

Derivatives of Eigenvalues

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Let's assume we work in a vector space with a non-degenerate symmetric bilinear pairing $\langle \cdot, \cdot \rangle$, and a differential operator d satisfying the usual Leibniz's rule. We will use A to denote linear operator/matrix in a basis $A = (a_{ij})$ and \mathbf{w} for a vector $w = (w_i)$.

Assuming L is not defect at the considered eigenvalue (multiplicity = 1), we have two equations $L\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{u}^\dagger L = \lambda\mathbf{u}^\dagger$, where λ is the eigenvalue, \mathbf{v} is the (right) eigenvector, and \mathbf{u} the left eigenvector. (Note that a left eigenvector of a matrix A is a (right) eigenvector of the adjoint A^\dagger .)

Now differentiate the first (right) eigenvector equation gives us

$$\begin{aligned} d(\lambda\mathbf{v}) &= d(L\mathbf{v}) \\ d\lambda\mathbf{v} + \lambda d\mathbf{v} &= dL\mathbf{v} + Ld\mathbf{v} \end{aligned}$$

Pair the above with the left eigenvectors yields

$$\begin{aligned} \langle \mathbf{u}, d\lambda\mathbf{v} \rangle + \langle \mathbf{u}, \lambda d\mathbf{v} \rangle &= \langle \mathbf{u}, dL\mathbf{v} \rangle + \langle \mathbf{u}, Ld\mathbf{v} \rangle \\ d\lambda\langle \mathbf{u}, \mathbf{v} \rangle + \langle \lambda\mathbf{u}, d\mathbf{v} \rangle &= \langle \mathbf{u}, dL\mathbf{v} \rangle + \langle \mathbf{u}, Ld\mathbf{v} \rangle \\ d\lambda\langle \mathbf{u}, \mathbf{v} \rangle &= \langle \mathbf{u}, dL\mathbf{v} \rangle + \langle L^\dagger\mathbf{u} - \lambda\mathbf{u}, d\mathbf{v} \rangle \\ d\lambda &= \frac{\langle \mathbf{u}, dL\mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{v} \rangle} \end{aligned}$$

Now if the operator L is self-adjoint (e.g. symmetric if we are over \mathbb{R}), we then have $\mathbf{u} = \mathbf{v}$, thus

$$d\lambda = \frac{\langle \mathbf{v}, dL\mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} = \langle \mathbf{v}, dL\mathbf{v} \rangle$$

Or in another notation, we have $d\lambda = \mathbf{v}^\dagger dL\mathbf{v}$.

Same principles apply for the generalized eigenvalue problem. If we again have left eigenvector \mathbf{u} and (right) eigenvector \mathbf{v} such that $(L - \lambda M)\mathbf{v} = \mathbf{u}^\dagger(L - \lambda M) = 0$, we can derive

$$\begin{aligned} d(L - \lambda M)\mathbf{v} &= 0 \\ (dL - d\lambda M - \lambda dM)\mathbf{v} + (L - \lambda dM)d\mathbf{v} &= 0 \end{aligned}$$

Pairing with \mathbf{u} returns

$$\begin{aligned} \mathbf{u}^\dagger(dL - d\lambda M - \lambda dM)\mathbf{v} + \mathbf{u}^\dagger(L - \lambda dM)d\mathbf{v} &= 0 \\ \mathbf{u}^\dagger(dL - d\lambda M - \lambda dM)\mathbf{v} &= 0 \\ \mathbf{u}^\dagger dL\mathbf{v} - \mathbf{u}^\dagger \lambda dM\mathbf{v} &= \mathbf{u}^\dagger d\lambda M\mathbf{v} \\ d\lambda \mathbf{u}^\dagger M\mathbf{v} &= \mathbf{u}^\dagger dL\mathbf{v} - \lambda \mathbf{u}^\dagger dM\mathbf{v} \\ d\lambda &= \frac{\mathbf{u}^\dagger dL\mathbf{v} - \lambda \mathbf{u}^\dagger dM\mathbf{v}}{\mathbf{u}^\dagger M\mathbf{v}} \end{aligned}$$

Now if we assume both L and M to be self-adjoint, we will again have $\mathbf{u} = \mathbf{v}$. Then normalizing \mathbf{v} such that $\langle \mathbf{v}, M\mathbf{v} \rangle = 1$ gives $d\lambda = \mathbf{v}^\dagger dL\mathbf{v} - \lambda \mathbf{v}^\dagger dM\mathbf{v}$.