HW2 - 236201

313511602
931190987

HW2 - theory

Q1

a.

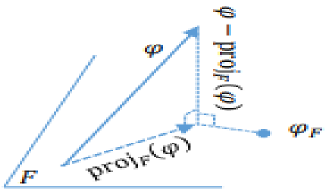
a)

First, we want to denote the n-term approximation of f in F using $Vec(\beta_{i_1}, \dots, \beta_{i_n})$.

Let us define this approximation as $\tilde{f}_n(t) = \sum_{i=i_1}^{i=i_n} f_i \beta_i(t)$ where $\{f_i\}_{i=i_1}^{i=i_n}$ are the representation coefficients.

The SE is:

$$\left\| f - \tilde{f} \right\|_2^2 = \left\| \underbrace{f - proj_F(f)}_{\perp F} + \underbrace{proj_F(f) - \tilde{f}_n}_{\in F} \right\|_2^2 \underbrace{=}_{\text{Pythagoras}} \underbrace{\| f - proj_F(f) \|_2^2}_{\text{constant}} + \underbrace{\left\| proj_F(f) - \tilde{f}_n \right\|_2^2}_{we\ want\ to\ miinimize}$$



At the draw $\varphi = f_n$.

As we can see, first element is a constant, so for minimizing the SE we must minimize the second element. We can see that choosing $\tilde{f} = proj_F(f)\ will\ give$ us zero, and because

$\left\| proj_F(f) - \tilde{f}_n \right\|_2^2 \geq 0$ it is the best choose for minimizing.

We also know from linear algebra $proj_F(f) = \sum_{i=i_1}^{i=i_n} \underbrace{< f, \beta_i >}_{\text{L2 is a Euclidean space}} \beta_i$.

With this knowledge we can say $\tilde{f}_n = \sum_{i=i_1}^{i=i_n} < f, \beta_i > \beta_i$

and SE is $\|f - proj_F(f)\|_2^2 \underbrace{=}_{Pythagoras} \|f\|_2^2 - \|proj_F(f)\|_2^2 = \underbrace{\int_{\mathbb{R}} |f|^2}_{f \in L^2(\mathbb{R},\mathbb{C})} - ||\left(\sum_{i=i_1}^{i=i_n} < f, \beta_i > \beta_i\right)||_2^2 \underbrace{=}_{(\beta_{i_1},\dots,\beta_{i_n})\text{ orthonormal functions}} \int_{\mathbb{R}} |f|^2 - \sum_{i=i_1}^{i=i_n} < f, \beta_i >^2$

We can see in the same way that for k-term approximation,

$\tilde{f_k}(t) = \sum_{i=i_1}^{i=i_k} < f(t), \beta_i(t) > \beta_i(t)$ where $\tilde{f_k}(t)$ is the k-term approximation of f in F.

So, we get $SE = \int_{\mathbb{R}} |f(t)|^2 - \sum_{i=i_1}^{i=i_k} < f(t), \beta_i(t) >^2$

Also, we can see from here that for minimizing the SE we have to take the k functions which $< f, \beta_i >$ are the largest (so $\sum_{i=i_1}^{i=i_k} < f, \beta_i >^2$ will be maximum) .

b)

The k functions for the best k-term approximation of f in F in the SE sense from $\binom{n}{k}$ possibilities are the k functions which $< f, \beta_i >$ are the largest. The reason is as we explain above for minimizing the SE we need to maximize $\sum_{i=i_1}^{i=i_k} < f, \beta_i >^2$.

We can get those by sorting all the $(< f, \beta_i >)^2$ large to small, then we will take the corresponding first k β_i's from this sorted list.

Therefore the $associeted\ SE = \int_{\mathbb{R}} |f(t)|^2 - \sum_{i=i_1}^{i=i_k} < f(t), \beta_i(t) >^2$ and choosing those $\beta_i's$ will minimize it the most.

It can happen that there will be $j_1 \neq j_2\ which\ \left(< f, \beta_{j_1} >\right)^2 = \left(< f, \beta_{j_2} >\right)^2$ where we chose β_{j_1} for our k-term approximation of f in F in the SE sense, as we explained above, but β_{j_2} isn't in this choice ($\beta_{j_2} should\ be\ the\ k+1\ choice$). We can see that choosing the same group of function but replacing $\beta_{j_1}\ with\ \beta_{j_2}$ will give us the same SE so the k-term approximation <u>isn't unique</u>.

b.

a)

As we saw at the last section we can say:

$f_n = \sum_{i=i_1}^{i=i_n} < f, \beta_i > \beta_i\ and\ \tilde{f_n} = \sum_{i=i_1}^{i=i_n} < f, \tilde{\beta_i} > \tilde{\beta_i}$

Where f_n will be n-approximation of f in F in the SE sense using $\beta_i's\ family$

and $\tilde{f_n}$ will be n-approximation of f in F in the SE sense using $\tilde{\beta_i}'s\ family$.

Both families are orthonormal bases of F (F is a subspace of E of finite dimension n).

So, we can say $f_n = proj_F(f) = \tilde{f_n}$, both n-approximation are the projections of f in F, which is a space of size n, means both n-approximation are the same.

b)

We can say now that because they are different finite families, the k-term approximation in this case will be apparently different because the projection will be different.

But, It's possible that $\beta_i' s\ family \neq \tilde{\beta}_i' s\ family$ but for $i = i_1, \ldots, i_k\ (k < n)\ \beta_i = \tilde{\beta}_i$ (as we saw above we have to choose the k biggest $< f, \beta_i >\ or\ < f, \tilde{\beta}_i >$).

In this case obviously the k-term approximation will be the same for both families, basically we can say that k-term approximation on each family probably won't be the same, but there is case that it will be.

Q2) Haar matrix and Walsh-Hadamard matrix

a.

i)

We need to show $H_4^* \cdot H_4 = I_{4\times 4} = H_4 \cdot H_4^*$, and therefore H_4 is unitary.

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii)

We could construct the orthonormal Haar functions $\{\psi_i^H(t)\}_{i=1}^4$ with direct transform:

$$(\psi_1^H(t), \psi_2^H(t), \psi_3^H(t), \psi_4^H(t)) = \sqrt{4} \cdot \left(1\Delta_1(t), 1\Delta_2(t), 1\Delta_3(t), 1\Delta_4(t)\right) H_4$$

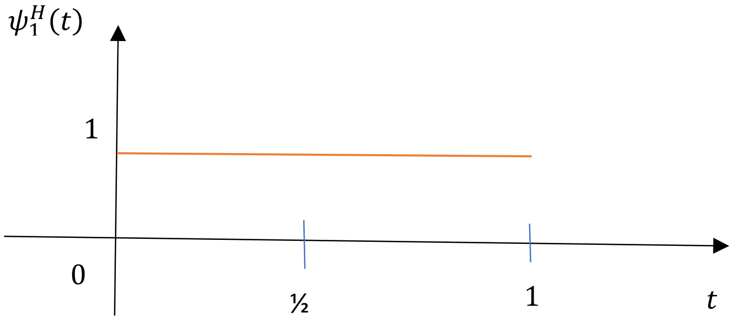
Where $\left(1\Delta_1(t), 1\Delta_2(t), 1\Delta_3(t), 1\Delta_4(t)\right)$ are the standard basis function and $\sqrt{4}$ is for normalization.

