Applications of Discrete Exterior Calculus on Exact Conservation FEM

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This paper surveys the applications of De Rham cohomology and discrete exterior calculus on linear elliptic PDEs. A code for 2D stokes system constructed in this paradigm is tested with known results.

Problem 1 *we would like to find the solution to the equation*

$$\Delta u^{(0)} = \mathrm{d}^*\mathrm{d}u^{(0)} = f^{(0)}$$

where $u^{(0)}, f^{(0)}$ *are 0-forms on 2D domain* Ω

Answer. Introducing the intermediate 1-form $p^{(1)}$

$$\Rightarrow \begin{cases} \mathrm{d}u^{(0)} = p^{(1)} \\ \mathrm{d}^*p^{(1)} = f^{(0)} \end{cases}$$

Taking inner product with test functions $v^{(0)}, w^{(1)}$, we get

$$\begin{cases} (w^{(1)}, \mathrm{d}u^{(0)})_\Omega = (w^{(1)}, p^{(1)})_\Omega \\ (v^{(0)}, \mathrm{d}^*p^{(1)})_\Omega = (v^{(0)}, f^{(0)})_\Omega \end{cases}$$

Performing integration by parts (from generalized Stokes theorem)

$$\begin{cases} (w^{(1)}, \mathrm{d}u^{(0)})_\Omega = (w^{(1)}, p^{(1)})_\Omega \\ (\mathrm{d}v^{(0)}, p^{(1)})_\Omega - \int_{\partial\Omega} v^{(0)} \wedge \star p^{(1)} = (v^{(0)}, f^{(0)})_\Omega \end{cases}$$

Next we approximate the forms in finite dimension spaces

$$\begin{aligned}
u_h^{(0)}(\xi^1, \xi^2) &= \sum_{i,j} \mathrm{u}_{i,j} P_{i,j}(\xi^1, \xi^2) \\
v_h^{(0)}(\xi^1, \xi^2) &= \sum_{i,j} \mathrm{v}_{i,j} P_{i,j}(\xi^1, \xi^2) \\
p_h^{(1)}(\xi^1, \xi^2) &= \sum_{i,j} \mathrm{p}^1_{i,j} L^1_{i,j}(\xi^1, \xi^2) + \sum_{i,j} \mathrm{p}^2_{i,j} L^2_{i,j}(\xi^1, \xi^2) \\
w_h^{(0)}(\xi^1, \xi^2) &= \sum_{i,j} \mathrm{w}^1_{i,j} L^1_{i,j}(\xi^1, \xi^2) d\xi^1 + \sum_{i,j} \mathrm{w}^2_{i,j} L^2_{i,j}(\xi^1, \xi^2) d\xi^2
\end{aligned}$$

where $P_{i,j} = N_i(x^1)N_j(x^2)$, $L^1_{i,j} = M_i(x^1)N_j(x^2)$, and $L^2_{i,j} = N_i(x^1)M_j(x^2)$ are the basis functions. Here $\mathrm{u}_{i,j}, \mathrm{v}_{i,j}, \mathrm{p}^k_{i,j}, \mathrm{w}^k_{i,j}$ are the nodal/edge values for each finite dimensional projections.

Here we can choose an appropriate reordering of the basis function for both P_i and L_i.

Substituting and rearranging terms give us

$$\begin{cases} \mathrm{M}_{(1)}\mathrm{D}_{(1,0)} \cdot \mathbf{u} - \mathrm{M}_{(1)} \cdot \mathbf{p} & = \mathbf{0} \\ \mathrm{D}_{(0,1)}\mathrm{M}_{(1)} \cdot \mathbf{p} - \int_{\partial\Omega} v^{(0)} \wedge \star p^{(1)} & = \mathrm{M}_{(0)}(f^{(0)}) \end{cases}$$

where $\mathrm{D}_{(1,0)}$ is the discrete matrix representation of divergence and $\mathrm{D}_{(0,1)} = \mathrm{D}^T_{(1,0)}$ is the dual operator. Furthermore, $\mathrm{M}_{(i)}$ is the mass matrix for the i-form basis functions integrated using (Gaussian) quadrature over the parent element domain. Specifically,

$$\mathrm{M}_{(0)} = \sum_{i,j} \int_\Omega P_i P_j \det J dx^1 \wedge dx^2$$

$$\mathrm{M}_{(1)} = \sum_{i,j} \int_\Omega L_i L_j \det J dx^1 \wedge dx^2$$

$$\mathrm{M}_{(0)}(f^{(0)}) = \sum_{i} \int_\Omega f^{(0)} P^k_i \det J dx^1 \wedge dx^2$$

