Project Report for Quantum Mechanics II (Spring 2023)

Optical Pumping Masers and Lasers

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Abstract: This report thoroughly investigates the fundamental quantum and physical principles behind lasers. We start with time-dependent perturbation theory and determine Einstein's coefficients to find the stimulated and spontaneous emission rates and the threshold condition for laser emission. This emission involves certain loss mechanisms hindering the attainment of the threshold condition. These associated losses are compensated for through the use of optical resonators.

Keywords: Time Dependent Perturbation Theory, Einstein's Coefficients, Gain Coefficient, Optical Resonators

Contents

1 Introduction

The invention of lasers has revolutionised technology - they have vast applications in medical physics, fibre optics, laser cutting and much more. The first were made in 1953, using microwave radiation, and were called masers (microwave amplification by stimulated emission of radiation). This principle was then extended to obtain the amplification of visible light in 1960. Since then, lasers spanning the whole spectral range have been developed.

The foundations of lasers are based on the fact that energy levels are discrete. If we could achieve a population inversion, electrons would move from a higher to a lower energy level, emitting a photon of a specific frequency. This photon then de-excites the rest through stimulated emission, resulting in an exponential increase in the number of emitted photons (light amplification).

Starting with two-level systems to understand this process at the quantum level, we first study how the probability of finding the electrons in these two levels changes with time under external influences. This change of probabilities results in transitions. These transitions include absorption, stimulated emission or spontaneous emission. For laser action, we require that stimulated emission exceeds the rest. Hence, we determine the rates of spontaneous and stimulated emission to maximize the rate of stimulated emission [1]. This is achieved through population inversion. We then discuss why it is impossible to have population inversion in two-level systems; hence, multi-level systems are required.

We also study the components of lasers and what losses might occur in the amplification process. From then on, we calculate the threshold power needed to ensure the emission of radiation and investigate ways to minimize these losses. While these radiations are distributed over a whole spectrum of frequencies, many applications of lasers require them to be concentrated over a very small frequency range. For that, we discuss single-mode lasers [2, 3].

2 Two Level Systems

Consider an unperturbed system where ψ_a and ψ_b are the eigenstates of Hamiltonian, $\hat{H}^{(0)}$ such that

$$\hat{H}^{(0)}\psi_a = E_a^{(0)}\psi_a\,, \qquad \hat{H}^{(0)}\psi_b = E_b^{(0)}\psi_b\,.$$

and these states are orthonormal. Any state can be thus represented as a linear combination of these. So, the state can be written as:

$$\psi(0) = c_a\psi_a + c_b\psi_b.$$

We are interested in the time dependence of this state. Time dependant Hamiltonian in an unperturbed state is:

$$\psi(t) = c_a\psi_a e^{-iE_at/\hbar} + c_b\psi_b e^{-iE_bt/\hbar}. \tag{2.1}$$

where c_a and c_b are the probability amplitudes of finding the system in state ψ_a and ψ_b respectively and total probability is 1:

$$|c_a|^2 + |c_b|^2 = 1$$

2.1 The Perturbed System

Suppose we have a time-dependent perturbation $\hat{H}'(t)$. The wavefunction can be written as in (2.1) but the coefficients c_a and c_b will also be functions of time.

$$\psi(t) = c_a(t)\psi_a e^{-iE_a t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar}.$$

For transitions to occur, the probabilities of finding the particle in these states must change. We are interested in finding how these probabilities vary and need to find these time-dependent coefficients. To determine these coefficients, the time-dependent Schrodinger equation needs to be solved.

$$\hat{H}\psi(t) = i\hbar\frac{\partial}{\partial t}\psi(t).$$

Hamiltonian for this perturbed system is $\hat{H} = \hat{H}^0 + \hat{H}'(t)$.

$$c_a(\hat{H}^0\psi_a)e^{-iE_a t/\hbar} + c_a(\hat{H}'(t)\psi_t)e^{-iE_a t/\hbar} + c_b(\hat{H}^0\psi_b)e^{-iE_b t/\hbar}$$
$$+c_b(\hat{H}'(t)\psi_t)e^{-iE_b t/\hbar} = i\hbar\frac{\partial}{\partial t}\left[c_a(t)\psi_a e^{-iE_a t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar}\right]$$

Solving,

$$c_a(\hat{H}^0\psi_a)e^{-iE_a t/\hbar} + c_a(\hat{H}'(t)\psi_t)e^{-iE_a t/\hbar} + c_b(\hat{H}^0\psi_b)e^{-iE_b t/\hbar}$$
$$+c_b(\hat{H}'(t)\psi_t)e^{-iE_b t/\hbar} = i\hbar\left[\frac{d}{dt}c_a\psi_a e^{-iE_a t/\hbar} + c_a\psi_a(-iE_a t/\hbar)e^{-iE_a t/\hbar}\right.$$
$$\left.+\frac{d}{dt}c_b\psi_b e^{-iE_b t/\hbar} + c_b\psi_b(-iE_b t/\hbar)e^{-iE_b t/\hbar}\right].$$

Cancelling terms we get,

$$c_a(\hat{H}'(t)\psi_a)e^{-iE_a t/\hbar} + c_b(\hat{H}'(t)\psi_b)e^{-iE_b t/\hbar} = i\hbar\left[\frac{d}{dt}c_a\psi_a e^{-iE_a t/\hbar} + \frac{d}{dt}c_b\psi_b e^{-iE_b t/\hbar}\right].$$

