagonal elements of the matrix, we get

$$\det(J-\lambda I)=0$$

$$\det(J-\lambda I)=det\begin{pmatrix}-\lambda & \cdots & \cdots & 0 & 1\\ 1 & -\lambda & \ddots & \ddots & 0\\ 0 & 1 & -\lambda & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 1 & -\lambda\end{pmatrix}$$

$$=-\lambda\cdot det\underbrace{\begin{pmatrix}-\lambda & \cdots & 0 & 0\\ 1 & -\lambda & \ddots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 1 & -\lambda\end{pmatrix}}_{lower\ triangular\ matrix}+(-1)^{n+1}\cdot 1\cdot det\underbrace{\begin{pmatrix}1 & -\lambda & \cdots & 0\\ 0 & 1 & -\lambda & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 1\end{pmatrix}}_{upper\ triangular\ matrix}$$

$$=(-\lambda)^n+(-1)^{n+1}$$

$$\Rightarrow(-\lambda)^n+(-1)^{n+1}=0$$

$$\Rightarrow\lambda^n=-(-1)=1$$

$$\lambda^n=1e^{i0}$$

Using the polar form, we denote λ by $\lambda=re^{i\theta}$, then we get

$$r^n e^{in\theta}=1e^{i0}$$

$$r^n=1\Rightarrow r=1$$

$$n\theta=0+2\pi k\ \text{ for } k=0,1,2,\dots,n-1$$

$$\theta=+\frac{2\pi k}{n}\ \text{for } k=0,1,2,\dots,n-1$$

Since every $n\times n$ matrix has exactly n complex eigenvalues, we get that the n eigenvalues of J are given by:

$$\lambda_k=e^{i\frac{2\pi k}{n}}\ \text{for } k=0,1,2,\dots,n-1$$

c. To complete the eigendecomposition of J, we need to compute the corresponding eigenvectors of the previously computed eigenvalues in (b):
We get

$$(J-\lambda_r I)V_r = 0 \text{ for } r = 0, \dots, n-1$$

$$\begin{pmatrix} -\lambda_r & \cdots & \cdots & 0 & 1 \\ 1 & -\lambda_r & \ddots & \ddots & 0 \\ 0 & 1 & -\lambda_r & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -\lambda_r \end{pmatrix} \begin{pmatrix} v_{r_1} \\ v_{r_2} \\ v_{r_3} \\ \vdots \\ v_{r_n} \end{pmatrix} = 0$$

$$\begin{pmatrix} -\lambda_r v_{r_1} + v_{r_n} \\ -\lambda_r v_{r_2} + v_{r_1} \\ -\lambda_r v_{r_3} + v_{r_2} \\ \vdots \\ -\lambda_r v_{r_n} + v_{r_{n-1}} \end{pmatrix} = 0$$

For $v_{r_1} = 1 = \lambda_r^0$ (since eigenvectors are up to multiplication by a scalar), we get

$$\begin{cases} v_{r_1} = 1 \\ v_{r_2} = \lambda_r^{-1} \\ v_{r_3} = \lambda_r^{-2} \\ \vdots \\ v_{r_n} = \lambda_r^{-(n-1)} = \lambda_r^1 \end{cases}$$

Thus, we get

$$V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ e^{-i\frac{2\pi}{n}} \\ e^{-i\frac{2\pi \cdot 2}{n}} \\ \vdots \\ e^{-i\frac{2\pi(n-1)}{n}} \end{pmatrix}, \dots, V_n = \begin{pmatrix} 1 \\ e^{-i\frac{2\pi(n-1)}{n}} \\ e^{-i\frac{2\pi(n-1)\cdot 2}{n}} \\ \vdots \\ e^{-i\frac{2\pi(n-1)\cdot(n-1)}{n}} \end{pmatrix}$$

Therefore, J is diagonalizable and can be decomposed to the form

$$J = PDP^{-1}$$

Where P is a matrix composed of the eigenvectors of J, D is the diagonal matrix constructed from the corresponding eigenvalues and P^{-1} is the matrix inverse of P.

Thus, P is of the form

$$P = \begin{pmatrix} | & | & \dots & | \\ V_1 & V_2 & \dots & V_n \\ | & | & \dots & | \end{pmatrix}$$

And D is of the form

$$D = \begin{pmatrix} \lambda_0 & \cdots & \cdots & 0 & 0 \\ 0 & \lambda_1 & \ddots & \ddots & 0 \\ 0 & 0 & \lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \lambda_{n-1} \end{pmatrix}$$

We can diagonalize J in a unitary basis, since we can get U by normalizing the matrix P such that $U = \frac{1}{\sqrt{n}} P$ (eigenvectors are up to multiplication by a scalar), U is symmetric and unitary since its rows and columns obey

$$\sum_{l=0}^{n-1} U^{kl} \left(U^{rl}\right)^* = \sum_{l=0}^{n-1} U^{(k-r)l} = \frac{1}{n} \sum_{l=0}^{n-1} e^{-\frac{i2\pi}{n}(k-r)l} = \begin{cases} 1 & k = r \\ 0 & k \neq r \end{cases}$$

and which makes the following statement true

$$U^* J U = \begin{pmatrix} \lambda_0 & \cdots & \cdots & 0 & 0 \\ 0 & \lambda_1 & \ddots & \ddots & 0 \\ 0 & 0 & \lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \lambda_{n-1} \end{pmatrix}$$

d. Considering the general circulant matrix $H = \begin{pmatrix} h_0 & h_{n-1} & h_{n-2} & \cdots & h_1 \\ h_1 & h_0 & h_{n-1} & \ddots & h_2 \\ h_2 & h_1 & h_0 & \ddots & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_0 \end{pmatrix}$.

And recalling that J is given by $J = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}$.

