Gravitational Collapse

Course Project for PHY 442: General Relativity

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Abstract

Once the star (greater than 1.3 times the sun's mass) has exhausted its nuclear fuel and thermal energy, it collapses under its own gravitational forces. In this report, solutions of Einstein's equations, under the collapse of a symmetric spherical shell, are presented that describe the dynamics. The star's surface is of special interest and requires a background knowledge of hypersurfaces, so a section is dedicated to an introduction to hypersurfaces. The critical aspects of gravitational collapse are derived and discussed.

Contents

1 Introduction

When stars have exhausted their nuclear fuel, they contract inwards and become unstable. The instability then results in an implosion in which the star collapses inwards. In an idealized spherical collapse, the star falls through the horizon, forming a black hole. After passing through the horizon, a singularity is reached. These are the general aspects of gravitational collapse that result in the formation of black holes.

In this report, one idealized model of gravitational collapse is discussed. The star's surface divides spacetime into two regions. The geometries and dynamics of the exterior and interior regions during the collapse are analyzed. The star's surface is modelled as a thin spherical shell—the shell partitions spacetime into an exterior and an interior region. The shell acts as a discontinuity, so discussing how to join the exterior and interior region smoothly is important.

To understand what is happening at the star's surface, an introduction to hypersurfaces is necessary. So, the report is divided into two sections. The first section serves as an introduction to the extrinsic and intrinsic geometry of hypersurfaces. Then, the smooth junction condition between these geometries is discussed. In the second section, a simple model of gravitation collapse is presented. Equipped with the basics of hypersurfaces, the star's surface is analyzed. Although the model seems simplified, qualitative aspects of the collapse still apply to general, more complicated collapses. This is where the significance of the collapse of a ball of dust lies. This seemingly simple model is analytically solvable, and critical aspects of the collapse apply to other non-idealized collapses in general [5].

2 An introduction to Hypersurfaces

An n dimensional hypersurface Σ is a subspace embedded in an n+1 dimensional spacetime manifold. A hypersurface is defined by an implicit equation of the form $\phi(x^\alpha) = 0$ or by parametric equations $x^\alpha = x^\alpha(y^a)$. x^α are the coordinates on the spacetime manifold and y^a are the coordinates of the hypersurface. In the case of a 3-dimensional hypersurface embedded in 4-dimensional spacetime, $y^a = (y^1, y^2, y^3)$. It can either be timelike, spacelike or null. In this report, only the first two cases are discussed.

2.1 Embeddings: Normal Vector, Tangent Vector and the Induced Metric

There are some aspects of the geometry of the hypersurface when looking at it as an embedding in 4D spacetime. These are the normal and tangent vectors and the induced metric.

The vector $\partial_\alpha \phi$ is normal to the hypersurface since on the hypersurface $\phi(x^\alpha) = 0$ and its value changes only in the direction orthogonal to Σ. For null surfaces, the normal vector is null. In spacelike and timelike Σ, a unit normal n_α can also be defined.

$$n^\alpha n_\alpha = \varepsilon \equiv \begin{cases} +1 & \text{if } \Sigma \text{ is spacelike} \\ -1 & \text{if } \Sigma \text{ is timelike} \end{cases}$$

With these definitions n_α is given by:

$$n_\alpha = \frac{\varepsilon \partial_\alpha \phi}{|g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi|^{1/2}}.$$

The numerator comes from the normalization condition and ε because the sign of unit normal was fixed from the way it was defined.

From the parametric equation $x^\alpha = x^\alpha(y^a)$, the vectors:

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a},$$

are the vectors tangent to the curves in Σ.

To get the metric intrinsic to the hypersurface, the line element is restricted only to the displacement on the

Σ, so:

$$\begin{aligned}
\mathrm{d}s^2_\Sigma &= g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\
&= g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a}\ \mathrm{d}y^a\right)\left(\frac{\partial x^\beta}{\partial y^b}\ \mathrm{d}y^b\right) \\
&= g_{\alpha\beta}e^\alpha_a e^\beta_b\ \mathrm{d}y^a\ \mathrm{d}y^b \\
&= h_{ab}\ \mathrm{d}y^a\ \mathrm{d}y^b,
\end{aligned}$$

where

$$h_{ab} = g_{\alpha\beta}e^\alpha_a e^\beta_b.$$

is the induced metric, or first fundamental form, of the hypersurface. It is a scalar with respect to transformations $x^\alpha \to x^{\alpha'}$ of the spacetime coordinates (notice the sum over Greek indices), but it behaves as a tensor under transformations $y^a \to y^{a'}$ of the hypersurface coordinates. Such objects will be referred to as 3-tensors. These relations are only for non-null surfaces. Also, note the relation:

$$g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + h^{ab}e^\alpha_a e^\beta_b.$$

The relation can be easily verified by taking the dot product with the normal vector as:

$$\begin{aligned}
g^{\alpha\beta}n_\alpha &= \varepsilon n^\alpha n^\beta n_\alpha + h^{ab}e^\alpha_a e^\beta_b n_\alpha \\
n^\beta &= \varepsilon^2 n^\beta.
\end{aligned}$$

This relation is also called the completion relation for the linearly independent normal and tangent vectors [1].

2.2 Instrinsic Geometry of Hypersurfaces

In the case of embedded hypersurfaces, its geometry has two aspects. One is its intrinsic geometry, and the other aspect is looking at it as an embedding beyond its intrinsic geometry. Its embedding into the spacetime gives rise to an extrinsic geometry, including features on how it is embedded into the spacetime [2].

One of the essential intrinsic properties is the intrinsic covariant derivative. Before introducing that, it is necessary to be familiar with tangent tensors. Tensors defined only on the hypersurface $A^{\alpha\beta\cdots}$ are purely tangential to the hypersurface. If they were defined on points other than the hypersurface, they would also have normal components, so their tangentiality follows from their definition. So, they can be decomposed as:

$$A^{\alpha\beta\dots} = A^{ab\dots}\frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b}\cdots = A^{ab\dots}e^\alpha_a e^\beta_b \dots.$$

From this decomposition, it is explicit that $A^{\alpha\beta\cdots}$ is in the direction of basis vectors only and is tangential. Its dot product with the unit normal is then:

$$A^{\alpha\beta\cdots}n_\alpha = \ A^{ab\cdots}e^\alpha_a e^\beta_b \dots n_\alpha = 0,$$

confirming that is purely tangential to Σ.
An arbitrary tensor can be projected onto Σ such that only its tagential components remain. So, the projected tensor can be written in terms of basis vectors which are tagential to the hypersurface.

$$A_{\alpha\beta\dots}e^\alpha_a e^\beta_b \dots = A_{ab\dots}$$

$A_{ab\dots}$ is a three tensor whose indices can be raised using the induced metric $h_{am}h_{bn}\dots A^{mn\cdots}$. $A_{ab\dots}$ is a tensor under the transformation of intrinsic coordinates y^a and a scalar under extrinsic transformations. So, it is a three tensor.