Taking inner product with state ψ_a:

$$c_a\langle\psi_a|\hat{H}'|\psi_a\rangle e^{-iE_a t/\hbar} + c_b\langle\psi_a|\hat{H}'|\psi_b\rangle e^{-iE_b t/\hbar} = i\hbar\frac{d}{dt}c_a e^{-iE_a t/\hbar}. \tag{2.2}$$

Defining,

$$\hat{H}'_{ij} = \langle\psi_i|\hat{H}'|\psi_j\rangle.$$

Simplifying equation (2.2),

$$\frac{d}{dt}c_a = \frac{-i}{\hbar}\left[c_a\hat{H}'_{aa} + c_b\hat{H}'_{ab}e^{-i(E_b-E_a)t/\hbar}\right].$$

Diagonal elements of perturbed Hamiltonian vanish, as the perturbation, is quite small and the diagonal elements of our full Hamiltonian come only from unperturbed Hamiltonian. Moreover, since ψ_a and ψ_b are the eigenstates of the unperturbed Hamiltonian, the expectation value of the perturbed Hamiltonian i.e, $\langle\psi_a|\hat{H}'|\psi_a\rangle$ and $\langle\psi_b|\hat{H}'|\psi_b\rangle$ will be zero and the perturbation does not change the energy of the system. So, we are left with,

$$\frac{d}{dt}c_a = \frac{-i}{\hbar}\left[c_b\hat{H}'_{ab}e^{-i(E_b-E_a)t/\hbar}\right].$$

Similarly taking inner product of (2.2) with ψ_b:

$$c_a \langle\psi_b|\hat{H}'|\psi_b\rangle e^{-iE_at/\hbar} + c_b\langle\psi_b|\hat{H}'|\psi_a\rangle e^{-iE_bt/\hbar} = i\hbar\frac{d}{dt}c_be^{-iE_bt/\hbar}.$$

Simplifying,

$$\frac{d}{dt}c_b = \frac{-i}{\hbar}\left[c_a\hat{H}'_{ba}e^{-i(E_b-E_a)t/\hbar}\right].$$

Assuming $E_b > E_a$, and defining $\omega_o = \frac{E_b-E_a}{\hbar}$:

$$\frac{d}{dt}c_a = -\frac{i}{\hbar}c_b\hat{H}'_{ab}e^{-i\omega_ot}, \qquad \frac{d}{dt}c_b = -\frac{i}{\hbar}c_a\hat{H}'_{ba}e^{i\omega_ot}. \tag{2.3}$$

2.2 Time Dependent Perturbation Theory

For small $\hat{H}'$, we can find the coefficients by successive approximation using (2.3). Suppose a particle starts out in state ψ_a, at t = 0, the coefficients are:

$$c_a(0) = 1, \qquad c_b(0) = 0.$$

So, the zeroth order approximation is:

$$c_a^{(0)}(t) = 1, \qquad c_b^{(0)}(t) = 0.$$

The first-order approximation can be obtained by inserting the zeroth order values in (2.3), as:

$$\frac{dc_a^{(1)}}{dt} = 0, \qquad c_a^{(1)} = 1$$

To find, c_b inserting the value from zeroth order approximation:

$$\frac{dc_b^{(1)}}{dt} = -\frac{i}{\hbar}\hat{H}'_{ba}e^{i\omega_ot}.$$

$$c_b^{(1)} = -\frac{i}{\hbar}\int_0^t \hat{H}'_{ba}(t')e^{i\omega_ot'}dt'.$$

And this process can be repeated for higher-order approximations. For multi-level systems,

$$c_a^{(2)}(t)e^{-iE_at/\hbar} = e^{-iE_at/\hbar} + (-\frac{i}{\hbar})^2\int_0^t\int_0^{t'} e^{-iE_a(t'-t)/\hbar}H'_{ab}(t')e^{-iE_b(t''-t')/\hbar}H'_{ba}(t'')e^{-iE_a(t'')/\hbar}dt''dt'$$

This is the second-order approximation. The first term describes that the system remains in state ψ_a the entire time. Reading the second term from right to left illustrates that the system transitions from ψ_a to ψ_b at some t'' and then moves back from b to a, after time t''. It is a beautiful explanation of this approximation which is worth noting.

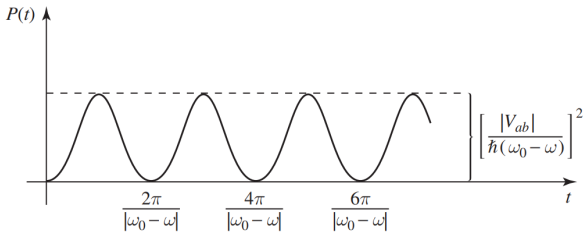

Figure 1. Transition probability as a function of time [1].