Using the results of (a) we get:

$$H = h_1 J + h_2 J^2 + \cdots + h_{n-1} J^{n-1} + h_0 J^n = h_0 I_n + h_1 J + h_2 J^2 + \cdots + h_{n-1} J^{n-1}$$

Thus, the polynomial P on matrix J is given by

$$P(J) = h_0 I_n + h_1 J + h_2 J^2 + \cdots + h_{n-1} J^{n-1}$$

Therefore, $H = P(J)$

e. We got that $H = P(J)$, by applying this we get that for all eigenvalue of matrix J, λ_k when $k = 1, \dots, n$, $P(\lambda_k)$ is an eigenvalue of H with the same eigenvector.
To see that we first show that if λ is an eigenvalue of matrix A then λ^k is the eigenvalue of the matrix A^k, since λ is the eigenvalue of A then $AV = \lambda V$ as V is the corresponding eigenvector. The eigenvalue of A^k is computed by

$$A^k V = A^{k-1}(AV) = A^{k-1}(\lambda V) = \lambda A^{k-1} V = \lambda A^{k-2}(AV) = \lambda^2 A^{k-2} V = \cdots = \lambda^k V$$

This result indicates that λ^k is the eigenvalue of A^k with the same eigenvector of the eigenvalue λ of A.
Now, let's compute the eigenvalues of H

$$HV_r = \left(\sum_{k=0}^{n-1} h_k J^k\right) V_r = \sum_{k=0}^{n-1} h_k \left(J^k V_r\right)$$

Using the results above, we get

$$HV_r = \left(\sum_{k=0}^{n-1} h_k J^k\right) V_r = \sum_{k=0}^{n-1} h_k \left(J^k V_r\right) = \sum_{k=0}^{n-1} h_k \left(\lambda_r^k V_r\right) = \left(\sum_{k=0}^{n-1} h_k \lambda_r^k\right) V_r = P(\lambda_r) V_r$$
$$= \lambda_{H_r} V_r$$

Therefore, the eigenvalue $\lambda_{H_r}$of H with V_r as corresponding eigenvector is given by
$$\lambda_{H_r} = P(\lambda_r) = P\left(e^{i\frac{2\pi(r-1)}{n}}\right) = h_1 e^{i\frac{2\pi r}{n}} + h_2 e^{i\frac{2\pi r \cdot 2}{n}} + \cdots + h_{n-1} e^{i\frac{2\pi r(n-1)}{n}} + h_0 \quad \text{for } r = 0,1,2, \dots, n-1$$
H is diagonalizable by the same unitary matrix $U = \frac{1}{\sqrt{n}} \begin{pmatrix} | & | & \dots & | \\ V_1 & V_2 & \dots & V_n \\ | & | & \dots & | \end{pmatrix}$ with the eigenvalues computed above, meaning it is diagonalizable in a unitary basis.

f. We want to show that the diagonalization basis matrix B can be chosen as the DFT^* matrix:
We saw earlier that H is diagonalizable in unitary basis using the basis matrix U. Since U is symmetric and each one of its elements is given by $W^{*kr} = e^{-\frac{i2\pi}{n}kr}$ for $k, r = 0, \dots, n-1$, therefore its conjugate is the conjugate of the Discrete Fourier Transform matrix, given that H is diagonalizable using U and since $U = DFT$, we get
$$H = B \Lambda B^*$$
$$H \underbrace{=}_{H \text{ is real}} \overline{H} = \overline{DFT^* \Lambda DFT} \underbrace{=}_{DFT \text{ is symmetric}} \overline{\overline{DFT^T}\Lambda DFT} \underbrace{=}_{DFT \text{ is symmetric}} DFT \overline{\Lambda} DFT^*$$
In short, both the DFT and its complex conjugate can be chosen as the diagonalization basis matrix and it's up to the choice of the eigenvalues (we can choose them to be $\lambda_k = e^{-i\frac{2\pi k}{n}}$ for $k = 0,1,2, \dots, n-1$ or $\lambda_k = e^{i\frac{2\pi k}{n}}$ for $k = 0,1,2, \dots, n-1$).

g. Let B be the diagonalization basis matrix and let's denote $\sqrt{n}B = U$, then we get $U^*HU = \Lambda$

By applying matrix transpose operation on both sides and considering the symmetry of U and U^* and the fact that Λ is diagonal matrix, we get

$$(U^*HU)^T = \Lambda^T$$
$$\Rightarrow UH^TU^* = \Lambda$$

After multiplying both side by U from the right, we get

$$UH^T = \Lambda U$$

$$U\begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-1} \\ h_{n-1} & h_0 & h_1 & \cdots & h_{n-2} \\ h_{n-2} & h_{n-1} & h_0 & \cdots & h_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1 & h_2 & h_3 & \cdots & h_0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ 1 & \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

Since both sides are equal, we get

$$\sqrt{n}B\begin{pmatrix} h_0 \\ h_{n-1} \\ h_{n-2} \\ \vdots \\ h_1 \end{pmatrix} = \begin{pmatrix} \lambda_0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix}$$

Therefore, we get

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} = \sqrt{n}B\begin{pmatrix} h_0 \\ h_{n-1} \\ h_{n-2} \\ \vdots \\ h_1 \end{pmatrix}$$

h. We want to show that given the two circulant matrices $H_1, H_2 \in \mathbb{R}^{n\times n}$, H_1 and H_2 commute, meaning $H_1H_2 = H_2H_1$:

Let B be the diagonalization basis matrix and let's denote $\sqrt{n}B = U$, hence $U^*HU = \Lambda$.

Let's assume the following

$$H_1 = U\Lambda_1U^*$$
$$H_2 = U\Lambda_2U^*$$

Then we get

$$H_1H_2 = U\Lambda_1U^*U\Lambda_2U^*$$

Since U is unitary matrix, we get $U^*U = I$, therefore

$$H_1H_2 = U\Lambda_1U^*U\Lambda_2U^* = U\Lambda_1\Lambda_2U^*$$

Considering that $\Lambda_1$and Λ_2 are diagonal matrices meaning they commute, we get

$$H_1H_2 = U\Lambda_1\Lambda_2U^* = U\Lambda_2\Lambda_1U^*$$

By using the fact $U^*U = I$ again we get

$$H_1H_2 = U\Lambda_2\Lambda_1U^* = U\Lambda_2U^*U\Lambda_1W^* = H_2H_1$$
Therefore, H_1 and H_2 commute.

