

Operational calculus for Fourier transform on the group $GL(2, \mathbb{R})$ and problem about action of overalgebra in Plancherel decomposition

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Consider the Fourier transform on the group $GL(2, \mathbb{R})$ of real 2×2 -matrices. We show that Fourier-images of polynomial differential operators on $GL(2, \mathbb{R})$ are differential-difference operators with coefficients meromorphic in parameters of representations. Expressions for operators contain shifts in imaginary direction with respect to the integration contour in the Plancherel formula. We present explicit formulas for images of partial derivations and multiplications by coordinates.

The Plancherel formula for $SL(2, \mathbb{C})$ for obtained in by I. M. Gelfand and M. I. Naimark in 1947, for $SL(2, \mathbb{R})$ by Harich-Chandra in 1951, see [4], [5]. Later there were published many Plancherel formulas for Fourier transforms on non-commutative locally compact groups and homogeneous spaces (see a collection of references in [18]). The spherical transform for Riemannian symmetric spaces and for Bruhat-Tits buildings had numerous continuations in mathematics, however a destiny of the majority of Plancherel formulas seems strange, they are heavy impressive results, which are difficult for usage. The corresponding integral transforms have properties that immediately follow from their definition. Efforts of a further expansion are faced with difficulties, also there is a few number of explicit calculations of Fourier-images of functions.

An example of a nontrivial transformation of differential operators for an $SL(2, \mathbb{R})$ -related Fourier transform was done in [13], images of certain first-order differential operators are differential-difference operators with differentiations of the order two in a space variable and shifts in imaginary direction in a parameter variable. Several correspondences of the same type were obtained by V. F. Molchanov and the author in [8]–[10], [16]–[17]. The present paper contains a simple corollary of calculations of [17], however our statement (Theorems 1-2) is essentially stronger and easier for a further usage than the statement of [17].

1. The principal series of representations of the group $GL(2, \mathbb{R})$. Consider the group $GL(2, \mathbb{R})$ of real invertible matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of order 2. For $\mu \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}_2$ we define the function $x^{\mu//\varepsilon}$ on $\mathbb{R} \setminus 0$ by

$$x^{\mu//\varepsilon} := |x|^\mu \operatorname{sgn}(x)^\varepsilon.$$

Denote $\Lambda := \mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C} \times \mathbb{Z}_2$. For each element $(\mu_1, \varepsilon_1; \mu_2, \varepsilon_2)$ of Λ we define a representation $T_{\mu, \varepsilon}$ of $GL_2(\mathbb{R})$ in the space of functions on \mathbb{R} by

$$\begin{aligned} T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varphi(t) &= \\ &= \varphi\left(\frac{b+td}{a+tc}\right) \cdot (a+tc)^{-1+\mu_1-\mu_2//\varepsilon_1-\varepsilon_2} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{1/2+\mu_2//\varepsilon_2}. \end{aligned}$$

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This formula determines the (non-unitary) *principal series of representations* of $\mathrm{GL}(2, \mathbb{R})$. To be definite, we define a representation $T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}$ in the space $C_{\mu_1 - \mu_2, \varepsilon_1 - \varepsilon_2}^\infty(\mathbb{R})$ of smooth functions f on \mathbb{R} satisfying the additional condition on asymptotics at infinity: the function

$$T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(t) = f(-t^{-1})(-t)^{-1 + \mu_1 - \mu_2 // \varepsilon_1 - \varepsilon_2}$$

must be smooth at 0. This space is invariant with respect to the whole group $\mathrm{GL}(2, \mathbb{R})$, see [3], Subsect. VII.1.2.

If $\mu_1 - \mu_2 - 1 \notin 2\mathbb{Z} + (\varepsilon_1 - \varepsilon_2)$, then representations $T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}$ and $T_{\mu_2, \varepsilon_2; \mu_1, \varepsilon_1}$ are irreducible and equivalent (see, e.g., [3], Subsect. VII.2.2). The intertwining operator is given by

$$A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} f(t) := \int_{\mathbb{R}} (t - s)^{-1 - \mu_1 + \mu_2 // \varepsilon_1 - \varepsilon_2} f(s) ds.$$

The integral converges if $\mathrm{Re}(\mu_2 - \mu_1) > 0$ and is holomorphic in the parameters μ_1, μ_2 . It has a meromorphic continuation to the whole domain $\mathbb{C} \times \mathbb{C}$, for $\varepsilon_1 - \varepsilon_2 = 0$ possible poles are on the complex $\mu_2 - \mu_1 = -1, -3, \dots$; for $\varepsilon_1 - \varepsilon_2 = 1$ on $\mu_2 - \mu_1 = 0, 2, 4, \dots$.