2.3 Sinusoidal Perturbations

Sinusoidal perturbations are used to model electromagnetic perturbations and quantum mechanical systems. Since perturbations governing laser phenomenon are sinusoidal, it is essential to consider these to understand lasers. Now, the Hamiltonian for a sinusoidal perturbation is:

$$\hat{H}'(\mathbf{r},t) = V(\mathbf{r},t)\cos\omega t.$$

and

$$\hat{H}'_{ab} = V_{ab}\cos\omega t,$$

where $V_{ab} = \langle\psi_a|V|\psi_b\rangle$. First order approximation for such a perturbation is:

$$\begin{aligned} c_b(t) \approx -\frac{i}{\hbar}V_{ba}\int_0^t \cos\omega t e^{i\omega_o t'}dt' &= -\frac{iV_{ba}}{2\hbar}\int_0^t e^{i(\omega_o+\omega)t'} + e^{i(\omega_o-\omega)t'}dt' \\ &= \frac{V_{ba}}{2\hbar}\left[\frac{e^{i(\omega_o+\omega)t}-1}{\omega_o+\omega} + \frac{e^{i(\omega_o-\omega)t}-1}{\omega_o-\omega}\right]. \end{aligned}$$

Assuming that driving frequencies are very close to the transition frequency ω_0, that is, $|\omega_0+\omega| >> |\omega_0-\omega|$, the first term in the expression is dropped, and we are left with

$$\begin{aligned} c_b(t) &\approx -\frac{V_{ba}}{2\hbar}\frac{e^{i(\omega_o-\omega)t/2}}{\omega_o-\omega}[e^{i(\omega_o-\omega)t/2} - e^{-i(\omega_o-\omega)t/2}] \\ &= -\frac{iV_{ba}}{\hbar}\frac{\sin(\omega_0-\omega)t/2}{\omega_0-\omega}e^{i(\omega_o-\omega)t/2}. \end{aligned}$$

Taking its amplitude squared gives the probability of finding the particle in state ψ_b,

$$P_{a\to b} = |c_b|^2 = \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2(\omega_0-\omega)t/2}{(\omega_0-\omega)^2}. \tag{2.4}$$

The probability of transition oscillates sinusoidally with time; see figure 1. The maximum probability $\frac{|V_{ab}|^2}{\hbar^2(\omega_0-\omega)^2}$, is much less than 1, as the perturbation is small. At $t_n = \frac{2n\pi}{\omega_0-\omega}$, the probability of finding the particle in state ψ_b is zero; hence the particle is back in the lower state ψ_a. The transition probability is greatest when the driving frequency is close to the transition frequency, as illustrated in figure 2.

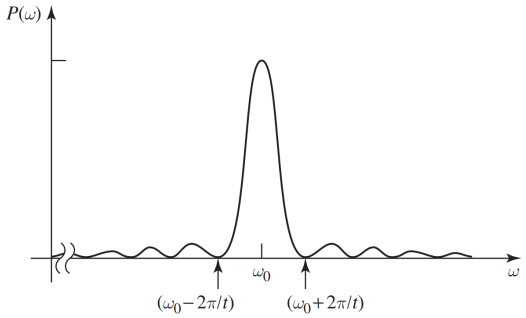

Figure 2. Transition probability as a function of ω_o [1].

3 Emission and Absorption of Radiation

Consider an atom exposed to electromagnetic radiation. For visible light, spatial variations can be ignored as the wavelength of light is much larger than the size of the atom. Interaction of the atoms with the magnetic fields is much weaker, so the perturbation is mainly due to the oscillating electric field.

$$\mathbf{E} = E_0 \cos \omega t \hat{k}.$$

Hamiltonian for an atom in the presence of electromagnetic fields polarized along the z-axis is,

$$\hat{H}' = -qE_0 z cos(\omega t),$$

where q is the charge on the electron. Assuming the nucleus is stationary and heavy, the wavefunction of electrons concerns us here.

$$\hat{H}'_{ba} = q\langle\psi_b|z|\psi_a\rangle E_0 z cos(\omega t),$$

$$\wp = q\langle\psi_b|z|\psi_a\rangle.$$

It represents the off-diagonal elements of the z-component of the dipole moment operator.

$$\hat{H}'_{ba} = \wp E_0 z cos(\omega t).$$

ψ_a is an even or odd function of z, and in both cases, $z|\psi_a|^2$ is odd in either case and integrates to zero, hence the diagonal matrix elements of $\hat{H}'$ vanish as discussed in section 2.1. So, this perturbation is sinusoidal, as already discussed in section 2.3.

$$V_{ba} = -\wp E_0. \tag{3.1}$$

3.1 Absorption, Stimulated Emission and Spontaneous Radiation

Using the transition probability given by (2.4) and inserting V_{ba} from (3.1), the transition probability becomes

$$P_{a\to b} = \left(\frac{|\wp|E_0}{\hbar}\right)^2 \frac{\sin^2\left[(\omega_0 - \omega)t/2\right]}{(\omega_0 - \omega)^2}. \tag{3.2}$$

Equation (3.2) gives the probability of transition for a particle that started in ψ_a to the upper state ψ_b This process is **absorption** in which a particle absorbs energy $E_b - E_a = \hbar\omega_0$ and transitions to ψ_b. If energy $\hbar\omega_0$ is provided to the atom that started in ψ_b, it transits to ψ_a and the transition probability is

$$P_{b\to a} = \left(\frac{|\wp|E_0}{\hbar}\right)^2 \frac{\sin^2(\omega_0-\omega)t/2}{(\omega_0-\omega)^2} \tag{3.3}$$

If the particle is in the upper state and exposed to electromagnetic radiation, it makes a transition to the lower state, and this process is called **stimulated emission**. Electromagnetic fields gain energy $\hbar\omega_0$. If you shine one photon, two photons come out. One is the original photon the atom is exposed to, and the other is the photon emitted due to the transition. This results in amplification. If there are many electrons in the upper state and they are hit with a photon, it results in a chain reaction resulting in an enormous number of photons. All the photons are emitted at virtually the same instant and same frequency. This is the principle behind Lasers. To achieve this, having more electrons in the upper state is essential. This is achieved through a process called **population inversion**. If it is not achieved, when you shine a photon on an atom, the probability that the result will be an emission or absorption is precisely the same. This will result in no amplification. There is another process called **spontaneous emission**. Without any applied electric field, the atom makes a transition from the upper state to the lower state. It is stimulated emission, in essence, since some radiations always stimulate the transition.

