Introduction to Data Processing and representation

236201

HW4

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Theory

1)

a.

$$H = \begin{bmatrix} -5/2 & 4/3 & -1/12 & 0 & \cdots & 0 & -1/12 & 4/3 \\ 4/3 & -5/2 & 4/3 & -1/12 & 0 & \cdots & 0 & -1/12 \\ -1/12 & 4/3 & -5/2 & 4/3 & \ddots & 0 & \vdots & 0 \\ 0 & -1/12 & 4/3 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & -1/12 \\ -1/12 & 0 & \vdots & 0 & \ddots & \ddots & \ddots & 4/3 \\ 4/3 & -1/12 & 0 & \cdots & 0 & -1/12 & 4/3 & -5/2 \end{bmatrix}$$

b.

H is a circulant matrix $\rightarrow$ *it is diagonalizable by the DFT matrix*

$H = [DFT]\Lambda[DFT]^*$

The first row of H

$$H_0 = \begin{bmatrix} -5/2 \\ 4/3 \\ -1/12 \\ 0 \\ \vdots \\ 0 \\ -1/12 \\ 4/3 \end{bmatrix}$$

$$\Lambda = \sqrt{M}\,[DFT]^{*}H_0 = \begin{bmatrix} \sum_{l=0}^{M-1} W_M^{0l}\,H_0 \\ \sum_{l=0}^{M-1} W_M^{1l}\,H_0 \\ \vdots \\ \sum_{l=0}^{M-1} W_M^{(M-1)l}\,H_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sum_{l=0}^{M-1} W_M^{1l}\,H_0 \\ \vdots \\ \sum_{l=0}^{M-1} W_M^{(M-1)l}\,H_0 \end{bmatrix}$$

The rank of H is $M-1$ *as* Λ_0 *is the only zero eigenvalue.*
Since H is a symmetric matrix all its eigenvalues are real.

The pseudo – inverse of H:

$$H^{\dagger} = [DFT]\Lambda^{\dagger}[DFT]^{*}$$

$$\textit{and}\ \Lambda^{\dagger} = diag\left(\Lambda^{\dagger}\right) = \begin{bmatrix} 0 \\ \dfrac{1}{\sum_{l=0}^{M-1} W_M^{1l}\,H_0} \\ \vdots \\ \dfrac{1}{\sum_{l=0}^{M-1} W_M^{(M-1)l}\,H_0} \end{bmatrix}$$

c.

φ *cannot perfectly recovered using the previously provided inverse filter because matrix H has a null space which is not invertible. Any signal that contains a component in the null space of H will be unrecoverable.*

If we take, for example, a signal which is a scaled version of the eigenvector corresponding to the eigenvalue 0 $\left(\textit{the first row of } [DFT]^{*}\right)$ *by a factor of* $\alpha \neq 0$

$$\varphi_{unrecoverable} = \alpha \cdot \begin{bmatrix} W_M^0 \\ W_M^0 \\ \vdots \\ W_M^0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \vdots \\ \alpha \end{bmatrix}$$

$Therefore,$

$$H \cdot \varphi_{unrecoverable} = 0$$
$$\varphi_{unrecoverable} \neq H^{\dagger} \cdot H \cdot \varphi_{unrecoverable} = 0 \quad \forall \alpha \neq 0$$

2)

a.

$The\ expected\ value\ of\ the\ i_{th} \leqslant \frac{N}{2}\ component\ is:$

$$E\{\varphi_i\} = E\{\varphi_i|K = i\}P(K = i) + E\{\varphi_i|K \neq i\}P(K \neq i) =$$

$$= E\{M + L_1\}P(K = i) + E\{M\}P(K \neq i) =$$

$$= (E\{M\} + E\{L_1\})P(K = i) + E\{M\}P(K \neq i) = 0$$

$The\ expected\ value\ of\ the\ i_{th} > \frac{N}{2}\ component\ is:$

$$E\{\varphi_i\} = E\{\varphi_i|K = i\}P(K = i) + E\{\varphi_i|K \neq i\}P(K \neq i) =$$

$$= E\{M + L_2\}P(K = i) + E\{M\}P(K \neq i) =$$

$$= (E\{M\} + E\{L_2\})P(K = i) + E\{M\}P(K \neq i) = 0$$

$\rightarrow$

$The\ mean\ signal\ E\{\varphi\} = 0$

b.

$We\ will\ denote\ r_{i,j}\ as\ the\ i, j\ element\ of\ R_{\varphi} = E\left\{\varphi\varphi^T\right\}$

$For\ i \neq j:$

$r_{i,j} = E\{\varphi_i, \varphi_j\} =$

$= E\{\varphi_i, \varphi_j|K = i\}P(K = i) + E\{\varphi_i, \varphi_j|K = j\}P(K = j) + E\{\varphi_i, \varphi_j|K \neq i, j\}P(K \neq i, j) =$

$= E\{(M + L)M\}P(K = i) + E\{M(M + L)\}P(K = j) + E\left\{M^2\right\}P(K \neq i, j)$

$E\{M\} = E\{L\} = 0$ *and* $M,\ L$ *are independent* $\rightarrow$

$r_{i,j} = \left(E\left\{M^2\right\} + E\{M\}E\{L\}\right)P(K = i) + \left(E\left\{M^2\right\} + E\{M\}E\{L\}\right)P(K = j) + E\left\{M^2\right\}P(K \neq i, j) =$

$= E\left\{M^2\right\}(P(K = i) + P(K = j) + P(K \neq i, j)) =$

$= E\left\{M^2\right\} = c$

For $i = j$:
$r_{i,i} =$
$= E\left\{\varphi_i^2|K = i\right\}P(K = i) + E\left\{\varphi_i^2|K \neq i\right\}P(K \neq i) =$

$= E\left\{(M + L)^2\right\}P(K = i) + E\left\{M^2\right\}P(K \neq i) =$

$= \left(E\left\{M^2\right\} + 2E\{M\}E\{L\} + E\left\{L^2\right\}\right)P(K = i) + E\left\{M^2\right\}P(K \neq i)$

Splitting for $i \leqslant \frac{N}{2},\ i > \frac{N}{2}$:

$r_{i,i} \quad E\left\{M^2\right\} + \frac{E\left\{L_1^2\right\}}{N} = c + a \qquad \text{for } i \leqslant \frac{N}{2}$

$r_{i,i} \quad E\left\{M^2\right\} + \frac{E\left\{L_2^2\right\}}{N} = c + b \qquad \text{for } i > \frac{N}{2}$

c.

For the PCA matrix to be DFT^*, R_φ *should be diagonalizable by the* DFT^*.

$This\ will\ occur\ if\ and\ only\ if\ R_\varphi\ is\ circulant,\ and\ in\ this\ case\ it\ will\ happen\ if\ a = b.$
$Notably,\ c\ can\ be\ of\ any\ value.$

3)

a.

$We\ will\ denote\ r_{i,j}\ as\ the\ i,j\ element\ of\ R_\varphi = E\left\{\varphi\varphi^T\right\}$

$For\ i \neq j, \left(j+\frac{N}{2}\right) \bmod N:$

$$r_{i,j} = E\{\varphi_i\varphi_j\} =$$

$$= E\{\varphi_i\varphi_j|K=i\}P(K=i) + E\{\varphi_i\varphi_j|K=j\}P(K=j) + E\{\varphi_i\varphi_j|K\neq i,j\}P(K\neq i,j) =$$

$$= E\{(M+L)M\}P(K=i) + E\{(M+L)M\}P(K=j) + E\left\{M^2\right\}P(K\neq i,j)$$

$Since\ M, L\ are\ independent$

$$r_{i,j} = \left(E\left\{M^2\right\} + E\{M\}E\{L\}\right)P(K=i) + \left(E\left\{M^2\right\} + E\{M\}E\{L\}\right)P(K=j) + E\left\{M^2\right\}P(K\neq i,j)$$

$E\{M\} = 0 \quad and \quad E\left\{M^2\right\} = c \rightarrow$
$$r_{i,j} = E\left\{M^2\right\}(P(K=i) + P(K=j) + P(K\neq i,j)) = E\left\{M^2\right\} = c$$

$For\ i = \left(j+\frac{N}{2}\right) \bmod N:$
$$r_{i,j} = E\{\varphi_i\varphi_j\} =$$

$$= E\{\varphi_i\varphi_j|K=i\}P(K=i) + E\{\varphi_i\varphi_j|K=j\}P(K=j) + E\{\varphi_i\varphi_j|K\neq i,j\}P(K\neq i,j) =$$

$$= E\left\{(M+L)^2\right\}P(K=i) + E\left\{(M+L)^2\right\}P(K=j) + E\left\{M^2\right\}P(K\neq i,j) =$$

$$= \left(E\left\{M^2\right\} + 2E\{M\}E\{L\} + E\left\{L^2\right\}\right)P(K = i) + \left(E\left\{M^2\right\} + 2E\{M\}E\{L\} + E\left\{L^2\right\}\right)P(K = j) +$$
$$E\left\{M^2\right\}P(K \neq i, j)$$

$$E\{M\} = E\{L\} = 0 \ \rightarrow$$

$$r_{i,j} = E\left\{M^2\right\} + E\left\{L^2\right\}(P(K = i) + P(K = j))$$

$$Since\ i = \left(j + \frac{N}{2}\right) \mathrm{mod}\, N\ and\ K \sim Uniform\ over\ the\ integers \left\{1, \ldots, \frac{N}{2}\right\} \ \rightarrow$$
$$either\ P\left(K = \left(j + \frac{N}{2}\right) \mathrm{mod}\, N\right) = 0\ or\ P(K = j) = 0 \ \rightarrow$$

$$(P(K = i) + P(K = j)) = \frac{2}{N} \quad \rightarrow$$

$$r_{i,j} = E\left\{M^2\right\} + E\left\{L^2\right\}(P(K = i) + P(K = j)) = c + \frac{N}{2}(1 - c)\frac{2}{N} = 1$$

$$For\ i = j:$$

$$r_{i,j} = E\{\varphi_i \varphi_j\} =$$

$$= E\left\{\varphi_i^2 | K = i\right\}P(K = i) + E\left\{\varphi_i^2 | K \neq i\right\}P(K \neq i) =$$

$$= E\left\{(M + L)^2\right\}P(K = i) + E\left\{M^2\right\}P(K \neq i) =$$

$$= \left(E\left\{M^2\right\} + 2E\{M\}E\{L\} + E\left\{L^2\right\}\right)P(K = i) + E\left\{M^2\right\}P(K \neq i) =$$

$$= E\left\{M^2\right\} + E\left\{L^2\right\}P(K = i) =$$

$$= c + \frac{N}{2}(1 - c)\frac{2}{N} = 1$$

$r_{i,j} = r_{0,(j-i)\bmod N} \implies R_\varphi$ *is a circulant matrix*

b.