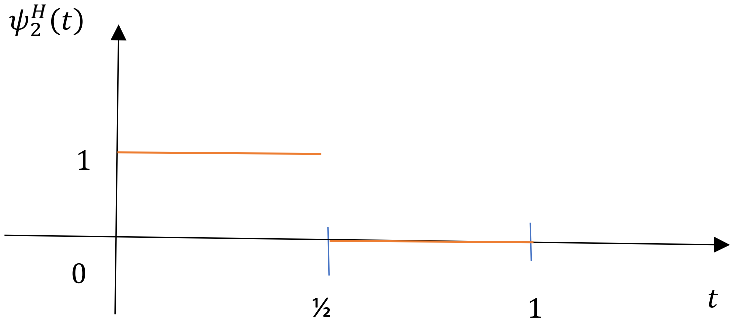
By that, we get:

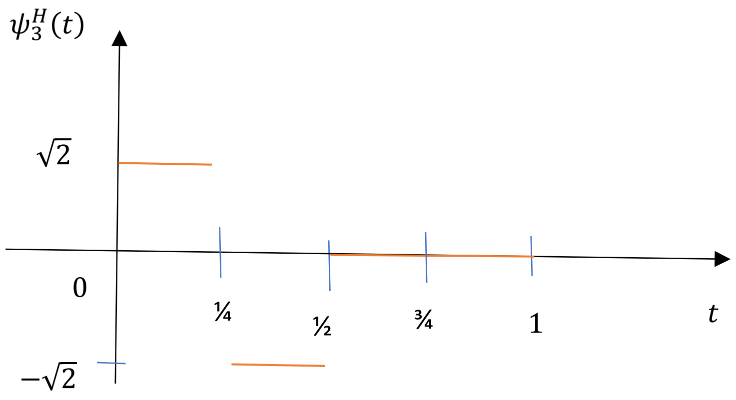
$$(\psi_1^H(t), \psi_2^H(t), \psi_3^H(t), \psi_4^H(t)) = 2 \cdot \left(1\Delta_1(t), 1\Delta_2(t), 1\Delta_3(t), 1\Delta_4(t)\right) \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix} =$$

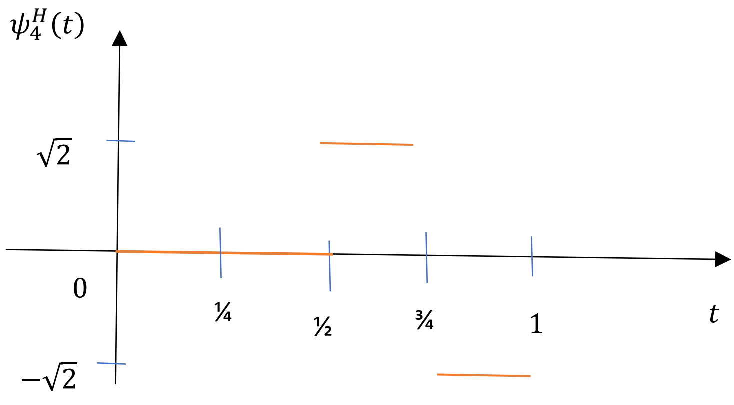
$$\left(1_{[0,1]}(t)\,, 1_{[0,\frac{1}{2})}(t) + \left(-1_{[\frac{1}{2},1]}(t)\right), \sqrt{2}_{[0,\frac{1}{4})}(t) + (-\sqrt{2}_{[\frac{1}{4},\frac{1}{2})}(t)) + 0_{[\frac{1}{2},1]}(t), 0_{[0,\frac{1}{2})}(t) + \sqrt{2}_{[\frac{1}{2},\frac{3}{4})}(t) + (-\sqrt{2}_{[\frac{3}{4},1]}(t))\right)$$

$$\begin{pmatrix} \psi_1^H(t) \\ \psi_2^H(t) \\ \psi_3^H(t) \\ \psi_4^H(t) \end{pmatrix} = \begin{pmatrix} 1_{[0,1]}(t) \\ 1_{[0,\frac{1}{2})}(t) + \left(-1_{[\frac{1}{2},1]}(t)\right) \\ \sqrt{2}_{[0,\frac{1}{4})}(t) + (-\sqrt{2}_{[\frac{1}{4},\frac{1}{2})}(t)) + 0_{[\frac{1}{2},1]}(t) \\ 0_{[0,\frac{1}{2})}(t) + \sqrt{2}_{[\frac{1}{2},\frac{3}{4})}(t) + (-\sqrt{2}_{[\frac{3}{4},1]}(t)) \end{pmatrix}$$









<u>Note</u>: outside the boundary of $\boldsymbol{t} \in [\mathbf{0}, \mathbf{1}]$, $\psi_i^H(t)$ isn't define.

iii)

The approximation of ϕ using the Haar basis is $\tilde{\phi}(t) = \sum_{i=1}^{4} \phi_i^{opt}\, \psi_i^H(t)$ and from the tutorial we know the best coefficients for each i in the MSE sense are $\phi_i^{opt} =< \phi(t), \psi_i^H(t) >$ so,

$$\tilde{\phi}(t) = \sum_{i=1}^{4} < \phi(t), \psi_i^H(t) > \psi_i^H(t)$$

1 . $< \phi(t), \psi_1^H(t) >=< a + bcos(2\pi t) + c \cdot cos^2(\pi t), 1_{[0,1]}(t) > = \int_0^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]^* \, 1_{[0,1]}(t)dt = \int_0^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]\, dt = a \int_0^1 [1]\, dt + b \int_0^1 [cos(2\pi t)]\, dt + c \int_0^1 [cos^2(\pi t)]\, dt = a + 0 + c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_0^1 = a + \frac{c}{2}$

2. $< \phi(t), \psi_2^H(t) >=< a + bcos(2\pi t) + c \cdot cos^2(\pi t), 1_{[0,\frac{1}{2})}(t) + \left(-1_{[\frac{1}{2},1]}(t)\right) > =$

$\int_0^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]^* \left(1_{[0,\frac{1}{2})}(t) + \left(-1_{[\frac{1}{2},1]}(t)\right)\right) dt \underbrace{=}_{linearity}$

$\int_0^{\frac{1}{2}} [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]\, dt - \int_{\frac{1}{2}}^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]\, dt = \left(\frac{a}{2} + 0 + c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_0^{\frac{1}{2}}\right) - \left(\frac{a}{2} + 0 + c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_{\frac{1}{2}}^1\right) = \frac{a}{2} + \frac{c}{4} - \frac{a}{2} - \frac{c}{4} = 0$

3. $< \phi(t), \psi_3^H(t) >=< a + bcos(2\pi t) + c \cdot cos^2(\pi t), \sqrt{2}_{[0,\frac{1}{4})}(t) + \ (-\sqrt{2}_{[\frac{1}{4},\frac{1}{2})}(t)) + 0_{[\frac{1}{2},1]}(t) > =$

$\int_0^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]^* \left(\sqrt{2}_{[0,\frac{1}{4})}(t) + \ (-\sqrt{2}_{[\frac{1}{4},\frac{1}{2})}(t)) + 0_{[\frac{1}{2},1]}(t)\right) dt \underbrace{=}_{linearity}$

$\sqrt{2} \int_0^{\frac{1}{4}} [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]\, dt - \sqrt{2} \int_{\frac{1}{4}}^{\frac{1}{2}} [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]\, dt + 0 \cdot \ldots =$

$\sqrt{2}\left(\frac{a}{4} + \frac{b}{2\pi}[\sin(2\pi t)]_0^{\frac{1}{4}} + c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_0^{\frac{1}{4}}) - \sqrt{2}(\frac{a}{4} + \frac{b}{2\pi}[\sin(2\pi t)]_{\frac{1}{4}}^{\frac{1}{2}} + c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_{\frac{1}{4}}^{\frac{1}{2}})\right) =$

$\sqrt{2}\left(0 + \frac{b}{\pi} + \frac{c}{2\pi}\right) = \frac{\sqrt{2}}{\pi}(b + \frac{c}{2})$

4. $<\phi(t),\psi_4^H(t)> = < a + bcos(2\pi t) + c\cdot cos^2(\pi t), 0_{[0,\frac{1}{2})}(t) + \sqrt{2}_{[\frac{1}{2},\frac{3}{4})}(t) + (-\sqrt{2})_{[\frac{3}{4},1]}(t) > =$

$$\int_0^1 [a + bcos(2\pi t) + c\cdot cos^2(\pi t)]^* \left(0_{[0,\frac{1}{2})}(t) + \sqrt{2}_{[\frac{1}{2},\frac{3}{4})}(t) + (-\sqrt{2})_{[\frac{3}{4},1]}(t)\right) dt \underbrace{=}_{linearity}$$

$$0\cdot \ldots + \sqrt{2}\int_{\frac{1}{2}}^{\frac{3}{4}} [a + bcos(2\pi t) + c\cdot cos^2(\pi t)]\, dt - \sqrt{2}\int_{\frac{3}{4}}^{1} [a + bcos(2\pi t) + c\cdot cos^2(\pi t)]\, dt =$$

$$\sqrt{2}\left(\frac{a}{4} + \frac{b}{2\pi}[\sin(2\pi t)]_{\frac{1}{2}}^{\frac{3}{4}} + \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_{\frac{1}{2}}^{\frac{3}{4}}\right) - \sqrt{2}\left(\frac{a}{4} + \frac{b}{2\pi}[\sin(2\pi t)]_{\frac{3}{4}}^{1} + \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_{\frac{3}{4}}^{1}\right) =$$

$$\sqrt{2}\left(0 - \frac{b}{\pi} - \frac{c}{2\pi}\right) = -\frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)$$