If $\mu_1 = i\tau_1, \mu_2 = i\tau_2 \in i\mathbb{R}$, then a representation $T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}$ is unitary in $L^2(\mathbb{R})$ (they are called representations of the *unitary principal series*).

2. Fourier transform in the complex domain. Consider the space $C_0^\infty(\mathrm{GL}(2, \mathbb{R}))$ of compactly supported smooth functions on $\mathrm{GL}(2, \mathbb{R})$. For $F \in C_0^\infty(\mathrm{GL}(2, \mathbb{R}))$ we define its Fourier transform as a function sending $(\mu_1, \varepsilon_1, \mu_2, \varepsilon_2)$ to an operator in $C_{\mu_1 - \mu_2, \varepsilon_1 - \varepsilon_2}^\infty(\mathbb{R})$ given by

$$T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F)f(t) = \int_{\mathrm{GL}_2(\mathbb{R})} F_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(g) T(g) dg,$$

where dg denotes the Haar measure on $\mathrm{GL}_2(\mathbb{R})$, its density with respect to the standard Lebesgue measure on the space of matrices is $(ad - bc)^{-2}$.

We denote the space of operator-valued functions on Λ obtained in this way by \mathcal{E} . A precise description of \mathcal{E} (a *Paley–Wiener theorem* for $\mathrm{SL}(2, \mathbb{R})$) was obtained in [1], we only present some remarks.

Elements of \mathcal{E} satisfy a symmetry condition

$$A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F) = T_{\mu_2, \varepsilon_2; \mu_1, \varepsilon_1}(F) A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}.$$

For any $F \in C_0^\infty(\mathrm{GL}(2, \mathbb{R}))$ the operators $T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F)$ can be represented as integral operators

$$T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F)f(t) = \int_{\mathbb{R}} K_F(t, s | \mu_1, \varepsilon_1; \mu_2, \varepsilon_2) f(s) ds,$$

where

$$\begin{aligned} K_F(t, s | \mu_1, \varepsilon_1; \mu_2, \varepsilon_2) &= \\ &= \iiint_{\mathbb{R}^3} F(u - tv, su - stv - tw, v, sv + w) u^{-3/2 + \mu_1 // \varepsilon_1} w^{-3/2 + \mu_2 // \varepsilon_2} du dv dw. \end{aligned}$$

It is easy to see that a support of the integrand is a bounded set having empty intersections with planes $u = 0, w = 0$. Therefore the function K_F is holomorphic in

$\mu_1, \mu_2 \in \mathbb{C}$ and smooth in $s, t \in \mathbb{R}$. Also it satisfies some conditions on asymptotics at infinity: the kernels

$$\begin{aligned} K(t, s|\mu_1, \varepsilon_1; \mu_1, \varepsilon_1) &= K(-1/t, s|\mu_1, \varepsilon_1; \mu_1, \varepsilon_1) = (-t)^{\mu_1 - \mu_2 - 1} \varepsilon_1^{-\varepsilon_2}; \\ K(t, s|\mu_1, \varepsilon_1; \mu_1, \varepsilon_1) &= K(t, -1/s|\mu_1, \varepsilon_1; \mu_1, \varepsilon_1) = (-s)^{-\mu_1 + \mu_2 - 1} \varepsilon_1^{-\varepsilon_2}; \\ K(t, s|\mu_1, \varepsilon_1; \mu_1, \varepsilon_1) &= K(-1/t, -1/s|\mu_1, \varepsilon_1; \mu_1, \varepsilon_1) = (t/s)^{\mu_1 - \mu_2 - 1} \varepsilon_1^{-\varepsilon_2} \end{aligned}$$

admit smooth continuations to the whole plane $(t, s) \in \mathbb{R}^2$ (see a discussion of a similar case $\text{SL}(2, \mathbb{C})$ in [3], §IV.1).

3. Immediate properties of the Fourier transform. Denote by $F_1 * F_2$ the convolution on $\text{GL}(2, \mathbb{R})$. For any $F_1, F_2 \in C_0^\infty(\text{GL}(2, \mathbb{R}))$ we have

$$T(F_1)T(F_2) = T(F_1 * F_2).$$

This is a general fact for locally compact groups with two-side invariant Haar measure, see e.g., [7], §10.

