Gravitational Waves from Compact Binaries: Foundations and Waveform Construction

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Abstract

Gravitational waves provide a unique lens into the universe, revealing the dynamics of systems like binary black holes and neutron stars. This thesis begins with a primer on gravitational radiation, exploring linearized gravity, gravitational wave solutions, and the mechanisms of wave production. We examine energy loss due to the emission of gravitational radiation and its impact on the orbital period of binary pulsars.

We then delve into the Post-Newtonian (PN) formalism, highlighting its role in constructing analytic waveforms for binary inspirals. By balancing accuracy and computational efficiency, the PN approach aids gravitational wave detection, parameter estimation, and tests of general relativity in strong-field regimes. Finally, we explore spin-precessing binaries, presenting analytic solutions to spin and angular momentum evolution equations in the absence of radiation reaction. These systems exhibit rich waveform structures and are critical for improving waveform modeling and breaking parameter degeneracies.

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To Ammi, Baba, Hassan and Hussain

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Introduction

Gravitational waves (GWs) are ripples in the fabric of spacetime generated by dynamic mass distributions. First predicted by Einstein in 1916 as a consequence of general relativity, their indirect effects were first observed in the 1970s through the orbital decay of the Hulse–Taylor binary pulsar[11]. However, it was not until 2015 that the Laser Interferometer Gravitational-Wave Observatory (LIGO) made the first direct detection of gravitational waves from a binary black hole merger[2]. This event opened a new era of observational astronomy, allowing us to probe the universe's most extreme relativistic phenomena.

Unlike electromagnetic radiation, GWs propagate through spacetime without significant interaction with matter. This allows them to carry crucial information about the systems that generate them—systems often electromagnetically dark, obscured, or at cosmological distances. Among the most prolific sources of GWs are compact binary systems, pairs of neutron stars (BNS), black holes (BBH), or neutron star black hole binaries (NSBH) locked in tight inspirals. The detection of gravitational waves from these systems has made it possible to measure fundamental properties such as the masses and spins of compact objects and has offered an unprecedented opportunity to test general relativity in the strong-field regime.

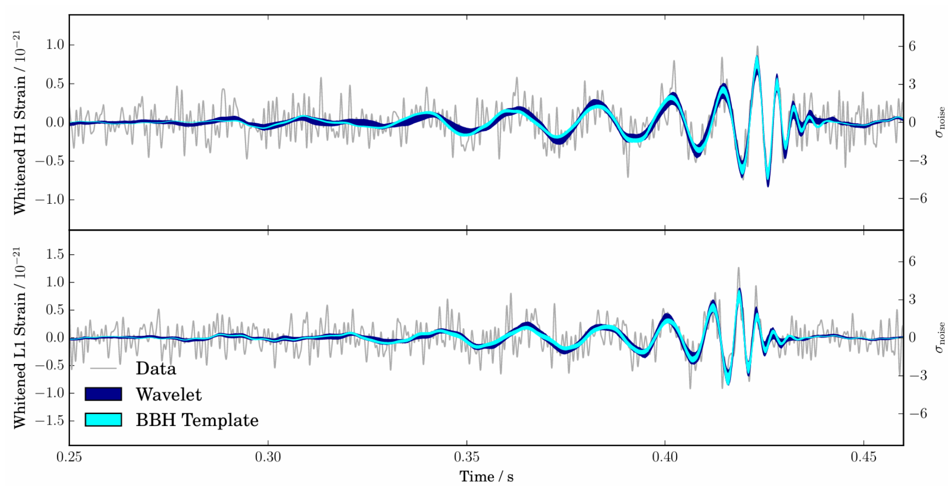

Figure 1: The first gravitational wave event GW150914 detected by LIGO overlaid with numerical relativity waveforms[3].

Modeling the gravitational radiation from compact binaries is essential for detecting and interpreting GW signals. Accurate waveform templates are used in matched filtering to extract weak signals from noisy data. These waveforms must capture binaries' orbital dynamics and radiation emission with high precision. While numerical relativity provides solutions to the full Einstein equations and is used to construct these waveforms, it is computationally expensive. This is where the post-Newtonian (PN) framework becomes invaluable.

The PN formalism expands Einstein's equations perturbatively in powers of velocity, allowing one to build accurate analytic waveforms for the inspiral phase of compact binaries. This framework is particularly useful for constructing models of quasicircular binaries and has also been extended to systems with spin-precession. Although PN waveforms carry approximations and break down near merger, they remain computationally efficient and interpretable, making them ideal for large-scale template banks and parameter estimation.

In chapter 1, we begin by reviewing the theoretical underpinnings of gravitational wave physics. Starting with linearized gravity, we introduce gauge transformations and gravitational waves' physical degrees of freedom. This leads to a derivation of plane wave solutions, a derivation of the quadrupole formula that describes the gravitational radiation. We then derive the quadrupole formula for gravitational wave emission by compact sources and study the rate of energy loss from binaries. The chapter concludes with an overview of interferometric detection techniques and a brief note on possible sources, including gravitational wave backgrounds and primordial signals.

Chapters 2-4 focus on the dynamics and gravitational radiation from compact binaries. Chapter 2 starts by modeling the inspiral of a binary in the Newtonian approximation, calculating waveforms, orbital evolution, and energy loss. Chapter 3 then provides a systematic review of the PN formalism, including its assumptions, derivations, and treatment of divergences in the presence of point particles. We discuss the construction of the multipole moments and flux expressions required for waveform generation.

Chapter 4 presents the construction of the PN waveform. We calculate the energy and flux up to 3.5PN order and derive the TaylorF2 waveform in the Fourier domain. This waveform family is widely used for its simplicity and low computational cost. We then study TaylorF2's behavior across BNS, NSBH, and BBH systems, plotting the resulting waveforms and exploring their amplitude and phase evolution. Though efficient, TaylorF2 has limitations for high-mass or highly asymmetric systems. We discuss these constraints in detail and summarize overlaps with numerical waveforms[6].

Spin-precession introduces rich structure into waveforms, especially when spin angular momentum is misaligned with the orbital plane. Chapter 5 explores the evolution of spin and orbital angular momentum vectors in such systems. We present an analytic solution to the precession equations without radiation reaction. Spin-precessing systems are an active area of research, and this chapter lays the foundation for further extensions, including radiation reaction and full modeling.

Throughout this thesis, we aim to understand the physical mechanisms that shape gravitational waveforms and to construct efficient, analytic models that balance interpretability, accuracy, and computational efficiency. Compact binaries remain one of the most powerful tools for testing general relativity, constraining nuclear equations of state, and probing the astrophysical history of black holes and neutron stars.

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A Primer on Gravitational Radiation[7,11]

This chapter delves into the theoretical foundations, sources, and observational implications of gravitational radiation. We begin by exploring how gravitational waves arise in general relativity. Terms like *dynamic mass distributions* and *ripples in spacetime* are often used to describe these phenomena, and here, we develop the mathematical framework behind such descriptions. By expanding the Minkowski metric with a small perturbation, we glimpse the rich structure of gravitational waves and their far-reaching consequences, laying the groundwork for more advanced topics in this field.

In Section 1.1, we introduce the framework of linearized gravity by perturbing the Minkowski metric and studying how such perturbations transform under coordinate changes. Then, in Section 1.2, we analyze the degrees of freedom of the perturbed metric, identify the physical components of the gravitational field, and introduce gauge choices, analogous to those in electromagnetism, that help simplify calculations.

Section 1.3 focuses on solving the Einstein equations in the linearized regime to obtain gravitational wave solutions, including a discussion of their polarization states. We then apply these solutions to compact astrophysical sources and derive the metric for radiation from binary systems in Section 1.4. The energy carried by gravitational waves causes a gradual loss of orbital energy, which is discussed in Section 1.5. This effect was historically crucial; observations of binary pulsars provided the first indirect evidence of gravitational waves. We briefly touch on this in the context of binary systems.

Finally, Section 1.6 briefly overviews how gravitational waves are detected using interferometric techniques.

This chapter largely follows Sean Carroll's book *Spacetime and Geometry* and his lecture notes[7].

1.1 Linearized Gravity and Gauge Transformations

When discussing the weak-field limit of Einstein's equations in the Newtonian approximation, we assume that the gravitational field is weak and sourced by static (or slowly moving) matter. To move beyond this, we continue to consider a weak field but now allow it to vary with time. This relaxation enables the study of dynamical spacetimes, particularly gravitational radiation. Importantly, we place no restriction on the motion of test particles, allowing us to investigate wave-like solutions of the field equations within the linear regime.

If the gravitational field is weak, the metric can be decomposed as a small perturbation around the Minkowski background:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $|h_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. We will linearize the Einstein field equations by expanding them to first order in $h_{\mu\nu}$, which is why this framework is called linearized gravity. The inverse metric is:

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}.$$

We raise and lower indices using the Minkowski metric, as the corrections will be in higher order.

We aim to derive the equations of motion satisfied by the perturbation $h_{\mu\nu}$. To do so, we expand the Einstein tensor to linear order. This requires computing the Christoffel symbols and the Riemann tensor. We begin with the Christoffel symbols:

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\ &= \frac{1}{2} \eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}).\end{aligned}$$

The Riemannian curvature tensor is given by

$$\begin{aligned} R^{\lambda}_{\nu\rho\sigma} &= \partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \partial_\sigma \Gamma^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\rho\mu}\Gamma^{\mu}_{\sigma\nu} - \Gamma^{\lambda}_{\sigma\mu}\Gamma^{\mu}_{\rho\nu} \\ \eta_{\mu\lambda} R^{\lambda}_{\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma \Gamma^{\lambda}_{\nu\rho}, \end{aligned}$$

where in the second equation we lowered an index for convenience. We have dropped the terms of order Γ^2 because they are of second order in perturbation. In this expression, we replace the Christoffel symbols

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho \left[\frac{1}{2}\eta^{\lambda\alpha}(\partial_\nu h_{\sigma\alpha} + \partial_\sigma h_{\alpha\nu} - \partial_\alpha h_{\nu\sigma})\right] - \eta_{\mu\lambda}\partial_\sigma \left[\frac{1}{2}\eta^{\lambda\alpha}(\partial_\nu h_{\rho\alpha} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho})\right] \\ &= \frac{1}{2}\delta^{\alpha}_{\mu}\partial_\rho \left[(\partial_\nu h_{\sigma\alpha} + \partial_\sigma h_{\alpha\nu} - \partial_\alpha h_{\nu\sigma})\right] - \frac{1}{2}\delta^{\alpha}_{\mu}\partial_\sigma \left[(\partial_\nu h_{\rho\alpha} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho})\right] \\ &= \frac{1}{2}\partial_\rho \left[(\partial_\nu h_{\sigma\mu} + \partial_\sigma h_{\mu\nu} - \partial_\mu h_{\nu\sigma})\right] - \frac{1}{2}\partial_\sigma \left[(\partial_\nu h_{\rho\mu} + \partial_\rho h_{\mu\nu} - \partial_\mu h_{\nu\rho})\right] \\ &= \frac{1}{2}(\partial_\rho\partial_\nu h_{\sigma\mu} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\rho\mu} + \partial_\sigma\partial_\mu h_{\nu\rho}). \end{aligned} \tag{1.1}$$

Contracting the curvature tensor using the Minkowski metric, we get the Ricci tensor:

$$\begin{aligned} \eta^{\rho\mu} R_{\mu\nu\rho\sigma} &= \frac{1}{2}\eta^{\rho\mu}(\partial_\rho\partial_\nu h_{\sigma\mu} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\rho\mu} + \partial_\sigma\partial_\mu h_{\nu\rho}) \\ R_{\nu\sigma} &= \frac{1}{2}(\partial_\rho\partial_\nu h^{\rho}_{\sigma} - \partial_\rho\partial^\rho h_{\nu\sigma} - \partial_\sigma\partial_\nu h^{\rho}_{\rho} + \partial_\sigma\partial_\mu h^{\mu}_{\nu}) \\ &= \frac{1}{2}(\partial_\rho\partial_\nu h^{\rho}_{\sigma} + \partial_\sigma\partial_\mu h^{\mu}_{\nu} - \Box h_{\nu\sigma} - \partial_\sigma\partial_\nu h), \end{aligned}$$

where $h^{\rho}_{\rho} = h$ and $\partial_\rho\partial^\rho = \Box$.

Taking trace, we get the Ricci scalar:

$$
\begin{aligned}
\eta^{\nu\sigma} R_{\nu\sigma} &= \frac{1}{2}\eta^{\nu\sigma}(\partial_\rho \partial_\nu h^\rho_\sigma + \partial_\sigma \partial_\mu h^\mu_\nu - \Box h_{\nu\sigma} - \partial_\sigma \partial_\nu h) \\
R &= \frac{1}{2}(\partial_\rho \partial_\nu h^{\rho\nu} + \partial_\sigma \partial_\mu h^{\mu\sigma} - \Box h - \Box h) \\
&= \partial_\rho \partial_\nu h^{\rho\nu} - \Box h.
\end{aligned}
$$

Finally, the Einstein tensor takes the form

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \\
&= \frac{1}{2}(\partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho \partial_\lambda h^{\rho\lambda} - \eta_{\mu\nu}\Box h).
\end{aligned} \tag{1.2}
$$

The Einstein field equations are

$$
G_{\mu\nu} = 8\pi G T_{\mu\nu},
$$

where $T_{\mu\nu}$ is the energy-momentum tensor. In the weak-field limit, where $|h_{\mu\nu}| \ll 1$, the stress-energy tensor must also be small. We therefore consider only the leading order (zeroth order in $h_{\mu\nu}$) contribution to $T_{\mu\nu}$, and ignore higher order corrections. In particular, we will focus on vacuum spacetimes, where $T_{\mu\nu} = 0$, simplifying the field equations to

$$
G_{\mu\nu} = 0.
$$

We now have the linearized field equation in a form that we can solve. But before that, note that there might be multiple spacetimes where the metric can be written as the Minkowski metric plus a perturbation, with different perturbation terms. So, the decomposition

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is not unique. In fact, one can always perform a coordinate transformation that changes the form of the perturbation without altering the underlying physics. This gauge freedom introduces ambiguity when solving Einstein's equation. Because the perturbation is not uniquely defined, an infinitesimal change in it changes its form but not the physics we are interested in. The perturbation will have components that are not physical but just artefacts of coordinate choice. This results in undetermined equations because different gauge choices explain the same physical phenomena. To eliminate this ambiguity, we need to manage gauge freedom to isolate the physical content described by the equations.

To solve this issue of gauge invariance, we impose additional conditions that restrict our coordinate choices by choosing a reference frame. Selecting a reference frame breaks the gauge invariance of relativity under general coordinate transformations, but it is necessary to understand the physical content of a field theory. Linearized theory can be thought of as one that governs the behaviour of tensor fields on a flat background. Consider a background spacetime M_b and a physical spacetime M_p such that a diffeomorphism $\phi : M_b \to M_p$ exists between them. The two spacetimes have different tensor fields defined on them. On M_b, we have defined the Minkowski metric, and on M_p, we have some arbitrary metric $g_{\alpha\beta}$ satisfying the Einstein equation. Since a map exists between the two manifolds, we can move tensors back and forth between the two. This setup is illustrated in Figure 1.1. Our linearized theory should take place on M_b, so we are interested in pulling back $(\phi^* g)_{\mu\nu}$ of the physical metric. The perturbation can be defined as:

$$h_{\mu\nu} = (\phi^* g)_{\mu\nu} - \eta_{\mu\nu}.$$

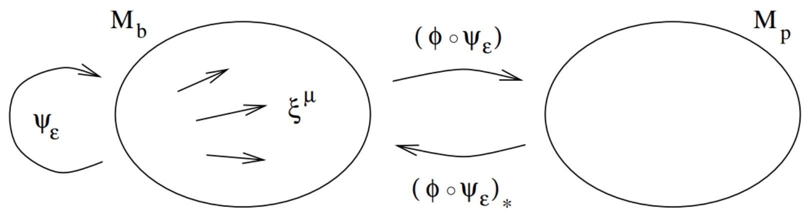


Figure 1.1: Diffeomorphisms Ψ_ϵ generated by vector field ξ^μ on a background spacetime M_b[7].

If the gravitational fields on M_p are weak, then for some ϕ, the perturbation will be small $|h_{\mu\nu}| \ll 1$. So, we focus only on such diffeomorphisms. From this, it can also be seen that $h_{\mu\nu}$ will obey the linearized Einstein equation on M_b.

Consider a vector field $\xi^\mu(x)$ on M_b. This generates diffeomorphism $\psi_\epsilon : M_b \to M_b$. For very small ϵ, $\phi \circ \psi_\epsilon$ will be very small. So, one can define several perturbations parametrized by ϵ:

$$\begin{aligned} h^{(\epsilon)}_{\mu\nu} &= [(\phi \circ \psi_\epsilon)^* g]_{\mu\nu} - \eta_{\mu\nu} \\ &= [\psi^*_\epsilon(\phi^* g)]_{\mu\nu} - \eta_{\mu\nu}. \end{aligned}$$

Plugging in the previous relation:

$$\begin{aligned} h^{(\epsilon)}_{\mu\nu} &= \psi^*_\epsilon(h+\eta)_{\mu\nu} - \eta_{\mu\nu} \\ &= \psi^*_\epsilon(h_{\mu\nu}) + \psi^*_\epsilon(\eta_{\mu\nu}) - \eta_{\mu\nu} \\ &= \psi^*_\epsilon(h_{\mu\nu}) + \epsilon\left[\frac{\psi^*_\epsilon(\eta_{\mu\nu}) - \eta_{\mu\nu}}{\epsilon}\right] \\ &= h_{\mu\nu} + \epsilon\mathcal{L}_\zeta \eta_{\mu\nu}, \end{aligned}$$

where the first term was expanded to the lowest order, and the second term gave us the metric's Lie derivative along the vector field ξ_μ. The Lie derivative can be written as $\mathcal{L}_\zeta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$.

$$h^{(\epsilon)}_{\mu\nu} = h_{\mu\nu} + 2\epsilon\partial_{(\mu}\xi_{\nu)}. \tag{1.3}$$

This is called a gauge transformation in linearized theory. It represents all transformations that satisfy the condition that the perturbation must be small. These metric perturbations denote physically equivalent spacetimes under which our linearized theory is invariant. To see this, we find that under the transformation defined by Eq. (1.3), the Riemannian tensor varies as:

$$\begin{aligned}\delta R_{\mu\nu\rho\sigma} =& \frac{1}{2}\left(\partial_\rho\partial_\nu\partial_\mu\xi_\sigma + \partial_\rho\partial_\nu\partial_\sigma\xi_\mu + \partial_\sigma\partial_\mu\partial_\nu\xi_\rho + \partial_\sigma\partial_\mu\partial_\rho\xi_\nu\right.\\ &\left.-\partial_\sigma\partial_\nu\partial_\mu\xi_\rho - \partial_\sigma\partial_\nu\partial_\rho\xi_\mu - \partial_\rho\partial_\mu\partial_\nu\xi_\sigma - \partial_\rho\partial_\mu\partial_\sigma\xi_\nu\right)\\ =& \, 0.\end{aligned}$$

So, the transformations leave the Riemannian tensor and, consequently, Einstein's equations invariant. The gauge transformations do not change the functional form of observables; this is termed gauge invariance.

1.2 Analyzing the Degrees of Freedom

We could go on to solve Einstein's equation, but first, we will look for further physical insights. Therefore, we choose a fixed inertial coordinate in the background Minkowski spacetime and decompose the components of the metric perturbation according to their transformation properties under spatial rotation.

$h_{\mu\nu}$ is a symmetric $(0,2)$ tensor. Under spatial rotations, the 00 component of $h_{\mu\nu}$ is scalar, the components $0i$ and $i0$ are equal and form a three-vector:

$$h_{00} = -2\Phi,$$
$$h_{0i} = w_i.$$

ij components form a spatial tensor. This symmetric part can be broken down into a trace and a traceless part, which are irreducible representations of the rotation group. These irreducible representations transform independently of each other under spatial rotations. The decomposition is written as

$$h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$$

where Ψ contains the trace of h_{ij} and s_{ij} is traceless:

$$\Psi = -\frac{1}{6}\delta^{ij}h_{ij}$$
$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right) = \frac{1}{2}\left(h_{ij} - \frac{1}{3}h^l_l\delta_{ij}\right)$$

In terms of these components, the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is written as:

$$ds^2 = -(1+2\Phi)dt^2 + w_i(dtdx^i + dx^i dt) + \left[(1-2\Psi)\delta_{ij} + 2s_{ij}\right]dx^i dx^j. \quad (1.4)$$

This is just a convenient notation.

To understand the physical interpretation of the fields appearing in the metric, we consider the motion of test particles given by the geodesic equation. For the geodesic equation, we first find Christoffel symbols for the metric:

$$
\begin{aligned}
\Gamma^0_{00} &= \partial_0 \Phi, \\
\Gamma^i_{00} &= \partial_i \Phi + \partial_0 w_i, \\
\Gamma^0_{j0} &= \partial_j \Phi, \\
\Gamma^i_{j0} &= \partial_{[j} w_{i]} + \frac{1}{2}\partial_0 h_{ij}, \\
\Gamma^0_{jk} &= -\partial_{(j} w_{k)} + \frac{1}{2}\partial_0 h_{jk}, \\
\Gamma^i_{jk} &= \partial_{(j} h_{k)i} - \frac{1}{2}\partial_i h_{jk}.
\end{aligned} \tag{1.5}
$$

Here, parentheses denote symmetrization and square brackets denote antisymmetrization of indices. In these expressions, we have used the symmetric tensor h_{ij} rather than s_{ij}, the traceless part and trace Ψ, because they appeared in the form $h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$, making it convenient to use this form. The distinction will become important once we take traces to get to the Ricci tensor and Einstein's equation.

By decomposing the metric under rotation, we have fixed the inertial frame. So, it is convenient to express the four-momentum $p^\mu = dx^\mu/d\lambda$ (where $\lambda = \tau/m$ if the particle is massive) in terms of the energy E and three-velocity $v^i = dx^i/dt$, as

$$
p^0 = \frac{dt}{d\lambda} = E, \quad p^i = Ev^i.
$$

The geodesic equation in terms of four-momentum becomes:

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma = 0.$$

Dividing it by E:

$$\begin{aligned}\frac{dp^\mu}{Ed\lambda} &= -\frac{\Gamma^\mu_{\rho\sigma} p^\rho p^\sigma}{E} \\ \frac{dp^\mu}{dt} &= -\Gamma^\mu_{\rho\sigma} \frac{p^\rho p^\sigma}{E}.\end{aligned}$$

Writing out the $\mu = 0$ component, which gives the evolution of energy,

$$\begin{aligned}\frac{dE}{dt} &= -\Gamma^0_{\rho\sigma} \frac{p^\rho p^\sigma}{E} \\ &= -\Gamma^0_{00} \frac{p^0 p^0}{E} - 2\Gamma^0_{j0} \frac{p^j p^0}{E} - \Gamma^0_{jk} \frac{p^j p^k}{E} \\ &= -\partial_0 \Phi E - 2\partial_j \Phi E v^j + (\partial_{(j} w_{k)} - \frac{1}{2}\partial_0 h_{jk}) E v^j v^k \\ &= -E \left[\partial_0 \Phi + 2\partial_j \Phi v^j - \left(\partial_{(j} w_{k)} - \frac{1}{2}\partial_0 h_{jk}\right) v^j v^k\right].\end{aligned}$$

The spatial components of the geodesic equation are:

$$\begin{aligned}\frac{dp^i}{dt} &= -\Gamma^i_{\rho\sigma} \frac{p^\rho p^\sigma}{E} \\ &= -\Gamma^i_{00} \frac{p^0 p^0}{E} - 2\Gamma^i_{j0} \frac{p^j p^0}{E} - \Gamma^i_{jk} \frac{p^j p^k}{E} \\ &= -E \left[\partial_i \Phi + \partial_0 w_i + 2(\partial_{[j} w_{i]} + \frac{1}{2}\partial_0 h_{ij}) v^j + (\partial_{(j} h_{k)i} - \frac{1}{2}\partial_i h_{jk}) v^j v^k\right].\end{aligned}$$

Now, to interpret these physically, we define *gravito-electric* and *gravito-magnetic* three-vector fields,

$$G^i = -\partial_i \Phi - \partial_0 w_i, \tag{1.6}$$

$$H^i = (\nabla \times \vec{w})^i = \epsilon^{ijk} \partial_j w_k. \tag{1.7}$$

These are analogous to the definition of electric and magnetic fields in terms of scalar potential V and vector potential $\vec{A}$. Using Eq. (1.6) and (1.7), the geodesic equation becomes

$$\frac{dp^i}{dt} = E \left[G^i + (\vec{v} \times H)^i - 2\partial_0 h_{ij} v^j + (\partial_{(j} h_{k)i} - \frac{1}{2} \partial_i h_{jk}) v^j v^k \right].$$

The first two terms describe how scalar and vector perturbations Φ and w_i affect a test particle moving along a geodesic. The first two terms are analogous to the Lorentz force law. The next terms are coupled to the spatial perturbation h_{ij}. Their relative importance will depend on the physical situation at hand.