H_1H_2 is a circulant matrix, we show this using the results of section (d).
We saw that H_1 and H_2 are given by polynomial expression of the matrix J, meaning
$$H_1 = P_1(J)$$
$$H_2 = P_2(J)$$
Hence,
$$H_1H_2 = P_1(J)P_2(J) = \sum_{k=0}^{n-1} h_{1_k}J^k \sum_{r=0}^{n-1} h_{2_r}J^r = \sum_{k=0}^{n-1}\sum_{r=0}^{n-1} h_{1_k}h_{2_r}J^{k+r}$$

As we saw earlier, J^{k+r} is a circulant matrix, thus $h_{1_k}h_{2_r}J^{k+r}$ is a circulant matrix two, and since the outcome of adding two circulant matrices is also a circulant matrix we get that H_1H_2 is a circulant matrix too.

i. We want to compute DFT^k for $k \in \mathbb{N}$:
Let's first check the behavior of DFT^k for $k = 2$, we get

$$DFT^2 = \frac{1}{\sqrt{n}}\begin{pmatrix} w^{*0\cdot 0} & w^{*1\cdot 0} & w^{*2\cdot 0} & \cdots & w^{*(n-1)\cdot 0} \\ w^{*0\cdot 1} & w^{*1\cdot 1} & w^{*2\cdot 1} & \cdots & w^{*(n-1)\cdot 1} \\ w^{*0\cdot 2} & w^{*2\cdot 1} & w^{*2\cdot 2} & \cdots & w^{*(n-1)\cdot 2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w^{*0\cdot(n-1)} & w^{*1\cdot(n-1)} & w^{*2\cdot(n-1)} & \cdots & w^{*(n-1)\cdot(n-1)} \end{pmatrix}$$

$$\cdot\frac{1}{\sqrt{n}}\begin{pmatrix} w^{*0\cdot 0} & w^{*1\cdot 0} & w^{*2\cdot 0} & \cdots & w^{*(n-1)\cdot 0} \\ w^{*0\cdot 1} & w^{*1\cdot 1} & w^{*2\cdot 1} & \cdots & w^{*(n-1)\cdot 1} \\ w^{*0\cdot 2} & w^{*2\cdot 1} & w^{*2\cdot 2} & \cdots & w^{*(n-1)\cdot 2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w^{*0\cdot(n-1)} & w^{*1\cdot(n-1)} & w^{*2\cdot(n-1)} & \cdots & w^{*(n-1)\cdot(n-1)} \end{pmatrix}$$
Hence,
$$DFT^2_{kr} = \frac{1}{n}\sum_{l=0}^{n-1} w^{*lk}w^{*rl} = \frac{1}{n}\sum_{l=0}^{n-1} e^{-\frac{i2\pi lk}{n}}e^{-\frac{i2\pi lr}{n}} = \frac{1}{n}\sum_{l=0}^{n-1} e^{-\frac{i2\pi l(k+r)}{n}}$$
$$= \begin{cases} 1 & (k+r)\ mod\ n = 0 \\ 0 & else \end{cases}$$
And we get
$$DFT^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \vdots & \iddots & 1 \\ 0 & \vdots & 0 & \iddots & 0 \\ \vdots & 0 & 1 & \iddots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

We notice that this is the vertically flipped form of J, let's denote it by FJ.
Now for $k = 4$ we get

$$DFT^4 = DFT^2 + DFT^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \vdots & ⋰ & 1 \\ 0 & \vdots & 0 & ⋰ & 0 \\ \vdots & 0 & 1 & ⋰ & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \vdots & ⋰ & 1 \\ 0 & \vdots & 0 & ⋰ & 0 \\ \vdots & 0 & 1 & ⋰ & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = I_{n\times n}$$

To calculate DFT^3 we rely on the fact that $DFT \cdot DFT^* = I, thus$ for $k = 3$ we get

$$DFT^3 = DFT^3 \cdot I = DFT^3 \cdot DFT \cdot DFT^* = DFT^4 \cdot DFT^* = I \cdot DFT^*$$

Relying on the results we got above, we can conclude that for $k \in \mathbb{N}$ DFT^k is given by:

$$DFT^k = \begin{cases} I_{n\times n} & k \bmod 4 = 0 \\ DFT & k \bmod 4 = 1 \\ FJ & k \bmod 4 = 2 \\ DFT^* & k \bmod 4 = 3 \end{cases}$$

j. We want to prove that a convolution of n-dimensional signals can be computed by point-wise multiplication of the signals in the Fourier domain, up to a normalization.
The convolution of x and y can be computed using a circulant matrix X built from x (the same as H by with the elements of x) then multiplying by y as follows

$$z = x \otimes y = Xy$$

We get

$$(DFT)z = (DFT)Xy$$

Considering $(DFT^*)(DFT) = I$ we get

$$(DFT)z = (DFT)Xy = (DFT)X(DFT^*)(DFT)y$$

Considering that $(DFT)X(DFT^*)$ is a diagonal matrix and hence symmetric, we get

$$(DFT)z = (DFT)Xy = (DFT)X(DFT^*)(DFT)y = (DFT^*)X(DFT) \cdot (DFT)y$$

Relying on the results of (g), we get

$$\sqrt{n}(DFT)y = \begin{pmatrix} \lambda_0^y \\ \lambda_1^y \\ \lambda_2^y \\ \vdots \\ \lambda_{n-1}^y \end{pmatrix}$$

$$(DFT)y = \frac{1}{\sqrt{n}}\begin{pmatrix} \lambda_0^y \\ \lambda_1^y \\ \lambda_2^y \\ \vdots \\ \lambda_{n-1}^y \end{pmatrix}$$