Next, the Lie algebra $\mathfrak{gl}(2)$ acts in $C^\infty(\text{GL}(2, \mathbb{R}))$ by right-invariant vector fields, we write formulas for the standard generators:

$$\begin{aligned} (1) \quad e_{11}^r &= -a \frac{\partial}{\partial a} - b \frac{\partial}{\partial b}, & e_{12}^r &= -c \frac{\partial}{\partial a} - d \frac{\partial}{\partial b}, \\ (2) \quad e_{21}^r &= -a \frac{\partial}{\partial c} - b \frac{\partial}{\partial d}, & e_{22}^r &= -c \frac{\partial}{\partial c} - d \frac{\partial}{\partial d}. \end{aligned}$$

Also, $\mathfrak{gl}(2)$ acts in $C^\infty(\text{GL}(2, \mathbb{R}))$ by left-invariant vector fields

$$\begin{aligned} (3) \quad e_{11}^l &= a \frac{\partial}{\partial a} + c \frac{\partial}{\partial c}, & e_{12}^l &= a \frac{\partial}{\partial b} + c \frac{\partial}{\partial d}, \\ (4) \quad e_{21}^l &= b \frac{\partial}{\partial a} + d \frac{\partial}{\partial c}, & e_{22}^l &= b \frac{\partial}{\partial b} + d \frac{\partial}{\partial d}. \end{aligned}$$

Thus we get an action of the Lie algebra $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$ in $C^\infty(\text{GL}(2, \mathbb{R}))$. The corresponding operators in the Fourier-image are given by

$$\begin{aligned} (5) \quad E_{11}^r &= -t \frac{\partial}{\partial t} - (1/2 - \mu_1), & E_{12}^r &= \frac{\partial}{\partial t}, \\ (6) \quad E_{21}^r &= -t^2 \frac{\partial}{\partial t} + (-1 + \mu_1 - \mu_2)t, & E_{22}^r &= t \frac{\partial}{\partial t} + (1/2 + \mu_2), \end{aligned}$$

and

$$\begin{aligned} (7) \quad E_{11}^l &= -s \frac{\partial}{\partial s} - (1/2 + \mu_1), & E_{12}^l &= \frac{\partial}{\partial s}, \\ (8) \quad E_{21}^l &= -s^2 \frac{\partial}{\partial s} + (-1 - \mu_1 + \mu_2)s, & E_{22}^l &= s \frac{\partial}{\partial s} + (1/2 - \mu_2). \end{aligned}$$

4. Correspondence of differential operators. We want to evaluate images of differential operators under the Fourier transform, i.e. for a differential operator D in $C_0^\infty(\text{GL}(2, \mathbb{R}))$ we wish to find a transformation $\Theta(D)$ in \mathcal{E} such that

$$K_{DF} = \Theta(D)K_F$$

for all F .

Define the following shift operators

$$\begin{aligned} V_1 K(t, s|\mu_1, \varepsilon_1; \mu_2, \varepsilon_2) &= K(t, s|\mu_1 + 1, \varepsilon_1 + 1; \mu_2, \varepsilon_2); \\ V_2 K(t, s|\mu_1, \varepsilon_1; \mu_2, \varepsilon_2) &= K(t, s|\mu_1, \varepsilon_1; \mu_2 + 1, \varepsilon_2 + 1). \end{aligned}$$

Theorem 1. a) *The Fourier-images of operators of multiplication by functions $a, b, c, d, (ad - bc)^{-1}$ in $C_0^\infty(\mathrm{GL}(2, \mathbb{R}))$ are the following operators in \mathcal{E} :*

$$(9) \quad a \longleftrightarrow V_1 - \frac{t}{\mu_1 - \mu_2} \left(\frac{\partial}{\partial t} V_1 + \frac{\partial}{\partial s} V_2 \right);$$

$$(10) \quad b \longleftrightarrow sV_1 - tV_2 - \frac{st}{\mu_1 - \mu_2} \left(\frac{\partial}{\partial t} V_1 + \frac{\partial}{\partial s} V_2 \right);$$

$$(11) \quad c \longleftrightarrow \frac{1}{\mu_1 - \mu_2} \left(\frac{\partial}{\partial t} V_1 + \frac{\partial}{\partial s} V_2 \right);$$

$$(12) \quad d \longleftrightarrow V_2 + \frac{s}{\mu_1 - \mu_2} \left(\frac{\partial}{\partial t} V_1 + \frac{\partial}{\partial s} V_2 \right);$$

