Project Report for Quantum Field Theory (Spring 2024)

# On Decay Rate of Bound Muons

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Abstract: Muons are of special interest because of experimental discrepancies which do not match the standard model predictions, which motivates the study of the decay rate of bound muons. For this, the Dirac equation in a Coulomb field is solved to find the ground state wavefunctions of hydrogen-like atoms. The decay rate of the bound muon is calculated, and the extreme relativistic and non-relativistic limits are explored.

## Contents

## 1 Introduction

While the discovery of the Higgs Boson completed the standard model, there are some indications of new physics beyond the standard model. For example, the discovery of neutrino oscillations had to be accommodated into the standard model later on. The study of muons is important because they are at the center of the existing discrepancies between the standard model and experiments, such as the anomalous magnetic moment of the muon and B-meson decays whose experimental data is consistently below the standard model predictions. These discoveries and experiments hint toward Charged Lepton Flavor Violation (CLFV). Lepton Flavor Violation has not been observed to date. Why lepton flavour is conserved while quark flavour is not is still a mystery to us. One way to explain these discrepancies is the idea that lepton flavour may also not be conserved. So, we search for the decay of muons into electrons to observe CLFV. Lepton flavour violation can be observed in such a decay because muons, although unstable, have a relatively greater lifetime than tauons. Moreover, producing muons in labs is accessible. is e Such experiments looking or Mu2e decay are being conducted at Fermilab and COMET [1].

In this report, we first solve the Dirac equation for a free particle to get familiar with gamma matrices, and we find the free particle wave solution of the Dirac equation. We then solve the Dirac equation in the presence of an external Coulomb potential to find wavefunctions for 1S state of hydrogen like atoms. The analytic solutions only exist for a Coulomb potential - the Dirac equation has to be solved numerically for other electroweak interactions.

We then find the decay rates of the free and bound muon. We first study the decay of free muons for a smoother transition into calculating the decay rate of bound muons. In Mu2e experiments at Fermilab, we look for muon to electron conversion, where muon is bound by an atom. Hence, the decay rate of bound muons is integral to calculations done in experiments. The values obtained are then investigated in extreme relativistic and non relativistic limits.
This report largely follows the thesis, Bound State Phenomena: Positronium Molecule and Muon Decays, and the electron Magnetic Moment by Muhammad Mubasher [2].

## 2 Bound State Wavefunction

In this section, we solve the free particle Dirac equation and Bound State wavefunction and 1S wavefunction of Hydrogen like atoms which will be used in finding bound muon Decay rate.

### 2.1 Dirac Equation

To generalize Schrodinger's equation to a relativistic invariant equation, writing Hamiltonian in terms of linear spacetime derivative in $c = \hbar = 1$ units:

$$i\partial_t \psi = (\alpha.\mathbf{p} + \beta m)\psi. \tag{2.1}$$

where $\alpha$ and $\beta$ are $4 \times 4$ traceless Hermitian matrices with the following properties:

$$\{\alpha_i, \alpha_i\} = 2\delta_{ij} I,$$
$$\{\alpha_i, \beta\} = 0,$$
$$\alpha_i^2 = \beta^2 = I.$$

These matrices act on $\psi$, so $\psi$ is now a column vector. The components of column vector $\psi$ are Dirac spinors. One of the possible representation of these matrices is:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

$\sigma_i$ are Pauli matrices. In terms of $4 \times 4$ gamma matrices these matrices are:

$$\gamma^0 = \beta, \qquad \gamma^i = \beta\alpha^i.$$

Properties of gamma matrices are given in Appendix A. Multiplying Eq. (2.1) with $\beta$, we get:

$$\begin{aligned} i\beta\partial_t\psi = (\beta\alpha.\mathbf{p} + \beta^2 m)\psi \\ (i\beta\partial_t - \beta\alpha.\mathbf{p} + m)\psi = 0 \\ (i\gamma^0\partial_t - \gamma^i p_i - m)\psi = 0 \\ (i\gamma^\mu\partial_\mu - m)\psi = 0 \\ (i\not{\partial} - m)\psi = 0, \end{aligned}$$

where in last line used Feynman slash notation $\not{\partial} = \gamma^\mu\partial_\mu$.
The free particle plane wave solution of Dirac equation have the form:

$$\psi(x) = u(p)e^{ip\cdot x},$$
$$\psi(x) = v(p)e^{ip\cdot x},$$

where $p$ is four momentum so $p \cdot x = p^\mu x_\mu$ and $u$ and $\nu$ are four component Dirac spinors that satisfy:

$$(\not{p} - m)u(p) = 0,$$
$$(\not{p} - m)v(p) = 0,$$

where we used $i\partial_\mu = p_\mu$ [3].
Dirac spinors have the form:

$$u_\uparrow(p) = N \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}, \quad u_\downarrow(p) = N \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix},$$

where $u_\uparrow(p)$ is for a spin up particle and $u_\downarrow(p)$ is for a spin down particle. N is the normalization constant.

$$v_\uparrow(p) = N \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}, \quad v_\downarrow(p) = N \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix},$$

where $v_\uparrow(p)$ is for a spin up antiparticle and $v_\downarrow(p)$ is for a spin down antiparticle. N is the normalization constant.

### 2.2 Dirac Equation in the Presence of Coulomb Potential

To get Dirac equation for a particle in an external field:

$$\begin{aligned} &\partial^\mu \to D^\mu = \partial^\mu + i\alpha A^\mu \\ &p^\mu \to p^\mu - qA^\mu, \end{aligned}$$

where $q$ is the charge on the particle. Doing this transformation in the Dirac equation gives:

$$[\gamma^\mu(p_\mu - qA_\mu) - m]\psi = 0,$$

multiplying this expression with $\beta$:

$$\begin{aligned} [\gamma^\mu\beta(p_\mu - qA_\mu) - m\beta]\psi &= 0 \\ [\gamma^0\beta(p_0 - qA_0) + \gamma^i\beta(p_i - qA_i) - m\beta]\psi &= 0 \\ [\gamma^0\beta(p_0 - qA_0) - \gamma^i\beta(p_i - qA_i) - m\beta]\psi &= 0 \\ [I(p_0 - qA_0) - \boldsymbol{\alpha}(p_i - qA_i) - m\beta]\psi &= 0, \end{aligned}$$

where in the last line used, $\gamma^\mu\beta = (I, \boldsymbol{\alpha})$. Rewriting $p_0$ as $i\partial_t$:

$$\begin{aligned} [(i\partial_t - qA_0) - \boldsymbol{\alpha}(p_i - qA_i) - m\beta]\psi &= 0 \\ i\partial_t\psi &= [qA_0\psi + \boldsymbol{\alpha}(p_i - qA_i) \ + m\beta]\psi. \end{aligned}$$

From here, one can read off the Dirac Hamiltonian as:

$$\hat{H} = qA_0 \ + \boldsymbol{\alpha}(p_i - qA_i) \ + m\beta.$$

In the presence of an external potential $A = (V(r), 0)$, Dirac Hamiltonian reduces to:

$$\hat{H} = \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta + qV(r),$$

where $V(r) = -\frac{qZ}{r}$ is the potential due to nucleus with charge $Z$ and the particle is at distance r. Due to the spherical symmetry of the potential, Hamiltonian commutes with the angular momentum operator $\hat{J}$ and the parity operator $\hat{P}$. So, the corresponding wave functions are written in terms of spinors $\phi(x)$ and $\chi(x)$ as:

$$\psi(x) = \begin{pmatrix} \phi(x) \\ \chi(x). \end{pmatrix}$$

With this, the eigenvalue equation becomes:

$$\left(\boldsymbol{\alpha} \cdot \mathbf{p} + m\beta + \frac{\alpha Z}{r}\right) \begin{pmatrix} \phi(x) \\ \chi(x). \end{pmatrix} = E \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix},$$

where $\alpha$ is the fine structure constant that quantifies the strength of electromagnetic interaction between elementary charged particles. For the solutions of this equation, we make the following ansatz:

$$\begin{aligned} \phi(x) &= ig(r)\Omega_{jlm}\left(\frac{\mathbf{r}}{r}\right), \\ \chi(x) &= -f(r)\Omega_{jl'm}\left(\frac{\mathbf{r}}{r}\right), \end{aligned}$$

where

$$l' = 2j - l = \begin{cases} 2\left(l + \frac{1}{2}\right) - l = l + 1 \\ 2\left(l - \frac{1}{2}\right) - l = l - 1 \end{cases}$$

The value $l' = l - 1$ must be excluded, because no total angular momentum $j = 1 + \frac{1}{2}$ can be constructed by $l' = l - 1$ and $S = 1/2$.

$$\begin{aligned} (\boldsymbol{\sigma}.\mathbf{p})\phi &= (\boldsymbol{\sigma}.\mathbf{p}ig(r))\Omega_{jlm} + ig(r)\boldsymbol{\sigma}.\mathbf{p}\Omega_{jlm} \\ &= g'(r)\boldsymbol{\sigma}.\frac{\mathbf{r}}{r} + ig(r)\boldsymbol{\sigma}.\mathbf{p}\Omega_{jlm} \end{aligned} \tag{2.2}$$

Now making use of the following relation between spinors:

$$\boldsymbol{\sigma}.\frac{\mathbf{r}}{r}\Omega_{jlm} = \Omega_{jl'm} \tag{2.3}$$

and using the identity, $(\boldsymbol{\sigma}.\mathbf{A})(\boldsymbol{\sigma}.\mathbf{B}) = \mathbf{A}.\mathbf{B} + i\boldsymbol{\sigma}.(\mathbf{A} \times \mathbf{B})$, with these:

$$\begin{aligned} \sigma.\mathbf{p}\Omega_{jlm} &= (\sigma.\mathbf{p})(\boldsymbol{\sigma}.\frac{\mathbf{r}}{r})\Omega_{jl'm} \\ &= \left(\mathbf{p}.\frac{\mathbf{r}}{r} + i\boldsymbol{\sigma} \cdot \left(\mathbf{p} \times \frac{\mathbf{r}}{r}\right)\right)\Omega_{jl'm}. \end{aligned}$$

Using $p = i\partial_i$ and $L = \mathbf{r} \times \mathbf{p}$, we can further simplify as:

$$\begin{aligned} \sigma.\mathbf{p}\Omega_{jlm} &= -\frac{i}{r}(2 + \boldsymbol{\sigma}.\mathbf{L})\Omega_{jl'm} \\ &= -\frac{i}{r}(1 + \kappa)\Omega_{jl'm} \end{aligned} \tag{2.4}$$

where,

$$\kappa = \begin{cases} -(l+1) & \text{for } j = l + \frac{1}{2} \\ l & \text{for } j = l - \frac{1}{2} \end{cases}$$

With these, Eq. (2.4), (2.3) and (2.2) takes the form:

$$
\begin{aligned}
(\sigma.\mathbf{p})\phi &= -\Omega_{jl'm}\left(g' + \frac{1+\kappa}{r}g\right), \\
(\sigma.\mathbf{p})\chi &= -i\Omega_{jlm}\left(f' - \frac{\kappa-1}{r}f\right).
\end{aligned}
$$

The second expression is derived similarly. Plugging these in the eigenvalue equation, angular function cancelled out and following differential equations are obtained [4]:

$$
\begin{aligned}
g'(r) + (1+\kappa)\frac{g(r)}{r} - \left[E + m + \frac{\alpha Z}{r}\right] f(r) = 0 \\
f'(r) + (1-\kappa)\frac{f(r)}{r} + \left[E - m + \frac{\alpha Z}{r}\right] g(r) = 0
\end{aligned}
$$

This expression can be simplified with the substitution $G = rg$ and $F = rf$:

$$
\begin{aligned}
G' + \frac{\kappa}{r}G - \left[E + m + \frac{\alpha Z}{r}\right] F = 0. \\
F' - \frac{\kappa}{r}F + \left[E - m + \frac{\alpha Z}{r}\right] G = 0.
\end{aligned}
$$

With shorthand notations $\alpha Z = \alpha_z$, and $E \pm m = \alpha_\pm$,

$$
\begin{aligned}
G' - \frac{\kappa}{r}G - [\alpha_+ + \frac{\alpha_z}{r}]F = 0 \\
F' + \frac{\kappa}{r}F + [\alpha_- + \frac{\alpha_z}{r}]G = 0.
\end{aligned}
$$