To analyze the degrees of freedom, we begin with the field equations expressed in terms of the metric components:

$$R_{0j0l} = \partial_j \partial_l \Phi + \partial_0 \partial_{(j} w_{l)} - \frac{1}{2} \partial_0 \partial_0 h_{jl},$$

$$R_{0jkl} = \partial_j \partial_{[k} w_{l]} - \partial_0 \partial_{[k} h_{l]j},$$

$$R_{ijkl} = \partial_j \partial_{[k} h_{l]i} - \partial_i \partial_{[k} h_{l]j},$$

with other components related by symmetries of the curvature tensor. Contracting using

$\eta^{\mu\nu}$ twice to obtain the Ricci tensor,

$$
\begin{aligned}
R_{00} &= \nabla^2\Phi + \partial_0\partial_k w^k + 3\partial_0^2\Psi, \\
R_{0j} &= -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s_j^k, \\
R_{ij} &= -\partial_i\partial_j(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + \Box\Psi\delta_{ij} - \Box s_{ij} + 2\partial_k\partial_{(i}s_{j)},
\end{aligned}
$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the three-dimensional flat space Laplacian. Since the Ricci tensor involves contractions, trace-free s_{ij} and trace part Ψ of the spatial perturbations have entered the equations. Finally, the Einstein tensor takes the form,

$$
\begin{aligned}
G_{00} =& 2\nabla^2\Psi + \partial_k\partial_l s^{kl}, && (1.8) \\
G_{0j} =& -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s_j^k, && (1.9) \\
G_{ij} =& \left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\Phi - \Psi) + \delta_{ij}\partial_0\partial_k w^k - \partial_0\partial_{(i}w_{j)} \\
& + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k - \delta_{ij}\partial_k\partial_l s^{kl}. && (1.10)
\end{aligned}
$$

To analyze the degrees of freedom of the gravitational field, we start with $G_{00} = 8\pi G T_{00}$:

$$\nabla^2\Psi = 8\pi T_{00} - \frac{1}{2}\partial_k\partial_l s^{kl}.$$

This equation for Ψ includes no time derivatives. Knowing T_{00} and s_{ij} at any time determines Ψ, provided appropriate spatial boundary conditions are imposed. Therefore, Ψ is not a propagating degree of freedom. Next component, $G_{0j} = 8\pi G T_{00}$:

$$(\delta_{jk}\nabla^2 - \partial_j\partial_k)w^k = -16\pi G T_{0j} + 4\partial_0\partial_j\Psi + 2\partial_0\partial_k s_j^k.$$

This is an equation for w^k with no time derivatives. Knowing T_{0j} and strain (from which we can find Ψ), we can find w^k. Next the G_{ij} part is

$$\left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)\Phi = 8\pi G T_{ij} + (\delta_{ij}\nabla^2 - \partial_i\partial_j - 2\delta_{ij}\partial_0^2)\Psi - \delta_{ij}\partial_0\partial_k w^k + \partial_0\partial_{(i}w_{j)} \\ + \Box s_{ij} - 2\partial_k\partial_{(i}s_{j)}{}^k - \delta_{ij}\partial_k\partial_l s^{kl}.$$

Again, no time derivatives act on Φ, which is determined as a function of other fields.

Therefore, the only propagating degrees of freedom in Einstein's equations are those contained in the strain tensor s_{ij}. We will soon discover that the strain is used to describe gravitational waves. The other components of the perturbation $h_{\mu\nu}$ are determined by the strain tensor.

1.2.1 Gauge Transformations

Under gauge transformations $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ generated by the vector field ξ^μ, the fields change as

$$\begin{aligned}
\Phi &\to \Phi + \partial_0\xi^0, \\
w_i &\to w_i + \partial_0\xi^i - \partial_i\xi^0, \\
\Psi &\to \Psi - \frac{1}{3}\partial_i\xi^i, \\
s_{ij} &\to s_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}\partial_k\xi^k\delta_{ij}.
\end{aligned}$$

Let's discuss some well-known gauge choices. The **transverse gauge** is analogous to the

Coulomb gauge of electromagnetism. First, we fix the strain to be spatially transverse as,

$$\partial_i s^{ij} = 0,$$

and ξ^i satisfy:

$$\nabla^2 \xi^j + \frac{1}{3}\partial_j \partial_i \xi^i = -2\partial_i s^{ij}.$$

To determine value of ε^0 we fix

$$\partial_i w^i = 0,$$

by choosing ξ^0 to satisfy:

$$\nabla^2 \xi^0 = \partial_i w^i + \partial_0 \partial_i \xi^i.$$

None of the conditions satisfied by the vector field completely fix its value. Because they are second-order differential equations, we need boundary conditions. In this gauge Einstein tensor becomes

$$G_{00} = 2\nabla^2 \Psi, \tag{1.11}$$

$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + 2\partial_0 \partial_j \Psi, \tag{1.12}$$

$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i \partial_j)(\Phi - \Psi) - \partial_0 \partial_{(i} w_{j)} + 2\delta_{ij}\partial_0^2 \Psi - \Box s_{ij}. \tag{1.13}$$

Another gauge is the **synchronous gauge**. In this gauge, we set $\Phi = 0$ and choose ε^0 to satisfy $\partial_0 \xi^0 = -\Phi$. Now to choose ξ^i we set $w^i = 0$ and choose $\partial_0 \xi^i = -w^i + \partial_i \xi^0$.

We also have another gauge **Lorenz/harmonic gauge**, where we set:

$$\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h = 0.$$

An additional decomposition of the metric perturbation becomes possible if we consider tensor fields – tensors defined at every point. This brings out the physical degrees of freedom more directly. A vector field can be decomposed into a transverse part $w^i_\perp$ and a longitudinal part $w^i_\|$:

$$w^i = w^i_\perp + w^i_\|.$$

As usual, the transverse vector is divergence-less $\partial_i w^i_\perp \;=\; 0$, and the longitudinal vector is curl-free. Due to this property, a transverse vector can be represented as a curl of some other vector ξ^i and a longitudinal vector as a divergence of a scalar vector:

$$w^i_\perp = \epsilon^{ijk}\partial_j\xi_k, \quad w_{\|j} = \partial_i\lambda.$$

This decomposition of vector fields is also invariant under spatial rotations. The scalar λ represents one degree of freedom while ξ^i has 2 since the choice is not unique and we can make gauge transformations of the form $\xi_i + \partial_i w$.

Similarly the traceless symmetric tensor, strain, s^{ij} can be decomposed into a transverse part $s^{ij}_\perp$, a solenoidal part s^{ij}_S and a longitudinal part $s^{ij}_\|$,

$$s^{ij} = s^{ij}_\perp + s^{ij}_S + s^{ij}_\|.$$

Again, the transverse part is divergence-less, while the divergence of the solenoidal part is a transverse vector, and the divergence of the longitudinal part is a longitudinal vector:

$$\partial_i s_{\perp}^{ij} = 0,$$
$$\partial_i \partial_j s_S^{ij} = 0,$$
$$\epsilon^{jkl} \partial_k \partial_i s_{\| j}^{i} = 0.$$

This implies that the longitudinal part can be derived from a scalar field θ, and the solenoidal part from a transverse vector ζ^i:

$$s_{\| ij} = \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \theta,$$
$$s_{Sij} = \partial_{(i} \zeta_{j)},$$

where $\partial_i \zeta^i = 0$.

Thus, the longitudinal component describes 1 degree of freedom, the solenoidal part 2, and the transverse-traceless part accounts for the remaining 2.

With this decomposition of tensor fields, we have written the ten component perturbation $h_{\mu\nu}$ in terms of four scalars Φ, Ψ, λ, θ, with one degree of freedom each, two transverse vectors ξ^i, ζ^i with two degrees of freedom each and one transverse tensor $s_{\perp}^{ij}$ with two degrees of freedom.

1.3 Gravitational Wave Solutions

Now we study freely propagating degrees of freedom of the gravitational field, requiring no local sources. Weak field equations, Eqs. (1.8), (1.9) and (1.10) become:

$$\nabla^2 \Psi = 0. \tag{1.14}$$

With well-behaved boundary conditions, i.e., no singularities and fields go to zero at infinity, Eq. (1.14) implies $\Psi = 0$.
Substituting Eq. (1.14) into Eq. (1.9), we get

$$\nabla^2 w_j = 0 \implies w_j = 0,$$

Taking trace of Eq. (1.10) and using the above results, we get

$$\nabla^2 \Phi = 0 \implies \Phi = 0.$$

The traceless part of Eq. (1.10) gives

$$\Box s_{ij} = 0.$$

We have set all other degrees of freedom to zero, and s_{ij} is transverse, so we are essentially working in a traceless transverse gauge. The perturbation metric is:

$$h_{\mu\nu}^{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & s_{11} & s_{12} & s_{13} \\ 0 & s_{12} & s_{22} & s_{23} \\ 0 & s_{13} & s_{23} & s_{33} \end{bmatrix}$$

$h_{\mu\nu}^{TT}$ is traceless, spatial and transverse. It is more conventional to write the equation of motion in terms of the perturbed metric, so it becomes:

$$\Box h_{\mu\nu}^{TT} = 0.$$

This is a wave equation and has plane wave solutions

$$h_{\mu\nu}^{TT} = C_{\mu\nu} e^{ik_\sigma x^\sigma},$$

k^σ is the wave vector. $C_{\mu\nu}$ is a symmetric $(0, 2)$, traceless and spatial tensor, therefore we can write,

$$C_{0\mu} = 0; \qquad \eta^{\mu\nu} C_{\mu\nu} = 0.$$

Let's verify whether the plane wave solutions satisfy the wave equation:

$$
\begin{aligned}
\Box h^{TT}_{\mu\nu} &= 0 \\
\partial^{\alpha}[\partial_{\alpha}(C_{\mu\nu}e^{ik_{\sigma}x^{\sigma}})] &= 0 \\
\partial^{\alpha}(C_{\mu\nu}e^{ik_{\sigma}x^{\sigma}}ik_{\alpha}) &= 0 \\
-C_{\mu\nu}e^{ik_{\sigma}x^{\sigma}}k_{\alpha}k^{\alpha} &= 0 \\
-h^{TT}_{\mu\nu}k^{\alpha}k_{\alpha} = 0.
\end{aligned}
$$

We cannot set $h_{\mu\nu}$ to zero, hence, we set

$$k_{\alpha}k^{\alpha} = 0.$$

Since the wave vector for a plane wave solution is null, gravitational waves propagate at the speed of light, at least, loosely speaking, within the context of general relativity and without modified gravity effects. Wave vector is of the form $k^{\sigma} = (\omega, k^{i})$, where ω is the wave frequency. For the wave vector to be null, we should have:

$$
\begin{aligned}
k^{\sigma}k_{\sigma} &= 0 \\
\implies \omega^2 &= k^{i}k_{i}
\end{aligned}
$$

This theoretical expectation was spectacularly confirmed by the joint detection of gravitational waves (GW170817) and a short gamma-ray burst (GRB 170817A). The observed time delay between the two signals was only 1.74 ± 0.05 seconds despite both signals having traveled over 130 million light-years. This implies that the speed of gravitational waves differs from the speed of light by no more than one part in 10^{15}, placing extremely tight constraints on a wide range of modified gravity theories that predict otherwise [14]. To ensure that the perturbation is transverse, we need:

$$\partial_\mu h^{\mu\nu}_{TT} = 0$$
$$iC^{\mu\nu} k_\mu e^{ik_\sigma x^\sigma} = 0 \implies k_\mu C^{\mu\nu} = 0.$$

So, the wave vector must be orthogonal to $C^{\mu\nu}$.

To make the solution more explicit, we choose a wave travelling in the z-direction:

$$k^\sigma = (\omega, 0, 0, \omega).$$

For $C_{\mu\nu}$ to be orthogonal to the chosen wave vector, we should have $C_{3\nu} = 0$. In general, we can write:

$$C_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As stated earlier, $C_{\mu\nu}$ is symmetric and traceless. So, in this gauge, for a plane wave travelling in the z direction, two components C_{11} and C_{12} completely characterise the wave.

To understand the physical effects of a passing gravitational wave, we consider the motion of test particles under its influence. However, solving for a single particle's motion only tells us about its coordinate behavior along a world line, which can be made to appear stationary (to first order) in an appropriately chosen transverse-traceless (TT) gauge. Therefore, to extract the wave's coordinate-independent and physically meaningful effects, we must instead study the relative acceleration between neighboring free-falling particles. The geodesic deviation equation governs this relative motion and encodes the tidal distortions caused by the gravitational wave, precisely the effect that gravitational wave detectors like LIGO measure. This leads us to consider the geodesic deviation equation,

$$\frac{D^2}{d\tau^2}S^\mu = R^\mu_{\nu\rho\sigma}T^\nu T^\rho S^\sigma. \tag{1.15}$$

Consider some nearby particles whose four velocities are described by a single vector field $U^\mu(x)$ and separation vector S^μ, the deviation equation (1.15) becomes:

$$\frac{D^2}{d\tau^2}S^\mu = R^\mu_{\nu\rho\sigma}U^\nu U^\rho S^\sigma.$$

We evaluate the right-hand side of this equation to first order in $h^{TT}_{\mu\nu}$. If we take the particles to be slowly moving, we can set the four velocity as $U^0 = 1$, with the rest of the components being 0, plus corrections of order $h^{TT}_{\mu\nu}$ and higher than that. But on RHS, we also have Riemann tensor, which is of first order in perturbation, so we take only the zeroth order in velocity, ignoring the higher order corrections.

$$U^\nu = (1, 0, 0, 0).$$

This way we can keep our equations only to the first order in perturbation. So, we only need to find $R^{\mu}_{00\sigma}$. Using Eq. (1.1):

$$R_{\mu 00\sigma} = \frac{1}{2}(\partial_0\partial_0 h^{TT}_{\sigma\mu} - \partial_0\partial_\mu h^{TT}_{0\sigma} - \partial_\sigma\partial_0 h^{TT}_{0\mu} + \partial_\sigma\partial_\mu h^{TT}_{00}).$$

Using $h^{TT}_{00} = h^{TT}_{\mu 0} = 0$, we get

$$R_{\mu 00\sigma} = \frac{1}{2}\partial_0\partial_0 h^{TT}_{\sigma\mu}.$$

To the lowest order, for slowly moving particles we have $\tau = t$, with these, the geodesic equation becomes:

$$\frac{\partial^2}{\partial t^2}S^{\mu} = \frac{1}{2}S^{\sigma}\frac{\partial^2}{\partial t^2}h^{TT\mu}_{\sigma}. \tag{1.16}$$

For a wave travelling in x^3 direction, only S^1 and S^2 components will be affected. It is because test particles are only disturbed in directions perpendicular to the wave propagation, just like in electromagnetism, the direction of the wave is perpendicular to **E** and **B** fields. Recall that two numbers C_{11} and C_{12} characterize this wave. Renaming those

$$h_+ = C_{11},$$

$$h_\times = C_{12}.$$

$C_{\mu\nu}$ becomes:

$$C_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.17}$$

To consider the effects of the two components separately, we first set $h_\times = 0$. From Eq (1.16) we have:

$$\frac{\partial^2}{\partial t^2} S^1 = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2}(h_+ e^{ik^\sigma_\sigma}),$$
$$\frac{\partial^2}{\partial t^2} S^2 = -\frac{1}{2} S^2 \frac{\partial^2}{\partial t^2}(h_+ e^{ik^\sigma_\sigma}).$$

These have perturbative solutions, expanding those to the lowest order in h:

$$S^1 = \left(1 + \frac{1}{2} h_+ e^{ik_\sigma k^\sigma}\right) S^1(0),$$
$$S^2 = \left(1 - \frac{1}{2} h_+ e^{ik_\sigma k^\sigma}\right) S^2(0).$$

So, the particles separated in x^1 direction will oscillate in x^1 direction, and the ones separated in x^2 direction will oscillate in x^2. So, if we have particles in a ring in the x-y direction. They will bounce back and forth in the shape of a + as shown in Figure 1.2.

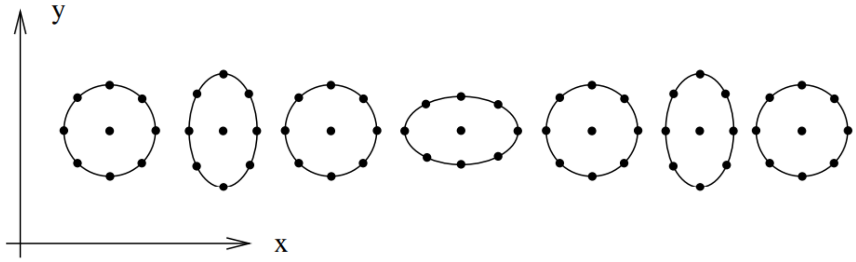


Figure 1.2: Particles displaced in a + pattern due to passing gravitational wave with + polarization[7].

Now we set $h_+ = 0$

$$\frac{\partial^2}{\partial t^2}S^1 = \frac{1}{2}S^2\frac{\partial^2}{\partial t^2}(h_\times e^{ik_\sigma^\sigma})$$
$$\frac{\partial^2}{\partial t^2}S^2 = -\frac{1}{2}S^1\frac{\partial^2}{\partial t^2}(h_\times e^{ik_\sigma^\sigma}),$$

with solutions:

$$S^1 = S^1(0) + S^2(0)\frac{1}{2}h_\times e^{ik_\sigma k^\sigma},$$
$$S^2 = S^2(0) - \frac{1}{2}h_\times e^{ik_\sigma k^\sigma}S^1(0).$$

In this case, the particles oscillate in a $\times$ pattern as shown in Figure 1.3.

h_+ and $h_\times$ measure two independent modes of linear polarization of the gravitational waves. These are known as 'plus' and 'cross' polarizations. Using these two polarizations,

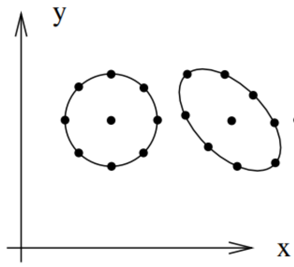


Figure 1.3: Particles displaced in a $\times$ pattern due to passing gravitational wave with $\times$ polarization[7].

we can construct right and left-handed circularly polarized modes:

$$h_R = \frac{1}{\sqrt{2}}(h_+ + ih_\times),$$
$$h_L = \frac{1}{\sqrt{2}}(h_+ - ih_\times).$$

It is interesting to note here that these polarization states of classical gravitational waves can be related to the kind of particles we will find upon quantization. If we know how a field behaves under spatial rotations (like polarization properties of the field), we can find out the spin of particles we will get upon quantization. An electromagnetic field has two independent polarizations. A single polarization mode is invariant under a 360 ° rotation in the x-y plane. So, quantizing this field gives a massless spin-1 particle. The neutrino is described by a field that picks up a minus sign under such a rotation, and it has spin $\frac{1}{2}$. The general rule is that the spin S is related to the angle θ under which the polarization modes are invariant by $S = 360°/\theta$. The gravitational field travels at the speed of light, so it should lead to massless particles. Polarization modes described above are invariant under rotations of $180°$ in the x-y plane, so they should lead to spin-2 particles upon quantization. These are called gravitons and have not been detected.

We have discussed the gravitational wave solutions to the perturbed metric in the vacuum, and now we will discuss the solution in the presence of sources.

1.4 Production of Gravitational Waves

In this section, we discuss the generation of gravitational radiation by matter sources. Accordingly, we consider Einstein's equations in the presence of matter, where the stress-energy tensor $T_{\mu\nu}$ does not vanish. In this regime, the metric perturbation generally includes non-zero scalar and vector components in addition to the transverse-traceless (TT) tensor modes. Therefore, we cannot assume a purely transverse-traceless form of the solution a priori. We retain the full perturbation and solve the field equations near the source. However, in the radiation zone, far from the source, we can adopt the transverse-traceless gauge to extract the observable gravitational wave strain.

We can still make more simplifications. We start with defining trace-reversed perturbation:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}. \tag{1.18}$$

To see why it is called trace-reversed, when we take the trace of Eq. (1.18):

$$\bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = -h.$$

We can reconstruct the original perturbation from the trace-reversed perturbation using Eq. (1.18). If we are far from sources, in a vacuum, we can adopt the traceless transverse gauge in which the trace-reversed and the original perturbations coincide:

$$\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}.$$

We are still free to perform gauge transformations. Under an infinitesimal coordinate change, the trace-reversed metric transforms as:

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} - \partial_\lambda \xi^\lambda \eta_{\mu\nu}.$$

And by choosing a gauge vector ξ_μ that satisfies $\Box\xi_\mu = -\partial_\lambda \bar{h}^\lambda_\mu$, we can impose the Lorenz gauge condition:

$$\partial_\mu \bar{h}^{\mu\nu} = 0.$$

However, the original perturbation is not transverse; instead, we have:

$$\begin{aligned} \partial_\mu \bar{h}^{\mu\nu} &= \partial_\mu h^{\mu\nu} - \frac{1}{2}\partial_\mu h \eta^{\mu\nu} = 0 \\ \implies \partial_\mu h^{\mu\nu} &= \frac{1}{2}\partial^\nu h. \end{aligned}$$

Using the trace-reversed tensor in Eq. (1.2) and the Lorenz gauge condition, we get:

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}.$$

This expression is much simpler than the one we get using the original perturbation. The

Einstein equation in this gauge is:

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{1.19}$$

This is a wave equation for each component of the trace-reversed metric perturbation, describing how spacetime responds dynamically to the presence of matter. We can find its solution using a Green's function, just like we do in electromagnetism.

Green's function, $G(x^\sigma - y^\sigma)$, for the d'Alembertian operator $\Box$ is the solution of the wave equation in the presence of a point source:

$$\Box_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma). \tag{1.20}$$

$\Box_x$ denotes the d'Alembertian with respect to x^σ coordinates. Using the Green's function, the general solution to Eq. (1.19) is

$$\bar{h}_{\mu\nu}(x^\sigma) = -16\pi G \int G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) d^4y. \tag{1.21}$$

This expression can be readily verified by taking the d'Alembertian. Eq. (1.20) has retarded and advanced solutions depending on whether they represent waves traveling forward or backward in time. On physical grounds, we are only interested in the retarded Green's function. It represents a cumulative effect on the current signals due to the past configuration of points under consideration. The retarded solution is given by:

$$G(x^\sigma - y^\sigma) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}\delta[|\mathbf{x} - \mathbf{y}| - (x^0 - y^0)]\theta(x^0 - y^0)), \tag{1.22}$$

where $\mathbf{x}$ denotes $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{y}$ denotes $\mathbf{y} = (y^1, y^2, y^3)$. The function $\theta(x^0 - y^0)$ equals 1 when $x^0 > y^0$ and zero otherwise. Substituting Eq. (1.22) in Eq. (1.21),

$$
\begin{aligned}
\bar{h}_{\mu\nu}(x^\sigma) &= -16\pi G \int -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\delta[|\mathbf{x}-\mathbf{y}| - (x^0 - y^0)]\theta(x^0 - y^0))T_{\mu\nu}(y^\sigma)d^4y \\
&= 4G \int -\frac{1}{|\mathbf{x}-\mathbf{y}|}T_{\mu\nu}(t - |\mathbf{x}-\mathbf{y}|, \mathbf{y})d^3y.
\end{aligned}
$$

In the second line, y^0 integration using the Dirac delta function was performed, where $t = x^0$. We also define retarded time $t_r = t - |\mathbf{x}-\mathbf{y}|$. The result says that the disturbance in the gravitational field at $t, \mathbf{x}$ is a sum of the influences from the energy momentum sources at point $(t_r, \mathbf{x}-\mathbf{y})$ on the past light cone.

We have a general solution now, let's consider a situation where the gravitational radiation is emitted by a far-away isolated source made up of non-relativistic matter. Before moving forward, we first set the conventions for the Fourier transforms. Given a function of spacetime, we are interested in its Fourier and inverse Fourier transform with respect to time only,

$$
\begin{aligned}
\widetilde{\phi}(\omega, \mathrm{x}) &= \frac{1}{\sqrt{2\pi}}\int dt e^{i\omega t}\phi(t, \mathrm{x}), \\
\phi(t, \mathrm{x}) &= \frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega t}\tilde{\phi}(\omega, \mathrm{x}).
\end{aligned}
$$

Using this convention, we take the Fourier transform of the metric perturbation as follows:

$$
\begin{aligned}
\widetilde{\bar{h}}_{\mu\nu}(\omega, \mathrm{x}) &= \frac{1}{\sqrt{2\pi}} \int dt e^{i\omega t} \bar{h}_{\mu\nu}(t, \mathrm{x}) \\
&= \frac{4G}{\sqrt{2\pi}} \int dt d^3 y e^{i\omega t} \frac{T_{\mu\nu}(t - |\mathrm{x} - \mathrm{y}|, \mathrm{y})}{|\mathrm{x} - \mathrm{y}|} \\
&= \frac{4G}{\sqrt{2\pi}} \int dt_r d^3 y e^{i\omega t_r} e^{i\omega |\mathrm{x}-\mathrm{y}|} \frac{T_{\mu\nu}(t_r, \mathrm{y})}{|\mathrm{x} - \mathrm{y}|} \\
&= 4G \int d^3 y e^{i\omega |\mathrm{x}-\mathrm{y}|} \frac{\widetilde{T}_{\mu\nu}(\omega, \mathrm{y})}{|\mathrm{x} - \mathrm{y}|}.
\end{aligned}
$$

In the first equation, we used the definition of the Fourier transform. In the third line, the integration variable was changed from t to t_r. Then we noticed that we have the Fourier transform of $T_{\mu\nu}(t_r, y)$, so we used the definition there.

Now, assume that the source is isolated, far away, and non-relativistic (slowly moving). We can consider the source centered at r with its parts at $r + \delta r$ such as $r \ll 1$. Radiations emitted by the source will be at low frequencies such that $\delta r \ll w^{-1}$. Using these approximations, we can replace $e^{i\omega|\mathbf{x}-\mathbf{y}|}/|\mathbf{x} - \mathbf{y}|$ with $e^{i\omega r}/r$ as

$$
\widetilde{\bar{h}}_{\mu\nu}(\omega, \mathrm{x}) = 4G \frac{e^{i\omega|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} \int d^3 y \widetilde{T}_{\mu\nu}(\omega, \mathrm{y}). \tag{1.23}
$$

Lorenz gauge condition in the Fourier space becomes

$$
\begin{aligned}
&\partial_\mu \widetilde{\bar{h}}^{\mu\nu} \\
&\partial_\mu \widetilde{\bar{h}}^{\mu\nu} = \int dt \partial_\mu \bar{h}^{\mu\nu} e^{i\omega t} = i\omega \int dt h^{0\nu} e^{i\omega t} + \int dt \partial_i h^{i\nu} e^{i\omega t} = 0 \\
\Longrightarrow \quad &\widetilde{\bar{h}}^{0\nu} = -\frac{i}{\omega} \partial_i \widetilde{\bar{h}}^{i\nu}.
\end{aligned}
$$

So, if we find the space-like components of the Fourier transform of the perturbation, we can find the time-like components using the above expression. We first work with a purely spatial component $\tilde{\bar{h}}^{ij}$, from which we find $\tilde{\bar{h}}^{0j}$, and then we can find $\tilde{\bar{h}}^{00}$ from $\tilde{\bar{h}}^{j0}$. We begin to solve the equation. (1.23) with integration by parts:

$$\int d^3y \widetilde{T}^{ij}(\omega, \mathrm{y}) = \int \partial_k \left(y^i \widetilde{T}^{kj} \right) d^3y - \int y^i \left(\partial_k \widetilde{T}^{kj} \right) d^3y.$$

The first term is a surface integral that will vanish when integrated as the source is isolated. The second integral can be related to $\widetilde{T}^{0j}$ if we the use Fourier-space version of energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$, as

$$-\partial_k \widetilde{T}^{k\mu} = i\omega \widetilde{T}^{0\mu}.$$

With this substitution we have:

$$\begin{aligned}
\int d^3y \widetilde{T}^{ij}(\omega, \mathrm{y}) &= i\omega \int y^i \widetilde{T}^{0j} d^3y \\
&= \frac{i\omega}{2} \int \left(y^i \widetilde{T}^{0j} + y^j \widetilde{T}^{0i} \right) d^3y \\
&= \frac{i\omega}{2} \int \left[\partial_l \left(y^i y^j \widetilde{T}^{0l} \right) - y^i y^j \left(\partial_l \widetilde{T}^{0l} \right) \right] d^3y \\
&= -\frac{\omega^2}{2} \int y^i y^j \widetilde{T}^{00} d^3y.
\end{aligned}$$

In the second line, we used the fact that the left-hand side is symmetric in i and j. In the third line, we used integration by parts again, and the first term vanishes at the boundary.