$$(13) \quad (ad - bc)^{-1} \longleftrightarrow V_1^{-1} V_2^{-1}.$$

b) *The operators $\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}$ correspond to:*

$$(14) \quad \frac{\partial}{\partial a} \longleftrightarrow \left(\frac{3}{2} - \mu_1 \right) V_1^{-1} + \frac{s}{\mu_1 - \mu_2} \left(\left(\frac{3}{2} - \mu_1 \right) \frac{\partial}{\partial s} V_1^{-1} + \left(\frac{3}{2} - \mu_2 \right) \frac{\partial}{\partial t} V_2^{-1} \right);$$

$$(15) \quad \frac{\partial}{\partial b} \longleftrightarrow -\frac{1}{\mu_1 - \mu_2} \left(\left(\frac{3}{2} - \mu_1 \right) \frac{\partial}{\partial s} V_1^{-1} + \left(\frac{3}{2} - \mu_2 \right) \frac{\partial}{\partial t} V_2^{-1} \right);$$

$$(16) \quad \frac{\partial}{\partial c} \longleftrightarrow \left(\frac{3}{2} - \mu_1 \right) t V_1^{-1} - \left(\frac{3}{2} - \mu_2 \right) s V_2^{-1} + \frac{st}{\mu_1 - \mu_2} \left(\left(\frac{3}{2} - \mu_1 \right) \frac{\partial}{\partial s} V_1^{-1} + \left(\frac{3}{2} - \mu_2 \right) \frac{\partial}{\partial t} V_2^{-1} \right);$$

$$(17) \quad \frac{\partial}{\partial d} \longleftrightarrow \left(\frac{3}{2} - \mu_2 \right) V_2^{-1} - \frac{t}{\mu_1 - \mu_2} \left(\left(\frac{3}{2} - \mu_1 \right) \frac{\partial}{\partial s} V_1^{-1} + \left(\frac{3}{2} - \mu_2 \right) \frac{\partial}{\partial t} V_2^{-1} \right).$$

PROOF. A derivation of the formula for $(ad - bc)^{-1}$ is straightforward. Other statements follow from the table of formulas [17], Subsection 2.5. The table contains a correspondence for 8 pencils of operators of type $A + \sigma B$, where σ is a parameter. Substituting $\tau = -1 + \sigma$ and considering correspondences of constant terms and terms of the first order in τ , we found the desired correspondences.

Arguments of [17] can be easily transformed to direct proofs of our formulas, but this does not essentially simplify calculations. \square

As an immediate corollary of Theorem 1 we get the following statement.

Theorem 2. *Let D be a differential operator on the group $\mathrm{GL}(2, \mathbb{R})$ admitting a polynomial expression in $a, b, c, d, (ad - bc)^{-1}, \frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}$. Then its Fourier-image is a finite sum of the form*

$$\Theta(D) = \sum_{k,l} Q_{kl} \left(t, s, \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) V_1^k V_2^l,$$

where $k, l \in \mathbb{Z}$, and $Q_{kl}(\cdot)$ are polynomial expressions with rational coefficients depending on μ_1, μ_2 with poles at lines $\mu_1 - \mu_2 = m \in \mathbb{Z}$.

REMARK. Similar statements hold for $\mathrm{GL}(2, \mathbb{C})$, in fact formulas for correspondence of differential and differential-difference operators in this case are same, see [17]. \square

5. The problem about overalgebra. In [13] there was formulated the following question.

Problem. *Let G be a semisimple Lie group, \mathfrak{g} its Lie algebra, $H \subset G$ a subgroup, and ρ a unitary representation of G . Assume that a restriction of ρ to H admits an explicit spectral decomposition. To transfer the action of the whole Lie algebra \mathfrak{g} in the spectral decomposition.*

In [13] there was obtained an explicit solution of the problem for a tensor product of a highest weight representation of $\mathrm{SL}(2, \mathbb{R})$ and the dual lowest weight representation². The overalgebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ act by differential-difference operators, including second derivatives. Now there is a collection of solved problems of this type, see [8]–[10], [16], [17]. In all the cases we observe differential-difference operators, in all cases we have shifts in imaginary directions. Results of [16] show that orders of partial derivatives increases with growth of a rank of a group. In my opinion, now there are reasons to hope that such problems are always solvable (as far as we can explicitly solve the restriction problem).

Question about the action of overalgebra makes sense only if we are able to obtain an explicit spectral decomposition for a restriction problem. Now a big zoo of such decompositions is known. We do not refer it and only indicate several types of problems, which are interesting from our point of view.