By that we got:

$$\tilde{\phi}(t) = \left(a + \frac{c}{2}\right)\psi_1^H(t) + 0\cdot\psi_2^H(t) + \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\psi_3^H(t) - \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\psi_4^H(t) = \left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{[0,\frac{1}{4})} - \left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{[\frac{1}{4},\frac{1}{2})} \left(a + \frac{c}{2} - \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{[\frac{1}{2},\frac{3}{4})} + \left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{[\frac{3}{4},1]},$$

Associated MSE is:

$$\int_0^1 \left(\phi(t) - \tilde{\phi}(t)\right)^2 dt = \int_0^1 \phi^2(t)\, dt - 2\int_0^1 \phi(t)\,\tilde{\phi}(t) dt + \int_0^1 \tilde{\phi}^2(t)\, dt \underbrace{=}_{as\ \ we\ saw\ in\ class} \int_0^1 \phi^2(t)\, dt - \sum_{i=1}^4 \left(\phi_i^{opt}\right)^2 dt =$$

$$\overbrace{\int_0^1 [a + bcos(2\pi t) + c\cdot cos^2(\pi t)]^2\, dt}^{*} - \overbrace{\sum_{i=1}^4 \left(<\phi(t),\psi_i^H(t)>\right)^2}^{**} dt$$

$$*= \int_0^1 [a^2 + 2abcos(2\pi t) + 2ac\cdot cos^2(\pi t) + b^2\cos^2(2\pi t) + 2bc\cdot cos(2\pi t)cos^2(\pi t) + c^2cos^4(\pi t)]dt \underbrace{=}_{linearity} a^2\int_0^1 dt + 2ab\int_0^1 cos(2\pi t)dt + 2ac\int_0^1 cos^2(\pi t)dt +$$

$$b^2\int_0^1 \cos^2(2\pi t)\, dt + 2bc\int_0^1 cos(2\pi t)cos^2(\pi t)dt + c^2\int_0^1 cos^4(\pi t)dt = a^2 + 0 + ac + \frac{b^2}{2} + \frac{bc}{2} + \frac{3c^2}{8}$$

$$**= \left(a + \frac{c}{2}\right)^2 + 0^2 + \left(\frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\right)^2 + \left(-\frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\right)^2 = a^2 + ac + \frac{c^2}{4} + \frac{4}{\pi^2}\left(b^2 + bc + \frac{c^2}{4}\right) =$$

$$a^2 + ac + c^2\left(\frac{1}{4} + \frac{1}{\pi^2}\right) + \frac{4b^2}{\pi^2} + \frac{4bc}{\pi^2}$$

, By that

$$MSE = a^2 + ac + \frac{b^2}{2} + \frac{bc}{2} + \frac{3c^2}{8} \left(a^2 + ac + c^2\left(\frac{1}{4} + \frac{1}{\pi^2}\right) + \frac{4b^2}{\pi^2} + \frac{4bc}{\pi^2}\right) = b^2\left(\frac{1}{2} - \frac{4}{\pi^2} +\right) bc\left(\frac{1}{2} - \frac{4}{\pi^2}\right) + c^2\left(\frac{1}{8} - \frac{1}{\pi^2}\right($$

iv)

First, let us sort the coefficients by the squared coefficients $\left(\phi_i^{opt}\right)^2 = \left(<\phi(t),\psi_i^H(t)>\right)^2$ *in decreasing order, we can simply sort the absolute values of each:*

$$|\phi_1^{opt}| = a + \frac{c}{2} > |\phi_4^{opt}| = \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right) = |\phi_3^{opt}| > |\phi_2^{opt}| = 0$$

Because $a + \frac{c}{2} \underbrace{\geq}_{a\geq b\geq 0 \text{ and } c\geq 0} b + \frac{c}{2}$ *and* $\frac{\sqrt{2}}{\pi} < 1$ *and for each* $x \in \mathbb{R}, |x| \geq 0$

As we saw in class, for k-term approximation $MSE = \int_0^1 \phi^2(t)\,dt - \sum_{i=1}^{k}\left(\phi_i^{opt}\right)^2 dt$ *and for minimizing it we have to choose the first k functions with the biggest squared projections (maximize* $\sum_{i=1}^{k}\left(\phi_i^{opt}\right)^2$ *will minimize the MSE).*

1. ***The best 1-term approximation(k=1)***

$$\tilde{\phi}_1(t) = \phi_1^{opt} \cdot \psi_1^H(t) = \left(a + \frac{c}{2}\right) \cdot 1_{[0,1]}(t)$$

2. ***The best 2-term approximation(k=2)***

$\tilde{\phi}_2(t) = \phi_1^{opt} \cdot \psi_1^H(t) + \phi_4^{opt}\psi_4^H(t) = \left(a + \frac{c}{2}\right) \cdot 1_{[0,1]}(t) - \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\left(0_{\left[0,\frac{1}{2}\right)}(t) + \sqrt{2}_{\left[\frac{1}{2},\frac{3}{4}\right)}(t) + \left(-\sqrt{2}_{\left[\frac{3}{4},1\right]}(t)\right)\right) = \left(a + \frac{c}{2}\right)_{\left[0,\frac{1}{2}\right)}(t) + \left(a + c\left(\frac{1}{2} + \frac{1}{\pi}\right) + \frac{2b}{\pi}\right)_{\left(\frac{1}{2},\frac{3}{4}\right)} + \left(a + c\left(\frac{1}{2} - \frac{1}{\pi}\right) - \frac{2b}{\pi}\right)_{\left[\frac{3}{4},1\right]}$

3. ***The best 3-term approximation(k=3)***

We can see $\phi_2^{opt} = 0$ *, so 3-term approximation will be the same as 4-term approximation, and they both will be the same as we saw at section (iii) because this Haar basis is 4-dimension. So, we get*

$\tilde{\phi}_3(t) = \tilde{\phi}_4(t) = \left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left[0,\frac{1}{4}\right)} + \left(a + \frac{c}{2} - \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left[\frac{1}{4},\frac{1}{2}\right)} + \left(a + \frac{c}{2} - \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left[\frac{1}{2},\frac{3}{4}\right)} + \left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left[\frac{3}{4},1\right]}$

v)

$$a = \frac{1}{\pi}, b = 1, c = \frac{3}{2}$$

First, let us sort the coefficients by the squared coefficients $\left(\phi_i^{opt}\right)^2 = \left(<\phi(t), \psi_i^H(t)>\right)^2$ *in decreasing order, we can simply sort by the absolute values of each:*

$$\phi_1^{opt} = a + \frac{c}{2} = \frac{1}{\pi} + \frac{3}{4} > |\phi_4^{opt}| = \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right) = \frac{7\sqrt{2}}{4\pi} = |\phi_3^{opt}| > |\phi_2^{opt}| = 0$$

1. ***The best 1-term approximation(k=1)***

$\tilde{\phi}_1(t) = \phi_1^{opt} \cdot \psi_1^H(t) = \left(\frac{1}{\pi} + \frac{3}{4}\right) \cdot 1_{[0,1]}(t) = 1.068 \cdot 1_{[0,1]}(t)$

2. ***The best 2-term approximation(k=2)***

$\tilde{\phi}_2(t) = \phi_1^{opt} \cdot \psi_1^H(t) + \phi_4^{opt}\psi_2^H(t) = \left(a + \frac{c}{2}\right)_{\left[0,\frac{1}{2}\right)}(t) + \left(a + c\left(\frac{1}{2} + \frac{1}{\pi}\right) + \frac{2b}{\pi}\right)_{\left(\frac{1}{2},\frac{3}{4}\right)} + \left(a + c\left(\frac{1}{2} - \frac{1}{\pi}\right) - \frac{2b}{\pi}\right)_{\left[\frac{3}{4},1\right]} = (1.068)_{\left[0,\frac{1}{2}\right)}(t) + (2.182)_{\left(\frac{1}{2},\frac{3}{4}\right)} - (0.0458)_{\left[\frac{3}{4},1\right]}$

3. *The best 3-term approximation(k=3), as we said 3-term approximation = 4-term approximation*

$$\tilde{\phi}_3(t) = \tilde{\phi}_4(t) = \left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left[0,\frac{1}{4}\right)} + \left(a + \frac{c}{2} - \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left[\frac{1}{4},\frac{1}{2}\right)} + \left(a + \frac{c}{2} - \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left[\frac{1}{2},\frac{3}{4}\right)} + \left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left[\frac{3}{4},1\right]} = (2.182)_{\left[0,\frac{1}{4}\right)} + (-0.0458)_{\left[\frac{1}{4},\frac{1}{2}\right)} + (-0.0458)_{\left[\frac{1}{2},\frac{3}{4}\right)} + (2.182)_{\left[\frac{3}{4},1\right]} = (2.182)_{\left[0,\frac{1}{4}\right)} + (-0.0458)_{\left(\frac{1}{4},\frac{3}{4}\right)} + (2.182)_{\left[\frac{3}{4},1\right]}$$

b.

i)

We need to show $W_4^* \cdot W_4 = I_{4\times 4} = W_4 \cdot W_4^*$, and therefore H_4 is unitary.

