(co)fiber sequences and $\pi_3(S^2)$

Feng Ling

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This note attempts to give an intuitive explanation of cofiber and fiber sequences in as basic of a language as possible without losing rigor[1]. Then the classical application of the fiber sequence induced long exact sequence to the calculation of $\pi_3(S^2)$ is justified. In essence this is a spelled out guide for sections 8.1-5 and 9.3 of May's A Concise Course in Algebraic Topology.

1 Slogans and Motivations

We want to study structures of new spaces. And one obvious way to go about it is to study maps from spaces that we know a lot about to these unknown ones.

But maps can behave very badly in the sense of arbitrariness even if we assume strong conditions like continuity. For example, we can still have space filling curves.

Thus we really want to work with nice maps that carry over properties and knowledge we desire. Intuitively, the simplest of maps are inclusions/injections and projections/surjections. As a result, given an arbitrary map, we would like to compare how similar they are to a suitable notion of injective and surjective maps[2] in our working category of spaces.

2 Based Spaces and Based Maps

The category of spaces that we will be working in are the based spaces. They are nice[3] topological spaces with an added distinguished point called its base point (from now on denoted by $*$), like how a group has the identity as a distinguished element.[4]

And we call maps between based spaces based if it respects the base point, i.e. $f : X \to Y$ is based if $f(*_X) = *_Y$.

Luckily the function space of based maps $\mathrm{Maps}(X, Y)$ for X, Y based spaces is also a based space with base point the constant map sending everything to base point of Y. [5]

Turns out we can have a natural adjunction homeomorphism[6]

$$\mathrm{Maps}(X, \mathrm{Maps}(Y, Z)) \cong \mathrm{Maps}(X \wedge Y, Z) \tag{1}$$

[1] or the Explain Like I'm an Analyst (ELIA) style, no disrespect to analysis. Thus many jargon is deferred to the footnotes.

[2] i.e. the cofibrations and the (Hurewicz) fibration.

[3] compactly generated

[4] Category is pointed, has a zero object, namely the space with just the base point.

[5] the category of (nice) based spaces is Cartesian closed with respect to function maps.

[6] recognizable as an everyday adjunction in a different notation, namely $Hom(X, \mathrm{Maps}_Y(Z)) \cong Hom(\wedge_Y(X), Z)$, where Hom are morphisms in the category of based spaces.

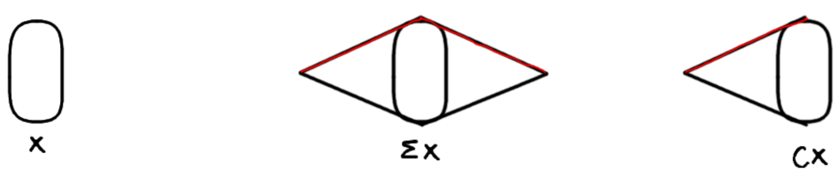
We know the LHS, and the RHS has $X \wedge Y := (X \times Y)/(X \vee Y)$[7]. here $\times$ denote the usual Cartesian product with base point $(*_X, *_Y)$. And $(X \vee Y)$ is space X, Y glued at their base point with no other changes, i.e. $(*_X, y) \cup (x, *_Y)$ for all $x \in X, y \in Y$ with base point also at $(*_X, *_Y)$. And the thing behind the slash is the quotient, in other words, they are squeezed to form the base point of the resulting space.

Another important property of smash product is that it is also (anti)symmetric with respect to the two input spaces. In other words, $X \wedge Y = -Y \wedge X \cong X \wedge Y$.

Definition. Below I is the unit interval $[0, 1]$ and S^1 is the circle[8]

1. The loop space of X is $\Omega X :=$ Maps(S^1, X), with base point the constant loop about $*_X$.
2. The suspension of X is $\Sigma X := X \wedge S^1$, with base point indicated in red below.
3. The path space[9] of X is $PX :=$ Maps(I, X), again with base point the constant map of $*_X$.
4. The cone of X is $CX := (X \times I)/(X \times \{0\})$, with base point indicated in red below.