- For any classical Lie group $G = \mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C}), \mathrm{GL}(n, \mathbb{H}), \mathrm{O}(p, q), \mathrm{U}(p, q), \mathrm{Sp}(p, q), \mathrm{Sp}(2n, \mathbb{R}), \mathrm{Sp}(2n, \mathbb{C}), \mathrm{O}(n, \mathbb{C}), \mathrm{SO}^*(2n)$ the regular representation of $G \times G$ in $L^2(G)$ can be obtained as a restriction from a certain unitary representation of a certain overgroup $\tilde{G} \supset G \times G$, namely, $\tilde{G} = \mathrm{GL}(2n, \mathbb{R}), \mathrm{GL}(2n, \mathbb{C}), \mathrm{GL}(2n, \mathbb{H}), \mathrm{O}(p+q, p+q), \mathrm{U}(p+q, p+q), \mathrm{Sp}(p+q, p+q), \mathrm{Sp}(4n, \mathbb{R}), \mathrm{Sp}(4n, \mathbb{C}), \mathrm{O}(2n, \mathbb{C}), \mathrm{SO}^*(4n)$ respectively. More generally, the representation of a classical group H in L^2 on a (pseudo-Riemannian) symmetric space H/L can be obtained as a restriction of a certain unitary representation of a certain overgroup \tilde{H} . For details and precise formulations, see [12]. Formulas for $\mathrm{GL}(2, \mathbb{R})$ mentioned in the proof of Theorem 1 were obtained as a solution of the overalgebra problem for restrictions of representations of degenerate principal series of $\mathrm{GL}(4, \mathbb{R})$ to $\mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R})$.

- Restrictions of holomorphic representations to symmetric subgroups, see [14].
- Howe dual pairs, [6], see an example below.
- Numerous problems on rank one groups and also problems related to maximally degenerate principal and complementary series (and their further degenerations); here several problems about the action of an overalgebra are solved, see [13], [8]–[10].
- Problems of restriction $\mathrm{GL}(n, \mathbb{C}) \downarrow \mathrm{GL}(n-1, \mathbb{C})$ ([16]) and similar problems for other series of classical groups. In this case explicit spectral decompositions are unknown, but apparently they can be obtained.

May be, the question exists for non-semisimple groups, but known interesting examples of spectral decompositions are related to semisimple groups or groups similar to semisimple (below we have such groups in examples).

6. The Weil representation. Consider the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, i.e., the group of block $(n+n) \times (n+n)$ real matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying the

²Problem of decomposition of a tensor product $\rho_1 \otimes \rho_2$ of unitary representations of a group G is a special case of a restriction problem, namely we restrict a representation of $G \times G$ to the diagonal G .

condition

$$g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g^t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the symbol t denotes the transposition. Its Lie algebra $\mathfrak{sp}(2n)$ consists of real matrices of the form

$$(18) \quad \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^t \end{pmatrix}, \quad \text{where } \beta = \beta^t, \gamma = \gamma^t.$$

The matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a^{t-1} \end{pmatrix}$ form a subgroup in $\mathrm{Sp}(2n, \mathbb{R})$, it is isomorphic to $\mathrm{GL}(n, \mathbb{R})$.

We also consider the Heisenberg group Heis_{2n+1} consisting of block matrices of size $(1 + n + 1)$ of the form $\begin{pmatrix} 1 & v & \lambda \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}$.

These groups have standard representations in the space $L^2(\mathbb{R}^n)$, we briefly describe them on the level of Lie algebras. The collection of differential operators

$$x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, 1$$

is closed with respect to the commutator and form a representation of the Heisenberg Lie algebra \mathfrak{heis}_{2n+1} .

The collection of operators (they are quadratic expressions in operators of the Heisenberg algebra)

$$x_k x_l, \quad x_k \frac{\partial}{\partial x_l} + \delta_{kl}, \quad \frac{\partial^2}{\partial x_k \partial x_l}$$

also is closed with respect to the commutator and forms the symplectic Lie algebra $\mathfrak{sp}(2n)$. This representation is called the *Weil representation* or *oscillator representation* (the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ itself acts by integral operators, see, e.g., [15]).

The operators $x_k \frac{\partial}{\partial x_l} + \delta_{kl}$ generate a representation of the Lie algebra $\mathfrak{gl}(n) \subset \mathfrak{sp}(2n)$; the corresponding representation of the Lie group $\mathrm{GL}(n, \mathbb{R})$ is the standard representation of $\mathrm{GL}(n, \mathbb{R})$ in $L^2(\mathbb{R}^n)$.