R_φ *is a symmetric circulant matrix* $\rightarrow$ *its eigenvalues are real and its eigenvector are the columns of the DFT.*

We will examine the i elelment of vector v_j *(the* $j-th$ *column of DFT),* W_N^{ij}

$$(R_\varphi \cdot v_j)_i = c\sum_{k=0}^{N-1} W^{k,j} - c\left(W^{i,j} + W^{i+\frac{N}{2},j}\right) + 1\cdot\left(W^{i,j} + W^{i+\frac{N}{2},j}\right) =$$

$$= c\sum_{k=0}^{N-1} W^{k,j} + (1-c)\left(W^{i,j} + W^{i+\frac{N}{2},j}\right)$$

$$= \begin{cases} cN + (1-c)(1+1) = 2 + cN - 2c, & j = 0 \\ 0, & j \text{ is odd} \\ (1-c)\cdot 2W^{i,j}, & j \text{ is even } j \neq 0 \end{cases}$$

When we multiply the matrix row with the eigenvector,
we multiply all entrys of the eigenvector by c besides two entries which we multiply by 1.

For $j = 0$, *meaning the first row of the DFT (a vector of* $1's$*), we get that each of the output vector is* $2 + Nc - 2c$, *meaning that the eigenvalue is that.*
For $j \neq 0$, *the sum in the expression evaluates to zero, and we are left with the right part of of the expression.*
As N is even:

For odd J, $W^{i,j} = -W^{i+\frac{N}{2},j} \rightarrow$ *the entire expression is zero*

For even J, $W^{i,j} = W^{i+\frac{N}{2},j} \rightarrow$ *we get that each entry is* $2(1-c)$ *times the entry in the eigenvector.*
$\implies$
The vector has one eigenvalue of $2 + cN - 2c$, $\frac{N}{2}$ *zero eigenvalues*

and $\left(\frac{N}{2} - 1\right)$ *eigenvalues of* $2(1-c)$.

4)

a

$$\varphi^* = \mathcal{H}\varphi + n$$

The correletion matrix of φ^*

$$R_{\varphi^*} = E\left\{\varphi^* {\varphi^*}^T\right\} =$$

$$= E\{(\mathcal{H}\varphi + n)(\mathcal{H}\varphi + n))^T\} =$$

$$= E\left\{(\mathcal{H}\varphi + n)\left(\varphi^T\mathcal{H}^T + n^T\right)\right\} =$$

$$= E\left\{\mathcal{H}\varphi\varphi^T\mathcal{H}^T + \mathcal{H}\varphi n^T + n\varphi^T\mathcal{H}^T + nn^T\right\} =$$

$$= \mathcal{H}R_\varphi\mathcal{H}^T + \mathcal{H}E\left\{\varphi n^T\right\} + E\left\{n\varphi^T\right\}\mathcal{H}^T + R_n$$

The noise and random signal are independent $\rightarrow$

$$E\left\{\varphi n^T\right\} = E\left\{n\varphi^T\right\} = 0 \quad \rightarrow$$

$$\boxed{R_{\varphi^*} = \mathcal{H}R_\varphi\mathcal{H}^T + R_n}$$

b

$R_n = \sigma_n^2\, I$ *where* $\sigma_n > 0 \quad \rightarrow$

$$R_{\varphi^*} = \mathcal{H}R_\varphi\mathcal{H}^T + R_n = \mathcal{H}R_\varphi\mathcal{H}^T + \sigma_n^2\, I$$

R_φ *is positive semi – definite matrix* $\rightarrow$ $\mathcal{H}R_\varphi\mathcal{H}^T$ *is also positive semi – definite matrix*
Since $\sigma_n > 0 \rightarrow \sigma_n^2\, I$ *is positive definite matrix*
$\rightarrow$
R_{φ^*} *is positive definite as it the sum of*

$positive\ semi-definite\ matrix\ \left(\mathcal{H}R_{\varphi}\mathcal{H}^T\right)\ and\ positive\ definite\ matrix\ \left(\sigma_n^2\ I\right)\ \rightarrow$
$R_{\varphi^*}\ is\ invertible\ matrix$

$$W_{filter}\ R_{\varphi^*} = R_{\varphi}\mathcal{H}^T$$
$$W_{filter}\left(\mathcal{H}R_{\varphi}\mathcal{H}^T + \sigma_n^2\ I\right) = R_{\varphi}\mathcal{H}^T\ \rightarrow$$

$$\boxed{W_{filter} = R_{\varphi}\mathcal{H}^T\left(\mathcal{H}R_{\varphi}\mathcal{H}^T + \sigma_n^2\ I\right)^{-1} = R_{\varphi}\mathcal{H}^T R_{\varphi^*}^{-1}}$$

c

$$A = [DFT]\Lambda[DFT]^*$$

$$a_{i,j} = \sum_{k=0}^{N-1} W_N^{-ik}\cdot\lambda_k\cdot W_N^{kj} = \sum_{k=0}^{N-1}\lambda_k\cdot W_N^{k(j-i)}$$

$$By\ definition\ W_N^l = W_N^{(l)\bmod N}\ \rightarrow$$

$$a_{i,j} = \sum_{k=0}^{N-1}\lambda_k\cdot W_N^{k(j-i)} = \sum_{k=0}^{N-1} W_N^{-0k}\cdot\lambda_k\cdot W_N^{k((j-i)\bmod N)} = a_{0,(j-i)\bmod N}\ \rightarrow$$

$A\ is\ a\ circulant\ matrix$

d

$A\ system\ is\ shift\ invariant\ if\ and\ only\ if\ the\ system\ matrix\ is\ a\ circulant\ matrix$
$(was\ shown\ in\ class).\ For\ example,\ if\ we\ take$

$$R_{\varphi} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\mathcal{H} = I$$

$$\sigma_n = 1$$

$$W_{filter} = R_\varphi \mathcal{H}^T \left(\mathcal{H} R_\varphi \mathcal{H}^T + \sigma_n^2\, I\right)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot I \cdot \left(I \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot I + \ I\right)^{-1} =$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \frac{1}{4} \cdot \begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix} \rightarrow W_{filter} \textit{ is not a circulant matrix}$$

e

Let $A,\ B$ be circulant matrices.

$if\ A, B$ *circulant matrices* $\rightarrow \boxed{A^* \text{ and } B^* \text{ circulant matrices}}$

$$A^{-1} = \left([DFT]\Lambda_A[DFT]^*\right)^{-1} = [DFT]\Lambda_A^{-1}[DFT]^* \quad \rightarrow \boxed{A^{-1} \text{ is a circulant matrix}}$$

$$AB = [DFT]\Lambda_A[DFT]^* \cdot [DFT]\Lambda_B[DFT]^* = [DFT]\Lambda_A\Lambda_B[DFT]^* \quad \rightarrow \boxed{AB \text{ is a circulant matrix}}$$

$$A + B = \ [DFT]\Lambda_A[DFT]^* + [DFT]\Lambda_B[DFT]^* = [DFT](\Lambda_A + \Lambda_B)[DFT]^* \quad \rightarrow$$
$$\boxed{A + B \text{ is a circulant matrix}}$$

$$\boxed{\sigma_n^2\ I \text{ is a ciculant matrix for every constant } \sigma_n} \quad \Longrightarrow$$

If R_φ *and* $\mathcal{H}$ *are circulant,* $W_{filter} = R_\varphi \mathcal{H}^T \left(\mathcal{H} R_\varphi \mathcal{H}^T + \sigma_n^2\ I\right)^{-1}$ *is a circulant as combination of operators preserving circulant property.*