We define **quadrupole moment tensor** of the energy density of the source,

$$I^{ij}(t) = \int y^i y^j T^{00}(t, \mathrm{y}) d^3 y,$$

a constant tensor on each surface of constant time. In terms of the Fourier transform of the quadrupole moment, our solution takes the form

$$\tilde{\bar{h}}_{ij}(\omega, \mathrm{x}) = -2G\omega^2 \frac{e^{i\omega r}}{r} \tilde{I}_{ij}(\omega).$$

We now take the inverse Fourier transform to recover the original perturbation

$$\begin{aligned}
\bar{h}_{ij}(\omega, \mathrm{x}) &= -\frac{1}{\sqrt{2\pi}} 2G \int dw \frac{e^{i\omega(t-r)}}{r} \omega^2 \tilde{I}_{ij}(\omega) \\
&= -\frac{1}{\sqrt{2\pi}} \frac{2G}{r} \int dw e^{i\omega(t_r)} \omega^2 \tilde{I}_{ij}(\omega) \\
&= \frac{1}{\sqrt{2\pi}} \frac{2G}{r} \frac{d^2}{dt^2} \int dw e^{i\omega(t_r)} \tilde{I}_{ij}(\omega) \\
&= \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}(t_r). \qquad (1.24)
\end{aligned}$$

This is called the **quadrupole formula**. The expression reads that the gravitational radiation produced by a non-relativistic source is proportional to the second derivative of the quadrupole moment of energy density at a point where the past light cone of the observer intersects the source. If we compare this with the electric field, the leading contribution to the electromagnetic field comes from the changing dipole moment of the charge density. This difference is due to the nature of gravity. Dipole moment changes due to the changing charge density in the case of electromagnetism. In the case of gravitation, it corresponds to the motion of energy density. Oscillation of the center of mass of an isolated system violates the conservation of momentum. Meanwhile, the oscillation of the dipole moment is permissible by the laws of physics. Quadrupole moment measures the shape of the system and is generally smaller than the dipole moment. For this reason and due to the weak coupling of matter to gravity, gravitational radiation is typically much weaker than electromagnetic radiation.

Gravitational Radiation from a Binary Star System

Now we apply the quadrupole formula to find that gravitational radiation is emitted by a binary star system. Imagine two stars, each of mass M, in a circular orbit in the $x^1 - x^2$ plane at a distance r from their common center of mass as shown in Figure 1.4.

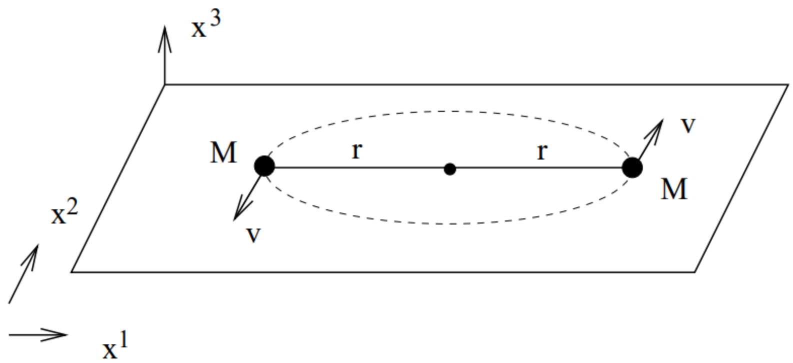


Figure 1.4: Binary star system in $x^1 - x^2$ plane[7].

Treating the system in Newtonian approximation, we can equate the gravitational force to the outward centrifugal force:

$$\frac{GM^2}{(2r)^2} = \frac{Mv^2}{r}.$$

Rearranging this expression to get velocity:

$$v = \left(\frac{GM}{4r}\right)^{1/2}. \tag{1.25}$$

Time period of the orbit is:

$$T = \frac{2\pi r}{v}.$$

Using T and v, we can find a more useful quantity, the angular frequency

$$\Omega = \frac{2\pi}{T} = \left(\frac{GM}{4r^3}\right)^{1/2}. \tag{1.26}$$

In terms of the angular frequency, the explicit path of the left star, star a is:

$$x_a^1 = R\cos\Omega t, \quad x_a^2 = R\sin\Omega t,$$

and of star b is

$$x_b^1 = -R\cos\Omega t, \quad x_a^2 = -R\sin\Omega t,$$

Energy density, if we consider the stars as point masses, becomes

$$T^{00}(t, \mathbf{x}) = M\delta(x^3)[\delta(x^1 - R\cos\Omega t)\delta(x^2 - R\sin\Omega t) + \delta(x^1 + R\cos\Omega t)\delta(x^2 + R\sin\Omega)t].$$

Now we can find the quadrupole moment tensor as

$$\begin{aligned}
I^{11} &= \int x^1 x^1 T^{00}(t, \mathrm{x}) d^3x \\
&= M(R\cos\Omega t)(R\cos\Omega t) + M(-R\cos\Omega t)(-R\cos\Omega t) \\
&= 2MR^2\cos^2\Omega t \\
&= MR^2(1+\cos 2\Omega t).
\end{aligned}$$

Similarly, we find the rest of the components of the quadrupole moment tensor:

$$\begin{aligned}
I^{22} = 2MR^2\sin^2\Omega t &= MR^2(1-\cos 2\Omega t), \\
I^{21} = I^{12} = 2MR^2\cos\Omega t\sin\Omega t &= MR^2\sin 2\Omega t, \\
I^{i3} = I^{3i} = 0.
\end{aligned}$$

Substituting the quadrupole moment tensor in Eq. (1.24), we find the components of metric perturbation:

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{8GM}{r}\Omega^2 R^2 \begin{bmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.27}$$

Using the Lorentz condition, we can find the remaining components of the perturbation.

We have derived the metric perturbation due to a binary star system. Next, we shall quantify the energy lost due to the emission of gravitational radiation.

1.5 Energy Loss due to Gravitational Radiation[7,16]

Unlike other field theories, gravity does not admit a well-defined, coordinate-independent local energy-momentum tensor for the gravitational field. In linearized gravity, however, we can attempt to define an effective energy-momentum tensor for the perturbations — a concept that has no exact analogue in full general relativity. In scalar field theory and electromagnetism, the energy-momentum tensors are quadratic in the fields and can be defined unambiguously. To construct something similar in gravity, we must expand the Einstein equations to second order in the metric perturbation, since gravitational energy is expected to manifest at that order.

Earlier, when analyzing the effect of gravitational waves on test particles, we assumed that they move along geodesics — a result that normally follows from the covariant conservation of energy-momentum. In the linearized theory, this conservation is expressed as $\partial_\mu T^{\mu\nu} = 0$, implying geodesic motion in the flat background spacetime, not the full curved geometry.

To define the gravitational energy content properly, we now proceed to expand both the metric and the Ricci tensor to second order.

$$
\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}, \\
R_{\mu\nu} &= R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}.
\end{aligned}
$$

$R^{(1)}_{\mu\nu}$ is of the same order as $h^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ is of order $\left(h^{(1)}_{\mu\nu}\right)^2$. In a flat background, the zeroth-order Einstein equation is trivially solved using $R^{(0)}_{\mu\nu}$. The first-order vacuum equation is:

$$R^{(1)}_{\mu\nu}[h^{(1)}] = 0. \tag{1.28}$$

The above equation determines the metric perturbation to the first order. The second-order perturbation is determined by:

$$R^{(1)}_{\mu\nu}[h^{(2)}] + R^{(2)}_{\mu\nu}[h^{(1)}] = 0.$$

$R^{(1)}_{\mu\nu}[h^{(2)}]$ are the parts of the expanded Ricci tensor that are linear in metric perturbation as given in section 2.1, but applied to the second order perturbation. Similarly, $R^{(2)}_{\mu\nu}[h^{(1)}]$ indicates the parts that are of the second order in Ricci tensor applied to the first order perturbation as:

$$\begin{aligned} R^{(2)}_{\mu\nu} =& \frac{1}{2} h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} + \frac{1}{4}\left(\partial_\mu h_{\rho\sigma}\right)\partial_\nu h^{\rho\sigma} + \left(\partial^\sigma h^\rho{}_\nu\right)\partial_{[\sigma} h_{\rho]\mu} - h^{\rho\sigma}\partial_\rho\partial_{(\mu} h_{\nu)\sigma} \\ &+ \frac{1}{2}\partial_\sigma\left(h^{\rho\sigma}\partial_\rho h_{\mu\nu}\right) - \frac{1}{4}\left(\partial_\rho h_{\mu\nu}\right)\partial^\rho h - \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right)\partial_{(\mu} h_{\nu)\rho}. \end{aligned} \tag{1.29}$$

Now considering the vacuum equation $G_{\mu\nu} = 0$ at second order

$$R^{(1)}_{\mu\nu}[h^{(2)}]\frac{1}{2} - \eta^{\rho\sigma} R^{(1)}_{\rho\sigma}[h^{(2)}]\eta_{\mu\nu} = 8\pi G t_{\mu\nu}, \tag{1.30}$$

where $t_{\mu\nu}$ is defined as:

$$t_{\mu\nu} \equiv -\frac{1}{8\pi G}\left\{R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta^{\rho\sigma} R^{(2)}_{\rho\sigma}[h^{(1)}]\eta_{\mu\nu}\right\}. \tag{1.31}$$

We made use of Eq. (1.28) in the resulting Eq. (1.30). We have moved the terms of the form

$R^{(2)}_{\rho\sigma}[h^{(1)}]$ to the right-hand side, relabeling those as the energy-momentum tensor for first-order perturbation. The resulting $t_{\mu\nu}$ is a symmetric tensor, quadratic in the perturbation, and represents how the perturbations affect the spacetime metric, just like the matter energy moment tensor. $t_{\mu\nu}$ is also conserved in the flat background sense:

$$\partial_\mu t^{\mu\nu} = 0,$$

which follows from the Bianchi identity $\partial_\mu G^{\mu\nu}$.

There are some limitations in the interpretation of $t_{\mu\nu}$ as an energy-momentum tensor. It turns out that it is not invariant under gauge transformations. To overcome this, we average the energy-momentum tensor over several wavelengths — denoted by angle brackets — effectively smoothing out local fluctuations to isolate physically meaningful, gauge-invariant quantities. This averaging procedure filters out rapidly oscillating terms, particularly total derivatives, which vanish under the average:

$$\langle \partial_\mu (X) \rangle = 0.$$

With this, the product rule becomes:

$$\langle \partial_\mu (AB) \rangle = 0 \implies \langle B \partial_\mu (A) \rangle = -\langle A \partial_\mu (B) \rangle.$$

Using this expression will simplify our calculations. Now we move on to calculating the energy moment tensor, using Eq. (1.29). We also removed the label on the perturbation because we were only interested in the first-order metric perturbation. To make our calculations simpler, we move to a transverse traceless gauge, recalling $\partial^\mu h_{\mu\nu}^{TT} = 0, h^{TT} = 0$. We can use the transverse traceless gauge in the vacuum. Using the gauge conditions, Eq. (1.29) is simplified to

$$
\begin{aligned}
R_{\mu\nu}^{(2)\mathrm{TT}} =& \frac{1}{2} h_{\mathrm{TT}}^{\rho\sigma} \partial_\mu \partial_\nu h_{\rho\sigma}^{\mathrm{TT}} + \frac{1}{4} \left(\partial_\mu h_{\rho\sigma}^{\mathrm{TT}}\right) \partial_\nu h_{\mathrm{TT}}^{\rho\sigma} + \frac{1}{2} \eta^{\rho\lambda} \left(\partial^\sigma h_{\rho\nu}^{\mathrm{TT}}\right) \partial_\sigma h_{\lambda\mu}^{\mathrm{TT}} \\
& - \frac{1}{2} \left(\partial^\sigma h_{\rho\nu}^{\mathrm{TT}}\right) \partial^\rho h_{\sigma\mu}^{\mathrm{TT}} - h_{\mathrm{TT}}^{\rho\sigma} \partial_\rho \partial_{(\mu} h_{\nu)\sigma}^{\mathrm{TT}} + \frac{1}{2} h_{\mathrm{TT}}^{\rho\sigma} \partial_\sigma \partial_\rho h_{\mu\nu}^{\mathrm{TT}}.
\end{aligned}
$$

Now we average over several wavelengths and use the product rule condition, the first two terms add up, and the last three terms cancel out, so we're left with:

$$
\left\langle R_{\mu\nu}^{(2)\mathrm{TT}} \right\rangle = -\frac{1}{4} \left\langle \left(\partial_\mu h_{\rho\sigma}^{\mathrm{TT}}\right) \left(\partial_\nu h_{\mathrm{TT}}^{\rho\sigma}\right) + 2\eta^{\rho\lambda} \left(\Box h_{\rho\nu}^{\mathrm{TT}}\right) h_{\lambda\mu}^{\mathrm{TT}} \right\rangle .
$$

$\Box h_{\rho\nu} = 0$ is the equation of motion in vacuum, so the expression that we're left with is

$$
\left\langle R_{\mu\nu}^{(2)\mathrm{TT}} \right\rangle = -\frac{1}{4} \left\langle \left(\partial_\mu h_{\rho\sigma}^{\mathrm{TT}}\right) \left(\partial_\nu h_{\mathrm{TT}}^{\rho\sigma}\right) \right\rangle .
$$

Now, using this simplified Ricci tensor, we update the Ricci scalar

$$
\begin{aligned}
\langle \eta^{\mu\nu} R_{\mu\nu}^{(2)\mathrm{TT}} \rangle &= -\frac{1}{4} \left\langle (\eta^{\mu\nu} \partial_\mu h_{\rho\sigma}^{\mathrm{TT}})(\partial_\nu h_{\mathrm{TT}}^{\rho\sigma}) \right\rangle \\
&= -\frac{1}{4} \left\langle (\partial^\nu h_{\rho\sigma}^{\mathrm{TT}})(\partial_\nu h_{\mathrm{TT}}^{\rho\sigma}) \right\rangle \\
&= \frac{1}{4} \left\langle (\partial^\nu \partial_\nu h_{\rho\sigma}^{\mathrm{TT}})(h_{\mathrm{TT}}^{\rho\sigma}) \right\rangle \\
&= 0.
\end{aligned}
$$

Inserting the obtained expressions for Ricci tensor and Ricci scalar in the expression for $t_{\mu\nu}$

(1.31), we get

$$t_{\mu\nu} = \frac{1}{32\pi G} \left\langle \left(\partial_\mu h^{\mathrm{TT}}_{\rho\sigma}\right) \left(\partial_\nu h^{\rho\sigma}_{\mathrm{TT}}\right) \right\rangle . \tag{1.32}$$

If we had not simplified the calculations by using the transverse traceless gauge, we'd have obtained:

$$\begin{aligned} t_{\mu\nu} = \frac{1}{32\pi G} \langle\, (\partial_\mu h_{\rho\sigma}) (\partial_\nu h^{\rho\sigma}) - \frac{1}{2} (\partial_\mu h) (\partial_\nu h) \\ - (\partial_\rho h^{\rho\sigma}) (\partial_\mu h_{\nu\sigma}) - (\partial_\rho h^{\rho\sigma}) (\partial_\nu h_{\mu\sigma}) \rangle \, , \end{aligned}$$

which would have been a more complicated way to extract physically meaningful results. Energy momentum for a plane wave of the form,

$$h^{TT}_{\mu\nu} = C_{\mu\nu} sin(k_\lambda x^\lambda).$$

using Eq. (1.32) is

$$t_{\mu\nu} = \frac{1}{32\pi G} k_\mu k_\nu C_{\rho\sigma} C^{\rho\sigma} \langle \cos^2(k_\lambda \lambda^\lambda) \rangle. \tag{1.33}$$

Averaging $\cos^2$ over several wavelengths, we get:

$$\langle \cos^2(k_\lambda \lambda^\lambda) \rangle = \frac{1}{2}.$$

Considering the wave to be moving along the z-axis, the wave vector is

$$k_\lambda = (-\omega, 0, 0, \omega).$$

We have the expression for $C_{\rho\sigma}$ in Eq. (1.17), which we use to find:

$$C_{\rho\sigma}C^{\rho\sigma} = 2(h_+^2 + h_\times^2).$$

Physical observables are more commonly expressed in terms of frequency $f = \omega/2\pi$, so for energy momentum tensor using the above expressions in (1.33) we obtain:

$$t_{\mu\nu} = \frac{\pi}{8G} f^2 (h_+^2 + h_\times^2) \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Typical frequencies of gravitational wave sources lie between 10^{-4} and 10^4 Hz, with strain amplitudes of order 10^{-22}. The corresponding energy flux carried in the z-direction is given by

$$-T_{0z} = 10^{-4}\left(\frac{f}{Hz}\frac{\text{erg}}{\text{cm}^2.\text{s}}\right).$$

In a gravitational wave detector, this is roughly the amount of energy that could be deposited in each centimeter square every second. This is a large amount of energy flux. For comparison, peak electromagnetic flux from a supernova at cosmological distance is approximately 10^{-9} erg/cm^2/s and lasts for months. However, the gravitational wave signal only lasts for milliseconds.

Now that we have the gravitational wave energy-momentum tensor, we can use it to compute the energy radiated by a system through gravitational waves. The total energy carried by gravitational radiation at a fixed moment in time is defined by integrating the energy density over a spatial hypersurface Σ of constant time:

$$E = \int_{\Sigma} t_{00} d^3x,$$

and the total energy radiated through all space is given by:

$$\Delta E = \int P dt,$$

where the power, P, is given by:

$$P = \int_{S^2_\infty} t_{0\nu} n^\nu r^2 d\Omega. \tag{1.34}$$

This integral is over a two-sphere that extends to infinity, and n^μ is a unit vector normal to the surface of the two sphere. In polar coordinates t, r, θ, ϕ, the normal vector is:

$$n^\mu = (0, 1, 0, 0).$$

We want to calculate the power P radiated by a source emitting gravitational radiation. The issue is that the expression for the energy-momentum tensor (1.32) is in terms of transverse-traceless perturbation, while the quadrupole formula (1.24) is in terms of trace-reversed perturbation. So, we need to first convert the trace reversed form into TT gauge to insert that into $t_{\mu\nu}$.

We start by introducing the projection tensor:

$$P_{ij} = \delta_{ij} - n_i n_j.$$

It projects the tensor components to a surface orthogonal to n^i. In case of our normal vector, P_{ij} will project onto the infinite two-sphere. If we have a symmetric spatial tensor, and we want to construct a transverse traceless tensor, we can do so by using the projection vector as:

$$X_{ij}^{TT} = \left(P_i^k P_j^l - \frac{1}{2} P_{ij} P^{kl}\right) X_{kl}. \tag{1.35}$$

Taking the trace of the above tensor gives $X_i^i = 0$, and it is also transverse as

$$\begin{aligned}
\partial_i \left(P^{ik} P^{jl} - \frac{1}{2} P^{ij} P^{kl}\right) X_{kl} &= (P^{jl}\partial_i(\delta^{ik} - n^i n^k) + P^{ik}\partial_i(\delta^{jl} - n^j n^l) \\
&\quad - \frac{1}{2}(P^{kl}\partial_i(\delta^{ij} - n^i n^j) + P^{ij}\partial_i(\delta^{kl} - n^k n^l))X_{kl} \\
&= 0.
\end{aligned}$$

Because the projection tensor is traceless, the trace-reversed tensor will also be the same as the original perturbation, and using the quadrupole formula (1.24) we get,

$$h_{ij}^{TT} = \bar{h}_{ij}^{TT} = \frac{2G}{r} \frac{d^2 I_{ij}^{TT}}{dt^2}(t - r). \tag{1.36}$$

A more convenient quantity than the quadrupole moment is the reduced quadrupole moment

$$J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl} I_{kl}.$$

It is just the traceless part of the quadrupole moment. With reduced quadrupole moment, Eq. (1.36) becomes:

$$h_{ij}^{TT} = \bar{h}_{ij}^{TT} = \frac{2G}{r}\frac{d^2 J_{ij}^{TT}}{dt^2}(t-r).$$

Power involves the factor $t_{0\mu}n^{\mu} = t_{0r}$, and for that we need derivatives of the metric perturbation with respect to time and r:

$$\begin{aligned}
\partial_0 h_{ij}^{TT} &= \frac{2G}{r}\frac{d^3 J_{ij}^{TT}}{dt^3}, \\
\partial_r h_{ij}^{TT} &= -\frac{2G}{r}\frac{d^3 J_{ij}^{TT}}{dt^3} - \frac{2G}{r^2}\frac{d^2 J_{ij}^{TT}}{dt^2} \\
&\sim \frac{2G}{r}\frac{d^3 J_{ij}^{TT}}{dt^3}.
\end{aligned}$$

r^{-2} term was dropped because it falls off rapidly in the $r \to \infty$ limit. With these results, t_{0r} becomes:

$$t_{0r} = -\frac{G}{8\pi r^2}\left\langle \left(\frac{d^3 J_{ij}^{TT}}{dt^3}\right)\left(\frac{d^3 J_{TT}^{ij}}{dt^3}\right)\right\rangle$$

Now, before substituting this into the expression for power, we have to convert back to J_{ij}

from the transverse traceless part. From Eq. (1.35),

$$
\begin{aligned}
X^{TT}_{ij}X^{ij}_{TT} &= \left(P^k_i P^l_j - \frac{1}{2}P_{ij}P^{kl}\right)X_{kl}\left(P^{im}P^{jn} - \frac{1}{2}P^{ij}P^{mn}\right)X_{mn} \\
&= [(\delta^k_i - n_i n^k)(\delta^l_j - n_j n^l) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta^{kl} - n^k n^l)]X_{kl} \\
&\quad [(\delta^{im} - n^i n^m)(\delta^{jn} - n^j n^n) - \frac{1}{2}(\delta^{ij} - n^i n^j)(\delta^{mn} - n^m n^n)]X_{mn} \\
&= [(X_{il} - n_i n^k X_{kl})(X_{kj} - X_{kl}n_j n^l) - \frac{1}{2}(X_{kl}\delta_{ij} - X_{kl}n_i n_j)(X - X_{kl}n^k n^l)] \\
&\quad [(X^i_n - X_{mn}n^i n^m)(X^j_m - X_{mn}n^j n^n) - \frac{1}{2}(X_{mn}\delta^{ij} - X_{mn}n^i n^j)(X - X_{mn}n^m n^n)].
\end{aligned}
$$

After some algebra we get

$$
X^{TT}_{ij}X^{ij}_{TT} = X_{ij}X^{ij} - 2X^j_i X^{ik}n_j n_k + \frac{1}{2}X^{ij}X^{kl}n_i n_j n_k n_l - \frac{1}{2}X^2 + XX^{ij}n_i n_j.
$$

For J_{ij}, the expression becomes:

$$
J^{TT}_{ij}J^{ij}_{TT} = J_{ij}J^{ij} - 2J^j_i J^{ik}n_j n_k + \frac{1}{2}J^{ij}J^{kl}n_i n_j n_k n_l,
$$

where we have used the fact that $J^i_i = 0$. Now we can write the power given by Eq. (1.34) as

$$
P = \frac{G}{8\pi}\int_{S^2_\infty}\left\langle \frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3} - 2\frac{d^3J^j_i}{dt^3}\frac{d^3J^ik}{dt^3}n_j n_k + \frac{1}{2}\frac{d^3J^{ij}}{dt^3}\frac{d^3J^{kl}}{dt^3}n_i n_j n_k n_l \right\rangle d\Omega.
$$

Quadrupole tensors are independent of the angular coordinates because they are integrals over all of space. So, we take them outside the integral and use the following identities to solve the expression for power:

$$
\begin{aligned}
\int d\Omega &= 4\pi, \\
\int n_i n_j d\Omega &= \frac{4\pi}{3}\delta_{ij}, \\
\int n_i n_j n_k n_l d\Omega &= \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).
\end{aligned}
$$

We evaluate the power as

$$
\begin{aligned}
P = &-\frac{G}{8\pi}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}\int_{S^2_\infty} d\Omega - 2\frac{d^3 J_i^j}{dt^3}\frac{d^3 J^{ik}}{dt^3}\int_{S^2_\infty} n_j n_k d\Omega \right. \\
&\left. +\frac{1}{2}\frac{d^3 J^{ij}}{dt^3}\frac{d^3 J^{kl}}{dt^3}\int_{S^2_\infty} n_i n_j n_k n_l d\Omega \right\rangle \\
&= -\frac{G}{8\pi}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}4\pi - 2\frac{d^3 J_i^j}{dt^3}\frac{d^3 J^{ik}}{dt^3}\frac{4\pi}{3}\delta_{jk} \right. \\
&\left. +\frac{1}{2}\frac{d^3 J^{ij}}{dt^3}\frac{d^3 J^{kl}}{dt^3}\frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{G}{8\pi}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}4\pi - 2\frac{d^3 J_i^j}{dt^3}\frac{d^3 J_j^i}{dt^3}\frac{4\pi}{3} + 0 + \frac{2\pi}{15}\frac{d^3 J_k^j}{dt^3}\frac{d^3 J_j^k}{dt^3} + \frac{1}{2}\frac{d^3 J_l^j}{dt^3}\frac{d^3 J_j^l}{dt^3}\frac{4\pi}{15}\right\rangle \\
&= -\frac{G}{8\pi}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}4\pi - \frac{12\pi}{5}\frac{d^3 J_i^j}{dt^3}\frac{d^3 J_j^i}{dt^3}\right\rangle \\
&= -\frac{G}{8\pi}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}4\pi - \delta_{im}\delta^{in}\frac{12\pi}{5}\frac{d^3 J^{jm}}{dt^3}\frac{d^3 J_{jn}}{dt^3}\right\rangle \\
&= -\frac{G}{8\pi}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}4\pi - \delta_m^n\frac{12\pi}{5}\frac{d^3 J^{jm}}{dt^3}\frac{d^3 J_{jn}}{dt^3}\right\rangle \\
&= -\frac{G}{5}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}\right\rangle .
\end{aligned}
$$

The reduced quadrupole moment J_{ij} is evaluated at a retarded time $t_r = t - r$. The negative sign represents that the radiating source will be losing energy.