Therefore, we get

$$(DFT)z = (DFT^*)X(DFT)(DFT)y = \begin{pmatrix} \lambda_0^x & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1^x & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2^x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1}^x \end{pmatrix}\begin{pmatrix} \lambda_0^y \\ \lambda_1^y \\ \lambda_2^y \\ \vdots \\ \lambda_{n-1}^y \end{pmatrix} =$$

$$\sqrt{n}\begin{pmatrix} \lambda_0^x \\ \lambda_1^x \\ \lambda_2^x \\ \vdots \\ \lambda_{n-1}^x \end{pmatrix} \odot \begin{pmatrix} \lambda_0^y \\ \lambda_1^y \\ \lambda_2^y \\ \vdots \\ \lambda_{n-1}^y \end{pmatrix} = \sqrt{n}(DFT)x \odot (DFT)y$$

2. Fourier Transform

a. $f(t)$ and $g(t)$ are two given functions, with convolution denoted by $h(t)$ and given by

$$h(t) = f(t) * g(t)$$

We want to find $f(t-1) * g(t+1)$ in terms of $h(t)$

We start by denoting $f(t-1) = \tilde{f}(t)$ and $g(t+1) = \tilde{g}(t)$, then we get

$$f(t-1) * g(t+1) = \tilde{f}(t) * \tilde{g}(t) = \int_{-\infty}^{\infty} \tilde{f}(\tilde{\tau})\tilde{g}(t-\tilde{\tau})\, d\tilde{\tau}$$

$$= \int_{-\infty}^{\infty} f(\tilde{\tau}-1)g(t-\tilde{\tau}+1)\, d\tilde{\tau} = \int_{-\infty}^{\infty} f(\tilde{\tau}-1)g\big(t-(\tilde{\tau}-1)\big)\, d\tilde{\tau}$$

By changing the integral variable $\tilde{\tau}$ by denoting $\tilde{\tau}-1=\tau$ and hence $d\tilde{\tau} = d\tau$, we get

$$f(t-1) * g(t+1) = \int_{-\infty}^{\infty} f(\tilde{\tau}-1)g\big(t-(\tilde{\tau}-1)\big)\, d\tilde{\tau} = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau$$

$$= f(t) * g(t) = h(t)$$

b. $f(t)$ and $g(t)$ are two given functions, we want to show that the following condition holds

$$\int_{-\infty}^{\infty} f(t)g(-t)\, dt = \int_{-\infty}^{\infty} \mathcal{F}(u)\mathcal{G}(u)\, du$$

Where $\mathcal{F}(u)$ and $\mathcal{G}(u)$ are the Fourier transform of $f(t)$ and $g(t)$ respectively.

We get

$$\int_{-\infty}^{\infty} \mathcal{F}(u)\mathcal{G}(u)\, du = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\tau)e^{-i2\pi u\tau}d\tau \int_{-\infty}^{\infty} g(\eta)e^{-i2\pi u\eta}d\eta\right)du$$

$$\underbrace{=}_{Fubini's\ theorem} \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)e^{-i2\pi u\tau}g(\eta)e^{-i2\pi u\eta}d\tau d\eta\right)du$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(\eta)e^{-i2\pi u(\tau+\eta)}d\tau d\eta\right)du$$

By changing the variables and denoting $\eta + \tau = t$, hence $\eta = t - \tau$ and $d\eta = dt$. We get

$$\int_{-\infty}^{\infty} \mathcal{F}(u)\mathcal{G}(u)\,du = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(\eta)e^{-i2\pi u(\tau+\eta)}d\tau d\eta\right)du$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(t-\tau)e^{-i2\pi ut}d\tau dt\right)du$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-i2\pi ut}dt \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right)du$$

$\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$ is the definition of convolution of the two functions $f(t)$ and $g(t)$, meaning

$$h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

Now we get

$$\int_{-\infty}^{\infty} \mathcal{F}(u)\mathcal{G}(u)\,du = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-i2\pi ut}dt \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right)du$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(t)e^{-i2\pi ut}dt\right)du$$

$\int_{-\infty}^{\infty} h(t)e^{-i2\pi ut}dt = H(u)$ is the Fourier transform of $h(t)$, therefore we get

$$\int_{-\infty}^{\infty} \mathcal{F}(u)\mathcal{G}(u)\,du = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(t)e^{-i2\pi ut}dt\right)du = \int_{-\infty}^{\infty} H(u)\,du = \int_{-\infty}^{\infty} H(u)\,e^{2\pi iu\cdot 0}du$$

$\int_{-\infty}^{\infty} H(u)\,e^{2\pi iu\cdot 0}du$ is the projection of $h(t=0)$ to the Fourier family, as seen in the lecture, or the inverse Fourier transform, also $h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$

Thus, $h(0) = \int_{-\infty}^{\infty} f(\tau)g(-\tau)d\tau = \int_{-\infty}^{\infty} f(t)g(-t)dt$

$$\int_{-\infty}^{\infty} \mathcal{F}(u)\mathcal{G}(u)\,du = \int_{-\infty}^{\infty} H(u)\,e^{2\pi iu\cdot 0}du = h(0) = \int_{-\infty}^{\infty} f(t)g(-t)\,dt$$

We got $\int_{-\infty}^{\infty} \mathcal{F}(u)\mathcal{G}(u)\,du = \int_{-\infty}^{\infty} f(t)g(-t)\,dt$ as required.