To see how tangent tensors are differentiated, take the simplest tangent vector and the associated projected vector:

$$A^\alpha = A^a e^\alpha_a, \quad A_a = A_\alpha e^\alpha_a.$$

Instrinsic corvariant derivative of A_a is defined as:

$$A_{a|b} = \nabla_\beta (A_\alpha) e^\alpha_a e^\beta_b.$$

So, it is the projection of covariant derivative of A^α onto the hypersurface.

$$\begin{aligned}
\nabla_\beta A_\alpha e^\alpha_a e^\beta_b &= \nabla_\beta (A_\alpha e^\alpha_a) e^\beta_b - A_\alpha \nabla_\beta (e^\alpha_a) e^\beta_b \\
&= \nabla_\beta A_a e^\beta_b - \nabla_\beta (e_{a\gamma}) g^{\alpha\gamma} A_\alpha e^\beta_b \\
&= \partial_\beta A_a e^\beta_b - \nabla_\beta (e_{a\gamma}) A^\gamma e^\beta_b \\
&= \partial_\beta A_a \frac{\partial x^\beta}{\partial y^b} - \nabla_\beta (e_{a\gamma}) A^c e^\gamma_c e^\beta_b \\
&= \partial_b A_a - \Gamma_{cab} A^c.
\end{aligned}$$

In the last line, defined:

$$\Gamma_{cab} = \nabla_\beta (e_{a\gamma}) e^\gamma_c e^\beta_b.$$

Now, consider:

$$\Gamma_{cab} A^c = \mathrm{h}_{mc} \Gamma^m_{ab} A^c = \Gamma^m_{ab} A^m.$$

With this, the expression for intrinsic covariant derivative becomes:

$$A_{a|b} = \partial_b A_a - \Gamma^m_{ab} A^m.$$

So, the expression for intrinsic covariant derivative is the same as of usual covariant derivative in spacetime, except of course for the difference that here these are three tensors. The connection coefficients are such that covariant derivative of the induced metric tensor is zero:

$$\begin{aligned}
h_{ab|c} &\equiv \nabla_\gamma h_{\alpha\beta} e^\alpha_a e^\beta_b e^\gamma_c \\
&= \nabla_\gamma (g_{\alpha\beta} - \varepsilon n_\alpha n_\beta) e^\alpha_a e^\beta_b e^\gamma_c \\
&= (\nabla_\gamma g_{\alpha\beta} - \varepsilon (n_\beta \nabla_\gamma n_\alpha + n_\alpha \nabla_\gamma n_\beta)) e^\alpha_a e^\beta_b e^\gamma_c \\
&= 0.
\end{aligned}$$

Last line follows from $n_\alpha e^\alpha_a = 0$. So, the connection coefficients are:

$$\Gamma_{cab} = \frac{1}{2}(\partial_b h_{ca} + \partial_a h_{cb} - \partial_c h_{ab}).$$

2.3 Extrinsic Curvature

$A_{a|b} = \nabla_\beta (A_\alpha) e^\alpha_a e^\beta_b$ is purely the tangental component of the vector $\nabla_\beta (A_\alpha) e^\beta_b$, its time to analyze if this vector has a normal component.

$$\begin{aligned}
\nabla_\beta (A_\alpha) e^\beta_b &= g^\alpha_\mu \nabla_\beta (A^\mu) e^\beta_b \\
&= (\varepsilon n^\alpha n_\mu + h^{am} e^\alpha_a e_{m\mu}) \nabla_\beta A^\mu e^\beta_b \\
&= \varepsilon n^\alpha (n_\mu \nabla_\beta A^\mu e^\beta_b) + e^\alpha_a (h^{am} e_{m\mu} \nabla_\beta A^\mu e^\beta_b)
\end{aligned}$$

The first term is the normal component while the second one is the tagential component of $\nabla_\beta(A_\alpha)e^\beta_b$.

$$\begin{aligned}
\nabla_\beta(A_\alpha)e^\beta_b &= \varepsilon n^\alpha[\nabla_\beta(A^\mu n_\mu)e^\beta_b - \nabla_\beta(n_\mu)A^\mu e^\beta_b] + e^\alpha_a[h^{am}\nabla_\beta A^\mu g^{\gamma\mu}e_{m\mu}e^\beta_b] \\
&= -\varepsilon n^\alpha \nabla_\beta(n_\mu)A^\mu e^\beta_b + e^\alpha_a h^{am}\nabla_\beta A^\mu e^\gamma_\mu e^\beta_b \\
&= -\varepsilon \nabla_\beta(n_\mu)A^\mu e^\beta_b n^\alpha + h^{am}A_{m|b}e^\alpha_a \\
&= -\varepsilon \nabla_\beta(n_\mu)A_a e^\mu_a e^\beta_b n^\alpha + A^a_{|b}e^\alpha_a
\end{aligned}$$

Defining the extrinsic curvature:

$$K_{ab} \equiv \nabla_\beta(n_\mu)e^\mu_a e^\beta_b \tag{2.1}$$

$$\nabla_\beta(A_\alpha)e^\beta_b = A^a_{|b}e^\alpha_a - \varepsilon A^a K_{ab}n^\alpha$$

The instrinsic covariant derivative gives the tangential component of the vector while the extrinsic curvature tensor gives the normal component of the tensor. Note that the normal component is zero when the extrinsic curvature vanishes. Contracting the intrinsic curvature gives:

$$\begin{aligned}
K \equiv h^{ab}K_{ab} &= h^{ab}\nabla_\beta(n_\mu)e^\mu_a e^\beta_b \\
&= \nabla_\alpha n^\alpha.
\end{aligned}$$

So, it is the divergence of the normal vector. The hypersurface is convex if $K > 0$ because there is a positive divergence and is concave if $K < 0$ [1].

2.4 Junction Condition and Thin Shells

Sometimes such a situation occurs where hypersurface Σ partitions spacetime into an exterior region call it $\mathscr{V}^+$ with a metric $g^+_{\alpha\beta}$ on it and an interior region $\mathscr{V}^-$ with a metric $g^-_{\alpha\beta}$. There must be some conditions to be put on the metrics so the regions $\mathscr{V}^+$ and $\mathscr{V}^-$ are joined smoothly at the hypersurface. If the regions are smoothly sewn together, then the metrics $g^+_{\alpha\beta}$ and $g^-_{\alpha\beta}$ are a solution to Einstein field equations.
To make the analysis simpler, assume that the same coordinates y^a prametrize both $\mathscr{V}^+$ and $\mathscr{V}^-$. If the coordinates on $\mathscr{V}^+$ are x^α_+ and on $\mathscr{V}^-$ are x^α_-, then continuous coordinates x^α can be installed on both regions (see Figure 1). A set of non intersecting geodesics is passing through Σ and their intersection at Σ is normal

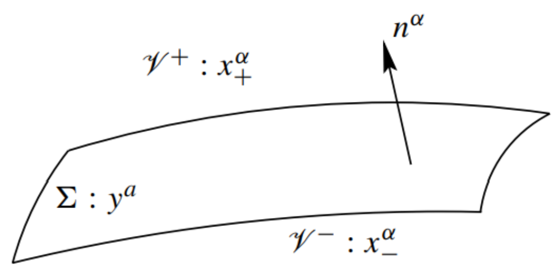

Figure 1: Two regions of spacetime divided by a hypersurface [1].

to the surface. These geodesics are parametrized by l, proper distance or proper time, and it is chosen such that $l = 0$ at Σ. Take it to be positive in $\mathscr{V}^+$ and negative in $\mathscr{V}^-$. There is a point P described in terms of x^α such that $l(x^\alpha)$ is the proper distance along the geodesic from Σ to P. Displacement away from Σ along one of the geodesics is $dx^\alpha = n^\alpha dl$, as it will be along l in the normal direction. Then,

$$\begin{aligned}
n_\alpha dx^\alpha &= n_\alpha n^\alpha dl \\
&= \varepsilon dl \implies n_\alpha = \varepsilon \partial_\alpha l.
\end{aligned}$$