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Problem 2 *we would like to find the solution to the equation*

$$\Delta u^{(2)} = \mathrm{d}\mathrm{d}^* u^{(2)} = f^{(2)}$$

where $u^{(2)}, f^{(2)}$ *are 2-forms on 2D domain* Ω

Answer. Introducing the intermediate 1-form $p^{(1)}$

$$\Rightarrow \begin{cases} \mathrm{d}^* u^{(2)} = p^{(1)} \\ \mathrm{d}p^{(1)} = f^{(2)} \end{cases}$$

Taking inner product with test functions $v^{(1)}, w^{(2)}$, we get

$$\begin{cases} (v^{(1)}, \mathrm{d}^* u^{(2)})_\Omega = (v^{(1)}, p^{(1)})_\Omega \\ (w^{(2)}, \mathrm{d}p^{(1)})_\Omega = (w^{(2)}, f^{(2)})_\Omega \end{cases}$$

Performing integration by parts (from generalized Stokes theorem)

$$\begin{cases} (\mathrm{d}v^{(1)}, u^{(2)})_\Omega - \int_{\partial\Omega} v^{(1)} \wedge \star u^{(2)} = (v^{(1)}, p^{(1)})_\Omega \\ (w^{(2)}, \mathrm{d}p^{(1)})_\Omega = (w^{(2)}, f^{(2)})_\Omega \end{cases}$$

Next we approximate the forms in finite dimension spaces

$$\begin{aligned}
u_h^{(2)}(\xi^1,\xi^2) &= \sum_{i,j} \mathrm{u}_{i,j} S_{i,j}(\xi^1,\xi^2) d\xi^1 \wedge d\xi^2 \\
p_h^{(1)}(\xi^1,\xi^2) &= \sum_{i,j} \mathrm{p}^1_{i,j} L^1_{i,j}(\xi^1,\xi^2) d\xi^1 + \sum_{i,j} \mathrm{p}^2_{i,j} L^2_{i,j}(\xi^1,\xi^2) d\xi^2 \\
v_h^{(1)}(\xi^1,\xi^2) &= \sum_{i,j} \mathrm{v}^1_{i,j} L^1_{i,j}(\xi^1,\xi^2) d\xi^1 + \sum_{i,j} \mathrm{v}^2_{i,j} L^2_{i,j}(\xi^1,\xi^2) d\xi^2 \\
w_h^{(2)}(\xi^1,\xi^2) &= \sum_{i,j} \mathrm{w}_{i,j} S_{i,j}(\xi^1,\xi^2) d\xi^1 \wedge d\xi^2
\end{aligned}$$

where $P_{i,j} = N_i(\xi^1)N_j(\xi^2)$, $L^1_{i,j} = M_i(\xi^1)N_j(\xi^2)$, and $L^2_{i,j} = N_i(\xi^1)M_j(\xi^2)$ are the basis functions. Here $\mathrm{u}_{i,j}, \mathrm{w}_{i,j}, \mathrm{p}^k_{i,j}, \mathrm{v}^k_{i,j}$ are the nodal/edge values for each finite dimensional projections.

Here we can choose an appropriate reordering of the basis function for both P_i and L_i.

Substituting and rearranging terms give us

$$\begin{cases} \mathrm{D}_{(1,2)}\mathrm{M}_{(2)} \cdot \mathbf{u} - \mathrm{M}_{(1)} \cdot \mathbf{p} & = \int_{\partial\Omega} v^{(1)} \wedge \star u^{(2)} \\ \mathrm{M}_{(2)}\mathrm{D}_{(2,1)} \cdot \mathbf{p} & = \mathrm{M}_{(2)}(f^{(2)}) \end{cases}$$

where $\mathrm{D}_{(1,0)}$ is the discrete matrix representation of divergence and $\mathrm{D}_{(0,1)} = \mathrm{D}^T_{(1,0)}$ is the dual operator. Furthermore, $\mathrm{M}_{(i)}$ is the mass matrix for the i-form basis functions integrated using (Gaussian) quadrature over the parent element domain. Specifically,