3.2 Incoherent Perturbations

In a system with electromagnetic waves having a range of frequencies, we can use the energy density of electromagnetic waves given by

$$u = \frac{\epsilon E_0^2}{2}.$$

to rewrite (3.3) as,

$$P_{b\to a} = \frac{2|\wp|^2 u}{\hbar^2\epsilon_0}\frac{sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}.$$

The energy density 'u' is not constant. Therefore we take its contribution of it as $\rho(\omega)d\omega$, which leads to the probability:

$$P_{b\to a} = \frac{2|\wp|^2}{\hbar^2\epsilon_0}\int_0^\infty \rho(\omega)\frac{sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}d\omega.$$

We know

$$\int_{-\infty}^{\infty}\frac{sin^2(x)}{x^2}dx = \pi,$$

Using this and the substitution $x = (\omega_0-\omega)t/2$, we get

$$P_{b\to a} = \frac{\pi}{\hbar^2\epsilon_0}|\wp|^2\rho(\omega_0)t.$$

The probability of a transition is directly proportional to time - now that we have a range of frequencies, the probabilities are non-oscillating, and the transition rate becomes time-independent!

$$R_{b\to a} = \frac{dP}{dt} = \frac{\pi}{\hbar^2\epsilon_0}|\wp|^2\rho(\omega_0). \tag{3.4}$$

To generalise the equation to include radiations coming from all directions, we take the average of polarizations from all directions and $|\wp|^2$ becomes $|\wp.\hat{\mathbf{n}}|^2$. In spherical coordinates,

$$\wp.\hat{\mathbf{n}} = \wp cos\theta.$$

$$|\wp.\hat{\mathbf{n}}|^2_{avg} = \frac{1}{4\pi}\int |\wp|^2 cos^2\theta sin\theta d\theta d\phi = \frac{1}{3}|\wp|^2.$$

Therefore, the transition rate (3.4) can now be generalised as

$$R_{b\to a} = \frac{\pi}{3\hbar^2\epsilon_0}|\wp|^2\rho(\omega_0) = B_{ba}\rho(\omega_0).$$

such that

$$B_{ba} = \frac{\pi}{3\hbar^2\epsilon_0}|\wp|^2. \tag{3.5}$$

B_{ba} is the rate of stimulated transitions per unit energy density from ψ_b to ψ_a.

3.3 Spontaneous Emission

We can now find the rate of stimulated emission using (3.5). To find a similar expression for the rate of spontaneous emission, we consider a system in dynamic equilibrium with N_a electrons in the state ψ_a and N_b electrons in the state ψ_b. Particles in ψ_b can either undergo spontaneous or stimulated emission: however, N_b remains constant. Hence,

$$\frac{dN_b}{dt} = B_{ab}N_a\rho(\omega_0) - AN_b - B_{ba}N_b\rho(\omega_0) = 0.$$

Rearranging this gives us,

$$\rho(\omega_0) = \frac{AN_b}{B_{ab}N_a - B_{ba}N_b} = \frac{A}{B_{ab}\frac{N_a}{N_b} - B_{ba}}. \tag{3.6}$$

Since the particles are at thermal equilibrium, their number is proportional to the **Boltzmann factor**. Therefore,

$$\frac{N_a}{N_b} = e^{(E_b - E_a)/k_BT} = e^{\hbar\omega_0/k_BT}.$$

Plugging this is in (3.6),

$$\rho(\omega_0) = \frac{A}{e^{\hbar\omega_0/k_BT}B_{ab} - B_{ba}}. \tag{3.7}$$

This formula must be equal to the expression for $\rho(\omega)$ given by Planck's blackbody formula:

$$\rho(\omega) = \frac{\hbar\omega_0^3}{\pi^2c^3}\frac{1}{e^{\hbar w/k_BT} - 1}. \tag{3.8}$$

Comparing the equation (3.6) and (3.8) makes it clear that B_{ab} must be equal to B_{ba}. Hence, (3.7) simplifies to

$$\rho(\omega) = \frac{A}{B_{ba}} \frac{1}{e^{\hbar\omega_0/k_B T} - 1}.$$

This implies that

$$\frac{A}{B_{ba}} = \frac{\hbar\omega_0^3}{\pi^2 c^3}.$$

We can now find A using (3.5),

$$A = \frac{\pi|\tilde{\wp}|^2}{3\epsilon_0\hbar^2} | \frac{\hbar\omega_0^3}{\pi^2 c^3}$$
$$A = \frac{\omega_0^3|\tilde{\wp}|^2}{3\epsilon_0\hbar\pi c^3}.$$

Therefore,

$$\frac{A}{B} = \frac{\hbar\omega_0^3}{\pi^2 c^3}.$$

A and B are known as **Einstein's A and B coefficients** where A is the rate of spontaneous transitions per unit energy density while B is the rate of stimulated transitions per unit energy density.
For a laser, we want the stimulated emission to exceed spontaneous emission (as discussed in section 3.1). As frequency increases, spontaneous emission increases, making it harder to make a laser and vice versa. This is why the first lasers (MASERS) were made using microwave radiation which has one of the lowest frequencies in the electromagnetic spectrum.