Introducing Heaviside distribution function $\Theta(l)$.

$$\Theta(l) = \begin{cases} 0 & \text{if } l < 0 \\ 1 & \text{if } l > 0 \\ \text{indeterminate} & \text{if } l = 0 \end{cases}$$

Its properties that will be useful in later calculations:

$$\Theta(-l)\Theta(l) = 0, \quad \Theta^2(l) = \Theta(l), \quad \frac{d\Theta(l)}{dl} = \delta(l),$$

$\delta(l)$ is the Dirac distribution. $\Theta(l)\delta(l)$ is not a valid distribution. If A is a tensor defined on both sides of the hypersurface, then the jump of tensor across the hypersurface is:

$$[A] = A(\mathscr{V}^+)|_\Sigma - A(\mathscr{V}^-)|_\Sigma.$$

For the normal and tagential vectors:

$$\begin{aligned} [n^\alpha] &= \varepsilon\frac{dl}{dx^\alpha}|_\Sigma - \varepsilon\frac{dl}{dx^\alpha}|_\Sigma = 0. \\ [e^\alpha_a] &= 0. \end{aligned}$$

These follow because l, x^α and y^a are continuous [1].

2.4.1 First Junction Condition

Expressing the metric as:

$$g_{\alpha\beta} = \Theta(l)g^+_{\alpha\beta} + \Theta(-l)g^-_{\alpha\beta}.$$

From this expression, one can immediately see that for $l < 0$, the metric is $g^+_{\alpha\beta}$ and for $l > 0$, $g^-_{\alpha\beta}$, all expressed in terms of x^α. To see whether it is a solution to Einstein field equation, one needs to check if its derivatives are continuous across the surface, because from derivative of the metric one constructs the Riemannian curvature tensor to get to the Einstein field equations. So, checking the continuity of all these expressions involved in the Einstein Field equation one by one.

$$\begin{aligned} \partial_\gamma g_{\alpha\beta} &= \partial_\gamma(\Theta(l)g^+_{\alpha\beta} + \Theta(-l)g^-_{\alpha\beta}) \\ &= \Theta(l)\partial_\gamma g^+_{\alpha\beta} + \Theta(-l)\partial_\gamma g^-_{\alpha\beta} + g^+_{\alpha\beta}\partial_\gamma\Theta(l) + g^-_{\alpha\beta}\partial_\gamma\Theta(-l) \\ &= \Theta(l)\partial_\gamma g^+_{\alpha\beta} + \Theta(-l)\partial_\gamma g^-_{\alpha\beta} + g^+_{\alpha\beta}\frac{\partial}{x^\gamma}\frac{dl}{dl}\Theta(l) + g^-_{\alpha\beta}\frac{\partial}{x^\gamma}\frac{dl}{dl}\Theta(-l) \\ &= \Theta(l)\partial_\gamma g^+_{\alpha\beta} + \Theta(-l)\partial_\gamma g^-_{\alpha\beta} + \varepsilon\delta(l)n_\gamma g^+_{\alpha\beta} - \varepsilon\delta(l)n_\gamma g^-_{\alpha\beta} \\ &= \Theta(l)\partial_\gamma g^+_{\alpha\beta} + \Theta(-l)\partial_\gamma g^-_{\alpha\beta} + \varepsilon\delta(l)[g_{\alpha\beta}]. \end{aligned}$$

When computing Christoffel symbols to compute the Riemannian curvature tensor, the terms involve products of metric tensor with its derivatives. Such terms when computed with the given metric $g_{\alpha\beta}$ give terms that involve the product $\Theta(l)\delta(l)$ and the connections would not be well defined. To avoid this, continuity of metric across Σ is imposed, $[g_{\alpha\beta}] = 0$, so that the first derivative becomes well defined. Recall, that the metric was in terms of x^α, so to make this statement coordinate independent:

$$\begin{aligned} [g_{\alpha\beta}] &= 0 \\ [g_{\alpha\beta}]e^\alpha_a e^\beta_b &= 0 \\ [g_{\alpha\beta}e^\alpha_a e^\beta_b] &= 0 \\ [h_{ab}] &= 0. \end{aligned}$$

Second last equation follows from $[e^\alpha_a] = 0$. So, the induced metric must be smooth across Σ and it is the same on both sides. This is the first junction condition [1].

2.4.2 Second Junction Condition

In order to get to the second junction condition, one needs to investigate into the potential discontinuities in the Riemannian curvature tensor.

$$
\begin{aligned}
\Gamma^{\alpha}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\mu}[\partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\beta\mu} - \partial_\mu g_{\beta\gamma}] \\
&= \frac{1}{2}[\Theta(l) g^{\alpha\mu +} + \Theta(-l) g^{\alpha\mu -}][\Theta(l)\partial_\beta g^{+}_{\gamma\mu} + \Theta(-l)\partial_\beta g^{-}_{\gamma\mu} + \Theta(l)\partial_\gamma g^{+}_{\beta\mu} + \Theta(-l)\partial_\gamma g^{-}_{\beta\mu} - \Theta(l)\partial_\mu g^{+}_{\beta\gamma} - \Theta(l)\partial_\mu g^{-}_{\beta\gamma}] \\
&= \Theta(l)\Gamma^{+\alpha}_{\beta\gamma} + \Theta(-l)\Gamma^{\alpha}_{\beta\gamma}.
\end{aligned}
$$

Last line follows from $\Theta(l)\Theta(-l) = 0$.

$$
\begin{aligned}
\partial_\delta \Gamma^{\alpha}_{\beta\gamma} &= \partial_\delta \Theta(l) \Gamma^{+\alpha}_{\beta\gamma} + \partial_\delta \Theta(-l) \Gamma^{-\alpha}_{\beta\gamma} \\
&= \Theta(l)\partial_\delta \Gamma^{+\alpha}_{\beta\gamma} + \Theta(-l)\partial_\delta \Gamma^{-\alpha}_{\beta\gamma} + \Gamma^{+\alpha}_{\beta\gamma}\partial_\delta\Theta(l) + \Gamma^{-\alpha}_{\beta\gamma}\partial_\delta\Theta(-l) \\
&= \Theta(l)\partial_\delta \Gamma^{+\alpha}_{\beta\gamma} + \Theta(-l)\partial_\delta \Gamma^{-\alpha}_{\beta\gamma} + \varepsilon\delta(l)[\Gamma^{\alpha}_{\beta\gamma}]n_\delta.
\end{aligned}
$$