Coming back to the binary star system with each star at a distance r from the center, first we find the reduced quadrupole moment to be:

$$
J_{ij} = \frac{Mr^2}{3}\begin{pmatrix} (1+3\cos 2\Omega t) & 3\sin 2\Omega t & 0 \\ 3\sin 2\Omega t & (1-3\cos 2\Omega t) & 0 \\ 0 & 0 & -2 \end{pmatrix},
$$

and the required time derivatives are:

$$
\frac{d^3 J_{ij}}{dt^3} = 8Mr^2\Omega^3\begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Using the expression for power that we just derived, we get:

$$P = -\frac{128}{5} G M^2 r^4 \Omega^6, \tag{1.37}$$

and using the expression for the frequency of a binary star system, Eq. (1.26), we get:

$$P = -\frac{2}{5} \frac{G^4 M^5}{r^5}.$$

Energy loss during the emission of gravitational radiation has been measured, and the results are consistent with the prediction of general relativity. In 1974, Hulse and Taylor discovered a binary system in which one of the stars is a Pulsar. Pulsars provide very accurate clocks, and by observing the change in their gravitational periods, a passing gravitational wave can be detected. So let's discuss how the orbital period is affected by energy loss.

Energy loss in a Binary Pulsar[11]

Gravitational radiation reduces the energy and angular momentum of an orbiting binary system and changes the orbital period P. We evaluate the rate of change of orbital period in a binary pulsar system. We assume that their orbit is circular and the pulsars are of equal masses M, with orbital radius R and speed V. Assuming that the system is non-relativistic, its Newtonian energy (in G = 1 units) is:

$$E_{\text{newt}} = 2\left(\frac{1}{2} M v^2\right) - \frac{M^2}{2R}.$$

Using Eq. (1.25), we get

$$E_{\text{newt}} = -\frac{M^2}{4R}.$$

And R is related to orbital time period P as:

$$\begin{aligned} 2\pi R &= vP = \sqrt{\frac{M}{4R}}P \\ R &= \left(\frac{MP^2}{16\pi^2}\right)^{1/3}. \end{aligned}$$

With this, the energy becomes:

$$\begin{aligned} E_{\text{newt}} &= -\frac{1}{4}M\left(\frac{4\pi M}{P}\right)^{2/3}, \\ \frac{dE}{dt} &= -\frac{1}{4}M\frac{d\left(\frac{4\pi M}{P}\right)^{2/3}}{dt}. \end{aligned}$$

Substituting Eq. (1.37) and expressing R in terms of orbital period, we get the expression for the rate of decrease of orbital period as:

$$\frac{dP}{dt} = -\frac{96}{5}\pi 4^{1/3}\left(\frac{2\pi M}{P}\right)^{2/3}. \tag{1.38}$$

This remarkable prediction of general relativity was confirmed through pulsar timing observations, providing indirect evidence for the existence of gravitational waves.

Having now discussed their theoretical production and energy loss, we turn to how gravitational waves are measured on Earth.

The bulk motion of large masses produces gravitational waves. As a simple example, consider a binary star system where the mass of each star is M, with orbital radius R. Using the Newtonian formulae for the orbital parameters will suffice for an order-of-magnitude estimate. The relevant parameters are the Schwarzschild radius, $R_s = 2GM/c^2$, the orbital radius R, and the distance r between the observer and the binary system. We are restoring the factors of c for comparison with experiments. In terms of the relevant parameters, the frequency of the orbit and of the gravitational waves produced is:

$$f = \frac{\Omega}{2\pi} \approx \frac{c\sqrt{R_s}}{10R^{3/2}}.$$

From the metric perturbation Eq. (1.27), we find the approximate amplitude:

$$h \approx \frac{R_s}{rR}.$$

To see what this implies, consider a black hole merger. We take both black holes to be of 10 solar masses, and the system is at a distance from an observer $r \approx 100$ Mpc. R is ten times their Schwarzschild radii:

$$R_s \sim 10^6\text{cm},$$

$$R \sim 10^7\text{cm},$$

$$r \sim 10^{26}\text{cm}.$$

Frequency and amplitude for such a source are:

$$f \sim 10^2 \mathrm{s}^{-1}, \quad h \sim 10^{-21}$$

So, to detect these, we need instruments sensitive to the frequency of 100 Hz and strains of order 10^{-21} or less.

One technique for detecting gravitational waves is interferometry. A passing gravitational wave slightly perturbs the motions of freely falling masses. If we have two test masses separated by a distance L, the change in their distance will be approximately:

$$\frac{\delta L}{L} \sim h.$$

If these test bodies are separated by order of kilometers, we would need sensitivity to changes of the order

$$\delta L \sim 10^{-16} \left(\frac{h}{10^{-21}} \right) \left(\frac{L}{\mathrm{km}} \right) .\mathrm{cm}.$$

For comparison, the size of a typical Fermi nucleus is 10^{-13} cm. So, we need to detect changes in distances much smaller than what the test particles would be made of.

One possible way to measure such small perturbations is using Laser Interferometers. Consider the setup in Figure 1.5. A laser is directed at a beam splitter, which sends the beam to two tubes of length L. At the end of the tubes are test masses represented by mirrors suspended from pendulums. These are partially reflective, so a typical photon is reflected around 100 times before returning to the beam splitter, which is then directed to a photodiode. The system is set up so that the returning beams destructively interfere if the

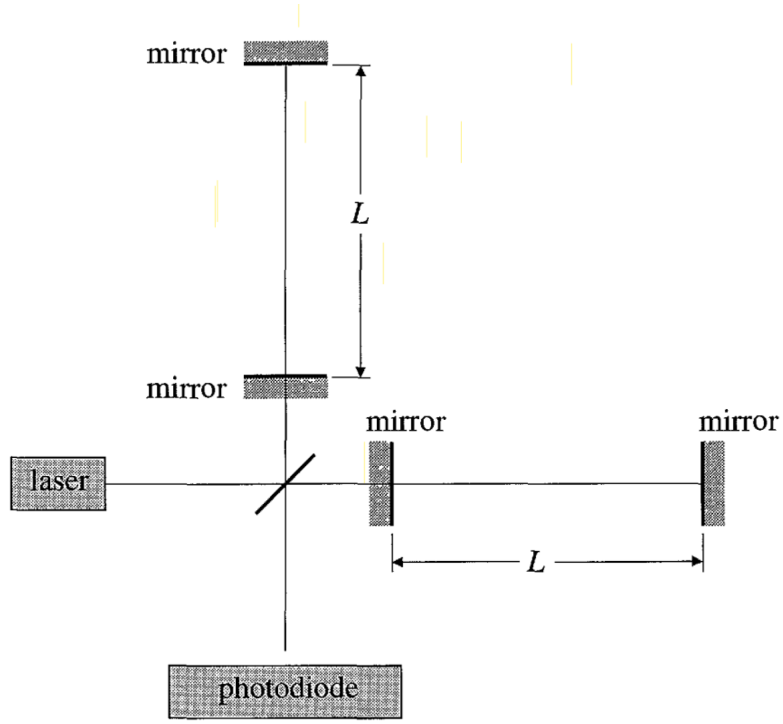


Figure 1.5: Schematic diagram of a GW interferometer.

test masses are perfectly stationary. No signal is sent to the photodiode. A passing gravitational wave will perturb the length, leading to a phase shift, and the waves will no longer destructively interfere. During 100 round trips, the accumulated phase shift will be

$$\delta\phi \sim 200\left(\frac{2\pi}{\lambda}\right)\delta L \sim 10^{-9}.$$

The factor of 200 represents that the phase shift from the two arms adds up. This is a very small phase shift and can be measured if the number of photons N is sufficiently large, particularly if $\sqrt{N} > \delta\phi$.

Terrestrial observatories are limited due to fundamental noise sources. Space-based observatories such as Laser Interferometer Space Antenna, LISA, will be more sensitive to frequencies in the range of 10^{-2} Hz because their implementation will be dramatically different. The potential terrestrial noise source is seismic noise, which dominates at low frequencies. At high frequencies, the dominant noise source is photon shot noise. At intermediate frequencies, thermal noise dominates. Noise from gravitation gradients due to atmospheric pressure is irreducible, too. Satellite observatories are free from such limitations. So, the fundamental limitation is expected to come from measuring changes in distances between the spacecraft and from non-gravitational accelerations of the spacecraft in which the test masses are suspended.

1.6.1 A Brief Overview of Possible Sources

Compact binaries are a source of gravitational waves, and they can be detected by ground-based observatories when they are close to coalescence. Another source is the non spherically symmetric collapse of massive stars that gives rise to supernovae. Their detection can be coordinated with the observation of supernovae by radio telescopes. Although they produce very small amplitude waves, rotating neutron stars are one source that advanced detectors can detect. The evolution of the gravitational wave signal from a solar-mass black hole orbiting another such black hole can be tracked. Such information will allow for precise mapping of the spacetime metric.

Other than these localized sources, there is also a possibility of gravitational wave backgrounds. These waves would have been generated in the early universe, with a smoothly varying power spectrum as a function of frequency. Such waves are currently impossible to detect even by possible space-based observatories. Primordial waves generated by a violent phase transition lie in a band potentially observable by space-based observatories. Additionally, pulsar timing arrays (PTAs) are sensitive to gravitational waves in the nanohertz regime. They provide a window into stochastic backgrounds from cosmological and astrophysical sources such as supermassive black hole binaries or early-universe phase transitions.

Next, we discuss gravitational waves from compact binaries as they are the most promising source for ground-based detectors, and the first detection made by LIGO was also of a binary neutron star.

2

Compact Binary Dynamics and Gravitational Radiation [10]

We have already encountered gravitational waves from a compact binary of equal-mass components. In this chapter, we analyze binaries with unequal masses and derive the gravitational wave strain, key orbital parameters, and the energy loss due to radiation, within the

Newtonian approximation. This allows us to understand the basic mechanisms of inspiral and radiation without the complexities of full general relativity.

Later in this chapter, we will motivate the need for a more accurate treatment of relativistic binaries and introduce the post-Newtonian (PN) approximation to improve Newtonian dynamics systematically. This sets the stage for Chapter 3, where we delve into the details of PN formalism.

2.1 Binary Dynamics in the Newtonian Limit

Consider a binary system in circular orbits in an x-y plane. As we know, gravitational waves carry energy, so the binary loses energy and the orbit cannot remain circular, but we assume they stay quasicircular. The rest masses of the objects, m_1 and m_2 in the binary are such that $m_1 \geq m_2$. If we consider a very large orbit, we can model the individual binary components as point masses. For a binary with separation r and orbital frequency ω, in the center of mass system location of the objects is:

$$\mathrm{x}_1 = \frac{m_2 r}{M}(\cos wt, \sin wt, 0), \tag{2.1}$$

$$\mathrm{x}_2 = -\frac{m_1}{m_2}\mathrm{x}_1, \tag{2.2}$$

where $M = m_1 + m_2$. Assuming point masses, their mass density is:

$$\rho(t, \mathrm{r}) = m_1\delta(\mathrm{x} - \mathrm{x}_1(t)) + m_2\delta(\mathrm{x} - \mathrm{x}_2(t)) = T^{00}.$$

The mass quadrupole moment of the system is:

$$I^{ij} = m_1 x_1^i(t) x_1^j(t) + m_2 x_2^i(t) x_2^j(t),$$

When differentiated twice,

$$\frac{d^2 I^{ij}}{dt^2} = m_1(\ddot{x}_1^i(t)x_1^j(t) + x_1^i(t)\ddot{x}_1^j(t) + 2\dot{x}_1^i(t)\dot{x}_1^j) + m_2(\ddot{x}_2^i(t)x_2^j(t) + x_2^i(t)\ddot{x}_2^j(t) + 2\dot{x}_2^i(t)\dot{x}_2^j(t)).$$

Using the coordinates from Eq. (2.1) and (2.1), we find:

$$\begin{aligned}
\frac{d^2 I^{11}}{dt^2} &= m_1\left(\ddot{x}_1^1(t)x_1^1(t) + x_1^1(t)\ddot{x}_1^1(t) + 2\dot{x}_1^1(t)^2\right) + m_2\left(\ddot{x}_2^1(t)x_2^1(t) + x_2^1(t)\ddot{x}_2^1(t) + 2\dot{x}_2^1(t)^2\right) \\
&= -2r^2\omega^2\cos(2\omega t)\left(\frac{m_1 m_2^2 + m_2 m_1^2}{M^2}\right) \\
&= -2r^2\omega^2\cos(2\omega t)\frac{m_1 m_2}{M}.
\end{aligned}$$

Similarly, calculating the rest of the components and using the quadrupole formula (1.24) (with $G = 1$), we write the metric perturbation as

$$h_{ij} = -4\frac{M\omega^2\eta r^2}{d}\begin{pmatrix} \cos 2\omega t_r & \sin 2\omega t_r & 0 \\ \sin 2\omega t_r & -\cos 2\omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $t_r = t - d$ and the symmetric mass ratio η is:

$$\eta = \frac{m_1 m_2}{M^2}.$$

Again, this tensor is traceless, and it is also transverse for waves traveling in the z-direction. An observer (essentially a detector) will not necessarily be along the z-axis. So, for more general scenarios, h_{ij} has to be projected to a two-dimensional subspace perpendicular to the direction of propagation to remain in a transverse traceless gauge. So, we can project it using Eq. (1.35) to a traceless transverse gauge.

Recall that the two independent components of the metric represent the two polarizations. Depending on the orientation of an L-shaped interferometer, it will be sensitive to a linear combination of the two polarizations. Denoting this observable signal by h, for the case of the compact binary signal, it can be written as:

$$h(t, \boldsymbol{\theta}) = \mathcal{A}\frac{M\nu^2\eta}{d}(\cos^2 2\omega t + \phi_0),$$

where $v = \omega r$ is the relative velocity, ϕ_0 is a constant phase, and $\mathcal{A}$ is the amplitude factor; the latter two depend on the geometry of the source and the detector. $\boldsymbol{\theta} = M, \nu, \eta, d, \omega, \phi_0, \mathcal{A}$ is a vector of parameters that describe the orbital motion and the orientation of the source. All these parameters are not independent; the velocity is related to mass and orbital radius using Kepler's third law as:

$$v^2 = \frac{M}{r} \implies v^3 = M\omega. \tag{2.3}$$

This relation allows us to express all relevant quantities, such as the waveform frequency and the gravitational wave power emitted by the binary, in terms of the orbital velocity or frequency.

Since the gravitational radiation is sourced by the time-varying quadrupole moment of the system, the total power radiated, often referred to as the gravitational wave luminosity, can be related to the strain amplitude and its time derivative. The luminosity, defined as the flux of gravitational wave energy integrated over a sphere surrounding the source and averaged over an orbital period, scales as

$$\mathcal{L} \propto |d\dot{h}|^2 \propto M^2\nu^4\eta^2\omega^2 = \eta^2\nu^{10},$$

where we used Kepler's law to rewrite the expression in terms of velocity parameters. An exact expression for the power radiated by a binary system, derived from the quadrupole formula, is given by Eq. (1.37):

$$P = \frac{32}{5}\eta^2\nu^{10}. \tag{2.4}$$

Here, P represents the total power (luminosity) emitted as gravitational waves, integrated over all directions and averaged over an orbital cycle.

Now we can find the energy of the binary using Newton's law,

$$E_N = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2\right) - \frac{m_1 m_2}{r}.$$

We can express this expression in terms of the symmetric mass ratio to make it more concise. In the center-of-mass frame, the velocities of the two masses can be written in terms of the total mass $M = m_1 + m_2$ and the relative velocity v of the binary system as follows

$$
\begin{aligned}
v_1 &= \frac{m_2}{M} v, \\
v_2 &= \frac{m_1}{M} v.
\end{aligned}
$$

Now, substituting these velocities into the kinetic energy expression, we get

$$
\begin{aligned}
E_{\text{kin}} &= \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2\right) \\
&= \frac{1}{2}\left(m_1\left(\frac{m_2}{M}v\right)^2 + m_2\left(\frac{m_1}{M}v\right)^2\right) \\
&= \frac{1}{2}v^2\left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2}\right) \\
&= \frac{1}{2}v^2\left(\frac{m_1 m_2 (m_1 + m_2)}{M^2}\right).
\end{aligned}
$$

Since, $M = m_1 + m_2$, kinetic energy simplifies

$$
E_{\text{kin}} = \frac{1}{2}\frac{m_1 m_2}{M} v^2.
$$

Substituting the symmetric mass ratio into the expression:

$$
E_{\text{kin}} = \frac{1}{2} M \eta v^2.
$$

With this expression and Newton's law of gravity, we find the total energy of the binary:

$$\begin{aligned}
E &= \frac{M\eta}{2}v^2 - \frac{m_1 m_2}{r} \\
&= \frac{M\eta}{2}v^2 - \frac{M^2\eta}{r} \\
&= \frac{M\eta}{2}v^2 - M\eta v^2 \\
&= -\frac{M\eta}{2}v^2. \qquad (2.5)
\end{aligned}$$

Using the expression for the power radiated by this binary system, Eq. (2.4) and the total energy of the binary, Eq. (2.5), the energy lost by the binary equals the energy carried away by gravitational waves. This energy balance allows us to determine how the orbital velocity evolves with time,

$$\begin{aligned}
\frac{dE(t)}{dt} &= P(t) \\
\frac{dE(v)}{dv}\frac{dv}{dt} &= P \\
\frac{dv}{dt} &= \frac{P}{dE/dv} = \frac{32}{5M}\eta v^9(t) \\
\int_v^0 \frac{dv}{v^9} &= \frac{32}{5M}\eta \int_t^{t_c} dt \\
\frac{1}{8}v^{-8} &= \frac{32}{5M}\eta(t_c - t) \\
v &= \sqrt[8]{\frac{5M}{256\eta(t_c - t)}} \\
v &= \frac{1}{2}\sqrt[8]{\frac{5M}{\eta(t_c - t)}},
\end{aligned}$$

where t_c is the coalescence time, at which the velocity diverges, although the assumptions we made are not valid all the way up to t_c. As the separation shrinks, the orbital velocity increases rapidly, approaching divergence as $t \to t_c$. Although our assumptions break down near t_c, the general behavior, an accelerating inspiral, is physically expected. The evolution of $v(t)$ is shown in Figure 2.1, exhibiting the steep increase in orbital speed as the binary nears merger.

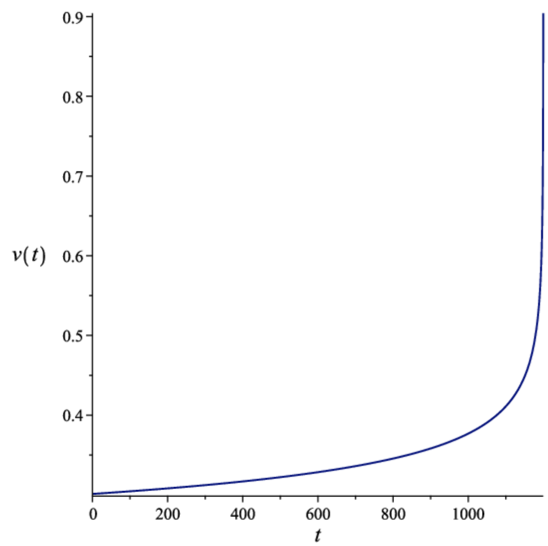


Figure 2.1: Evolution of relative velocity.

To find how the separation of the objects in the binary changes with time, we relate r with v as:

$$r = \frac{M}{v^2} = 4\sqrt[4]{\frac{\eta(t_c - t)}{5M}}.$$

At $t = 0$, the binary's separation is

$$r_0 = \frac{M}{v^2} = 4M\sqrt[4]{\frac{\eta(t_c)}{5M}}.$$

From this expression, we can find t_c

$$
\begin{aligned}
r_0^4 &= 4^4 M^4 \frac{\eta t_c}{5M}, \\
t_c &= \frac{5r_0^4}{256\eta M^3}.
\end{aligned}
$$

The orbital frequency given by Eq. (2.3) now becomes

$$
\omega(t) = \frac{v^3}{M} = \frac{1}{8}\left(\frac{5}{M^{3/8}\eta(t_c - t)}\right)^{3/8} = \frac{1}{8\mathcal{M}^{5/8}}\left(\frac{5}{(t_c - t)}\right)^{3/8},
$$

where

$$
\mathcal{M} = M\eta^{3/5},
$$

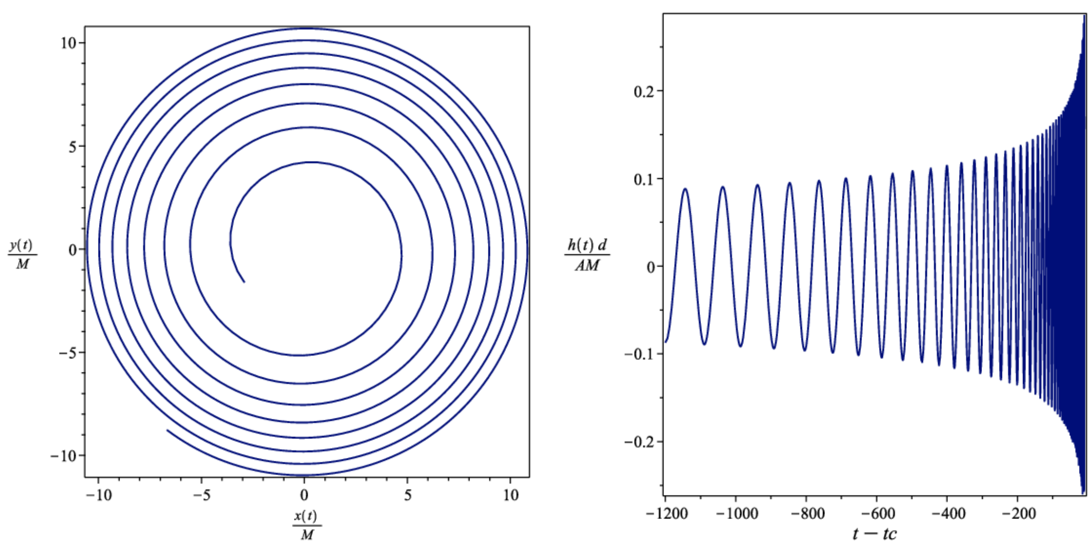
is the chirp mass.

The orbital phase $\phi(t)$ tracks the position of the binary in its orbit, and it directly controls the oscillations of the gravitational wave signal. This makes it crucial to accurately model and match waveforms to observed data. The orbital frequency ω describes how fast the binary components revolve around each other. The dominant gravitational wave frequency observed is twice this orbital frequency because the quadrupole radiation repeats every half orbit. Using the expression for orbital frequency, we can find the orbital phase:

$$
\begin{aligned}
\phi(t) &= \int \omega(t) dt \\
&= \frac{1}{8\mathcal{M}^{5/8}} \int \left(\frac{5}{(t_c - t)} \right)^{3/8} dt \\
&= -\frac{1}{8\mathcal{M}^{5/8}} \left(5^{3/8} \right) \frac{8}{5} \left(t_c - t \right)^{5/8} + \phi_c \\
&= \phi_c - \left(\frac{t_c - t}{5\mathcal{M}} \right)^{5/8} .
\end{aligned}
$$

The integration constant ϕ_c is the phase at coalescence time t_c. This result shows that the chirp mass predominantly governs the inspiral of the binary.

The binary separation vector $(r \cos \phi, r \sin \phi)$, plotted in Figure $2.2a$, traces a nearly circular orbit typical of the Newtonian inspiral approximation. The gravitational wave strain $h(t)$, shown in Figure $2.2b$, exhibits the expected oscillatory pattern driven by this orbital motion.



(a) Inspiral track of a Newtonian binary.

(b) Scaled gravitational wave signal $h(t)$.

2.2 Towards the Post-Newtonian Expansion

We have illustrated the behaviour of a compact binary in the Newtonian limit. We have two main approaches to include relativistic, non-linear gravitational effects: a full numerical solution of Einstein's equations (numerical relativity), or an analytic approximation scheme known as the Post-Newtonian (PN) expansion.

In the following, we review the Post-Newtonian formalism, which systematically extends Newtonian results by incorporating corrections in powers of (v/c). This allows us to capture important non-linear effects without solving the full equations numerically.

3

A Review of Post-Newtonian (PN) Formalism[15,2]

The Post-Newtonian (PN) formalism offers a systematic and iterative framework to refine Newtonian solutions by incorporating relativistic corrections. Its strength lies in yielding analytic expressions that can be expanded to any desired order in powers of velocity. By expanding the metric and solving it iteratively using the PN formalism, we can construct non-linear solutions that allow us to obtain analytical waveforms for compact binary systems. This section largely follows Maggiore's book on gravitational waves,[15] and Luc Blanchet's review[2].

So far, we have assumed a flat background spacetime, where the contribution of sources to the overall curvature is negligible. However, many astrophysical systems of interest, such as compact binaries, are governed by strong gravitational forces, where the motion of the source and the curvature of spacetime are closely intertwined. For a binary with total mass M, we have $(v/c)^2 = 2GM/c^2d = R_s/d$, where R_s is the Schwarzschild radius. The ratio R_s/d characterizes the strength of the gravitational field near the source and incorporates relativistic corrections due to the source's velocity.

Assuming a flat background amounts to working in the Newtonian limit. But we must go beyond Newtonian theory to accurately describe systems where self-gravity significantly affects the background curvature, such as inspiraling relativistic binaries. This is where the Post-Newtonian (PN) formalism becomes essential. It systematically incorporates relativistic corrections to Newtonian gravity, allowing us to construct more accurate waveforms to extract gravitational wave signals from experimental data.

3.1 Slowly Moving Weakly Self-Gravitating Sources

Slowly moving, weakly self-gravitating sources are characterized by values of $(v/c)^2$ both small and comparable in magnitude, but not close to 1. In such cases, post-Newtonian (PN) formalism provides a valid and accurate description. However, as R_s/d gets close to 1, we enter the regime of strong gravitational fields where PN expansions break down, and strong-field methods may be required.

Consider slowly moving and weakly self-gravitating sources, for which (v/c) (both bulk and internal velocities of compact objects) and R_s/d are sufficiently small. However, during the last stages of coalescence of binaries, they can reach values of v/c as high as $1/2$ and are very relativistic objects. In that case, we will need the result to a very high order in v/c. However, we can only expand to a certain order when doing an analytic expansion of the metric. This is an unavoidable limitation of the PN formalism. Beyond these high velocities, during the late inspiral phase of the compact binary coalescence, we have to resort to numerical methods of solving the Einstein equations. We use these as expansion parameters, and they are generally related by $v/c \approx (R_s/d)^{1/2}$. We also assume the energy-momentum tensor $T^{\mu\nu}$ has a spatially compact support, i.e., it can be enclosed in a time-like world tube $r \leq d$, and the matter distribution inside the source is smooth and infinitely differentiable.