We solve these coupled differential equations for the limit $r \to \infty$. We then have:

$$
\begin{aligned}
G' - [\alpha_+]F = 0 \\
F' + [\alpha_-]G = 0.
\end{aligned}
$$

Differentiating the first equation with respect to $r$ once again:

$$
G'' = \alpha_+ F'
$$

Plugging this into the second equation and solving:

$$
\begin{aligned}
\frac{G''}{\alpha_+} + (\alpha_-)G = 0 \\
G'' + (\alpha_+\alpha_-)G = 0
\end{aligned}
$$

$$
G'' = -(\alpha_+\alpha_-)G \tag{2.5}
$$

$$
\implies G \propto Ae^{-\rho} \tag{2.6}
$$

where $\rho = \sqrt{-\alpha_+\alpha_-}r = \sqrt{m^2 - E^2}r$, and the positive exponent solution is discarded because it is not normalisable. Doing the same, we can get an uncoupled equation for F:

$$
F'' = (\alpha_+\alpha_-)F \tag{2.7}
$$

$$
\implies F \propto e^{-\rho}. \tag{2.8}
$$

Let us now postulate an ansatz:

$$F(\rho) = f(\rho)e^{-\rho}, \quad G(\rho) = g(\rho)e^{-\rho}$$

The functions $f$ and $g$ defined now are distinct from the ones used above. Plugging our ansatz into the coupled equations, we get:

$$\left(\frac{d}{d\rho} - 1 + \frac{\kappa}{\rho}\right) f - \left(\sqrt{\frac{\alpha_-}{\alpha_+}} - \frac{\alpha_z}{\rho}\right) g = 0 \tag{2.9}$$

$$\left(\frac{d}{d\rho} - 1 - \frac{\kappa}{\rho}\right) g - \left(\sqrt{\frac{\alpha_+}{\alpha_-}} + \frac{\alpha_z}{\rho}\right) f = 0 \tag{2.10}$$

We now investigate the limit $r \to 0 \implies \rho \to 0$:

$$\left(\frac{d}{d\rho} + \frac{\kappa}{\rho}\right) f + \frac{\alpha_z}{\rho} g = 0$$

$$\left(\frac{d}{d\rho} - \frac{\kappa}{\rho}\right) g - \frac{\alpha_z}{\rho} f = 0$$

Decoupling the equations, we obtain:

$$\left(\rho\frac{d^2}{d\rho^2}\frac{d}{d\rho} - \frac{\kappa^2 - \alpha_z^2}{\rho}\right) g = 0$$

$$\left(\rho\frac{d^2}{d\rho^2}\frac{d}{d\rho} - \frac{\kappa^2 - \alpha_z^2}{\rho}\right) f = 0$$

The solutions to these equations are given as:

$$f = \rho^s \phi_f, \quad g = \rho^s \phi_f,$$

where $s = \sqrt{\kappa^2 - \alpha_z^2}$ and the positive exponent has been chosen since $\rho$ is small, and hence the negative exponent solution is not normalisable. This is our small $r$ asymptotic form. We now postulate that the remaining factors $\phi_{f,g}$ are a power series:

$$\phi_f = \sum_{a=0} f_a \rho^a, \quad \phi_g = \sum_{a=0} g_a \rho^a, \quad f_0 g_0 = 0$$

We can now factor out the asymptotic form and plug the series solution into our Eqs. (2.9). Simplifying our result, we get

$$\sum_{a=0}^{\infty} \left[(s + a + \kappa)f_a - f_{a-1} + \frac{\alpha_-}{q} g_{a-1} + \alpha_z g_a\right] \rho^{s+a-1} = 0$$

$$\sum_{a=0}^{\infty} \left[(s + a - \kappa)g_a - g_{a-1} - \frac{\alpha_+}{q} f_{a-1} - \alpha_z f_a\right] \rho^{s+a-1} = 0$$

where $q = \sqrt{m^2 - E^2} = \sqrt{-\alpha_+\alpha_-}$.
The coefficients of each $\rho^i$ must be zero:

$$(s + a + \kappa)f_a - f_{a-1} + \frac{\alpha_-}{q} g_{a-1} + \alpha_z g_a = 0 \tag{2.11}$$

$$(s + a - \kappa)g_a - g_{a-1} - \frac{\alpha_+}{q} f_{a-1} - \alpha_z f_a = 0 \tag{2.12}$$

Multiplying the first equation with $\alpha_+$, and the second with $q$, we get:

$$\alpha_+[(s + a + \kappa)f_a - f_{a-1} + \alpha_z g_a] - q g_{a-1} = 0$$
$$q[(s + a + \kappa)g_a - \alpha_+ f_{a-1} - \alpha_z f_a] - q g_{a-1} = 0$$

where $q = \sqrt{m^2 - E^2} = \sqrt{-\alpha_+\alpha_-}$. We can now subtract the second equation from the first to obtain:

$$[\alpha_+(s + a + \kappa) + q\alpha_z]f_a + [\alpha_+\alpha_z - q(s + a + \kappa)]g_a = 0$$

Rewriting which gives,

$$f_a = \frac{q(s + a + \kappa) - \alpha_+\alpha_z}{\alpha_+(s + a + \kappa) + q\alpha_z} g_a \tag{2.13}$$

This equation along with Eqs. (2.11, 2.12) can be used to investigate the behavior of both series when $a$ is large. Taking the large $a$ limit of all equations individually:

$$a f_a - f_{a-1} + \frac{\alpha_-}{q} g_{a-1} + \alpha_z g_a \approx 0$$
$$a g_a - g_{a-1} - \frac{\alpha_+}{q} f_{a-1} - \alpha_z f_a \approx 0$$
$$f_a = \frac{q}{\alpha_+} g_a$$

Using the third expression in the first two equations, we can derive the following expressions that give us the large $a$ behavior of the series:

$$-2f_{a-1} + \left[a + \alpha_z \frac{\alpha_+}{q}\right] f_a \approx 0 \quad -2g_{a-1} + \left[a - \alpha_z \frac{q}{\alpha_+}\right] g_a \approx 0$$

In the large $a$ limit, this is further reduced to:

$$f_a \approx \frac{2}{a} f_{a-1} \quad g_a \approx \frac{2}{a} g_{a-1}$$
$$f_a \approx \frac{2^a}{a!} f_0 \quad g_a \approx \frac{2^a}{a!} g_0$$

In this limit, our series solution then becomes:

$$\sum_{a=0}^{\infty} f_a \rho^a = \sum_{a=0}^{\infty} \frac{2^a}{a!} f_0 \rho^a = \sum_{a=0}^{\infty} \frac{(2\rho)^a}{a!} f_0 = f_0 e^{2\rho}$$
$$\sum_{a=0}^{\infty} g_a \rho^a = \sum_{a=0}^{\infty} \frac{2^a}{a!} g_0 \rho^a = \sum_{a=0}^{\infty} \frac{(2\rho)^a}{a!} g_0 = g_0 e^{2\rho}$$

This behavior is not normalizable since we have obtained a positive exponential in the limit $r \to \infty$. Therefore, we require that both of the series must terminate at some positive integer $a_{\text{max}}$. The imposition of this termination means that:

$$f_{a_{\text{max}}+1} = g_{a_{\text{max}}+1} = 0.$$

Plugging this into the equations Eq. (2.11) for $a = a_{\text{max}} + 1$,

$$\begin{aligned}
(s+a+\kappa) f_{a_{\text{max}}+1} - f_{a_{\text{max}}} + \frac{\alpha_-}{q} g_{a_{\text{max}}} + \alpha_z g_{a_{\text{max}}+1} = 0 &\implies f_{a_{\text{max}}} = \frac{\alpha_-}{q} g_{a_{\text{max}}}, \\
(s+a-\kappa) g_{a_{\text{max}}+1} - g_{a_{\text{max}}} - \frac{\alpha_+}{q} f_{a_{\text{max}}} - \alpha_z f_{a_{\text{max}}+1} = 0 &\implies g_{a_{\text{max}}} = -\frac{\alpha_+}{q} f_{a_{\text{max}}}.
\end{aligned}$$

We can now find the ratio of $f_{a_{\text{max}}}$ and $g_{a_{\text{max}}}$, and compare it with the answer we obtain from Eq. (2.13).

$$\frac{f_{a_{\text{max}}}}{g_{a_{\text{max}}}} = \frac{q\,(s + a_{\text{max}} - \kappa) - \alpha_+ \alpha_z}{\alpha_+ (s + a_{\text{max}} + \kappa) + q\alpha_z} = \frac{\alpha_- g_{a_{\text{max}}}}{q g_{a_{\text{max}}}} = \frac{\alpha_-}{q}$$

We can now solve for energy:

$$\begin{gathered}
\frac{q\,(s + a_{\text{max}} - \kappa) - \alpha_+ \alpha_z}{\alpha_+ (s + a_{\text{max}} + \kappa) + q\alpha_z} = \frac{\alpha_-}{q} \\
-q\,(s + a_{\text{max}} + \kappa) + \alpha_- \alpha_z + \alpha_+ \alpha_z - q\,(s + a_{\text{max}} - \kappa) = 0 \\
E^2 \alpha_z - q\,(s + a_{\text{max}}) = 0
\end{gathered}$$

where in the last step we used $\alpha_+ + \alpha_- = E^2 + m^2 + E^2 - m^2 = 2E^2$. We can now square both sides, and using the fact that $q^2 = m^2 - E^2$, solve for E.

$$\begin{aligned}
E^2 \alpha_z^2 &= \left(m^2 - E^2\right) \left(\sqrt{\kappa^2 - \alpha_z^2} + a_{\text{max}}\right)^2 \\
E^2 \left(\alpha_z^2 + \left(\sqrt{\kappa^2 - \alpha_z^2} + a_{\text{max}}\right)^2\right) &= m^2 \left(\sqrt{\kappa^2 - \alpha_z^2} + a_{\text{max}}\right)^2
\end{aligned}$$

$$E = m \left[ \frac{\left(\sqrt{\kappa^2 - \alpha_z^2} + a_{\text{max}}\right)^2}{\alpha_z^2 + \left(\sqrt{\kappa^2 - \alpha_z^2} + a_{\text{max}}\right)^2} \right]^{1/2} = m \left[ 1 + \left( \frac{\alpha_z^2}{\sqrt{\kappa^2 - \alpha_z^2} + a_{\text{max}}^2} \right)^2 \right]^{-1/2} \tag{2.14}$$

We define $n = a_{\text{max}}$ as our principle quantum number,

$$E_{\kappa n} = m_e \left[ 1 + \left( \frac{\alpha_z}{n + \sqrt{\kappa^2 - \alpha_z^2}} \right)^2 \right]^{-1/2}$$

This is the relativistic energy of a particle that is confined in a Coulomb potential [4]. Let us now find the ground state wave function. Setting $n = 0$ and $\kappa = -1$, we get $s = \sqrt{1 - \alpha_z^2} \equiv \gamma, E = m\sqrt{1 - \alpha_z^2}, q = \sqrt{-\alpha_+ \alpha_-} = \sqrt{m^2 - E^2} = m\alpha_z, \rho = m\alpha_z r$. Then,

$$\begin{aligned}
F &= e^{\rho} \rho^s f_0 = e^{-m\alpha_z r} \left(m\alpha_z r\right)^{\gamma} f_0 \\
G &= e^{\rho} \rho^s g_0 = e^{-m\alpha_z r} \left(m\alpha_z r\right)^{\gamma} g_0,
\end{aligned}$$

and our total wavefunction becomes:

$$\psi = N e^{-m\alpha_Z r} \left(m\alpha_Z r\right)^{\gamma-1} \begin{pmatrix} \chi_r \\ i\frac{1-\gamma}{\alpha_Z}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}}{r}\chi_r \end{pmatrix},$$

Here $\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}}{r}$ is a pseudo scalar and we have used the fact that $A(\theta,\phi) = Y_0^0\chi_r$.
We can find the normalization constant $N$ using the normalization condition: $1 = \int_{-\infty}^{+\infty} |\psi|^2$.
Solving this integral, we find that N comes out to be:

$$N = \frac{2^{\gamma-1}}{\sqrt{\pi}} \left(m\alpha_Z r\right)^{3/2} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}},$$

We can now finally write down the normalized wave functions for the ground state of (hydrogen-like) atoms:

$$\psi_{n=1,j=1/2,\uparrow}(r,\theta,\phi) = \frac{(2m\alpha_Z)^{3/2}}{\sqrt{4\pi}} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}} \left(2m\alpha_Z r\right)^{\gamma-1} e^{-m\alpha_Z r} \begin{bmatrix} 1 \\ 0 \\ \frac{i(1-\gamma)}{\alpha_Z} C_\theta \\ \frac{i(1-\gamma)}{\alpha_Z} S_\theta e^{i\phi} \end{bmatrix},$$

$$\psi_{n=1,j=1/2,\downarrow}(r,\theta,\phi) = \frac{(2m\alpha_Z)^{3/2}}{\sqrt{4\pi}} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}} \left(2m\alpha_Z r\right)^{\gamma-1} e^{-m\alpha_Z r} \begin{bmatrix} 0 \\ 1 \\ \frac{i(1-\gamma)}{\alpha} S_\theta e^{-i\phi} \\ -\frac{i(1-\gamma)}{\alpha} C_\theta \end{bmatrix}.$$

We will be using these for finding bound muon decay rate. This is valid for hydrogen like atoms, where we remove all other electrons and we just have one electron or muon in 1S. More generally, the $\phi$ functions would take the form of Laguerre polynomials. However, we are only concerned with the ground state here because bound state of muon is formed by replacing the ground state electron with a muon and removing all other electrons present in the atom.