Now, the Riemannian curvature tensor using the form of Christoffel symbols and the derivatives computed above:

$$
\begin{aligned}
R^{\alpha}_{\beta\gamma\delta} &= \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\nu}_{\beta\delta} - \Gamma^{\alpha}_{\nu\delta}\Gamma^{\nu}_{\beta\gamma} \\
&= \Theta(l)\partial_\gamma \Gamma^{+\alpha}_{\beta\delta} + \Theta(-l)\partial_\gamma \Gamma^{-\alpha}_{\beta\delta} + \varepsilon\delta(l)[\Gamma^{\alpha}_{\beta\delta}]n_\gamma - \Theta(l)\partial_\delta \Gamma^{+\alpha}_{\beta\gamma} - \Theta(-l)\partial_\delta \Gamma^{-\alpha}_{\beta\gamma} - \varepsilon\delta(l)[\Gamma^{\alpha}_{\beta\gamma}]n_\delta \\
&\quad + (\Theta(l)\Gamma^{+\alpha}_{\nu\gamma} + \partial_\delta\Theta(-l)\Gamma^{-\alpha}_{\nu\gamma})(\Theta(l)\Gamma^{+\nu}_{\beta\delta} + \partial_\delta\Theta(-l)\Gamma^{-\nu}_{\beta\delta}) \\
&\quad + (\Theta(l)\Gamma^{+\alpha}_{\nu\delta} + \partial_\delta\Theta(-l)\Gamma^{-\alpha}_{\nu\delta})(\Theta(l)\Gamma^{+\nu}_{\beta\gamma} + \partial_\delta\Theta(-l)\Gamma^{-\nu}_{\beta\gamma}) \\
&= \Theta(l)R^{+\alpha}_{\beta\gamma\delta} + \Theta(-l)R^{+\alpha}_{\beta\gamma\delta} + \delta(l)[\varepsilon([\Gamma^{\alpha}_{\beta\delta}]n_\gamma - \Gamma^{\alpha}_{\beta\gamma}]n_\delta)] \\
&= \Theta(l)R^{+\alpha}_{\beta\gamma\delta} + \Theta(-l)R^{+\alpha}_{\beta\gamma\delta} + \delta(l)A^{\alpha}_{\beta\gamma\delta}.
\end{aligned}
$$

In the last line, defined:

$$
A^{\alpha}_{\beta\gamma\delta} = \varepsilon([\Gamma^{\alpha}_{\beta\delta}]n_\gamma - [\Gamma^{\alpha}_{\beta\gamma}]n_\delta). \tag{2.2}
$$

The last term in the Riemannian curvature tensor presents a singularity in the curvature at $\delta(l)$ (since at $l = 0$ the curvature tensor blows up and is not well defined).
Now, to investigate the tensor $A^{\alpha}_{\beta\gamma\delta}$, first $[\Gamma^{\alpha}_{\beta\delta}]$ should be looked into. It comes from the derivatives of the metric. Earlier, the condition of smoothness was imposed on the metric, which implies that the tangential derivatives of the metrics must be continuous too. So, if partial derivative of the metric across Σ is discontinuous, it must be so in the normal direction. Calling this discontinuouity $\kappa_{\alpha\beta}$,

$$
\begin{aligned}
[\partial_\gamma g_{\alpha\beta}] &= \kappa_{\alpha\beta} n_\gamma \\
[\partial_\gamma g_{\alpha\beta}] n^\gamma &= \kappa_{\alpha\beta} n_\gamma n^\gamma \\
\kappa_{\alpha\beta} &= \varepsilon[\partial_\gamma g_{\alpha\beta}] n^\gamma
\end{aligned} \tag{2.3}
$$

Using Eq. (2.3),

$$
\begin{aligned}
[\Gamma^{\alpha}_{\beta\gamma}] &= \frac{1}{2} g^{\alpha\mu}([\partial_\beta g_{\gamma\mu}] + [\partial_\gamma g_{\beta\mu}] - [\partial_\mu g_{\beta\gamma}]) \\
&= \frac{1}{2}(\kappa^{\alpha}_{\beta} n_\gamma + \kappa^{\alpha}_{\gamma} n_\beta - \kappa_{\beta\gamma} n^\alpha).
\end{aligned} \tag{2.4}
$$

Substituting this result in Eq. (2.2),

$$
\begin{aligned}
A^{\alpha}_{\beta\gamma\delta} &= \varepsilon((\frac{1}{2}(\kappa^{\alpha}_{\beta} n_\delta + \kappa^{\alpha}_{\delta} n_\beta - \kappa_{\beta\delta} n^\alpha))n_\gamma - (\frac{1}{2}(\kappa^{\alpha}_{\beta} n_\gamma + \kappa^{\alpha}_{\gamma} n_\beta - \kappa_{\beta\gamma} n^\alpha))n_\delta) \\
&= \frac{\varepsilon}{2}(\kappa^{\alpha}_{\delta} n_\beta n_\gamma - \kappa^{\alpha}_{\gamma} n_\beta n_\delta - \kappa_{\beta\delta} n^\alpha n_\gamma + \kappa_{\beta\gamma} n^\alpha n_\delta).
\end{aligned}
$$

Explicitly, this is the part which resulted in discontinuity in the Riemannian curvature tensor. To see the resulting continuity in the Ricci Tensor, contracting on the first and third indices:

$$
\begin{aligned}
A_{\beta\delta} = A^{\gamma}_{\beta\gamma\delta} &= \frac{\varepsilon}{2}(\kappa^{\gamma}_{\delta} n_\beta n_\gamma - \kappa^{\gamma}_{\gamma} n_\beta n_\delta - \kappa_{\beta\delta} n^\gamma n_\gamma + \kappa_{\beta\gamma} n^\gamma n_\delta) \\
&= \frac{\varepsilon}{2}(\kappa_{\gamma\delta} n_\beta n^\gamma - \kappa n_\beta n_\delta - \kappa_{\beta\delta}\varepsilon + \kappa_{\beta\gamma} n^\gamma n_\delta) \\
A_{\alpha\beta} &= \frac{\varepsilon}{2}(\kappa_{\mu\alpha} n_\beta n^\mu + \kappa_{\mu\beta} n^\mu n_\alpha - \kappa n_\alpha n_\beta - \varepsilon\kappa_{\alpha\beta}).
\end{aligned}
$$

In the last line just relabeled indices to bring it into a more familiar form. Contracting again, to get the discontinuous part of the Ricci scalar:

$$
A \equiv A^{\alpha}_{\alpha} = \varepsilon(\kappa_{\mu\nu} n^\mu n^\nu - \varepsilon\kappa).
$$

where $\kappa^{\alpha}_{\alpha} \equiv \kappa$. Now, the discontinuous δ part of the Einstein field equations can be obtained from $A_{\alpha\beta}$ and A :

$$
\begin{aligned}
S_{\alpha\beta} &\equiv A_{\alpha\beta} - \frac{1}{2} A g_{\alpha}\beta \qquad (2.5) \\
16\pi\varepsilon S_{\alpha\beta} &= \kappa_{\mu\alpha} n_\beta n^\mu + \kappa_{\mu\beta} n^\mu n_\alpha - \kappa n_\alpha n_\beta - \varepsilon\kappa_{\alpha\beta} - (\kappa_{\mu\nu} n^\mu n^\nu - \varepsilon\kappa) g_{\alpha\beta}.
\end{aligned}
$$