$(W_4^* = W_4)$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii)

We could construct the orthonormal WH functions $\{\psi_i^W(t)\}^4{}_{i=1}$ with direct transform:

$$(\psi_1^W(t), \psi_2^W(t), \psi_3^W(t), \psi_4^W(t)) = \sqrt{4} \cdot \left(1\Delta_1(t), 1\Delta_2(t), 1\Delta_3(t), 1\Delta_4(t)\right)W_4$$

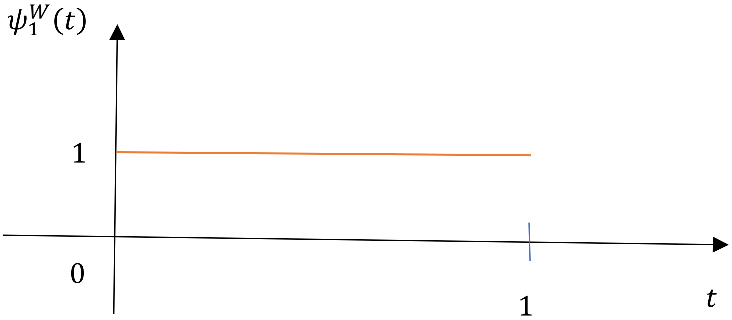
Where $\left(1\Delta_1(t), 1\Delta_2(t), 1\Delta_3(t), 1\Delta_4(t)\right)$ are the standard basis function and $\sqrt{4}$ is for normalization.

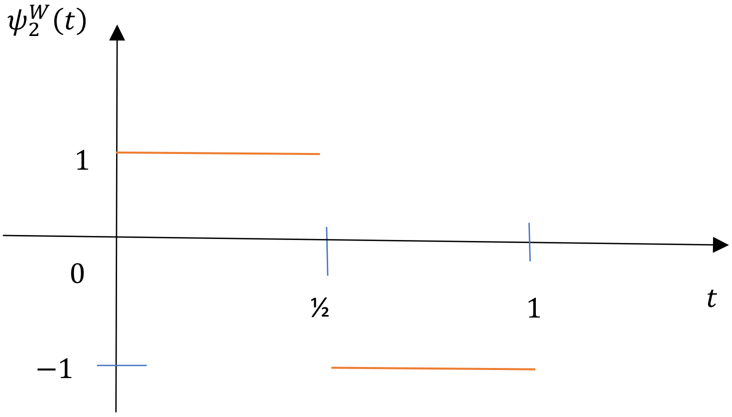
By that, we get:

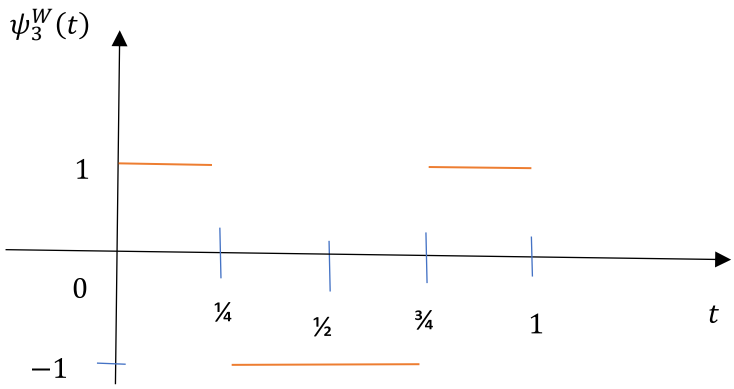
$$(\psi_1^W(t), \psi_2^W(t), \psi_3^W(t), \psi_4^W(t)) = \sqrt{4} \cdot \left(1\Delta_1(t), 1\Delta_2(t), 1\Delta_3(t), 1\Delta_4(t)\right)\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} =$$

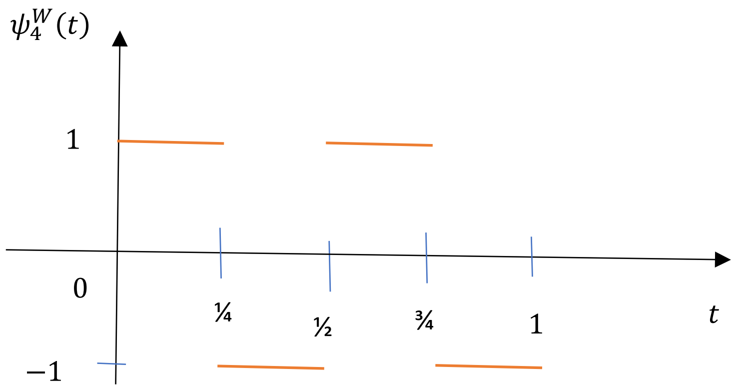
$$\left(1_{[0,1]}(t)\,, 1_{\left[0,\frac{1}{2}\right)}(t) + \left(-1_{\left(\frac{1}{2},1\right]}(t)\right), 1_{\left[0,\frac{1}{4}\right)}(t) + \left(-1_{\left(\frac{1}{4},\frac{3}{4}\right)}(t)\right) + 1_{\left(\frac{3}{4},1\right]}(t), 1_{\left[0,\frac{1}{4}\right)}(t) + \left(-1_{\left(\frac{1}{4},\frac{2}{4}\right)}(t)\right) + 1_{\left(\frac{2}{4},\frac{3}{4}\right)}(t) + \left(-1_{\left(\frac{3}{4},1\right]}(t)\right)\right)$$

$$\begin{pmatrix} \psi_1^W(t) \\ \psi_2^W(t) \\ \psi_3^W(t) \\ \psi_4^W(t) \end{pmatrix} = \begin{pmatrix} 1_{[0,1]}(t) \\ 1_{\left[0,\frac{1}{2}\right)}(t) + \left(-1_{\left(\frac{1}{2},1\right]}(t)\right) \\ 1_{\left[0,\frac{1}{4}\right)}(t) + \left(-1_{\left(\frac{1}{4},\frac{3}{4}\right)}(t)\right) + 1_{\left(\frac{3}{4},1\right]}(t) \\ 1_{\left[0,\frac{1}{4}\right)}(t) + \left(-1_{\left(\frac{1}{4},\frac{2}{4}\right)}(t)\right) + 1_{\left(\frac{2}{4},\frac{3}{4}\right)}(t) + \left(-1_{\left(\frac{3}{4},1\right]}(t)\right) \end{pmatrix}$$









<u>Note</u>: outside the boundary of $\boldsymbol{t} \in [\mathbf{0}, \mathbf{1}]$, $\psi_i^W(t)$ isn't define.

iii)

The approximation of ϕ using the WH basis is $\tilde{\phi}(t) = \sum_{i=1}^{4} \phi_i^{opt}\, \psi_i^W(t)$ and from the tutorial we know the best coefficients for each i in the MSE sense are $\phi_i^{opt} =< \phi(t), \psi_i^W(t) >$ so,

$$\tilde{\phi}(t) = \sum_{i=1}^{4} < \phi(t), \psi_i^W(t) > \psi_i^W(t)$$

$\mathbf{1}\,. < \phi(t), \psi_1^W(t) >=< a + bcos(2\pi t) + c \cdot cos^2(\pi t), 1_{[0,1]}(t) > = \int_0^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]^* \, 1_{[0,1]}(t)dt = \int_0^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]\, dt = \, a \int_0^1 [1]\, dt + b \int_0^1 [cos(2\pi t)]\, dt + c \int_0^1 [cos^2(\pi t)]\, dt = a + 0 + c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_0^1 = a + \frac{c}{2}$

$\mathbf{2}. < \phi(t), \psi_2^W(t) >=< a + bcos(2\pi t) + c \cdot cos^2(\pi t), 1_{[0,\frac{1}{2})}(t) + \left(-1_{[\frac{1}{2},1]}(t)\right) > =$