## 3 Decay Rate of a Free Muon

In this section we will calculate the decay rate of a free muon to a free electron. The tree level Feynman diagram for this decay is pictured in Fig. 1. To calculate the decay rate, we divide the decay into two steps: the muon decaying into and electron and some boson A,

$$\mu^- \to e^- A, \tag{3.1}$$

followed by the boson decaying into an electron anti-neutrino and muon-neutrino as in Fig. 2,

$$A \to \nu_\mu \bar{\nu}_e \tag{3.2}$$

as illustrated in Fig. 3.
For our purposes, it is sufficient to calculate the decay rate for the first half of the process

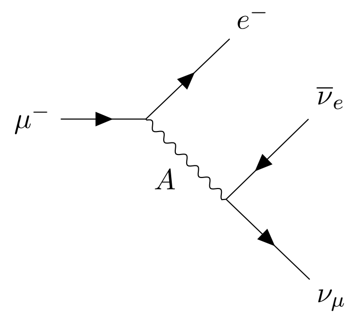


**Figure 1**. The tree level Feynman diagram of the decay of a muon into an electron.

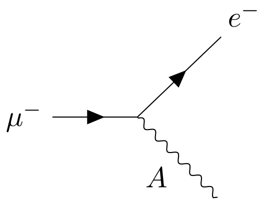


**Figure 2**. Muon decaying into an electron and boson.

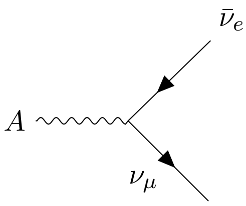


**Figure 3**. Boson decaying into an electron anti-neutrino and muon neutrino.

(i.e. 3.1).
Experimentally, the neutrinos are much more difficult to observe than the electron, so we are usually interested in the decay rate as a function of just the electron energy. Hence, the total decay rate can then be found using:

$$\Gamma(\mu^- \to e^- \nu_\mu \bar{\nu}_e) = \frac{256\pi}{g^2 m_\mu} \Gamma_0 \int_0^{z_{max}} \Gamma(\mu^- \to e^- A) z^3 dz \tag{3.3}$$

where $\Gamma_0 = \frac{G_\Gamma^2 m_\mu^2}{192\pi^3}$ is the decay rate of a muon in limit in which the electron is massless. The decay rate $\Gamma(\mu^- \to e^- A)$ can be calculated using the following:

$$\Gamma = \frac{1}{2m_\mu} \int d\Pi_{\text{LIPS}} |\langle \mathcal{M} \rangle|^2. \tag{3.4}$$

Here $\mathcal{M}$ is the usual amplitude written down using the Feynman rules for Quantum Electrodynamics (QED). We integrate over the Lorentz invariant phase space:

$$d\Pi_{\text{LIPS}} = \frac{d^3\boldsymbol{p}_e}{(2\pi)^3 2E_e} \frac{d^3\boldsymbol{p}_A}{(2\pi)^3 2E_A} (2\pi)\delta\left(m_\mu - E_e - E_A\right) (2\pi)^3 \delta^3\left(\boldsymbol{p}_e + \boldsymbol{p}_A\right),$$

which tells us the number of available states in the phase space for final state particles. It depends solely on the kinematics, i.e. masses, energies and momenta.
We begin by writing down the amplitude of the decay:

$$\mathcal{M} = \bar{u}_e\left(p_e\right)\left(ig\gamma^\mu L\right)\epsilon_\mu\left(p_A\right) u_\mu,$$

where $L = \frac{1-\gamma^5}{2}$ is the left-handed Weyl spinor. To calculate $|\langle\mathcal{M}\rangle|^2$, we make use of Casimir's trick and the properties of gamma matrices listed in Appendix A:

$$
\begin{aligned}
|\langle\mathcal{M}\rangle|^2 &= \frac{1}{2}\sum_{\text{spins}}\left[\bar{u}_e\left(p_e\right)\left(\frac{ig}{\sqrt{2}}\gamma^\mu L\right)u_\mu\left(\not{p}_\mu\right)\right]\left[\bar{u}_e\left(\not{p}_e\right)\left(\frac{-ig}{\sqrt{2}}\gamma^\nu L\right)u_\mu\left(p_\mu\right)\right]^*\epsilon_\mu\left(p_A\right)\epsilon_\nu\left(p_A\right)^* \\
&= \frac{1}{2}\cdot\mathrm{Tr}\left[\left(\frac{ig}{\sqrt{2}}\gamma^\mu L\right)\left[\not{p}_\mu + m_\mu\right]\left(\frac{-ig}{\sqrt{2}}\gamma^\nu L\right)\left[\not{p}_e + m_e\right]\right]\cdot\left[-g_{\mu\nu} + \frac{p_{A_\mu}p_{A_\nu}}{p_A^2}\right] \\
&= \frac{g^2}{16}\cdot\mathrm{Tr}\left[\gamma^\mu\left(1-\gamma^5\right)\left[\not{p}_\mu + m_\mu\right]\gamma^\nu\left(1-\gamma^5\right)\left[\not{p}_e + m_e\right]\right]\cdot\left[-g_{\mu\nu} + \frac{p_{A_\mu}p_{A_\nu}}{p_A^2}\right] \\
&= \frac{g^2}{8}\cdot\mathrm{Tr}\left[\gamma^\mu\not{p}_\mu\gamma^\nu\left(1-\gamma^5\right)\left[\not{p}_e + m_e\right]\right]\cdot\left[-g_{\mu\nu} + \frac{p_{A_\mu}p_{A_\nu}}{p_A^2}\right] \\
&= \frac{g^2}{8}\cdot\mathrm{Tr}\left[\gamma^\mu\not{p}_\mu\gamma^\nu\not{p}_e + \gamma^\mu\not{p}_\mu\gamma^\nu m_e - \gamma^\mu\not{p}_\mu\gamma^\nu\gamma^5\not{p}_e - \gamma^\mu\not{p}_\mu\gamma^\nu m_e\right]\cdot\left[-g_{\mu\nu} + \frac{p_{A_\mu}p_{A_\nu}}{p_A^2}\right]
\end{aligned}
$$

where the simplification in the second last line is owing to the facts that $\{\gamma^\nu, \gamma^5\} = 0$, $(1-\gamma^5)(1+\gamma^5) = 0$ and $(1-\gamma^5)^2 = 2(1-\gamma^5)$. There is a sum over spins because we are finding the average amplitude. The second and fourth terms in the trace are zero since they contain an odd number of $\gamma^\mu$ matrices. We can take the $-g_{\mu\nu}$ and $p_{A_\mu}p_{A_\nu}$ terms inside the trace to evaluate the other two, since they are independent of the $\gamma$ matrices. Taking in $-g_{\mu\nu}$, the first term becomes:

$$\mathrm{Tr}[-\gamma^\mu\not{p}_\mu\gamma_\mu\not{p}_e] = \mathrm{Tr}[2\not{p}_\mu\not{p}_e] = 8p_\mu \cdot p_e$$

Similarly, the third term becomes:

$$\mathrm{Tr}[-\gamma^\mu\not{p}_\mu\gamma_\mu\gamma^5\not{p}_e] = \mathrm{Tr}[2\not{p}_\mu\gamma^5\not{p}_e] = 0,$$

where the last equality again follows from the fact that we have an odd number of $\gamma^\mu$ matrices. Taking in $p_{A_\mu}p_{A_\nu}$, the first term becomes:

$$\mathrm{Tr}\left[\not{p}_A\not{p}_\mu\not{p}_A\not{p}_e\right] = 8\left(p_A\cdot p_\mu\right)\left(p_A\cdot p_e\right) - 4p_A^2\left(p_\mu\cdot p_e\right)$$

The third time just goes to zero.

We now have the full trace and can write down the total amplitude:

$$|\langle\mathcal{M}\rangle|^2 = \frac{g^2}{2}\left(p_\mu\cdot p_e + \frac{2\left(p_A\cdot p_\mu\right)\left(p_A\cdot p_e\right)}{p_A^2}\right).$$

In the rest frame of the muon, $p_\mu = (m_\mu, \mathbf{0})$ and $p_\mu = p_A + p_e$. We can hence write:

$$|\langle\mathcal{M}\rangle|^2 = \frac{g^2}{2}\left(m_\mu E_e + \frac{2\left(m_\mu^2 - E_e m_\mu\right)\left(E_e m_\mu - m_e^2\right)}{m_A^2}\right).$$

We are now ready to integrate using Eq. (3.4).

$$\begin{aligned}\Gamma =&\frac{g^2}{4m_\mu}\int\frac{d^3\boldsymbol{p}_e}{(2\pi)2E_e}\frac{d^3\boldsymbol{p}_A}{(2\pi)2E_A}\delta\left(m_\mu-E_e-E_A\right)\delta^3\left(\mathbf{p}_e+\mathbf{p}_A\right)\\&\left(m_\mu E_e+\frac{2\left(m_\mu^2-E_e m_\mu\right)\left(E_e m_\mu-m_e^2\right)}{m_A^2}\right)\\=&\frac{g^2}{16\pi m_\mu}\int d^3\boldsymbol{p}_e\frac{\delta\left(m_\mu-E_e-\sqrt{m_A^2+\boldsymbol{p}_e^2}\right)}{4\pi E_e\cdot\sqrt{m_A^2+\boldsymbol{p}_e^2}}\\&\left(m_\mu E_e+\frac{2\left(m_\mu^2-E_e m_\mu\right)\left(E_e m_\mu-m_e^2\right)}{m_A^2}\right)\end{aligned}$$

The integral over $\boldsymbol{p}_e$ is evaluated in spherical coordinates so $\boldsymbol{p}_e^3\to\boldsymbol{p}_e^2\sin\theta d|\boldsymbol{p}_e|d\theta d\phi$. The angular integration just gives us $4\pi$ and the decay rate becomes:

$$\begin{aligned}\Gamma =\ &\frac{g^2}{16\pi m_\mu}\int d\left|\boldsymbol{p}_e\right|\boldsymbol{p}_e^2\frac{\delta\left(m_\mu-\sqrt{m_e^2+\boldsymbol{p}_e^2}-\sqrt{m_A^2+\boldsymbol{p}_e^2}\right)}{\sqrt{m_e^2+\boldsymbol{p}_e^2}\cdot\sqrt{m_A^2+\boldsymbol{p}_e^2}}\\&\left(m_\mu E_e+\frac{2\left(E_e m_\mu-m_e^2\right)\left(m_\mu^2-m_\mu E_e\right)}{m_A^2}\right).\end{aligned}$$