Lifetime of an Excited State

If we have a large number of atoms in the excited state ψ_b, they will move into ψ_a over time through spontaneous emission.

$$dN_b dt = -AN_b$$

$$N_b(t) = N_b(0)e^{-AN_b}$$

A can be said to be the loss factor and we can define the mean lifetime τ of a state as

$$\tau = \frac{1}{A} \tag{3.9}$$

Selection Rules

The matrix elements for the perturbation are given by

$$\langle n^{'} l^{'} m^{'} | \hat{\mathbf{r}} | nlm \rangle$$

We can define two raising and lowering operators as follows:

$$\hat{\mathbf{r}}\pm \equiv \hat{\mathbf{r}}_x \pm i\hat{\mathbf{r}}_y$$

We know $\hat{\mathbf{r}}$ follows the following commutation relation:

$$[\hat{\mathbf{L}}_i, \hat{\mathbf{r}}_j] = i\hbar\epsilon_{ijk}\hat{\mathbf{V}}_k. \tag{3.10}$$

Using this commutation relation,

$$[\hat{\mathbf{L}}_z, \hat{\mathbf{r}}_\pm] = \pm\hbar\hat{\mathbf{r}}_\pm.$$

and therefore

$$\begin{aligned}\pm\hbar\langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}_\pm}|nlm\rangle &= \langle n^{'}l^{'}m^{'}|\hat{\mathbf{L}}_z\hat{\mathbf{r}}_\pm|nlm\rangle - \langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_\pm\hat{\mathbf{L}}_z|nlm\rangle \\ &= \hbar m^{'}\langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_\pm|nlm\rangle - \hbar m\langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_\pm|nlm\rangle.\end{aligned}$$

Rearranging this gives us:

$$[m^{'} - (m \pm 1)]\langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_\pm|nlm\rangle = 0.$$

For the matrix elements to be non-zero, $m^{'} = m \pm 1$.
Now, again from (3.10),

$$[\hat{\mathbf{L}}_z, \hat{\mathbf{r}}_z] = 0$$

Hence,

$$\begin{aligned}\langle n^{'}l^{'}m^{'}|[\hat{\mathbf{L}}_z, \hat{\mathbf{r}}_z]|nlm\rangle &= 0 \\ \langle n^{'}l^{'}m^{'}|\hat{\mathbf{L}}_z\hat{\mathbf{r}}_z|nlm\rangle - \langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_z\hat{\mathbf{L}}_z|nlm\rangle &= 0 \\ \hbar m^{'}\langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_z|nlm\rangle - \hbar m\langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_z|nlm\rangle &= 0 \\ (m^{'} - m)\langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_z|nlm\rangle &= 0\end{aligned}$$

This result implies that for the matrix elements to be non-zero, $m^{'} = m$. Combining the two results, we obtain the selection rule for m:

$$\Delta m \equiv m^{'} - m = 0 \text{ or } \pm 1.$$

We can repeat this process of sandwiching the commutators between two states for the commutators of $\hat{\mathbf{r}}_z$ and $\hat{\mathbf{r}}_\pm$ with the raising and lowering operators of L. Doing this for $[\hat{\mathbf{L}}_\pm, \hat{\mathbf{r}}_\pm] = 0$, $[\hat{\mathbf{L}}_\pm, \hat{\mathbf{r}}_z] = \mp\hbar\hat{\mathbf{r}}_\pm$ and $[\hat{\mathbf{L}}_\pm, \hat{\mathbf{r}}_\mp] = \pm 2\hbar\hat{\mathbf{r}}_z$, we obtain the following equations:

$$\begin{aligned}\langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_+|nlm\rangle &= -\sqrt{2}C^{l\ 1\ l^{'}}_{m1m}\langle n^{'}l^{'}||r||nl\rangle \\ \langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_-|nlm\rangle &= \sqrt{2}C^{l\ 1\ l^{'}}_{m-1m}\langle n^{'}l^{'}||r||nl\rangle \\ \langle n^{'}l^{'}m^{'}|\hat{\mathbf{r}}_z|nl\rangle &= C^{l\ 1\ l^{'}}_{m0m}\langle n^{'}l^{'}||r||nl\rangle\end{aligned}$$

The coefficients $C^{j_1 j_2 J}_{m_1 m_2 M}$ are the **Clebsch-Gordan coefficients**. These are zero unless $M = m_1 + m_2$, and unless $J = j_1 + j_2, j_1 + j_2 - 1, ..., |j_1 - j_2|$. Therefore, the matrix elements are non zero only if $\Delta l \equiv l^{'} - l = 0$ or ± 1.

However, **Laporte's rule** says that the matrix elements are zero when $l^{'} + l$ is even - this is always the case when $\Delta l = 0$. Hence, we obtain the following selection rules for which the matrix elements are non zero:

$$\Delta l \equiv l^{'} - l = \pm 1.$$
$$\Delta m \equiv m^{'} - m = 0 \text{ or } \pm 1.$$

When these elements are zero, the transition is said to be **forbidden** as selection rules are not followed: such a state is called a **metastable** state [1]

Multi-Level Systems

In two-level systems, if there are more electrons in the state 1 (lower state), absorption dominates when photons shine on it. When electrons are pumped to the higher energy state 2, the two states become equally populated and reach equilibrium. Now, stimulated emission and absorption will occur simultaneously. Any further attempts to provide energy will result in equal emission and absorption rates. So, photons are neither emitted nor absorbed. This is called two-level **saturation**. Photons are just transmitted unchanged through this system. So, it is not possible to achieve population inversion in two-level systems.