$\int_0^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]^* \left(1_{[0,\frac{1}{2})}(t) + \left(-1_{[\frac{1}{2},1]}(t)\right)\right) dt \underbrace{=}_{linearity}$

$\int_0^{\frac{1}{2}} [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]\, dt - \int_{\frac{1}{2}}^{1} [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]\, dt = \left(\frac{a}{2} + 0 + c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_0^{\frac{1}{2}}\right) - \left(\frac{a}{2} + 0 + c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_{\frac{1}{2}}^{1}\right) = \frac{a}{2} + \frac{c}{4} - \frac{a}{2} - \frac{c}{4} = 0$

3. $< \phi(t), \psi_3^W(t) > = < a + bcos(2\pi t) + c \cdot cos^2(\pi t), 1_{\left[1,\frac{1}{4}\right)}(t) + \ (-1_{\left(\frac{1}{4},\frac{3}{4}\right)}(t)) + 1_{\left(\frac{3}{4},1\right]}(t) > =$
$\int_0^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]^* \, (1_{\left[1,\frac{1}{4}\right)}(t) + \ (-1_{\left(\frac{1}{4},\frac{3}{4}\right)}(t)) + 1_{\left(\frac{3}{4},1\right]}(t))dt \underbrace{=}_{linearity}$

$\int_0^{\frac{1}{4}} [a + bcos(2\pi t) + c \cdot cos^2(\pi t)] \, dt - \int_{\frac{1}{4}}^{\frac{3}{4}} [a + bcos(2\pi t) + c \cdot cos^2(\pi t)] \, dt + \int_{\frac{3}{4}}^{1} [a +$
$bcos(2\pi t) + c \cdot cos^2(\pi t)] \, dt = \Bigg(\frac{a}{4} + \frac{b}{2\pi} [\sin(2\pi t)]_0^{\frac{1}{4}} + c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2} \right]_0^{\frac{1}{4}} - \frac{a}{2} - \frac{b}{2\pi} [\sin(2\pi t)]_{\frac{1}{4}}^{\frac{3}{4}} -$
$c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2} \right]_{\frac{1}{4}}^{\frac{3}{4}} + \frac{a}{4} + \frac{b}{2\pi} [\sin(2\pi t)]_{\frac{3}{4}}^{1} + c \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2} \right]_{\frac{3}{4}}^{1} \Bigg) = \frac{2}{\pi} (b + \frac{c}{2})$

4. $< \phi(t), \psi_4^W(t) > = < a + bcos(2\pi t) + c \cdot cos^2(\pi t), 1_{\left[1,\frac{1}{4}\right)}(t) + \ (-1_{\left(\frac{1}{4},\frac{2}{4}\right)}(t)) + 1_{\left(\frac{1}{2},\frac{3}{4}\right)}(t) +$
$(-1_{\left(\frac{3}{4},1\right]}(t)) > = \int_0^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]^* \Big(1_{\left[1,\frac{1}{4}\right)}(t) + \ (-1_{\left(\frac{1}{4},\frac{2}{4}\right)}(t)) + 1_{\left(\frac{1}{2},\frac{3}{4}\right)}(t) +$
$(-1_{\left(\frac{3}{4},1\right]}(t)) \Big) \, dt \underbrace{=}_{linearity}$

$\int_0^{\frac{1}{4}} [a + bcos(2\pi t) + c \cdot cos^2(\pi t)] \, dt - \int_{\frac{1}{4}}^{\frac{1}{2}} [a + bcos(2\pi t) + c \cdot cos^2(\pi t)] \, dt + \int_{\frac{1}{2}}^{\frac{3}{4}} [a +$
$bcos(2\pi t) + c \cdot cos^2(\pi t)] \, dt - \int_{\frac{3}{4}}^{1} [a + bcos(2\pi t) + c \cdot cos^2(\pi t)] \, dt = \Bigg(\frac{a}{4} + \frac{b}{2\pi} [\sin(2\pi t)]_0^{\frac{1}{4}} +$
$\left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2} \right]_0^{\frac{1}{4}} - \frac{a}{4} - \frac{b}{2\pi} [\sin(2\pi t)]_{\frac{1}{4}}^{\frac{1}{2}} - \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2} \right]_{\frac{1}{4}}^{\frac{1}{2}} + \frac{a}{4} + \frac{b}{2\pi} [\sin(2\pi t)]_{\frac{1}{2}}^{\frac{3}{4}} + \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2} \right]_{\frac{1}{2}}^{\frac{3}{4}} - \frac{a}{4} -$
$\frac{b}{2\pi} [\sin(2\pi t)]_{\frac{3}{4}}^{1} - \left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2} \right]_{\frac{3}{4}}^{1} \Bigg) = 0$

By that we got:

$\tilde{\phi}t) = \left(a + \frac{c}{2}\right) \psi_1^H(t) + 0 \cdot \psi_2^H(t) + \frac{2}{\pi} \left(b + \frac{c}{2}\right) \psi_3^H(t) - 0 \cdot \psi_4^H(t) = \left(a + \frac{c}{2} + \frac{2}{\pi} \left(b + \frac{c}{2}\right)\right)_{0,\frac{1}{4})} +$
$\left(a + \frac{c}{2} - \frac{2}{\pi} \left(b + \frac{c}{2}\right)\right)_{,\frac{1}{4}\frac{1}{2})} \left(a + \frac{c}{2} - \frac{2}{\pi} \left(b + \frac{c}{2}\right)\right)_{,\frac{1}{2}\frac{3}{4})} \left(a + \frac{c}{2} + \frac{2}{\pi} \left(b + \frac{c}{2}\right)\right)_{[\frac{3}{4} [1,}$

Associated MSE is:

$\int_0^1 \left(\phi(t) - \tilde{\phi}(t) \right)^2 \, dt = \int_0^1 \phi^2(t) \, dt - 2 \int_0^1 \phi(t) \, \tilde{\phi}(t) dt +$
$\int_0^1 \tilde{\phi}^2(t) \, dt \underbrace{=}_{as \ \ we \ saw \ in \ class} \int_0^1 \phi^2(t) \, dt - \sum_{i=1}^4 \left(\phi_i^{opt}\right)^2 dt =$
$\overbrace{\int_0^1 [a + bcos(2\pi t) + c \cdot cos^2(\pi t)]^2 \, dt}^{*} - \overbrace{\sum_{i=1}^4 \left(< \phi(t), \psi_i^W(t) >\right)^2}^{**} dt$

$* = \int_0^1 [a^2 + 2abcos(2\pi t) + 2ac \cdot cos^2(\pi t) + b^2 \cos^2(2\pi t) + 2bc \cdot cos(2\pi t)cos^2(\pi t) +$
$c^2 cos^4(\pi t)]dt \underbrace{=}_{linearity} a^2 \int_0^1 dt + 2ab \int_0^1 cos(2\pi t)dt + 2ac \int_0^1 cos^2(\pi t)dt +$
$b^2 \int_0^1 \cos^2(2\pi t) \, dt + 2bc \int_0^1 cos(2\pi t)cos^2(\pi t)dt + c^2 \int_0^1 cos^4(\pi t)dt = a^2 + 0 + ac + \frac{b^2}{2} + \frac{bc}{2} +$
$\frac{3c^2}{8}$

$$**= \left(a+\frac{c}{2}\right)^2 + 0^2 + \left(\frac{2}{\pi}\left(b+\frac{c}{2}\right)\right)^2 + (0)^2 = a^2 + ac + \frac{c^2}{4} + \frac{4}{\pi^2}\left(b^2 + bc + \frac{c^2}{4}\right) = a^2 + ac + c^2\left(\frac{1}{4} + \frac{1}{\pi^2}\right) + \frac{4b^2}{\pi^2} + \frac{4bc}{\pi^2}$$

, By that

$$MSE = a^2 + ac + \frac{b^2}{2} + \frac{bc}{2} + \frac{3c^2}{8} \left(a^2 + ac + c^2\left(\frac{1}{4} + \frac{1}{\pi^2}\right) + \frac{4b^2}{\pi^2} + \frac{4bc}{\pi^2}\right) b^2\left(\frac{1}{2} - \frac{4}{\pi^2}\right) + bc\left(\frac{1}{2} - \frac{4}{\pi^2}\right) c^2\left(\frac{1}{8} - \frac{1}{\pi^2}\right)$$