To evaluate this integral, we decompose the delta function using the identity:

$$\delta(f(x))=\sum_i\frac{\delta(x-x_i)}{|f'(x_i)|}$$

where $x_i$ are the roots of $f(x)$. This allows us to rewrite our delta function as follows:

$$\delta\left(m_\mu-\sqrt{m_e^2+\boldsymbol{p}_e^2}-\sqrt{m_A^2+\boldsymbol{p}_e^2}\right)=\frac{\sqrt{m_e^2+p^2}\cdot\sqrt{m_A^2+p^2}}{|p|\left(\sqrt{m_e^2+p^2}+\sqrt{m_A^2+p^2}\right)}\delta\left(\boldsymbol{p}_e-p\right),\tag{3.5}$$

where the roots are

$$\pm p=\pm\frac{\sqrt{m_A^4+m_e^4+m_\mu^4-2m_e^2m_A^2-2m_\mu^2m_A^2-2m_e^2m_\mu^2}}{2m_\mu}=\frac{\lambda^{1/2}(m_\mu,m_e,m_A)}{2m_\mu}.$$

We discard the negative root since we are integrating over $|p|$. Since the two delta functions in Eq. (3.5) are imposing equivalent conditions, we find that the sum in the denominator is just $m_\mu$. We hence get:

$$\frac{\delta\left(m_\mu-\sqrt{m_e^2+\boldsymbol{p}_e^2}-\sqrt{m_A^2+\boldsymbol{p}_e^2}\right)}{\sqrt{m_e^2+\boldsymbol{p}_e^2}\cdot\sqrt{m_A^2+\boldsymbol{p}_e^2}}=\frac{1}{|p|m_\mu}\delta\left(\boldsymbol{p}_e-p\right),$$

Plugging this into the integral and integrating, we find

$$\Gamma=\frac{g^2}{16\pi m_\mu}\frac{\lambda^{1/2}(m_\mu,m_e,m_A)}{2m_\mu^2}\cdot\left(m_\mu E_e+\frac{2\left(E_e m_\mu-m_e^2\right)\left(m_\mu^2-m_\mu E_e\right)}{m_A^2}\right).$$

Let's now define $\delta = m_e/m_\mu$. Furthermore, we can see that the minimum mass of A, $m_A$, is 0, and the maximum mass it can have is the mass it has when it's momentum is zero, and hence $m_A = E_A = E_\mu - E_e = m_\mu - m_e$ in the rest frame of a free muon. This motivates us to define

$$m_A = z m_\mu,$$

where $0 < z < z_{max} = (1-\delta)$.
In terms of our new variables, $\lambda^{1/2}(m_\mu, m_e, m_A) = m_{\mu^2}\lambda^{1/2}(1,\delta^2,z^2)$. Hence, we we define

$$q(z,\delta) = \frac{1}{2}\lambda^{1/2}(1,\delta^2,z^2)m_\mu$$

and. Rewriting the decay rate in terms of our new variables, we obtain:

$$\begin{aligned}\Gamma &= \frac{g^2}{16\pi}q(z)\cdot\left(\frac{E_e z^2 m_\mu^2 + 2\left(E_e m_\mu^2 - E_e^2 m_\mu - m_e^2 m_\mu + m_e^2 E_e\right)}{z^2 m_\mu^3}\right)\\ &= \frac{g^2}{16\pi}\frac{q(z)}{z^2}\cdot\left(\frac{E_e}{m_\mu}z^2 + 2\left(\frac{E_e}{m_\mu}(1+\delta^2) - \frac{E_e^2}{m_\mu^2}\right) - 2\delta^2\right)\end{aligned}$$

For the decay $\mu \to eA$, we can find the energy of the electron using conservation of momentum and applying laws of cosines. We find $E_e = \frac{m_\mu^2 + m_e^2 - m_A^2}{2m_\mu}$ which, in our new variables, gives us

$$\frac{E_e}{m_\mu} = \frac{1}{2}\left(\delta^2 + 1 - z^2\right).$$

Substituting this into our expression, we get:

$$\begin{aligned}\Gamma &= \frac{g^2}{32\pi}\frac{q(z)}{z^2}\left[(\delta^2 z^2 + z^2 - z^4 + (\delta^2 + 1 - z^2)\left(1+\delta^2+z^2\right) - 4\delta^2\right],\\ &= \frac{g^2}{32\pi}\frac{q(z)}{z^2}\left(1 + z^2 - 2z^4 + \delta^2 z^2 - 2\delta^2 + \delta^4\right)\end{aligned}$$

We are now finally ready to calculate the decay rate for $\mu^- \to e^-\bar{\nu}_e\nu_\mu$ using Eq.(3.3).

$$\begin{aligned}\Gamma\left(\mu^- \to e^-\bar{\nu}_e\nu_\mu\right) &= \frac{8}{m_\mu}\Gamma_0\int_0^{z_{\max}} z\cdot q(z)\cdot\left(1 + z^2 - 2z^4 + \delta^2 z^2 - 2\delta^2 + \delta^4\right)dz\\ &= 4\Gamma_0\int_0^{1-\delta}\lambda^{1/2}(1,\delta^2,z^2)\cdot\left(z + z^3 - 2z^5 + \delta^2 z^3 - 2\delta^2 z + \delta^4 z\right)dz\\ &= \Gamma_0\left[1 - 8\delta^2 - 24\log\delta + 8\delta^6 - \delta^8\right]\end{aligned}$$

where $\Gamma_0$ is the decay rate in the massless limit of the electron. It is clearly recovered when we take $m_e \to 0 \implies \delta \to 0$.

## 4 Bound Muon

Bound state arises because of Coloumb interaction of muon with heavy charged particles such as atomic nuclei. The atomic bound state is formed by replacing the electron of hydrogen atoms by a muon. When a muon comes close to the 1S state of a muonic atom, it can either be captured by the nucleus or it can decay. For heavy nuclei, capture dominates the decay. Due to the higher mass of muons as compared to electrons, the atomic orbits of muon states are very small. The decay rate of bound muon is significantly different from the free muon decay rate due to nuclear effects. We also assume that the nucleus has no spin.

### 4.1 Decay Rate

Consider the following process:

$$(Z\mu^-) \to (Ze^-)A, \tag{4.1}$$

$(Z\mu^-)$ represents the bound state of muon with the atomic number Z that decays into a bound electron. A is a boson that then decays as:

$$A \to \nu_\mu \bar{\nu}_e$$

It is convenient to divide the decay into these two processes. From Eq. (4.1), A will have maximum mass when its momentum $q_A$ is zero and its mass is then $m_A = E_\mu - E_e$. Its mass will be zero i.e $m_A = 0$, when $k_A = \frac{E_\mu - E_e}{m_\mu}$. Introducing a dimensionless paramater, $z = m_A/m_\mu$. Corresponding the cases discussed above, $z_{max} = \frac{E_\mu - E_e}{m_\mu}$ and $z_{min} = 0$. In case of $z_{max}$, the neutrinos resulting from decay of A, will move in opposite directions and for $z_{min}$, they propagate in the same direction. The decay rate of the two processes $(Z\mu^-) \to (Ze^-)A$ and $A \to \nu_\mu \bar{\nu}_e$ are related using z as:

$$\Gamma((Z\mu^-) \to (Ze^-))\nu_\mu \bar{\nu}_e = \frac{256\pi}{g^2 m_\mu} \Gamma_0 \int_0^{z_{\max}} \Gamma((Z\mu^-) \to (Ze^-)A) z^3 dz,$$

where $\Gamma_0$ is the decay rate of free muon, g is the weak coupling constant. We are interested in finding the bound muon decay rate to the free electron decay rate ratio:

$$\frac{\Gamma}{\Gamma_0} = \frac{256\pi}{g^2 m_\mu} \int_0^{z_{\max}} \Gamma((Z\mu^-) \to (Ze^-)A) z^3 dz, \tag{4.2}$$

Now there are two situations:

1. The spin of muon and electron is the same, i.e the spin does not flip.
2. The spin of muon and electron is opposite, i.e. the spin flips.

We find the amplitude of the cases in the following sections.

#### 4.1.1 The Spin Non Flip Case

Amplitude for the case where both electron and muon have the same spin state:

$$\mathcal{M} = \frac{g}{\sqrt{2}} \int d^3 r e^{i\vec{q}\cdot\vec{r}} \bar{\Phi}_e(\vec{r}) \not{\epsilon}^{\lambda_A *} L \Phi_\mu(\vec{r})$$

Where $\Phi(\vec{r})$ is the position space wavefunction.

$$\begin{aligned}
\Phi(\boldsymbol{r}) &= f(r)u_{\uparrow/\downarrow}, \\
u_{\uparrow/\downarrow} &= \not{\rho}\phi_{\uparrow/\downarrow}, \\
f(r) &= \tfrac{(2m_\mu\alpha_Z)^{\gamma+1/2}}{\sqrt{4\pi}}\sqrt{\tfrac{1+\gamma}{2\Gamma(1+2\gamma)}}r^{\gamma-1}\exp\left(-m_\mu\alpha_Z r\right), \\
\phi_{\uparrow/\downarrow} &= \begin{pmatrix}\chi_{\uparrow/\downarrow}\\ 0\end{pmatrix}, \\
\rho^\mu &= (1, ia\hat{r}), & \chi_\uparrow &= \begin{pmatrix}1\\0\end{pmatrix}, \\
a &= \tfrac{1-\gamma}{\alpha_Z}, & \chi_\downarrow &= \begin{pmatrix}0\\1\end{pmatrix}.
\end{aligned}$$

Now, the amplitude in which muon and electron both are in spin up state:

$$\mathcal{M}_{\uparrow\to\uparrow} = \frac{g}{\sqrt{2}}\int d^3r e^{i\vec{q}\cdot\vec{r}} f_e(r)f_\mu(r)(\bar{u}_e^\uparrow\gamma^u\epsilon_\mu^{\lambda_{A^*}}Lu_\mu^\uparrow). \tag{4.3}$$

The spin of muon and the electron is the same so the third component of the polarization vector of boson A is zero, since there should be no component of the polarization vector in the z direction. Boson is polarized in longitudnal direction so, its helicty $\lambda = 0$. To write polarization we use $\epsilon_\mu(\vec{k}, 0) = \frac{1}{m}\left(|\vec{k}|, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta\right)$, since for longitudnal polarization $\lambda = 0$. In case of A traveling along z direction, $\theta = 0$, and this reduces to $\epsilon_\mu(\vec{k}, 0) = \frac{1}{m}\left(k_A, 0, 0, E_A\right)$.

$$\begin{aligned}
\epsilon_\mu^{0_{A^*}} &= \frac{1}{m_A}(k_A, 0, 0, E) \\
&= \frac{1}{z}\left(\frac{k_A}{m_\mu}, 0, 0, \frac{E_\mu - E_e}{m_\mu}\right) \\
&= \frac{1}{z}(k_A, 0, 0, z_{max})
\end{aligned}$$

If the boson moves in negative z direction, then $\epsilon_\mu^{\lambda_{A^*}} = \frac{1}{z}(k_A, 0, 0, -z_{max})$. There are two ways to solve the problem:

1. We keep the initial state muon fixed and take into account the two polarization states of A.

2. We keep the polarization of A fixed and take into account both spin projects.

We do the latter.
In Eq. (4.3), leaving the constants out, them term in brackets is $\bar{u}_e^\uparrow\gamma^\mu\epsilon_\mu^{\lambda_{A^*}}(1-\gamma_5)u_\mu^\uparrow = \mathrm{Tr}(\bar{u}_e^\uparrow\gamma^u(1-\gamma_5)u_\mu^\uparrow)\cdot\epsilon_\mu^{\lambda_{A^*}}$. $\bar{u}_e^\uparrow = \bar{\phi}\not{\rho} = \phi^{\dagger\uparrow}\gamma^0\not{\rho}$, $\phi^{\dagger\uparrow}\phi^\uparrow = \frac{1+\gamma^0}{2}\frac{\gamma^5+\gamma^3}{2}\gamma^5$. With these the trace becomes:

$$\begin{aligned}
\epsilon_\mu^{\lambda_{A^*}}\cdot\mathrm{Tr}(\bar{u}_e^\uparrow\gamma^u(1-\gamma_5)u_\mu^\uparrow) &= \epsilon_\mu^{\lambda_{A^*}}\cdot\mathrm{Tr}[\not{\rho}\phi^{\dagger\uparrow}\phi^\uparrow\gamma^0\not{\rho}'\gamma^\mu(1-\gamma_5)] \\
&= \epsilon_\mu^{\lambda_{A^*}}\cdot\mathrm{Tr}\left[\not{\rho}\frac{1+\gamma^0}{2}\frac{\gamma^5+\gamma^3}{2}\gamma^5\gamma^0\not{\rho}'\gamma^\mu(1-\gamma_5)\right]
\end{aligned}$$

$$\mathrm{Tr}\left[\frac{1}{4}\not{\rho}\gamma^5\gamma^0\not{\rho}'\gamma^\mu(1-\gamma_5)\right] = \mathrm{Tr}\left[-\frac{1}{4}\not{\rho}\gamma^5\gamma^0\not{\rho}'\gamma^\mu\gamma_5\right] = \mathrm{Tr}\left[\frac{1}{4}\not{\rho}\gamma^0\not{\rho}'\gamma^\mu\right] = \rho^0\cdot\rho'^\mu + \rho'^\mu\cdot\rho^0 - \eta^{0\mu}(\rho\cdot\rho')$$

$$\mathrm{Tr}\left[\frac{1}{4}\not{\rho}\gamma^0\gamma^3\gamma^5\gamma^0\not{\rho}'\gamma^\mu(1-\gamma_5)\right] = \mathrm{Tr}\left[\frac{1}{4}\not{\rho}\gamma^0\gamma^3\gamma^0\not{\rho}'\gamma^\mu\right] = \rho^3\cdot\rho'^\mu + \rho'^\mu\cdot\rho^3 - \eta^{3\mu}(\rho\cdot\rho')$$

