Gravitational Waves from Compact Binaries

Mid Project Report for PHY 491 A: Sproj I

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Abstract

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Provide some introduction, motivation here.

2 A Primer on Gravitational Radiation

2.1 Linearized Gravity and Gauge Transformations

When discussing weak field of Einstein equations in order to consider the Newtonian limit, we assume that the sources are static and the gravitational field is weak. But here we extend this and use the weak field limit assuming that the field is weak but varies with time. There is no restriction on the motion of test particles, which allows the discussion of gravitational radiation, where field varies with time and the deflection of light which describes the motion of fast moving particles in such a field. If the gravitational field is weak, metric can be decomposed as the sum of Minkowski metric and a small perturbation as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu} \ll 1$. We will expand the resulting equations of motion to first order in $h_{\mu\nu}$, that is why this theory is called linearized theory. Inverse metric is:

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$$

We raise and lower indices using the Minkowski metric as the corrections will of be higher order. The goal is to find equations of motion obeyed by the perturbation. Since the perturbation is small, we examine the Einstein equations to the first order. To get to that, we would need Christoffel symbols and the Riemannian curvature Tensor. Starting with Christoffel symbols:

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}),\\ &= \frac{1}{2}g^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu})\end{aligned}$$

Reimannian curvature tensor is given by:

$$\begin{aligned}R^{\lambda}_{\nu\rho\sigma} &= \partial_\rho\Gamma^{\lambda}_{\nu\sigma} - \partial_\sigma\Gamma^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\rho\mu}\Gamma^{\mu}_{\sigma\nu} - \Gamma^{\lambda}_{\sigma\mu}\Gamma^{\mu}_{\rho\nu}\\ \eta_{\mu\lambda}R^{\lambda}_{\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho\Gamma^{\lambda}_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma\Gamma^{\lambda}_{\nu\rho},\end{aligned}$$

where in the second equation we lowered an index for convenience. We have dropped the Γ^2 terms because they are of second order in perturbation. In this expression, we replace the Christoffel symbols:

$$\begin{aligned}R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho\left[\frac{1}{2}\eta^{\lambda\alpha}(\partial_\nu h_{\sigma\alpha} + \partial_\sigma h_{\alpha\nu} - \partial_\alpha h_{\nu\sigma})\right] - \eta_{\mu\lambda}\partial_\sigma\left[\frac{1}{2}\eta^{\lambda\alpha}(\partial_\nu h_{\rho\alpha} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho})\right]\\ &= \frac{1}{2}\delta^{\alpha}_{\mu}\partial_\rho\left[(\partial_\nu h_{\sigma\alpha} + \partial_\sigma h_{\alpha\nu} - \partial_\alpha h_{\nu\sigma})\right] - \frac{1}{2}\delta^{\alpha}_{\mu}\partial_\sigma\left[(\partial_\nu h_{\rho\alpha} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho})\right]\\ &= \frac{1}{2}\partial_\rho\left[(\partial_\nu h_{\sigma\mu} + \partial_\sigma h_{\mu\nu} - \partial_\mu h_{\nu\sigma})\right] - \frac{1}{2}\partial_\sigma\left[(\partial_\nu h_{\rho\mu} + \partial_\rho h_{\mu\nu} - \partial_\mu h_{\nu\rho})\right]\\ &= \frac{1}{2}(\partial_\rho\partial_\nu h_{\sigma\mu} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\rho\mu} + \partial_\sigma\partial_\mu h_{\nu\rho}) \qquad (2.1)\end{aligned}$$

Contracting the curvature tensor using the Minkowski metric, we get Ricci Tensor:

$$\begin{aligned}\eta^{\rho\mu}R_{\mu\nu\rho\sigma} &= \frac{1}{2}\eta^{\rho\mu}(\partial_\rho\partial_\nu h_{\sigma\mu} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\rho\mu} + \partial_\sigma\partial_\mu h_{\nu\rho})\\ R_{\nu\sigma} &= \frac{1}{2}(\partial_\rho\partial_\nu h^{\rho}_{\sigma} - \partial_\rho\partial^{\rho} h_{\nu\sigma} - \partial_\sigma\partial_\nu h^{\rho}_{\rho} + \partial_\sigma\partial_\mu h^{\mu}_{\nu})\\ &= \frac{1}{2}(\partial_\rho\partial_\nu h^{\rho}_{\sigma} + \partial_\sigma\partial_\mu h^{\mu}_{\nu} - \Box h_{\nu\sigma} - \partial_\sigma\partial_\nu h),\end{aligned}$$

where $h^\rho_\rho = h$ and $\partial_\rho \partial^\rho = \Box$. Taking trace, we get the Ricci scalar:

$$\begin{aligned}
\eta^{\nu\sigma} R_{\nu\sigma} &= \frac{1}{2}\eta^{\nu\sigma}(\partial_\rho\partial_\nu h^\rho_\sigma + \partial_\sigma\partial_\mu h^\mu_\nu - \Box h_{\nu\sigma} - \partial_\sigma\partial_\nu h) \\
R &= \frac{1}{2}(\partial_\rho\partial_\nu h^{\rho\nu} + \partial_\sigma\partial_\mu h^{\mu\sigma} - \Box h - \Box h) \\
&= \partial_\rho\partial_\nu h^{\rho\nu} - \Box h
\end{aligned}$$

Finally, Einstein tensor takes the form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \tag{2.2}$$

$$= \frac{1}{2}(\partial_\sigma\partial_\mu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \partial_\mu\partial_\nu - \Box h_{\mu\nu} - \eta_{\mu\nu}\Box h) \tag{2.3}$$

The field equation is $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $T_{\mu\nu}$ is the stress energy tensor calculated to the zeroth order in $h_{\mu\nu}$. Higher-order corrections can be ignored because in the weak-field limit, the magnitude of the stress-energy tensor must be small. The amount of energy-momentum tensor itself must also be very small in order to apply the weak field limit. So, we will be concerned with the vacuum equation.

We have the linearized field equation which we can solve. But before that, note that there might be multiple spacetimes where the metric can be written as the Minkowski metric plus a perturbation and the perturbation will be different. So, the decomposition $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is not unique. We can choose a reference frame where this decomposition holds over a large region of space.

To solve this issue of gauge invariance, we restrict our reference frame. Choosing a reference frame breaks the gauge invariance of relativity under general coordinate transformations but is necessary to understand the physical content of a field theory. Linearized theory can be thought of as one that governs the behavior of tensor fields on a flat background. Consider a background space time M_b and a physical spacetime M_p such that there exists a diffeomorphism $\phi : M_b \to M_p$ between them. But they have different tensor fields defined on them. On M_b, we have defined the Minkowski metric and on M_p we have some arbitrary metric $g_{\alpha\beta}$ satisfying Einstein equation. Since, there exists a map between the two manifolds, we can move tensors back and forth between the two. Our linearized theory should take place on M_b so we are interested in pull back $(\phi^* g)_{\mu\nu}$ of the physical metric. The perturbation can be defined as:

$$h_{\mu\nu} = (\phi^* g)_{\mu\nu} - \eta_{\mu\nu}.$$

If the gravitational fields on M_p are weak, then for some ϕ the perturbation will be small $|h_{\mu\nu}| \ll 1$. So we focus only on such diffeomorphisms. From this, it can also be seen that $h_{\mu\nu}$ will obey the linearized Einstein equation on M_b.

Consider a vector field $\xi^\mu(x)$ on M_b. This generates diffeomorphism $\psi_\epsilon : M_b \to M_b$. For very small ϵ, $\phi \circ \psi_\epsilon$ will be very small. So, one can define a number of perturbations parametrized by ϵ:

$$\begin{aligned}
h^{(\epsilon)}_{\mu\nu} &= [(\phi \circ \psi_\epsilon)^* g]_{\mu\nu} - \eta_{\mu\nu} \\
&= [\psi^*_\epsilon(\phi^* g)]_{\mu\nu} - \eta_{\mu\nu}
\end{aligned}$$

Plugging in the previous relation:

$$\begin{aligned}
h^{(\epsilon)}_{\mu\nu} &= \psi^*_\epsilon(h + \eta)_{\mu\nu} - \eta_{\mu\nu} \\
&= \psi^*_\epsilon(h_{\mu\nu}) + \psi^*_\epsilon(\eta_{\mu\nu}) - \eta_{\mu\nu} \\
&= \psi^*_\epsilon(h_{\mu\nu}) + \epsilon\left[\frac{\psi^*_\epsilon(\eta_{\mu\nu}) - \eta_{\mu\nu}}{\epsilon}\right] \\
&= h_{\mu\nu} + \epsilon \mathcal{L}_\zeta \eta_{\mu\nu}
\end{aligned}$$

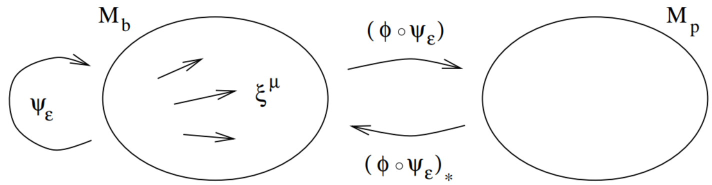

Figure 1: Diffeomorphisms Ψ_ϵ generated by vector field ξ^μ on a background spacetime M_b

where the first term was expanded to the lowest order and second terms gave us the Lie derivative. Lie derivative of the metric along the vector field ξ_μ. The Lie derivative can be written as $\mathcal{L}_\zeta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$.

$$h_{\mu\nu}^{(\epsilon)} = h_{\mu\nu} + 2\epsilon\partial_{(\mu}\xi_{\nu)}. \tag{2.4}$$

This is called a gauge transformation in linearized theory. It represents all such transformations which satisfy the condition that the perturbation must be small. These metric perturbations denote physically equivalent spacetimes under which our linearized theory is invariant. To see this, we find that the under the transformation defined by Eq. 2.4, Reimanniann tensor varies as:

$$\begin{aligned}\delta R_{\mu\nu\rho\sigma} =& \frac{1}{2}\left(\partial_\rho\partial_\nu\partial_\mu\xi_\sigma + \partial_\rho\partial_\nu\partial_\sigma\xi_\mu + \partial_\sigma\partial_\mu\partial_\nu\xi_\rho + \partial_\sigma\partial_\mu\partial_\rho\xi_\nu\right.\\ &\left.-\partial_\sigma\partial_\nu\partial_\mu\xi_\rho - \partial_\sigma\partial_\nu\partial_\rho\xi_\mu - \partial_\rho\partial_\mu\partial_\nu\xi_\sigma - \partial_\rho\partial_\mu\partial_\sigma\xi_\nu\right)\\ =&0.\end{aligned}$$

So, the transformations leave the Reimannian tensor and consequently Einstein's equations invariant. The gauge transformations do not change the functional form of observables; this is termed as gauge invariance.