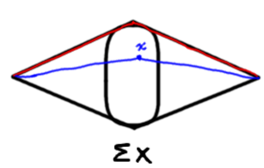
Here are some pictures illustrating the above definitions.



Proposition 2.1. *Clearly substituting S^1 in place of Y in Equation 1, we get*

$$Maps(X, \Omega Z) \cong Maps(\Sigma X, Z) \tag{2}$$

In fact, we can see a proof of this adjunction pictorially as below.



[7]It was brought to my attention that the choice of notation has to do with how a cross $\times$ is formed by a $\vee$ sitting on top of a $\wedge$, fulfilling the usual intuition of quotients.

[8]Turns out homotopically, the contractible spaces and the circle are two spaces that we know really well.

[9]Old literature adhereing to Serre might use the notation EX. This is also distinguished from Y^I in May as the paths in PX do not have to start at the base point while the other one does. This is important as X^I is clearly contractible, while the other is not necessarily so.

3 Cofiber and Fiber Sequences

Now we want to construct the cofiber and fiber sequences. The cofiber sequence gives us concrete information on how related is our arbitrary map $f : X \to Y$ is to a nice inclusion, and the fiber sequence to a nice surjection.

Definition. The cofiber sequence of map $f : X \to Y$ is

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma Cf \longrightarrow \Sigma\Sigma X \longrightarrow \cdots \tag{3}$$

where i is an inclusion map and Cf is called the (homotopy) cofiber of the map f. The sequence continues indefinitely to the right with one more suspension operator every 3 steps.

The cofiber Cf can be defined as $CX \cup_f Y$. This notation means that we glue the cone of X and Y along the image of X under f. Pictorially,

We can also express this relation via a (commuting) pushout square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "{(id,1)}"] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "i"] \\ CX \arrow[r] & X \cup_f Y \end{tikzcd}$$

Crucially, we are calling the above one cofiber sequence because (homotopically) we are doing the same thing to our map every step.

Lemma 3.1. $\Sigma X \simeq Ci$

Proof. $\square$

Based on our previous adjuction relation, we can really see a dual picture as well.

Definition. The fiber sequence of map $f : X \to Y$ is

$$\cdots \longrightarrow \Omega\Omega Y \longrightarrow \Omega Ff \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow Ff \xrightarrow{\pi} X \xrightarrow{f} Y \tag{4}$$

where π is a projection map and Ff is called the (homotopy) fiber of the map f. The sequence continues indefinitely to the left with one more looping operator every 3 steps.

The fiber Ff is defined to be $X \times_f PY$[10]. This notation means that $X \times_f PY = \{(x, \gamma) \in X \times PY : f(x) = \gamma(1)\}$.

The same information is conveyed in the (commuting) pullback square below.

$$\begin{tikzcd}
X \times_f PY \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & PY \arrow[d, "ev(1)"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

We are calling this one fiber sequence because (homotopically) we are doing the same thing to our map every step as well.

Lemma 3.2. $\Omega Y \simeq F\pi$

Proof. $\square$

4 Long Exact Sequences of Homotopic Maps

Definition. The homotopy classes of maps from space X to Y is defined as

$$[X, Y] := \text{Maps}(X, Y)/\simeq$$

[10]the fibered product

Theorem 4.1. *For a fiber sequence of $f : X \to Y$ and an arbitrary based space Z, we have the following a long exact sequence (LES) of maps up to homotopy:*

$$\cdots \longrightarrow [Z, \Omega Ff] \longrightarrow [Z, \Omega X] \longrightarrow [Z, \Omega Y] \longrightarrow [Z, Ff] \xrightarrow{\pi^*} [Z, X] \xrightarrow{f^*} [Z, Y]$$

Here a LES means that at successive stage $Z' \xrightarrow{i} Z \xrightarrow{j} Z''$ we have $j(z) = *_{Z''}$ if and only if there is a z' such that $z = i(z')$. [11]

Proof. We separate the proof into 3 parts.