The rest of the terms give zero. Using these, we get:

$$\begin{aligned}
\epsilon_\mu^{\lambda_A *}\cdot(\bar{u}_e^\uparrow\gamma^u(1-\gamma_5)u_\mu^\uparrow) &= \epsilon_\mu^{\lambda_A *}\cdot[1+a^2,\ldots,\ldots,-(1-a^2+2a^2\cos^2\theta)] \\
&= \frac{1}{z}[k_A,0,0,z_{max}]\cdot[1+a^2,\ldots,\ldots,-(1-a^2+2a^2\cos^2\theta)] \\
&= \frac{1}{z}(k_a(1+a^2)+z_{max}(1-a^2+2a^2\cos^2\theta))
\end{aligned} \tag{4.4}$$

Now, for the case where a spin down muon decays into spin down electron Eq. (4.3) will stay the same and only spin of $\bar{u}_e$ and $u_\mu$ changes. Computing the term in brackets for the spin down case:

$$\begin{aligned}
\bar{u}_e^\downarrow\gamma^u\epsilon_\mu^{\lambda_A *}(1-\gamma_5)u_\mu^\downarrow &= \mathrm{Tr}(\bar{u}_e^\downarrow\gamma^\mu(1-\gamma_5)u_\mu^\downarrow)\cdot\epsilon_\mu^{\lambda_A *} \\
&= \epsilon_\mu^{\lambda_A *}\cdot\mathrm{Tr}\left[\not{\rho}\frac{\gamma^5-\gamma^3}{2}\frac{1-\gamma^0}{2}\gamma^5\gamma^0\not{\rho}'\gamma^\mu(1-\gamma_5)\right] \\
&= \epsilon_\mu^{\lambda_A *}\cdot[1+a^2,\ldots,\ldots,(1-a^2+2a^2\cos^2\theta)] \\
&= \frac{1}{z}(k_a(1+a^2)-z_{max}(1-a^2+2a^2\cos^2\theta))
\end{aligned} \tag{4.5}$$

Eq. (4.4) and Eq. (4.5), differ only by the sign of second of the second term only, everything else in the amplitudes is the same, it convenient to define:

$$N_a = \frac{\mathcal{M}_{\uparrow\to\uparrow}+\mathcal{M}_{\downarrow\to\downarrow}}{2}$$
$$N_b = \frac{\mathcal{M}_{\uparrow\to\uparrow}-\mathcal{M}_{\downarrow\to\downarrow}}{2}$$

Moreover, note that we are intrested in average amplitudes and they relate to these variable as:

$$N_a^2 + N_b^2 = |\langle\mathcal{M}_{\uparrow\to\uparrow}\rangle|^2 + |\langle\mathcal{M}_{\downarrow\to\downarrow}\rangle|^2. \tag{4.6}$$

This relation will be useful in computing the final decay rate later. Now, $N_a$, using the definitions above:

$$N_a = \frac{g}{2(2)}\frac{2k_A}{\sqrt{2}z}\left(1+a^2\right)\int d^3r e^{i\boldsymbol{q}\cdot\boldsymbol{r}} f_e(r)f_\mu(r),$$

where

$$f_e(r)f_\mu(r) = \frac{\left(4m_\mu^2\alpha_Z^2\delta\right)^{\gamma+1/2}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}r^{2\gamma-2}e^{-m_\mu(1+\delta)\alpha_Z r},$$

with $\delta = m_e/m_\mu$.

$$\begin{aligned}
N_a &= \frac{g\left(1+a^2\right)k_A}{2\sqrt{2}z}\frac{\left(4m_\mu^2\alpha_Z^2\delta\right)^{\gamma+1/2}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\int_0^{2\pi}d\phi\int_0^\infty r^2dr\int_{-1}^1 e^{iqrC_\theta}dC_\theta r^{2\gamma-2}e^{-m_\mu(1+\delta)\alpha_Z r} \\
&= \frac{g\left(1+a^2\right)k_A}{\sqrt{2}z}\frac{\left(4m_\mu^2\alpha_Z^2\delta\right)^{\gamma+1/2}}{4}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\int_0^\infty dr\frac{2\sin(qr)}{qr}r^{2\gamma}e^{-m_\mu(1+\delta)\alpha_Z r}
\end{aligned}$$

Using the integral:

$$\int_0^\infty ds e^{-s} s^{a-1} \sin(ps) = \Gamma(a)(1+p^2)^{-a/2} \sin\big(a \tan^{-1} p\big)$$

$$\int_0^\infty dr \sin(qr) r^{2\gamma-1} e^{-m_\mu(1+\delta)\alpha_Z r} = \frac{(m_\mu(1+\delta)\alpha_Z)^{1-2\gamma}}{m_\mu(1+\delta)\alpha_Z} \int_0^\infty ds \sin\bigg(\frac{qs}{m_\mu(1+\delta)\alpha_Z}\bigg) s^{2\gamma-1} e^{-s}$$

$$= (m_\mu(1+\delta)\alpha_Z)^{-2\gamma}\Gamma(2\gamma)\left(1+\left(\frac{q}{m_\mu(1+\delta)\alpha_Z)}\right)^2\right)^{-\gamma} \sin\bigg(2\gamma \tan^{-1}\frac{q}{m_\mu(1+\delta)\alpha_Z)}\bigg)$$

In the second line, we did a change of variable $s = m_\mu(1+\delta)\alpha_Z r \implies r = \frac{s}{m_\mu(1+\delta)\alpha_Z}$. Defining $k = \frac{k_A}{\alpha_Z(1+\delta)}$ and the dimension-less momentum of the boson $A$, $k_A = \frac{q}{m_\mu}$.

$$= (m_\mu(1+\delta)\alpha_Z)^{-2\gamma}\Gamma(2\gamma)\left(1+k^2\right)^{-\gamma} \sin\big(2\gamma \tan^{-1} k\big)$$

Now $N_a$ becomes:

$$N_a = \frac{g\left(1+a^2\right)k_A}{\sqrt{2}z} \frac{\left(4m_\mu^2\alpha_Z^2\delta\right)^{\gamma+1/2}}{4} \frac{1+\gamma}{2\Gamma(1+2\gamma)} (m_\mu(1+\delta)\alpha_Z)^{-2\gamma}\Gamma(2\gamma)\left(1+k^2\right)^{-\gamma} \sin\big(2\gamma \tan^{-1} k\big)$$
$$= \sqrt{2}g\frac{k_A}{2z}\left(1+a^2\right)\frac{1+\gamma}{8}\left(\frac{4\delta}{(1+\delta)^2}\right)^{\gamma+\frac{1}{2}}\left[1+k^2\right]^{-\gamma}\frac{\Gamma[2\gamma]}{\Gamma(1+2\gamma)k}\sin\left[2\gamma\tan^{-1}(k)\right]$$

We can find $N_b$ in a similar way,

$$N_b = \frac{gk_A}{2\sqrt{2}z} \frac{\left(4m_\mu^2\alpha_Z^2\delta\right)^{\gamma+1/2}}{4\pi} \frac{1+\gamma}{2\Gamma(1+2\gamma)} \int_0^{2\pi} d\phi d\int_0^\infty dr \int_{-1}^1 e^{iqrC_\theta}\left(1-a^2+2a^2\cos^2\theta\right) dC_\theta r^{2\gamma} e^{-m_\mu(1+\delta)\alpha_Z r}$$

The $\theta$ integral is:

$$\begin{aligned}\int_{-1}^1 e^{iqrC_\theta}\left(1-a^2+2a^2\cos^2\theta\right) dC_\theta &= \int_{-1}^1 e^{iqrC_\theta}\left(1-a^2\right) dC_\theta \\ &\quad + 2a^2\int_{-1}^1 e^{iqrC_\theta}\left(\cos^2\theta\right) dC_\theta \\ &= \left(1-a^2\right)\frac{2\sin(qr)}{qr} + 2a^2\left[2\frac{\sin qr}{qr} - 4\frac{\sin qr}{(qr)^3} + 2\frac{\cos qr}{(qr)^2}\right] \\ &= \left(1+a^2\right)\frac{2\sin(qr)}{qr} + 2\frac{\cos qr}{(qr)^2}\end{aligned}$$

Using the same integral we get:

$$N_b = \frac{gz_{\max}}{\sqrt{2}z}\left[\frac{4\delta}{(1+\delta)^2}\right]^{\gamma+\frac{1}{2}}\frac{1+\gamma}{4}\Bigg[4a^2\Bigg\{\frac{\Gamma[2\gamma-1]}{\Gamma(1+2\gamma)k^2}\left(1+k^2\right)^{-\gamma+\frac{1}{2}}\cos\left[(2\gamma-1)\tan^{-1}k\right]$$
$$-\left(1+k^2\right)^{-\gamma+1}\frac{\Gamma[2\gamma-2]}{\Gamma(1+2\gamma)k^3}\sin\left[(2\gamma-2)\tan^{-1}k\right]\Bigg\}$$
$$+\left(1+a^2\right)\left(1+k^2\right)^{-\gamma}\frac{\Gamma[2\gamma]}{\Gamma(1+2\gamma)k}\sin\left[2\gamma\tan^{-1}k\right]\Bigg]$$

Introducing the notation:

$$S_n \equiv \frac{1+\gamma}{8}\left(\frac{4\delta}{(1+\delta)^2}\right)^{\gamma+1/2}\frac{\Gamma[1+2\gamma-n]}{\Gamma(1+2\gamma)k^n}\left[1+k^2\right]^{\frac{n-1}{2}-\gamma}\sin\left[(2\gamma-n+1)\tan^{-1}(k)\right], \tag{4.7}$$

$$C_n \equiv \frac{1+\gamma}{8}\left(\frac{4\delta}{(1+\delta)^2}\right)^{\gamma+1/2}\frac{\Gamma[1+2\gamma-n]}{\Gamma(1+2\gamma)k^n}\left[1+k^2\right]^{\frac{n-1}{2}-\gamma}\cos\left[(2\gamma-n+1)\tan^{-1}(k)\right]. \tag{4.8}$$

Expressing $N_a$ and $N_b$ in this notation,

$$\begin{aligned} N_a &= \sqrt{2}\frac{k_A}{z}g\left(1+a^2\right)S_1, \\ N_b &= \sqrt{2}\frac{z_{\max}}{z}g\left[4a^2\left(C_2-S_3\right)+\left(1+a^2\right)S_1\right]. \end{aligned}$$

The squared sum of these two expressions gives the contribution from the spin non-flip part toward the decay rate of the bound muon as given by Eq. (4.6).