2.2 Degrees of Freedom

We could not go on to solve the Einstein equation but we first will get further physical insights. So, we choose a fixed interial coordinate in background Minkowski spacetime and decompose the components of the metric perturbation according to their transformation properties under spatial rotation.
$h_{\mu\nu}$ is a symmetric (0, 2) tensor. Under spatial rotations, the 00 component of $h_{\mu\nu}$ is scalar, $0i = i0$ forms a three-vector as:

$$h_{00} = -2\Phi$$
$$h_{0i} = w_i$$

ij components form a spatial tensor. This symmetric part can be broken down into a trace and a traceless part, which are the irreducible representations of the rotation group (O3) right???. These irreducible representations transform independently of each other under spatial rotations as. The decomposition is written as:

$$h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$$

where Ψ contains the trace of h_{ij} and s_{ij} is traceless:

$$\Psi = -\frac{1}{6}\delta^{ij}h_{ij} = -\frac{1}{6}h_i^i \tag{2.5}$$

$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right) = \frac{1}{2}\left(h_{ij} - \frac{1}{3}h_l^l\delta_{ij}\right) \tag{2.6}$$

The metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is written as:

$$ds^2 = -(1+2\Phi)dt^2 + w_i(dtdx^i + dx^i dt) + [(1-2\Psi)\delta_{ij} + 2s_{ij}]dx^i dx^j \tag{2.7}$$

This is just a convenient notation.
To understand the physical interpretation of the fields appearing in metric, we consider the motion of test particles given by the geodesic equation. The crystoffel symbols for the metric are:

$$\begin{aligned}
\Gamma^0_{00} &= \partial_0 \Phi \\
\Gamma^i_{00} &= \partial_i \Phi + \partial_0 w_i && (2.8) \\
\Gamma^0_{j0} &= \partial_j \Phi && (2.9) \\
\Gamma^i_{j0} &= \partial_{[j} w_{i]} + \frac{1}{2}\partial_0 h_{ij} && (2.10) \\
\Gamma^0_{jk} &= -\partial_{(j} w_{k)} + \frac{1}{2}\partial_0 h_{jk} && (2.11) \\
\Gamma^i_{jk} &= \partial_{(j} h_{k)i} - \frac{1}{2}\partial_i h_{jk}. && (2.12)
\end{aligned}$$

In these expressions, we have used the symmetric tensor h_{ij} rather than s_{ij}, traceless part and Ψ, they appeared in the form $h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$, so it was convenient to use this form. The distinction will become important once we start taking traces to get to the Ricci tensor and Einstein's equation. By decomposition of the metric under rotations, we fixed the inertial frame. So, it is convenient to express the four-momentum $p^\mu = dx^\mu/d\lambda$ (where $\lambda = \tau/m$ if the particle is massive) in terms of the energy E and three-velocity $v^i = dx^i/dt$, as

$$p^0 = \frac{dt}{d\lambda} = E, \quad p^i = Ev^i$$

Geodesic equation in terms of four momentum becomes:

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma = 0$$

Dividing it be E:

$$\begin{aligned}
\frac{dp^\mu}{d\lambda E} &= -\frac{\Gamma^\mu_{\rho\sigma} p^\rho p^\sigma}{E} \\
\frac{dp^\mu}{dt} &= -\Gamma^\mu_{\rho\sigma}\frac{p^\rho p^\sigma}{E}
\end{aligned}$$

Writing out the $\mu = 0$ gives the evolution of energy:

$$\begin{aligned}
\frac{dE}{dt} &= -\Gamma^0_{\rho\sigma}\frac{p^\rho p^\sigma}{E} \\
&= -\Gamma^0_{00}\frac{p^0 p^0}{E} - 2\Gamma^0_{j0}\frac{p^j p^0}{E} - \Gamma^0_{jk}\frac{p^j p^k}{E} \\
&= -\partial_0 \Phi E - 2\partial_j \Phi E v^j + (\partial_{(j} w_{k)} - \frac{1}{2}\partial_0 h_{jk}) E v^j v^k \\
&= -E\left[\partial_0 \Phi + 2\partial_j \Phi v^j - \left(\partial_{(j} w_{k)} - \frac{1}{2}\partial_0 h_{jk}\right) v^j v^k\right]
\end{aligned}$$

The spatial components of the geodesic equation are:

$$\begin{aligned}
\frac{dp^i}{dt} &= -\Gamma^i_{\rho\sigma}\frac{p^\rho p^\sigma}{E} \\
&= -\Gamma^i_{00}\frac{p^0 p^0}{E} - 2\Gamma^i_{j0}\frac{p^j p^0}{E} - \Gamma^i_{jk}\frac{p^j p^k}{E} \\
&= -E\left[\partial_i\Phi + \partial_0 w_i + 2(\partial_{[j}w_{i]} + \frac{1}{2}\partial_0 h_{ij})v^j + (\partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk})v^j v^k\right]
\end{aligned}$$

Now to interpret these physically, we define gravito-electric and gravito-magnetic three vector fields,

$$\begin{aligned}
G^i &= -\partial_i\Phi - \partial_0 w_i \\
H^i &= (\nabla \times \vec{w})^i = \epsilon^{ijk}\partial_j w_k
\end{aligned}$$

These are analogous to definition of electric and magnetic fields in terms of scalar potential V and vector potential $\vec{A}$ With this, geodesic equation becomes:

$$\frac{dp^i}{dt} = E\left[G^i + (\vec{v}\times H)^i - 2\partial_0 h_{ij}v^j + (\partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk})v^j v^k\right]$$

The first two terms describe how test particle moving along a geodesic is affected by scalar and vector perturbations Φ and w_i. The first two terms are analogous to Lorentz force law. The nexy terms are coupled to the spatial perturbation h_ij. Their relative importance will depend on the physical situation at hand. Now we find field equations for the metric: SHOW WORKING

$$\begin{aligned}
R_{0j0l} &= \partial_j\partial_l\Phi + \partial_0\partial_{(j}w_{l)} - \frac{1}{2}\partial_0\partial_0 h_{jl} \\
R_{0jkl} &= \partial_j\partial_{[k}w_{l]} - \partial_0\partial_{[k}h_{l]j} \\
R_{ijkl} &= \partial_j\partial_{[k}h_{l]i} - \partial_i\partial_{[k}h_{l]j},
\end{aligned}$$

with other components related by symmetries of the curvature tensor. Using $\eta^{\mu\nu}$ to obtain the Ricci tensor,

$$\begin{aligned}
R_{00} &= \nabla^2\Phi + \partial_0\partial_k w^k + 3\partial_0^2\Psi \\
R_{0j} &= -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s^k_j \\
R_{ij} &= -\partial_i\partial_j(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + \Box\Psi\delta_{ij} - \Box s_{ij} + 2\partial_k\partial_{(i}s_{j)}
\end{aligned}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the three-dimensional flat Laplacian. Since the Ricci tensor involves contractions, the trace-free s_ij and trace parts Ψ of the spatial perturbations now enter the equation. Finally, the Einstein tensor,

$$\begin{aligned}
G_{00} =& 2\nabla^2\Psi + \partial_k\partial_l s^{kl} \\
G_{0j} =& -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s^k_j \\
G_{ij} =& \left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\Phi - \Psi) + \delta_{ij}\partial_0\partial_k w^k - \partial_0\partial_{(i}w_{j)} \\
&+ 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} + 2\partial_k\partial_{(i}{s_{j)}}^k - \delta_{ij}\partial_k\partial_l s^{kl}
\end{aligned}$$

Now, we analyze the degrees of freedom of the gravitational fields. We start with $G_{00} = 8\pi G T_{00}$:

$$\nabla^2\Psi = 8\pi T_{00} - \frac{1}{2}\partial_k\partial_l s^{kl}$$

This equation for Ψ includes no time derivatives. Knowing T_{00} and s_{ij} at any time determines Ψ (of course, spatial boundary conditions are there). So, Ψ is not a PROPAGATING? degree of freedom. Next $G_{0J} = 8\pi G T_{00}$:

$$(\delta_{jk}\nabla^2 - \partial_j\partial_k)w^k = -16\pi G T_{0j} + 4\partial_0\partial_j\Psi + 2\partial_0\partial_k s_j^k$$

This is an equation for w^k with no time derivatives. Knowing T_{0j} and strain (from which we can find Ψ), we can find w^k. Now, the G_{ij} part is:

$$\left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)\Phi = 8\pi G T_{ij} + (\delta_{ij}\nabla^2 - \partial_i\partial_j - 2\delta_{ij}\partial_0^2)\Psi - \delta_{ij}\partial_0\partial_k w^k + \partial_0\partial_{(i}w_{j)} + \Box s_{ij} - 2\partial_k\partial_{(i}s_{j)}{}^k - \delta_{ij}\partial_k\partial_l s^{kl}$$

Again, there are no time derivatives acting on Φ which is determined as a function of other fields.
Therefore, the only PROPAGATING DEGREES of freedom in Einstein's equations are those in the strain tensor s_{ij}. WE WILL FIND OUT, that its used to describe gravitational waves. The other components of the perturbation $h_{\mu\nu}$ are determined in terms of the strain vector.

2.2.1 Gauge Transformations

Under gauge tranformations $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ generated by the vector field ξ^μ, the fields change by: WORK IT OUT

$$\begin{aligned}
\Phi &\to \Phi + \partial_0\xi^0 \\
w_i &\to w_i + \partial_0\xi^i - \partial_i\xi^0 \\
\Psi &\to \Psi - \frac{1}{3}\partial_i\xi^i \\
s_{ij} &\to s_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}\partial_k\xi^k\delta_{ij}
\end{aligned}$$

Let's discuss some well known gauge choices. The transverse gauge is analogous to Coulomb gauge of electromagnetism. First we fix the strain to be spatially transverse as,

$$\partial_i s^{ij} = 0$$

and ξ^i satisfy:

$$\nabla^2\xi^j + \frac{1}{3}\partial_j\partial_i\xi^i = -2\partial_i s^{ij}$$

To determine value of ε^0 we fix:

$$\partial_i w^i = 0$$

, by choosing ξ^0 to satisfy:

$$\nabla^2\xi^0 = \partial_i w^i + \partial_0\partial_i\xi^i$$

None of the conditions satisfied by the vector field completely fix its value. Because they are second order differential equations, we need boundary conditions. In this guage the Einstein's tensor becomes: WORK OUT

$$G_{00} = 2\nabla^2\Psi \tag{2.13}$$

$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + 2\partial_0\partial_j\Psi \tag{2.14}$$

$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} \tag{2.15}$$

Another guage is synchronous gauge. In this gauge we set $\Phi = 0$ and choose ε^0 to satisfy $\partial_0\xi^0 = -\Phi$. Now to choose ξ^i we set $w^i = 0$ and choose $\partial_0\xi^i = -w^i + \partial_i\xi^0$. We also have another gauge Lorenz/harmonic gauge where we set:

$$\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h = 0$$

An additional decomposition of the metric perturbation, there are additional decompositions possible if we consider tensor fields. This brings out the physical degrees of freedom. A vector field can be decomposed into a transverse part $w^i_\perp$ and a longitudnal part $w^i_\parallel$:

$$w^i = w^i_\perp + w^i_\parallel$$

As usual the transverse vector is divergenceless $\partial_i w^i_\perp = 0$, and longitudnal vector is curl free ϵ. Due to this property, a transverse vector can be represented as a curl of some other vector ξ^i and a longitudnal vector as a divergence of a scalar vector:

$$w^i_\perp = \epsilon^{ijk}\partial_j\xi_k, \quad w_{\parallel j} = \partial_i\lambda$$

