On the Decay of Bound Muons

Project Presentation - PHY 539

Aiman Nauman
Tayyaba Noureen

12 May 2024

An Overview

Motivation

- Muons are the center of discrepancies (anamolous magnetic moment, B-meson decays).
- Charged Lepton Flavor Violation (CLFV) as a way to explain these discrepancies.
- Muons are a good candidate for observing CLFV, so we look for muons to electrons decay.

Dirac Equation in the Presence of Coulomb potential

- Dirac Hamiltonian in the presence of an external potential :

$$\hat{H} = \boldsymbol{\alpha} \cdot \mathbf{p} + m\beta + qV(r)$$

In terms of 4×4 gamma matrices α_i and β matrices are :

$$\gamma^0 = \beta, \qquad \gamma^i = \beta\alpha^i.$$

- With this, the eigenvalue equation becomes :

$$\left(\boldsymbol{\alpha} \cdot \mathbf{p} + m\beta - \frac{\alpha Z}{r}\right) \begin{pmatrix} \phi(x) \\ \chi(x). \end{pmatrix} = E \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix},$$

wave functions are written in terms of spinors $\phi(x)$ and $\chi(x)$, as $\psi(x) = \begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix}$.

Dirac Equation in the Presence of Coulomb potential

- Total wavefunction :

$$\psi = Ne^{-m\alpha_Z r}\,(m\alpha_Z r)^{\gamma-1}\begin{pmatrix}\chi_r\\ i\frac{1-\gamma}{\alpha_Z}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}}{r}\chi_r\end{pmatrix},$$

- Wave functions for the ground state of (hydrogen-like) atoms :

$$\psi_{n=1,j=1/2,\uparrow} = \frac{(2m\alpha_Z)^{3/2}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}}\,(2m\alpha_Z r)^{\gamma-1}\,e^{-m\alpha_Z r}\times\begin{pmatrix}1\\ 0\\ \dfrac{i(1-\gamma)}{\alpha_Z}C_\theta\\ \dfrac{i(1-\gamma)}{\alpha_Z}S_\theta e^{i\phi}\end{pmatrix},$$

$$\psi_{n=1,j=1/2,\downarrow} = \frac{(2m\alpha_Z)^{3/2}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}}\,(2m\alpha_Z r)^{\gamma-1}\,e^{-m\alpha_Z r}\times\begin{pmatrix}0\\ 1\\ \dfrac{i(1-\gamma)}{\alpha}S_\theta e^{-i\phi}\\ -\dfrac{i(1-\gamma)}{\alpha}C_\theta\end{pmatrix},$$

where C_θ and S_θ are shorthand notations for $\cos\theta$ and $\sin\theta$, respectively.

Decay Rate of Free Muon

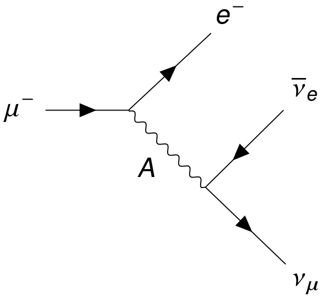

Figure – Muon decay.

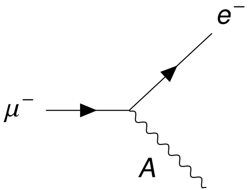

Figure – Muon decaying into an electron and A

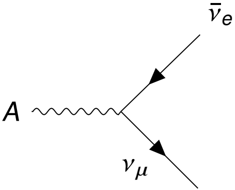

Figure – A decaying into an electron anti-neutrino and muon neutrino.

Decay Rate of Free Muon

The total decay rate is calculated using :

$$\Gamma(\mu^- \to e^- \nu_\mu \bar{\nu}_e) = \frac{256\pi}{g^2 m_\mu} \Gamma_0 \int_0^{z_{max}} \Gamma(\mu^- \to e^- A) z^3 dz,$$

- where

$$\Gamma(\mu^- \to e^- A) = \frac{1}{2m_\mu} \int d\Pi_{\mathrm{LIPS}} |\langle \mathcal{M} \rangle|^2,$$

- Γ_0 is the decay rate of the free muon in the limit $m_e \to 0$, $z = \dfrac{m_A}{m_\mu}$,

and $0 < z < z_{max} = 1 - \delta$

Decay Rate of a Free Muon

The amplitude for the decay is :

$$\mathcal{M} = \bar{u}_e\left(p_e\right)\left(ig\gamma^{\mu}L\right)\epsilon_{\mu}\left(p_A\right)u_{\mu},$$

We can use Casimir's trick to calculate the average amplitude squared. In the rest frame of the muon, it becomes :

$$|\langle\mathcal{M}\rangle|^2 = \frac{g^2}{2}\left(m_{\mu}E_e + \frac{2\left(m_{\mu}^2 - E_e m_{\mu}\right)\left(E_e m_{\mu} - m_e^2\right)}{m_A^2}\right).$$

Decay Rate of Free Muon

Using the amplitude squared in the decay rate formula, we find that the final decay rate of the free muon is :

$$\Gamma = \Gamma_0 \left[1 - 8\delta^2 - 24 \log \delta + 8\delta^6 - \delta^8\right],$$

where $\delta = \dfrac{m_e}{m_\mu}$.