### 4.1.2 The Spin Flip Part

There are two cases in which spin of the muon is flipped. If a spin up muon decays into a spin down electron, the amplitude is:

$$\mathcal{M}_{\uparrow\to\downarrow} = \frac{g}{\sqrt{2}}\int d^3r e^{i\vec{q}\cdot\vec{r}} f_e(r) f_\mu(r)(\bar{u}_e^{\downarrow}\gamma^u\epsilon_\mu^{\lambda_A *}Lu_\mu^{\uparrow}). \tag{4.9}$$

Calculating $(\bar{u}_e^{\uparrow}\gamma^u\epsilon_\mu^{\lambda_A *}Lu_\mu^{\downarrow})$ first:

$$\begin{aligned} \left[\bar{u}_e^{\downarrow}\gamma^\mu\epsilon_\mu^{\lambda_A *}\left(1-\gamma^5\right)u_\mu^{\uparrow}\right] &= \epsilon_\mu^{\lambda_A *}\,\mathrm{Tr}\left[\rho\phi_\uparrow\phi_\downarrow^\dagger\gamma^0\rho\gamma^\mu\left(1-\gamma_5\right)\right] \\ &= \rho^\alpha\rho^{\beta'}\epsilon_\mu^{\lambda_A *}\,\mathrm{Tr}\left[\gamma^\alpha\frac{1+\gamma^0}{2}\frac{\gamma^1+i\gamma^2}{2}\gamma^5\gamma^0\gamma^\beta\gamma^\mu\left(1-\gamma_5\right)\right] \\ &= \epsilon_\mu^{\lambda_A *}\left[0,-\left(1-a^2\right)+2iax\sin\theta\cos\phi,-i\left(1-a^2\right)+2xia\sin\theta\sin\phi,\ldots\right] \end{aligned}$$

Expression for $\phi_\uparrow\phi_\downarrow^\dagger$ in terms of gamma matrices is given in Appendix A2. We defined $x = (ia\sin\theta\cos\phi - a\sin\theta\sin\phi)$ and in this case of spin flip polarization must have the form $\epsilon_\mu^{\lambda_A *} = \epsilon_\mu^{\lambda_A *}(\hat{z},+1) = \frac{1}{\sqrt{2}}[0,-1,i,0]$. Taking the dot product with the polarization vector, we get:

$$\begin{aligned} \bar{u}_e^{\downarrow}\gamma^\mu\epsilon_\mu^{\lambda_A *}\left(1-\gamma_5\right)u_\mu^{\uparrow} &= -\frac{1}{\sqrt{2}}\left[2\left(1-a^2\right)-2iax\sin\theta\cos\phi-2ax\sin\theta\sin\phi\right]. \\ &= -\frac{1}{\sqrt{2}}[2\left(1-a^2\right)-2ia(ia\sin\theta\cos\phi-a\sin\theta\sin\phi)\sin\theta\cos\phi \\ &\quad -2a(ia\sin\theta\cos\phi-a\sin\theta\sin\phi)\sin\theta\sin\phi] \\ &= -\frac{1}{\sqrt{2}}2(1-a^2cos^2\theta), \end{aligned}$$

where in second line we substituted the value of x and then simplified it. Substituting in Eq. (4.9):

$$
\begin{aligned}
|\mathcal{M}_{\uparrow\to\downarrow}| &= \frac{g}{2\sqrt{2}\sqrt{2}} \frac{\left(4m_\mu^2\alpha_Z^2\delta\right)^{\gamma+1/2}}{4\pi} \frac{1+\gamma}{2\Gamma(1+2\gamma)} \int d\phi dr e^{iqrC_\theta} dC_\theta r^{2\gamma} e^{-m_\mu(1+\delta)\alpha_Z r} \\
&\quad 2[1-a^2cos^2\theta] \\
&= \frac{g}{4} \frac{\left(4m_\mu^2\alpha_Z^2\delta\right)^{\gamma+1/2}}{4} \frac{1+\gamma}{2\Gamma(1+2\gamma)} \int dr e^{iqrC_\theta} dC_\theta r^{2\gamma} e^{-m_\mu(1+\delta)\alpha_Z r} \\
&\quad 4\left[1-a^2\cos^2\theta\right]
\end{aligned}
$$

Performing the $\theta$ integral:

$$
\begin{aligned}
\int_{-1}^{1} e^{iqrC_\theta}\left(1-a^2\cos^2\theta\right) dC_\theta &= \int_{-1}^{1} e^{iqrC_\theta} dC_\theta - a^2 \int_{-1}^{1} e^{iqrC_\theta}\left(\cos^2\theta\right) dC_\theta \\
&= \frac{2\sin(qr)}{qr} - a^2\left[2\frac{\sin qr}{qr} - 4\frac{\sin qr}{(qr)^3} + 2\frac{\cos qr}{(qr)^2}\right] \\
&= \left(1-a^2\right)\frac{2\sin(qr)}{qr} - a^2\left[-4\frac{\sin qr}{(qr)^3} + 2\frac{\cos qr}{(qr)^2}\right]
\end{aligned}
$$

Substituting this back and performing the r integral (the integration follows same steps as before gives:

$$
\begin{aligned}
\mathcal{M}_{\uparrow\to\downarrow} &= 2g\left[\frac{4\delta}{(1+\delta)^2}\right]^{\gamma+\frac{1}{2}} \frac{1+\gamma}{8}\left[2a^2\left\{\frac{\Gamma(2\gamma-2)}{k^3}\left(1+k^2\right)^{-\gamma+1}\sin\left[2(\gamma-1)\tan^{-1}k\right]\right.\right. \\
&\quad \left. -\frac{\Gamma(2\gamma-1)}{k^2}\left(1+k^2\right)^{-\gamma+\frac{1}{2}}\cos\left[(2\gamma-1)\tan^{-1}k\right]\right\} \\
&\quad \left. +\left(1-a^2\right)q(2\gamma)\Gamma(2\gamma)\left(1+k^2\right)^{-\gamma}\sin\left[2\gamma\tan^{-1}k\right]\right] \\
&= g\left[4a^2\left(C_2-S_3\right) - 2\left(1-a^2\right)S_1\right] \equiv F_a
\end{aligned}
$$

Now consider the case where a spin down muon decays into a spin up electron:

$$
\mathcal{M}_{\downarrow\to\uparrow} = \frac{g}{\sqrt{2}}\int d^3 r e^{i\vec{q}\cdot\vec{r}} f_e(r) f_\mu(r) (\bar{u}_e^{\uparrow}\gamma^u\epsilon_\mu^{\lambda_A *} L u_\mu^{\downarrow}). \tag{4.10}
$$

$$
\begin{aligned}
\left[\bar{u}_e^{\uparrow}\gamma^\mu\epsilon_\mu^{\lambda_A *}\left(1-\gamma_5\right)u_\mu^{\downarrow}\right] &= \epsilon_\mu^{\lambda_A *}\,\mathrm{Tr}\left[\rho\phi_\downarrow\phi_\uparrow^\dagger\gamma^0\rho\gamma^\mu\left(1-\gamma_5\right)\right] \\
&= \rho^\alpha\rho^{\beta'}\epsilon_\mu^{\lambda_A *}\,\mathrm{Tr}\left[\gamma^\alpha\frac{\gamma^1-i\gamma^2}{2}\frac{\gamma^5-\gamma^3}{2}\gamma^0\gamma^\beta\gamma^\mu\left(1-\gamma_5\right)\right] \\
&= \epsilon_\mu^{\lambda_A *}[\ldots, -2ia\cos\theta, -2a\cos\theta, \ldots]
\end{aligned}
$$

where $\epsilon_\mu^{\lambda_A *} = \epsilon_\mu^{\lambda_A *}(\hat{z}, -1) = \frac{1}{\sqrt{2}}[0, 1, i, 0]$.

$$
\left[\bar{u}_e^{\uparrow}\gamma^\mu\epsilon_\mu^{\lambda_A *}\left(1-\gamma_5\right)u_\mu^{\downarrow}\right] = \frac{1}{\sqrt{2}}\left[2ia\cos\theta + 2ia\cos\theta\right] = \frac{1}{\sqrt{2}}4ia\cos\theta
$$

Substituting in Eq. (4.10) and after performing the $d\phi$ integral we get:

$$\begin{aligned}
\mathcal{M}_{\downarrow\to\uparrow} &= \frac{g}{2}\frac{\left(4m_\mu^2\alpha_Z^2\delta\right)^{\gamma+1/2}}{4}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\int dre^{iqrC_\theta}dC_\theta r^{2\gamma}e^{-m_\mu(1+\delta)\alpha_Z r}\left[4iaC_\theta\right] \\
&= \frac{g}{2}\left[\frac{4\delta}{(1+\delta)^2}\right]^{\gamma+\frac{1}{2}}\frac{1+\gamma}{8}8a\left[\frac{\Gamma(2\gamma)}{k}\left(1+k^2\right)^{-\gamma}\cos\left[(2\gamma)\tan^{-1}k\right]\right. \\
&\left.-\frac{\Gamma(2\gamma-1)}{k^2}\left(1+k^2\right)^{-\gamma+\frac{1}{2}}\sin\left[(2\gamma-1)\tan^{-1}k\right]\right] \\
&= -4ga\left(S_2-C_1\right)\equiv F_b.
\end{aligned}$$

The decay rate is

$$\begin{aligned}
\Gamma\left(\left(Z\mu^-\right)\to\left(Ze^-\right)A\right) &= \int d\Pi_{\text{LIPS}}|\langle\mathcal{M}\rangle|^2 \\
&= \frac{q}{2\pi}|\langle\mathcal{M}\rangle|^2 \\
&= \frac{q}{2\pi}\left(N_a^2+N_b^2+F_a^2+F_b^2\right) \\
&= \frac{m_\mu}{2\pi}k_A\left(N_a^2+N_b^2+F_a^2+F_b^2\right).
\end{aligned}$$

Substituting this value in Eq. (4.2) ratio of bound muon decay to the decay in the limit of massless electron becomes:

$$\frac{\Gamma\left(\left(Z\mu^-\right)\to\left(Ze^-\right)\nu_\mu\bar{\nu}_e\right)}{\Gamma_0} = \frac{128}{g^2}\int_0^{z_{\max}}\left(N_a^2+N_b^2+F_a^2+F_b^2\right)k_A z^3 dz$$

Redefining the $N_a$, $N_b$, $F_a$, $F_b$ as $g^2 N_a$ and similarly for the rest, we can get rid of the $g^2$ in the denominator we get:

$$\frac{\Gamma\left(\left(Z\mu^-\right)\to\left(Ze^-\right)\nu_\mu\bar{\nu}_e\right)}{\Gamma_0} = 128\int_0^{z_{\max}}\left(N_a^2+N_b^2+F_a^2+F_b^2\right)k_A z^3 dz \tag{4.11}$$

where

$$N_a \equiv \sqrt{2}\frac{k_A}{z}\left(1+a^2\right)S_1, \tag{4.12}$$

$$N_b \equiv \sqrt{2}\frac{z_{\max}}{z}\left[4a^2\left(C_2-S_3\right)+\left(1+a^2\right)S_1\right] \tag{4.13}$$

$$F_a \equiv \left[4a^2\left(C_2-S_3\right)-2\left(1-a^2\right)S_1\right] \tag{4.14}$$

$$F_b \equiv 4a\left(S_2-C_1\right) \tag{4.15}$$

This result is much simpler than the results of decays previously published [6]. Fig. 4, shows the contributions of spin flip and non flip parts to the total decay rate. Decay rate vanishes at non relativistic case, $\gamma = 1$, as expected and interestingly at extreme relativistic case $\gamma = 0$.