This decomposition of vector fields is also invariant under spatial rotations. The scalar represents one degree of freedom while ξ^i has 2, since the choice is not unique $\xi_i + \partial_i w$.
Similarly the traceless symmetric tensor, strain, s^{ij} can be decomposed into a transverse part $s^{ij}_\perp$, a solenoidal part s^{ij}_S and a longitudnal part $s^{ij}_\parallel$,

$$s^{ij} = s^{ij}_\perp + s^{ij}_S + s^{ij}_\parallel$$

Again, the transverse part is divergencless, while divergence of the solenoidal part is a transverse vector and divergence of longitudnal part is a longitudnal vector,

$$\partial_i s^{ij}_\perp = 0$$
$$\partial_i\partial_j s^{ij}_S = 0$$
$$\epsilon^{jkl}\partial_k\partial_i s^i_{\parallel j} = 0$$

This implies that the longitudnal part can be derived from a scalar field θ and the solenoidal part can be derived from a transverse vector ζ^i:

$$s_{\parallel ij} = (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\theta$$
$$s_{Sij} = \partial_{(i}\zeta_{j)},$$

where $\partial_i\zeta^i = 0$.
So, the longitudnal part descrives a single degree of freedom, solenoidal 2 and transverse describes the remaining 2.
With this decomposition of tensor fields, we have written the ten component perturbation $h_{\mu\nu}$ in terms of four scalars $\Phi, \Psi, \lambda, \theta$, with one degree of freedom each, two transverse vectors ξ^i, ζ^i with two degrees of freedom each and one transverse tensor $s^{ij}_\perp$ with two degrees of freedom.

2.3 Newtonian Fields and Photon Trajectories

In the Newtonian limit (weak field, static non relativistic), Einstein's equations reduce to Poisson equation. This is a special case of general week field limit equations. Here, we extend that and allow the test paticles to move at any velocity. We model static gravitating sources by dust, a perfect fluid for which pressure is 0. Working in the rest frame of dust, the energy-momentum tensor becomes:

$$T = \rho U_\mu U_\nu = \begin{bmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

We were able to use the simply Lorentz transform into the rest frame because of the background Minkowski metric. We cannot deal with multiple sources with large relative velocities in such a framework. We go back to the Einstein's equations in transverse gauge given by 2.13 and drop the time derivative terms (static sources) to get:

$$\nabla^2 \Psi = 4\pi G\rho \tag{2.16}$$

$$\nabla^2 w_j = 0 \tag{2.17}$$

$$(\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \nabla^2 s_{ij} = 0 \tag{2.18}$$

Looking for solutions that are non singular and vanish as infinity, only the fields sourced by rhs, i.e ρ will be non zero. So, the second equation in 2.18 (not having a source term on rhs), implies $w^i = 0$. Taking trace of the third equation:

$$(\delta_i^i\nabla^2 - \partial^i\partial_i)(\Phi - Psi) - \nabla^2 s_i^i = 0$$
$$2\nabla^2(\Phi - \Psi) = 0$$

This implies, that the two scalar potentials are equal,

$$\Phi = \Psi.$$

Plugging this into the last equation of 2.18 gives:

$$\nabla^2 s_{ij} = 0,$$

which implies $s_{ij} = 0$. With these simplifications, the perturbed metric in 2.7 becomes:

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j \tag{2.19}$$

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \tag{2.20}$$

$$h_{\mu\nu} = \begin{bmatrix} -2\Phi & & & \\ & -2\Phi & & \\ & & 2\Phi & \\ & & & -2\Phi \end{bmatrix}$$

By the first equation in 2.18, potential obeys the Poisson equation:

$$\nabla^2 \Phi = 4\pi G\rho$$

.

Now consider a massless particle in this geometry. So, we solve the perturbed geodesic equation for a null geodesic. Similarly to considering the perturbation as a field defined on a flat background metric, we decompose the geodesic into a background path plus a perturbation:

$$x\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda)$$

where $x^{(0)\mu}$ is the straight null path. We find quantities along the background path to find the perturbed path. For this to be valid, we assume that the potential is not much different along the background and the true geodesic $x\mu(\lambda)$. So we require $x^{(i)\mu}(\lambda)\partial_i\Phi \ll \Phi$ (only concerned with the spacial component because the source is static). Or we could consider very short paths along which $x^{(i)\mu}$ will be small. We will derive the true equations that we will have to integrate on the actual path $x\mu(\lambda)$. Let the wave vector of the background path be k^μ and the derivative of deviation vector be l^μ:

$$k^\mu \equiv \frac{x^{(0)\mu}}{d\lambda}, \quad l^\mu \equiv \frac{x^{(1)\mu}}{d\lambda}$$

Null trajectory is:

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

We solve this order-by-order. At zeroth order, we just have $g_{\mu\nu} = \eta_{\nu\nu}$, and the wave vector of background path, so $\eta_{\nu\nu}k^\mu k^\nu = 0$. Rewriting:

$$(k^0)^2 = (\vec{k})^2 \equiv k^2.$$

At first order, we have:

$$(\eta_{\mu\nu} + h_{\mu\nu})(k^\mu + l^\mu k^\nu + l^\nu) = 0 \tag{2.21}$$
$$\eta_{\mu\nu}k^\mu l^\nu + \eta_{\mu\nu}k^\nu l^\mu + h_{\mu\nu}k^\mu k^\nu = 0 \tag{2.22}$$
$$2\eta_{\mu\nu}k^\mu l^\nu + h_{\mu\nu}k^\mu k^\nu = 0 \tag{2.23}$$
$$2(-kl^0 + \vec{l}k) + (-2\Phi)(k^2 + k^2) = 0 \tag{2.24}$$
$$-kl^0 + \vec{l}k = 2k^2\Phi \tag{2.25}$$

The geodesic equation is:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

We first find the christoffel symbols by setting $w^i = 0$ and $h_{ij} = -2\Phi\delta_{ij}$ in Eq. 2.12,

$$\Gamma^0_{0i} = \Gamma^i_{00} = \partial_i\Phi$$
$$\Gamma^i_{jk} = \delta_{jk}\partial_i\Phi - \delta_{ik}\partial_j\Phi - \delta_{ij}\partial_k\Phi$$

Now, solving the geodesic equation order-by-order. At zeroth order, we just find that $x^{(0)\mu}$ is a straight trajectory (because Christoffel symbols are first order so we have $\frac{d^2x^{(0)\mu}}{d\lambda^2} = 0$). At first order, we have:

$$\frac{dl^\mu}{d\lambda} = -\Gamma^\mu_{\rho\sigma}k^\rho k^\sigma$$

Note that we don't have factors of l^μ because Γ is already of first order. The temporal component of first order geodesic equation is:

$$\frac{dl^0}{d\lambda} = -\Gamma^0_{\rho\sigma}k^\rho k^\sigma \tag{2.26}$$
$$\frac{dl^0}{d\lambda} = -2\Gamma^0_{0i}k^0 k^i \tag{2.27}$$
$$\frac{dl^0}{d\lambda} = -2k^0\partial_i\Phi k^i \tag{2.28}$$
$$\frac{dl^0}{d\lambda} = -2k(\vec{k}\cdot\nabla\Phi) \tag{2.29}$$

The spatial components:

$$\begin{aligned}
\frac{dl^i}{d\lambda} &= -\Gamma^i_{\rho\sigma}k^\rho k^\sigma \\
&= -\Gamma^i_{00}k^0 k^0 - \Gamma^i_{jk}k^j\ k^k \\
&= -\partial_i\Phi k^2 - (\delta_{jk}\partial_i\Phi - \delta_{ik}\partial_j\Phi - \delta_{ij}\partial_k\Phi)k^j k^k \\
&= -\partial_i\Phi k^2 - (k_k k^k\partial_i\Phi - k_i k^j\partial_j\Phi - k_i k^k\partial_k\Phi) \\
&= -2k^2\partial_i\Phi + 2k_i k\cdot\nabla\Phi \\
&= -2k^2\nabla_\perp\Phi
\end{aligned}$$

where $\nabla_\perp$ is the gradient transverse to the path (background path), defined as:

$$\begin{aligned}\nabla_\perp &\equiv \nabla\Phi - \nabla_\parallel\Phi \\ &= \nabla\Phi - k^{-2}(\vec{k}\cdot\nabla\Phi)\vec{k}\end{aligned}$$

(checking if one of the christoffel symbols is correct

$$\begin{aligned}\Gamma^i_{jk} &= \partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk}. \\ &= \frac{1}{2}\partial_j h_{ki} + \frac{1}{2}\partial_k h_{ji} - \frac{1}{2}\partial_i h_{jk} \\ &= -\partial_j\Phi\delta_{ki} - \partial_k\Phi\delta_{ji} + \partial_i\Phi\delta_{jk}\end{aligned}$$

)
To the first order in Φ, $\vec{l}$ is orthogonal to the the original wave vector $\vec{k}$. To see this, we integrate the $\mu = 0$ component:

$$\begin{aligned}l^0 &= \int \frac{dl^0}{d\lambda} d\lambda \\ &= 2k\int (\vec{k}\cdot\nabla\Phi) d\lambda \\ &= -2k\int \left(\frac{dx^i}{d\lambda}\partial\right) d\lambda \\ &= -2k\int partial_i\Phi dx^i \\ &= -2k\Phi\end{aligned}$$

where we demanded that $l^0 = 0$, when $\Phi = 0$. Plugging this into 2.25:

$$l^i k_i = kl^0 + 2k^2\Phi = 0,$$

so, to first order, wave vector is orthogonal to the perturbation wave vector.
We define **Deflection angle**, $\hat{\alpha}$ the amount by which the original spatial wave vector is deflected as it propagates from a source to the observer. It is a 2d vector in a plane perpendicular to $\vec{k}$. From figure (it is the derivative (tangent to the curve) of the deflection vector, so change in deflection vector wrt to the magnitude of original propagation vector k), it is:

$$\hat{\alpha} = -\frac{\Delta\vec{l}}{k}$$

Minus sign is there because the observer is looking backward along the trajectory of photon. $\Delta\vec{l}$ is the rotation of wave vector calculated asL

$$\begin{aligned}\Delta\vec{l} &= \int \frac{d\vec{l}}{d\lambda} d\lambda \\ &= -2k^2\int \vec{\nabla}_\perp\Phi d\lambda\end{aligned}$$