To overcome this, **three-level systems** are used. Suppose the three levels are 1 (ground state), 3 and 2. At equilibrium $N_1 > N_3 > N_2$. Through pumping, electrons are excited to the higher energy state 3. Still, $N_1 > N_2$ means no population inversion between these two states. State 2 has a small lifetime (τ_2 is of the order nanoseconds), due to which electrons decay into state 3, which has a longer lifetime. Such a state is called a **metastable state**. Population inversion is achieved between states 3 and 1 due to spontaneous emission. When spontaneous emission occurs, a photon is emitted, stimulating the emission of other photons in state 3. This again stimulates the emission of photons, which in turn stimulates more photons and hence the number of photons emitted increases exponentially. There are now electrons populating state 3 as a result of decay in state 2, and hence a population inversion is achieved. An example of this is the Ruby laser. This process is illustrated in figure 3.

In **four-level systems** there are two metastable states. Say the states are 1, 2, 3 and 4. When pumped, electrons from the lower state 1 move to state 2, which has a very short lifetime (τ_2 is of the order nanoseconds). They rapidly transition to state 3 without emitting any radiation from this state. State 3 has a much longer lifetime, i.e, $\tau_1 >> \tau_2$. Now, there are no electrons in state 4 while state 3 is populated due to the decay of electrons from state 2. Hence, comparing 3 and 4, we see that population inversion is achieved easily. Now, when an electron from state 3 undergoes spontaneous emission, a cascade of photons is generated, as discussed previously. τ_4 is also much smaller, resulting in rapid decay of atoms to state 1. Now, electrons from state 1 are again pumped and moved to state 2. The process continues, resulting in a continuous laser. The three-level system resulted in pulsed radiation emission (requiring more pumping energy). Four-level systems also utilise

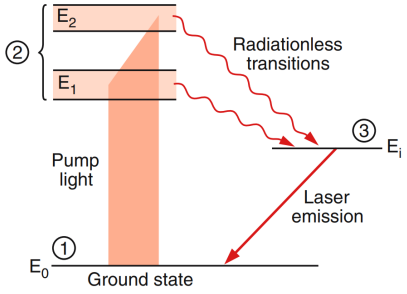

Figure 3. A three-level system (in Ruby Laser) [3]

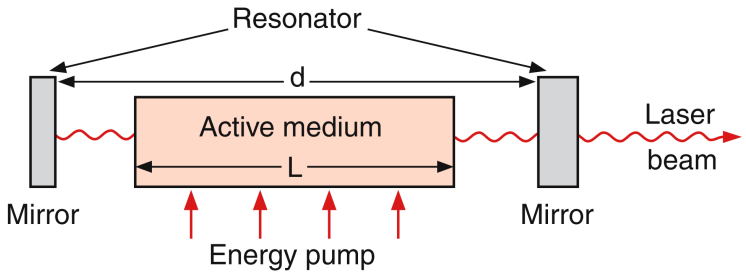

Figure 4. Basic illustration of a laser [3]

lower pump energy, making it easier to achieve population inversion between states 3 and 4. An example of such a four-level system is a He-Ne laser [2].

4 Physical Principle Behind Lasers

Components of Lasers

1. **Optical Resonator**

   It is the cavity in the laser between the mirrors. It stores the fluorescence emitted in a few modes of the radiation field. The modes of electromagnetic radiation within the resonator cavity dictate the resonant frequencies at which the cavity will resonate.

2. **Active Medium**

   It is the light amplifier placed in the resonator. For example, He-Ne is the active medium in the He-Ne laser. Emitted photons travel through this medium, are reflected by mirrors, and pass through it repeatedly.

3. **Energy Pump**

   Supplies the energy for the generation of population inversion.

   A schematic setup of lasers is shown in figure 4.

4.1 Gain Coefficient

Placing the active medium between two mirrors of cross-sectional area A, we can send light through the medium to amplify it. However, there are some energy losses due to energy absorbed by the medium.

Let the intensity of light upon entering the medium be $I(z)$. After traversing a distance dz through the medium, the intensity becomes $I(z+dz)$. Assuming dz is small enough, we can use Taylor approximation to write

$$I(z+dz) = \frac{\partial I}{\partial z}dz + I(z).$$

The energy absorbed and emitted will be

$$E_{abs} = B\rho(\omega).$$
$$E_{emt} = B\rho(\omega)N_b A dz.$$

Hence, the net energy emitted is given by

$$E = B\rho(\omega)A dz \Delta N. \tag{4.1}$$

where

$$\Delta N = N_b - N_a.$$

The net energy emitted is related to the change in intensity of the light by

$$E = \frac{\partial I}{\partial z}dzA. \tag{4.2}$$

combining equation (4.1) and (4.2), we get

$$\frac{\partial I}{\partial z} = B\rho(\omega)\Delta N. \tag{4.3}$$

We can write the intensity in terms of ρ as

$$I = \rho(\omega)v. \tag{4.4}$$

The speed of light in the cavity is given by

$$v = \frac{c}{n} \tag{4.5}$$

where n is the refractive index of the cavity. Hence, we can rewrite (4.3) using and (4.4) as

