Derivatives of Eigenvalues

Feng Ling

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Let's assume we work in an $\mathbb{R}$ vector space with a non-degenerate symmetric bilinear pairing $\langle \cdot, \cdot \rangle$, and a differential operator d satisfying the usual Leinbiz's rule. We will use A to denote linear operator/matrix in a basis $A = (a_{ij})$ and $\mathbf{w}$ for a vector $w = (w_i)$.

If we have $\langle \mathbf{v}, \mathbf{v} \rangle = 1$ ($\mathbf{v}$ has unit norm), then $d\langle \mathbf{v}, \mathbf{v} \rangle = 2\langle \mathbf{v}, d\mathbf{v} \rangle = 0$ follows. Then the eigenvalue equation (assuming L diagonalizable) gives us

$$\begin{aligned}
\lambda \mathbf{v} &= \mathsf{L}\mathbf{v} \\
d(\lambda \mathbf{v}) &= d(\mathsf{L}\mathbf{v}) \\
d\lambda \mathbf{v} + \lambda d\mathbf{v} &= d\mathsf{L}\mathbf{v} + \mathsf{L}d\mathbf{v} \\
\langle \mathbf{v}, d\lambda \mathbf{v} \rangle + \langle \mathbf{v}, \lambda d\mathbf{v} \rangle &= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathbf{v}, \mathsf{L}d\mathbf{v} \rangle \\
d\lambda \langle \mathbf{v}, \mathbf{v} \rangle + \lambda \langle \mathbf{v}, d\mathbf{v} \rangle &= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathbf{v}, \mathsf{L}d\mathbf{v} \rangle \\
d\lambda &= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathbf{v}, \mathsf{L}d\mathbf{v} \rangle
\end{aligned}$$

Now if the operator L is self-adjoint (matrix is symmetric since we are over $\mathbb{R}$), we have

$$\begin{aligned}
d\lambda &= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathsf{L}\mathbf{v}, d\mathbf{v} \rangle \\
&= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + \langle \lambda \mathbf{v}, d\mathbf{v} \rangle = \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + \lambda \langle \mathbf{v}, d\mathbf{v} \rangle \\
&= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle
\end{aligned}$$

Written is usual matrix multiplication notation, this is just $d\lambda = \mathbf{v}^T d\mathsf{L}\mathbf{v}$.

We can make our lives slightly messier by talking about the generalized eigenvalue equation

$$\lambda \mathsf{M}\mathbf{v} = \mathsf{L}\mathbf{v}$$

In this case, if we assume the condition $\langle \mathbf{v}, \mathsf{M}\mathbf{v} \rangle = 1$, then we have $d\langle \mathbf{v}, \mathsf{M}\mathbf{v} \rangle = \langle d\mathbf{v}, \mathsf{M}\mathbf{v} \rangle + \langle \mathbf{v}, d\mathsf{M}\mathbf{v} \rangle + \langle \mathbf{v}, \mathsf{M}d\mathbf{v} \rangle = 0$. Then

$$\begin{aligned}
d(\lambda \mathsf{M}\mathbf{v}) &= d(\mathsf{L}\mathbf{v}) \\
d\lambda \mathsf{M}\mathbf{v} + \lambda d\mathsf{M}\mathbf{v} + \lambda \mathsf{M}d\mathbf{v} &= d\mathsf{L}\mathbf{v} + \mathsf{L}d\mathbf{v} \\
\langle \mathbf{v}, d\lambda \mathsf{M}\mathbf{v} \rangle + \langle \mathbf{v}, \lambda d\mathsf{M}\mathbf{v} \rangle + \langle \mathbf{v}, \lambda \mathsf{M}d\mathbf{v} \rangle &= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathbf{v}, \mathsf{L}d\mathbf{v} \rangle \\
d\lambda \langle \mathbf{v}, \mathsf{M}\mathbf{v} \rangle - \lambda \langle d\mathbf{v}, \mathsf{M}\mathbf{v} \rangle &= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathbf{v}, \mathsf{L}d\mathbf{v} \rangle \\
d\lambda &= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathbf{v}, \mathsf{L}d\mathbf{v} \rangle + \lambda \langle d\mathbf{v}, \mathsf{M}\mathbf{v} \rangle
\end{aligned}$$

If we again assume L to be self-adjoint, we get

$$\begin{aligned}
d\lambda &= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathsf{L}\mathbf{v}, d\mathbf{v} \rangle + \lambda \langle d\mathbf{v}, \mathsf{M}\mathbf{v} \rangle \\
&= \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle + 2\lambda \langle d\mathbf{v}, \mathsf{M}\mathbf{v} \rangle
\end{aligned}$$

Or in matrix form $d\lambda = \mathbf{v}^T d\mathsf{L}\mathbf{v} + 2\lambda (d\mathbf{v})^T \mathsf{M}\mathbf{v}$.

This result is rather unsatisfactory given that $d\mathbf{v}$ appears on the right hand side. But if we further assume that M is self-adjoint, then the eigenvector norm condition reduces to $2\langle d\mathbf{v}, \mathsf{M}\mathbf{v} \rangle + \langle \mathbf{v}, d\mathsf{M}\mathbf{v} \rangle = 0$. This gives $d\lambda = \mathbf{v}^T d\mathsf{L}\mathbf{v} - \lambda v^T d\mathsf{M}\mathbf{v}$.