$$\mathrm{M}_{(1)} = \sum_{i,j} \int_\Omega L_i L_j g^{i,j} \det J d\xi^1 \wedge d\xi^2$$

$$\mathrm{M}_{(2)} = \sum_{i,j} \int_\Omega S_i S_j \frac{1}{\det J} d\xi^1 \wedge d\xi^2$$

$$\mathrm{M}_{(0)}(f^{(0)}) = \sum_{i} \int_\Omega f^{(0)} P_i^k \det J d\xi^1 \wedge d\xi^2$$

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Problem 3 *We would like to find the solution to the 2D incompressible stokes flow problem with unit viscosity on 2D domain* Ω

$$\Delta u^{(1)} + \mathrm{d}^* p^{(2)} = f^{(1)}$$

$$\mathrm{d}u^{(1)} = 0$$

[. Manufactured Solution] The analytic solution. Given solutions

$$\begin{aligned}
u^{(1)} &= u^x \mathrm{d}x^1 + u^y \mathrm{d}y \\
p^{(2)} &= p\, \mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

we can calculate the corresponding vorticity and forcing term

$$\omega^{(0)} = \omega$$
$$f^{(1)} = f_1 \mathrm{d}x + f_2 \mathrm{d}y$$

for a k-form $\alpha = g^I \mathrm{d}x^I$, $\mathrm{d}g^I \mathrm{d}x^I = \sum_{i=1}^n \frac{\partial g^i}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I$ $\mathrm{d}^* = (-1)^{n(k+1)+1} \star \mathrm{d}\star = (-1)^{2k+1} \star \mathrm{d}\star$ for $n = 2$.

$$\star(1) = \mathrm{d}x \wedge \mathrm{d}y$$

Substituting into the stokes system, we get

$$\begin{aligned}
f^{(1)} &= -\star \mathrm{d} \star \mathrm{d}(u^x \mathrm{d}x + u^y \mathrm{d}y) - \star\mathrm{d}\star(p\, \mathrm{d}x \wedge \mathrm{d}y) \\
&= -\star \mathrm{d} \star (\frac{\partial u^x}{\partial x^1} \mathrm{d}x \wedge \mathrm{d}y + \frac{\partial u^x}{\partial x^2} \mathrm{d}y \wedge \mathrm{d}y + \frac{\partial u^y}{\partial x^1} \mathrm{d}x \wedge \mathrm{d}x + \frac{\partial u^y}{\partial x^2} \mathrm{d}y \wedge \mathrm{d}x) - \star\mathrm{d}(p) \\
&= -\star \mathrm{d}(\frac{\partial u^x}{\partial x^1} - \frac{\partial u^y}{\partial x^2}) - (\frac{\partial p}{\partial x^1} \mathrm{d}y + \frac{\partial p}{\partial x^2} \mathrm{d}x) \\
&= -\star (\frac{\partial}{\partial x^1} \frac{\partial u^x}{\partial x^1} \mathrm{d}x + \frac{\partial}{\partial x^2} \frac{\partial u^x}{\partial x^1} \mathrm{d}y - \frac{\partial}{\partial x^1} \frac{\partial u^y}{\partial x^2} \mathrm{d}x - \frac{\partial}{\partial x^2} \frac{\partial u^y}{\partial x^2} \mathrm{d}y) - \frac{\partial p}{\partial x^1} \mathrm{d}y - \frac{\partial p}{\partial x^2} \mathrm{d}x \\
&= -(\frac{\partial}{\partial x^2} \frac{\partial u^x}{\partial x^1} - \frac{\partial}{\partial x^2} \frac{\partial u^y}{\partial x^2} + \frac{\partial p}{\partial x^2}) \mathrm{d}x - (\frac{\partial}{\partial x^1} \frac{\partial u^x}{\partial x^1} - \frac{\partial}{\partial x^1} \frac{\partial u^y}{\partial x^2} + \frac{\partial p}{\partial x^1}) \mathrm{d}y
\end{aligned}$$

Specifically for $u = -\cos(2\pi x)\sin(2\pi y)\mathrm{d}x - \sin(2\pi x)\cos(2\pi y)\mathrm{d}y$, we get ■

Answer. ■