We want to figure out a way to compute systematically the corrections to the linearized theory in powers of v/c. We first distinguish between the near zone and the far zone. If ω_s is the frequency of motion inside the source and d is the source size, then the velocities inside the source are $v \approx \omega_s d$, and the frequency of the source will also be $\omega \approx \omega_s$ and $\lambda = c/\omega \implies \lambda \approx \frac{c}{v} d$. For non-relativistic sources

$$\lambda \gg d.$$

The near zone is the region where $r \ll \lambda$. And the exterior near zone is the region:

$$d < r \ll \lambda.$$

In the near zone, retardation effects are limited, and we have static potentials. We will see that PN formalism is the correct approximation to use here. The far zone is the region where $r \gg \lambda$.

We might think that the expansion has two independent aspects. First, we must determine the GR correction to the equations of motion to the desired order in v/c, and the second is to compute GWs emitted by these sources. However, things are much more complicated than that. These two aspects are intertwined. Thus, at the leading order, emission of GWs costs energy, which is lost by the sources, which, after a certain order, will backreact on the sources, affecting their equations of motion. Moreover, due to the nonlinearity of general relativity, the gravitational field is itself a source for GW generation, and GWs computed to a particular order are sources for GW production at a higher order. This complicates a full-fledged formalism for computing the output of GWs in powers of v/c.

3.2 PN Expansion of Einstein Equations

Let's begin by analyzing the lowest-order PN corrections, neglecting the back reaction (back reaction does not contribute to the equations of motion at the PN first order). Assuming the source is non-relativistic, we introduce a small parameter:

$$\epsilon \approx \frac{R_s}{d}^{1/2} \approx \frac{v}{c}.$$

We also require that $|T^{ij}|/T^{00} = O(\epsilon^2)$ i.e, the source be weakly stressed. Thus, energy density is the main contributor to the gravitational effects at the leading order. We expand the metric and the stress-energy tensor in powers of ϵ. Neglecting radiation emission, a classical system under conservative forces is invariant to time reversal. Under time reversal, g_{00} and g_{ij} are even. Velocity changes sign so g_{00} and g_{ij} can contain only even powers of v. However, g_{0j} are odd, so they contain odd powers of velocity. By inspection of Einstein's equations, one finds that, to work consistently to a given order in ϵ, if we expand g_{00} up to order ϵ^n, we must also expand g_{0i} up to order ϵ^{n-1} and g_{ij} up to ϵ^{n-2}. Furthermore, the expansion of g_{0i} starts from $O\left(\epsilon^3\right)$. Thus, the metric is expanded as follows,

$$\begin{aligned}
g_{00} &= -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + {}^{(6)}g_{00} + \ldots \\
g_{0i} &= {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \ldots \\
g_{ij} &= \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \ldots
\end{aligned} \tag{3.1}$$

where ${}^{(n)}g_{\mu\nu}$ denotes the terms of order ϵ^n in the expansion of $g_{\mu\nu}$. It reduces to the Minkowski metric at the first order, as expected. Similarly, we expand the energy-momentum tensor of matter,

$$\begin{aligned}
T^{00} &= {}^{(0)}T^{00} + {}^{(2)}T^{00} + \ldots \\
T^{0i} &= {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + \ldots \\
T^{ij} &= {}^{(2)}T^{ij} + {}^{(4)}T^{ij} + \ldots
\end{aligned} \tag{3.2}$$

Stresses T^{ij} are second-order relativistic corrections in a non-relativistic system, and we also required the stresses to be small compared to the energy density.

We now substitute these in Einstein's equation and equate the terms of the same order in ϵ. To determine the order, we must also consider that for slow-moving sources, time derivatives of the metric generated by this source are smaller than the spatial derivative and are related as:

$$\frac{\partial}{\partial t} = O(v)\frac{\partial}{\partial x^i}. \tag{3.3}$$

So, the d'Alembertian operator becomes:

$$\Box^2 = [(1 + O(\epsilon^2))]\nabla^2.$$

Therefore the spatial derivative is the leading order term and the lowest order solution is in terms of instantaneous potentials and retardation effects are small corrections.

In PN expansion, we are effectively trying to compute some quantity $F(t - r/c)$, such as a given component of a metric which is a function of retarded time,

$$F\left(t - \frac{r}{c}\right) = F(t) - \frac{r}{c}\dot{F}(t) + \frac{r^2}{2c^2}\ddot{F}(t) + \dots \tag{3.4}$$

Each derivative of F carries a factor of ω, the frequency of the radiation emitted. And $\omega/c = 1/\lambda$, so Eq. (3.4) is essentially an expansion in powers of r/λ; therefore, a PN expansion is only valid in the near zone $r \ll \lambda$ and breaks down in the far/radiation zone $r \gg \lambda$. We will return to the breakdown in the far region later, as a naive expansion in the far region will lead to divergences. Therefore, PN expansion is a tool that can be used to compute the gravitational field in the near region. Still, it must be supplemented by a different treatment to find the fields in the radiation zone.

3.3 Newtonian Limit

In the Newtonian limit, we keep $g_{00} = 1 + {}^{(2)}g_{00}, g_{0i} = 0$, and $g_{ij} = \delta_{ij}$. We can obtain the equation of motion of a test particle with velocity v, in a gravitational field, from the geodesic equation:

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{3.5}$$

In a weak gravitational field, we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and, in the limit of low velocities, the proper time τ is the same, to lowest order, as the coordinate time t. Furthermore, $dx^0/dt = c$ while $dx^i/dt = O(v)$. Then, the leading term in v/c is obtained by setting $\mu = \nu = 0$ in Eq. (3.5),

$$\begin{aligned}\frac{d^2x^i}{dt^2} &\simeq -c^2\Gamma^i_{00} \\ &= c^2\left(\frac{1}{2}\partial^i h_{00} - \partial_0 h^i_0\right).\end{aligned}$$

Recall that the time derivative of the metric is of higher order, so we have:

$$\begin{aligned}\frac{d^2x^i}{dt^2} &\simeq -c^2\Gamma^i_{00} \\ &= c^2\frac{1}{2}\partial^i h_{00}.\end{aligned}$$

Writing $h_{00} = -2\phi$ and defining $U = -c^2\phi$, we recover the Newtonian equation of motion. U is the sign-reversed gravitational potential. The equation of motion corresponds to the Newtonian potential U with $v^2/c^2 \sim O(U)$. Comparing this with Eq. (3.5), we see that the leading-order term of the metric is given by $g_{00} = -1 + 2U/c^2$, while corrections to g_{ij} and g_{0i} are of order $O(v^2/c^2)$. For a photon, both g_{00} and g_{ij} contribute to the deviation from flat spacetime. The gravitational potential U in the harmonic gauge is expressed as:

$$U(t, \mathrm{x}) = \frac{G}{c^2} \int d^3x' \frac{T^{00}(t, \mathrm{x}')}{|\mathrm{x} - \mathrm{x}'|}.$$

Here, we have established that $g_{00} = 1 + {}^{(2)}g_{00}$, $g_{0i} = 0$ and $g_{ij} = \delta_{ij}$ give the Newtonian limit. Therefore, we can infer that ${}^{(4)}g_{00}$, ${}^{(3)}g_{0i}$, and ${}^{(4)}g_{ij}$ provide the metric expansion to the 1PN order, and the corresponding next order terms contribute to the 2PN order and so on. Next, we consider the expansion of the metric to 1PN order.

3.4 The 1PN Order

First, we choose a gauge condition because that will simplify the equations. We choose the harmonic gauge:

$$\partial_\mu(\sqrt{-g} g^{\mu\nu}) = 0.$$

In principle, expanding the Einstein equations by inserting the expansions is now easy. We use Eqs. (3.1), and the gauge condition to simplify the equations.

At the Newtonian order, we get

$$\nabla^2[{}^{(2)}g_{00}] = -\frac{8\pi G}{c^4}\, {}^{(0)}T^{00}. \tag{3.6}$$

Similarly, expanding to 1PN order, we get:

$$\nabla^2 \left[{}^{(2)}g_{ij}\right] = -\frac{8\pi G}{c^4}\delta_{ij}\,{}^{(0)}T^{00}, \tag{3.7}$$

$$\nabla^2 \left[{}^{(3)}g_{0i}\right] = \frac{16\pi G}{c^4}\,{}^{(1)}T^{0i}, \tag{3.8}$$

$$\begin{aligned}\nabla^2 \left[{}^{(4)}g_{00}\right] &= \partial_0^2 \left[{}^{(2)}g_{00}\right] + g_{ij}\partial_i\partial_j \left[{}^{(2)}g_{00}\right] - \partial_i \left[{}^{(2)}g_{00}\right] \partial_i \left[{}^{(2)}g_{00}\right] \\ &\quad - \frac{8\pi G}{c^4}\left\{ {}^{(2)}T^{00} + {}^{(2)}T^{ii} - 2\,{}^{(2)}g_{00}\,{}^{(0)}T^{00}\right\}.\end{aligned} \tag{3.9}$$

The solution to Eq. (3.6), with the boundary condition that it vanishes at infinity, is

$$g_{00} = -2\phi,$$

$$\phi(t, \mathrm{x}) = -\frac{G}{c^4}\int \frac{T^{00}(t, \mathrm{x}')}{|\mathrm{x} - \mathrm{x}'|} d^3x'.$$

Similarly, the 1PN order equations, Eq. (3.8) and Eq. (3.7) are solved as

$${}^{(2)}g_{ij} = -2\phi\delta_{ij}, \tag{3.10}$$

$${}^{(2)}g_{0i} = \varsigma_i, \tag{3.11}$$

where

$$\varsigma(t, \mathrm{x}) = -\frac{4G}{c^4}\int \frac{T^{0i}(t, \mathrm{x}')}{|\mathrm{x} - \mathrm{x}'|} d^3x'. \tag{3.12}$$

Now we solve Eq. (3.9):

$$\nabla^2\left[{}^{(4)}g_{00}\right] = \partial_0^2(-2\phi) + 4\phi\delta_{ij}\partial_i\partial_j(\phi) - \partial_i(-2\phi)\partial_i(-2\phi)$$
$$- \frac{8\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii} + 4\phi\,{}^{(0)}T^{00}\right\}.$$

Using the vector identity

$$\partial^i\phi\partial_i\phi = \frac{1}{2}\nabla^2(\phi^2) - \phi\nabla^2\phi,$$

we get:

$$\begin{aligned}
\nabla^2\left[{}^{(4)}g_{00}\right] &= -2\partial_0^2(\phi) - 4\left(\frac{1}{2}\nabla^2(\phi^2) - \phi\nabla^2\phi\right) + 4\phi\nabla^2\phi \\
&\quad - \frac{8\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii} + 4\phi\,{}^{(0)}T^{00}\right\}, \\
&= -4\left(\frac{1}{2}\nabla^2(\phi^2) - \phi\nabla^2\phi\right) + 4(\phi)\nabla^2(\phi) - 2\partial_0^2(\phi) - 8\phi\nabla^2(\phi) - \frac{8\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii}\right\} \\
&= -2\nabla^2(\phi^2) - 2\partial_0^2(\phi) - \frac{8\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii}\right\} \\
&= -2\nabla^2(\phi^2 + \psi),
\end{aligned}$$

where

$$\nabla^2\psi = \partial_0^2(\phi) + \frac{4\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii}\right\}, \tag{3.13}$$

which has the solution

$$\psi = \int \frac{d^3x'}{|\mathrm{x} - \mathrm{x}'|}\left\{\frac{1}{4\pi}\partial_0^2(\phi) + \frac{G}{c^4}\left[{}^{(2)}T^{00} + {}^{(2)}T^{ii}\right]\right\}.$$

ϕ and ς^i are not independent. Moreover, their explicit expressions also show that they are related through the conservation of the stress-energy tensor.

ϕ, ψ, and ε are instantaneous potentials as their value depends on stress-energy tensor at the same time (retardation effects are small corrections of O(ϵ^2)). We can re-express the solutions in terms of retarded potentials to understand them better and use them to compute higher-order potentials. Expanding g_{00} to 1PN order we have:

$$\begin{aligned} g_{00} &= -1 - 2\phi - 2(\phi^2 + \psi) + O(\epsilon^6) \\ &= -1 - 2(\phi + \psi) - 2\phi^2 + O(\epsilon^6). \end{aligned}$$

Replacing ϕ^2 with $(\phi + \psi)^2$ because ψ is already of higher order, and the additional terms will be beyond 1PN order. Introducing

$$V = -c^2(\phi + \psi), \tag{3.14}$$

with this expression, the solution for g_{00} to 1PN order is

$$\begin{aligned} g_{00} &= -1 + \frac{2V}{c^2} - \frac{2V^4}{c^4} + O\left(\frac{1}{c^6}\right) \\ &= -e^{-\frac{2V}{c^2}} + O\left(\frac{1}{c^6}\right). \end{aligned}$$

The potential satisfies the relation:

$$\nabla^2\phi = \frac{4\pi G}{c^4}\,{}^{(0)}T^{00},$$

while $\nabla^2\psi$ is given by Eq. (3.13), combining the two

$$\nabla^2(\phi+\psi) = \partial_0^2(\phi) + \frac{4\pi G}{c^4}\left\{ {}^{(0)}T^{00} + {}^{(2)}T^{00} + {}^{(2)}T^{ii} \right\}. \tag{3.15}$$

Again writing $\partial_0^2\phi = \partial_0^2(\phi+\psi)$, rearranging the above equation, we get

$$\begin{aligned}\Box V &= -\frac{4\pi G}{c^4}\left[{}^{(0)}T^{00} + {}^{(2)}T^{00} + {}^{(2)}T^{ii} \right] \\ &= \frac{4\pi G}{c^4}\left[T^{00} + T^{ii}\right].\end{aligned}$$

We also define active gravitational mass density

$$\sigma = \frac{1}{c^2}[T^{00} + T^i_i],$$

with this, 1PN equation for g_{00} becomes

$$\Box V = -4\pi G\sigma.$$

Now V can be written as the retarded integral:

$$V(t,\mathrm{x}) = -\frac{G}{c^4}\int d^3x' \frac{\sigma(t - |\mathrm{x}-\mathrm{x}'|/c, x')}{|\mathrm{x}-\mathrm{x}'|} \tag{3.16}$$

The retarded potential can be expanded in terms of instantaneous potentials by expanding $\sigma(t - |\mathrm{x}-\mathrm{x}'|/c, x')$ for small retardation effects.

$$\sigma(t - |\mathrm{x}-\mathrm{x}'|/c, x') = \sigma(t, x') - \frac{\mathrm{x}-\mathrm{x}'}{c}\partial_t\sigma + \ldots$$

Similarly, for g_{ij} and g_{0j} we can write them in terms of V. We also replace ε^i with V^i, since retardation effects are already of higher order, in Eq. (3.11) use the active mass current density,

$$\sigma_i \equiv \frac{1}{c} T^{0i}.$$

Expressing in terms of σ_i, the 1PN solution to the g_{0j} equation is

$$V_i(t, \mathrm{x}) = -\frac{G}{c^4} \int d^3x' \frac{\sigma_i(t - |\mathrm{x} - \mathrm{x}'|/c, x')}{|\mathrm{x} - \mathrm{x}'|}. \tag{3.17}$$

Similarly, in the expansion for g_{ij}, we can replace $-c\phi^2$ with V, which is of higher order. Summarizing the 1PN solution in terms of V and V_i:

$$\begin{aligned}
g_{00} &= -1 + \frac{2V}{c^2} - \frac{2V^4}{c^4} + O\left(\frac{1}{c^6}\right), \\
g_{0i} &= -\frac{4}{c^3} V_i + O\left(\frac{1}{c^5}\right), \\
g_{0j} &= \delta_{ij}\left(1 + \frac{2}{c^2} V\right) + O\left(\frac{1}{c^4}\right).
\end{aligned}$$

To 1PN order, energy-momentum enters only in two combinations σ and σ_i. Due to redefinitions, we have been able to express the metric succinctly.
Also note that at a large distance r from the source, we can expand the potentials V and V_i using

$$\frac{1}{\mathrm{x} - \mathrm{x}'} = \frac{1}{r} + \frac{\mathrm{x}.\mathrm{x}'}{r^3} + \ldots \tag{3.18}$$

Until now, PN expansion seems straightforward, as we can exp metric to the desired PN order and use the results. However, this straightforward expansion we have just discussed suffers from some limitations.

3.5 Limitations and Divergences in the PN expansion

We are trying to iteratively solve equations that have the form:

$$\Box h = S_{\mu\nu}(h),$$

$S_{\mu\nu}$ is the source term that depends on the energy-momentum tensor and $h_{\mu\nu}$. We could do an expansion of the form:

$$h_{\mu\nu} = {}^{(0)}h_{\mu\nu} + {}^{(1)}h_{\mu\nu} + {}^{(2)}h_{\mu\nu} + \dots$$

At zeroth order, we set ${}^{(0)}h_{\mu\nu} = 0$ and then solve the equation of the form

$$\nabla^2[\,{}^{(1)}h_{\mu\nu}] = (\text{matter sources}).$$

This equation is then integrated by using Green's function. Then, at the next iteration, we have an equation of the form

$$\nabla^2[\,{}^{(2)}h_{\mu\nu}] = (\text{matter sources}) + (\text{terms that depend on } {}^{(1)}h_{\mu\nu}), \tag{3.19}$$

which again we will solve using Green's function and the Poisson integral. Beyond 1PN order, the resulting Poisson integrals are *necessarily divergent.* Even if the source is compact, the second term in equation Eq. (3.19) extends over all space, raising an issue of convergence at infinity. Moreover, when we expand the potentials to a higher order l, the factors $(\mathrm{x}.\mathrm{x}')^l$ that come from expanding $\frac{1}{\mathrm{x}-\mathrm{x}'}$ diverge at large x'. The Poisson integral does not necessarily give the correct non-divergent solution to the Poisson equation.

Another problem is that the expansion of the retarded potential diverges. Our solutions are of the form:

$$h_{\mu\nu} = \frac{1}{r} F_{\mu\nu}(t - r/c).$$

For $r/c \ll 1$, the expansion

$$\frac{1}{r} F_{\mu\nu}(t - r/c) = \frac{1}{r} F_{\mu\nu}(t) - \frac{1}{c} \dot{F}_{\mu\nu}(t) - \frac{r}{2c^2} \ddot{F}_{\mu\nu}(t) + \dots,$$

blows up as $r \to \infty$. So, we cannot use PN expansion at large distances from the source. So, we use the PN expansion only in the near zone and use a different formalism in the regions far from sources. This method is called **matched asymptotic expansion**. Before discussing this formalism, we will do some redefinitions in the next section to simplify our calculations.

3.6 The Relaxed Einstein Equations

First, we recast the Einstein equations in a form that will be convenient. We define a field $\mathsf{h}^{\alpha\beta}$ as

$$\mathsf{h}^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta}. \tag{3.20}$$

This is an exact definition where we have given up the assumption of keeping $h^{\alpha\beta}$ small. In the limit that $\mathsf{h}_{\alpha\beta}$ is small $-g = 1 + h$ and we can make the simplification

$$\begin{aligned} -\mathsf{h}^{\alpha\beta} &\approx \eta^{\alpha\beta} - \sqrt{1+h}(\eta^{\alpha\beta} - h^{\alpha\beta}) \\ &= h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h. \end{aligned}$$

So, it has reduced to the trace-reversed perturbation, but with an overall negative sign. Harmonic gauge condition becomes

$$\partial_\beta \mathsf{h}^{\alpha\beta} = 0.$$

In the harmonic gauge, the Einstein equations take the form

$$\Box \mathsf{h}^{\alpha\beta} = \frac{16\pi G}{c^4} \tau^{\alpha\beta}. \tag{3.21}$$

where $\tau^{\alpha\beta}$ is defined as

$$\tau^{\alpha\beta} \equiv (-g) T^{\alpha\beta} + \frac{c^4}{16\pi G} \Lambda^{\alpha\beta},$$

$T^{\alpha\beta}$ is the matter energy-momentum tensor. The tensor $\Lambda^{\alpha\beta}$ does not depend on the matter variables and is defined by

$$\Lambda^{\alpha\beta} = \frac{16\pi G}{c^4}(-g) t_{LL}^{\alpha\beta} + \left(\partial_\nu \mathrm{h}^{\alpha\mu} \partial_\mu \mathrm{h}^{\beta\nu} - \mathrm{h}^{\mu\nu} \partial_\mu \partial_\nu \mathrm{h}^{\alpha\beta}\right), \tag{3.22}$$

where $t_{LL}^{\alpha\beta}$ is the Landau-Lifshitz energy-momentum pseudotensor:

$$\begin{aligned}\frac{16\pi G}{c^4}(-g)t_{LL}^{\alpha\beta} =& g_{\lambda\mu}g^{\nu\rho}\partial_\nu \mathrm{h}^{\alpha\lambda}\partial_\rho \mathrm{h}^{\beta\mu} + \frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}\partial_\rho \mathrm{h}^{\lambda\nu}\partial_\nu \mathrm{h}^{\rho\mu} \\ &- g_{\mu\nu}\left(g^{\lambda\alpha}\partial_\rho \mathrm{h}^{\beta\nu} + g^{\lambda\beta}\partial_\rho \mathrm{h}^{\alpha\nu}\right)\partial_\lambda h^{\rho\mu} \\ &+ \frac{1}{8}\left(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu}\right)\left(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau}\right)\partial_\lambda \mathrm{h}^{\nu\tau}\partial_\mu \mathrm{h}^{\rho\sigma}.\end{aligned}$$

Since $t_{LL}^{\alpha\beta}$ depends on the metric $g_{\mu\nu}$, it is a highly non-linear function of $\mathrm{h}_{\mu\nu}$. Using the harmonic gauge condition, we see that the last term in Eq. (3.22) can be expressed as

$$\partial_\nu \mathrm{h}^{\alpha\mu}\partial_\mu \mathrm{h}^{\beta\nu} - \mathrm{h}^{\mu\nu}\partial_\mu\partial_\nu \mathrm{h}^{\alpha\beta} = \partial_\mu\partial_\nu\left(\mathrm{h}^{\alpha\mu}\mathrm{h}^{\beta\nu} - \mathrm{h}^{\mu\nu}\mathrm{h}^{\alpha\beta}\right).$$

Defining

$$\chi^{\alpha\beta\mu\nu} = \frac{c^4}{16\pi G}\left(\mathrm{h}^{\alpha\mu}\mathrm{h}^{\beta\nu} - \mathrm{h}^{\mu\nu}\mathrm{h}^{\alpha\beta}\right).$$

Using this definition, we can also rewrite Eq. (3.21) as

$$\Box h^{\alpha\beta} = +\frac{16\pi G}{c^4}\left[(-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta}\right) + \partial_\mu\partial_\nu\chi^{\alpha\beta\mu\nu}\right],$$

Since the Einstein equations impose the covariant energy conservation, Eq. (3.21) along with the harmonic gauge condition, is completely equivalent to the Einstein field equations. The gauge condition implies:

$$\partial_\beta \tau^{\alpha\beta} = 0.$$

Eq. (3.21) alone does not constrain the dynamics of matter variables. An arbitrary time-dependent $T^{\alpha\beta}$ would satisfy the Eq. (3.21). This is why the ten components of Eq. (3.21) are called relaxed Einstein equations, as the requirement that matter variables follow equations of motion has been relaxed. Eq. (3.21) has the solution of the form

$$\mathsf{h}^{\alpha\beta} = -\frac{4G}{c^4}\int \frac{d^3x'}{|\mathrm{x}-\mathrm{x}'|}\tau^{\alpha\beta}(t-|\mathrm{x}-\mathrm{x}'|/c,\mathrm{x}').$$

We can then impose the gauge condition on this solution. Comparing this solution with the one in linearized theory Eq. (1.21), here $\tau^{\alpha\beta}$ is itself a function of h and its derivatives. So, the above equation is just an *integro-differential* equation, practically an unsolvable equation. Therefore, we have to employ approximation methods. In the near region, we can use PN expansion and retardation effects are small, while in the far region, we will have gravitational waves and retardation effects will be significant. The source term $\tau^{\alpha\beta}$ extends over all spacetime. Naively trying to solve or expand this integral results in divergences, as we discussed earlier. It is time to discuss the PN formalism that relieves the theory of these limitations.

3.7 The Blanchet-Damour Approach

For a self-gravitating, slowly moving source, two length scales are important: the size of the source d, which in the case of a binary is its orbital radius, and the length $\mathcal{R}$ boundary of the near zone. Near zone extends to $\mathcal{R} \gg d$. PN approximation breaks down in the far zone $r \gg \mathcal{R}$. Outside the source $r > d$, the energy momentum tensor is zero, and the contribution to $\tau^{\alpha\beta}$ comes from the field itself. If the field inside the source is weak, then at

$r = d$ the spacetime would not differ much from flat spacetime, and away from the source, it will start approaching Minkowski spacetime as r increases. Therefore, in the region $d < r < \infty$, we can use Post Post-Minkowskian approximation. PN expansion is valid in the region $0 < r < \mathcal{R}$, so the two expansions overlap in the region $d < r < \mathcal{R}$. In the Blanchet-Damour approach, we use PN expansion in the near region, post-Minkowskian expansion outside the source, and match these two in the overlapping region.

3.7.1 Post-Minkowskian Expansion

Outside the source, we solve the vacuum Einstein equations. If we are considering a weak source, in the first approximation metric is the Minkowski metric $\eta^{\alpha\beta}$. At the distance r, we give expansions in terms of R_s/r, where $R_s = 2Gm/c^2$, and m is the mass of the system. R_s is proportional to G, so we can also expand the metric in powers of G as

$$\sqrt{-g}g^{\alpha\beta} = \eta^{\alpha\beta} + G\mathsf{h}_1^{\alpha\beta} + G^2\mathsf{h}_2^{\alpha\beta}.$$

So, we have

$$\mathsf{h}^{\alpha\beta} = \sum_{n=1}^{\infty} G^n h_n^{\alpha\beta}.$$

Plugging it in relaxed Einstein equations (3.21), and setting $T^{\alpha\beta} = 0$:

$$\Box\mathsf{h}^{\alpha\beta} = \Lambda^{\alpha\beta}[\mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3 ... \mathsf{h}_{n-1}]. \tag{3.23}$$

We expand tensor $\Lambda^{\alpha\beta}$ in powers of $\mathsf{h}^{\alpha\beta}$ and can write

$$\Lambda^{\alpha\beta} = N^{\alpha\beta}[h, h] + M^{\alpha\beta}[h, h, h] + L^{\alpha\beta}[h, h, h, h] + O(h^5),$$

where the coefficients can be by expanding $\Lambda^{\alpha\beta}$ in powers of $\mathsf{h}^{\alpha\beta}$.