Joining \mathfrak{heis}_{2n+1} and $\mathfrak{sp}(2n)$ we get a representation of a semidirect product $\mathfrak{sp}(2n) \ltimes \mathfrak{heis}_{2n+1}$ of Lie algebras, and a representation of the corresponding group $\mathrm{Sp}(2n, \mathbb{R}) \ltimes \mathrm{Heis}_{2n+1}$.

Recall that $(\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{R}))$ is an example of Howe dual pair (see [6]). We do not need a definition of such pairs and only present minimal data necessary for our purposes.

Consider a symplectic group $\mathrm{Sp}(2n^2, \mathbb{R})$ and its subgroup $\mathrm{GL}(n^2, \mathbb{R})$. We realize \mathbb{R}^{n^2} as a space $\mathrm{Mat}(n, \mathbb{R})$ of real matrices of $n \times n$. The group $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ acts on $\mathrm{Mat}(n, \mathbb{R})$ by

$$(g_1, g_2) : X \mapsto g_1^{-1} X g_2, \quad \text{where } g_1, g_2 \in \mathrm{GL}(n, \mathbb{R}).$$

Therefore, we have an homomorphism $i : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \rightarrow \mathrm{GL}(n^2, \mathbb{R})$. We restrict the Weil representation of $\mathrm{Sp}(2n^2, \mathbb{R})$ to the subgroup $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ (or, more precisely, we consider the composition of i and the Weil representation). Clearly, this restriction is the natural representation of $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ in $L^2(\mathrm{Mat}(n, \mathbb{R}))$. On the other hand the group $\mathrm{GL}(n, \mathbb{R})$ is a dense open subset

in $\text{Mat}(n, \mathbb{R})$ and we can identify $L^2(\text{Mat}(n, \mathbb{R}))$ with the space L^2 on $\text{GL}(n, \mathbb{R})$ with respect to the Haar measure $\det(X)^{-n-1} dX$, the identification is given by the formula

$$f \mapsto |\det(X)|^{(n+1)/2} f(X),$$

the factor $\det(X)^{-(n+1)/2}$ is necessary to make the operator unitary. There arise a question:

To obtain the action of the Lie algebra $\mathfrak{sp}(2n^2) \ltimes \mathfrak{heis}_{2n^2+1}$ in the spectral decomposition of $L^2(\text{GL}(n, \mathbb{R}))$.

Clearly, Theorem 1 gives a solution for $n = 2$.

7. Action of $\mathfrak{sp}(8)$ in the space \mathcal{E} . Return to the notation a, b, c, d for coordinates on $\text{GL}(2, \mathbb{R})$ and $\text{Mat}(2, \mathbb{R})$. The Heisenberg algebra acts on $\text{Mat}(2, \mathbb{R})$ by the operators

$$ia, ib, ic, id, \frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}.$$

We identify $L^2(\text{Mat}(2, \mathbb{R}))$ and L^2 on the group by

$$f \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} |ad - bc|.$$

The coordinate operators ia, ib, ic, id under this identification remain the same, the partial derivatives transform to

$$\begin{aligned} \partial_a &:= \frac{\partial}{\partial a} - \frac{d}{ad - bc}; & \partial_b &:= \frac{\partial}{\partial b} + \frac{c}{ad - bc}; \\ \partial_c &:= \frac{\partial}{\partial c} + \frac{b}{ad - bc}; & \partial_d &:= \frac{\partial}{\partial d} - \frac{a}{ad - bc}. \end{aligned}$$

This gives us the action of the Heisenberg algebra \mathfrak{heis}_9 . The operators in \mathcal{E} corresponding to ∂_a, ∂_b , etc. can be easily evaluated. A straightforward calculation shows that they can be obtained from formulas (14)–(17) by a substitution $\mu_1 \rightarrow \mu_1 + 1, \mu_2 \rightarrow \mu_2 + 1$ (this is equivalent to a formal substitution $\frac{3}{2} \mapsto \frac{1}{2}$ to expressions (14)–(17)).