In the second equation, multiplied the equation with $16\pi\varepsilon$ to get rid of coefficients on rhs. Computing $S_{\alpha\beta} n^\beta$ gives 0, so it is tangent to the surface. So, tangent vector can be expressed as:

$$
S^{\alpha\beta} = S^{ab} e^{\alpha}_{a} e^{\beta}_{b},
$$

where $S_{ab} = S_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b}$ is a three-tensor. Using Eq. (2.5):

$$
\begin{aligned}
16\pi S_{ab} &= -\kappa_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} - \varepsilon\left(\kappa_{\mu\nu} n^\mu n^\nu - \varepsilon\kappa\right) h_{ab} \\
&= -\kappa_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} - \kappa_{\mu\nu}\left(g^{\mu\nu} - h^{mn} e^{\mu}_{m} e^{\nu}_{n}\right) h_{ab} + \kappa h_{ab} \\
&= -\kappa_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} - \kappa h_{ab} - \kappa_{\mu\nu} h^{mn} e^{\mu}_{m} e^{\nu}_{n} h_{ab} + \kappa h_{ab} \\
&= -\kappa_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} + h^{mn}\kappa_{\mu\nu} e^{\mu}_{m} e^{v}_{n} h_{ab}. \qquad (2.6)
\end{aligned}
$$

where second line follows from completeness relation. Using the expression of covariant derivative:

$$
\begin{aligned}
[n_{\alpha;\beta}] &= [\partial_\beta n_\alpha] - [\Gamma^{\gamma}_{\alpha\beta}] n_\gamma \\
[n_{\alpha;\beta}] &= -[\Gamma^{\gamma}_{\alpha\beta} n_\gamma] \\
&= -\frac{1}{2}\left(\kappa_{\gamma\alpha} n_\beta + \kappa_{\gamma\beta} n_\alpha - \kappa_{\alpha\beta} n_\gamma\right) n^\gamma \\
&= \frac{1}{2}\left(\varepsilon\kappa_{\alpha\beta} - \kappa_{\gamma\alpha} n_\beta n^\gamma - \kappa_{\gamma\beta} n_\alpha n^\gamma\right),
\end{aligned}
$$

where in first equation, since n_α is smooth across the surface, its derivative must be smooth as well. In second line used, Eq. (2.4). Covariant derivative of the normal vector was computed to relate the extrinsic curvature tensor to S_{ab}. Now, the exstrinsic curvature:

$$
[K_{ab}] = [n_{\alpha;\beta}]\, e^{\alpha}_{a} e^{\beta}_{b} = \frac{\varepsilon}{2}\kappa_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b}.
$$

$$
[h^{ab} K_{ab}] \equiv [K] = \frac{\varepsilon}{2}[h^{ab}\kappa_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b}]
$$

Substituting these results into Eq. 2.6:

$$S_{ab} = -\frac{\varepsilon}{8\pi}\left([K_{ab}] - [K]h_{ab}\right).$$

This expression relates the discontinuity in Einstein equation to the jump in exstrinsic curvature on both sides of Σ.

The results are independent of the assumptions made in the start that coordinates x^α is installed on both sides and independent of $x^\alpha_\pm$ [1].

With this introduction to junction conditions, it is time to discuss gravitational collapse.

3 Collapse of a Star with Uniform Density and Zero Pressure

Gravitational collapse occurs when the star runs out of its fuel and internal pressures cannot sustain it and collapses inwards [3]. Consider a star that is static and is spherically symmetric. The star is also non-radiating. Outside the star is vacuum, and Schwarzschild solution to the Einstein field equations is valid. Inside the star is some complicated geometry. Assume that the pressures inside the star are zero $P = 0$, then the star has some non-zero density ρ; such a spherically symmetric, pressureless, non-radiating fluid is called dust. So, here, a model collapse of dust, more famously known as the Oppenheimer-Snydert Collapse, is presented.

3.1 Exterior Geometry and its Dynamics

As discussed earlier, the star is static and spherically symmetric. To describe the geometry of stars outside one looks for spherically symmetric solutions of Einstein equations. Birkhoff's theorem suggests that the solution is given by the Schwarzschild metric.

Birkhoff's theorem states that the Schwarzschild solution is the most general spherically symmetric vacuum solution of Einstein's equations. The external region of any non-radiating, neutral, spherically symmetric star fulfils both conditions, i.e. the external region is a vacuum and is spherically symmetric. Hence, the exterior geometry is Schwarzschild geometry. It can easily be understood intuitively. Suppose a star is initially static and perturbed such that it collapses radially inward. During this collapse, the exterior geometry is unchanged due to spherical symmetry. All the gravitational influences of the collapse propagate inwards. So, more and more external Schwarzschild geometry is revealed during the collapse.

So, for now, we focus on the exterior geometry and interior geometry will be discussed later. Writing down the Schwarzschild solution:

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2,$$

where $f = 1 - 2M/r$, M is the mass of the collapsing star. $r = 2M$ is the gravitational or Schwarzschild radius. $r = 2M$ is the coordinate singularity which can be removed by transforming coordinates, while $r = 0$ is a physical singularity where the Schwarzschild solution is no longer valid. Because, there is no pressure inside the star, the surface of star free falls along radial geodesics in the exterior Schwarzschild geometry. So, first finding the geodesic equation. Geodesics extremize the action so:

$$\delta \int ds = 0$$

$$\delta \int \sqrt{g_{\alpha\beta}\dot{x^\alpha}\dot{x}^\beta d\lambda} = 0,$$

where $\dot{}$ denotes $\frac{d}{d\lambda}$, λ is an auxillary parameter. So, the Lagrangian takes the form:

$$\begin{aligned}\mathscr{L} &= g_{\alpha\beta}\dot{x^{\alpha}}\dot{x}^{\beta} \\ &= -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{1}{\left(1-\frac{2M}{r}\right)}\dot{x}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2.\end{aligned}$$

Solving the Euler Lagrange for θ:

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial\theta} &= \frac{d}{d\lambda}\frac{\partial\mathscr{L}}{\partial\dot{\theta}} \\ \sin(2\theta)\dot{\theta}^2 &= 2r^2\ddot{\theta} + 4r\dot{r}\dot{\theta}.\end{aligned}$$

Turns out $\theta = \pi/2$ is a solution to the above equation. This also implies that the motion along the geodesics is restricted to the $\theta = \pi/2$ plane. With this the Lagrangian can be simplified as:

$$\mathscr{L} = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{1}{\left(1-\frac{2M}{r}\right)}\dot{r}^2 + r^2\dot{\phi}^2.$$

It can be noted that the lagrangian is idependent of t and θ, so these are conserved quantities as:

$$\frac{\partial\mathscr{L}}{\partial t} = 0 = \frac{d}{d\lambda}\frac{\partial\mathscr{L}}{\partial\dot{t}} \implies \frac{\partial\mathscr{L}}{\partial\dot{t}} = \text{const.} = -2E$$

where E is the constant energy per unit mass of the test particle, given by:

$$E = \left(1-\frac{2M}{r}\right)\dot{t} \tag{3.1}$$

$$\frac{\partial\mathscr{L}}{\partial\phi} = 0 = \frac{d}{d\lambda}\frac{\partial\mathscr{L}}{\partial\dot{\phi}} \implies \frac{\partial\mathscr{L}}{\partial\dot{\phi}} = \text{const.} = 2L,$$

where L is angular momentum per unit mass given by:

$$L = r^2\dot{\theta}.$$

Moreover, $\mathscr{L}$ is itself a conserved quantity $\frac{d\mathscr{L}}{d\lambda} = 0$ which implies that:

$$\mathscr{L} = \varepsilon\begin{cases}-1 & \text{for timelike geodesics} \\ 0 & \text{for null geodesics} \\ +1 & \text{for spacelike geodesics}\end{cases}$$

Replacing all these results:

$$\frac{1}{2}\dot{r}^2 + \left[\varepsilon\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}\right] = \frac{1}{2}\left[E^2 + \varepsilon\right] \tag{3.2}$$