Question 3

a. $$\phi^F = \sqrt{2N}\begin{bmatrix} \sum_{k=0}^{2N-1} W^{*k\cdot 0}\cdot \phi(k) \\ \sum_{k=0}^{2N-1} W^{*k\cdot 1}\cdot \phi(k) \\ \sum_{k=0}^{2N-1} W^{*k\cdot 2}\cdot \phi(k) \\ \vdots \\ \sum_{k=0}^{2N-1} W^{*k\cdot (2N-1)}\cdot \phi(k) \end{bmatrix} =$$

$$\sqrt{2N}\begin{bmatrix} W^{*0\cdot 0} + \frac{W^{*0}}{2} + \frac{W^{*(2N-1)\cdot 0}}{2} \\ W^{*0\cdot 1} + \frac{W^{*1}}{2} + \frac{W^{*(2N-1)\cdot 1}}{2} \\ W^{*0\cdot 2} + \frac{W^{*2}}{2} + \frac{W^{*(2N-1)\cdot 2}}{2} \\ \vdots \\ W^{*0\cdot (2N-1)} + \frac{W^{*(2N-1)}}{2} + \frac{W^{*(2N-1)\cdot(2N-1)}}{2} \end{bmatrix} =$$

$$\sqrt{2N}\begin{bmatrix} 1 + \frac{W^{*0}}{2} + \frac{W^{0}}{2} \\ 1 + \frac{W^{*1}}{2} + \frac{W^{1}}{2} \\ 1 + \frac{W^{*2}}{2} + \frac{W^{2}}{2} \\ \vdots \\ 1 + \frac{W^{*(2N-1)}}{2} + \frac{W^{(2N-1)}}{2} \end{bmatrix}$$

b. For $k \in [0, \ldots, 2N-1]$:

$$\gamma^F(k) = \frac{1}{\sqrt{2N}} \cdot \sum_{n=0}^{2N-1} W_{2N}^{*n\cdot k} \cdot \gamma_n =$$

$$\frac{1}{\sqrt{2N}} \cdot \sum_{n=0}^{N-1} W_{2N}^{*(2n)\cdot k} \cdot \psi_n =$$

$$\frac{1}{\sqrt{2N}} \cdot \sum_{n=0}^{N-1} W_{N}^{*n\cdot k} \cdot \psi_n =$$

$$\frac{1}{\sqrt{2}} \cdot \psi^F\big(k\ (mod\ N)\big)$$

So:

$$\gamma^F = \frac{1}{\sqrt{2}} \cdot [\psi_0^F, \psi_1^F, \dots, \psi_{N-1}^F, \psi_0^F, \psi_1^F, \dots, \psi_{N-1}^F]^T$$

c. For $k \in [0, \dots, 2N-1]$:

$$(\gamma * \phi)(k) =$$

$$\sum_{m=0}^{2N-1} \phi(m) \cdot \gamma(k-m) =$$

$$\phi(0) \cdot \gamma(k-0) + \phi(1) \cdot \gamma(k-1) + \phi(2N-1) \cdot \gamma(k-2N+1) =$$

$$\gamma(k) + \frac{\gamma(k-1)}{2} + \frac{\gamma(k-2N+1)}{2} =$$

$$\gamma(k) + \frac{\gamma(k-1)}{2} + \frac{\gamma(k+1)}{2} = \begin{cases} \psi\left(\frac{k}{2}\right), & k \text{ is even} \\ \frac{\psi\left(\frac{k-1}{2}\right)}{2} + \frac{\psi\left(\frac{k+1}{2}\right)}{2}, & k \text{ is odd} \end{cases}$$

d. For $k \in [0, \dots, 2N-1]$:

$$h^F(k) = \frac{1}{\sqrt{2N}} \cdot \sum_{n=0}^{2N-1} W_{2N}^{*n \cdot k} \cdot h_n =$$

$$\frac{1}{\sqrt{2N}} \cdot \sum_{n=0}^{N-1} W_{2N}^{*(2n) \cdot k} \cdot \psi_n + \frac{1}{\sqrt{2N}} \cdot \sum_{n=0}^{N-1} W_{2N}^{*(2n+1) \cdot k} \cdot \frac{\psi_n + \psi_{n+1}}{2} =$$

$$\frac{1}{\sqrt{2}} \cdot \psi^F\big(k \ (mod\ N)\big) + \frac{1}{\sqrt{8N}} \cdot \left(\sum_{n=0}^{N-1} W_{2N}^{*(2n+1) \cdot k} \cdot \psi_n + \cdot \sum_{n=0}^{N-1} W_{2N}^{*(2n+1) \cdot k} \cdot \psi_{n+1} \right) =$$

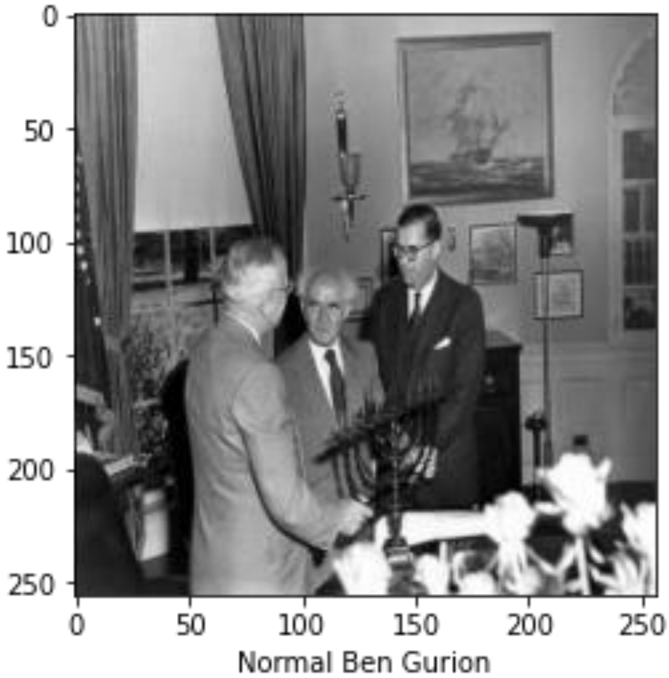
$$\frac{1}{\sqrt{2}} \cdot \psi^F\big(k \ (mod\ N)\big) + \frac{1}{\sqrt{8N}} \cdot \left(W_{2N}^{*k} \sum_{n=0}^{N-1} W_{2N}^{*(2n) \cdot k} \cdot \psi_n + \frac{1}{W_{2N}^{*k}} \sum_{n=0}^{N-1} W_{2N}^{*(2n) \cdot k} \cdot \psi_n \right) =$$