First, let us sort the coefficients by the squared coefficients $\left(\phi_i^{opt}\right)^2 = \left(< \phi(t), \psi_i^W(t) >\right)^2$ *in decreasing order, we can simply sort the absolute values of each:*

$$|\phi_1^{opt}| = a + \frac{c}{2} > |\phi_3^{opt}| = \frac{2}{\pi}\left(b+\frac{c}{2}\right) > |\phi_4^{opt}| = 0 = |\phi_2^{opt}|$$

Because $a + \frac{c}{2} \underbrace{\geq}_{a\geq b\geq 0 \text{ and } c\geq 0} b + \frac{c}{2}$ *and* $\frac{2}{\pi} < 1$ *and for each* $x \in \mathbb{R}, |x| \geq 0$

As we saw in class, for k-term approximation $MSE = \int_0^1 \phi^2(t)\,dt - \sum_{i=1}^{k}\left(\phi_i^{opt}\right)^2 dt$ *and for minimizing it we have to choose the first k functions with the biggest squared projections (maximize* $\sum_{i=1}^{k}\left(\phi_i^{opt}\right)^2$ *will minimize the MSE).*

1. ***The best 1-term approximation(k=1)***

$$\tilde{\phi}_1(t) = \phi_1^{opt} \cdot \psi_1^W(t) = \left(a+\frac{c}{2}\right) \cdot 1_{[0,1]}(t)$$

2. ***The best 2-term approximation(k=2)***

We can see $\phi_4^{opt} = 0 = \phi_2^{opt}$ *, so 2-term approximation will be the same as 3/4-term approximation, and all of them will be the same as we saw at section (iii) because this WH basis is 4-dimension. So, we get*

$$\tilde{\phi}_2(t) = \phi_1^{opt} \cdot \psi_1^W(t) + \phi_3^{opt}\psi_3^W(t) = \left(a+\frac{c}{2}\right) \cdot 1_{[0,1]}(t) + \frac{2}{\pi}\left(b+\frac{c}{2}\right)\left(1_{\left[0,\frac{1}{4}\right)}(t) + \left(-1_{\left(\frac{1}{4},\frac{3}{4}\right)}(t)\right) + 1_{\left(\frac{3}{4},1\right]}(t)\right) = \left(a+\frac{c}{2}+\frac{2}{\pi}\left(b+\frac{c}{2}\right)\right)_{\left[0,\frac{1}{4}\right)}(t) + \left(a+\frac{c}{2}-\frac{2}{\pi}\left(b+\frac{c}{2}\right)\right)_{\left(\frac{1}{4},\frac{3}{4}\right)} + \left(a+\frac{c}{2}+\frac{2}{\pi}\left(b+\frac{c}{2}\right)\right)_{\left[\frac{3}{4},1\right]}$$

v)

$$a = \frac{1}{\pi}, b = 1, c = \frac{3}{2}$$

First, let us sort the coefficients by the squared coefficients $\left(\phi_i^{opt}\right)^2 = \left(< \phi(t), \psi_i^H(t) >\right)^2$ *in decreasing order, we can simply sort by the absolute values of each:*

$$|\phi_3^{opt}| = \frac{2}{\pi}\left(b+\frac{c}{2}\right) = \frac{7}{2\pi} > \phi_1^{opt} = a + \frac{c}{2} = \frac{1}{\pi} + \frac{3}{4} > |\phi_4^{opt}| = 0 = |\phi_2^{opt}|$$

1. *The best 1-term approximation(k=1)*

$\tilde{\phi}_1(t) = \phi_3^{opt} \cdot \psi_3^W(t) = \frac{2}{\pi}\left(b + \frac{c}{2}\right) \cdot (1_{\left[0,\frac{1}{4}\right)}(t) + \ (-1_{\left(\frac{1}{4},\frac{3}{4}\right)}(t)) + 1_{\left(\frac{3}{4},1\right]}(t)) = 1.114 \cdot 1_{\left[0,\frac{1}{4}\right)}(t) - 1.114 \cdot (-1)_{\left(\frac{1}{4},\frac{3}{4}\right)}(t) + 1.114 \cdot 1_{[\frac{3}{4},1]}(t)$

2. *The best 2-term approximation(k=2), as we said 2-term approximation = 3/4-term approximation*

$\tilde{\phi}_2(t) = \ \phi_3^{opt} \cdot \psi_3^W(t) + \phi_1^{opt}\psi_3^W(t) = \left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left[0,\frac{1}{4}\right)}(t) + \left(a + \frac{c}{2} - \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left(\frac{1}{4},\frac{3}{4}\right)} + \left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)_{\left[\frac{3}{4},1\right]} = (2.182)_{\left[0,\frac{1}{4}\right)}(t) - (0.04578)_{\left(\frac{1}{4},\frac{3}{4}\right)} + (2.182)_{[\frac{3}{4},1]}$

$$H_1 = [1],$$
$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$
$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

matrix hadamard

Q3

a.

We will use induction :

Base H_1 *can be written as* $H_1 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ *where* $\lambda_{2^1} = \dfrac{1}{\sqrt{2}}$ $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

H_1 *is symmetric and real*

$$H_1H_1^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I_{2^1\times 2^1} \rightarrow$$

therefore, H_1 *is also unitary*

step : *we assume the condition applys for* $H_{2^{n-1}}$

$$By\ definition:\ H_{2^n} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix}$$

$$H_{2^n}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix}^T =$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}}^T & H_{2^{n-1}}^T \\ H_{2^{n-1}}^T & -H_{2^{n-1}}^T \end{bmatrix} \underset{\uparrow}{=} \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} = H_{2^n}$$

$$by\ induction\ H_{2^{n-1}}^T\ is\ symmetric\ \rightarrow\ H_{2^{n-1}}^T = H_{2^{n-1}}$$

Therefore, H_{2^n} *is symmetric*
Because $H_{2^{n-1}}$ *is real, it can be seen that* H_{2^n} *is also real*

$$H_{2^n}H_{2^n}^T = H_{2^n}H_{2^n} = \frac{1}{2}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} 2H_{2^{n-1}}^2 & 0 \\ 0 & 2H_{2^{n-1}}^2 \end{bmatrix} = \begin{bmatrix} H_{2^{n-1}}^2 & 0 \\ 0 & H_{2^{n-1}}^2 \end{bmatrix} \underset{\uparrow}{=} I_{2^n \times 2^n}$$

$$by\ induction\ H_{2^{n-1}}^2 = I_{2^{n-1} \times 2^{n-1}}$$

Therefore, H_{2^n} *is unitary*

$$H_{2^n} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} =$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \lambda_{2^{n-1}} A_{n-1} & \lambda_{2^{n-1}} A_{n-1} \\ \lambda_{2^{n-1}} A_{n-1} & -\lambda_{2^{n-1}} A_{n-1} \end{bmatrix} =$$

$$= \frac{\lambda_{2^{n-1}}}{\sqrt{2}}\begin{bmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & -A_{n-1} \end{bmatrix}$$

$$and\ we\ will\ denote\ \ \lambda_{2^n} = \frac{\lambda_{2^{n-1}}}{\sqrt{2}},\ A_n = \begin{bmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & -A_{n-1} \end{bmatrix}$$

and they hold the conditions.

We conclude, H_{2^n} *is real, symmetric, unitary and can be formulated as* λ_{2^n} *and* A_n *for all* $n \in \mathbb{N}$.