The first term in the brackets is defined to be effective potential V_{eff} and the term on the R.H.S. in brackets the effective energy $\mathscr{E}_{eff}$. Now, from the geodesic equation one can find proper time of observer falling in along the radial geodesic, the motion of surface of dust, and the coordinate time of free fall. For proper time for an observer freely falling with the surface of star along a radial geodesic (in which case Eq. (3.2) is parameterized by τ and $L = 0$):

$$\tau = \int d\tau = \int \frac{dr}{(2M/r - 2M/R_i)^{1/2}},$$

where $R_i \equiv \frac{2M}{1-E^2}$ is the initial radius where the comoving observer has zero velocity. The motion along radial geodesics follows the cycloid motion. Free fall of a particle towards a Newtonian center of attraction is described by the cycloid motion where it relates t and r, here it relates τ and r. Parametric equations of cycloid have the form:

$$x = a(\eta - \sin\eta).$$
$$y = a(1 - \cos\eta),$$

where the parameter η goes from 0 to 2π and so does x [4]. a is the maximum y displacement. For relation between r and τ in terms of parameter η is shown in the Fig. 2. The cycloid is generated when the circle

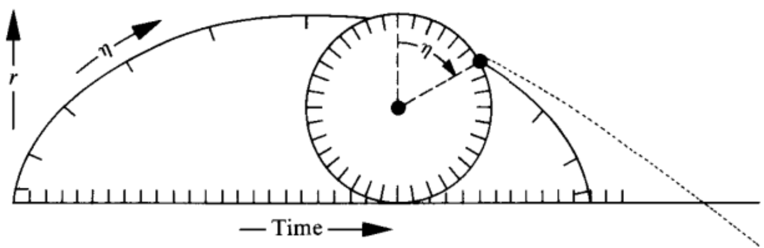

Figure 2: Cycloid giving relation between Schwarzschild coordinate r and proper time for a particle falling with the surface of star [3].

turns through the angle η. Reparametrizing these equations so that r goes from $r = R_i$ to $r = 0$, so $\eta \to \eta + \pi$ gives:

$$r = \frac{R_i}{2}(1 + \cos\eta). \tag{3.3}$$

$$\tau = \frac{R_i}{2}\left(\frac{R}{2M}\right)^{1/2}(\eta + \sin\eta). \tag{3.4}$$

Now η goes from $0 \to \pi$.
Now, to get the coordinate time replace $\frac{dr}{d\tau} = \frac{dr}{dt}\frac{1}{\sqrt{1-2M/r}}$ in the expression for proper time, to get the general radial or non radial motion:

$$t = \int dt = \int \frac{1}{(2M/r - 2M/R_i)^{1/2}} \frac{dr}{\sqrt{1 - 2M/r}},$$

Solving this integral gives:

$$t = 2M \ln\left[\frac{(R_i/2M - 1)^{1/2} + \tan\eta/2}{(R_i/2M - 1)^{1/2} - \tan\eta/2}\right] + 2M(R_i/2M - 1)^{1/2}(\eta + (R_i/4M)(\eta + \sin\eta)) \tag{3.5}$$

Plotting Eq. (3.3), (3.4) and (3.5) gives Fig. 3.
Figure follows the collapse described by the equations in exterior Schwarzschild coordinates. It takes finite amount of proper time of the observer falling with the surface of dust as r goes from $r = R_i$ to $r = 0$ as η goes from 0 to π:

$$\Delta\tau = \pi\sqrt{\frac{R_i^3}{8M}}.$$

While, t becomes asymptotic to $R = 2M$, so for an external observer it takes an infinite amount of time for the surface of star to reach from $R = 2M$ to $r = 0$ while for the co-falling observer it takes a finite amount of time to reach the singularity. The worldline of this observer is plotted in Fig. 4. From the figure, following things can be noted:

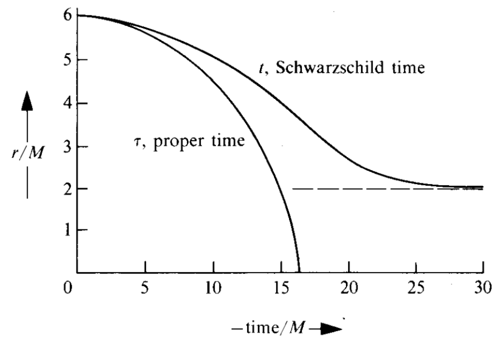

Figure 3: Free fall described by coordinate time t (a far away observer), and τ (comoving observer) [3].

1. The trajectory of the infalling observer must lie within the light cones so, once the observer has crossed the Schwarzschild radius $R = 2M$, also called the event horizon, it is the observers fate to fall into singularity.

2. Since the observer cannot send the signals that travel faster than the speed of light, so any information the observer sends must lie on or within the light cone. Once the observer crosses the Schwarzschild radius, no information can be sent out.

3. Since, no information can be sent out and no light escapes from event horizon, an external observer can never see the star after its surface crosses the Schwarzschild radius.

Also note from equation 3.5 that the coordinate time slows down as it approaches the Schwarzschild radius, so to an external observer the collapse seems to be happening slowly and it takes an infinite amount of time to reach the Schwarzschild radius [3].

3.2 Interior Geometry and its Dynamics

There is no unique choice for the interior geometry of dust. The simplest of the interiors is the one that is homogenous (since density is uniform inside) and isotropic everywhere inside the star except of course the surface. Homogenous and Isotropic geometry means that the spacetime everywhere is the same. Freidmann solution satisfies these conditions. Since initially the star has maximal radius and then it falls inward, so the closed Freidmann solution with $k = +1$ is chosen. This is one of the solutions of Einsteins that satisfies these conditions set for simplicity in the interior region.

$$ds^2 = -dt^2 + a^2(\tau)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)].$$

This expression describes the dynamics of interior geometry as described by radius a as a function of time. Using comoving hyperspherical coordinates, whose origin is at the center of the star, metric in the Freedman form becomes:

$$ds^2 = -d\tau^2 + a^2(\tau)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)].$$

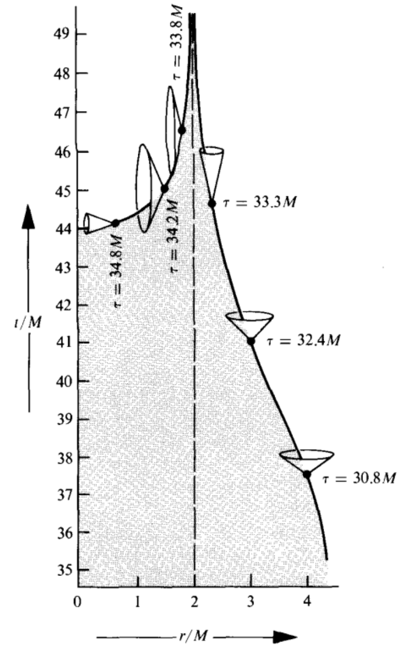

Figure 4: Gravitational collapse of dust in Schwarzschild coordinates. Grey region is star's interior. The curve is worldline of comoving observer in Schwarzschild coordinates.