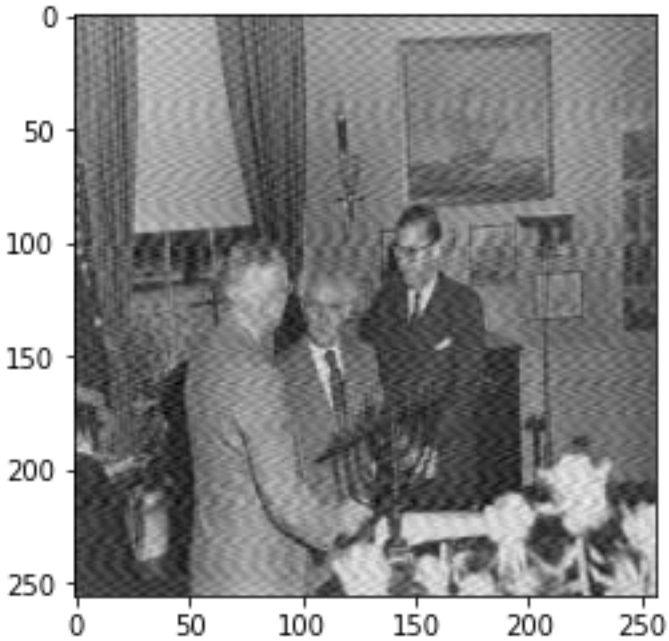
$$\frac{1}{\sqrt{2}} \cdot \psi^F\big(k \ (mod\ N)\big) + \frac{W_{2N}^{*k} + \frac{1}{W_{2N}^{*k}}}{\sqrt{8}} \cdot \left(\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} W_{2N}^{*(2n) \cdot k} \cdot \psi_n \right) =$$

$$\frac{W_{2N}^{*k} + W_{2N}^{k} + 2}{\sqrt{8}} \cdot \psi^F\big(k \ (mod\ N)\big)$$

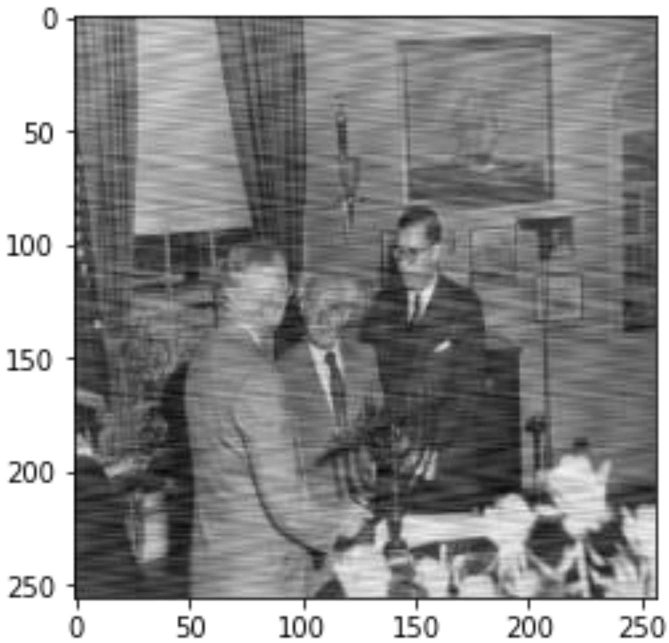
Implementation

a.

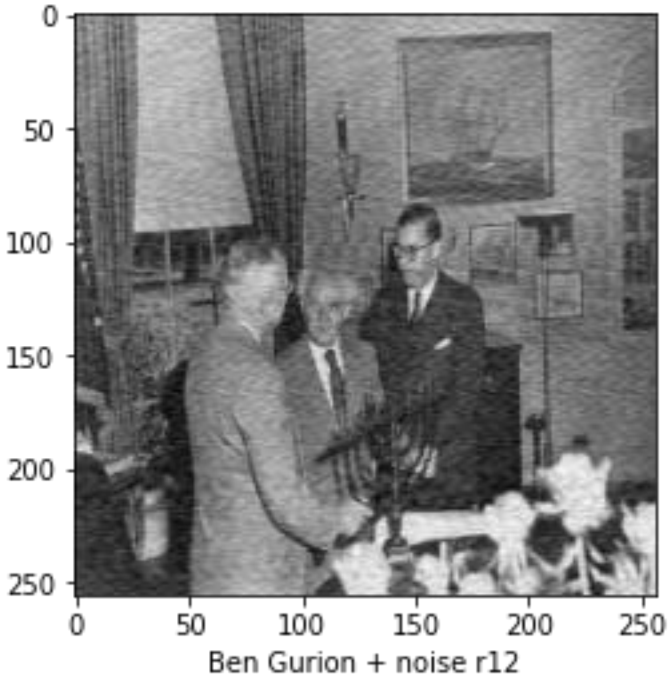




Ben Gurion + noise r1



Ben Gurion + noise r2



b. We use the index t instead of i, so as not to confuse with the imaginary number i.
We will define: $u = f \cdot n$.

$$\phi^F_{I^{noisy}_{t,j}} = \phi^F_{I^{noisy}_t}(j) = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot I_t^{noisy}(k) =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot \left(I_t(k) + A_t\cdot\cos(2\pi f k + \varphi_t)\right) =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot I_t(k) + \frac{1}{2\sqrt{n}}\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot A_t\cdot\left(e^{\frac{i2\pi}{n}uk+i\varphi_t} + e^{-\frac{i2\pi}{n}uk-i\varphi_t}\right) =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot I_t(k) + \frac{1}{2\sqrt{n}}\sum_{k=0}^{n-1} A_t\cdot W^{*j\cdot k}\cdot\left(W^{u\cdot k}\cdot e^{i\varphi_t} + W^{-u\cdot k}\cdot e^{-i\varphi_t}\right) =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot I_t(k) + \frac{A_t e^{i\varphi_t}}{2\sqrt{n}}\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot W^{u\cdot k} + \frac{A_t e^{-i\varphi_t}}{2\sqrt{n}}\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot W^{-u\cdot k}$$