$b1.$

We will consider the two possible scenarios, according to if the end of s_1 *is the same as the start of* s_2 *or not.*

$$S(s_1 s_2) = \begin{cases} S(s_1) + S(s_2) + 1 & \text{where the meeting point is different} \\ S(s_1) + S(s_2) & \text{where the meeting point is the same} \end{cases}$$

$b2.$

We will use induction to prove the theoram:

base: For $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ *it is clear that* $S(r_1) = 0,\ S(r_2) = 1$, *therefore satisfies the ensamble equality*

step: we will assume that the equality holds for $H_{2^{n-1}}$

we will denote r_i *to be the* $i-th$ *row of* $H_{2^{n-1}}$

Considering the structure $H_{2^n} = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix}$, *we will denote the* $i-th$ *row of* H_{2^n} *as* R_i

We can see that $S(R_i) = S(r_i r_i),\ S(R_{i+2^{n-1}}) = S(r_i(-r_i))$ *for* $1 \leqslant i \leqslant 2^{n-1}$

According to the previous substraction:
$S(R_i) = S(r_i r_i) = 2S(r_i) + \delta_i$ *where* δ_i *is* 0 *or* 1 *depending on the meeting point in between the two concatenated rows*

Since putting a minus sign for each entry in a row does not change the number of signs

$changes,\ we\ get\ that\ S(r_i) = S(-r_i),\ and\ therefore:$
$S(R_{i+2^{n-1}}) = S(r_i(-r_i)) = 2S(r_i) + (1-\delta_i)$

$Therefore,\ the\ rows\ of\ H_{2^n}\ satisfy:$

$\{S(R_1), S(R_{1+2^{n-1}}), S(R_2), S(R_{2+2^{n-1}}),..., S(R_{2^{n-1}}), S(R_{2^n})\} =$
$\{2S(r_1), 2S(r_1)+1, 2S(r_2), 2S(r_2)+1,..., 2S(r_{2^{n-1}}), 2S(r_{2^{n-1}})+1\}$
$and\ by\ the\ induction\ assumption\ we\ get:$
$\{S(R_1), S(R_2),.., S(R_{2^n})\} = \left\{0, 1,..., 2^n - 1\right\}$

Q 4

$First\ we\ will\ prove\ a\ few\ properties\ of\ Kronecker\ product$

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \cdots & \cdots & \cdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} B = \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1} & \cdots & b_{p,q} \end{bmatrix} C = \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ \cdots & \cdots & \cdots \\ c_{p,1} & \cdots & c_{p,q} \end{bmatrix}$$

First property $-\ \mathbf{A\otimes(B+C) \stackrel{?}{=} A\otimes B + A\otimes C}$

$A\otimes(B+C)$

$$A\otimes\begin{bmatrix} b_{1,1}+c_{1,1} & \cdots & b_{1,q}+c_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1}+c_{p,1} & \cdots & b_{p,q}+c_{p,q} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{1,1}\cdot\begin{bmatrix} b_{1,1}+c_{1,1} & \cdots & b_{1,q}+c_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1}+c_{p,1} & \cdots & b_{p,q}+c_{p,q} \end{bmatrix} & \cdots & a_{1,n}\cdot\begin{bmatrix} b_{1,1}+c_{1,1} & \cdots & b_{1,q}+c_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1}+c_{p,1} & \cdots & b_{p,q}+c_{p,q} \end{bmatrix} \\ a_{m,1}\cdot\begin{bmatrix} b_{1,1}+c_{1,1} & \cdots & b_{1,q}+c_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1}+c_{p,1} & \cdots & b_{p,q}+c_{p,q} \end{bmatrix} & & a_{m,n}\cdot\begin{bmatrix} b_{1,1}+c_{1,1} & \cdots & b_{1,q}+c_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1}+c_{p,1} & \cdots & b_{p,q}+c_{p,q} \end{bmatrix} \end{bmatrix}$$

$$A \otimes B \ = \begin{bmatrix} a_{1,1} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \\ b_{p,1} & & b_{p,q} \end{bmatrix} & \cdots & a_{1,n} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \\ B_{p,1} & & b_{p,q} \end{bmatrix} \\ \\ a_{m,1} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \\ b_{p,1} & & b_{p,q} \end{bmatrix} & & a_{m,n} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \\ b_{p,1} & & b_{p,q} \end{bmatrix} \end{bmatrix}$$

$$A \otimes C = \begin{bmatrix} a_{1,1} \cdot \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ \\ c_{p,1} & & c_{p,q} \end{bmatrix} & \cdots & a_{1,n} \cdot \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ \\ c_{p,1} & & c_{p,q} \end{bmatrix} \\ \\ a_{m,1} \cdot \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ \\ c_{p,1} & & c_{p,q} \end{bmatrix} & & a_{m,n} \cdot \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ \\ c_{p,1} & & c_{p,q} \end{bmatrix} \end{bmatrix}$$

$$A \otimes B + A \otimes C =$$

$$= \begin{bmatrix} a_{1,1} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \\ b_{p,1} & & b_{p,q} \end{bmatrix} & \cdots & a_{1,n} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \\ B_{p,1} & & b_{p,q} \end{bmatrix} \\ \\ a_{m,1} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \\ b_{p,1} & & b_{p,q} \end{bmatrix} & & a_{m,n} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \\ b_{p,1} & & b_{p,q} \end{bmatrix} \end{bmatrix} +$$

$$\begin{bmatrix} a_{1,1} \cdot \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ \\ c_{p,1} & & c_{p,q} \end{bmatrix} & \cdots & a_{1,n} \cdot \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ \\ c_{p,1} & & c_{p,q} \end{bmatrix} \\ \\ a_{m,1} \cdot \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ \\ c_{p,1} & & c_{p,q} \end{bmatrix} & & a_{m,n} \cdot \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ \\ c_{p,1} & & c_{p,q} \end{bmatrix} \end{bmatrix} =$$

$$
\begin{bmatrix}
a_{1,1}\begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ & & \\ B_{p,1} & & b_{p,q} \end{bmatrix} + a_{1,1}\begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ & & \\ c_{p,1} & & c_{p,q} \end{bmatrix} & \cdots & a_{1,n}\begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ & & \\ b_{p,1} & & b_{p,q} \end{bmatrix} + a_{1,n}\begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ & & \\ c_{p,1} & & c_{p,q} \end{bmatrix} \\
a_{m,1}\begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ & & \\ b_{p,1} & & b_{p,q} \end{bmatrix} + a_{m,1}\begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ & & \\ c_{p,1} & & c_{p,q} \end{bmatrix} & & a_{m,n}\begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ & & \\ b_{p,1} & & b_{p,q} \end{bmatrix} + a_{m,n}\begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ & & \\ c_{p,1} & & c_{p,q} \end{bmatrix}
\end{bmatrix} =
$$

$$
= \begin{bmatrix}
a_{1,1}\left(\begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ & & \\ b_{p,1} & & b_{p,q} \end{bmatrix} + \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ & & \\ c_{p,1} & & c_{p,q} \end{bmatrix}\right) & \cdots & a_{1,n}\left(\begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ & & \\ b_{p,1} & & b_{p,q} \end{bmatrix} + \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ & & \\ c_{p,1} & & c_{p,q} \end{bmatrix}\right) \\
a_{m,1}\left(\begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ & & \\ b_{p,1} & & b_{p,q} \end{bmatrix} + \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ & & \\ c_{p,1} & & c_{p,q} \end{bmatrix}\right) & & a_{m,n}\left(\begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ & & \\ b_{p,1} & & b_{p,q} \end{bmatrix} + \begin{bmatrix} c_{1,1} & \cdots & c_{1,q} \\ & & \\ c_{p,1} & & c_{p,q} \end{bmatrix}\right)
\end{bmatrix} =
$$

$$
= \begin{bmatrix}
a_{1,1} \cdot \begin{bmatrix} b_{1,1}+c_{1,1} & \cdots & b_{1,q}+c_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1}+c_{p,1} & \cdots & b_{p,q}+c_{p,q} \end{bmatrix} & \cdots & a_{1,n} \cdot \begin{bmatrix} b_{1,1}+c_{1,1} & \cdots & b_{1,q}+c_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1}+c_{p,1} & \cdots & b_{p,q}+c_{p,q} \end{bmatrix} \\
a_{m,1} \cdot \begin{bmatrix} b_{1,1}+c_{1,1} & \cdots & b_{1,q}+c_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1}+c_{p,1} & \cdots & b_{p,q}+c_{p,q} \end{bmatrix} & & a_{m,n} \cdot \begin{bmatrix} b_{1,1}+c_{1,1} & \cdots & b_{1,q}+c_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1}+c_{p,1} & \cdots & b_{p,q}+c_{p,q} \end{bmatrix}
\end{bmatrix} =
$$

$$
= A \otimes \begin{bmatrix} b_{1,1}+c_{1,1} & \cdots & b_{1,q}+c_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1}+c_{p,1} & \cdots & b_{p,q}+c_{p,q} \end{bmatrix} =
$$

$$
= A \otimes (B+C) \ \rightarrow \boxed{\mathbf{A \otimes (B+C) \ = A \otimes B \ + A \otimes C}}
$$

Second property – $(\mathbf{A \otimes B})^{\mathbf{T}} \stackrel{?}{=} \mathbf{A}^{\mathbf{T}} \otimes \mathbf{B}^{\mathbf{T}}$