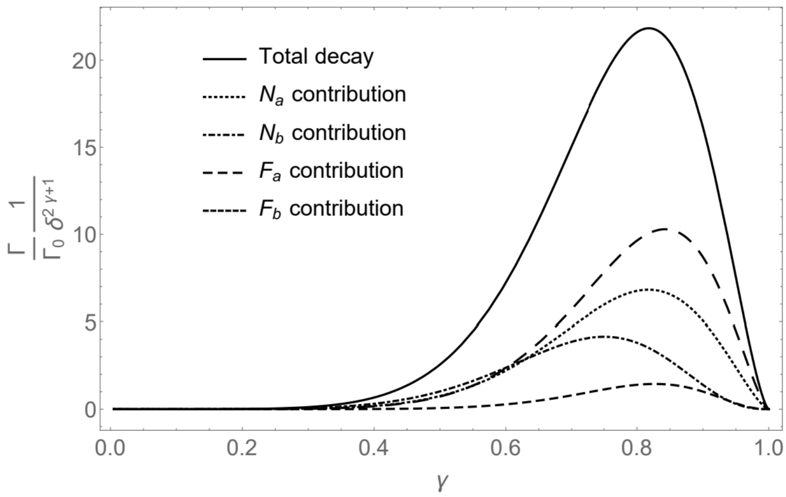


**Figure 4**. Decay rate of bound muon plotted as a function $\gamma$

### 4.2 The Extreme Non Relativistic Limit

We will now attempt to find the decay rate in the extreme non-relativistic limit, i.e. when $\alpha_Z \to 0$, and hence $\gamma = \sqrt{1-\alpha_Z^2} \to 1$. We will also take the limit that $m_e << m_\mu \to \delta = \frac{m_e}{m_\mu} \to 0$.
Let us begin by finding expressions for $C_n$ and $S_n$ (defined in Eq. (4.7))in this limit. In this limit, the term

$$\frac{1+\gamma}{8}\left(\frac{4\delta}{(1+\delta)^2}\right)^{\gamma+1/2} \to \frac{1}{4}(4)^{3/2}\delta^{\gamma+1/2}(1+\delta)^{-2\gamma-1} \approx 2\delta^{\gamma+1/2} \tag{4.16}$$

To evaluate $N_a, N_b, F_a$ and $F_b$, we need to find $C_1, C_2, S_1, S_2$ and $S_3$. Using the following facts:

$$\cos\left[a\tan^{-1}k\right] = \frac{a}{\sqrt{k^2+1}}$$
$$\sin\left[2\tan^{-1}k\right] = \frac{2k}{1+k^2}$$
$$\Gamma(n) = (n-1)!$$

in the definitions Eq. (4.12), we find:

$$C_1 = \frac{1}{k}\delta^{\gamma+1/2}\left[1+k^2\right]^{-3/2}.$$
$$C_2 = \frac{1}{k^2}\delta^{\gamma+1/2}[1+k^2]^{-1}$$
$$S_1 = 2\delta^{\gamma+1/2}\left[1+k^2\right]^{-2}$$
$$S_2 = \frac{1}{k}\delta^{\gamma+1/2}[1+k^2]^{-1}$$
$$S_3 = 0$$

Plugging these into our expressions for $N_{a,b}$ and $F_{a,b}$, we obtain:

$$N_a \to 2\sqrt{2}\delta^{\gamma+\frac{1}{2}}\frac{k_A}{z}\left(1+a^2\right)\frac{1}{\left(1+k^2\right)^2}$$

$$N_b \to 2\sqrt{2}\delta^{\gamma+\frac{1}{2}}\frac{z_{\max}}{z}\frac{1}{k^2}\left(\frac{1}{k^2+1}\right)^2\left(2a^2+3a^2k^2+k^2\right).$$

$$F_a \to \frac{4\delta^{\gamma+\frac{1}{2}}}{k^2\left(1+k^2\right)^2}\left(a^2+2a^2k^2-k^2\right),$$

$$F_b \to 8\delta^{\gamma+\frac{1}{2}}a\frac{k}{\left(1+k^2\right)^2}.$$

### 4.3 Non Relativistic Limit upto $\alpha_z^3$

In this section we will find the decay rate of the muon in the limit $\alpha_z \to 0$. We hence have:

$$a = \frac{1-\gamma}{\alpha_z} \to \frac{1}{2}\alpha_z, \quad k_A \to \alpha_z k, \quad z_{\max} \to (1-\delta).$$

Furthermore, we know $k \to \frac{\sqrt{1-z^2}}{\alpha_z}$as we take $z_{max}^2 \to 1$.
We can now move towards calculating the individual contributions of $N_{a,b}$ and $F_{a,b}$ to the decay rate according to Eq. (4.11).
Writing $z_{max}$ as $z_m$ for convenience, we have for $N_a$ :

$$\begin{aligned}
\left(\frac{\Gamma}{\Gamma_0}\right)_{N_a} &= 128\int_0^{z_m}\left(8\delta^{2\gamma+1}\frac{k_A^2}{z^2}\frac{\left(1+a^2\right)^2}{\left(1+k^2\right)^4}\right)k_A z^3 dz \\
&= 128\int_0^{z_m}\left(\frac{1}{2}\delta^{2\gamma+1}\alpha_z^3 k^3\frac{\left(4+\alpha_z^2\right)^2}{\left(1+k^2\right)^4}\right)z dz \\
&= O(\alpha_z^4)
\end{aligned}$$

Hence this term has no contribution up to the order $\alpha_z^3$.
For $N_b$,

$$\begin{aligned}
\left(\frac{\Gamma}{\Gamma_0}\right)_{N_b} &= 128\int_0^{z_m}\left(8\delta^{2\gamma+1}\frac{z_m^2}{z^2k^4}\frac{1}{(1+k^2)^4}(2a^2+3a^2k^2+k^2)^2\right)k_A z^3 dz \\
&= 128\int_0^{z_m} 8\delta^{2\gamma+1}\frac{z_m^2}{k^3}\alpha_z\frac{1}{(1+k^2)^4}\left(2\frac{\alpha_z^2}{4}+3\frac{\alpha_z^2}{4}k^2+k^2\right)^2 z dz \\
&= 128\int_0^{z_m}\left(8\delta^{2\gamma+1}\frac{z_m^2}{k}\alpha_z\frac{1}{(1+k^2)^4}\left(\alpha_z^2+3\frac{\alpha_z^2}{2}k^2+k^2\right)\right)z dz \\
&= 128\times 8\delta^{2\gamma+1}z_m^2\int_0^{z_m}\frac{\alpha_z z}{k(1+k^2)^4}\left(\alpha_z^2+3\frac{\alpha_z^2}{2}k^2+k^2\right)dz \\
&= 128\times 8\delta^{2\gamma+1}z_m^2\int_0^{z_m} z\alpha_z^{10}\frac{\left(\alpha_z^2+3(z_m^2-z^2)+\frac{z_m^2-z^2}{\alpha_z^2}\right)}{\sqrt{z_m^2-z^2}(\alpha_z^2+z_m^2-z^2)^4}dz \\
&\approx 128\times 8\delta^{2\gamma+1}\left(\frac{3\alpha_z^3}{48}16\tan^{-1}\left(\frac{z_m}{\alpha_z}\right)\right) \\
&\approx 32\delta^{2\gamma+1}\pi\alpha_z^3
\end{aligned}$$

For $F_a$,

$$
\begin{aligned}
\left(\frac{\Gamma}{\Gamma_0}\right)_{F_a} &= 128\int_0^{z_m}\left(\frac{16\delta^{2\gamma+1}}{k^4(1+k^2)^4}(a^2+2a^2k^2-k^2)^2\right)k_A z^3 dz \\
&\approx 128\int_0^{z_m}\frac{16\delta^{2\gamma+1}\alpha_z}{k(1+k^2)^4}\left(-\frac{\alpha_z^2}{2}-\alpha_z^2k^2+k^2\right)z^3dz \\
&= 128\times 16\delta^{2\gamma+1}\int_0^{z_m}\alpha_z\frac{z^3}{k(1+k^2)^4}\left(-\frac{\alpha_z^2}{2}-\alpha_z^2k^2+k^2\right)dz \\
&= 128\times 16\delta^{2\gamma+1}\int_0^{z_m}\frac{\alpha_z^{10}z^3\left(-\frac{\alpha_z^2}{2}-z_m^2+z^2+\frac{z_m^2-z^2}{\alpha_z^2}\right)}{\sqrt{z_m^2-z^2}(\alpha_z^2+z_m^2-z^2)^4}dz \\
&= 128\times 16\delta^{2\gamma+1}\left(\alpha_z^3\frac{z_m^2}{16}\tan^{-1}\left(\frac{z_m}{\alpha_z}\right)\right) \\
&\approx 64\delta^{2\gamma+1}\pi\alpha_z^3
\end{aligned}
$$

For $F_b$,

$$
\begin{aligned}
\left(\frac{\Gamma}{\Gamma_0}\right)_{F_b} &= 128\int_0^{z_m}\left(8\delta^{\gamma+1/2}\frac{ak}{(1+k^2)^2}\right)^2 k_A z^3dz \\
&\approx 128\times 64\delta^{2\gamma+1}\alpha_z^3\int_0^{z_m}\frac{k^3z^3}{(1+k^2)^4}dz \\
&= 128\times 64\delta^{2\gamma+1}\alpha_z^3\int_0^{z_m}\alpha_z^5\frac{(1-z^2)^{3/2}z^3}{(\alpha_z^2+1-z^2)^4}dz
\end{aligned}
$$

Integrating this and keeping terms only up to $\alpha_z^3$,:

$$
\begin{aligned}
&= 128\times 64\delta^{2\gamma+1}\alpha_z^3\left(\frac{3\alpha_z^2z_m^6\tan^{-1}(z_m/\alpha_z)}{48(z_m^2+\alpha_z^2)^2}\right) \\
&\approx 128\times 64\delta^{2\gamma+1}\alpha_z^3\left(\frac{\alpha_z^2z_m^6\pi}{32}\right) \\
&= O(\alpha_z^5)
\end{aligned}
$$

We hence only have two contributing terms: $N_b$ and $F_a$. The total contribution is then:

$$
\begin{aligned}
\frac{\Gamma\left((Z\mu^-)\to(Ze^-)\,\nu_\mu\bar{\nu}_e\right)}{\Gamma_0} &= \left(\frac{\Gamma}{\Gamma_0}\right)_{N_b}+\left(\frac{\Gamma}{\Gamma_0}\right)_{F_a} \\
&= 64\delta^{2\gamma+1}\pi\alpha_z^3+32\delta^{2\gamma+1}\pi\alpha_z^3 \\
&= 96\delta^{2\gamma+1}\pi\alpha_z^3
\end{aligned}
$$

We can hence write the decay rate in the non relativistic limit upto $\alpha_z^3$ as:

$$
\frac{1}{\delta^{2\gamma+1}}\frac{\Gamma}{\Gamma_0} = 96\pi\alpha_z^3 \tag{4.17}
$$

### 4.4 Extreme Relativistic Limit

We now consider the solution in the extreme relativistic limit, i.e. $\alpha_z \to 1$ and hence $\gamma = \sqrt{1-\alpha_z^2} \to 0$. We can therefore use $\gamma$ as our expansion parameter. Again, since $\delta \ll 1$ we can write $k = k_A(1+\gamma)\alpha_Z$ as $k_A = k\alpha_Z$. The upper bound of the integral then becomes will change to $z_{max} = \gamma(1-\delta) \approx \gamma$ and $k_A = \sqrt{z_{max}^2 - z^2} \to \sqrt{\gamma^2 - z^2}$. We now need to find $C_n$ and $S_n$ in this limit. To do this, we expand the gamma functions as:

$$\frac{\Gamma(1+2\gamma-n)}{\Gamma(1+2\gamma)} = \frac{\Gamma(1+2\gamma-n)}{(2\gamma)(2\gamma-1)\cdots(1+2\gamma-n)} = \frac{1}{(2\gamma)(2\gamma-1)\cdots(1+2\gamma-n)},$$

and use $\gamma \to 0$ except in the first term:

$$\frac{\Gamma(1+2\gamma-n)}{\Gamma(1+2\gamma)} = (-1)^{n-1}\frac{1}{2\gamma(n-1)!}.$$

$$\frac{1+\gamma}{8}\left(\frac{4\delta}{(1+\delta)^2}\right)^{\gamma+\frac{1}{2}} = \frac{1+\gamma}{8}(4\delta)^{\gamma+\frac{1}{2}}(1+\delta)^{-2\gamma-1} \approx \frac{1}{4}\delta^{\gamma+\frac{1}{2}}.$$

We now find $C_n$ and $S_n$ in this limit as:

$$\begin{aligned}
C_1 &= \frac{1}{4k}\delta^{\gamma+1/2}\left[1+k^2\right]^{-1/2}.\\
C_2 &= -\frac{1}{4k^2}\delta^{\gamma+1/2}\\
S_1 &= \frac{1}{4k}\delta^{\gamma+1/2}\left[1+k^2\right]^{-1}\\
S_2 &= -\frac{1}{4k^2}\delta^{\gamma+1/2}[1+k^2]^{-1/2}\\
S_3 &= \frac{1}{8k^3}\delta^{\gamma+1/2}
\end{aligned}$$