Becuase the pressure is zero, other components of the energy momentum tensor $T^{\mu\nu}$ are zero and only T^{00} component is non zero, which gives the energy density.

$$G_{\tau\tau} = \frac{3}{a^2}(1+\dot{a}^2) = 8\pi T_{\tau\tau} = 8\pi\rho u_\tau u_\tau = 8\pi\rho$$

$$\frac{3}{a^2}\left(\frac{da}{d\tau}\right)^2 + \frac{6}{a^2} = 16\pi\rho. \tag{3.6}$$

$$\tag{3.7}$$

Calculation for this solution is shown in Appendix. Four velocity of the fluid in comoving coordinates is $u^\alpha = (1,0,0,0)$. The solution to this equation, a and τ are again related by the cycloid relation:

$$a = \frac{a_m}{2}(1+\cos\eta), \tag{3.8}$$

$$\tau = \frac{a_m}{2}(\eta + \sin\eta), \tag{3.9}$$

where a_m is the maximum radius at $\eta = 0$.
For dust, at any instant of time the density of star times its volume gives it mass:

$$\rho 2\pi^2 a^3 = M,$$

Define a_m, which is the initial maximum radius of star, substituting a_m in eq. 3.6:

$$\frac{3}{a_m^2}\frac{da_m}{d\tau}^2 + \frac{6}{a_m^2} = 16\pi\rho$$
$$\frac{6}{a_m^2} = 16\pi\rho \implies a_m = \frac{3}{8\pi\rho}$$
$$a_m = \frac{3\times 2\pi^2 a_m^3}{8\pi M} \implies a_m = \frac{4M}{3\pi}$$

In the last step substituted the value of ρ at the time of maximum expansion.
With this the density becomes:

$$\rho = \frac{4M}{3\pi}\frac{1}{2\pi^2 a^3} = \frac{3a_m}{8\pi a^3}$$
$$= \frac{3a_m}{8\pi}\left(\frac{a_m}{2}(1+\cos\eta)\right)^{-3} = \frac{3}{8\pi a_m^2}\left(\frac{1}{2}(1+\cos\eta)\right)^{-3}$$

This expression relates density to the initial radius of star. η goes from $0 \to \pi$, when $\eta = \pi$, note that the density of the star is not well defined $\rho \to \infty$. As $r \to 0$ very large mass is concentrated in a very small region. This results in formation of a singularity which is called black hole [3].
There is one issue with this interior solution, the star is no longer homogeneous and isotropic at its surface that lies at some $\chi = \chi_0$. For this interior solution to be actually valid, the junction conditions must be met.

3.3 Surface of the Star

For the two metrics to join smoothly at the surface junction conditions must be fulfilled. The hypersurface Σ coincides with the surface of the collapsing star located at $\chi = \chi_0$ in comoving coordinates. Rewriting the metric in the interior region $\mathscr{V}^-$:

$$ds_-^2 = -d\tau^2 + a^2(\tau)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)].$$

For calculating the induced metric, it is convenient to choose $y^a = (\tau, \theta, \phi)$. As seen from $\mathscr{V}^-$, the induced metric is:

$$h_{ab} = g_{\alpha\beta}e_a^\alpha e_b^\beta = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b}$$
$$h_{00} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^\beta}{\partial \tau} = g_{00}\frac{\partial x^0}{\partial \tau}\frac{\partial x^0}{\partial \tau} = -1$$
$$h_{11} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \theta}\frac{\partial x^\beta}{\partial \theta} = g_{22}\frac{\partial x^2}{\partial \theta}\frac{\partial x^2}{\partial \theta} = a^2(\tau)\sin^2\chi_0$$
$$h_{22} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \phi}\frac{\partial x^\beta}{\partial \phi} = g_{33}\frac{\partial x^3}{\partial \phi}\frac{\partial x^3}{\partial \phi} = a^2(\tau)\sin^2\chi_0 sin^2\theta$$
$$\implies ds_\Sigma^2 = -d\tau^2 + a^2(\tau)\sin^2\chi_0 d\Omega^2. \tag{3.10}$$

Due to the convenient choice of coordinates calculating the induced metric was very simple.
As seen from the exterior region Σ is described by parametric equations $r = R(\tau)$, $t = T(\tau)$, τ is the proper time for observers comoving with the hypersurface. Similarly the induced metric as seen from $\mathscr{V}^+$:

$$ds_\Sigma^2 = -(F\dot{T}^2 - F^{-1}\dot{R}^2)d\tau^2 + R^2(\tau)d\Omega^2, \tag{3.11}$$

where $F = 1 - 2M/R$. Because the induced metric must be the same as seen by both the regions, comparing Eq. (3.10) and (3.11),

$$R(\tau) = a(\tau)\sin\chi_0$$
$$F\dot{T}^2 - F^{-1}\dot{R}^2 = 1 \implies F\dot{T} = \sqrt{\dot{R}^2 + F} = \beta(R, \dot{R}). \tag{3.12}$$

The second junction condition is that the extrinsic curvature on the surface must match. The extrinsic curvature is given by:

$$K_{ab} = \nabla_\beta (n_\alpha) e^\alpha_a e^\beta_b.$$

From the chosen coordinates on Σ, one of the basis vectors is $e^\alpha_\tau = u^\alpha$ which is the four velocity of co-moving coordinates. To get the unit normal, using $n_\alpha u^\alpha = 0$ and $n_\alpha n^\alpha = 1$. As seen from $\mathscr{V}^-$, choosing $n^\chi > 0$, so that the normal is direction towards the exterior region. With this, $n^-_\alpha dx^\alpha = a d\chi \implies n_\alpha = (0, a, 0, 0)$. Now, the extrinsic curvature tensor:

$$\begin{aligned}
K_{ab} &= \nabla_\beta (n_\alpha) e^\alpha_a e^\beta_b \\
&= \partial_\beta n_\alpha e^\alpha_a e^\beta_b - \Gamma^\gamma_{\alpha\beta} n_\gamma e^\alpha_a e^\beta_b \\
&= \partial_\beta n_\chi e^\alpha_a e^\beta_b - \Gamma^\chi_{\alpha\beta} n_\chi e^\alpha_a e^\beta_b \\
&= 0 - \Gamma^\chi_{ab} n_\chi \\
K^-_{\tau\tau} &= -\Gamma^\chi_{\tau\tau} n_\chi = 0 &(3.13)\\
K^-_{\theta\theta} &= \Gamma^\chi_{\theta\theta} n_\chi = a \sin\chi_0 \cos\chi_0 &(3.14)\\
K^-_{\phi\phi} &= \Gamma^\chi_{\phi\phi} n_\chi = a \sin^2\theta \sin\chi_0 \cos\chi_0 &(3.15)
\end{aligned}$$

where, $\Gamma^\chi_{\tau\tau} = 0$, $\Gamma^\chi_{\theta\theta} = -\sin\chi\cos\chi$, $\Gamma^\chi_{\phi\phi} = -\sin^2\theta \sin\chi\cos\chi$. Rest of the components of the curvature tensor are zero.
As seen from $\mathscr{V}^+$, $u^\alpha = (\dot{T}, \dot{R}, 0, 0)$, normal vector satisfying the conditions mentioned earlier can be written as $n_\alpha = (-\dot{R}, \dot{T}, 0, 0)$, components of the extrinsic curvature are:

$$\begin{aligned}
K_{ab} &= \partial_\beta n_\alpha e^\alpha_a e^\beta_b - \Gamma^\gamma_{\alpha\beta} n_\gamma e^\alpha_a e^\beta_b, \\
K^+_{\theta\theta} &= \partial_\beta n_\alpha e^\alpha_\theta e^\beta_\theta - \Gamma^\gamma_{\alpha\beta} n_\gamma e^\alpha_\theta e^\beta_\theta, \\
&= -\Gamma^\tau_{\theta\theta} n_\tau e^\theta_\theta e^\theta_\theta - \Gamma^r_{\theta\theta} n_r e^\theta_\theta e^\theta_\theta, \\
&= -\Gamma^\tau_{\theta\theta}(-\dot{R}) - \Gamma^r_{\theta\theta}\dot{T} \\
&= RF\dot{T} = \beta R, &(3.16)
\end{aligned}$$

where $\Gamma^\tau_{\theta\theta} = 0$, $\Gamma^r_{\theta\theta} = -RF$.