In the massless limit of the electron's mass $(\delta \longrightarrow 0)$, we recover Γ_0 as expected.

Decay Rate of Bound Muon

- Formed by replacing electron of hydrogen atom by a muon.
- Due to higher mass, atomic orbits of muon states are small.
- Decay rate is significantly different due to nuclear effects.
- Can conveniently divide decay rate as :

$$(Z\mu^-) \rightarrow (Ze^-)A, \quad A \rightarrow \nu_\mu \bar{\nu_e}$$

- Decay rate of two processes is related as :

$$\Gamma((Z\mu^-) \rightarrow (Ze^-))\nu_\mu \bar{\nu_e} = \frac{256\pi}{g^2 m_\mu} \Gamma_0 \int_0^{z_{\text{max}}} \Gamma((Z\mu^-) \rightarrow (Ze^-)A) z^3 dz,$$

Amplitude

- Amplitude for bound state :

$$\mathcal{M} = \frac{g}{\sqrt{2}} \int d^3 r e^{i\vec{q}\cdot\vec{r}} \bar{\Phi}_e(\vec{r}) \epsilon^{\lambda_A *} L \Phi_\mu(\vec{r})$$

Where $\Phi(\vec{r})$ is the position space wavefunction.

$$\begin{aligned}
\Phi(\boldsymbol{r}) &= f(r) u_{\uparrow/\downarrow}, \\
u_{\uparrow/\downarrow} &= \rho \phi_{\uparrow/\downarrow}, \\
f(r) &= \frac{(2m_\mu \alpha_Z)^{\gamma+1/2}}{\sqrt{4\pi}} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}} r^{\gamma-1} \exp\left(-m_\mu \alpha_Z r\right), \\
\phi_{\uparrow/\downarrow} &= \left(\ \chi_{\uparrow/\downarrow} \ \right), \\
\rho^\mu &= (1, ia\hat{r}), \qquad \chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
a &= \frac{1-\gamma}{\alpha_Z}, \qquad \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

Spin Non-Flip Part

$$\mathcal{M}_{\uparrow\to\uparrow} = \frac{g}{\sqrt{2}} \int d^3 r e^{i\vec{q}\cdot\vec{r}} f_e(r) f_\mu(r) (\bar{u}_e^{\uparrow} \gamma^u \epsilon_\mu^{\lambda_{A^*}} L u_\mu^{\uparrow}).$$

•The term in the brackets :

$$\bar{u}_e^{\uparrow} \gamma^u \epsilon_\mu^{\lambda_{A^*}} (1-\gamma_5) u_\mu^{\uparrow} = \frac{1}{z}(k_a(1+a^2) + z_{max}(1-a^2+2a^2\cos^2\theta))$$

• For the case where spin down muon decays into a spin down electron :

$$\bar{u}_e^{\downarrow} \gamma^u \epsilon_\mu^{\lambda_{A^*}} (1-\gamma_5) u_\mu^{\downarrow} = \frac{1}{z}(k_a(1+a^2) - z_{max}(1-a^2+2a^2\cos^2\theta))$$

• Defining :

$$N_a = \frac{\mathcal{M}_{\uparrow\to\uparrow} + \mathcal{M}_{\downarrow\to\downarrow}}{2}$$

$$N_b = \frac{\mathcal{M}_{\uparrow\to\uparrow} - \mathcal{M}_{\downarrow\to\downarrow}}{2}$$

Spin Non-Flip Part

With these redefinitions we calculate :

$$N_a = \frac{g}{2(2)} \frac{2k_A}{\sqrt{2}z} \left(1 + a^2\right) \int d^3 r e^{i\boldsymbol{q}\cdot\boldsymbol{r}} f_e(r) f_\mu(r)$$
$$= \sqrt{2} g \frac{k_A}{2z} \left(1 + a^2\right) \frac{1+\gamma}{8} \left(\frac{4\delta}{(1+\delta)^2}\right)^{\gamma+\frac{1}{2}} \left[1 + k^2\right]^{-\gamma} \frac{\Gamma[2\gamma]}{\Gamma(1+2\gamma)k} \sin\left[2\gamma \tan^{-1}(k)\right]$$
$$= \sqrt{2} \frac{k_A}{z} g \left(1 + a^2\right) S_1$$

Introducing the notation :

$$S_n \equiv \frac{1+\gamma}{8} \left(\frac{4\delta}{(1+\delta)^2}\right)^{\gamma+1/2} \frac{\Gamma[1+2\gamma-n]}{\Gamma(1+2\gamma)k^n} \left[1+k^2\right]^{\frac{n-1}{2}-\gamma} \sin\left[(2\gamma - n + 1)\tan^{-1}(k)\right]$$