For $\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot W^{u\cdot k}$:
If $u = j$:
$\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot W^{u\cdot k} = \sum_{k=0}^{n-1} W^0 = n$
Else:
$\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot W^{u\cdot k} = \sum_{k=0}^{n-1}\left(W^{*j-u}\right)^k = \frac{\left(W^{*j-u}\right)^n-1}{\left(W^{*j-u}\right)-1} = \frac{\left(W^{*n}\right)^{j-u}-1}{\left(W^{*j-u}\right)-1} = \frac{(1)^{j-u}-1}{\left(W^{*j-u}\right)-1} = 0$

For $\sum_{k=0}^{n-1} W^{*j\cdot k}\cdot W^{-u\cdot k}$:
If $u = -j$:

$\sum_{k=0}^{n-1} W^{*j\cdot k} \cdot W^{-u\cdot k} = \sum_{k=0}^{n-1} W^{0} = n$

Else:

$\sum_{k=0}^{n-1} W^{*j\cdot k} \cdot W^{-u\cdot k} = \sum_{k=0}^{n-1} \left(W^{*-j-u}\right)^{k} = \frac{\left(W^{*-j-u}\right)^{n}-1}{\left(W^{*-j-u}\right)-1} = \frac{(1)^{-j-u}-1}{\left(W^{*-j-u}\right)-1} = 0$

So:

$\phi^{F}_{I^{noisy}_{t,j}} =$

$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} W^{*j\cdot k} \cdot I_t(k) + \frac{A_t e^{i\varphi_t}\sqrt{n}}{2} \cdot \delta_{j,f\cdot n} + \frac{A_t e^{-i\varphi_t}\sqrt{n}}{2} \cdot \delta_{j,-f\cdot n}$

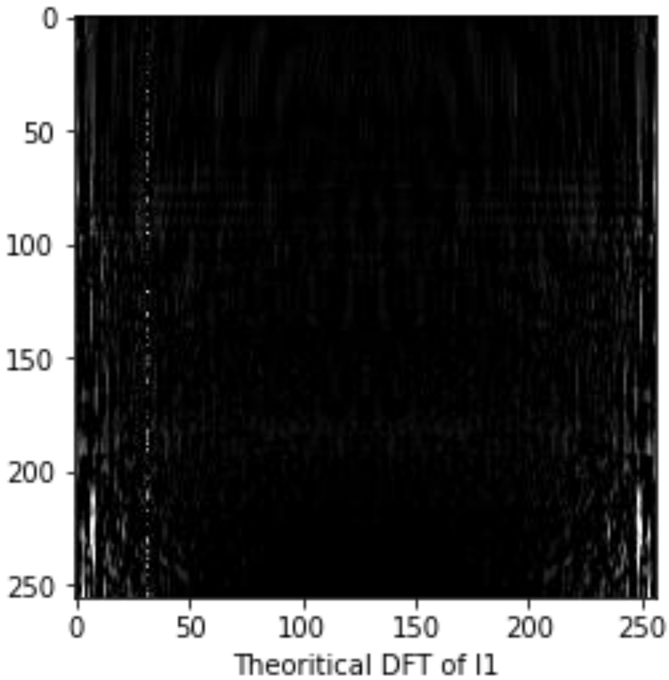
c. The weight of noise 1: w_1; The weight of noise 2: w_2

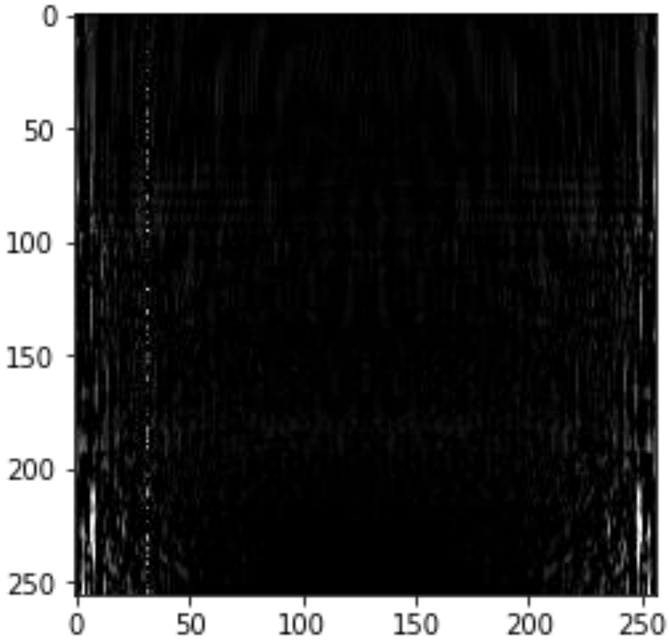
$\phi^{F}_{I^{noisy}_{t,j}} =$

$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} W^{*j\cdot k} \cdot \left(I_t(k) + \frac{w_1 \cdot A_t^{(1)} \cdot \cos\left(2\pi f^{(1)} k + \varphi_t^{(1)}\right)}{w_1+w_2} + \frac{w_2 \cdot A_t^{(2)} \cdot \cos\left(2\pi f^{(2)} k + \varphi_t^{(2)}\right)}{w_1+w_2}\right) =$