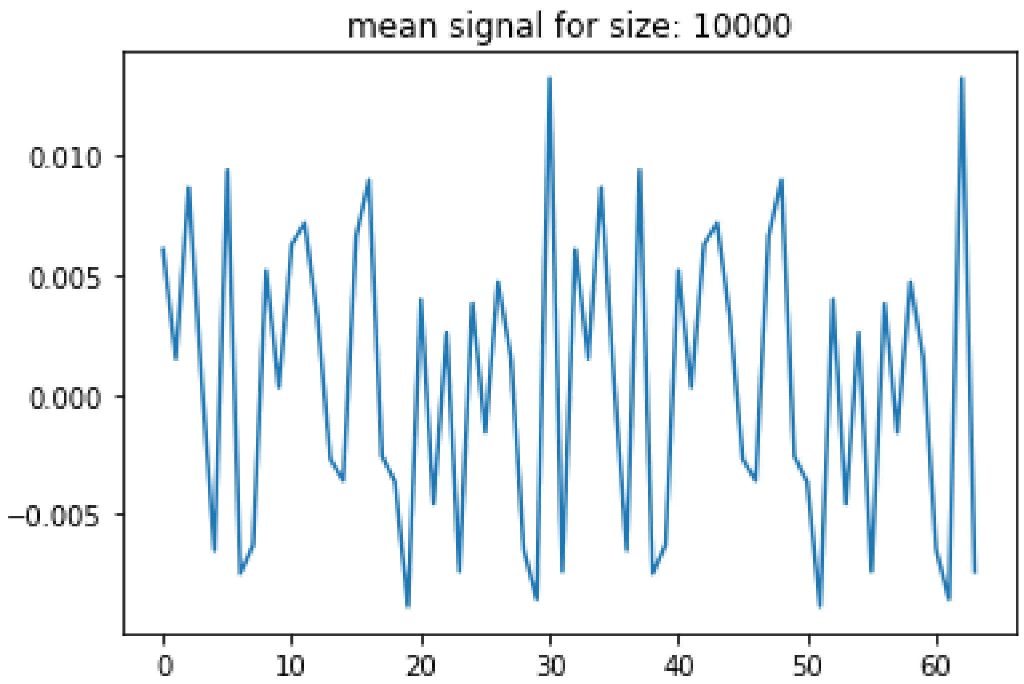
Introduction to Data Processing and Representation

Implementation

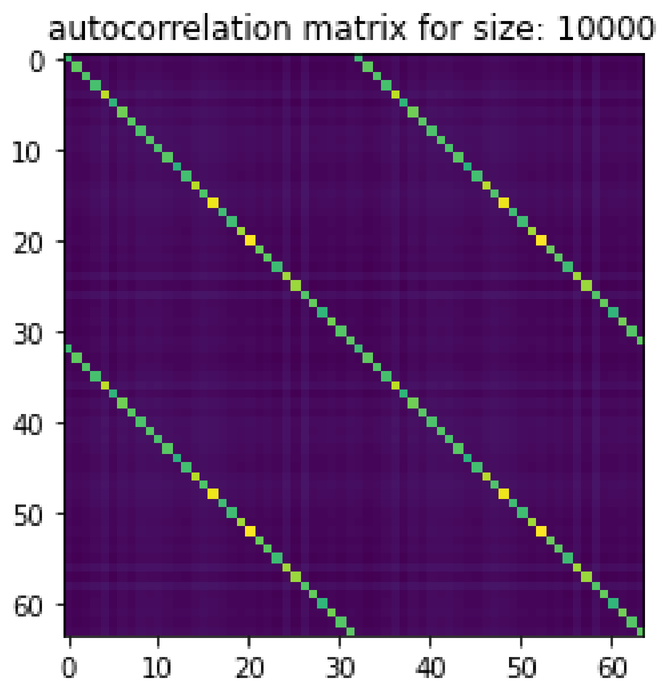
1)

a.

The empirical approximation of the mean signal:



And the empirical autocorrelation matrix is:

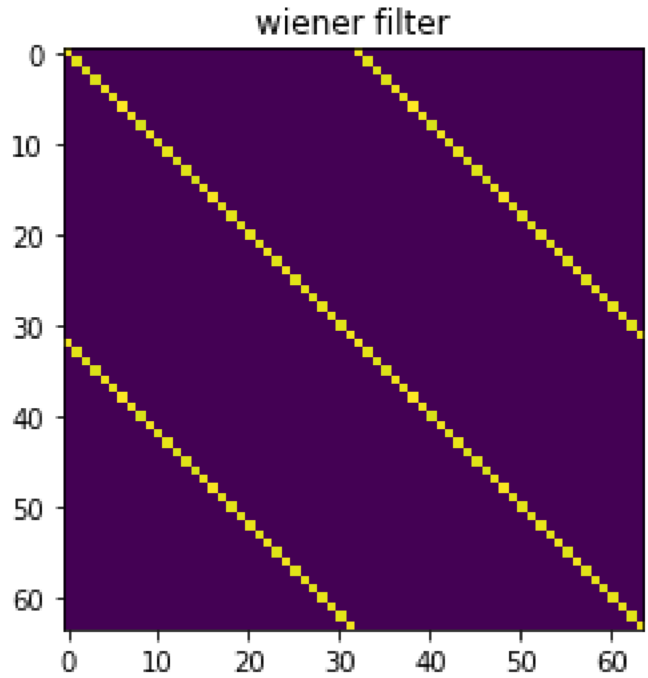


As we can see this sits fine with our solution at the third exercise because we are getting a circulant matrix with all entries equal to c=0.6 except for three diagonals which get 1. Moreover the empirical mean signal is very close to 0 as we computed in the third exercise.

The number of realizations we used was 10000, because we managed to approximate the empirical results well with this amount of realizations.

b.

The Weiner Filter matrix we got:



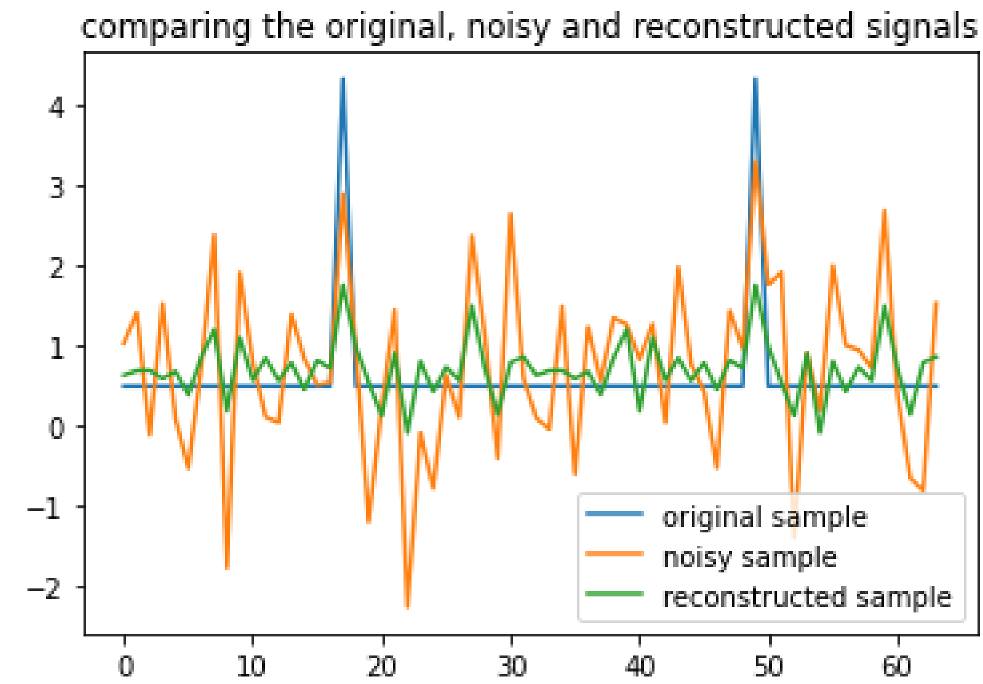
The structure is circulant and like the structure of the autocorrelation matrix as expected. This is from the formulae:

$$W=R_\phi H^T(HR_\phi H^T+(\sigma_n)^2 I)^{-1}$$

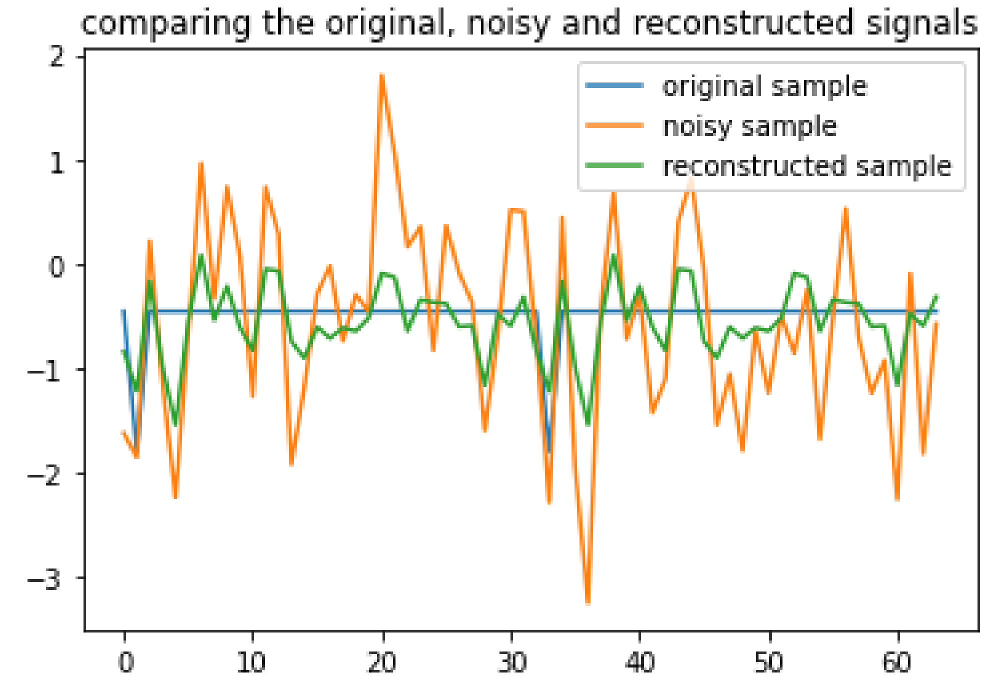
Whereas in our case H = I. Because R_ϕ and I are circulant we get that W is circulant.