This gives us an action of the Heisenberg algebra \mathfrak{heis}_9 in \mathcal{E} . The symplectic Lie algebra $\mathfrak{sp}(8)$ acts by quadratic expressions in generators of the Heisenberg algebra. The author does not know a nice general formula for all 36 generators of $\mathfrak{sp}(8)$, but for a given element its expression can be written. For instance,

$$\begin{aligned} c^2 &\longleftrightarrow \frac{1}{(\mu_1 - \mu_2)(\mu_1 - \mu_2 + 1)} \frac{\partial^2}{\partial t^2} V_1^2 + \\ &+ \frac{2}{(\mu_1 - \mu_2 - 1)(\mu_1 - \mu_2 + 1)} \frac{\partial^2}{\partial t \partial s} V_1 V_2 + \frac{1}{(\mu_1 - \mu_2)(\mu_1 - \mu_2 - 1)} \frac{\partial^2}{\partial s^2} V_2^2; \\ c\partial_b &\longleftrightarrow \frac{\mu_2 - \frac{1}{2}}{(\mu_1 - \mu_2 + 1)(\mu_1 - \mu_2)} \frac{\partial^2}{\partial t^2} V_1 V_2^{-1} + \\ &+ \frac{\mu_1 + \mu_2}{(\mu_1 - \mu_2 + 1)(\mu_1 - \mu_2 - 1)} \frac{\partial^2}{\partial t \partial s} + \frac{\mu_1 - \frac{1}{2}}{(\mu_1 - \mu_2)(\mu_1 - \mu_2 - 1)} \frac{\partial^2}{\partial s^2} V_1^{-1} V_2 \end{aligned}$$

$$\begin{aligned} \partial_b^2 &\longleftrightarrow \frac{(\mu_2 - \frac{3}{2})(\mu_2 - \frac{1}{2})}{(\mu_1 - \mu_2 + 1)(\mu_1 - \mu_2)} \frac{\partial^2}{\partial t^2} V_2^{-1} + \\ &+ \frac{2(\mu_2 - \frac{1}{2})(\mu_1 - \frac{1}{2})}{(\mu_1 - \mu_2 + 1)(\mu_1 - \mu_2 - 1)} \frac{\partial^2}{\partial t \partial s} V_1^{-1} V_2^{-1} + \frac{(\mu_1 - \frac{3}{2})(\mu_1 - \frac{1}{2})}{(\mu_1 - \mu_2)(\mu_1 - \mu_2 - 1)} \frac{\partial^2}{\partial t^2} V_1^{-2}. \end{aligned}$$

For some quadratic operators Fourier-images have a simpler form. Namely for generators (1)–(4) of the subalgebra $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$ we have operators (5)–(8). In particular,

$$(19) \quad a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} + c \frac{\partial}{\partial c} + d \frac{\partial}{\partial d} \longleftrightarrow -\mu_1 - \mu_2.$$

The operator corresponding to $(ad - bc)$ is given by (13). Also

$$(20) \quad \partial_a \partial_d - \partial_b \partial_c \longleftrightarrow (\mu_1 - \frac{1}{2})(\mu_2 - \frac{1}{2}) V_1^{-1} V_2^{-1}.$$

REMARK. These simple expressions are related to a structure of the Lie algebra $\mathfrak{sp}(8)$ as a $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -module. First, we notice that algebra of matrices (18) splits into a direct sum of subalgebras consisting of matrices $\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha^t \end{pmatrix}$, $\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$ respectively. These subalgebras are invariant with respect to the adjoint action of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Denote by V_k a k -dimensional irreducible $\mathfrak{sl}(2)$ -module. The action of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ on symmetric matrices β splits as

$$(V_3 \otimes V_3) \oplus (V_1 \otimes V_1),$$

the same structure has the space of symmetric matrices γ . The space of matrices α is

$$(V_2 \oplus V_2) \otimes (V_2 \oplus V_2) = (V_3 \otimes V_3) \oplus (V_3 \otimes V_1) \oplus (V_1 \otimes V_3) \oplus (V_1 \otimes V_1).$$

For 3 invariants (elements of the spaces $V_1 \otimes V_1$) we have formulas (13), (19), (20). Two summands $V_3 \otimes V_1$, $V_1 \otimes V_3$ correspond to the subalgebra $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Three long formulas above correspond to representatives of three modules $V_3 \otimes V_3$. \square

8. A toy example: the action of $SL(2, \mathbb{R})$ in functions on plane. Consider the natural action of the group $SL(2, \mathbb{R})$