$$\begin{aligned}\frac{\partial I}{\partial z} &= \frac{n\Delta N\pi^2 c^2}{\hbar\omega^3\tau}I(z).\\ &= G(\omega)I(z).\end{aligned} \tag{4.6}$$

where

$$G(\omega) = \frac{n\Delta N\pi^2 c^2}{\hbar\omega^3\tau}. \tag{4.7}$$

This constant is called the **gain coefficient**. Solving the differential equation in (4.6), we find

$$I(z) = I_0 e^{G(\omega)z}. \tag{4.8}$$

This equation makes it clear that for a laser to function, we want the gain coefficient to be positive, and hence, from (4.7), ΔN must be positive i.e. $N_b > N_a$. ΔN, and hence G, increase as pump power increases, but up to a threshold, after which they become constant. Light is emitted only after the threshold value of G is hit. Now, G becomes constant and further increase in pumping power causes the intensity of emission to increase [2, 3].

4.2 Accounting for Losses

Losses can occur due to multiple reasons. When light hits the mirror, all light is not reflected (1-R) of the light is transmitted through the mirror, where R is the reflectivity of the mirror. We can define a loss factor γ that to include all such losses such that the loss after one round-trip is given by

$$I(z) = I_0 e^{-\gamma}. \tag{4.9}$$

Taking (4.9) into account, (4.8) becomes

$$I(z) = I_0 e^{-(\gamma + G(\omega)z)}$$

For amplification to occur, i.e. $I(z) > I_0$, we need $\gamma + G(\omega)z < 0$. Using (4.7),

$$\gamma + z\frac{n\Delta N\pi^2 c^2}{\hbar\omega^3\tau} < 0$$
$$\Delta N < -\gamma\frac{\hbar\omega^3\tau}{zn\pi^2c^2}$$

Therefore, for light to be amplified for each round-trip we want,

$$\Delta N > \gamma\frac{\hbar\omega^3\tau}{zn\pi^2c^2} = \Delta N_{thr},$$

where ΔN_{thr} is the minimum difference between N_b and N_a required for amplification to occur.

The laser oscillation is then built up as discussed earlier in section 3.1.

4.3 Frequency Spectrum of Emission

Only precise frequencies can be excited in the cavity of a resonator of length 'L' as the nodes must exist at the ends of the resonator:

$$L = m\frac{\lambda}{2},\ m \in \mathbb{N}$$
$$\text{from (4.5)},\ = m\frac{c}{2nf}$$
$$= m\frac{c\pi}{n\omega}$$

This can be rewritten as

$$\omega = m\frac{c\pi}{nL},\ m \in \mathbb{N}$$

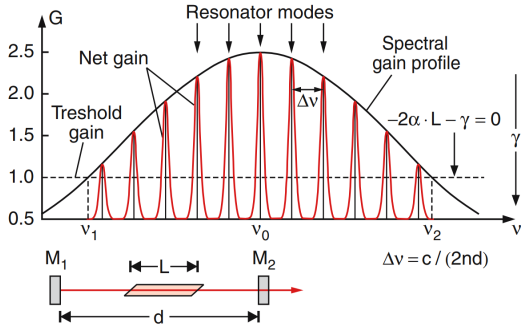

Figure 5. Net gain G(ν) for resonator modes within the gain profile of the active medium. [3]

Clearly, for a cavity of length 'L' only a certain set of frequencies can be emitted: there is finite distance between allowed frequencies of $\Delta\omega = \frac{\pi c}{nL}$.

From our Quantum mechanical discussion in 3.1, we already know that the energy difference in the population inversion between the excited and ground state determines the frequency of emitted photons - this frequency is fixed. For a laser to work we must make sure both conditions are fulfilled. However, it is important to recognise that while we know the precise frequency, a range of frequencies is still emitted due to the uncertainty principle - we never get a sharp spectral line but the graph is centred at w_0 and has a certain bandwidth as seen in figure 5. Along with the uncertainty principle, this range of frequencies can be accounted for by collisional broadening and doppler broadening (due to doppler shift since some atoms moving away and some towards the detector). Such a laser contains many modes and hence is called a **multi-mode laser**.

5 Optical Resonators

As the wavelength decreases, the number of modes within a doppler broadened spectral line increases. This results in much fewer photons per mode as the spontaneous emission inside the closed cavity is spread over multiple modes. The laser emission is hence also in multiple directions, resulting in the directionality of a laser being lost. To achieve oscillation on all the modes a very big amount of pumping power is needed which is why closed cavities are not suitable for optical resonators.

We consider the kth resonator mode with energy $E_k(t)$. Assuming there are no external inputs of energy, we can write a first order differential equation to get the time dependent stored energy:

$$\begin{aligned} \frac{dE_k}{dt} &= -\beta_k \cdot E_k, \\ E_k(t) &= E_k(0) \cdot e^{-\beta_k t}. \end{aligned} \tag{5.1}$$

Here B_k is the loss factor and can be interpreted as the average lifetime of a photon in the kth resonator mode. The loss factor per round-trip is hence

$$\gamma_k = \beta_k \cdot \frac{2d}{c}.$$

Now, we define a **quality factor** Q_k as

$$Q_k = -\frac{2\pi v \cdot W_k}{dW_k/dt}$$

using 5.1 we rewrite this as

$$Q_k = -\frac{2\pi v}{\beta_k}$$

We want emissions to be over just a few modes: i.e. Q-factor should be large and the losses should be small. This can be achieved using open resonators.