1. The last stage is exact.

$$\begin{tikzcd} Z \arrow[r, dotted] \arrow[rr, bend left] \arrow[dr] & Ff \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & PY \arrow[d, "ev(1)"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

2. Looping and fibering commutes[12].

   First we need to show $\Omega(PY) \cong P(\Omega Y)$. By passing through the adjunction, we can use the (anti)symmetry of smash products between I and S^1 and then back through the adjunction relation to get it.

   Because looping is a functor, we can apply it to our previous pullback square and retain it's universal property.

$$\begin{tikzcd} \Omega(Ff) \arrow[r] \arrow[d, "\Omega\pi"] \arrow[dr, phantom, "\lrcorner"] & \Omega(PY) \arrow[d] \\ \Omega X \arrow[r, "\Omega f"] & \Omega Y \end{tikzcd}$$

   But then by definition of the (homotopy) fiber, we have another pullback square,

$$\begin{tikzcd} F(\Omega f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P(\Omega Y) \cong \Omega(PY) \arrow[d] \\ \Omega X \arrow[r, "\Omega f"] & \Omega Y \end{tikzcd}$$

   Thus by universal property of pullback, we get $\Omega(Ff) \cong F(\Omega f)$.

3. show the triangle.

4. Finally with the technical lemma presented in the **Appendix** along with some inductive argument, we can prove the whole sequence is exact.

[11] One motivation for this exactness criterion is that it is sort of the easiest chain of maps we can think of; anything goes further than a stage is *exactly* the trivial map. One imprecise analogy can be made between this and the memoryless-ness of Markov chains, where states of ≥ 2 steps ago have trivial/no effects on the current state.

[12] This in fact generalizes to limits and limits or colimits and colimits commute.

□

We have a corresponding theorem about LES for the cofiber sequence. The only difference is that there we are mapping from our (cofiber) sequence of spaces to an arbitrary space Z[13]. The proofs are almost exact parallels of above (also contained in May section 8.4).

To make everything more practical, we want to translate this LES of based spaces to that of abelian groups, a category where everything is classified and exactness reduce to the normal notion of ker = im.

Proposition 4.2. *$[\Sigma\Sigma X, Y]$ is an abelian group for all based spaces X and Y.*

This can be reasoned as commuting two small (homotopy) squares inside a big (homotopy) square by selectively contracting parts to the constant (base point) map. It is done at the end of May section 8.2.

5 Facts about Circles and Spheres

First a fact that is easy to believe:

Proposition 5.1. $\Sigma S^n = S^{n+1}$

In low dimensions, a simple picture like the ones in secion 2 above will do. In general, we can see that S^n is exactly the equator of S^{n+1}.

Next we introduce the concept of "Hopf bundle" or "Hopf fibration." These are maps between spheres of certain dimensions with lower dimensional spheres as its fiber. It is a bundle in the sense that it admits a local trivialization into Cartesian products. The one of interest to us is the following

$$\begin{array}{ccc} f^{-1}(pt) \cong S^1 & \hookrightarrow & X = S^3 \\ & & \Big\downarrow f \\ & & Y = S^2 \end{array}$$

To be able to apply our fiber sequence machinery to the Hopf map, we need to know that up to homotopy, the actual fiber ($f^{-1}(pt) \cong S^1$) from this fiber bundle is the same as the fiber (Ff) that was used in the fiber sequence earlier[14].

Proposition 5.2. *$[S^m, S^m] \cong \mathbb{Z}$ for all $m \in \mathbb{Z}^{\geq 1}$.*

This homeomorphism is realized by the degree map. Intuitively we can see that wrapping spheres around itself different number of times should give non homotopic maps as the preimage of any point changes multiplicity.

[13] Contravariant instead of covariant.

[14] This require first knowing Hopf bundle is a (Hurewicz) fibration, then applying the lemma in Appendix again gives us what we want. Turns out that in general, a fiber bundle does not have to be a (Hurewicz) fibration.

Proposition 5.3. $[S^k, S^1] \cong 0$ *for all* $k \neq 1$.