$(A \otimes B)^T \ =$

$$= \begin{bmatrix} a_{1,1} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1} & \cdots & b_{p,q} \end{bmatrix} & \cdots & a_{1,n} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \cdots & \cdots & \cdots \\ b & \cdots & b_{p,q} \end{bmatrix} \\ & & \\ a_{m,1} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1} & \cdots & b_{p,q} \end{bmatrix} & & a_{m,n} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \cdots & \cdots & \cdots \\ b_{p,1} & \cdots & b_{p,q} \end{bmatrix} \end{bmatrix}^T =$$

$$= \left(\begin{bmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \cdots & \cdots & \cdots \\ a_{m,1}B & \cdots & a_{m,n}B \end{bmatrix} \in \Re^{mp \times nq} \right)^T =$$

*

$$= \begin{bmatrix} a_{1,1}B^T & \cdots & a_{m,1}B^T \\ \cdots & \cdots & \cdots \\ a_{1,n}B^T & \cdots & a_{m,n}B^T \end{bmatrix} \in \Re^{nq \times mp}$$

$$A^T \otimes B^T = \begin{bmatrix} a_{1,1} & \cdots & a_{m,1} \\ \cdots & \cdots & \cdots \\ a_{1,n} & \cdots & a_{m,n} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & \cdots & b_{p,1} \\ \cdots & \cdots & \cdots \\ b_{1,q} & \cdots & b_{p,q} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{1,1} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{p,1} \\ \cdots & \cdots & \cdots \\ b_{1,q} & \cdots & b_{p,q} \end{bmatrix} & \cdots & a_{m,1} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{p,1} \\ \cdots & \cdots & \cdots \\ b_{1,q} & \cdots & b_{p,q} \end{bmatrix} \\ & & \\ a_{1,n} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{p,1} \\ \cdots & \cdots & \cdots \\ b_{1,q} & \cdots & b_{p,q} \end{bmatrix} & & a_{m,n} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{p,1} \\ \cdots & \cdots & \cdots \\ b_{1,q} & \cdots & b_{p,q} \end{bmatrix} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{1,1}B^T & \cdots & a_{m,1}B^T \\ \cdots & \cdots & \cdots \\ a_{1,n}B^T & \cdots & a_{m,n}B^T \end{bmatrix} \in \Re^{nq \times mp} = (A \otimes B)^T \rightarrow \boxed{\mathbf{(A \otimes B)^T = A^T \otimes B^T}}$$

$* \ (a_{1,1}B)^T = B^T a_{1,1}^T$ *and since* $a_{1,1}$ *is a scalar* $\rightarrow$ *it is equal to its transpose* $\rightarrow$
$(a_{1,1}B)^T = a_{1,1}B^T$

Third property –
Let A $\in \Re^{m\times n}$, B $\in \Re^{r\times s}$, C $\in \Re^{n\times p}$ and D $\in \Re^{s\times t}$

(A⊗B)(C⊗B) $\stackrel{?}{=}$ AC⊗BD

$$(A\otimes B)(C\otimes B) =$$

$$= \begin{bmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \cdots & \cdots & \cdots \\ a_{m,1}B & \cdots & a_{m,n}B \end{bmatrix} \begin{bmatrix} c_{1,1}D & \cdots & c_{1,p}D \\ \cdots & \cdots & \cdots \\ c_{n,1}D & \cdots & c_{n,p}D \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{k=1}^{n} a_{1,k}c_{k,1}BD & \cdots & \sum_{k=1}^{n} a_{1,k}c_{k,p}BD \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} a_{m,k}c_{k,1}BD & \cdots & \sum_{k=1}^{n} a_{m,k}c_{k,p}BD \end{bmatrix} = AC\otimes BD \;\rightarrow\; \boxed{\mathbf{(A\otimes B)(C\otimes B) = AC\otimes BD}}$$

a.

$$H_{2^{n+1}} = \begin{bmatrix} H_{2^n} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^n} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{2^n}^T = \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix}^T =$$

$$= \begin{bmatrix} \left(H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix}\right)^T & \left(I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix}\right)^T \end{bmatrix} = \begin{bmatrix} \left(H_{2^{n-1}}^T \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) & \left(I_{2^{n-1}}^T \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \end{bmatrix} =$$

by property 2 $\boxed{\mathbf{(A\otimes B)^T = A^T\otimes B^T}}$
by induction assumption $H_{2^{n-1}}^T = H_{2^{n-1}}$

$$= \left[\left(H_{2^{n-1}} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \quad \left(I_{2^{n-1}} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \right] \neq \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix}$$

$\rightarrow$ *H_{2^n} is NOT symmetric*

Of note, in the following we will prove that H_{2^n} is orthogonal. Another option to prove that H_{2^n} is not symmetric ortogonal matrix is by showing that $H_{2^n} \cdot H_{2^n} \neq I_{2^n \times 2^n}$

b.

Let us use induction:

Base H_{2^1} can be written as $H_{2^1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

step: we shall assume $H_{2^{n-1}}$ is orthogonal

By definition: $H_{2^n} = \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix}$

$$H_{2^n}^T H_{2^n} = \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix} =$$

$$= \left[\left(H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \right)^T \quad \left(I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \right)^T \right] \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix} =$$

by property 2 $\boxed{(\mathrm{A} \otimes \mathrm{B})^{\mathrm{T}} = \mathrm{A}^{\mathrm{T}} \otimes \mathrm{B}^{\mathrm{T}}}$

$$= \left[\left(H_{2^{n-1}}^T \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \quad \left(I_{2^{n-1}}^T \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \right] \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix} =$$

$$= \left[\left(H_{2^{n-1}}^T \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \left(H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \right) \left(I_{2^{n-1}}^T \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \left(I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \right) \right] =$$

by property 3 $\boxed{(\mathrm{A} \otimes \mathrm{B})(\mathrm{C} \otimes \mathrm{B}) = \mathrm{AC} \otimes \mathrm{BD}}$

$$= \left[\left(\left(H^T_{2^{n-1}} H_{2^{n-1}}\right) \otimes \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \right) \right) \left(\left(I^T_{2^{n-1}} I_{2^{n-1}}\right) \otimes \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \right) \right) \right] =$$

by induction assumption $H^T_{2^{n-1}} H_{2^{n-1}} = I_{2^{n-1} \times 2^{n-1}}$

$$= \left[\left(I_{2^{n-1} \times 2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \left(I_{2^{n-1} \times 2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \right] =$$

by first property $A \otimes B + A \otimes C = A \otimes (B + C)$

$$= \left[I_{2^{n-1} \times 2^{n-1}} \otimes \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right] = 2I_{2^n \times 2^n} \rightarrow H_{2^n} \text{ is orthogonal}$$

$$\text{since } H^T_{2^n} H_{2^n} = \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix} = 2 \cdot I_{2^n \times 2^n} \rightarrow H_{2^n} \text{ is not unitary}$$

c.

$$\text{we will prove by induction that } H_{2^n} = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix} \text{ is a unitary matrix}$$

$$\text{Base } \widetilde{H}_{2^1} \text{ can be written as } H_{2^1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$\widetilde{H}_{2^1}$ *is a unitary matrix*

step: we shall assume $H_{2^{n-1}}$ *is unitary*

$$H^T_{2^n} H_{2^n} = \frac{1}{2} \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} H_{2^{n-1}} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ I_{2^{n-1}} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix} =$$

$$by\ the\ proof\ in\ "e"$$

$$= \frac{1}{2}\left[I_{2^{n-1}\times 2^{n-1}} \otimes \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right] = I_{2^n \times 2^n} \rightarrow H_{2^n}\ is\ unitary\ and\ the\ recursion$$

$$formula\ is\ H_{2^n} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}} \otimes [1 \ \ 1] \\ I_{2^{n-1}} \otimes [1 \ \ -1] \end{bmatrix} with\ base\ H_{2^1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

d. *See approval of propery* 2 *above*

e.

$$H^T_{2^n} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^{n-1}} \otimes [1 \ \ 1] \\ I_{2^{n-1}} \otimes [1 \ \ -1] \end{bmatrix}^T = \frac{1}{\sqrt{2}}\left[\left(H_{2^{n-1}} \otimes [1 \ \ 1]\right)^T \quad \left(I_{2^{n-1}} \otimes [1 \ \ -1]\right)^T \right] =$$

$$by\ property\ 2\ \boxed{(\mathrm{A}\otimes\mathrm{B})^{\mathrm{T}} = \mathrm{A}^{\mathrm{T}}\otimes\mathrm{B}^{\mathrm{T}}}$$

$$= \frac{1}{\sqrt{2}}\left[\left(H^T_{2^{n-1}} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \middle| \left(I_{2^{n-1}} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \right]$$