5.1 Open Optical Resonators

We use open optical resonators to concentrate induced emission onto just a few modes. Any suitable arrangement of optical mirrors can fulfill this condition. An example would be of an open optical resonator that contains two parallel optical mirrors with reflecivity R_1 and R_2 and diameter $2a$, separated by a distance d such that $d >> 2a$. Light travels back and forth between the mirrors, which reflect it. Due to reflection losses, the intensity per round trip decreases per round-trip as

$$I(2d) = I_0 R_1 R_2 = I_0 e^{-\gamma_r} \tag{5.2}$$

$$\text{where } \gamma_r = \ln(R_1 R_2) \tag{5.3}$$

Now, the mean lifetime of a photon τ is given by

$$\tau = \frac{1}{\gamma_r} T = \frac{2d}{c \cdot ln(R_1 R_2)} \tag{5.4}$$

5.2 Modes of Open Resonators

The modes of open resonators cannot be treated as a superposition of plane waves because their amplitude and phase is changing since there is a curvature on the wavefronts due to diffraction. Usually, circular mirrors are used, and the active medium has a circular cross-section. This cylindrical symmetry makes it convenient to use cylindrical coordinates, and we can use Kirchoff's diffraction theory and figure 6 to find the amplitude distribution across the n_{th} aperture,

$$A_n(x,y) = -\frac{i}{2\lambda} \int_{x'} \int_{y'} A_{n-1}(x', y') \frac{1}{\varrho} e^{-ik\varrho} \times (1 + cos\vartheta) dx' dy' \tag{5.5}$$

Analytical solutions for (5.5) are only available for very special cases and are called transverse electromagnetic (***TEM***) modes of an open resonator. $TEM_{x,y,z}$ where x,y,z are the number of nodes in the standing waves in respective directions. TEM modes with x = y = 0 are called **fundamental modes**.

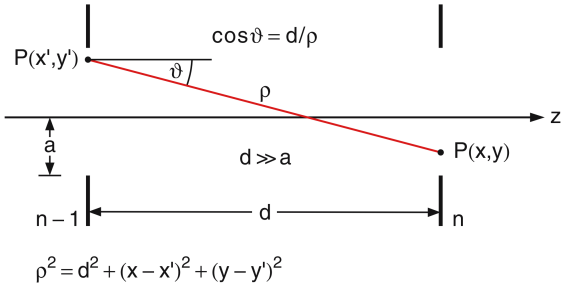

Figure 6. The n-1 and nth aperture of an open mode resonator. [3]

5.3 Single Mode Resonators

There are several ways of obtaining single-mode lasers. One of the ways is to shorten the resonator length L. When it reaches a value where the spacing between the modes becomes larger than half the spectral width of the gain profile at threshold time, laser oscillation at a single mode is achieved.

Another way is to insert additional frequency selective optical instruments into the resonators. Such an element can be a tiltable glass plate with reflecting surfaces of reflectivity R. It is called a Febry-Perotetalon. Transmission through it is given by:

$$T = \frac{1}{1 - F \sin^2 \delta/2} \tag{5.6}$$

where F is,

$$F = \frac{4R}{(1-R)^2}$$

and δ is the phase shift given by

$$\delta = 2\pi \Delta s/\lambda \tag{5.7}$$

This phase shift between two interfering laser beams with incidence angle α normal to the glass plate is given by,

$$\Delta s = 2t\sqrt{n^2 - \sin^2 \alpha}$$

From equation (5.6), it can been seen that $T = 1$ when $\delta = 2m\pi$ ($m \in \mathbb{N}$). From equation (5.7), $\lambda_m = \Delta s/m$, and all the frequencies $\nu_m = cm/\Delta s$ fulfill this condition. So by adjusting α, one of the frequencies can coincide with the allowed resonator oscillation frequency inside the gain profile. T is small for all the other frequencies, and hence, due to high reflectivity, the losses are larger than the gain and do not reach the oscillation threshold. Laser oscillates at a single mode if transverse modes are removed through high diffraction losses. This frequency, as it is equal to resonator eigenfrequency, is given by,

$$\nu_l = \nu_r = qc/2nd \tag{5.8}$$

where q is the mode number and n is the refractive index. Due to the inherent uncertainty in the frequency of emitted light, frequencies fluctuate. The mean spectral width of frequency

varies due to the fluctuation in refractive index or Δd of the length of resonators.

$$-\frac{\Delta\nu_L}{\nu_L} = \frac{\Delta n}{n} + \frac{\Delta d}{d} \tag{5.9}$$

These fluctuations can be limited by taking some measures. If one of the resonator mirrors is mounted on a piezocrystal. It changes its length when an external voltage is applied. If some part of the laser output is sent through Fabry Perot Interferometer (also called etalon), it reflects the change in frequency by changing the transmitted intensity. A photodiode gives the voltage reflecting this intensity change. It is then used to control the laser frequency. It can lower the bandwidth up to 1Hz. It can be further lowered by using more accurate and sophisticated mechanisms. The lower limit is given by the Schawlow-Townes formula:

$$\Delta\nu_l = \frac{\pi h\nu_L}{P_L}.\Delta(\nu_r)^2 \tag{5.10}$$

$\Delta\nu_r$ is resonator resonance for an empty resonator and P_l is the power output of the laser [2].

Acknowledgments

We would like to acknowledge the efforts of our TA here for his constant guidance. His constructive feedback helped us in improving the report maximally. We would also like to thank Dr. Rizwan Khalid whose lectures on Quantum Mechanics helped us a lot during our project.

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