Afterwards we denoised our noisy signals with the Weiner filter.

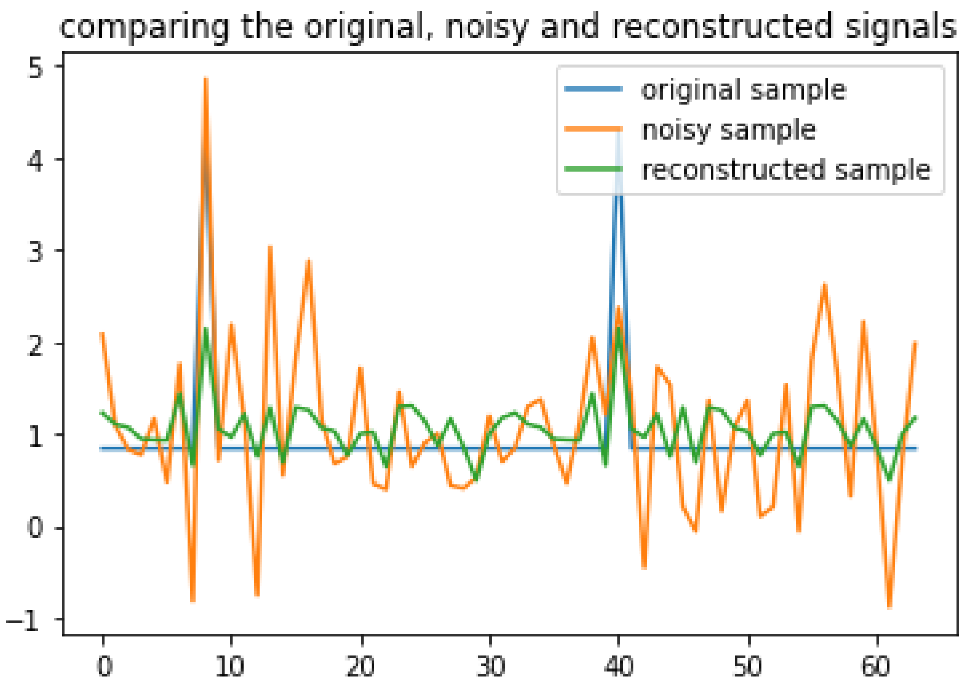
Some of the signals attached:




```
the mse for sample: 71 is: 0.33402077435404637
```




```
the mse for sample: 308 is: 0.1305365394581765
```

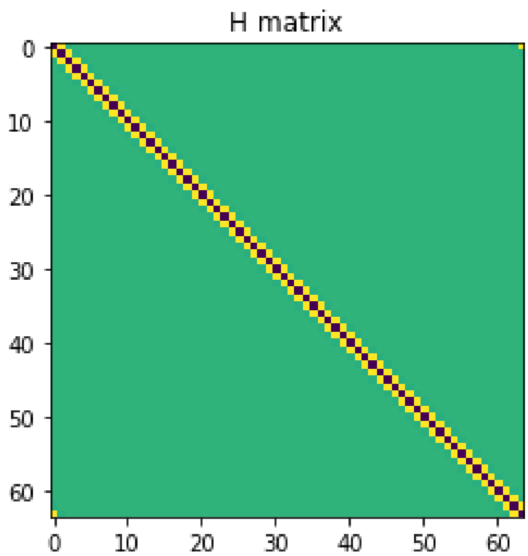



```
the mse for sample: 4007 is: 0.22276866744094342
```

As we can see we managed to denoise the signals well, but not to the point of $MSE=0$.
For each denoise realization, we computed the MSE with respect to the clean version of the same realization and averaged all the values. The average MSE that we got was 0.22932

c.

Here we present the matrix H:

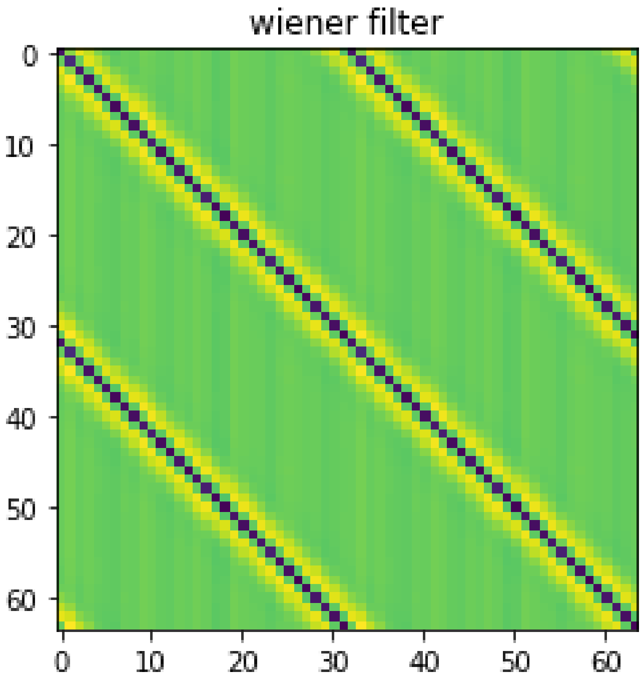


The autocorrelation matrix doesn’t change.

The Weiner filter for this problem is again computed by the formulae:

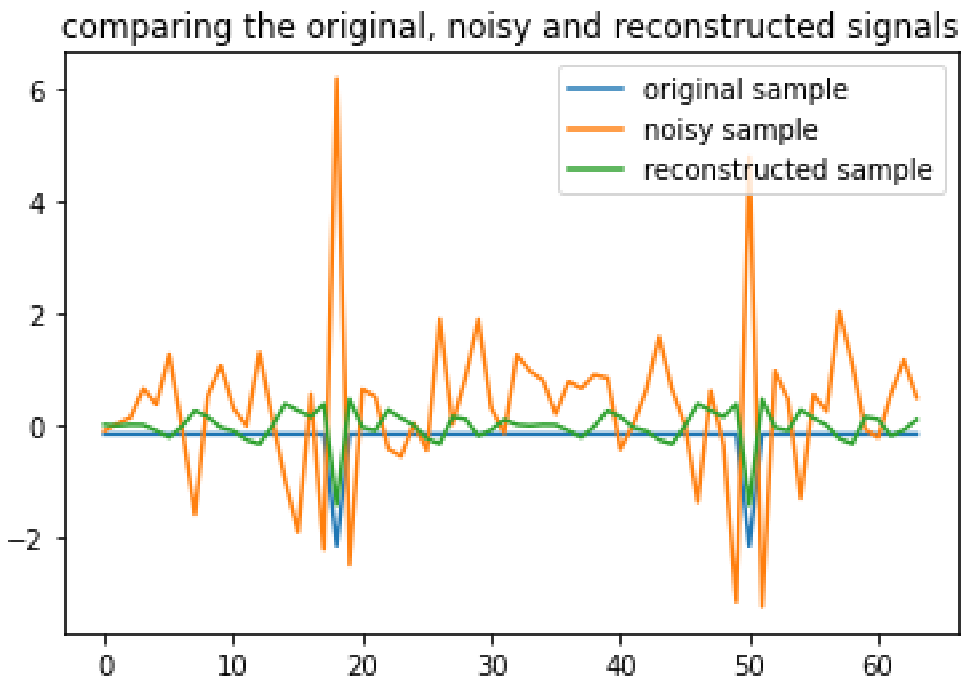
$$W=R_{\phi}H^{T}(HR_{\phi}H^{T}+(\sigma_{n})^{2}I)^{-1}$$

And the matrix itself is:

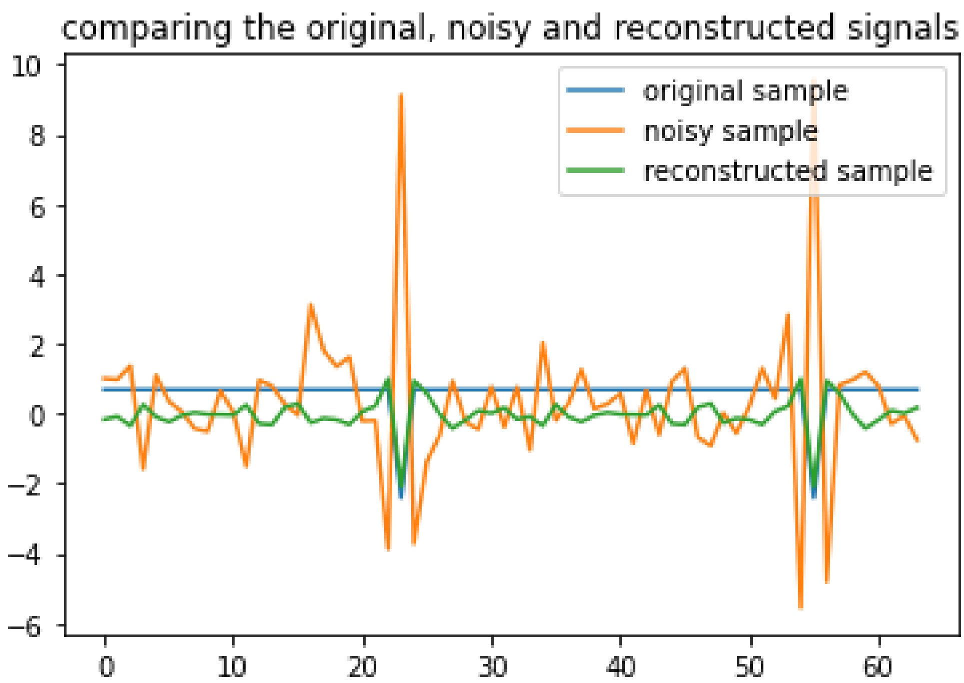


Again, it’s a circulant matrix as expected.