Similarly, we find :

$$N_b = \sqrt{2} \frac{z_{\max}}{z} g \left[4a^2 \left(C_2 - S_3\right) + \left(1 + a^2\right) S_1\right]$$

Total contribution from the spin non flip part :

$$N_a^2 + N_b^2 = |\langle \mathcal{M}_{\uparrow\to\uparrow} \rangle|^2 + |\langle \mathcal{M}_{\downarrow\to\downarrow} \rangle|^2.$$

The Spin Flip Part

Let's consider a spin up muon decaying into a spin down electron :

$$\mathcal{M}_{\uparrow\to\downarrow} = \frac{g}{\sqrt{2}} \int d^3 r e^{i\vec{q}\cdot\vec{r}} f_e(r) f_\mu(r) (\bar{u}_e^{\downarrow} \gamma^u \epsilon_\mu^{\lambda_A *} L u_\mu^{\uparrow}),$$

We find that we can write :

$$(\bar{u}_e^{\downarrow} \gamma^u \epsilon_\mu^{\lambda_A *} L u_\mu^{\uparrow}) = -\frac{2}{\sqrt{2}} (1 - a^2 cos^2 \theta)$$

Plugging this in and integrating, we obtain :

$$\mathcal{M}_{\uparrow\to\downarrow} = g \left[4a^2 \left(C_2 - S_3 \right) - 2 \left(1 - a^2 \right) S_1 \right] \equiv F_a$$

The Spin Flip Part

We can alternatively have a spin down muon decaying into a spin up electron :

$$\mathcal{M}_{\downarrow\to\uparrow} = \frac{g}{\sqrt{2}} \int d^3 r e^{i\vec{q}\cdot\vec{r}} f_e(r) f_\mu(r) (\bar{u}_e^{\uparrow} \gamma^u \epsilon_\mu^{\lambda_A *} L u_\mu^{\downarrow}).$$

Here,

$$(\bar{u}_e^{\uparrow} \gamma^u \epsilon_\mu^{\lambda_A *} L u_\mu^{\downarrow}) = \frac{1}{\sqrt{2}} 4ia \cos\theta$$

Plugging this into out integral and integrating, we obtain :

$$\mathcal{M}_{\downarrow\to\uparrow} = -4ga\,(S_2 - C_1) \equiv F_b.$$

The Total Decay Rate

We have now calculated the amplitude for all possibilities. Adding up all contributions, we can write the decay rate as :

$$\Gamma\left((Z\mu^-)\rightarrow(Ze^-)\,A\right)=\frac{m_\mu}{2\pi}k_A\left(N_a^2+N_b^2+F_a^2+F_b^2\right),$$

We can then write the total decay rate as a single integral :

$$\frac{\Gamma\left((Z\mu^-)\rightarrow(Ze^-)\,\nu_\mu\bar{\nu}_e\right)}{\Gamma_0}=128\int_0^{z_{\max}}\left(N_a^2+N_b^2+F_a^2+F_b^2\right)k_A z^3 dz,$$

Extreme Limits

- **Non Relativistic Limit**

In the limit $\alpha_Z \to 0$, we find that the first contribution to the decay rate comes at the order α_Z^3, where only N_b and F_a contribute.

$$\frac{1}{\delta^{2\gamma+1}}\frac{\Gamma}{\Gamma_0} = 96\pi\alpha_Z^3$$

- **Relativistic Limit**

We can do a similar calculation for the relativistic limit i.e. $\alpha_Z \to 1$. Here, the first contribution is from terms of the order γ^0, and the decay rate is :

$$\frac{1}{\delta^{2\gamma+1}}\frac{\Gamma}{\Gamma_0} = \frac{256}{15}$$

The Total Decay Rate

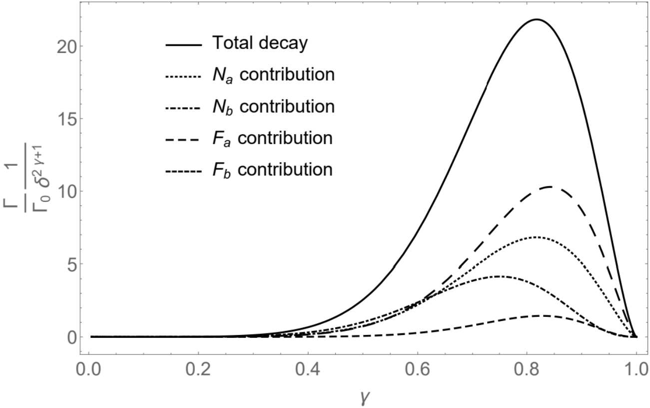

Figure – The total decay rate plotted against $\gamma = \sqrt{1 - \alpha_Z^2}$

Thank You !

Any Questions ?