This is true because the universal cover $\mathbb{R}$ of S^1, realized by the map $\phi : \mathbb{R} \to S^1 \subset \mathbb{C}$ by sending t to $\exp(2\pi\sqrt{-1}t)$, is contractible. Then by covering space theory, the lifting property[15] dictates that maps into S^1 can be lifted to maps into $\mathbb{R}$. Thus contractibility gives a homotopy of any map to the constant trivial map.

6 Punchline

Theorem 6.1. $\pi_3(S^2) = [S^3, S^2] \cong \mathbb{Z}$

Proof. Inspecting the LES induced by the fiber sequence of the Hopf bundle (i.e. a substitution of the total space $E = S^3$ for X, base space $B = S^2$ for Y, fiber bundle map p for f, fiber $F = S^1$ for Ff, and finally the space $Z = S^2 = \Sigma\Sigma S^0$), we get the following exact sequence in abelian groups

$$\cdots \longrightarrow [S^2, \Omega S^1] \longrightarrow [S^2, \Omega S^3] \longrightarrow [S^2, \Omega S^2] \longrightarrow [S^2, S^1] \longrightarrow \cdots$$

Applying the adjunction relation (Eq. 2) to first three terms and using proposition 5.2 on the first and last term and proposition 5.3 on the second term, we get a "short exact sequence"

$$[S^2, \Omega S^1] \cong [S^3, S^1] \cong 0 \longrightarrow [S^2, \Omega S^3] \cong [S^3, S^3] \cong \mathbb{Z} \longrightarrow [S^2, \Omega S^2] \cong [S^3, S^2] \longrightarrow [S^2, S^1] \cong 0$$

Therefore the injective-ness and surjective-ness of the middle map ensures that $[S^3, S^2] \cong \mathbb{Z}$. $\square$

This theorem shows that there is somehow a whole integral family of nontrivial maps from the 3-sphere to the 2-sphere up to homotopy, a fact that topologists would sometimes analogize to the 2-sphere having a "3-dimensional hole."

Acknowledgments

Many thanks to Prof. Andrew Blumberg and Ernest Fontes for putting up with my endless silly questions.

Appendix

Lemma. For a fibration $f : X \to Y$, we have $Ff \simeq f^{-1}(*_Y)$

Proof. Since we have a commuting square, the fibration gives us a lifting map

$$\begin{tikzcd}
Nf = X \times_f Y^I \arrow[r, "\pi"] \arrow[d, hook, "{(id,id,1)}"] & X \arrow[d, two heads, "f"] \\
Nf \times I \arrow[r, "g"] \arrow[ur, dotted, "\tilde{g}"] & Y
\end{tikzcd}$$

[15]fibration and CHP

where Nf is the mapping cocylinder, $g(x, \gamma, t) = \gamma(t)$, and π is the projection on first coordinate.

Next we can construct a continuous homotopy $H_t : Nf \to Nf$ via $H_t(x, \gamma) = (\tilde{g}(x, \gamma, t), \gamma|_{[0,t]})$. Restricting the domain to $Ff \subset Nf$, we note that when $t = 1$, we have $H_1(x, \gamma) = (x, \gamma)$ the identity map on Ff. When $t = 0$, it gives $H_0(x, \gamma) = (f^{-1}(\gamma(0) = *_Y), cst_{*_Y})$.

Now define $j : f^{-1}(*_Y) \to Ff$ via $j(x) = (x, cst_{*_Y})$, then the above homotopy is the following diagram

$$\begin{tikzcd} Ff \arrow[r, "\tilde{g}(-,-,0)"] \arrow[rr, bend right, "H_1 \simeq id"'] & f^{-1}(*_Y) \arrow[r, "j"] & Ff \end{tikzcd}$$

This demonstrates that we have a homotopic left inverse of j. Since we have projection onto the first coordinate as a right inverse, namely $\pi \circ j(x \in f^{-1}(*_Y)) = \pi(x, cst_{*_Y}) = x$, we have proved that $Ff \simeq f^{-1}(*_Y)$. $\square$