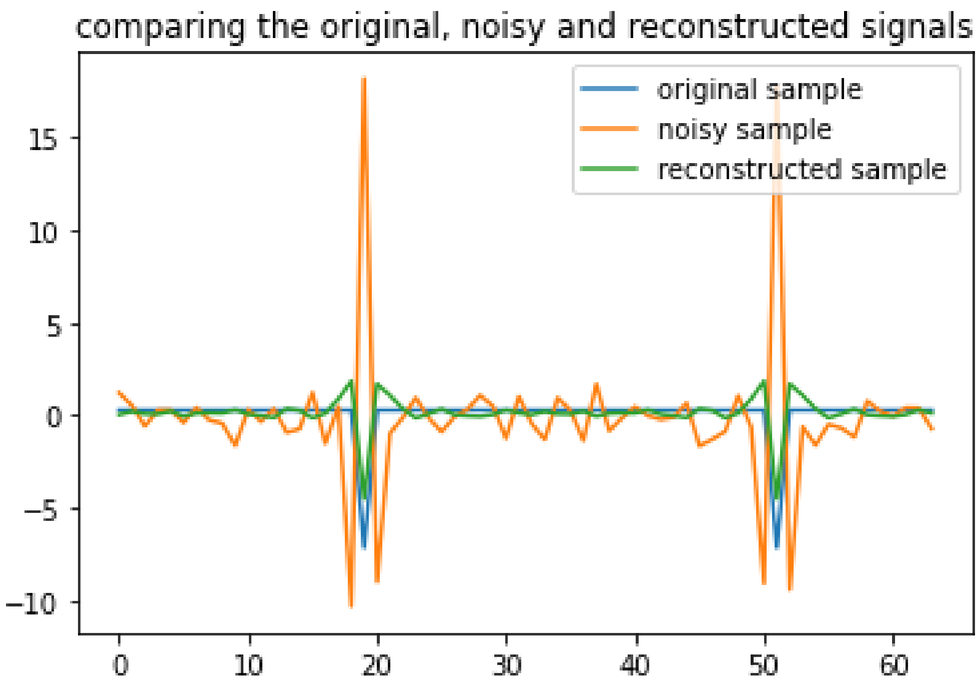
We again examine three samples:



the mse for sample: 71 is: 0.08949026240729514



the mse for sample: 308 is: 0.5271614821295815



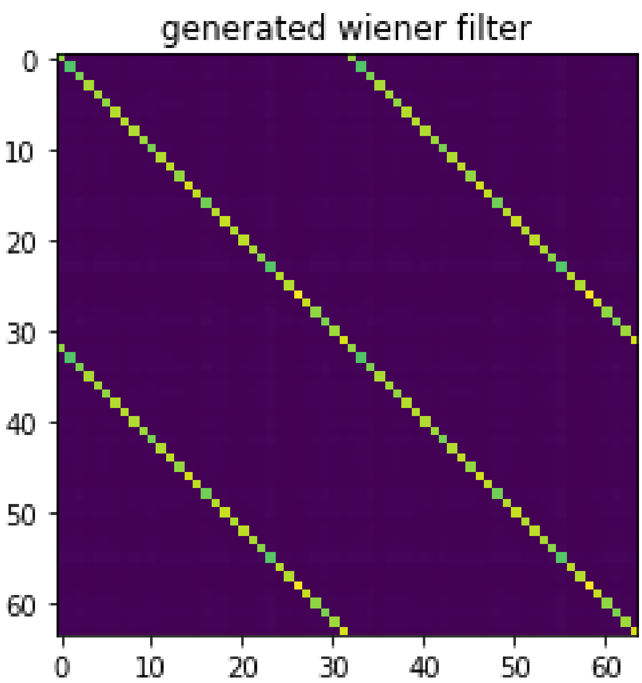

```
the mse for sample: 4007 is: 0.42955218934825595
```

We can see again that the filter worked well approximating all the edges and lowering the signals to a somewhat approximate clean level. The average MSE received was:

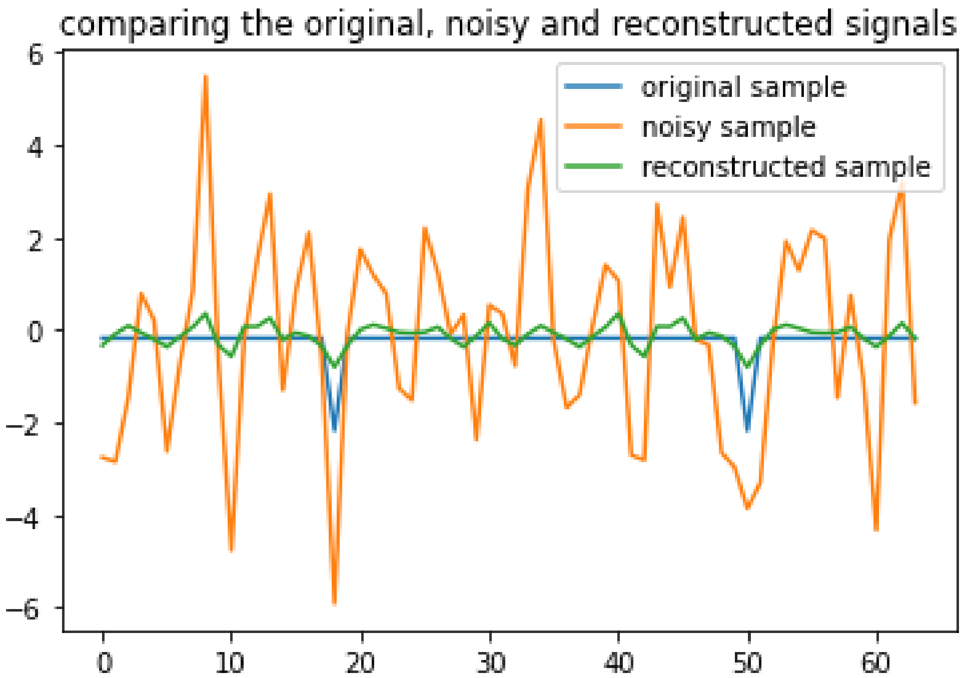
```
0.766576647971665
```

d.

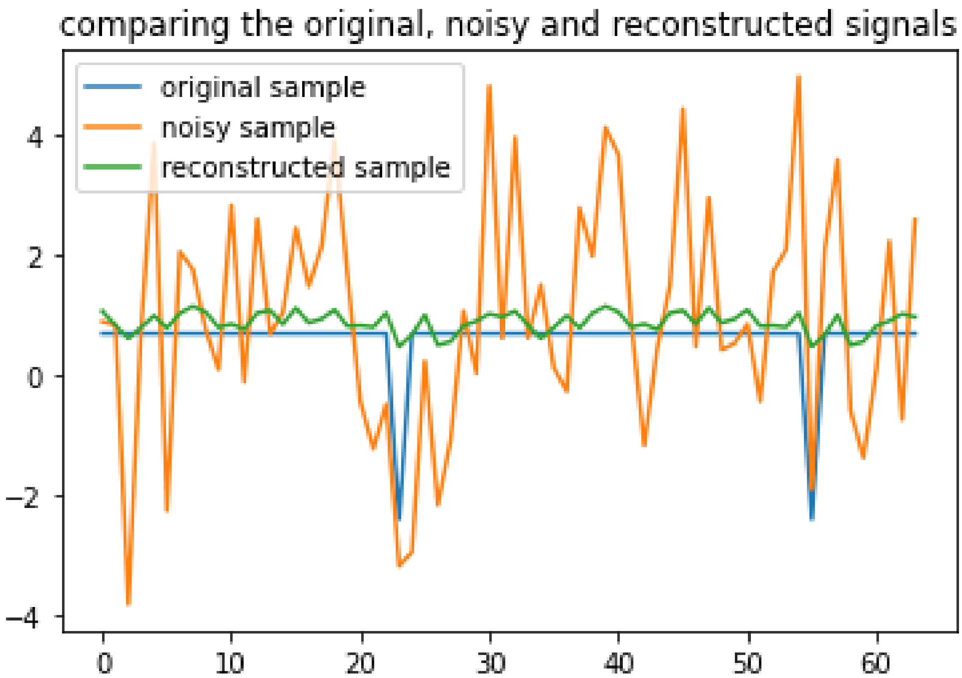
Setting $\sigma_n^2=5$ we get the following results:
For the first Weiner filter we can see that the values have been decreased:



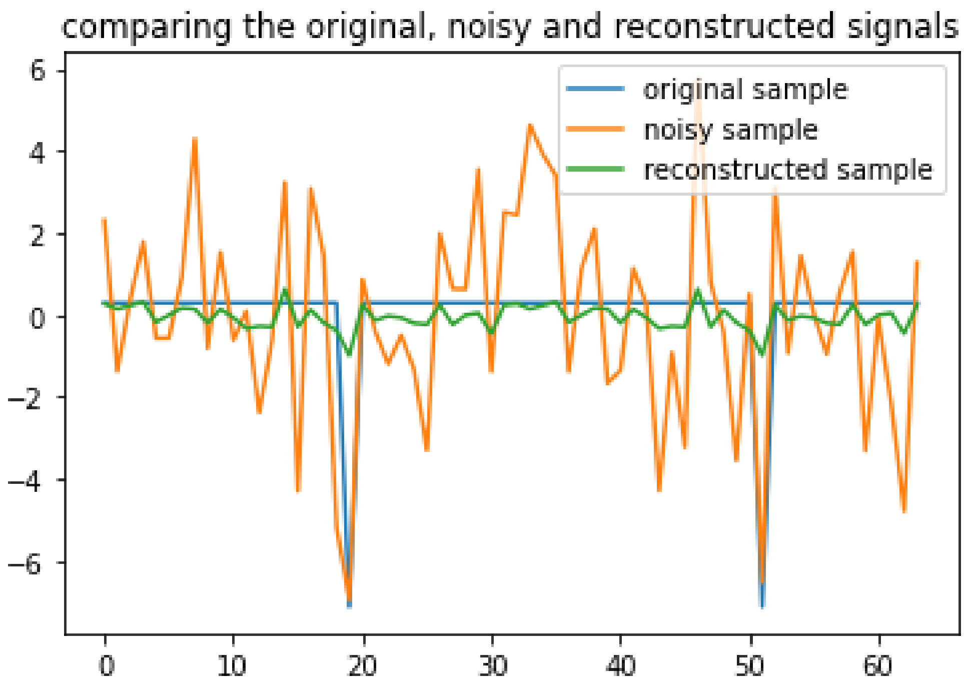
And the Samples are:




```
the mse for sample: 71 is: 0.1046399966323058
```



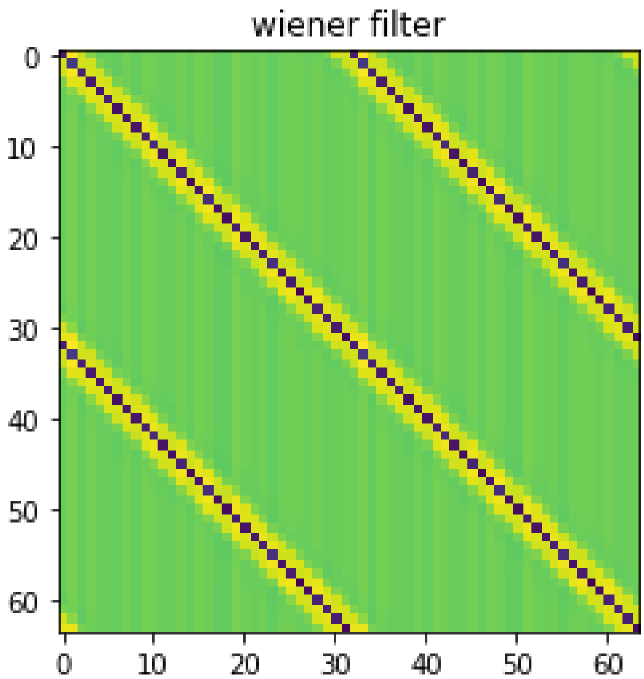

```
the mse for sample: 308 is: 0.32144790488100305
```



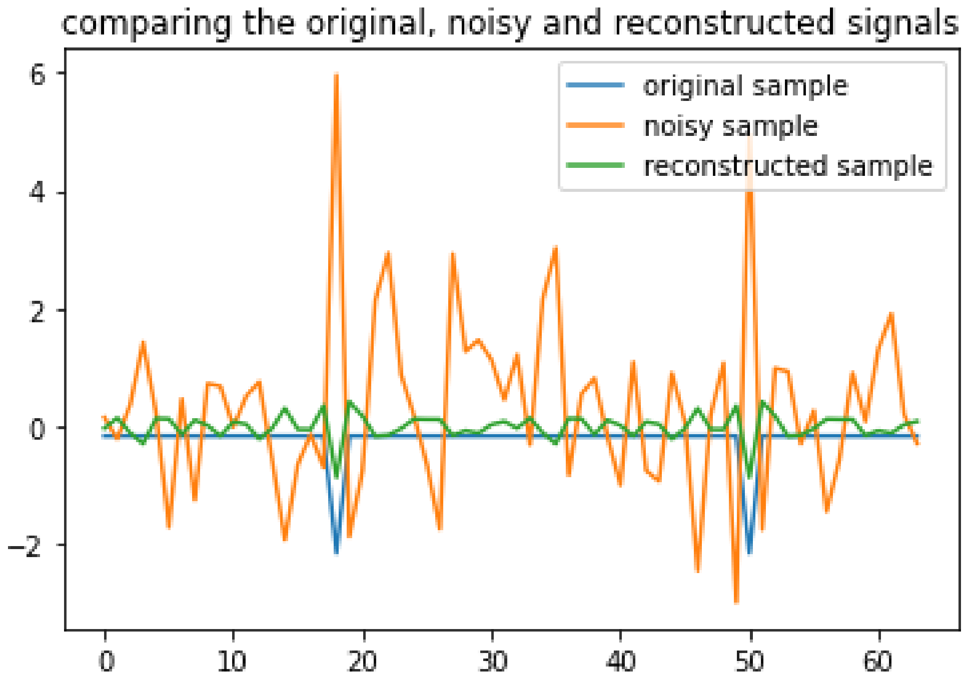

```
the mse for sample: 4007 is: 1.3252289208856771
```

And all the samples give the average MSE: 0.43

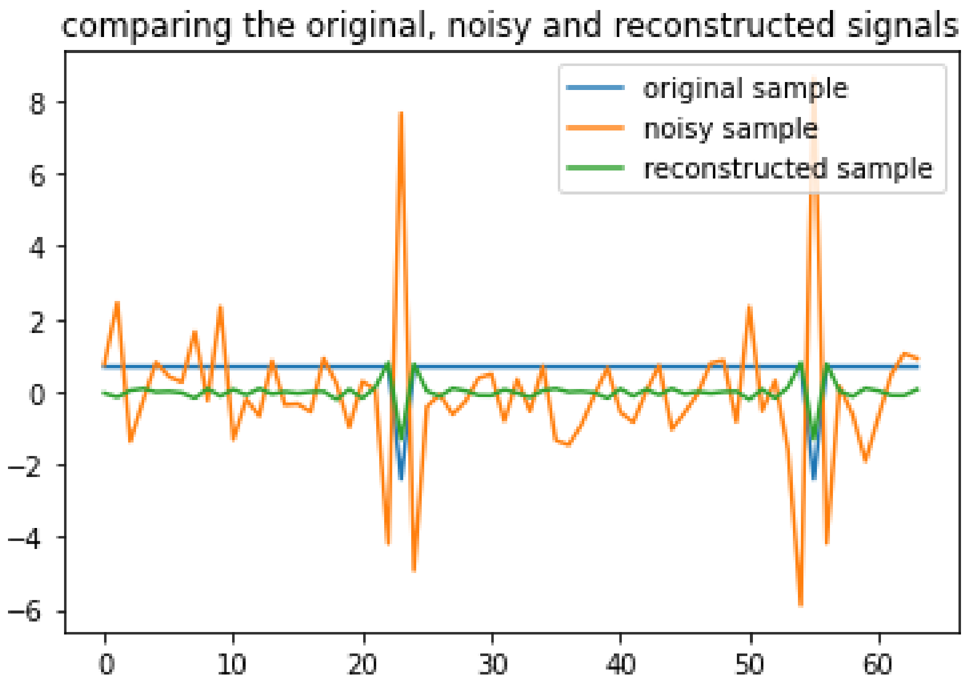
For transformation using H matrix: The Weiner Filter differs again because we use a different σ_n in our calculations of W.



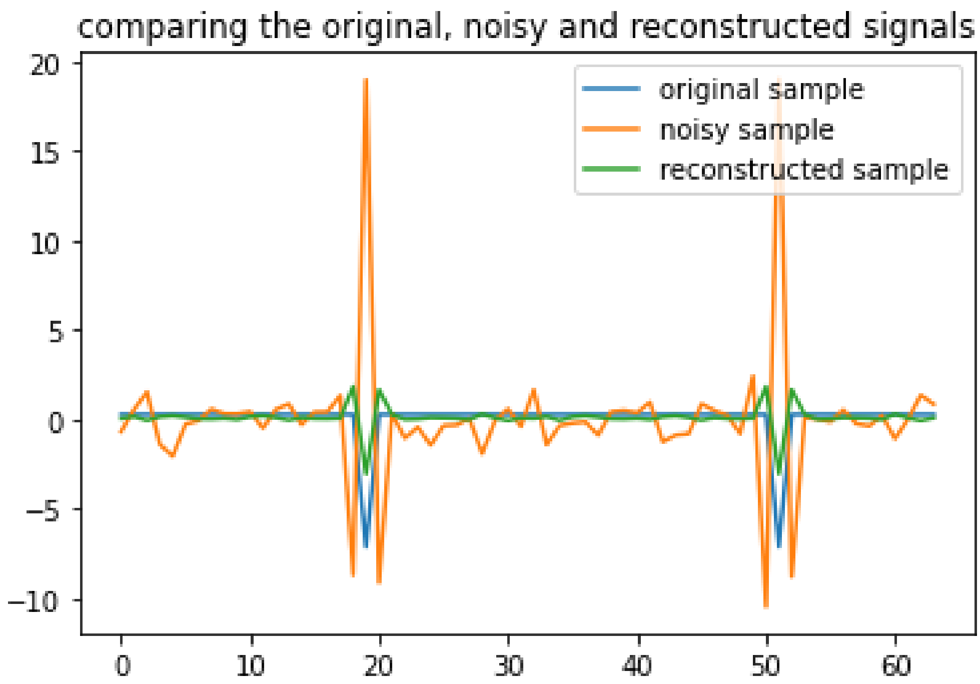
And the samples are:




```
the mse for sample: 71 is: 0.11073238137886912
```




```
the mse for sample: 308 is: 0.50711260487628
```