Changing the angle to deflection as $s = k\lambda$ (physical spatial distance transversed) as:

$$\hat{\alpha} = 2\int \vec{\nabla}_\perp\Phi ds \tag{2.30}$$

Lets evaluate the deflection angle by a point mass M. Suppose that the background path is along x direction. Impact parameter is defined by a transverse vector $\vec{b}$ pointing from the path to the mass at the point of closest approach ($alongx^{(0)}$). The potential is:

$$\phi = -\frac{GM}{r} = -\frac{GM}{(b^2 + x^2)^{\frac{1}{2}}}$$

its transverse gradienti is:

$$\nabla_\perp \Phi = -\frac{GM}{(b^2 + x^2)^{\frac{3}{2}}}\vec{b}.$$

$$\begin{aligned}\hat{\alpha} &= 2GMb \int_{-\infty}^{\infty} \frac{dx}{(b^2 + x^2)^{3/2}} \\ &= 4\frac{GM}{b}\end{aligned}$$

ANOTHER OB

2.4 Gravitational Wave Solutions

Now we study freely propagating degrees of freedom of the gravitational field, requiring no local sources (fields in vacuum right?). Weak field einstein equations (00 component) become:

$$\nabla^2 \Psi = 0$$

with well behaved boundary conditions (no singularities, fields go to zero at infinity), $\Psi = 0$. $0j$ equation:

$$\nabla^2 w_j = 0,$$

implies $w_j = 0$. trace of ij equation and using the above results:

$$\nabla^2 \Phi = 0,$$

so, $\Phi = 0$. Traceless part of ij equation:

$$\Box s_{ij} = 0$$

we have set all other degrees of freedom zero and s_{ij} is transverse, so we are working in traceless transverse gauge. The perturbation metric is:

$$h_{\mu\nu}^{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2s_{ij} & \\ 0 & & & \end{bmatrix}$$

$h_{\mu\nu}^{TT}$ is traceless, spatial and transverse. It is more conventional to write the equation of motion in terms of the perturbed metric, so it is:

$$\Box h_{\mu\nu}^{TT} = 0$$

This wave equation has plane wave solutions,

$$h_{\mu\nu}^{TT} = C_{\mu\nu} e^{ik_\sigma x^\sigma}$$

k^σ is wave vector. $C_{\mu\nu}$ is a symmetric (0, 2), traceless and a spatial tensor.

$$\begin{aligned} C_{0\mu} &= 0 \\ \eta^{\mu\nu} C_{\mu\nu} &= 0 \end{aligned}$$

To verify whether the plane wave solutions satisfy the wave equation:

$$\begin{aligned}\Box h_{\mu\nu}^{TT} &= 0\\ \partial^{\alpha}[\partial_{\alpha}(C_{\mu\nu}e^{ik_{\sigma}x^{\sigma}})] &= 0\\ \partial^{\alpha}(C_{\mu\nu}e^{ik_{\sigma}x^{\sigma}}ik_{\alpha}) &= 0\\ -C_{\mu\nu}e^{ik_{\sigma}x^{\sigma}}k_{\alpha}k^{\alpha} &= 0\\ -h_{\mu\nu}^{TT}k^{\alpha}k_{\alpha} &= 0\end{aligned}$$

We cannot make $h_{\mu\nu} = 0$, we set:

$$k_{\alpha}k^{\alpha} = 0$$

Since, for plane waves solution to satisfy the wave equation, the wave vector must be null, loosely speaking, gravitational waves travel at speed of light. The vector of w waves is of the form $k^{\sigma} = (\omega, k^{i})$, where ω is the frequency of the wave. For wave vector to be null, we should have:

$$\begin{aligned}k^{\sigma}k_{\sigma} &= 0\\ \omega^{2} &= k^{i}k_{i}\end{aligned}$$

To ensure that the perturbation is transverse we need:

$$\begin{aligned}\partial_{\mu}h_{TT}^{\mu\nu} &= 0\\ iC^{\mu\nu}k_{\mu}e^{ik_{\sigma}x^{\sigma}} &= 0 \implies k_{\mu}C^{\mu\nu} = 0\end{aligned}$$

So, the wave vector must be orthogonal to $C^{\mu\nu}$.
To make the solution more explicit, we choose a wave traveling in z direction:

$$k^{\sigma} = (\omega, 0, 0, \omega)$$

For $C_{\mu\nu}$ to be orthogonal, we should have $C_{3\nu} = 0$. In general we can write:

$$C_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & C_{11} & C_{12} & 0\\ 0 & C_{12} & -C_{11} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Note that this is symmetric and traceless. So, in this gauge, for a plane wave traveling in the z direction, two components C_{11} and C_{12} completely characterize the wave.
To understand the physical effect of a gravitational wave, consider the motion of test particles under the influence of a wave. Solving for a single particle will only tell about the coordinates along the world line. We can find transverse traceless coordinates where that particle appears stationary, to the first order. So, we must consider relative motion of particles to get coordinate independent wave effects. We start with the geodesic deviation equation:

$$\frac{D^{2}}{d\tau^{2}}S^{\mu} = R^{\mu}_{\nu\rho\sigma}T^{\nu}T^{\rho}S^{\sigma}$$

Consider some nearby particles whose four velocities are describes by a single vector field $U^{\mu}(x)$ and separation vector S^{μ}, the deviation equation becomes:

$$\frac{D^{2}}{d\tau^{2}}S^{\mu} = R^{\mu}_{\nu\rho\sigma}U^{\nu}U^{\rho}S^{\sigma}$$

We evaluate the right hand side to firsr order in $h^{TT}_{\mu\nu}$. If we take the particles to be slowly moving, we can write it as a four velocity with $U^0 = 1$, with rest of components being 0, plus corrections of order $h^{TT}_{\mu\nu}$ and higher than that. But on rhs, we already have reimannian tensor which is first order in perturbation, so we take only the zeroth order in velocity ignoring the higher order corrections.

$$U^{\nu} = (1, 0, 0, 0)$$

So, we only need to find $R^{\mu}_{00\sigma}$. Using Eq. 2.1:

$$R_{\mu 00\sigma} = \frac{1}{2}(\partial_0\partial_0 h^{TT}_{\sigma\mu} - \partial_0\partial_\mu h^{TT}_{0\sigma} - \partial_\sigma\partial_0 h^{TT}_{0\mu} + \partial_\sigma\partial_\mu h^{TT}_{00})$$

Using, $h^{TT}_{00} = h^{TT}_{\mu 0} = 0$ we get:

$$R_{\mu 00\sigma} = \frac{1}{2}\partial_0\partial_0 h^{TT}_{\sigma\mu}$$

To the lowest order, for slowly moving particles we have $\tau = t$, with these the geodesic equation becomes:

$$\frac{\partial^2}{\partial t^2} S^{\mu} = \frac{1}{2} S^{\sigma} \frac{\partial^2}{\partial t^2} h^{TT\mu}_{\sigma}$$

For a wave traveling in x^3 direction, only S^1 and S^2 will be affected. It is because test particles are only distributed in directions perpendicular to the wave propagation just like in electromagnetism the direction of wave is perpendicular to E and B fields. Recall that two numbers C_{11} and C_{12} characterize this wave. Renaming those:

$$h_+ = C_{11}$$
$$h_\times = C_{12}$$

$C_{\mu\nu}$ becomes:

$$C_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.31}$$

To consider their effects separately, we first set $h_\times = 0$. We have:

$$\frac{\partial^2}{\partial t^2} S^1 = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2}(h_+ e^{ik^{\sigma}_{\sigma}})$$
$$\frac{\partial^2}{\partial t^2} S^2 = -\frac{1}{2} S^2 \frac{\partial^2}{\partial t^2}(h_+ e^{ik^{\sigma}_{\sigma}})$$

These have perturbative solutions, expanding those to lowest order in h:

$$S^1 = \left(1 + \frac{1}{2} h_+ e^{ik_\sigma k^\sigma}\right) S^1(0)$$
$$S^2 = \left(1 - \frac{1}{2} h_+ e^{ik_\sigma k^\sigma}\right) S^2(0)$$

So, the particles separated in x^1 direction will oscillate in x^1 direction and the ones separated in x^2 direction will oscillate in x^2. So, if we have particles in a ring in x-y direction. They will bounce back and forth in the

shape of a +. ADD FIGURE.
Now we set $h_+ = 0$

$$\frac{\partial^2}{\partial t^2}S^1 = \frac{1}{2}S^2\frac{\partial^2}{\partial t^2}(h_\times e^{ik^\sigma_\sigma})$$
$$\frac{\partial^2}{\partial t^2}S^2 = -\frac{1}{2}S^1\frac{\partial^2}{\partial t^2}(h_\times e^{ik^\sigma_\sigma})$$

with solutions:

$$S^1 = S^1(0) + S^2(0)\frac{1}{2}h_\times e^{ik_\sigma k^\sigma}$$
$$S^2 = S^2(0) - \frac{1}{2}h_\times e^{ik_\sigma k^\sigma}S^1(0)$$

In this case, the particles oscillate in a $\times$ pattern.
h_+ and $h_\times$ measure two independent modes of linear polarization of the gravitational waves. These are known as 'plus' and 'cross' polarizations. Using these we can construct right and left handed circularly polarized modes:

$$h_R = \frac{1}{\sqrt{2}}(h_+ + ih_\times)$$
$$h_L = \frac{1}{\sqrt{2}}(h_+ - ih_\times)$$

These polarization states of classical gravitational waves can be related to the kind of particles we will find upon quantization. If we know how a field behaves under spatial rotations (like polarization properties of the field), we can find out spin of particles we will get upon quantization. Electromagnetic field has two independent polarizations. A single polarization mode is invariant under a 360 deg rotation in x-y plane. So, quantizing this field gives a massless spin-1 particle. The neutrino is described by a field that picks up a minus sign under such a rotation and it has spin $\frac{1}{2}$. The general rule is the spin S is related to the angle θ under which the polarization modes are invariant by $S = 360^{\text{deg}}/\theta$. Gravitational field travels at the speed of light, so it should lead to massless particles. Polarization modes described above are invariant under rotations of 180^{deg} in x-y plane, so they should lead to spin-2 particles upon quantization. These are called gravitons and have not been detected.

2.5 Energy Loss due to Gravitational Radiation

Energy in gravitational field cannot be localized. In linearized gravity, we hope to derive an energy-momentum tensor for fluctuations in the perturbation. The energy momentum tensors for electromagnetism and scalar field theory are quadratic in the relevant fields. So, we must extend the weak field limit to second order in perturabtion. When we discussed the effect of the gravitational wave on test particles, we assumed that the test particles move along geodesics, which is derived from covariant energy conservation when actually $\partial_\mu T^{\mu\nu} = 0$, which means that the test particles move along geodesics in the flat background metric.