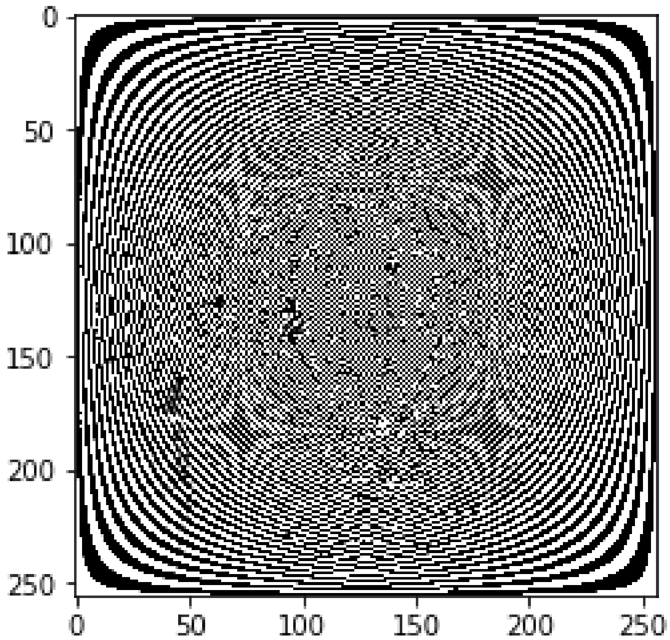
$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} W^{*j\cdot k} \cdot I_t(k) + \frac{w_1 A_t^{(1)} e^{i\varphi_t^{(1)}}\sqrt{n}}{2(w_1+w_2)} \cdot \delta_{j,f^{(1)}\cdot n} + \frac{w_2 A_t^{(2)} e^{i\varphi_t^{(2)}}\sqrt{n}}{2(w_1+w_2)} \cdot \delta_{j,f^{(2)}\cdot n} + \frac{w_1 A_t^{(1)} e^{-i\varphi_t^{(1)}}\sqrt{n}}{2(w_1+w_2)} \cdot \delta_{j,-f^{(1)}\cdot n} + \frac{w_2 A_t^{(2)} e^{-i\varphi_t^{(2)}}\sqrt{n}}{2(w_1+w_2)} \cdot \delta_{j,-f^{(2)}\cdot n}$

d.

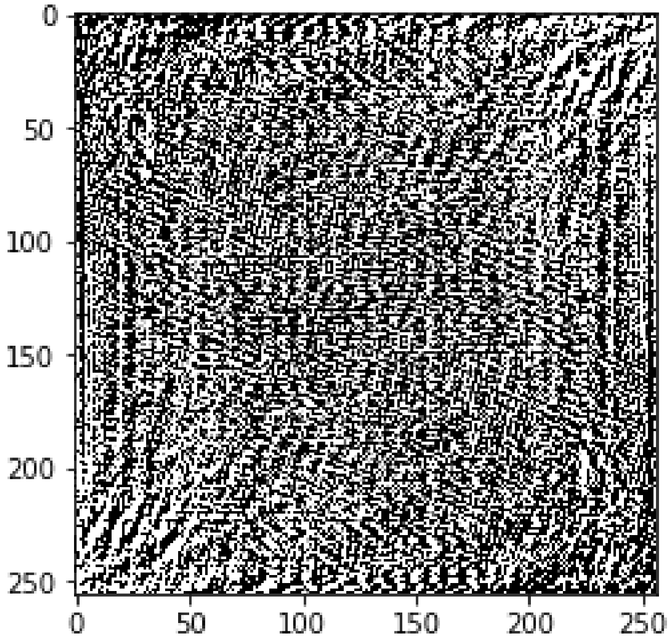




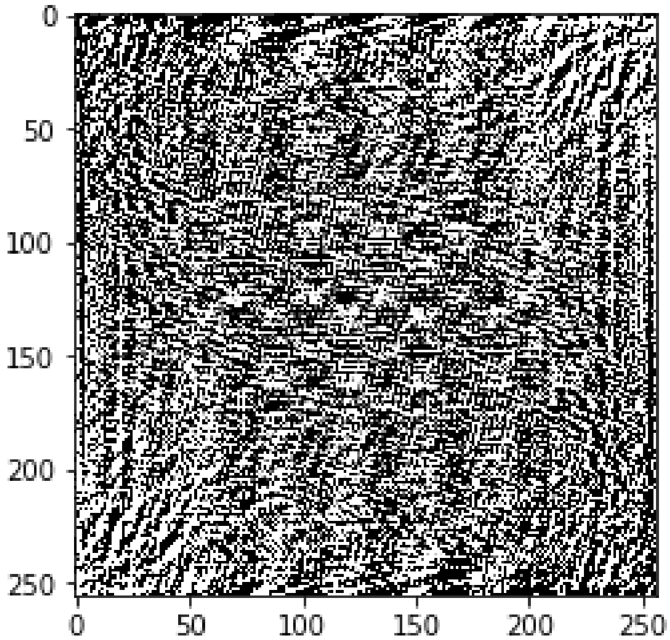
Theoritical DFT of I1



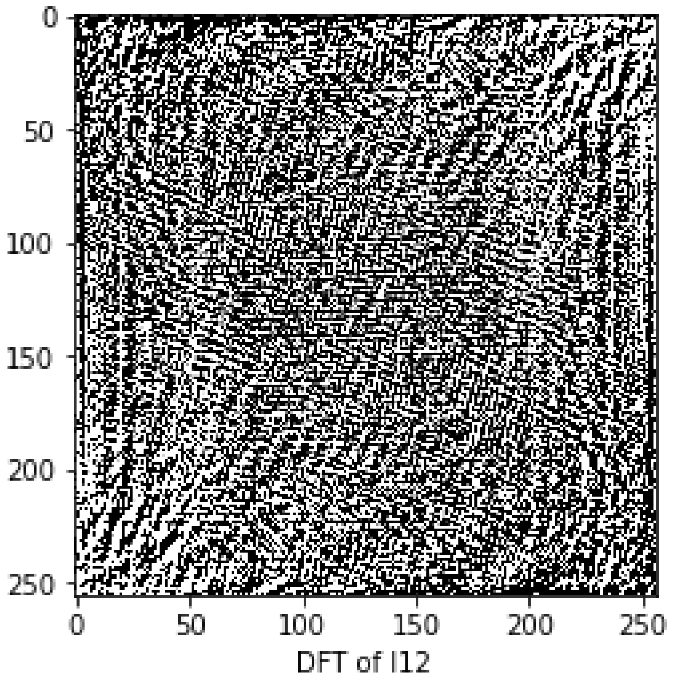
Theoritical DFT of I12



DFT of I1



DFT of I2



e.

MSE of I1 is: 2.7109850297764546
MSE of I2 is: 2.7002239091778053
MSE of I12 is: 359.91252877302543