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varphi(x, y) = \varphi(ax + cy, bx + dy)$$

on the space $C_0^\infty(\mathbb{R}^2)$ of smooth compactly supported functions on the plane \mathbb{R}^2 . For $\varphi(x, y) \in C_0^\infty(\mathbb{R}^2)$ we assign the function

$$(21) \quad F(u, \sigma, \varepsilon) = J\varphi(u, \sigma, \varepsilon) := \int_{\mathbb{R}} \varphi(t, tu) t^{-\sigma//\varepsilon} dt,$$

where $u \in \mathbb{R}$, $\sigma \in \mathbb{C}$, $\varepsilon = 0, 1$. For $\text{Re } \sigma < 1$ the integral converges and is holomorphic in σ , this function admits a meromorphic continuation to arbitrary σ with possible poles at $\sigma = 1, 2, 3, \dots$. Denote by \mathcal{M} the space of all functions F that can be obtained in this way. The action of $SL(2, \mathbb{R})$ transfers to the action

$$R \begin{pmatrix} a & b \\ c & d \end{pmatrix} F(u, \sigma, \varepsilon) = F\left(\frac{b + ud}{a + uc}, \sigma, \varepsilon\right) (a + uc)^{-1 + \sigma//\varepsilon}.$$

Next, define the inner product on the space \mathcal{M} by

$$\langle F_1, F_2 \rangle = \sum_{\varepsilon=1,2} \int_{\mathbb{R}} \int_{\mathbb{R}} F_1(u, i\tau, \varepsilon) \overline{F_2(u, i\tau, \varepsilon)} du d\tau.$$

Passing to the completion, we get a Hilbert space \mathcal{H} (a sum of two copies of $L^2(\mathbb{R} \times \mathbb{R})$). The operator J extends to a unitary operator $J : L^2(\mathbb{R}^2) \rightarrow \mathcal{H}$ (see, e.g., [2], §I.3), and this gives us a spectral decomposition of the representation T .

As we mentioned above, the representation T is a restriction of the Weil representation of $\mathrm{Sp}(4, \mathbb{R}) \ltimes \mathrm{Heis}_5$ to the subgroup $\mathrm{SL}(2, \mathbb{R})$. Our previous considerations suggest that we can transfer the action of the Lie algebra $\mathfrak{sp}(4) \ltimes \mathfrak{heis}_5$ to the space \mathcal{M} .

Define a shift operator

$$VF(u, \sigma, \varepsilon) = F(u, \sigma + 1, \varepsilon + 1).$$

We have the following correspondence of operators:

$$(22) \quad x \longleftrightarrow V^{-1};$$

$$(23) \quad y \longleftrightarrow uV^{-1};$$

$$(24) \quad \frac{\partial}{\partial x} \longleftrightarrow \sigma V - u \frac{\partial}{\partial u} V;$$

$$(25) \quad \frac{\partial}{\partial y} \longleftrightarrow u \frac{\partial}{\partial u} V.$$

Let us verify (25). By (21), we get

$$\frac{\partial}{\partial u} F(u, \sigma + 1, \varepsilon + 1) := \int_{\mathbb{R}} t \frac{\partial \varphi}{\partial u}(t, tu) t^{-\sigma+1//\varepsilon+1} dt = \int_{\mathbb{R}} \frac{\partial \varphi}{\partial u}(t, tu) t^{-\sigma//\varepsilon} dt,$$

this is the required result. A derivation of formulas (22) and (23) are straightforward. To verify (24) we write

$$(26) \quad \int_{\mathbb{R}} \frac{d}{dt}(\varphi(t, tu)) t^{-\sigma//\varepsilon} dt = \int_{\mathbb{R}} \frac{\partial \varphi}{\partial x}(t, tu) t^{-\sigma//\varepsilon} dt + \int_{\mathbb{R}} (tu) \frac{\partial \varphi}{\partial y}(t, tu) t \cdot t^{-\sigma//\varepsilon} dt.$$

For $\mathrm{Re} \sigma < 0$, integrating the left hand side by parts we get $\sigma F(u, \sigma + 1, \varepsilon + 1)$, by analytic continuation we extend this to arbitrary $\sigma \in \mathbb{C}$. The second term in the right hand side is

$$\int_{\mathbb{R}} \left(y \frac{\partial \varphi(x, y)}{\partial y} \right) \Big|_{x=t, y=tu} t \cdot t^{-\sigma+1//\varepsilon+1} dt.$$

By (23) and (25) it equals $u \frac{\partial}{\partial u} F(u, \sigma + 1, \varepsilon + 1)$. The first term in the right hand side of (26) is the J -image of $\frac{\partial}{\partial x}$.

So we have a possibility to evaluate the image of any differential operator, which is polynomial in $x, y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. In particular, we get formulas for the action of the symplectic Lie algebra $\mathfrak{sp}(4)$.

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