$$\begin{aligned}
K^+_{\phi\phi} &= \partial_\beta n_\alpha e^\alpha_\phi e^\beta_\phi - \Gamma^\gamma_{\alpha\beta} n_\gamma e^\alpha_\phi e^\beta_\phi, \\
&= -\Gamma^\tau_{\phi\phi} n_\tau e^\phi_\phi e^\phi_\phi - \Gamma^r_{\phi\phi} n_r e^\phi_\phi e^\phi_\phi, \\
&= -\Gamma^\tau_{\phi\phi}(-\dot{R}) - \Gamma^r_{\phi\phi}\dot{T} \\
&= \sin^2\theta RF\dot{T} = \sin^2\theta R\beta, &(3.17)
\end{aligned}$$

where $\Gamma^\tau_{\phi\phi} = 0$ and $\Gamma^r_{\phi\phi} = -\sin^2\theta RF$

$$\begin{aligned}
K^+_{\tau\tau} &= \partial\tau n_\alpha e^\alpha_\tau e^\tau_\tau - \Gamma^\mu_{\alpha\beta} n_\mu e^\alpha_\tau e^\beta_\tau \\
&= \partial\tau n_\tau e^\tau_\tau e^\tau_\tau + \partial\tau n_R e^R_\tau e^\tau_\tau - \Gamma^\mu_{\tau\tau} n_\mu - 2\Gamma^\mu_{\tau R} n_\mu e^\tau_\tau e^R_\tau - \Gamma^\mu_{RR} n_\mu e^R_\tau e^R_\tau \\
&= -\ddot{R} + \dot{R}\ddot{T} - \Gamma^R_{\tau\tau} n_R - 2\Gamma^\tau_{\tau R} n_\tau e^\tau_\tau e^R_\tau - \Gamma^R_{RR} n_R e^R_\tau e^R_\tau \\
&= -\ddot{R} + \dot{R}\ddot{T} + \frac{2M}{FR^2}\dot{R}\dot{R} - \frac{FM}{R^2}\dot{T} + \frac{M}{FR^2}\dot{T}\dot{R}\dot{R}
\end{aligned}$$

where $\Gamma^\tau_{\tau\tau} = 0$, $\Gamma^\tau_{RR} = 0$, $\Gamma^\tau_{\tau R} = \frac{M}{FR^2}$, $\Gamma^R_{\tau\tau} = \frac{FM}{R^2}$ and $\Gamma^R_{RR} = -\frac{M}{FR^2}$. Rest of the components of the extrinsic curvature tensor are zero as seen from outside.
Second junction condition requires $[K_{ab}] = 0$. Comparing Eq. (3.14), (3.16), (3.15), and (3.17) gives:

$$a \sin\chi_0 \cos\chi_0 = \beta R \implies \beta = \cos\chi_0,$$

where $R = a \sin \chi_0$ was given by the first junction condition. To compare $K_{\tau\tau}$, it is convenient to first raise the index using the induced metric $K^{\tau+}_{\tau} = \frac{\dot{\beta}}{\dot{R}}$, should be equal to $K^{\tau-}_{\tau} = 0$, so

$$\frac{\dot{\beta}}{\dot{R}} = 0 \implies \dot{\beta} = 0,$$

which should follow because β was found to be a constant term while imposing $[K_{\theta\theta}] = 0$ and $[K_{\phi\phi}] = 0$. Comparing Eq. (3.12) with the equation for energy conservation when moving along the Schwarzschild geodesics Eq. (3.1), $\implies$ $\beta = E$, so $\dot{\beta} = 0$ just gives us energy conservation parameter of the co-moving observer.

$$\beta = E = \cos \chi_0.$$

So, the requirement for smooth transition at the surface is that the hypersurface must be generated by geodesics of both exterior and interior geometries [1].

4 Conclusions

Analyzing the exterior geometry of the star, it is found that the co-moving observer falls into singularity after crossing the event horizon. External observers never see the surface of the star moving past the horizon. No information will be received from this region, called the black hole. Discussing the interior geometry, the energy density of the star is found to be approaching infinity as a singularity at $r = 0$ is reached. So, the collapsing star results in an unstable state. At the surface of star, the junction conditions must be met. For null hypersurfaces, conditions are different and can be found in [5]. The significance of the collapse of dust lies in the quantitative features of gravitational collapse obtained by exploring it. The general behaviour will also manifest in more complicated and realistic collapses. Roger Penrose proved that the black hole and spacetime singularities will always form in a realistic gravitational collapse [6]. So, the characteristics of this collapse can be generalized to more realistic collapses where the collapse is not spherically symmetric and radiating or internal pressures are present.

5 Appendix

FRW metric is given by:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right].$$

Non zero Christoffel symbols:

$$\begin{aligned}
&\Gamma^t_{rr} = \frac{a\dot{a}}{1 - kr^2}, \quad \Gamma^t_{\theta\theta} = r^2 a\dot{a}, \quad \Gamma^t_{\phi\phi} = r^2 a\dot{a} \sin^2 \theta, \\
&\Gamma^r_{tr} = \frac{\dot{a}}{a} \quad \Gamma^\theta_{t\theta} = \frac{\dot{a}}{a}, \quad \Gamma^\phi_{t\phi}, = \frac{\dot{a}}{a} \\
&\Gamma^r_{rr} = \frac{kr}{1 - kr^2}, \quad \Gamma^r_{\theta\theta} = (-1 + kr^2) r, \Gamma^r_{\phi\phi} = (-1 + kr^2) r \sin^2 \theta \\
&\Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\cos \theta \sin \theta, \quad \Gamma^\phi_{\phi\theta} = \cot \theta
\end{aligned}$$

Ricci tensor non zero components:

$$
\begin{aligned}
R_{tt} &= -\frac{3\ddot{a}}{a} \\
R_{rr} &= \frac{2k + 2\dot{a}^2 + a\ddot{a}}{1 - kr^2} \\
R_{\theta\theta} &= r^2(2k + 2\dot{a}^2 + a\ddot{a}) \\
R_{\phi\phi} &= r^2(2k + 2\dot{a}^2 + a\ddot{a}sin^2\theta)
\end{aligned}
$$

Ricci scalar:

$$
R = \frac{6(k + \dot{a}^2 + a\ddot{a})}{a^2}
$$

For k = 1, and comoving coordinates, using the Ricci tensor and Ricci scalar, $G_{\tau\tau}$:

$$
\begin{aligned}
G_{\tau\tau} &= R_{\tau\tau} - \frac{1}{2}g_{\tau\tau}R \\
&= -\frac{3\ddot{a}}{a} + \frac{3(1 + \dot{a}^2 + a\ddot{a})}{a^2} \\
&= \frac{3(1 + \dot{a}^2)}{a^2}
\end{aligned}
$$

and $\dot{a}$ denotes the derivative of a with respect to proper time.

References

[1] Eric Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*, Cambridge University Press (2004).

[2] T. Padmanabhan, *Lecture Notes on General Relativity*, newlecturesGR.pdf.

[3] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, Princeton University Press, 1973.

[4] Tom Roidt, *Cycloids and Paths*, pdx.edu.

[5] J. R. Oppenheimer and H. Snyder, *On continued gravitational contraction*, *Physical Review* 56, 455–459 (1939), PhysRev.56.455.

[6] S. W. Hawking and R. Penrose, *The singularities of gravitational collapse and cosmology*, royalsociety-publishing.org 1970.0021.