We hence get

$$\begin{aligned}
N_a &\to \frac{\sqrt{2}}{4}\delta^{\gamma+\frac{1}{2}}\frac{k_A}{z}\left(1+a^2\right)\frac{1}{(1+k^2)}\\
N_b &\to \frac{\sqrt{2}}{4}\delta^{\gamma+\frac{1}{2}}\frac{z_{\max}}{z}\left(-\frac{4a^2}{k^2}\frac{1}{2k^3} - \left(\frac{1+a^2}{k(k^2+1)}\right)\right).\\
F_a &\to \frac{\delta^{\gamma+\frac{1}{2}}}{4}\left(-\frac{4a^2}{k^2\sqrt{1+k^2}} - \frac{1}{2k^3} - \frac{2(1-a^2)}{k(1+k^2)}\right),\\
F_b &\to \delta^{\gamma+\frac{1}{2}}\left(-\frac{1}{k^2}[1+k^2]^{-1/2} - \frac{1}{k}\left[1+k^2\right]^{-1/2}\right).
\end{aligned}$$

Evaluating the decay rate in the limit $\gamma \to 0$, where terms of order $\gamma^0$ contribute, we get:

$$\frac{\Gamma}{\Gamma_0} = \frac{256}{15}\delta^{2\gamma+1}.$$

## 5 Extreme Non Relativistic Limit up to $\alpha_Z^7$

To expand the decay rate up to $\alpha_Z^7$, it is convenient to write the decay rate in terms of trigonometric functions. In the expressions for $S_n$ and $C_n$ (4.1.2), redefining $t = \tan^{-1} k$, such that $k^n$ can be written as $\tan^n(t)$ and $\frac{1}{[1+k^2]^{-\frac{n-1}{2}-\gamma}} = (\sec t)^{n-1-2\gamma}$, $S_n$ becomes:

$$S_n \equiv \frac{\Gamma[1+2\gamma-n](\sec t)^{n-1-2\gamma}}{\Gamma(1+2\gamma)\tan^n(t)} \sin\left[(2\gamma-n+1)t\right]$$

Also, redefining:

$$\begin{aligned}
N_a' &\equiv \sqrt{2}\frac{k_A}{z}\left(1+a^2\right)S_1, \\
N_b' &\equiv \sqrt{2}\frac{z_{\max}}{z}\left[4a^2\left(C_2-S_3\right)+\left(1+a^2\right)S_1\right] \\
F_a' &\equiv \left[4a^2\left(C_2-S_3\right)-2\left(1-a^2\right)S_1\right] \\
F_b' &\equiv 4a\left(S_2-C_1\right),
\end{aligned}$$

with these, the decay rate in Eq. (4.11), becomes:

$$\begin{aligned}
\frac{\Gamma\left((Z\mu^-)\to(Ze^-)\,\nu_\mu\bar{\nu}_e\right)}{\Gamma_0} &= 128\left(\frac{1+\gamma}{8}\right)^2\left(\frac{4\delta}{(1+\delta)^2}\right)^{2\gamma+1}\int_0^{z_{\max}}\left(N_a'^2+N_b'^2+F_a'^2+F_b'^2\right)k_A z^3 dz \\
&\equiv \left(\frac{4\delta}{(1+\delta)^2}\right)^{2\gamma+1}\int_0^{z_{\max}} f(\gamma,\delta,z) z dz, \qquad (5.1)
\end{aligned}$$

where

$$f(\gamma,\delta,z) = \left(\frac{1+\gamma}{8}\right)^2 128\left(N_a'^2+N_b'^2+F_a'^2+F_b'^2\right)k_A z^2$$

Recall, $k_A = (1+\delta)\alpha_Z k = (1+\delta)\alpha_Z\tan(t) \equiv \lambda\tan(t)$.
Changing the variable of integration in Eq. 5.1 as:

$$z^2 = {z_{max}}^2 - {k_A}^2 = {z_{max}}^2 - \lambda^2\tan^2(t)$$

$$z dz = -\lambda^2\tan(t)\sec^2 t dt$$

It can be seen that the limits change as:

$$z\to 0 \implies t\to\tan^{-1} z_{max} \equiv t_0 \quad z\to z_{max} \implies t\to 0.$$

Making these changes, the integral takes the form:

$$\frac{\Gamma}{\Gamma_0}\frac{1}{\delta^{2\gamma+1}}\int_0^{t_0} G(\delta,\gamma,t)dt, \qquad (5.2)$$

where $G(\delta,\gamma,t) = f\lambda^2\tan(t)\sec^2 t$. Now since $\alpha_z\to 0$, we expand $G(\delta,\gamma,t)$ around $\alpha_Z$:

$$\begin{aligned}
G(\delta,\gamma,t) &= f\lambda^2\tan(t)\sec^2 t \\
&= \left(\frac{1+\gamma}{8}\right)^2 128\left(N_a'^2+N_b'^2+F_a'^2+F_b'^2\right)k_A({z_{max}}^2-{k_A}^2)\lambda^2\tan(t)\sec^2 t
\end{aligned}$$

This expansion is done numerically and we find that only odd powera of $\alpha_Z$ occur:

$$G(\delta, \gamma, t) = \sum_{n=1}^{\infty} \alpha_Z^{2n+1} I_{2n+1}(\delta, t).$$

Expanding only till $\alpha_Z^7$:

$$G(\delta, \gamma, t) \approx \alpha_Z^3 I_3(\delta, t) + \alpha_Z^5 I_5(\delta, t) + \alpha_Z^7 I_7(\delta, t). \tag{5.3}$$

Making this substitution in Eq. (5.2), we get:

$$\frac{\Gamma}{\Gamma_0}\frac{1}{\delta^2\gamma + 1} \approx \left(\frac{4}{(1+\delta)^2}\right)^{2\gamma+1} \left[\int_0^{t_0} \alpha^3 Z dt I_3(\delta, t) + \alpha^5 Z dt I_5(\delta, t) + \alpha^7 Z dt I_7(\delta, t)\right]. \tag{5.4}$$

These decay rates in the extreme non relativistic limits are plotted in Fig. 5. The dotted dashed line shows contribution upto $\alpha_Z^3$ term, the dotted till $\alpha_Z^5$ and dashed line shows the contributions added upto $\alpha_Z^7$ term. These are plotted against exact numerical result. Similarly, decay rate in extreme relativistic limit is also plotted. We were not able to complete these calculations. They can be found in [2]. Expansion upto three terms of

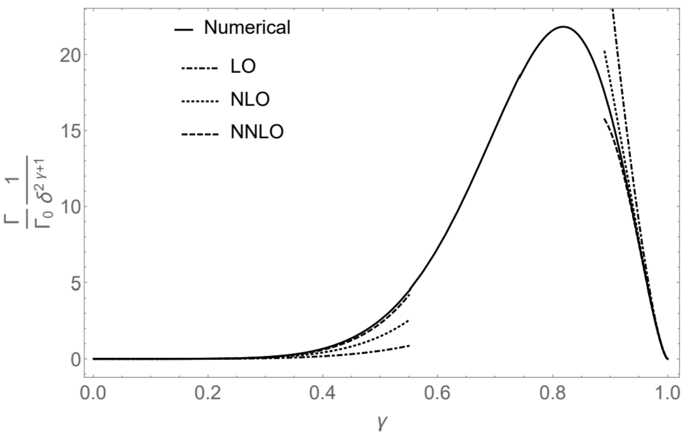


**Figure 5**. Decay rate in extreme relativistic and non relativistic limits [2].

the series is important because it is found that the values agree with the numerical result with an error of less than 1 percent for atomic number up to $Z = 43$.

## 6 Conclusions

We have calculated the decay rate for a free muon, and then a bound muon. We have been able to derive an expression for decay rate of bound muon as a single integral. Limiting cases were also explored, such as the extreme non-relativistic limit up to $\alpha_z^7$, for which the values found agreed with numerical results with an error of less than 1 percent, for atomic numbers up to Z=43.

# Appendix

## A Properties of Gamma Matrices

- $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I_4$
- $\left\{\gamma^5, \gamma^\mu\right\} = 0$
- Trace of the product of an odd number of $\gamma^\mu$ is zero
- Trace of $\gamma^5$ times the product of an odd number of $\gamma^\mu$ is zero
- $\mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$
- $\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)$
- $\mathrm{tr}\,\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) = -4i\epsilon^{\mu\nu\rho\sigma}$

**Feynman Slash Notation (and more trace properties)**

- $\not{a} \equiv \gamma^\mu a_\mu$
- $\not{a}\not{b} = [a \cdot b - i a_\mu \sigma^{\mu\nu} b_\nu]\, I_4$
- $\not{a}\not{a} = [a^\mu a^\nu \gamma_\mu\gamma_\nu]\, I_4 = \left[\frac{1}{2} a^\mu a^\nu\left(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\right)\right] I_4 = [\eta_{\mu\nu} a^\mu a^\nu]\, I_4 = a^2 I_4$
- $\mathrm{tr}\,(\not{a}\not{b}) = 4(a \cdot b)$
- $\mathrm{tr}\,(\not{a}\not{b}\not{c}\not{d}) = 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right]$
- $\mathrm{tr}\,(\gamma_5\not{a}\not{b}) = 0$
- $\mathrm{tr}\,(\gamma_5\not{a}\not{b}\not{c}\not{d}) = -4i\epsilon_{\mu\nu\rho\sigma} a^\mu b^\nu c^\rho d^\sigma$
- $\gamma_\mu\not{a}\gamma^\mu = -2\not{a}$
- $\gamma_\mu\not{a}\not{b}\gamma^\mu = 4(a \cdot b) I_4$
- $-\gamma^\mu \not{p}_\mu \gamma_\mu \not{p}_\nu = 2\not{p}_\mu \gamma^5 \not{p}_\nu$
- $\gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$ [7].

## B 4 × 4 Matrices in terms of Gamma-Matrices

The following matrix is used to find Dirac spinors in terms of gamma matrices.

$$A_{ij} = i \begin{pmatrix} 0 & .. & .. & .. & 0 \\ : & .. & .. & .. & : \\ 0 & .. & 1 & 0 & : \\ : & : & & : & : \\ 0 & .. & .. & .. & 0 \end{pmatrix}$$

$$
\begin{aligned}
&A_{ij} = i \\
&\frac{1+\gamma^0}{2}\frac{\gamma^5+\gamma^3}{2} = A_{13} \\
&\frac{1+\gamma^0}{2}\frac{\gamma^1+i\gamma^2}{2} = A_{14} \\
&\frac{\gamma^1-i\gamma^2}{2}\frac{\gamma^5-\gamma^3}{2} = A_{21} \\
&\frac{\gamma^5-\gamma^3}{2}\frac{1-\gamma^0}{2} = A_{24} \\
&\frac{\gamma^1+i\gamma^2}{2}\frac{1+\gamma^0}{2} = -A_{32} \\
&\frac{1-\gamma^0}{2}\frac{\gamma^5-\gamma^3}{2} = A_{31} \\
&\frac{\gamma^5+\gamma^3}{2}\frac{1+\gamma^0}{2} = A_{42} \\
&\frac{1-\gamma^0}{2}\frac{\gamma^1-i\gamma^2}{2} = -A_{41} \\
&\frac{1+\gamma^0}{2}\frac{\gamma^5+\gamma^3}{2}\gamma^5 = A_{11} \\
&\frac{1+\gamma^0}{2}\frac{\gamma^1+i\gamma^2}{2}\gamma^5 = A_{12} \\
&\frac{\gamma^1-i\gamma^2}{2}\frac{\gamma^5-\gamma^3}{2} = A_{23} \\
&\frac{\gamma^5-\gamma^3}{2}\frac{1-\gamma^0}{2}\gamma^5 = A_{22} \\
&\frac{\gamma^1+i\gamma^2}{2}\frac{1+\gamma^0}{2}\gamma^5 = -A_{34} \\
&\frac{1-\gamma^0}{2}\frac{\gamma^5-\gamma^3}{2}\gamma^5 = A_{33} \\
&\frac{\gamma^5+\gamma^3}{2}\frac{1+\gamma^0}{2}\gamma^5 = A_{44} \\
&\frac{1-\gamma^0}{2}\frac{\gamma^1-i\gamma^2}{2}\gamma^5 = -A_{43}
\end{aligned}
$$

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