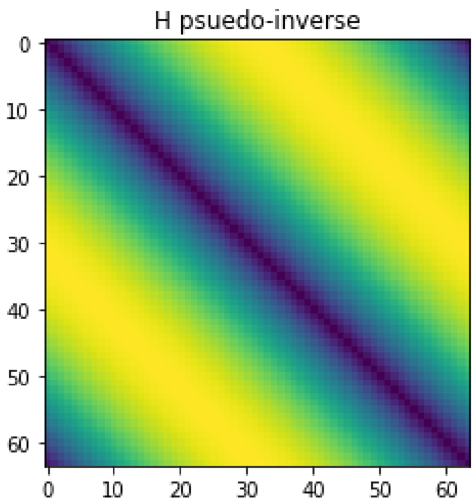

```
the mse for sample: 4007 is: 0.6982612877178969
```

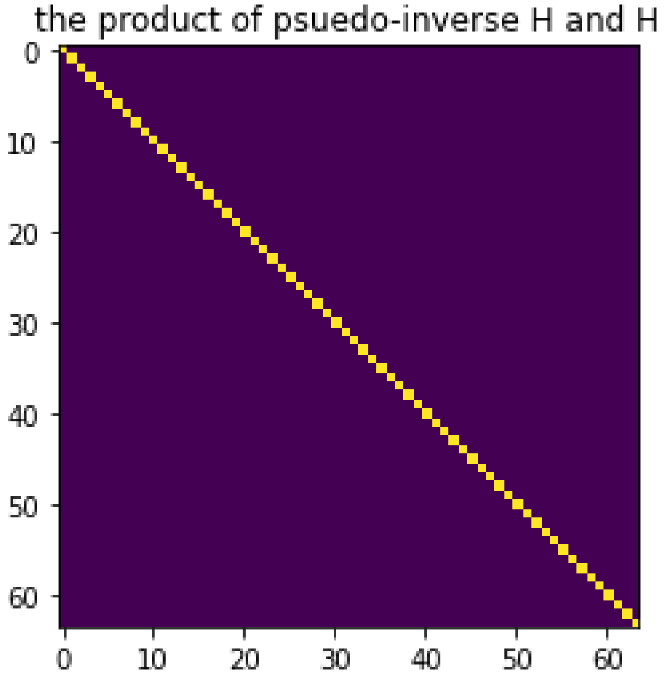
We can see that we are actually able to approximate our signal well.
And all the samples give the average MSE: `0.7992770905892264`.
Overall we can see that increasing the std we will have a harder time to approximate our original signal, because the error margin now is bigger.

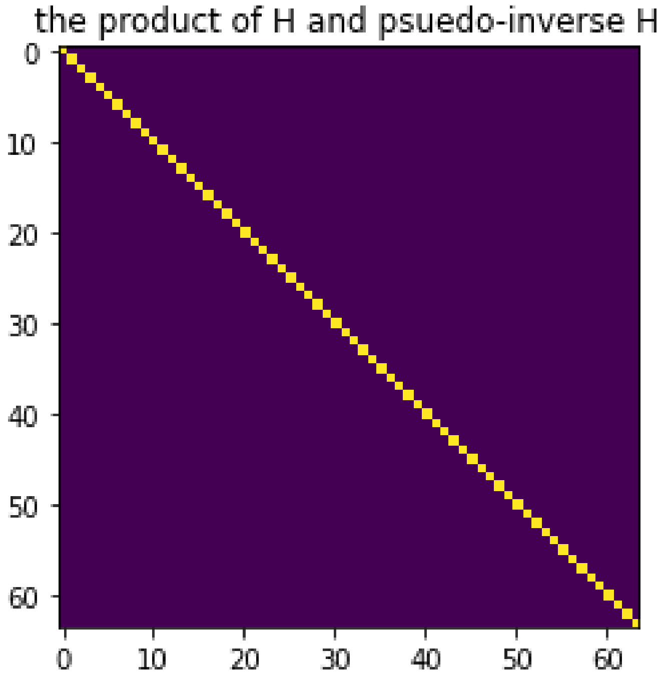
.e

Yes.

The selection of values and calculations are detailed in the code.

