(b) EVD for A: eigenvalues are $(0, 0, \underbrace{3.429}_{>0}, -3.429)$ ∴ unstable

(c) $\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\cdot\underline{w}^T\underline{x} = (\underline{A} + \underline{B}\,\underline{w}^T)\underline{x} = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -5.88 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11.76 & 0 \end{bmatrix} + \begin{pmatrix} 0 \\ 0.8 \\ 0 \\ -0.6 \end{pmatrix} (w_1\ w_2\ w_3\ w_4) \right) \underline{x}$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.8w_1 & 0.8w_2 & 0.8w_3-5.88 & 0.8w_4 \\ 0 & 0 & 0 & 1 \\ -0.6w_1 & -0.6w_2 & -0.6w_3+11.76 & -0.6w_4 \end{bmatrix} \underline{x}$$

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0.8w_1 & 0.8w_2 & 0.8w_3 & 0.8w_4 \\ 0 & 0 & 0 & 0 \\ -0.6w_1 & -0.6w_2 & -0.6w_3 & -0.6w_4 \end{matrix}$$

(d) plug in and solve the EVD, all $\mathrm{Real}(\sigma_i) < 0$

Robotics

Exercise 6

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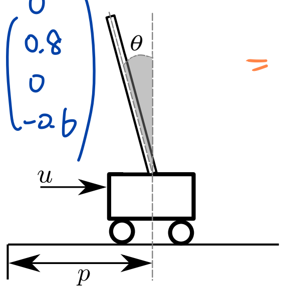
1 Stability (12 points)

(a) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -5.88 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11.76 & 0 \end{bmatrix}$ $B = \begin{pmatrix} 0 \\ 0.8 \\ 0 \\ -0.6 \end{pmatrix}$



$C = [B\ AB\ A^2B\ A^3B]$

$= \begin{pmatrix} 0 & 0.8 & 0 & 3.528 \\ 0.8 & 0 & 3.528 & 0 \\ 0 & -0.6 & 0 & 7.056 \\ -0.6 & 0 & 7.056 & 0 \end{pmatrix}$, $\mathrm{Rank}(C) = 4$ ∴ Controllabel

In the last exercise we calculated the local linearization of the cart-pole around $x^* = (0, 0, 0, 0)$. The solution is

$$\dot{x} = Ax + Bu\ , \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{c_2 g}{\frac{4}{3}l - c_2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{\frac{4}{3}l - c_2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ c_1 + \frac{c_1 c_2}{\frac{4}{3}l - c_2} \\ 0 \\ \frac{-c_1}{\frac{4}{3}l - c_2} \end{pmatrix}$$

$x = \begin{pmatrix} p \\ \dot{p} \\ q \\ \dot{q} \end{pmatrix}$ $\dot{x} = \begin{pmatrix} \dot{p} \\ \ddot{p} \\ \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} 0 \\ \ddot{p} \\ 0 \\ \ddot{q} \end{pmatrix}$

with $g = 9.8ms^2$ the gravitational constant, $l = 1m$ the pendulum length and constants $c_1 = (M_p + M_c)^{-1}$ and $c_2 = lM_p(M_p + M_c)^{-1}$ where $M_p = M_c = 1kg$ are the pendulum and cart masses respectively.

a) Consider the local linearization of the cart-pole. Is the system controllable? (2 P)

b) Consider the *uncontrolled* system (where there are no controls, $u = 0$). Perform a linear stability analysis. (Show whether the system is asymptotically stable, marginally stable, or unstable.) (2 P)

c) Consider a linear controller $u = w^\top x$ with 4 parameters $w \in \mathbb{R}^4$ for the cart-pole. What is the closed-loop linear dynamics $\dot{x} = \widehat{A}x$ of the system? (2 P)

d) Test if the controller with $w = (1.0000, 2.6088, 52.9484, 16.5952)$ (computed using ARE) is asymtotically stable. What are the eigenvalues? (2 P)

e) Given some eigenvalues $\lambda_1^\star, \lambda_2^\star, \lambda_3^\star, \lambda_4^\star$. Come up with a method that finds parameters w for these eigenvalues around $x^* = (0, 0, 0, 0)$. What could "good" eigenvalues be to achieve a "maximally stable" system (e.g., asymptotically stable with fastest convergence rate)? (2 P)

$(A + Bw^T - \lambda I)\,x = 0$

$\dot{x} = \hat{A}x$

f) Output the optimal parameters and test them on the cart-pole simulation. (2 P)

1. For python please install pygame and pyopengl (using *'python3 -m pip install pygame'* and *'python3 -m pip install pyopengl'*), then you can run: *'jupyter-notebook py/05-stability/05-stability.ipynb'*

2. For C++ run: *'cd cpp/05-stability'*, *'make'*, *'./x.exe'*

$\boxed{\hat{A}X = X\Lambda}$

$(A + Bw^T)X = X\Lambda$

$$= \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0.8w_1 & 0.8w_2-\lambda & 0.8w_3-5.88 & 0.8w_4 \\ 0 & 0 & -\lambda & 1 \\ -0.6w_1 & -0.6w_2 & -0.6w_3+11.76 & -0.6w_4-\lambda \end{bmatrix}$$

$$-\lambda \begin{pmatrix} 0.8w_2-\lambda & 0.8w_3-5.88 & 0.8w_4 \\ 0 & -\lambda & 1 \\ -0.6w_2 & -0.6w_3+11.76 & -0.6w_4-\lambda \end{pmatrix}$$

$$- \begin{pmatrix} 0.8w_1 & 0.8w_3-5.88 & 0.8w_4 \\ 0 & -\lambda & 1 \\ -0.6w_1 & -0.6w_3+11.76 & -0.6w_4-\lambda \end{pmatrix}$$

$$0.8w_1 \left((0.6w_4+\lambda)\lambda - (-0.6w_3+11.76) \right)$$

$$+ (-0.6w_1)\left(0.8w_3-5.88 + 0.8w_4\lambda \right)$$

$$\Delta = tr(\hat{A})^2 - 4 \cdot det(\hat{A})$$

$$= \left(\sum^{4} l_i\right)^2 - 4 \prod^{4} l_i \doteq 0, \quad s.t. \ l_i < 0$$

$$(-2+\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2 = 0$$

$$(\lambda_1+\lambda_2)^2 - 4(\lambda_1+\lambda_2) + 4 - 4\lambda_1\lambda_2 = 0$$

$$(\lambda_1-\lambda_2)^2 - 4(\lambda_1+\lambda_2) + 4 = 0$$

$$\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2$$