Now we solve the relaxed Einstein's equations order by order. Since $\Lambda^{\alpha\beta}$, is quadratic in $\mathsf{h}^{\alpha\beta}$, at first order we have

$$\Box \mathsf{h}_1^{\alpha\beta} = 0, \tag{3.24}$$

At higher orders, we get

$$\Box h_2^{\alpha\beta} = N^{\alpha\beta}[h_1, h_1], \tag{3.25}$$

$$\Box h_3^{\alpha\beta} = M^{\alpha\beta}[h_1, h_1, h_1] + N^{\alpha\beta}[h_1, h_2] + N^{\alpha\beta}[h_2, h_1], \tag{3.26}$$

and so on with the gauge conditions,

$$\partial_\beta \mathsf{h}_n^{\alpha\beta} = 0.$$

We first find general solutions to Eq. (3.24). The most general solution to (3.24), outside the source is written in terms of retarded multipolar waves:

$$\mathsf{h}_1^{\alpha\beta} = \sum_{l=0}^{\inf} \partial_L \left[\frac{1}{r} K_L^{\alpha\beta}(t - r/c) \right].$$

Where K_L are symmetric trace-free (STF) tensors briefly described in the appendix A. The term in bracket satisfies the wave equation because it is a function of retarded time, $\Box K_L^{\alpha\beta}(u)/r = 0$, where $u = t - r/c$. This solution is only acceptable in the region $r > d$, as it becomes singular at $r = 0$. We have not yet used the harmonic gauge condition.

So, we have ten independent components of the tensor, and using the gauge condition, the solution takes a *canonical* form parametrized by two moments $I_L(u)$ and $J_L(u)$

$$h_1^{\alpha\beta} = k_1^{\alpha\beta} + \partial^\alpha \phi_1^\beta + \partial^\beta \phi_1^\alpha - \eta^{\alpha\beta}\partial_\mu \phi_1^\mu.$$

$k_1^{\alpha\beta}$ depends on STF multipole moments $I_L(u)$ and $J_L(u)$, which are arbitrary functions of spacetime and satisfy the harmonic gauge condition. It is given by

$$\begin{aligned}
k_1^{00} &= -\frac{4}{c^2}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}\mathrm{I}_L(u)\right),\\
k_1^{0i} &= \frac{4}{c^3}\sum_{\ell\geqslant 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}\mathrm{I}^{(1)}_{iL-1}(u)\right) + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}\,\mathrm{J}_{bL-1}(u)\right)\right\},\\
k_1^{ij} &= -\frac{4}{c^4}\sum_{\ell\geqslant 2}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}\mathrm{I}^{(2)}_{ijL-2}(u)\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\epsilon_{ab(i}\mathrm{J}^{(1)}_{j)bL-2}(u)\right)\right\},
\end{aligned}$$

where $I^{(1)}$ denotes the first derivative of I with respect to u. The STF multipole moments are explicitly $I_L(u) = I, I_i, I_{ij}, \ldots$ and $J_L(u) = J, J_i, J_{ij}, \ldots$ and are called mass-type and current-type moments. They are *arbitrary* except for the conservation of monopole moments, which gives the mass of the source $I \equiv M$, total linear momentum $P_i \equiv I_i^{(1)} = 0$, and total angular momentum J_i, and these follow from the gauge condition. The system's center of mass is I_i, and we can set it to zero by choosing the origin of our coordinate system to coincide with it. These terms include the source's contribution to the waves at the linearized level. They are arbitrary and not yet specified in terms of the stress energy tensor. The vector ϕ_1^α gives some arbitrary linear gauge transformations, and they are given in terms of the moments W_L, X_L, Y_L, Z_L. This vector is given by

$$\varphi^0_{(1)} = \frac{4}{c^3}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}\,\mathrm{W}_L(u)\right),$$

$$\varphi^i_{(1)} = -\frac{4}{c^4}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_{iL}\left(\frac{1}{r}\mathrm{X}_L(u)\right)$$

$$-\frac{4}{c^4}\sum_{\ell\geqslant 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}\mathrm{Y}_{iL-1}(u)\right)+\frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}\mathrm{Z}_{bL-1}(u)\right)\right\}.$$

These moments W_L, X_L, Y_L, Z_L play a physical role at the non-linear level. Ignoring these right now later results in a metric depending only on I_L and J_L. The gauge vector is important because we want to construct a full linear solution to the Einstein equation; discarding it will give back the result we got in linear theory.

Iteration of the solution. Multipolar post-Minkowskian expansion

We have found the general solution of the first-order equation, and we want to use it to find the solution to Eq. (3.25), and then use these to determine the next-order solution and so on. The general problem is how to integrate Eq. (3.23) when the source term Λ_n is found by the previous iterations. We cannot use Green's function because that requires knowing Λ_n in all of space, while we are working only in the region outside the source. The multipole expansion is valid only for $d/r < 1$.

Blanchet and Damour found the appropriate function that solves the equation. First, we do not need the full expansion, because we are interested in computing the PN expansion to a given order, and only a few multipoles contribute there. So, we iterate a truncated multipole expansion of $\mathsf{h}_1^{\alpha\beta}$. The solution needs to have the same structure as the source term, irregular at $r = 0$, and satisfy the equation at $r > d$. So, we use a trick where we first regularize the source term by multiplying it with a factor r^B, where B is a complex number. $\Lambda_n^{\alpha\beta}$ is expanded to a multipolar order ℓ_{max}. If we have the source terms that go like $1/r^k$, maximal order of divergence is $k_{\max}$, the real part of B is large such that the source is regular at $r \to 0$, we can use the retarded integral operator:

$$I_n^{\alpha\beta}(B) \equiv \Box^{-1}(r^B \Lambda_n^{\alpha\beta}), \tag{3.27}$$

where $\Box^{-1}$ denotes the convolution with the Green's function,

$$\Box^{-1} f(t, \mathrm{x}) \equiv -\frac{1}{4\pi} \int_{R^3} \frac{d^3x'}{|\mathrm{x} - \mathrm{x}'|} f(t - |\mathrm{x} - \mathrm{x}'|)/c, \mathrm{x}',$$

where we have also imposed the condition that in some remote past $r \to \inf$, the field becomes stationary. This way, we have imposed the no incoming radiation condition. $I^{\alpha\beta}$ can be expanded when $B \to 0$ in the form of a Laurent series:

$$I_n^{\alpha\beta}(B) = \sum_{p=p_0}^{\infty} B^p l_{n,p}^{\alpha\beta},$$

where $p_0 \in Z$ and for $p_0 < 0$, we have poles. Applying $\Box$ to both sides and using Eq. (3.27), we have

$$r^B \Lambda_n^{\alpha\beta} = \sum_{p=p_0}^{\infty} B^p \Box l_{n,p}^{\alpha\beta}.$$

Writing $r^B = e^{B \log r}$ and doing an expansion

$$e^{B \log r} \Lambda_n^{\alpha\beta} = \sum_{p=p_0}^{\infty} B^p \Box l_{n,p}^{\alpha\beta}$$
$$\sum_{n=0}^{\infty} \frac{(B \log r)^n}{n!} \Lambda_n^{\alpha\beta} = \sum_{p=p_0}^{\infty} B^p \Box l_{n,p}^{\alpha\beta}.$$

Equating it for the same powers of B, we find that for $p_0 \leq p \leq -1, \Box l_{n,p}^{\alpha\beta} = 0$ and for $p \geq 0$,

$$\Box l_{n,p}^{\alpha\beta} = \frac{(logr)^p}{p!} \Lambda_n^{\alpha\beta},$$

for $p = 0$, let $u_n^{\alpha\beta} \equiv l_{n,p=0}^{\alpha\beta}$, we have $\Box u_n^{\alpha\beta} = \Lambda_n^{\alpha\beta}$.

So, the solution is given by the coefficient of B^0 in the Laurent expansion. This is called the finite part at $B = 0$ of the retarded integral and denoted as

$$u_n^{\alpha\beta} = FP_{B=0} \Box^{-1} [r^B \Lambda_n^{\alpha\beta}]. \tag{3.28}$$

This is one particular solution to the Eq. (3.23). The solution in Eq. (3.28) will not satisfy the harmonic gauge condition, so we look for a solution of the form:

$$\mathsf{h}_n^{\alpha\beta} = u_n^{\alpha\beta} + v_n^{\alpha\beta},$$

where $v_n^{\alpha\beta}$ is chosen such that $\partial_\alpha v_n^{\alpha\beta} = -\partial_\alpha u_n^{\alpha\beta}$. Therefore, Multipolar post-Minkowskian expansion provides a well-defined algorithm for computing the Minkowskian corrections to an arbitrary order. Now we return to finding the solution in the near region.

3.7.2 PN Expansion in the Near Region

In the near region, we already found the solution at 1PN order in terms of $g^{\alpha\beta}$. We first express that in terms $\mathsf{h}^{\alpha\beta}$, using the relaxed Einstein equations. We find,

$$
\begin{aligned}
\mathsf{h}^{00} &= -4V/c^2 + O(1/c^4), \\
\mathsf{h}^{0i} &= O(1/c^3), \\
\mathsf{h}^{ij} &= O(1/c^4).
\end{aligned}
$$

Now, plugging this solution into Eq. (3.21),

$$
\begin{aligned}
\Box h^{00} &= \frac{16\pi G}{c^4}\left(1+\frac{4V}{c^2}\right)T^{00} - \frac{14}{c^4}\partial_k V \partial_k V + O\left(\frac{1}{c^6}\right), \\
\Box h^{0i} &= \frac{16\pi G}{c^4}T^{0i} + O\left(\frac{1}{c^5}\right), \\
\Box h^{ij} &= \frac{16\pi G}{c^4}T^{ij} + \frac{4}{c^4}\left\{\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V\right\} + O\left(\frac{1}{c^6}\right).
\end{aligned}
$$

The solution to these equations:

$$
\begin{aligned}
h^{00} &= -\frac{4}{c^2}V + \frac{4}{c^4}\left(W - 2V^2\right) + O\left(\frac{1}{c^6}\right), \\
h^{0i} &= -\frac{4}{c^3}V_i + O\left(\frac{1}{c^5}\right), \\
h^{ij} &= -\frac{4}{c^4}W_{ij} + O\left(\frac{1}{c^6}\right),
\end{aligned}
$$

where W_{ij} is:

$$W_{ij}(t,\mathrm{x}) = G \int d^3x' \, \frac{1}{|\mathrm{x}-\mathrm{x}'|} \left[\sigma_{ij} + \frac{1}{4\pi G} (\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V) \right] \bigg|_{\mathrm{x}', t-|\mathrm{x}-\mathrm{x}'|/c}.$$

Therefore, by iterating over the solution, we have obtained equations at $O(1/c^6)$ similar to the equations that we had for $g_{\mu\nu}$ at 1PN order.

Multipolar PN Expansion

Multipolar PN expansion combines the PN expansion with multipole expansion. To 1PN order, we need the multipole expansion of the potentials V and V_i, which are written as

$$V(t,\mathrm{x}) = G \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{1}{r} F_L \left(t - \frac{r}{c} \right) \right],$$
$$V_i(t,\mathrm{x}) = G \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{1}{r} G_{iL} \left(t - \frac{r}{c} \right) \right],$$

F_L and G_{iL} can be expressed in terms of the source terms.

Multipole Expansion to an Arbitrary Order

We are interested in finding the PN solution at all orders. We can expand $\mathsf{h}_{\mu\nu}$ in the form:

$$\mathsf{h}^{\mu\nu} = \sum_{n=2}^{\infty} \frac{1}{c^n} {}^{(n)}\mathsf{h}^{\mu\nu},$$

where $1/c^n$ has been extracted to make the c dependence explicit. Similarly, we can expand the effective energy-momentum tensor:

$$\tau^{\mu\nu} = \sum_{n=-2}^{\infty} \frac{1}{c^n} {}^{(n)}\tau^{\mu\nu}.$$

Inserting it into the relaxed Einstein equation (3.21) and equating terms with the same power of c, we get a recursive set of equations:

$$\nabla^2[{}^{(n)}\mathsf{h}^{\mu\nu}] = 16\pi G[{}^{(n-4)}\tau^{\mu\nu}] + \partial_t^2{}^{(n-2)}\mathsf{h}^{\mu\nu}. \tag{3.29}$$

Recall that we cannot write solutions in the near zone using Green's function. A particular solution to the above set of equations is found using a variant of the technique discussed in the last section. Given a function, consider:

$$\left[\Delta^{-1}\left(r^B f\right)\right](\mathrm{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{d^3x'}{|\mathrm{x}-\mathrm{x}'|}|\mathrm{x}'|^B f(\mathrm{x}')$$

If B is large and negative, the integral is regular as $|\mathrm{x}'| \to \infty$. Doing the same Laurent expansion, the coefficient u of B^0 is denoted by $\mathrm{FP}_{B=0}$,

$$u = \mathrm{FP}_{B=0}\left\{\Delta^{-1}\left[r^B f\right]\right\}.$$

u satisfies $\nabla^2 u = f$ and is a well-defined inversion of the Laplacian. When the integral converges $\mathrm{FP}_{B=0}\left\{\Delta^{-1}\left[r^B f\right]\right\}$ is the same as $\nabla^{-1} f$. If we now expand to n-th order in the PN expansion, we denote that by an overbar as

$$\bar{\mathsf{h}}^{\mu\nu} = \sum_{m=2}^{n} \frac{1}{c^m} {}^{(m)}\mathsf{h}^{\mu\nu}.$$

The particular solution we have found can be written as

$$\bar{\mathsf{h}}^{\mu\nu} = \frac{16\pi G}{c^4} \mathcal{FP} \Box_{\mathrm{ret}}^{-1} \bar{\tau}^{\mu\nu}. \tag{3.30}$$

We add this to the general solution of the homogeneous wave equation, which has the form:

$$h_{\mathrm{hom}}^{\alpha\beta} = \frac{16\pi G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{R_L^{\alpha\beta}\left(t - \frac{r}{c}\right) - R_L^{\alpha\beta}\left(t + \frac{r}{c}\right)}{2r} \right],$$

where $R_L^{\alpha\beta}$ are arbitrary functions of retarded and advanced times. This antisymmetric combination removes any outgoing radiation. The antisymmetric combination ensures that the solution is regular at $r = 0$. Under time reversal, the equation is odd and therefore, it describes the radiation reaction.

Next, we match the expansions in the region of overlap.

3.7.3 Matching the Solutions

In the external region $d < r < \infty$ for $d/r < 1$, we found the solution in the form of a Minkowskian expansion. Multipole expansion is applicable, so we write the solutions in terms of multipole moments. All higher-order terms are determined through iteration in the form of a multipole expansion. In the region $0 < r < \mathcal{R}$ where $\mathcal{R}$ is the boundary of the near region, we found the solution in terms of a Post-Newtonian expansion.

Since we are considering slowly moving sources with $v \ll c$, we have $\mathcal{R} \gg d$, the region where PN approximation is valid, overlaps with the region where post-Minkowskian multipole expansion is valid. In the post-Minkowskian region solution is parametrized by multipole moments, which are yet to be determined. In the PN solution, we have energy energy-momentum tensor of the source. So, comparing these solutions in the overlapping region, we can fix the multipole moments in terms of the source terms.

In the overlap region $d < r < \mathcal{R}\, d/r < 1$, so we can do a multipolar PN expansion in powers of d/r. Also, post-Minkowskian expansion can be done in the same way as PN expansion in powers of v/c. When expanded (see Eq. (5.5) in Blanchet and Damour[4]), we get:

$$h_n^{00} = O\left(\frac{1}{c^{2n}}\right), \quad h_n^{0i} = O\left(\frac{1}{c^{2n+1}}\right), \quad h_n^{ij} = O\left(\frac{1}{c^{2n}}.\right) \tag{3.31}$$

So, to a given order in the PN expansion, we take a finite number of iterations of the post-Minkowskian solution. For example, at 2PN order, we want to compute the $O\left(\frac{1}{c^4}\right)$ correction, so we compute g_{00} to $O\left(\frac{1}{c^6}\right)$, g_{0i} to $O\left(\frac{1}{c^5}\right)$ and g_{ij} to $O\left(\frac{1}{c^4}\right)$. So, Eq. (3.31) shows that we need to compute h_n to order $n = 3$. Therefore, we do two iterations of the solution h_1. Comparing the PN expansion with the re-expanded post-Minkowskian expansion fixes the multipole moments in terms of the energy-momentum tensors. They were computed to an arbitrary order in the PN expansion by Blanchet and Damour [1]. I_L and J_L are given by:

$$
\begin{aligned}
\mathrm{I}_L(u) =& \mathcal{FP} \int d^3x \int_{-1}^{1} dz \left\{ \delta_l(z) \hat{x}_L \Sigma - \frac{4(2l+1)\delta_{l+1}(z)}{c^2(l+1)(2l+3)} \hat{x}_{iL} \Sigma_i^{(1)} \right. \\
&\left. + \frac{2(2l+1)\delta_{l+2}(z)}{c^4(l+1)(l+2)(2l+5)} \hat{x}_{ijL} \Sigma_{ij}^{(2)} \right\} (u + z|\mathrm{x}|/c, \mathrm{x}), \\
\mathrm{J}_L(u) =& \mathcal{FP} \int d^3x \int_{-1}^{1} dz \epsilon_{ab\langle i_l} \left\{ \delta_l(z) \hat{x}_{L-1\rangle a} \Sigma_b \right. \\
&\left. - \frac{(2l+1)\delta_{l+1}(z)}{c^2(l+2)(2l+3)} \hat{x}_{L-1\rangle ac} \Sigma_{bc}^{(1)} \right\} (u + z|\mathrm{x}|/c, \mathrm{x}).
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma &\equiv \frac{\bar{\tau}^{00} + \bar{\tau}^{ii}}{c^2}, \\
\Sigma_i &\equiv \frac{\bar{\tau}^{0i}}{c}, \\
\Sigma_{ij} &\equiv \bar{\tau}^{ij}.
\end{aligned}
$$

where $\bar{\tau}^{ii} \equiv \delta_{ij}\bar{\tau}^{ij}$, $\tau^{\mu\nu}$ is the effective stress energy tensor. The bar over a quantity denotes its PN expansion up to the required order. The function $\delta_l(z)$ is given by Eq. (A.8). Despite all complications of the nonlinear theory, the full nonlinear result for $\mathsf{h}_1^{\mu\nu}$ to all orders in the PN expansion is obtained from the result of linearized theory simply replacing $T^{\mu\nu}$ with $\tau^{\mu\nu}$ and inserting the $\mathcal{FP}$ prescription. We have determined an analytic expression for mass and current moments at all orders as a recipe to expand the metric to the desired order using analytic expressions.

The next step is to systematically expand the metric for a compact binary system beyond the Newtonian limit using the derived post-Newtonian multipole moments.

4

PN Waveform Construction and Signal Modeling[5]

In this chapter, we explore how post-Newtonian expansions translate into gravitational waveform models. We begin by computing the binary energy and gravitational radiation flux up to 3PN and 3.5PN orders, then compare the performance of various PN approximants, including the TaylorF2 waveform used in data analysis.

4.1 Post-Newtonian Energy (3PN) and Flux (3.5PN)

To determine the time evolution of a binary's orbital phase, we need the 3PN approximation. Binary's phase depends on the gravitational-wave flux through an energy balance equation. Therefore, we need to first find the energy flux.

We start with the mass-type and current-type multipole moments that are given by,

$$\begin{aligned} I_L(t) = \underset{B=0}{\mathrm{FP}} \int d^3\mathrm{x}|\tilde{\mathrm{x}}|^B \int_{-1}^{1} dz \Big\{ \delta_l(z)\hat{x}_L\Sigma - \frac{4(2l+1)}{c^2(l+1)(2l+3)}\delta_{l+1}(z)\hat{x}_{iL}\dot{\Sigma}_i \\ + \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\delta_{l+2}(z)\hat{x}_{ijL}\ddot{\Sigma}_{ij} \Big\} (\mathrm{x}, t+z|\mathrm{x}|/c), \end{aligned} \tag{4.1}$$

$$\begin{aligned} J_L(t) = \underset{B=0}{\mathrm{FP}}\, \varepsilon_{ab<i_l} \int d^3\mathrm{x}|\tilde{\mathrm{x}}|^B \int_{-1}^{1} dz \, \Big\{ \delta_l(z)\hat{x}_{L-1>a}\Sigma_b \\ - \frac{2l+1}{c^2(l+2)(2l+3)}\delta_{l+1}(z)\hat{x}_{L-1>ac}\dot{\Sigma}_{bc} \Big\} (\mathrm{x}, t+z|\mathrm{x}|/c), \end{aligned}$$

where $L = i_1 i_2 \cdots i_l$ is a multi-index composed of l indices; a product of l spatial vectors $x^i \equiv x_i$ is denoted $x_L = x_{i_1}x_{i_2}\cdots x_{i_l}$; the symmetric trace free (STF) part of that product is denoted using a hat: $\hat{x}_L = \mathrm{STF}\,(x_L)$, for instance $\hat{x}_{ij} = x_i x_j - \frac{1}{3}\delta_{ij}$, $\hat{x}_{ijk} = x_i x_j x_k - \frac{1}{5}(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij})$; the STF projection is also denoted using brackets surrounding the indices, e.g. $\hat{x}_{ij} \equiv x_{<ij>}$, $x_{<i}v_{j>} = \frac{1}{2}(x_i v_j + x_j v_i) - \frac{1}{3}\delta_{ij}x_k v_k$; ε_{ijk} denotes the usual Levi-Civita symbol ($\varepsilon_{000} = +1$); the dots refer to the time differentiation. The matter densities Σ, Σ_i and Σ_{ij} are evaluated at the position x and at time

$t + z|\mathrm{x}|/c$. The function $\delta_l(z)$ is given by

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}l!}\left(1-z^2\right)^l; \quad \int_{-1}^{1} dz\delta_l(z) = 1.$$

This function tends to the Dirac distribution when $l \to +\infty$. Each of the terms composing I_L and J_L is to be understood in the sense of post-Newtonian expansion, and computed using the (infinite) post-Newtonian series

$$\int_{-1}^{1} dz\delta_l(z)S(\mathrm{x}, t + z|\mathrm{x}|/c) = \sum_{j=0}^{\infty} \frac{(2l+1)!!}{2^j j!(2l+2j+1)!!}|\mathrm{x}|^{2j}\left(\frac{\partial}{c\partial t}\right)^{2j} S(\mathrm{x}, t) \tag{4.2}$$

We require the post-Newtonian-expanded metric and the associated matter distributions to compute the multipole moments. The metric components in harmonic coordinates are expanded up to 3PN as:

$$\begin{aligned}
\bar{h}_{00} &= -\frac{4}{c^2}V - \frac{2}{c^4}(W + 4V^2) - \frac{8}{c^6}\left(\hat{Z} + 2\hat{X} + V\hat{W} + \frac{4}{3}V^3\right) + O(1/c^8),\\
\bar{h}_{0i} &= -\frac{4}{c^3}V_i - \frac{8}{c^5}\left(\hat{R}_i V V_i\right) + O(1/c^7),\\
\bar{h}_{ij} &= -\frac{4}{c^4}\left(\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}\right) - \frac{16}{c^6}\left(\hat{Z}_{ij} - \frac{1}{2}\delta_{ij}\hat{Z}\right) + O(1/c^8).
\end{aligned}$$

The potentials $\hat{Z}_{ij}, \hat{R}_i, \hat{X}$ show up at 2PN order and are given in terms of derivatives V and $\hat{W}$. These potentials contribute to the source terms in the multipole integrals. The detailed construction of these terms is given in Blanchet et al. 2005.

The post-Newtonian expanded metric is substituted into the gravitational source pseudotensor, notably the term $\Lambda^{\mu\nu}$, which is expanded up to quartic order h^4. Using Eq. (4.2), the source multipole moments $I_L(t)$ and $J_L(t)$ are expressed as functionals of all the retarded potentials, and subsequently of the instantaneous potentials that are given by Poisson-type integrals. Some terms are transformed by integration by parts, carefully accounting for the analytic continuation factor $|\tilde{\mathrm{x}}|^B$. Surface terms vanish under analytic continuation (valid for large negative real parts of B).

While the Leibniz rule is valid for smooth ('fluid') sources, complications arise when dealing with singular expressions corresponding to point-like particles. In such cases, derivatives must be interpreted in the distributional sense, and the Leibniz rule is not generally satisfied. This leads to ambiguities that are discussed in Blanchet et al. 2005 To manage the complexity of the multipole moments, a systematic nomenclature is introduced:

- Contributions from the source densities Σ, Σ_i, Σ_{ij} are labeled as scalar (S), vector (V), and tensor (T) types, respectively.
- Each type is further subdivided based on the summation index j in Eq. (4.2):
  - SI denotes scalar terms from Σ with $j = 0$
  - SII, SIII, etc., correspond to higher j
  - VI, VII, etc., come from vector terms with different j
  - Similarly for tensor terms TI, TII, ...
- The full structure of the mass-type moment up to 3 PN order is:

$$I_L = \mathrm{SI} + \mathrm{SII} + \mathrm{SIII} + \mathrm{SIV} + \mathrm{VI} + \mathrm{VII} + \mathrm{VIII} + \mathrm{TI} + \mathrm{TII} + \mathcal{O}(7),$$

where we suppress the multi-index L on the individual terms for brevity.

- Each piece (e.g., SI) is further decomposed into individual terms, e.g., $\mathrm{SI}(1), \mathrm{SI}(2), \ldots$, numbered in order of appearance.

The explicit expressions of these terms are given in Blanchet et al. 2005. The expressions for the multipole moments are initially derived under the assumption of smooth (continuous) matter sources. When applying them to point-particle binaries, the integrals become divergent at the particle locations $\mathrm{x} \to \mathrm{y}_1(t), \mathrm{y}_2(t)$. To handle these divergences, the Hadamard regularization method is employed throughout the derivation. This prescription allows the finite part of the divergent integrals to be systematically extracted. A distributional framework adapted to this regularization was later developed to rigorously define such expressions.