Now we expand Einstein equation to the second order. We expand the metric and the Ricci tensor as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}$$
$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$$

$R^{(1)}_{\mu\nu}$ is of the same order as $h^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ is of order $\left(h^{(1)}_{\mu\nu}\right)^2$. In a flat background, the zeroth-order Einstein equation is trivially solved $R^{(0)}_{\mu\nu}$. The first order vacuum equation is:

$$R^{(1)}_{\mu\nu}[h^{(1)}] = 0 \tag{2.32}$$

The above equation determines the metric perturbation to the first order. The second order perturbation is determined by:

$$R^{(1)}_{\mu\nu}[h^{(2)}] + R^{(2)}_{\mu\nu}[h^{(1)}] = 0$$

$R^{(1)}_{\mu\nu}[h^{(2)}]$ are the parts of expanded Ricci tensor that are linear in metric perturbation as given in section 2, but applied to the second order perturbation. Similarly, $R^{(2)}_{\mu\nu}[h^{(1)}]$ indicates the parts of the second order in Ricci tensor applied to the first order perturbation as:

$$R^{(2)}_{\mu\nu} = \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} + \frac{1}{4}\left(\partial_\mu h_{\rho\sigma}\right)\partial_\nu h^{\rho\sigma} + \left(\partial^\sigma h^\rho{}_\nu\right)\partial_{[\sigma}h_{\rho]\mu} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} \tag{2.33}$$

$$+ \frac{1}{2}\partial_\sigma\left(h^{\rho\sigma}\partial_\rho h_{\mu\nu}\right) - \frac{1}{4}\left(\partial_\rho h_{\mu\nu}\right)\partial^\rho h - \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right)\partial_{(\mu}h_{\nu)\rho}, \tag{2.34}$$

Now considering the vacuum equation $G_{\mu\nu} = 0$ at second order:

$$R^{(1)}_{\mu\nu}[h^{(2)}]\frac{1}{2} - \eta^{\rho\sigma}R^{(1)}_{\rho\sigma}[h^{(2)}]\eta_{\mu\nu} = 8\pi G t_{\mu\nu}, \tag{2.35}$$

where $t_{\mu\nu}$ is defined as:

$$t_{\mu\nu} \equiv -\frac{1}{8\pi G}\left\{R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h^{(1)}]\eta_{\mu\nu}\right\}$$

We made use of Eq. 2.32 in the result 2.35. We have moved the terms of the $R^{(2)}_{\rho\sigma}[h^{(1)}]$ to the right hand side relabeling those as the energy momentum tensor for first order perturbation. The resulting $t_{\mu\nu}$ is a symmetric tensr, quadratic in the perturbation and represents how the perturbations affect the spacetime metric just like the matter energy moment tensor. $t_{\mu\nu}$ is also conserved in the flat background sense:

$$\partial_\mu t^{\mu\nu} = 0,$$

which follows from the Bianchi identity $\partial_\mu G^{\mu\nu}$.
There are some limitations in the interpretation of $t_{\mu\nu}$ as an energy momentum tensor. It is not invariant under gauge transformations (CHECKK!!). To avoid this limitation, we average the energy-momentum tensor over several wavelengths. This is denoted by angle brackets. By averaging over several wavelengths, we try to capture physical curvature in a small region to describe a gauge-invariant measure. So, derivatives will average to zero,

$$\langle\partial_\mu(X)\rangle = 0$$

With this the product rule becomes:

$$\langle\partial_\mu(AB)\rangle = 0 \implies \langle B\partial_\mu(A)\rangle = -\langle A\partial_\mu(B)\rangle.$$

This will simplify our calculations. Now we move onto calculating the energy moment tensor, using Eq. 2.34. We also remove the labels because we are interested in the first order metric perturbation. To make our calculations simpler we do those in a transverse tracless guage, recalling $\partial^\mu h^{TT}_{\mu\nu} = 0, h^{TT} = 0$. We can use the transverse traceless gauge in vacuum. Using the guage conditions, expressions are simplified to

$$R^{(2)\mathrm{TT}}_{\mu\nu} = \frac{1}{2}h^{\rho\sigma}_{\mathrm{TT}}\partial_\mu\partial_\nu h^{\mathrm{TT}}_{\rho\sigma} + \frac{1}{4}\left(\partial_\mu h^{\mathrm{TT}}_{\rho\sigma}\right)\partial_\nu h^{\rho\sigma}_{\mathrm{TT}} + \frac{1}{2}\eta^{\rho\lambda}\left(\partial^\sigma h^{\mathrm{TT}}_{\rho\nu}\right)\partial_\sigma h^{\mathrm{TT}}_{\lambda\mu}$$
$$- \frac{1}{2}\left(\partial^\sigma h^{\mathrm{TT}}_{\rho\nu}\right)\partial^\rho h^{\mathrm{TT}}_{\sigma\mu} - h^{\rho\sigma}_{\mathrm{TT}}\partial_\rho\partial_{(\mu}h^{\mathrm{TT}}_{\nu)\sigma} + \frac{1}{2}h^{\rho\sigma}_{\mathrm{TT}}\partial_\sigma\partial_\rho h^{\mathrm{TT}}_{\mu\nu}.$$

Now we average over several wavelengths and use the product rule condition, the first two terms add up and the last three terms cancel out, so we're left with:

$$\left\langle R_{\mu\nu}^{(2)\mathrm{TT}} \right\rangle = -\frac{1}{4} \left\langle \left(\partial_\mu h_{\rho\sigma}^{\mathrm{TT}}\right) \left(\partial_\nu h_{\mathrm{TT}}^{\rho\sigma}\right) + 2\eta^{\rho\lambda} \left(\Box h_{\rho\nu}^{\mathrm{TT}}\right) h_{\lambda\mu}^{\mathrm{TT}} \right\rangle .$$

$\Box h_{\rho\nu} = 0$, is the equation of motion in vacuum, so the expression that we're left with:

$$\left\langle R_{\mu\nu}^{(2)\mathrm{TT}} \right\rangle = -\frac{1}{4} \left\langle \left(\partial_\mu h_{\rho\sigma}^{\mathrm{TT}}\right) \left(\partial_\nu h_{\mathrm{TT}}^{\rho\sigma}\right) \right\rangle .$$

Now, to get the Ricci scalar:

$$\begin{aligned}
\langle \eta^{\mu\nu} R_{\mu\nu}^{(2)\mathrm{TT}} \rangle &= -\frac{1}{4} \left\langle (\eta^{\mu\nu} \partial_\mu h_{\rho\sigma}^{\mathrm{TT}})(\partial_\nu h_{\mathrm{TT}}^{\rho\sigma}) \right\rangle \\
&= -\frac{1}{4} \left\langle (\partial^\nu h_{\rho\sigma}^{\mathrm{TT}})(\partial_\nu h_{\mathrm{TT}}^{\rho\sigma}) \right\rangle \\
&= \frac{1}{4} \left\langle (\partial^\nu \partial_\nu h_{\rho\sigma}^{\mathrm{TT}})(h_{\mathrm{TT}}^{\rho\sigma}) \right\rangle \\
&= 0
\end{aligned}$$

inserting the obtained expressions for Ricci tensor and Ricci scalar in the expression for $t_{\mu\nu}$, we get

$$t_{\mu\nu} = \frac{1}{32\pi G} \left\langle \left(\partial_\mu h_{\rho\sigma}^{\mathrm{TT}}\right) \left(\partial_\nu h_{\mathrm{TT}}^{\rho\sigma}\right) \right\rangle \tag{2.36}$$

If we had not simplified the calculations by using the transeverse traceless guage, we'd have obtained: (CHECKK GUAGE INVARIANCE)

$$\begin{aligned}
t_{\mu\nu} = \frac{1}{32\pi G} \langle & \left(\partial_\mu h_{\rho\sigma}\right) \left(\partial_\nu h^{\rho\sigma}\right) - \frac{1}{2} \left(\partial_\mu h\right) \left(\partial_\nu h\right) \\
& - \left(\partial_\rho h^{\rho\sigma}\right) \left(\partial_\mu h_{\nu\sigma}\right) - \left(\partial_\rho h^{\rho\sigma}\right) \left(\partial_\nu h_{\mu\sigma}\right) \rangle .
\end{aligned}$$

Energy momentum for a plane wave of the form,

$$h_{\mu\nu}^{TT} = C_{\mu\nu} sin(k_\lambda x^\lambda)$$

is:

$$t_{\mu\nu} = \frac{1}{32\pi G} k_\mu k_\nu C_{\rho\sigma} C^{\rho\sigma} \langle \cos^2(k_\lambda \lambda^\lambda) \rangle .$$

Averaging the $\cos^2$ over several wavelengths, we get:

$$\langle \cos^2(k_\lambda \lambda^\lambda) \rangle = \frac{1}{2}.$$

Taking the wave to be moving along the z-axis:

$$k_\lambda = (-\omega, 0, 0, \omega)$$

We have the expression for $C_{\rho\sigma}$ Eq. 2.31, which we use to find:

$$C_{\rho\sigma} C^{\rho\sigma} = 2(h_+^2 + h_\times^2)$$

Observables are more commonly expressed in terms of frequency $f = \omega/2\pi$, so for energy momentum tensor we obtain:

$$t_{\mu\nu} = \frac{\pi}{8G} f^2 (h_+^2 + h_\times^2) \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Typical frequencies of gravitational wave sources lie between 10^{-4} and 10^{4} Hz, and amplitudes 10^{-22}. In z direction, the energy flux is:

$$-T_{0z} = 10^{-4}\left(\frac{f}{Hz}\frac{\text{erg}}{\text{cm}^2.\text{s}}\right)$$

So, in a gravitational wave detector, this is the amount of energy that could be deposited in each centimeter square every second. This is a large amount of energy flux. For comparison, peak electromagnetic flux from a supernova at cosmological distance is approximately 10^{-9} erg/cm^2/s and lasts for months. However, the gravitational wave signal only lasts for milliseconds.

Now that we have the gravitational wave energy momentum tensor, we can use it to calculate the amount of energy lost by a system emitting gravitational radiation. The total energy in gravitational wave on a surface of constant time Σ is defined to be:

$$E = \int_{\Sigma} t_{00} d^3x,$$

and the total energy radiated through aall space is given by:

$$\Delta e = \int P dt,$$

where P is given by:

$$P = \int_{S^2_\infty} t_{0\nu} n^\nu r^2 d\Omega.$$

The integral is over a two sphere that extends to infinity and n^μ is a unit vector normal to the surface of the two sphere. In polar coordinates t, r, θ, ϕ, the normal vector is:

$$n^\mu = (0, 1, 0, 0)$$

We want to calculate the power P. The issue is that expression for energy momentum tensor is in terms of transverse-tracless gauge, while quadrupole formula in terms of trace reversed perturbation. So, we need t0 first convert the trace reversed form into TT gauge to insert that into $t_{\mu\nu}$.