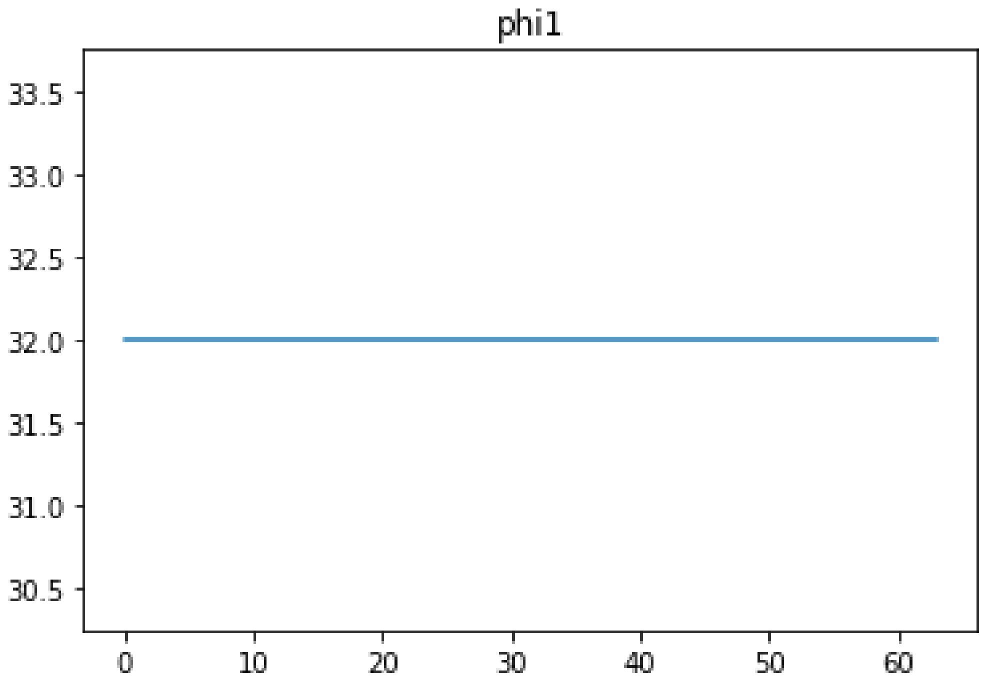


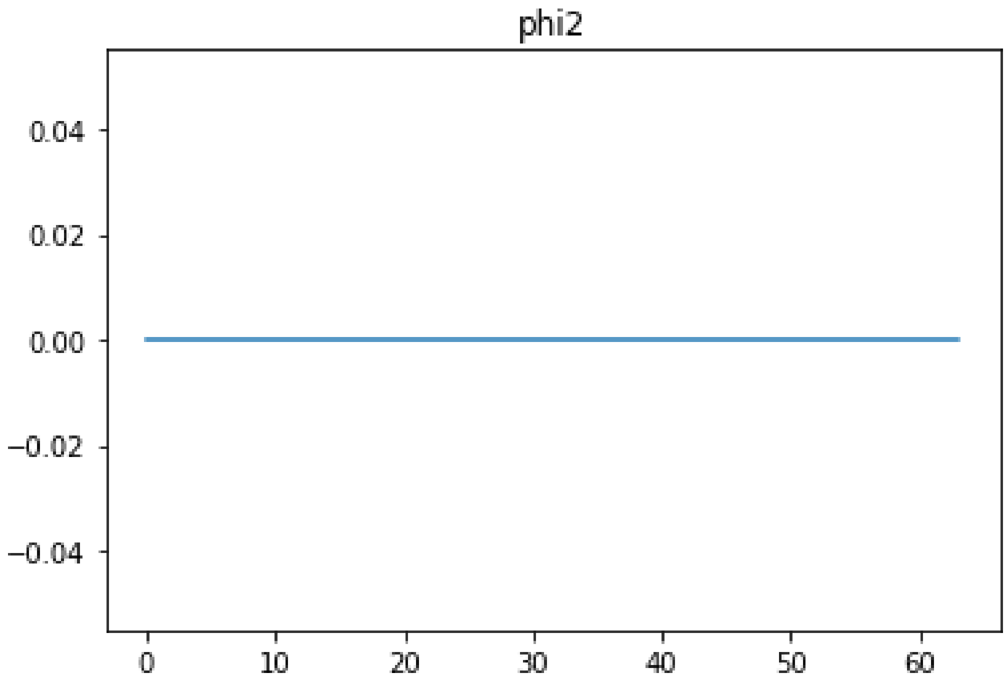


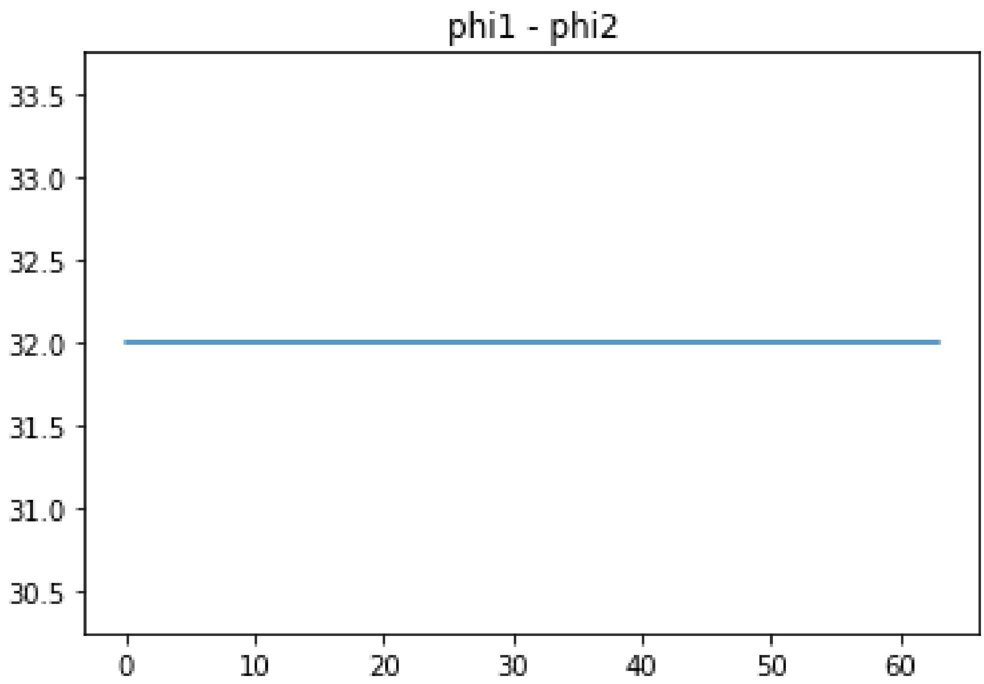


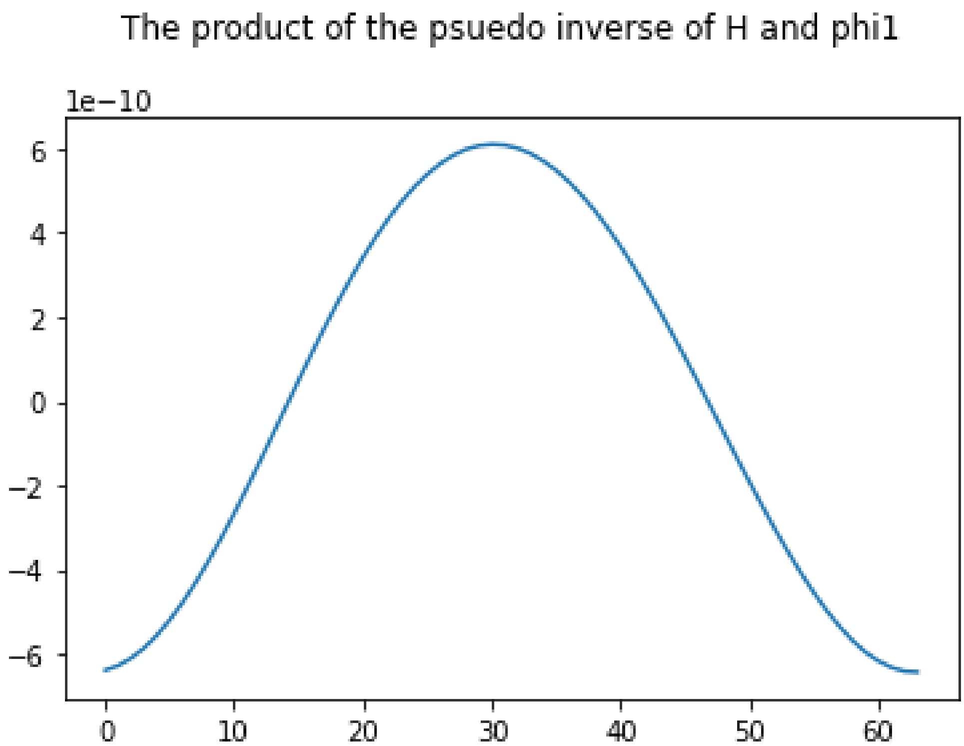
As we can see $H\dagger H$ = $HH\dagger$ Because those are symmetric matrices.

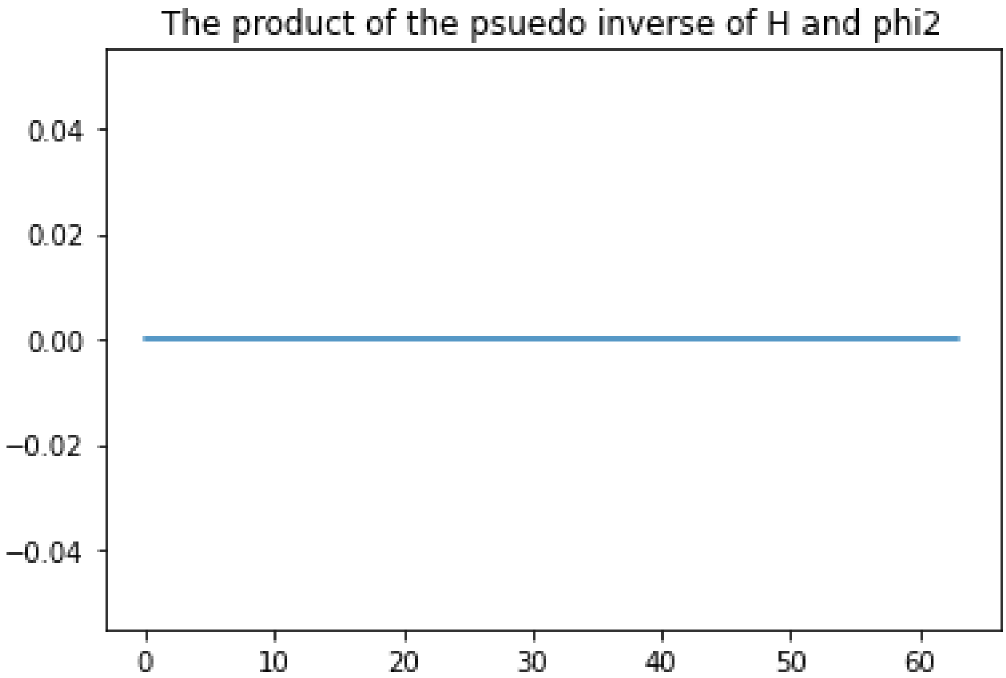
We will plot rest of the signals :

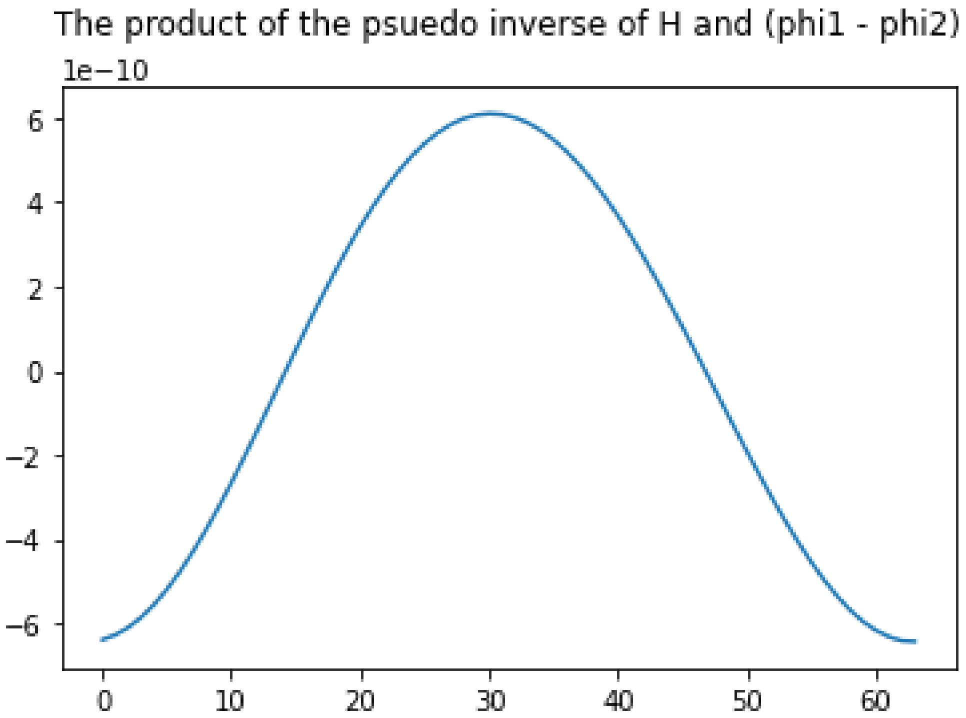





phi1 - phi2
33.5
33.0
32.5
32.0
31.5
31.0
30.5
0
10
20
30
40
50
60


The product of the psuedo inverse of H and phi1
1e−10
6
4
2
0
−2
−4
−6
0
10
20
30
40
50
60





Which fits the theory because the multiplications are almost zero (due to numerical errors) and the norms equal to zero too.