Title of this talk is (co) fiber seq. they are important tools in htpy theory, a kind of mathematics used to study shapes of spaces.

Goal of this talk is to demonstrate how these tools can be used to obtain the classical use some tools called (co) fiber seq. to show the surprising fact that there is a 3-dim "hole" in the 2 sphere.

1st  we want to say that we work in a class kind of spaces called based spaces category topological. there are just your everyday nice topological spaces with a distinguished pt. we write usually as $*$.    $X$ like circles.

$X, \exists *_X \in X.$

In this category. the natural maps between sp sp are the based maps where we require our special pt always go to the sp special point.

Def: $f: X \to Y$ is based if $f(*_X) = *_Y$.

We denote the maps from $X$ to $Y$ based spaces as $F(X, Y)$
Here we have an interesting adjunction. (prev. talk)

& luckily, this is also a based space with cst base point map as its base pt

Write this back work.
$$F(X \wedge -, Y) \cong F(X, F(\overline{*}, Y)).$$
$\uparrow$
homeom.

we know what RHS is. maps from space $X$ to space of maps $*, Y$.

the LHS is called a smash product.

Def: $X \wedge Y := X \times Y / X \vee Y$

Cartesian product.  Contracting the Right spaces. quotient

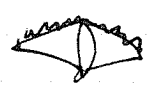we use this to denote two spaces joined at their basepoint.

given this def. the adjunction can be checked "easily".
(took me a while) "initially". tho.

Next a few more definitions:
$\Sigma X = X \wedge S^1$  circle.
$\Omega X = F(S^1, X)$  interval  space of loops in $X$.
$CX = X \wedge I / X \times \{0\}$
$PX = F(I, X).$  Space of paths in $X$

You will notice that the interval & circle shows up repeated b/c turns out these are really the 2 kind of spaces that we

(a)

Finally Def: $[X;Y] := F(X,Y)/$ cts deformation.

(for advanced, same as pass thru $\pi_0$ functor)
~ self

Ask Ernie.

(b) With these definition settled, we want to talk about the main subject of this talk: cofiber & fiber sequences.

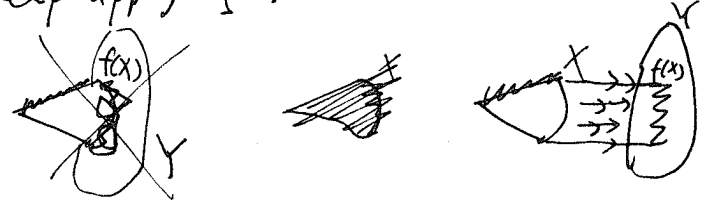given $f: X \to Y$ in our world/category. we have two sequences.

$$(\cdots \to \Omega X \to \Omega Y \to Ff \xrightarrow{\pi} X \xrightarrow{f} Y \xrightarrow{i} Cf \to \Sigma X \to \Sigma Y \to \cdots)$$

note self: say its 2 seq. & left keep applying.

Here $Cf$ — cofiber
$CX \cup_f Y$

$Ff$ — fiber     unfortunately, no good way to draw pictures so

$$X \times_f PY = \{ (x, \gamma) \in X \times PY \mid f(x) = \gamma(1) \}$$

$\uparrow$ cartesian prod

the reason we think these sequences are special b/c we can see it as performing the same operation over & over again.

for cofiber side: claim is   $X \to Y \to Cf$ ∴ using sat notation

$$X : Y : Cf :: Y : Cf : \Sigma X.$$

↙ one can prove this relation propagates indefinitely

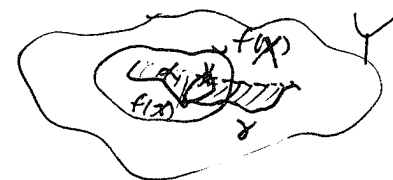for fiber side: we have   $Ff : X : Y :: \Omega Y : Ff : X.$

ask ernie
so is homotopy
eg. only

b/c $\gamma \in \Omega Y, \gamma(0) = \gamma(1) = *_Y$.

$$(Ff) \times_\pi PX = \{ ((x, \gamma), \alpha) \mid f(x) = \gamma(1), \pi(x, \gamma) = \alpha(1) \}$$
       $\Updownarrow$
       $x = \alpha(1)$.

again htpy eg. (ask ernie)

again one can prove this relation propagates this time to the left indefinitely.

Next the amazing fact is that these (co)fiber seq. give rise to the following long exact sequence. (in case cat person did not use)

For those who don't know what exact sequences means.
it is ⓐ a ~~sequence~~ chain of objects with maps from one to next.
ⓑ. the maps are "memoryless" ~~(maybe markov analogy)~~
as in $\cdots \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to \cdots$
$g \circ f(X') \Longleftrightarrow *_{X''}$ ~~exactly~~ thus 2 step ago are ~~trivial~~

insert ⓐ after.

$$[X, Z) \xrightarrow{f^*} [Y, Z] \to [Cf, Z] \to [\Sigma X, Z] \to \cdots$$

~~$[\Sigma X, Z] \to$~~

$$\to [Z, \Omega Y] \to [Z, Ff] \to [Z, X] \to [Z, Y]$$

now we shift gears to explain briefly what is a fiber bundle.

$$F \to E \qquad E|_U = U \times F. \qquad E \text{ are spaces that locally (open set)}$$
$$\downarrow p$$
$$B \qquad \qquad \qquad \qquad \text{looks like simple } \vee \text{ products}$$

this consumer is ~~the example~~ relevant to us b/c expand. $S^1 \to S^3 \to S^2$ the Hopf Bundle. cartesian
~~these~~ turns out conveniently for us, $F \to E \xrightarrow{p} B$ easy to prove it is intact

is ~~one realizable~~ htpyically same as $Fp \to E \xrightarrow{p} B$.

with other lemmas omitted.
which means that applying the LES above:

$$\to [Z, \Omega E] \to [Z, \Omega B] \to [Z, F] \to [Z, E] \to [Z, B].$$
$$\quad \text{||} \qquad\qquad \text{||} \qquad\qquad \text{use adjunction.}$$
$$[\Sigma Z, E] \to [\Sigma Z, B] \to \quad \cdots \quad \cdots$$

substitute $Z = S^1$ ~~yields~~ $F = S^1$, $E = S^3$, $B = S^2$ yields

$$\to [S^3, S^1] \to [S^3, S^3] \to [S^3, S^2] \to [S^2, S^1] \to [S^2, S^3] \to \cdots$$

( how we know from $p: \mathbb{R} \to S^1 \subset \mathbb{C}$    & $\mathbb{R} \simeq I$. contractible.
$t \mapsto \exp(2\pi i t)$.    $\simeq *$
that $[S^n, S^1] = \emptyset$ if $n \neq 1$. )

switch.
→ & heuristically it is pretty easy to believe $[S^n, S^n] \cong \mathbb{Z}$