While the above formalism works for smooth matter distributions, real compact objects are modeled as point particles, leading to singularities in the source integrals. These divergences are handled using the Hadamard regularization, which introduces two arbitrary constants u_1 and u_2 associated with the particles' locations. Additionally, the analytic continuation factor $|\tilde{\mathrm{x}}|^B = |\mathrm{x}/r_0|^B$ introduces an arbitrary scale r_0, though it cancels out in physically measurable quantities like the energy flux.

Despite the success of this approach, certain ambiguities remain at the 3PN order. Specifically, the 3PN mass quadrupole moment contains undetermined contributions that depend on u_1, u_2, and a third ambiguity arising from limitations of Hadamard regularization itself. These constants ultimately combine into a single undetermined parameter θ, which enters the third time derivative of the quadrupole moment, and hence the energy flux. This ambiguity is of the same nature as the constant λ that arises in the 3PN equations of motion, and the flux depends only on a particular combination of θ and λ.

The constant λ represents a genuine physical ambiguity tied to the incompleteness of Hadamard regularization at this order. In harmonic coordinates, λ has been related to the ADM-Hamiltonian parameter ω_{static} via

$$\lambda = -\frac{3}{11}\omega_{\text{static}} - \frac{1987}{3080}.$$

Dimensional regularization was used to fix $\omega_{\text{static}} = 0$, implying $\lambda = -\frac{1987}{3080}$. In contrast, other constants such as r_1' and r_2', which appear in the harmonic-coordinate equations of motion, are gauge-related and cancel out of gauge-invariant quantities like the binding energy for circular orbits.

These ambiguity parameters—such as ω_{static} in the equations of motion and θ in the flux—were uniquely fixed through dimensional regularization, which preserves the gauge and Lorentz invariance of general relativity. In particular, Blanchet, Damour, and Esposito-Farèse demonstrated that dimensional regularization yields finite, ambiguity-free results consistent with the post-Newtonian formalism at 3PN order[5]. This procedure thus acts as a powerful consistency check on the use of Hadamard regularization in previous computations.

Having addressed the instantaneous contributions and ambiguity resolution, we now turn to hereditary terms, which encapsulate non-local-in-time effects in the flux. These arise from the non-linear structure of general relativity and include:

- *Tails* — backscattering of gravitational waves off the curved spacetime generated by the total mass-energy of the system. These first appear at 1.5PN order.

- *Tails-of-tails* — a tail scattering off another tail, entering at 3PN.
- *Memory* — a net change in the waveform amplitude, often associated with nonlinear interactions, typically relevant at 2.5PN and higher.

These hereditary contributions introduce non-local-in-time terms and logarithmic dependencies in the flux, reflecting the influence of the past dynamics of the binary on its present radiation.

The 3.5PN energy and flux expressions are[6]:

$$\begin{aligned} E_{3PN}(v) = -\frac{1}{2}\nu v^2 \Bigg[1 &- \left(\frac{3}{4}+\frac{1}{12}\nu\right) v^2 - \left(\frac{27}{8}-\frac{19}{8}\nu+\frac{1}{24}\nu^2\right) v^4 \\ &- \left\{\frac{675}{64} - \left(\frac{34445}{576}-\frac{205}{96}\pi^2\right)\nu + \frac{155}{96}\nu^2 + \frac{35}{5184}\nu^3\right\} v^6 \Bigg], \end{aligned} \tag{4.3}$$

$$\begin{aligned} \mathcal{F}_{3.5PN}(v) = \frac{32}{5}\nu^2 v^{10} \Bigg[1 &- \left(\frac{1247}{336}+\frac{35}{12}\nu\right) v^2 + 4\pi v^3 \\ &- \left(\frac{44711}{9072}-\frac{9271}{504}\nu-\frac{65}{18}\nu^2\right) v^4 - \left(\frac{8191}{672}+\frac{583}{24}\nu\right)\pi v^5 \\ &+ \Bigg\{\frac{6643739519}{69854400}+\frac{16}{3}\pi^2-\frac{1712}{105}\gamma \\ &\quad + \left(\frac{41}{48}\pi^2-\frac{134543}{7776}\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3 - \frac{856}{105}\ln(16v^2)\Bigg\} v^6 \\ &- \left(\frac{16285}{504}-\frac{214745}{1728}\nu-\frac{193385}{3024}\nu^2\right)\pi v^7 \Bigg]. \end{aligned} \tag{4.4}$$

$\gamma = 0.57221\ldots$ is the Euler–Mascheroni constant. Here, ν denotes the symmetric mass ratio, which we previously referred to using the symbol η. The appearance of terms like $\ln(v)$, Euler's constant γ, and π^2 are hallmark signatures of non-linear and non-local effects such as tail and tail-of-tail interactions. These stem from integrals over the source's history and the curvature of spacetime, and do not arise from purely instantaneous dynamics.

4.2 Current Status of PN Approximants[6]

The post-Newtonian (PN) approximation is a perturbative expansion in the small parameter $v = (\pi M F)^{1/3}$, where F is the orbital frequency. Alternatively, we define $x = v^2$, which governs the PN expansion.

For compact binary inspirals, we are interested in modeling the orbital phase $\phi(t)$ accurately. A common assumption is the **adiabatic approximation**, which states that the inspiral occurs slowly, ensuring that the fractional change in orbital velocity per period is negligible. We assume that the orbit evolves slowly, so that the fractional change in the orbital velocity v over an orbital period is negligibly small, that is,

$$\frac{\dot{\omega}}{\omega^2} \ll 1.$$

This means that the luminosity in gravitational waves is determined by the change in orbital energy averaged over a period. For circular orbits, this allows one to use the energy balance equation:

$$\mathcal{F} = -\frac{d\mathcal{E}}{dt}, \quad \text{where} \quad \mathcal{E} = ME.$$

Using this, we can derive equations for the evolution of any binary parameter. For example, the orbital separation $r(t)$ evolves as:

$$\dot{r}(t) = \frac{\dot{\mathcal{E}}}{(d\mathcal{E}/dr)} = -\frac{\mathcal{F}}{(d\mathcal{E}/dr)}.$$

Using Kepler's law, the energy balance equation gives the evolution of the orbital phase:

$$\begin{aligned} \frac{d\phi}{dt} - \frac{v^3}{M} &= 0, \\ \frac{dv}{dt} + \frac{\mathcal{F}(v)}{ME'(v)} &= 0. \end{aligned}$$

Equivalently, integrating these equations, we obtain:

$$\begin{aligned} t(v) &= t_{\text{ref}} + M \int_v^{v_{\text{ref}}} \frac{dv}{\mathcal{F}(v)}, \\ \phi(v) &= \phi_{\text{ref}} + \int_v^{v_{\text{ref}}} dv\, v^3 \frac{E'(v)}{\mathcal{F}(v)}. \end{aligned}$$

Here, t_{ref} and ϕ_{ref} are integration constants, and v_{ref} is an arbitrary reference velocity.

Different PN families arise from how the ratio $\mathcal{F}(v)/E'(v)$ is treated in the orbital evolution equations. For example, TaylorT1, TaylorT2, TaylorT3, TaylorT4, and TaylorEt all represent different choices of how to expand or sum this series.

- **TaylorT1**: Directly integrates the phase evolution equation numerically.
- **TaylorT2**: Expresses $\phi(v)$ and $t(v)$ parametrically.
- **TaylorT3**: Inverts $t(v)$ to obtain $\phi(t)$ explicitly.
- **TaylorT4**: Expands $\mathcal{F}/E'$ before solving numerically.

- **TaylorF2**: Uses the Fourier domain via the Stationary Phase Approximation (SPA).

TaylorF2 is preferred at 3.5PN order for low-mass binaries ($M < 12M_{\odot}$), whereas high-mass binaries require Effective-One-Body (EOB) waveforms calibrated to numerical relativity.

4.3 TaylorF2 Approximant[6]

The most commonly used PN approximant in the Fourier domain is the TaylorF2 model. The TaylorF2 approximant is a Fourier-domain representation of the gravitational waveform using the stationary phase approximation (SPA). Using SPA, the waveform in the frequency domain is written as:

$$\tilde{h}^{\mathrm{SPA}}(f) = \frac{a(t_f)}{\sqrt{\dot{F}(t_f)}} e^{i[\psi_f(t_f) - \pi/4]},$$

where $\psi_F \equiv 2\pi f t - 2\phi(t)$.

To obtain the above frequency-domain representation of the waveform, we start by taking the Fourier transform of the time-domain waveform $h(t)$:

$$\tilde{h}(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} dt.$$

Since $h(t)$ is oscillatory and non-trivial, we use the stationary phase approximation (SPA) [12]. The main contribution to the Fourier integral comes from the stationary points of the phase, where the phase satisfies

$$\frac{d}{dt}(2\pi f t - 2\phi(t)) = 0,$$

which simplifies to the **stationary phase condition**:

$$\frac{d\phi}{dt} = \pi f.$$

This defines **stationary phase time** t_f, the time at which the gravitational wave frequency equals the Fourier frequency.

Coming back to the integral, if a function has the form:

$$B(t) = A(t)\cos(\phi(t)),$$

where the amplitude $A(t)$ varies much more slowly than the phase $\phi(t)$, its Fourier transform can be approximated as:

$$\tilde{B}(f) \approx \frac{1}{2}A(t)\left(\frac{df}{dt}\right)^{-1/2} e^{i(2\pi f t - \phi(t) - \pi/4)}.$$

The condition $d \ln A/dt \ll d\phi/dt$ ensures that the phase dominates over the amplitude, making the exponential term dominant. The factor $(df/dt)^{-1/2}$ arises from the Jacobian of the transformation when switching from t-dependence to f-dependence.

Applying the stationary phase approximation,

$$\tilde{h}^{\rm SPA}(f) = \frac{a(t_f)}{\sqrt{\dot{F}(t_f)}} e^{i[\psi_f(t_f) - \pi/4]}.$$

where $F(t_f)$ is given by the saddle point condition:

$$\frac{d\psi_f(t)}{dt} = 0 \quad \Rightarrow \quad F(t_f) = f.$$

Using the energy balance equation,

$$\frac{dE}{dt} = -\mathcal{F},$$

we rewrite dt in terms of frequency

$$\frac{dt}{df} = \frac{E'(f)}{\mathcal{F}(f)}.$$

The time t_f at which the gravitational-wave frequency $F(t)$ matches the Fourier variable f is

$$t_f = t_{\rm ref} + M \int_{v_f}^{v_{\rm ref}} \frac{E'(v)}{\mathcal{F}(v)} dv.$$

The phase is given by

$$\psi_f(t_f) = 2\pi f t - 2\phi(t). \tag{4.5}$$

Expressing the conditions we have from the energy balance equation in terms of ψ

$$\psi_f(t_f) = 2\pi f t_{\text{ref}} - \phi_{\text{ref}} + 2\int_{v_f}^{v_{\text{ref}}} (v_f^3 - v^3)\frac{E'(v)}{\mathcal{F}(v)} dv.$$

The post-Newtonian velocity parameter is defined as:

$$v_f \equiv (\pi M f)^{1/3}.$$

It is more efficient to use the differential form, so we solve the differential equations:

$$\frac{d\psi}{dt} - 2\pi t = 0, \tag{4.6}$$

$$\frac{dt}{df} + \frac{\pi M^2}{3v^2}\frac{E'(f)}{\mathcal{F}(f)} = 0. \tag{4.7}$$

The Fourier domain waveform containing only the most dominant mode is given:

$$\tilde{h}(f) = A f^{-7/6} e^{i\psi(f)}, \tag{4.8}$$

where $A \propto \mathcal{M}^{5/6} Q(\text{angles})/D$ and D is the distance to the binary. Solving the differential equations (4.7) and (4.6), using expressions for energy and luminosity as given in Eqs. (4.3) and (4.4), we get the phase $\psi(f)$ expanded to 3.5PN order

$$\begin{aligned}
\psi_{\mathrm{F2}}(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} \\
&+ \frac{3}{128\nu} v^{-5} \Bigg\{ 1 + \left(\frac{3715}{756} + \frac{55}{9}\nu \right) v^2 - 16\pi v^3 \\
&+ \left(\frac{15293365}{508032} + \frac{27145}{504}\nu + \frac{3085}{72}\nu^2 \right) v^4 \\
&+ \left(\frac{38645}{756} - \frac{65}{9}\nu \right) \pi \left(1 + 3\ln\frac{v}{v_{\mathrm{LSO}}} \right) v^5 \\
&+ \Bigg[\frac{11583231236531}{4694215680} - \frac{640}{3}\pi^2 - \frac{6848}{21}\gamma - \frac{6848}{21}\ln(4v) \\
&\quad + \left(-\frac{15737765635}{3048192} + \frac{2255}{12}\pi^2 \right) \nu + \frac{76055}{1728}\nu^2 - \frac{127825}{1296}\nu^3 \Bigg] v^6 \\
&+ \left(\frac{77096675}{254016} + \frac{378515}{1512}\nu - \frac{74045}{756}\nu^2 \right) \pi v^7 \Bigg\}. \qquad (4.9)
\end{aligned}$$

Here $v = (\pi M f)^{1/3}$ and $v_{\mathrm{LSO}} = 1/\sqrt{6}$ represent the velocity at the last stable orbit for a non-spinning binary.

Plugging this expression into Eq. (4.8), we get the Fourier domain waveform, and we can recover the time domain waveform by taking the inverse Fourier transform. The resulting waveform is shown in Figure 4.1.

4.3.1 Analysis Across Binary Types and Mass Ratios

To further explore the performance of the TaylorF2 approximant, we generated waveforms for different compact binary systems: BNS, NSBH, and BBH, with varying symmetric

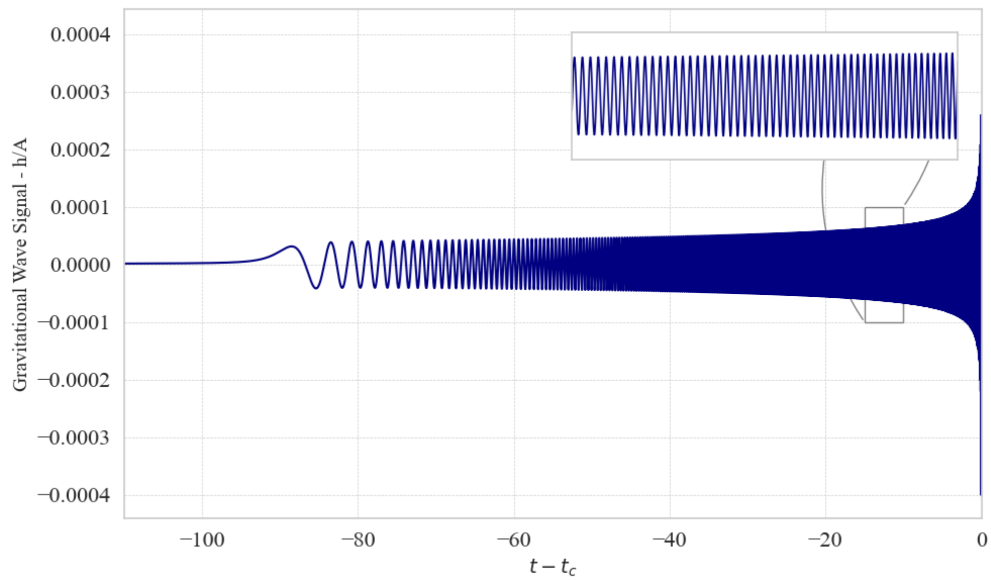


Figure 4.1: Gravitational wave signal at 3.5 PN order for a quasicircular binary with mass 12 $M_\odot$. The waveform shows the characteristic inspiral, increasing amplitude, and frequency as the binary components spiral closer together under gravitational radiation reaction.

mass ratios ν and component masses. These waveform templates use the 3.5PN expansion for the phase $\psi(f)$ in the frequency domain.

From Figure 4.2, we observe that as ν decreases (i.e., as mass asymmetry increases), the waveform duration shortens and the amplitude changes, reflecting a faster inspiral and earlier merger. The case $\nu = 3$ is unphysical but is included to illustrate how extending template banks to unphysical values of ν can improve overlaps with numerical waveforms[6]. The time-domain duration of the waveform depends on the frequency resolution, which is determined by the maximum frequency f_{max}. In this implementation, f_{max} is set by the velocity at the last stable orbit, v_{LSO}, using the relation $f_{\text{max}} = v_{\text{LSO}}^3/(\pi M)$, where M is the total mass of the binary.

As a result, low-mass systems like the BNS binary have a higher f_{max}, leading to a larger

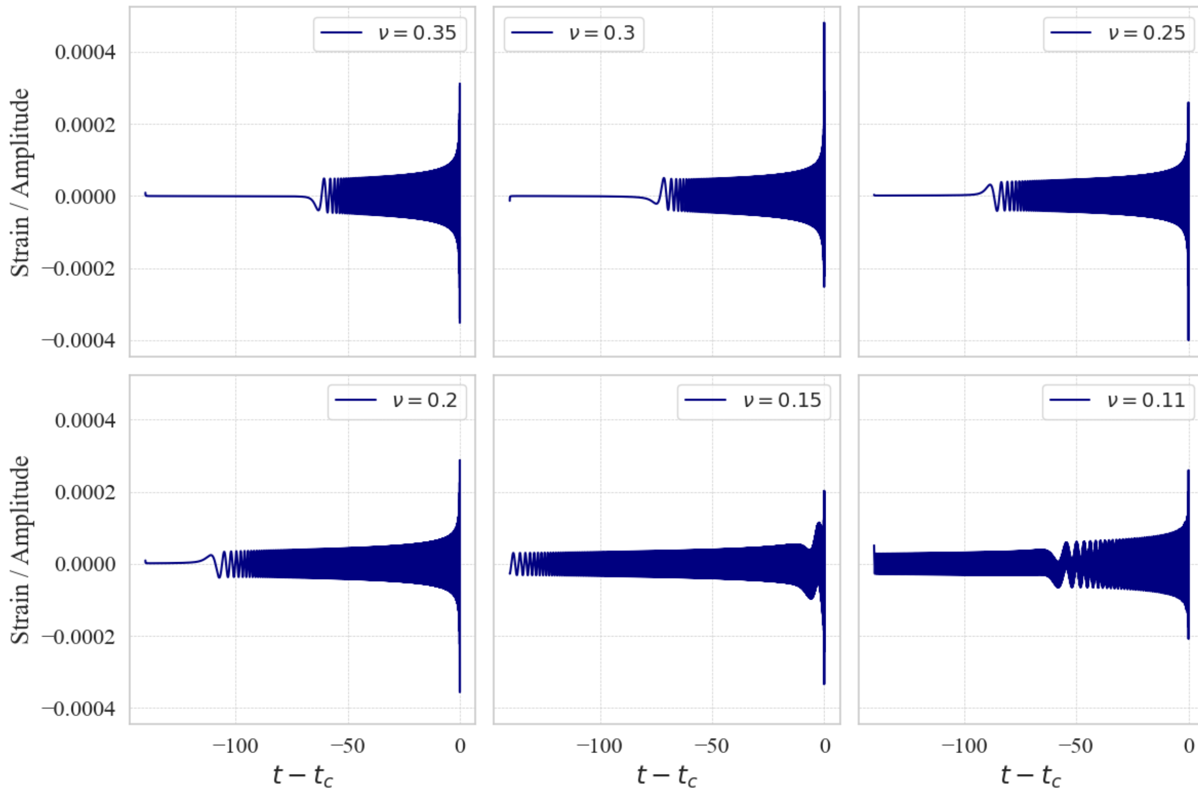


Figure 4.2: Waveforms for a constant total mass of $12M_{\odot}$ with varying symmetric mass ratio ν.

frequency spacing Δf, and hence a shorter time-domain duration $T = 1/\Delta f$. This is why the BNS waveform appears shorter in Figure 4.3, despite such systems physically inspiraling over a longer timescale. To partially compensate for this effect, I manually increased the frequency resolution for the neutron star system. Conversely, high-mass systems like the BBH have a lower f_{max}, resulting in finer frequency resolution and correspondingly longer time-domain waveforms.

This highlights an important artifact of the frequency-domain sampling: the apparent waveform duration does not directly reflect the physical inspiral time, but rather the numerical resolution set by f_{max}.

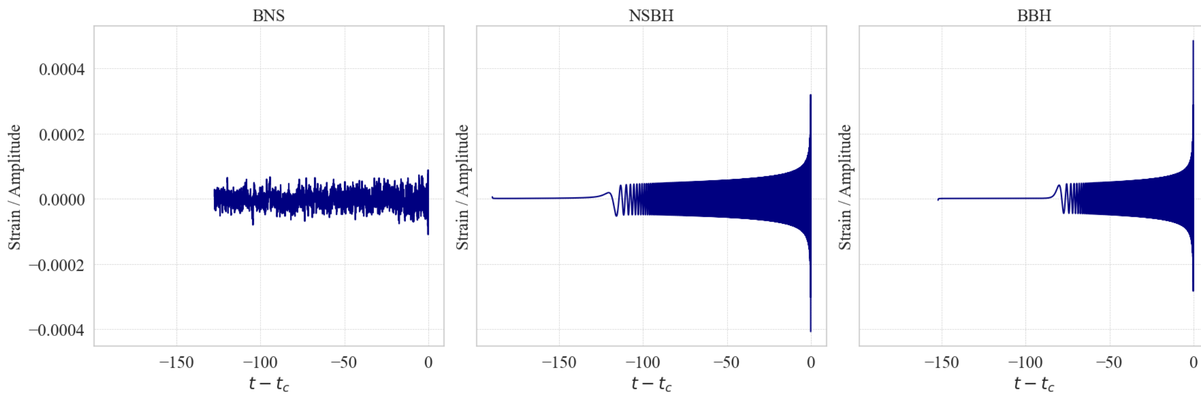


Figure 4.3: Waveforms for different binary types: BNS (1.38, 1.42) $M_{\odot}$, NSBH (2.1, 14) $M_{\odot}$, and BBH (7.8, 5.2) $M_{\odot}$.

4.3.2 Template Validity and Overlap with Numerical Relativity

Although we do not directly compare with numerical relativity (NR) waveforms in this work, insights from existing studies[6] provide valuable context on the performance of TaylorF2 templates. The accuracy of post-Newtonian waveforms is commonly assessed using two metrics: effectualness, which quantifies their ability to detect signals with variable parameters, and faithfulness, which evaluates how accurately they reproduce the true signal for a given set of parameters. Effectualness measures how well a template waveform can detect a true signal, allowing for variation in binary parameters (good for searches). Faithfulness measures how well a template matches the true signal when the binary parameters are fixed (important for parameter estimation).

- For unequal-mass binaries, better overlaps with NR can be achieved by allowing unphysical values of $\nu > 0.25$. This is particularly helpful in BNS systems, where the symmetric mass ratio may not fall within physical limits. Symmetric mass ratio was also set to 3 to get a more defined waveform.

- TaylorF2 templates at 3.5PN are more unfaithful for higher total masses. In the case of BBH with masses (9.5, 10.5) $M_{\odot}$, approximants disagree irrespective of PN order, making it difficult to rely on any particular approximant.
- The pseudo-4PN (p4PN) family of templates, with additional higher-order terms, has been recommended in literature for better matching NR results, especially in BNS systems.
- According to overlap studies, good mass choices for TaylorF2 with consistent overlaps (> 0.95) are:
  - BNS: (1.38, 1.42) $M_{\odot}$
  - NSBH: (1.4, 10) $M_{\odot}$
  - BBH: (4.8, 5.2) $M_{\odot}$
- Mass configurations where TaylorF2 is unreliable (overlaps not consistent > 0.95):
  - BBH: (9.5, 10.5) $M_{\odot}$.

Overall, from a computational perspective, TaylorF2 is the least expensive to generate and is recommended for search templates below a total mass of $12M_{\odot}$. For systems with total mass $> 12M_{\odot}$, TaylorF2 may still be **effectual** if the upper cut-off frequency is extended. For instance, for minimal match requirements < 0.95, TaylorF2 may be used up to $20M_{\odot}$ with effectualness around 0.90.

5

Gravitational Waves from Spin Precessing Binaries[6,13]

5.1 Overview of Precession in Binaries

When compact objects such as NS or BH are formed through supernova explosions, they are spinning rapidly due to asymmetries in the explosions. The spin of the compact objects would not necessarily be aligned with the orbital angular momentum. If the spin vectors of the compact objects are not aligned with the orbital angular momentum, these momenta will change direction, causing orbital plane precession around the total angular momentum of the system. Precession of the orbital plane introduces amplitude and phase modulations in the gravitational wave signal. These modifications can encode new information about the source, breaking degeneracies in parameter estimation. In the sensitivity band of ground-based detectors, gravitational waves typically arise from compact binaries with arbitrary spin configurations.

This precession also introduces mathematical catastrophes. When computing the Fourier transform, the Stationary Phase Approximation (SPA) breaks down because both the first and second derivatives of the phase vanish, leading to non-monotonic frequency evolution and the absence of a single dominant frequency component.

With misaligned spins, orbital equations of motion have been found to the 2.5PN order. However, they are complex differential equations and do not have a closed-form solution. There are four main representations of GW solutions for these equations. One is simple precession, in which you can find closed closed-form solution when only one object is spinning. Another method is effective one-body (EOB) formulation. The third way is to transform to a frame where precession is minimum, and then, after finding the solution, you can transform back to the original frame. However, this method introduces biases in parameter estimation. The fourth way is to use Multiple Scale Analysis (MSA). It is a technique to solve differential equations that have different characteristic time scales by expanding in the ratio of those time scales. For the precessing binary, the orbital time scale is much shorter than the precession time scale, which is much shorter than the radiation reaction

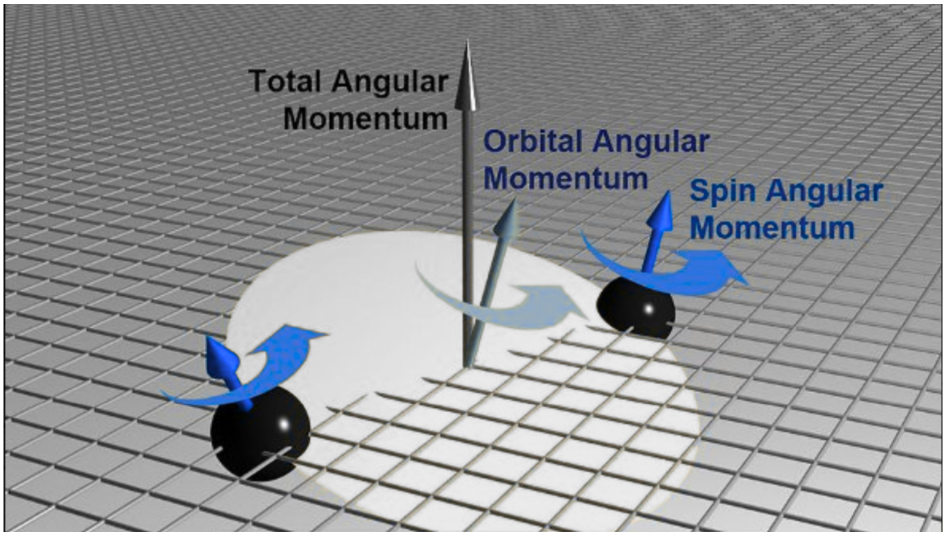


Figure 5.1: Illustration of spin and angular momentum vectors in a compact binary system. The figure shows the individual spin vectors of the two compact objects, the orbital angular momentum, and the total angular momentum resulting from their vector sum[8].

time scale. Using this approach, we solve the equations at the precession time scale and then add radiation reaction on top of it. Here we only find the analytic solution to the precession equations. This sets the stage for including radiation reaction and finally constructing the spin precessing waveform.