We start by introducing the projection tensor:

$$P_{ij} = \delta_{ij} - n_i n_j$$

It projects the tensor components to a surface orthogonal to n^i. In case of our normal vector, P_{ij} will project onto the infinite two sphere. If we have a symmetric spatial tensor, and we want to construct a transverse scalar tensor, we can do so by using the projection vector as:

$$X^{TT}_{ij} = \left(P^k_i P^l_j - \frac{1}{2}P_{ij}P^{kl}\right) X_{kl} \tag{2.37}$$

Taking the trace of above tensor gives, $X^i_i = 0$ and it is also transverse:

$$\begin{aligned}\partial_i \left(P^{ik}P^{jl} - \frac{1}{2}P^{ij}P^{kl}\right) X_{kl} &= (P^{jl}\partial_i(\delta^{ik} - n^i n^k) + P^{ik}\partial_i(\delta^{jl} - n^j n^l) \\ &\quad - \frac{1}{2}(P^{kl}\partial_i(\delta^{ij} - n^i n^j) + P^{ij}\partial_i(\delta^{kl} - n^k n^l))X_{kl} \\ &= 0\end{aligned}$$

So, the trace reversed tensor will also be the same as the original perturbation,

$$h^{TT}_{ij} = \bar{h}_{ij}^{TT} = \frac{2G}{r}\frac{d^2 I^{TT}_{ij}}{dt^2}(t-r) \tag{2.38}$$

A more convenient quantity than quadrupole moment is the reduced quadrupole moment:

$$J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}I_{kl},$$

it is just the traceless part of the quadrupole moment. So, eq. 2.38 becomes:

$$h_{ij}^{TT} = \bar{h}_{ij}^{TT} = \frac{2G}{r}\frac{d^2 J_{ij}^{TT}}{dt^2}(t-r)$$

Power involves the factor $t_{0\mu}n^\mu = t_{0r}$. And for that we need derivatives with respect to time and r:

$$\begin{aligned}
\partial_0 h_{ij}^{TT} &= \frac{2G}{r}\frac{d^3 J_{ij}^{TT}}{dt^3}, \\
\partial_r h_{ij}^{TT} &= -\frac{2G}{r}\frac{d^3 J_{ij}^{TT}}{dt^3} - \frac{2G}{r^2}\frac{d^2 J_{ij}^{TT}}{dt^2} \\
&\sim \frac{2G}{r}\frac{d^3 J_{ij}^{TT}}{dt^3}
\end{aligned}$$

r^{-2} term was dropped because it falls off rapidly in the $r \to \infty$ limit. With these results, t_{0r} becomes:

$$t_{0r} = -\frac{G}{8\pi r^2}\left\langle \left(\frac{d^3 J_{ij}^{TT}}{dt^3}\right)\left(\frac{d^3 J_{TT}^{ij}}{dt^3}\right)\right\rangle$$

Now, before moving substituting this back in the expression for power, we have to convert back the reduced quadrupole moment from the transverse traceless part. From Eq. 2.37,

$$\begin{aligned}
X_{ij}^{TT}X_{TT}^{ij} &= \left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)X_{kl}\left(P^{im}P^{jn} - \frac{1}{2}P^{ij}P^{mn}\right)X_{mn} \\
&= [(\delta_i^k - n_i n^k)(\delta_j^l - n_j n^l) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta^{kl} - n^k n^l)]X_{kl} \\
&\quad [(\delta^{im} - n^i n^m)(\delta^{jn} - n^j n^n) - \frac{1}{2}(\delta^{ij} - n^i n^j)(\delta^{mn} - n^m n^n)]X_{mn} \\
&= [(X_{il} - n_i n^k X_{kl})(X_{kj} - X_{kl}n_j n^l) - \frac{1}{2}(X_{kl}\delta_{ij} - X_{kl}n_i n_j)(X - X_{kl}n^k n^l)] \\
&\quad [(X_n^i - X_{mn}n^i n^m)(X_m^j - X_{mn}n^j n^n) - \frac{1}{2}(X_{mn}\delta^{ij} - X_{mn}n^i n^j)(X - X_{mn}n^m n^n)]
\end{aligned}$$

After some algebra we get:

$$X_{ij}^{TT}X_{TT}^{ij} = X_{ij}X^{ij} - 2X_i^j X^{ik}n_j n_k + \frac{1}{2}X^{ij}X^{kl}n_i n_j n_k n_l - \frac{1}{2}X^2 + XX^{ij}n_i n_j$$

For J_{ij}, the expression becomes:

$$J_{ij}^{TT}J_{TT}^{ij} = J_{ij}J^{ij} - 2J_i^j J^{ik}n_j n_k + \frac{1}{2}J^{ij}J^{kl}n_i n_j n_k n_l$$

where we always used that fact $J_i^i = 0$. Now we can write power as:

$$P = \frac{G}{8\pi}\int_{S_\infty^2}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3} - 2\frac{d^3 J_i^j}{dt^3}\frac{d^3 J^i k}{dt^3}n_j n_k + \frac{1}{2}\frac{d^3 J^{ij}}{dt^3}\frac{d^3 J^{kl}}{dt^3}n_i n_j n_k n_l\right\rangle d\Omega$$

Quadrupole tensors are independent of the angular coordinates (they are integral over all of space). So, we take them outside the integral and using the following identities:

$$\int d\Omega = 4\pi$$
$$\int n_i n_j d\Omega = \frac{4\pi}{3}\delta_{ij}$$
$$\int n_i n_j n_k n_l d\Omega = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Using these we evaluate the power as:

$$\begin{aligned}
P &= -\frac{G}{8\pi}\left\langle \frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3}\int_{S^2_\infty} d\Omega - 2\frac{d^3J_i^j}{dt^3}\frac{d^3J^{ik}}{dt^3}\int_{S^2_\infty} n_j n_k d\Omega \right.\\
&\left. +\frac{1}{2}\frac{d^3J^{ij}}{dt^3}\frac{d^3J^{kl}}{dt^3}\int_{S^2_\infty} n_i n_j n_k n_l d\Omega \right\rangle\\
&= -\frac{G}{8\pi}\left\langle \frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3}4\pi - 2\frac{d^3J_i^j}{dt^3}\frac{d^3J^{ik}}{dt^3}\frac{4\pi}{3}\delta_{jk} \right.\\
&\left. +\frac{1}{2}\frac{d^3J^{ij}}{dt^3}\frac{d^3J^{kl}}{dt^3}\frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \right\rangle\\
&= -\frac{G}{8\pi}\left\langle \frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3}4\pi - 2\frac{d^3J_i^j}{dt^3}\frac{d^3J_j^i}{dt^3}\frac{4\pi}{3} + 0 + \frac{2\pi}{15}\frac{d^3J_k^j}{dt^3}\frac{d^3J_j^k}{dt^3} + \frac{1}{2}\frac{d^3J_l^j}{dt^3}\frac{d^3J_j^l}{dt^3}\frac{4\pi}{15} \right\rangle\\
&= -\frac{G}{8\pi}\left\langle \frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3}4\pi - \frac{12\pi}{5}\frac{d^3J_i^j}{dt^3}\frac{d^3J_j^i}{dt^3} \right\rangle\\
&= -\frac{G}{8\pi}\left\langle \frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3}4\pi - \delta_{im}\delta^{in}\frac{12\pi}{5}\frac{d^3J^{jm}}{dt^3}\frac{d^3J_{jn}}{dt^3} \right\rangle\\
&= -\frac{G}{8\pi}\left\langle \frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3}4\pi - \delta_m^n\frac{12\pi}{5}\frac{d^3J^{jm}}{dt^3}\frac{d^3J_{jn}}{dt^3} \right\rangle\\
&= -\frac{G}{5}\left\langle \frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3} \right\rangle
\end{aligned}$$

Quadrupole moment is evaluated at a retarted time $t_r = t - r$. The negative sign represents that the radiating source will be losing energy. Coming back to the binary star system, first we find the reduced quandrupole moment to be:

$$J_{ij} = \frac{MR^2}{3}\begin{pmatrix} (1+3\cos 2\Omega t) & 3\sin 2\Omega t & 0 \\ 3\sin 2\Omega t & (1-3\cos 2\Omega t) & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

and the time derivatives are:

$$\frac{d^3J_{ij}}{dt^3} = 8MR^2\Omega^3\begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using the expression for power, we get:

$$P = -\frac{128}{5}GM^2R^4\Omega^6,$$

and using the expression for the frequency of a binary star system, we get:

$$P = -\frac{2}{5}\frac{G^4 M^5}{R^5}$$

Energy loss during the emission of gravitational system has been measured and the results are consistent with the prediction of general relativity.

2.5.1 Energy loss in a Binary Pulsar

Gravitation radiation reduces the energy and angular momentum on a orbiting binary system and change the orbital period P. We evaluate the rate of change of orbital period in a binary pulsar system. We assume that their orbit is circular and the pulsars are of equal masses M, with orbital radius R and speed V. Assuming that the system is non relativisitic, its Newtonian energy (in G = 1 units) is:

$$E_{\text{newt}} = 2\left(\frac{1}{2}MV^2\right) - \frac{M^2}{2R}$$

Using Eq. **??**, we get

$$E_{\text{newt}} = -\frac{M^2}{4R}$$

And R is related to orbital time period P as:

$$2\pi R = VP = \sqrt{\frac{M}{4R}}P$$
$$R = \left(\frac{MP^2}{16\pi^2}\right)^{1/3}$$

With this, the energy becomes:

$$E_{\text{newt}} = -\frac{1}{4}M\left(\frac{4\pi M}{P}\right)^{2/3}$$
$$\frac{dE}{dt} = -\frac{1}{4}M\frac{d\left(\frac{4\pi M}{P}\right)^{2/3}}{dt}$$

2.6 Detection of Gravitational Waves

Gravitational waves are produced by the bulk motion of large masses. As a simple example, consider a binary star system where the mass of each star is M, with orbital radius R. Using the Newtonian formulae for the orbital parameter will suffice for an order of magnitude estimate. The relevant parameters are Schwarzschild radius, $R_s = 2GM/c^2$, orbital radius R, and the distance r between the observer and the binary system. We are restoring the factors of c for comparison with experiments. In terms of the relevant parameters, the frequency of the orbit and of the gravitational waves produced is:

$$f = \frac{\Omega}{2\pi} \approx \frac{c\sqrt{R_s}}{10R^{3/2}}$$

From the metric perturbation, we find the approximate amplitude:

$$h \approx \frac{R_s}{rR}$$

To see what this implies, consider a black hole merger. We take both black holes to be 10 solar masses and the system is at $r \approx 100$ Mpc. R is ten times their Schwarzschild radii:

$$R_s \sim 10^6\text{cm}$$
$$R \sim 10^7\text{cm}$$
$$r \sim 10^{26}\text{cm}.$$

Frequenct and amplitude for such a source is:

$$f \sim 10^2\text{s}^{-1}, \quad h \sim 10^{-21}$$

So, to detect these we need instruments sensitive to the frequency of 100Hz and strains of order 10^{-21} or less.
One technique for detecting gravitational waves is interferometry. A passing gravitational wave slightly perturbs the motions of freely falling masses. If we have two test masses separated by a distance L, the change in their distance will be approximately:

$$\frac{\delta L}{L} \sim h$$

If these test bodies are separated by order of kilometers, we would need sensitivity to changes of the order:

$$\delta L \sim 10^{-16} \left(\frac{h}{10^{-21}}\right) \left(\frac{L}{km}\right) cm$$

For comparison, size of a typical Fermi nucleus is 10^{-13} cm. So, we need to detect changes in distances much smaller than what the test particles would be made of.

