Derivatives of Eigenvalues

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September, 2016

Let's assume we work in a vector space with a non-degenerate symmetric bilinear pairing $\langle \cdot, \cdot \rangle$, and a differential operator d satisfying the usual Leinbiz's rule. We will use A to denote linear operator/matrix in a basis $A = (a_{ij})$ and $\mathbf{w}$ for a vector $w = (w_i)$.

Assuming L is not defect at the considered eigenvalue (multiplicity = 1), we have two equations $\mathsf{L}\mathbf{v} = \lambda \mathbf{v}$ and $\mathbf{u}^\dagger \mathsf{L} = \lambda \mathbf{u}^\dagger$, where λ is the eigenvalue, $\mathbf{v}$ is the (right) eigenvector, and $\mathbf{u}$ the left eigenvector. (Note that a left eigenvector of a matrix A is a (right) eigenvector of the adjoint $\mathsf{A}^\dagger$.)

Now differentiate the first (right) eigenvector equation gives us

$$\begin{aligned} d(\lambda \mathbf{v}) &= d(\mathsf{L}\mathbf{v}) \\ d\lambda \mathbf{v} + \lambda d\mathbf{v} &= d\mathsf{L}\mathbf{v} + \mathsf{L} d\mathbf{v} \end{aligned}$$

Pair the above with the left eigenvectors yields

$$\begin{aligned} \langle \mathbf{u}, d\lambda \mathbf{v} \rangle + \langle \mathbf{u}, \lambda d\mathbf{v} \rangle &= \langle \mathbf{u}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathbf{u}, \mathsf{L} d\mathbf{v} \rangle \\ d\lambda \langle \mathbf{u}, \mathbf{v} \rangle + \langle \lambda \mathbf{u}, d\mathbf{v} \rangle &= \langle \mathbf{u}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathbf{u}, \mathsf{L} d\mathbf{v} \rangle \\ d\lambda \langle \mathbf{u}, \mathbf{v} \rangle &= \langle \mathbf{u}, d\mathsf{L}\mathbf{v} \rangle + \langle \mathsf{L}^\dagger \mathbf{u} - \lambda \mathbf{u}, d\mathbf{v} \rangle \\ d\lambda &= \frac{\langle \mathbf{u}, d\mathsf{L}\mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{v} \rangle} \end{aligned}$$

Now if the operator L is self-adjoint (e.g. symmetric if we are over $\mathbb{R}$), we then have $\mathbf{u} = \mathbf{v}$, thus

$$d\lambda = \frac{\langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} = \langle \mathbf{v}, d\mathsf{L}\mathbf{v} \rangle$$

Or in another notation, we have $d\lambda = \mathbf{v}^\dagger d\mathsf{L}\mathbf{v}$.

Same principles apply for the generalized eigenvalue problem. If we again have left eigenvector $\mathbf{u}$ and (right) eigenvector $\mathbf{v}$ such that $(\mathsf{L} - \lambda \mathsf{M})\mathbf{v} = \mathbf{u}^\dagger (\mathsf{L} - \lambda \mathsf{M}) = 0$, we can derive

$$\begin{aligned} d(\mathsf{L} - \lambda \mathsf{M})\mathbf{v} &= 0 \\ (d\mathsf{L} - d\lambda \mathsf{M} - \lambda d\mathsf{M})\mathbf{v} + (\mathsf{L} - \lambda d\mathsf{M}) d\mathbf{v} &= 0 \end{aligned}$$

Pairing with $\mathbf{u}$ returns

$$\begin{aligned} \mathbf{u}^\dagger (d\mathsf{L} - d\lambda \mathsf{M} - \lambda d\mathsf{M})\mathbf{v} + \mathbf{u}^\dagger (\mathsf{L} - \lambda d\mathsf{M}) d\mathbf{v} &= 0 \\ \mathbf{u}^\dagger (d\mathsf{L} - d\lambda \mathsf{M} - \lambda d\mathsf{M})\mathbf{v} &= 0 \\ \mathbf{u}^\dagger d\mathsf{L}\mathbf{v} - \mathbf{u}^\dagger \lambda d\mathsf{M}\mathbf{v} &= \mathbf{u}^\dagger d\lambda \mathsf{M}\mathbf{v} \\ d\lambda \mathbf{u}^\dagger \mathsf{M}\mathbf{v} &= \mathbf{u}^\dagger d\mathsf{L}\mathbf{v} - \lambda \mathbf{u}^\dagger d\mathsf{M}\mathbf{v} \\ d\lambda &= \frac{\mathbf{u}^\dagger d\mathsf{L}\mathbf{v} - \lambda \mathbf{u}^\dagger d\mathsf{M}\mathbf{v}}{\mathbf{u}^\dagger \mathsf{M}\mathbf{v}} \end{aligned}$$

Now if we assume both L and M to be self-adjoint, we will again have $\mathbf{u} = \mathbf{v}$. Then normalizing $\mathbf{v}$ such that $\langle \mathbf{v}, \mathsf{M}\mathbf{v} \rangle = 1$ gives $d\lambda = \mathbf{v}^\dagger d\mathsf{L}\mathbf{v} - \lambda \mathbf{v}^\dagger d\mathsf{M}\mathbf{v}$.