5.2 Evolution of Spin and Angular Momentum

A quasicircular binary with spinning compact objects has spin-orbit and spin-spin interactions that cause all the angular momenta to precess. Given that the time scale for orbital and precession is different, we can average over one orbit and get the precession equations that do not take radiation reaction into account. The evolution of these angular momenta is given by

$$
\begin{aligned}
\dot{\hat{\mathrm{L}}} = & \left\{ \left(2 + \frac{3}{2}q\right) - \frac{3}{2}\frac{v}{\eta} \right\} v^6 \left(\mathrm{S}_1 \times \hat{\mathrm{L}}\right) \\
& + \left\{ \left(2 + \frac{3}{2q}\right) - \frac{3}{2\eta}v \right\} v^6 \left(\mathrm{S}_2 \times \hat{\mathrm{L}}\right) + \mathcal{O}(v^7), \qquad (5.1) \\
\dot{\mathrm{S}}_1 = & \left\{ \eta \left(2 + \frac{3}{2}q\right) - \frac{3v}{2} \left[(q\mathrm{S}_1 + \mathrm{S}_2) \cdot \hat{\mathrm{L}} \right] \right\} v^5 \left(\hat{\mathrm{L}} \times \mathrm{S}_1\right) \\
& + \frac{v^6}{2} \mathrm{S}_2 \times \mathrm{S}_1 + \mathcal{O}(v^7), \qquad (5.2) \\
\dot{\mathrm{S}}_2 = & \left\{ \eta \left(2 + \frac{3}{2q}\right) - \frac{3v}{2} \left[\left(\frac{1}{q}\mathrm{S}_2 + \mathrm{S}_1 \right) \cdot \hat{\mathrm{L}} \right] \right\} v^5 \left(\hat{\mathrm{L}} \times \mathrm{S}_2\right) \\
& + \frac{v^6}{2} \mathrm{S}_1 \times \mathrm{S}_2 + \mathcal{O}(v^7). \qquad (5.3)
\end{aligned}
$$

S_1 and S_2 are the spin of compact objects, L is the orbital angular momenta. v is the PN expansion parameters related to orbital frequency as $v = \omega^{1/3}$, and $q = m_1/m_2$ where $m_1 \geq m_2$. These equations only include the conservational dynamics of the orbits. The radiation reaction evolves the magnitude of orbital angular momentum and leaves the magnitude of spins unchanged up to the current PN expansion. The magnitude of orbital angular momentum L is related to the orbital frequency and ω and v by

$$
\dot{v} = \frac{v^9}{3} \frac{1}{\sum_{n=0}^{7} [g_n + 3g_n^l \ln v] v^n}. \qquad (5.4)
$$

The coefficients g_n, g_n^l are functions of the symmetric mass ratio and inner products of the angular momenta. They are given in Appendix B.

The precession equations (5.1), (5.2), and (5.3) describe *conservative dynamics*. They take into account the change in direction of the angular momenta, using only the leading-order terms, i.e, spin-orbit interaction at 1.5PN order and spin-spin order correction at 2PN order. However, equation (5.4) governs the *dissipative dynamics* due to radiation reaction. Dissipative dynamics govern the spiral of the orbital, the loss of angular momentum, and the evolution of the GW frequency. In calculations and analysis, terms to 3.5PN order are kept. 3PN spin-spin precession is not included because it has not yet been calculated.

Conservative and dissipative dynamics evolve on different time scales. Eqs. (5.1), (5.2), and (5.3) evolve on precession timescale

$$T_{\mathrm{pr}} \equiv \frac{|\mathrm{S}_1|}{|\dot{\mathrm{S}}_1|} \approx v^{-5}.$$

while the dissipative dynamics evolve over the radiation time

$$T_{\mathrm{rr}} \equiv \frac{v}{\dot{v}} \sim v^{-8}.$$

Due to the difference in time scales, their ratio $T_{\mathrm{pr}}/T_{\mathrm{rr}} \sim v^3$ becomes a natural expansion parameter.

One way to solve these equations is to solve the precession equations, ignoring Eq. (5.4). Then you average the full equations with radiation reaction by integrating out the fast variations in spin precession results and focusing on the long-term evolution only. But here, rather than integrating, we use the ratio of precession time scales to expand the equations and solve the precession equations analytically. We then include radiation reaction effects by using Multiple Scale Analysis. MSA is a perturbation theory technique that we use to treat the radiation reaction as a slowly evolving perturbation on top of precession. As a result we find solutions to equations (5.1), (5.2), (5.3), and (5.4) as an expansion in $T_{\mathrm{pr}}/T_{\mathrm{rr}}$.

5.3 Analytic Solution to Precession Equations Ignoring Radiation Reaction

We can solve precession equations analytically by ignoring radiation reaction and making use of the conserved quantities of the system. The first solution was presented by Kesden et al. 2015.

There are a total of 9 degrees of freedom in a precessing binary, as each momentum $(\mathbf{L}, \mathbf{S}_1, \mathbf{S}_2)$ contributes 3 components. If we only consider the precession equation, then we have 7 conserved quantities that are the magnitude and direction of total angular momentum $J, \hat{J}$, the magnitudes of individual spins S_1, S_2, the magnitude of orbital angular momentum L, and mass-weighted effective spin ξ defined as

$$\xi \equiv (1+q)\mathbf{S}_1 \cdot \hat{\mathbf{L}} + (1+q^{-1})\mathbf{S}_2 \cdot \hat{\mathbf{L}}. \tag{5.5}$$

We represent these conserved quantities by $\lambda = (S_1, S_2, L, J, \hat{\mathbf{J}}, \xi)$. If we include radiation reaction, $S_1, S_2,$ and ξ are still conserved at the current PN order, and J, $\hat{J}$, and L evolve at the radiation reaction time scale. The remaining two degrees of freedom correspond to the choice of the coordinate system and to a dynamical quantity that changes with time. We choose it to be the magnitude of the total spin angular momentum $S = |\mathbf{S}_1 + \mathbf{S}_2|$.

Now we use these 7 conserved quantities to geometrically solve for 9 quantities. We express components of the momenta as a function of S in a chosen coordinate system. We then solve the differential equation for $S(t)$ to obtain the complete solution for the angular momenta.

We first solve for precession equations in a non-inertial frame, and then we can move to an inertial frame because the gravitational waveform is measured in the local inertial frame of the detector. Since $\hat{\mathrm{J}}$ is a conserved quantity, we can choose a frame where $\hat{\mathrm{J}} = \hat{\mathrm{z}}$. We choose x and y axes to rotate around the z-axis such that the orbital angular momentum vector L always lies in the x–z plane, thereby defining a non-inertial frame that co-precesses with L. The angle that L makes with respect to the z-axis is

$$\cos\theta_L = \hat{\mathrm{J}} \cdot \hat{\mathrm{L}} = \frac{J^2 + L^2 - S^2}{2JL}.$$

Using this we can express L as,

$$\mathrm{L}(S, \lambda) = L[\sin\theta_L, 0, \cos\theta_L]. \tag{5.6}$$

Total spin angular momentum in terms of J and L is

$$\mathrm{S}(S; \lambda) = \mathrm{J} - \mathrm{L} = [-L\sin\theta_L, 0, J - L\cos\theta_L].$$

In another frame with axes $\hat{\mathrm{z}}' = \hat{\mathrm{S}}$, $\hat{\mathrm{y}}' = \hat{\mathrm{y}}$ and $\hat{\mathrm{x}}' = \hat{\mathrm{y}}' \times \hat{\mathrm{z}}'$, we define angles (θ', ϕ') as shown in Figure 5.2. In this frame,

$$\mathrm{S_1}' = S_1[\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta'].$$

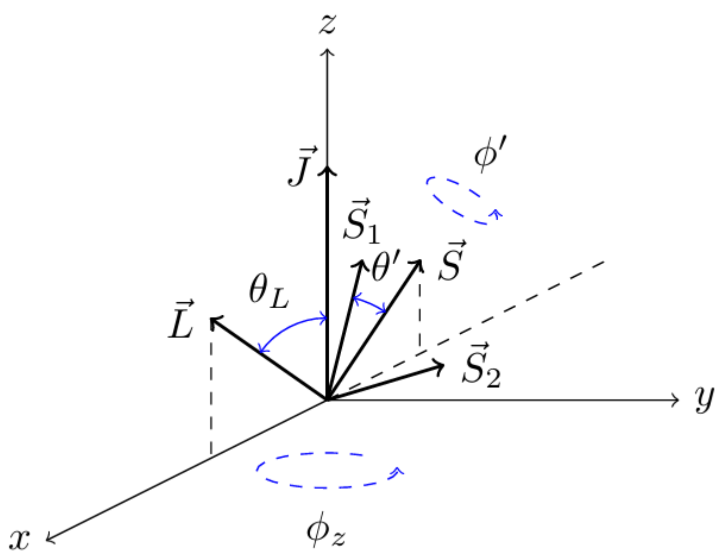


Figure 5.2: Configuration of angular momenta in a non-inertial precessing frame.

The angle $\cos\theta'$ in terms of spins is

$$\cos\theta' = \hat{\mathrm{S}}_1 \cdot \hat{\mathrm{S}} = \frac{S^2 + S_1^2 - S_2^2}{2SS_1}.$$

Using definition of ξ from Eq. 5.2, we find,

$$\cos\phi' = \frac{(J^2 - L^2 - S^2)\left[S^2(1+q)^2 - (S_1^2 - S_2^2)(1-q^2)\right] - 4qS^2L\xi}{(1-q^2)A_1A_2A_3A_4}, \qquad (5.7)$$

where

$$A_1 = \sqrt{J^2 - (L-S)^2}, \tag{5.8}$$

$$A_2 = \sqrt{(L+S)^2 - J^2}, \tag{5.9}$$

$$A_3 = \sqrt{S^2 - (S_1 - S_2)^2}, \tag{5.10}$$

$$A_4 = \sqrt{(S_1 + S_2)^2 - S^2}. \tag{5.11}$$

In the original unprimed system if we rotate S_1' around ŷ, then we get

$$\mathrm{S}_1(S;\lambda) = \mathbb{R}(\hat{\mathrm{y}}, \theta_S)\mathrm{S}_1', \tag{5.12}$$

where $\mathbb{R}(\hat{\mathrm{y}}, \theta_S)$ is a rotation around ŷ by an angle θ_S and since θ_s takes us back from $\hat{\mathrm{S}}$ aligned frame to $\hat{\mathrm{J}}$,

$$\cos\theta_S = \hat{\mathrm{S}} \cdot \hat{\mathrm{J}} = \frac{J^2 + S^2 - L^2}{2JS}. \tag{5.13}$$

Using the last two equations and the expression for S_1', we write S_1 in terms of J, S, and L. Once we have S_1 in the original unprimed system, then we can find S_2

$$\mathrm{S}_2(S;\lambda) = \mathrm{J} - \mathrm{L} - \mathrm{S}_1. \tag{5.14}$$

Eqs. (5.6), (5.12), and (5.14) determine angular momenta in a non-inertial frame as a function of s.

We have expressed S_1, S_2, and L in terms of S only. Substituting these expressions in precession equations (5.1), (5.2), and (5.3), we get the equation that describes the evolution of S

$$\left(\frac{dS^2}{dt}\right)^2 = -A^2(S^6 B + S^4 + CS^2 + D), \tag{5.15}$$

where A, B, C, and D are given in terms of quantities that depend on the radiation reaction time scale (see Appendix B). The roots of the right-hand side of Eq. (5.15) indicate the values of S^2 at which its derivative vanishes. We call roots of right hand side, S_+^2, S_-^2, which are the maximum and minimum of S^2 and the third root is S_3. Only two of the roots S_+^2, S_-^2 are real and correspond to physically bound spins. If at least two roots of a cubic polynomial are not real, then the equation S^2 will be physically bounded; otherwise, S^2 will increase or decrease with no bounds. When two roots of a cubic polynomial are real, then the third root must be real too, but it is mostly negative for physical systems and cannot correspond to any physical scenarios.
Writing Eq. (5.15) in terms of roots of the polynomial

$$S^2 = A(S^2 - S_+^2)(S^2 - S_-^2)(S^2 - S_3^2)).$$

This is a separable ODE involving a cubic polynomial whose solution is expressed in terms of elliptic Jacobi functions

$$S^2(t) = S_+^2 + (S_-^2 - S_+^2)\,\mathrm{sn}^2(\omega t; m).$$

In this equation, sn $= 0 \implies S^2 = S_+^2$ and sn = 1, $\implies S^2 = S_-^2$. So, S^2 oscillates between the two physical turning points. Therefore, this solution makes sense intuitively.

We now substitute this solution into the differential equation (5.15) to find out what ψ and m are. Define the ansatz

$$S^2(t) = S_+^2 + (S_-^2 - S_+^2)\,\mathrm{sn}^2(\psi; m), \tag{5.16}$$

where $\psi = \omega t$ for some function ω to be determined, and m is the elliptic modulus, which controls the shape of the Jacobi elliptic functions. It lies in the interval $[0, 1]$ and determines the nature of the periodicity of $\mathrm{sn}(\psi; m)$. For $m = 0$, sn reduces to a sine function, and for $m = 1$, it behaves like a hyperbolic tangent.

The Jacobi elliptic function $\mathrm{sn}(\psi; m)$ is the solution to the nonlinear differential equation:

$$\left(\frac{d}{d\psi}\mathrm{sn}(\psi; m)\right)^2 = (1 - \mathrm{sn}^2(\psi; m))(1 - m\,\mathrm{sn}^2(\psi; m)).$$

The functions $\mathrm{cn}(\psi; m)$ and $\mathrm{dn}(\psi; m)$ are defined respectively as:

$$\mathrm{cn}(\psi; m) = \sqrt{1 - \mathrm{sn}^2(\psi; m)}, \quad \mathrm{dn}(\psi; m) = \sqrt{1 - m\,\mathrm{sn}^2(\psi; m)}.$$

These satisfy the identity:

$$\left(\frac{d}{d\psi}\mathrm{sn}(\psi; m)\right) = \mathrm{cn}(\psi; m)\,\mathrm{dn}(\psi; m),$$

which is used in taking time derivatives of the ansatz for $S^2(t)$.

Taking the time derivative of the guessed solution,

$$\frac{dS^2}{dt} = (S_-^2 - S_+^2) \cdot 2\,\mathrm{sn}(\psi; m)\,\mathrm{cn}(\psi; m)\,\mathrm{dn}(\psi; m)\,\frac{d\psi}{dt}, \tag{5.17}$$

$$\left(\frac{dS^2}{dt}\right)^2 = 4(S_-^2 - S_+^2)^2\,\mathrm{sn}^2(\psi; m)\,\mathrm{cn}^2(\psi; m)\,\mathrm{dn}^2(\psi; m)\left(\frac{d\psi}{dt}\right)^2. \tag{5.18}$$

Now we use the identity for Jacobi elliptic functions:

$$\mathrm{cn}^2(\psi; m) = 1 - \mathrm{sn}^2(\psi; m), \qquad \mathrm{dn}^2(\psi; m) = 1 - m\,\mathrm{sn}^2(\psi; m).$$

Substituting these in, we get

$$\left(\frac{dS^2}{dt}\right)^2 = 4(S_-^2 - S_+^2)^2\left(\frac{d\psi}{dt}\right)^2\,\mathrm{sn}^2(\psi; m)\left(1 - \mathrm{sn}^2(\psi; m)\right)\left(1 - m\,\mathrm{sn}^2(\psi; m)\right). \tag{5.19}$$

From the guessed solution for S^2, we have

$$\mathrm{sn}^2(\psi; m) = \frac{S^2 - S_+^2}{S_-^2 - S_+^2},$$

which when substituted into Eq. (5.19) gives

$$\left(\frac{dS^2}{dt}\right)^2 = 4\left(\frac{d\psi}{dt}\right)^2 \cdot \frac{(S^2 - S_+^2)(S_-^2 - S^2)\left[(S_-^2 - S_+^2) - m(S^2 - S_+^2)\right]}{S_-^2 - S_+^2} \tag{5.20}$$

Matching with the original ODE in Eq. (5.15), we identify

$$A^2 = \frac{4\left(\frac{d\psi}{dt}\right)^2}{(S_-^2 - S_+^2)} \quad \Rightarrow \quad \frac{d\psi}{dt} = \frac{A}{2}\sqrt{S_+^2 - S_3^2},$$

and

$$m = \frac{S_-^2 - S_+^2}{S_3^2 - S_+^2}.$$

Thus, the final solution is

$$S^2(t) = S_+^2 + (S_-^2 - S_+^2)\,\mathrm{sn}^2\left(\frac{A}{2}\sqrt{S_+^2 - S_3^2}\,t; m\right). \tag{5.21}$$

This solution requires that $S_+^2 \neq S_3^2$, which is almost always the case because S_+^2 and S_3^2 are defined to be the largest and smallest roots, respectively. The only possible case when $S_3^2 = S_+^2$ is when $S^2 = S_+^2$, but then S^2 is constant, and there is no precession.

The phase $\psi(t)$ can be obtained by noticing that $\dot{\psi}$ is constant if we ignore radiation reaction, so that:

$$\psi(t) = \frac{A}{2}\sqrt{S_+^2 - S_3^2}\,t$$

The final ingredient we need in order to have a complete expression for all angular momenta as a function of time in a non-inertial frame precessing around the z-axis is the sign of $\sin\phi'$. We consider $S_1^y = A_1 \sin\theta' \sin\phi'$. Since $A_1 > 0$ and $\sin\theta' > 0$ (as $\theta' \in (0, \pi)$), the sign of S_1^y is governed entirely by the sign of $\sin\phi'$, i.e.,

$$\mathrm{sign}(S_1^y) = \mathrm{sign}(\sin\phi').$$

Similarly, from the ansatz $S^2(t) = S_+^2 + (S_-^2 - S_+^2)\,\mathrm{sn}^2(\psi; m)$, we computed

$$\frac{dS^2}{dt} \propto \mathrm{sn}(\psi; m)\,(S_-^2 - S_+^2)\mathrm{cn}(\psi; m)\,\mathrm{dn}(\psi; m),$$

and since $\mathrm{dn} > 0$ always and $S_+^2 > S_-^2$, the sign of $\frac{dS^2}{dt}$ is determined by

$$\mathrm{sign}\left(-\frac{dS^2}{dt}\right) = \mathrm{sign}(\mathrm{sn}) \cdot \mathrm{sign}(\mathrm{cn}).$$

By leveraging conserved quantities and geometric insights, we derived an analytic solution to the spin-precession equations for compact binaries in the absence of radiation reaction. The dynamics reduce to the evolution of a single parameter, the magnitude of the total spin, capturing the essential features of spin precession. This geometric approach not only simplifies the complex dynamics of spin-precessing binaries but also lays the groundwork for incorporating radiation reaction effects perturbatively via Multiple Scale Analysis. The next step is to incorporate radiation reaction effects and study how the precession dynamics couple to the inspiral, with the eventual goal of constructing accurate waveform models for generic spin configurations.

6

Conclusion

We explored the theoretical foundations and modeling techniques for gravitational wave signals from compact binaries. Starting from linearized gravity, we derived gravitational wave solutions, quantified the energy they carry, and discussed their production. We examined the influence of gravitational radiation on binary dynamics, particularly focusing on the orbital energy loss and the decrease in orbital period.

Using the Post-Newtonian approximation, we developed analytic expressions for waveform phase evolution up to 3.5PN order and constructed the TaylorF2 waveform model. We analyzed its accuracy and limitations across different binary configurations, BNS, BBH, and NSBH, emphasizing where the model works and where its assumptions break down.

Finally, we studied the effects of spin precession on binary dynamics by analyzing the evolution of spin and angular momentum vectors in the absence of radiation reaction. While we did not construct full spin-precessing waveforms, this analysis highlights the importance of including spin effects in waveform modeling for accurate parameter estimation and source characterization.

Analytic models, while limited in strong-field regimes, remain invaluable tools for gravitational wave data analysis due to their low computational cost. Future extensions of this work include incorporating full spin evolution, eccentricity, and comparisons with numerical relativity to further improve model accuracy.

A

Multipole Expansion and STF Notation

For the Poisson equation $\nabla^2\phi = -4\pi\rho$, the solution is written in terms of the Green's function as:

$$\phi(\mathrm{x}) = \int d^3y \frac{1}{|\mathrm{x}-\mathrm{y}|}/$$

A way of writing the multipole expansion is:

$$\begin{aligned}\frac{1}{|\mathrm{x}-\mathrm{y}|} &= \frac{1}{|\mathrm{x}|} - y^i\partial_i\frac{1}{|\mathrm{x}|} + \frac{1}{2}y^i y^j\partial_i\partial_j\frac{1}{|\mathrm{x}|} + \dots \\ &= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}y^{i_1}\dots y^{i_l}\partial_{i_1}\dots\partial_{i_l}\frac{1}{|\mathrm{x}|}.\end{aligned}$$

This expansion satisfies the Poisson equation. Now, removing traces from the terms in expansion, we get:

$$\phi(\mathrm{x}) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}Q_{i_1\dots i_l}\partial_{i_1}\dots\partial_{i_l}\frac{1}{|\mathrm{x}|}, \tag{A.1}$$

where

$$Q_{i_1\dots i_l} = \int d^3y\, y^{\langle i_1}\dots y^{i_l\rangle}\rho(\mathrm{y}), \tag{A.2}$$

where $y^{\langle i_1}\dots y^{i_l\rangle}$ denotes that we are taking the symmetric trace-free parts of the tensors $y^{i_1}\dots y^{i_l}$. Now, we introduce multi-index notation in which a tensor with l indices $i_1 i_2 ... i_l$ is labeled by a letter L,

$$F_L \equiv F_{i_1 i_2 \dots i_l}. \tag{A.3}$$

Similarly,

$$G_{iL} \equiv G_{i i_1 i_2 \dots i_l}.$$

If we have repeated L indices, then a summation over all indices $i_1 i_2 \ldots i_l$ is understood,

$$F_L G_L = \sum_{i_1 i_2 \ldots i_l} F_{i_1 i_2 \ldots i_l} G_{i_1 i_2 \ldots i_l}.$$

We use a hat to indicate a tensor that is symmetric and trace-free $\hat{K}_L$, or equivalently, write it as $K_{<L>}$. A symmetric trace free tensor has $2l+1$ independent components, therefore it is an irreducible representation of the rotation group SO(3).

Solution to the relativistic wave equation $\Box\phi = -4\pi\rho$ can be written as:

$$\phi(t, \mathrm{x}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{F_L\left(t - \frac{r}{c}\right)}{r} \right], \tag{A.4}$$

where F_L is an arbitrary function satisfying

$$\Box \left[\frac{F_L\left(t - \frac{r}{c}\right)}{r} \right] = 0, . \tag{A.5}$$

Because the set of tensors F_L is a complete representation of the rotation group, this solution is the most general solution. Comparing this solution to the solution found using the Green's function:

$$\phi(t, \mathrm{x}) = \int d^3x' \, \frac{1}{|\mathrm{x} - \mathrm{x}'|} \, \rho\left(t - \frac{|\mathrm{x} - \mathrm{x}'|}{c}, \mathrm{x}'\right), \tag{A.6}$$

we can find the relativistic multipoles:

$$F_L(u) = \int d^3y \, y^{\langle L \rangle} \int_{-1}^{1} dz \, \delta_l(z) \rho\left(u + z\frac{|\mathrm{y}|}{c}, \mathrm{y}\right). \tag{A.7}$$

δ_l is:

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l, \tag{A.8}$$

and it satisfies:

$$\int_{-1}^{1} dz \delta_l(z) = 1$$

and as $l \to \infty$, it approaches the Dirac delta function $\delta(z)$ [4].

B

Coefficients of $\dot{v}$ and Precession Solution

The coefficients of Eq. (5.4):

$$g_0 = \frac{1}{a_0},$$

$$g_2 = -\frac{a_2}{a_0},$$

$$
\begin{aligned}
g_3 &= -\frac{a_3}{a_0}, \\
g_4 &= -\frac{a_4 - a_2^2}{a_0}, \\
g_5 &= -\frac{a_5 - 2a_3a_2}{a_0}, \\
g_6 &= -\frac{a_6 - 2a_4a_2 - a_3^2 + a_2^3}{a_0}, \\
g_6^\ell &= -\frac{3b_6}{a_0}, \\
g_7 &= -\frac{a_7 - 2a_5a_2 - 2a_4a_3 + 3a_3a_2^2}{a_0},
\end{aligned}
$$

These coefficients a_i, b_i are given in Appendix A of[9].

The coefficients of Eq. (5.15):

$$
\begin{aligned}
A &= -\frac{3}{2\sqrt{\eta}} v^6 (1 - \xi v), \\
B &= \left(L^2 + S_1^2\right) q + 2L\xi - 2J^2 - S_1^2 - S_2^2 + \frac{L^2 + S_2^2}{q}, \\
C &= \left(J^2 - L^2\right)^2 - 2L\xi \left(J^2 - L^2\right) \\
&\quad - 2\frac{1-q}{q} \left(S_1^2 - qS_2^2\right) L^2 + 4\eta L^2 \xi^2 \\
&\quad - 2\delta m \left(S_1^2 - S_2^2\right) \xi L + 2\frac{1-q}{q} \left(qS_1^2 - S_2^2\right) J^2, \\
D &= \frac{1-q}{q} \left(S_2^2 - qS_1^2\right) \left(J^2 - L^2\right)^2 + \frac{\delta m^2}{\eta} \left(S_1^2 - S_2^2\right)^2 L^2 \\
&\quad + 2\delta m L\xi \left(S_1^2 - S_2^2\right) \left(J^2 - L^2\right).
\end{aligned}
$$

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