One possible way to measure such small perturbations is using Laser interferometers. Consider the setup in FIGURE. So, a laser is directed at a beam splitter, which sends the beam to two tubes of length L. At the end of the tubes, there are test masses represented by mirrors suspended from pendulums. These are partially reflective, so a typical photon is reflected around a 100 times before getting back to the beam splitter, which is then directed to a photodiode. The system is set up in a way that, if the test masses are perfectly stationary, the returning beams destructively interfere. No signal is sent to photodiode. A passing gravitational wave will perturb the length leading to a phase shift and the waves will no longer destructively interfere. During 100 round trips, the accumulated phases shift will be:

$$\delta\phi \sim 200 \left(\frac{2\pi}{\lambda}\right) \delta L \sim 10^{-9}$$

The factor of 200 respresents that the phase shift from two arms add up. This is a very small phase shift and can be measured if number of photon N is sufficiently large, in particular if $\sqrt{N} > \delta\phi$.

Terrestrial observatories are limited due to the fundamental noise sources. Space based observatories such as LISA are more sensitive to the frequencies in the range of 10^{-2} Hz because their implementation will be dramatically different.

Potential terrestrial noise source is seismic noise which has the dominant effect at low frequencies. At high frequencies, the dominant noise source is photon shot noise. At intermediate frequencies, thermal noise dominates. Noise from gravitation gradients due to atmospheric pressure is irreducible, too. Satellite observatories are free from such limitations. So, the fundamental limitation is measuring changes in distances between spacecrafts and from spacecraft's accelerations.

A brief overview of possible sources for gravitational waves. Compact binaries are a source of gravitational waves and they can be detected by ground bases observatories when they are close to coalesence. Another source is non spherically symmetric collapse of massive stars that gives rise to supernovae. Their detection can be coordinates with the observation of supernovae by radio telescopes. Rotating neutron stars, although

produce very small amplitude waves, but are one source that can be detected by advanced detectors. Evolution of gravitational wave signal from a solar-mass blackhole orbiting another such black hole can be tracked. Such information will allow for mapping of spacetime metric precisely.
Other than these localized sources, there is a possibility of gravitational wave backgrounds as well. These waves would have generated in the early universe, with a smoothly varying power spectrum as a function of frequency. Such waves are currently impossible to detect even by possible space based observatories. Primordial waves generated by a violent phase transition lie in a band potentially observable by space based observatories.

3 Gravitational Waves from a Compact Binary

We have already seen what gravitational waves from a compact binary with two compact objects of an equal mass look like. Here, we work with a compact binary of unequal size and derive, the gravitational wave strain, binary parameters, and energy lost by the binary in the Newtonian approximation. Later, we will extend those results using the Post Newtonian Approximation.

3.1 Binary Parameters in the Newtonian Limit

Consider a binary system in circular orbits in an x-y plane. As we know gravitational waves carry energy, so the binary loses energy and the orbit cannot remain circular but we assume that they remain quasicircular. The rest masses of the objects, m_1 and m_2 in the binary are such that $m_1 \geq m_2$. If we consider very large orbit, then we can model the individual binary components as point masses. For a binary with separation r and orbital frequency ω, in the center of mass system location of the objects is:

$$\mathbf{x}_1 = \frac{m_2 r}{M}(\cos wt, \sin wt, 0), \tag{3.1}$$

$$\mathbf{x}_2 = -\frac{m_1}{m2}\mathbf{x}_1, \tag{3.2}$$

where $M = m_1 + m_2$. Assuming point masses, their mass density is:

$$\rho(t, \mathbf{r}) = m_1 \delta(\mathbf{x} - \mathbf{x_1}(t)) + m_2 \delta(\mathbf{x} - \mathbf{x_2}(t)) = T^{00}$$

Mass quadrupole moment is:

$$I^{ij} = m_1 x_1^i(t) x_1^j(t) + m_2 x_2^i(t) x_2^j(t)$$

$$\begin{aligned}
I^{11} &= m_1 x_1^1(t) x_1^1(t) + m_2 x_2^1(t) x_2^1(t) \\
&= \frac{r^2 m_1 m_2^2}{M^2}\cos^2 \omega t + \frac{r^2 m_2 m_1^2}{M^2}\cos^2 \omega t \\
\frac{d^2 I^{11}}{dt^2} &= \frac{d^2 \cos^2 \omega t}{dt^2}\left(\frac{r^2 m_1 m_2^2}{M^2} + \frac{r^2 m_2 m_1^2}{M^2}\right) \\
&= -2\omega^2 \cos 2\omega t\left(\frac{m_1 m_2^2}{M^2} + \frac{r^2 m_2 m_1^2}{M^2}\right) \\
&= -2r^2\omega^2 \frac{m_1 m_2}{M}\cos 2\omega t
\end{aligned}$$

Its second derivative:

$$\frac{d^2 I^{ij}}{dt^2} = m_1(\ddot{x}_1^i(t) x_1^j(t) + x_1^i(t)\ddot{x}_1^j(t) + 2\dot{x}_1^i(t)\dot{x}_1^j) + m_2(\ddot{x}_2^i(t) x_2^j(t) + x_2^i(t)\ddot{x}_2^j(t) + 2\dot{x}_2^i(t)\dot{x}_2^j(t))$$

Using the coordinates, we find:

$$\begin{aligned}
\frac{d^2 I^{11}}{dt^2} &= m_1(\ddot{x}_1^1(t)x_1^1(t) + x_1^i(t)\ddot{x}_1^1(t) + 2\dot{x}_1^1(t)\dot{x}_1^1) + m_2(\ddot{x}_2^1(t)x_2^1(t) + x_2^1(t)\ddot{x}_2^1(t) + 2\dot{x}_2^1(t)\dot{x}_2^1(t)) \\
&= r^2 m_1(-2\frac{m_2^2}{M^2}\omega^2\cos^2\omega t - 2\frac{m_2^2}{M^2}\omega^2\sin^2\omega t) \\
&- r^2\frac{m_2 m_1^2}{M^2}(-2\omega^2\sin^2 wt - 2\omega^2\cos^2\omega t) \\
&= -2\frac{r^2 m_1 m_2^2}{M^2}\omega^2 - 2\frac{r^2 m_2 m_1^2}{M^2}\omega^2 \\
&= -2r^2\omega^2\frac{m_1 m_2}{M}
\end{aligned}$$

So, in the end we get:

$$h_{ij} = -4\frac{M\omega^2\eta r^2}{d}\begin{pmatrix} \cos 2\omega t_r & \sin 2\omega t_r & 0 \\ \sin 2\omega t_r & -\cos 2\omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $t_r = t - d$ and the symmetric mass ratio η is:

$$\eta = \frac{m_1 m_2}{M^2}$$

Again, this tensor is traceless and for waves travelling in z-direction is transverse as well. An observer (detector) will not necessarily be located along z axis. So, for more general scenarios h_{ij} has to be projected to a two dimensional subspace perpendicular to the direction of propagation to remain in a transverse traceless gauge. So, we can project it using 2.37 to a traceless transverse gauge.

Recall that the two independent components of the metric represent the two polarizations. Depending on the orientation of an L shaped inteferometer, it will be sensitive to a linear combination of the two polarizations. Denoting this observable signal by h, for the case of the compact binary signal, it can be written as:

$$h(t,\theta) = \mathcal{A}\frac{M\nu^2\eta}{d}(\cos^2 2\omega t + \phi_0),$$

where $v = \omega r$ is the relative velocity, ϕ_0 is a constant phase and $\mathcal{A}$ is the amplitude factor, the latter two depend on the geometry of the source and the detector. $\theta = M, \nu, \eta, d, \omega, \phi_0, \mathcal{A}$ is a vector of paramters that describe the orbital motion and the orientation of the source. All these parameters are not independent; the velocity is related to mass and orbital radius using Kepler's third law as:

$$v^2 = \frac{M}{r} \implies v^3 = M\omega$$

Power given by the EXPRESSION, and the Luminosity, the flux averaged over an orbit and integrated over a sphere of radius is:

$$\mathcal{L} \propto |d\dot{h}|^2 \propto M^2\nu^4\eta^2\omega^2 = \eta^2\nu^{10}$$

where we used the Kepler's law stated above. The energy carried by the gravitational law can be found using the expression for power in the previous section. Caclulation for a binary of unequal masses follows the same steps, we get:

$$P = \frac{M^2 r^4\eta^2\omega^6 4^2.2}{5} = \frac{32}{5}\eta^2\nu^{10}$$

Using Newton's law, total energy of the system is given by:

$$E_N = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2\right) - \frac{m_1 m_2}{r}$$

In the center-of-mass frame, the velocities of the two masses can be written in terms of the total mass $M = m_1 + m_2$ and the relative velocity v of the binary system:

$$
\begin{aligned}
v_1 &= \frac{m_2}{M} v, \\
v_2 &= \frac{m_1}{M} v.
\end{aligned}
$$

Now, substituting these velocities into the kinetic energy expression:

$$
\begin{aligned}
E_{\text{kin}} &= \frac{1}{2} \left(m_1 v_1^2 + m_2 v_2^2 \right) \\
&= \frac{1}{2} \left(m_1 \left(\frac{m_2}{M} v \right)^2 + m_2 \left(\frac{m_1}{M} v \right)^2 \right) \\
&= \frac{1}{2} v^2 \left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2} \right) \\
&= \frac{1}{2} v^2 \left(\frac{m_1 m_2 (m_1 + m_2)}{M^2} \right).
\end{aligned}
$$

Since $M = m_1 + m_2$, this simplifies to:

$$
E_{\text{kin}} = \frac{1}{2} \frac{m_1 m_2}{M} v^2.
$$

Substituting symmetric mass ratio into the expression:

$$
E_{\text{kin}} = \frac{1}{2} M \eta v^2.
$$

Subsituting this in the expression for total energy of the binary:

$$
\begin{aligned}
E &= \frac{M\eta}{2} v^2 - \frac{m_1 m_2}{r} \\
&= \frac{M\eta}{2} v^2 - \frac{M^2 \eta}{r} \\
&= \frac{M\eta}{2} v^2 - M\eta v^2 \\
&= -\frac{M\eta}{2} v^2
\end{aligned}
$$

Using the expression for the power radiated by this binary system and the energy of the binary, we can use the energy balance to find out how velocity changes with time:

$$\begin{aligned}
\frac{dE(t)}{dt} &= P(t) \\
\frac{dE(v)}{dv}\frac{dv}{dt} &= P \\
\frac{dv}{dt} &= \frac{P}{dE/dv} = \frac{32}{5M}\eta v^9(t) \\
\int_v^0 \frac{dv}{v^9} &= \frac{32}{5M}\eta \int_t^{t_c} dt \\
\frac{1}{8}v^{-8} &= \frac{32}{5M}\eta(t_c - t) \\
v &= \sqrt[8]{\frac{5M}{256\eta(t_c - t)}} \\
v &= \frac{1}{2}\sqrt[8]{\frac{5M}{\eta(t_c - t)}}
\end{aligned}$$

where t_c is the coalescence time, at which the velocity diverges, although the assumptions we made are not valid all the way up to t_c. To find how the separation changes with time, we relate r with v as:

$$r = \frac{M}{v^2} = 4\sqrt[4]{\frac{\eta(t_c - t)}{5M}}$$

At $t = 0$, the binary's separation is:

$$r_0 = \frac{M}{v^2} = 4M\sqrt[4]{\frac{\eta(t_c)}{5M}}$$

From this expression we can find t_c:

$$\begin{aligned}
r_0^4 &= 4^4 M^4 \frac{\eta t_c}{5M} \\
t_c &= \frac{5r_0^4}{256\eta M^3}
\end{aligned}$$

The orbital frequency:

$$\omega(t) = \frac{v^3}{M} = \frac{1}{8}\left(\frac{5}{M^{3/8}\eta(t_c - t)}\right)^{3/8} = \frac{1}{8\mathcal{M}^{5/8}}\left(\frac{5}{(t_c - t)}\right)^{3/8}$$

where

$$\mathcal{M} = M\eta^{3/5}$$

is the chirp mass. Using the expression for orbital frequency, we can find the orbital phase:

$$\begin{aligned}
\phi(t) &= \int \omega(t)dt \\
&= \frac{1}{8\mathcal{M}^{5/8}}\int\left(\frac{5}{(t_c - t)}\right)^{3/8} dt \\
&= -\frac{1}{8\mathcal{M}^{5/8}}\left(5^{3/8}\right)\frac{8}{5}(t_c - t)^{5/8} + \phi_c \\
&= \phi_c - \left(\frac{t_c - t}{5\mathcal{M}}\right)^{5/8}
\end{aligned}$$

The integration constant ϕ_c is the phase at coalescence time t_c. This result shows that the inspiral of the binary is predominantly governed by the chirp mass.

3.2 A Review of PN Formalism

- in radiation effects misner expands the newtonian potential in powers of r
- MTW discusess PPN formalism more which is more general and contains a set of PPN Parameters that can be specified arbitrarilly. It can be applied to various theories of gravity (what are major theories??). for example, one set of parameters gives PN limit of general relativity while another one describes Dicke Brans Jordan theory. It is used to test alternative theories of gravity by introducing parameters that describe deviations from GR and can be constrained through experiments.

Maggiore: So far, we have been assuming that the background spacetime is flat and sources contribute negligibly to the curvature. Interesting astrophysical systems are held together by strong gravitational forces where the velocity of source cannot be separated from the space-time curvature. For a compact binary system with total mass m we have $(v/c)^2 = 2GM/c^2d = R_s/d$. R_s/d is the measure of the strength of gravitational field near a source, including v/c corrections mean that we must consider the deviation of background from the flat space-time. Keeping the background spacetime flat means that we are really working with a Newtonian system, and to consider effects of the gravitating system on the background curvature, when dealing with a relativistic system, we must describe it by a Post-Newtonian formalism. PN formalism is particualrly important in predicting accurate waveforms to extract the GW signal of an inspiraling binary from experimental data.

3.2.1 slowly moving weakly self gravitating sources

slowly moving weakly self gravitating sources have $(v/c)^2$ and R_s/d which is comparable and none of them is too close to 1. They must be described by a post Newtonian formalism. Slowly moving weakly self-gravitating sources have $(v/c)^2$ and R_s/d which is comparable and none of them is too close to 1. They must be described by a post-Newtonian formalism. Consider slowly moving and weakly self gravitating sources which means (v/c) (bulk as well as internal velocities of compact objects) and R_s/d are sufficiently small. However, during the last stages of coordination of binaries they can reach high velocities as high as 1/2 and are relativistic objects. So, we will need the result to a very high order in v/c. We use these as expansion parameters and that they are generally related by $v/c \approx (R_s/d)^{1/2}$. We also assume the energy momentum tensor $T^{\mu\nu}$ has a spatially compact support i.e it can be enclosed in a time like world tube $r \leq d$ and the matter distribution inside the source is smooth and infinitely differentiable.

We want to understand how to compute corrections to the linearized theory in powers of v/c. We now distinguish the near and far zonne. If ω_s is the frequency of motion inside the source and d is the source size, then the velocities inside the source are $v \approx \omega_s d$, and frequency of source will also be of $\omega \approx \omega_s$, and $\lambda = c/\omega \implies \lambda \approx \frac{c}{v}d$. For non-relativistic sources:

$$\lambda \gg d$$

The near zone is the region where $r \ll \lambda$. And the exterior near zone is the region:

$$d < r \ll \lambda$$

n the near zone retardation effects are limited and we have static potentials. We will see that it is correct to use PN formalism here. The far zone is the region $r \gg \lambda$.

We might think that the expansion has two aspects. First, we must determine the gr correction to equation to the desired order in v/c and the second is that we compute GWs emitted by these sources. However, things are much more complicated than that. These two aspects are intertwined. Emission of GWs costs energy which is lost by the sources which after a certain order will back react on the sources affecting their equations of motion. Moreover, due to the non linearity of general relativity the gravitational field is itself a source for GW generation and GWs computed to a particular order are sources for GW production at a higher order. This makes a full fledged formalism for computing the production of GWs in powers of v/c necessarily complicated.

3.2.2 PN Expansion of Einstein Equations

Lets begin by analyzing the lowest-order PN corrections, neglecting the back reaction (back reaction does not occur at the PN first order either, and we will see this). Assuming the source is non relativistic, introduce a small parameter:

$$\epsilon \approx \frac{R_s}{d}^{1/2} \approx \frac{v}{c}$$

We also demand that $|T^{ij}|/T^{00} = O(\epsilon^2)$ i.e the source be weakly stressed. Thus, at the leading order energy density is the main contributer to the gravitational effects. We expand the metric and the stress energy tensor in powers of ϵ. A classical system under conservative forces is invariant to time reversal (neglecting radiation emission). Under time reversal g_{00} and g_{ij} are even under time reversal. Velocity changes sign so g_{00} and g_{ij} can contain only even powers of v. However g_{0j} are odd so they contain odd powers of velocity. By inspection of Einstein's equations one finds that, to work consistently to a given order in ϵ, if we expand g_{00} up to order ϵ^n we must also expand g_{0i} up to order ϵ^{n-1} and g_{ij} up to ϵ^{n-2}. Furthermore, the expansion of g_{0i} starts from $O\left(\epsilon^3\right)$. Thus the metric is expanded as follows

$$\begin{aligned} g_{00} &= -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + {}^{(6)}g_{00} + \ldots \\ g_{0i} &= {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \ldots \\ g_{ij} &= \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \ldots \end{aligned} \tag{3.3}$$

where ${}^{(n)}g_{\mu\nu}$ denotes the terms of order ϵ^n in the expansion of $g_{\mu\nu}$. It reduces to the minkowski metric at the first order. Similarly, we expand the energy-momentum tensor of matter,

$$\begin{aligned} T^{00} &= {}^{(0)}T^{00} + {}^{(2)}T^{00} + \ldots \\ T^{0i} &= {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + \ldots \\ T^{ij} &= {}^{(2)}T^{ij} + {}^{(4)}T^{ij} + \ldots \end{aligned} \tag{3.4}$$

Stresses T^{ij} are a second order relativistic corrections in a non relativistic system and we also required that the stresses be small as compared to the energy density.

We now substitute these in Einstein's equation and equate the terms that are of same order in ϵ. To determine the order, we must also consider that for slow moving sources time derivatives of the metric generated by this source are smaller than the spatial derivative and are related as:

$$\frac{\partial}{\partial t} = O(v)\frac{\partial}{\partial x^i} \tag{3.5}$$

So, the d'Alembertian operator operator becomes:

$$\Box^2 = [(1 + O(\epsilon^2))]\nabla^2$$

Therefore the spatial derivative is the leading order term and the lowest order solution is in terms of instantaneous potentials and retardation effects are small corrections. In PN expansion, we are effectively trying to compute some quantity $F(t - r/c)$, such as given component of a metric which is a function of retarded time.

$$F\left(t - \frac{r}{c}\right) = F(t) - \frac{r}{c}\dot{F}(t) + \frac{r^2}{2c^2}\ddot{F}(t) + \dots \tag{3.6}$$

Each derivative of F carries a factor of ω, which is the frequency of the radiation emitted. And $\omega/c = 1/\lambda$, so the eq 3.6 is essentially an expansion in powers of r/λ, therefore, a PN expansion is only valid in the near zone $r \ll \lambda$ and breaks down in the far/radiation zone $r \gg \lambda$. I Will come back to the breakdown in far region later (a naive expansion will lead to divergences). Therefore, PN expansion is a tool that can be used to compute the gravitational field in the near region but it must be supplemented by a different treatment to find the fields in the radiation zone.

3.2.3 Newtonian Limit

In the newtonian limit we keep $g_{00} = 1 + {}^{(2)}g_{00}, g_{0i} = 0$ and $g_{ij} = \delta_{ij}$ eq. (5.3). In fact, the equation of motion of a test particle with velocity v, in a gravitational field, is obtained from the geodesic equation

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3.7}$$

In a weak gravitational field we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and, in the limit of low velocities, the proper time τ is the same, to lowest order, as the coordinate time t. Furthermore, $dx^0/dt = c$ while $dx^i/dt = O(v)$. Then, the leading term in v/c is obtained setting $\mu = \nu = 0$ in eq. 3.7,

$$\begin{aligned}\frac{d^2x^i}{dt^2} &\simeq -c^2\Gamma^i_{00} \\ &= c^2\left(\frac{1}{2}\partial^i h_{00} - \partial_0 h^i_0\right)\end{aligned}$$

Recall that the time derivative of the metric is of higher order, so we have:

$$\begin{aligned}\frac{d^2x^i}{dt^2} &\simeq -c^2\Gamma^i_{00} \\ &= c^2\frac{1}{2}\partial^i h_{00}\end{aligned}$$

writing $h_{00} = -2\phi$, and defining $U = -c^2\phi$ we recover the newtonian eq of motion. U is the sign reversed gravitational potential. the equation of motion corresponds to the Newtonian potential U, with $v^2/c^2 \sim O(U)$. Comparing this with eq 3.7 we see that the leading-order term of the metric is given by $g_{00} = -1 + 2U/c^2$, while corrections to g_{ij} and g_{0i} are of order $O(v^2/c^2)$. For a photon, both g_{00} and g_{ij} contribute to the deviation from flat spacetime. The gravitational potential U in the de Donder gauge is expressed as:

$$U(t, \mathbf{x}) = \frac{G}{c^2}\int d^3x' \frac{T^{00}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

This approximation results in first post-Newtonian (1PN) corrections for both massive particles and photons.

3.2.4 The 1PN Order

First we choose a gauge condition because that will simplify the equations. We choose harmonic gauge:

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$$

In principle, it is now easy to expand Einstein equations by inserting eq. 3.3 and eq. 3.4 and use the gauge condition to simplify the equations.

At the Newtonian order, we have

$$\nabla^2[{}^{(2)}g_{00}] = -\frac{8\pi G}{c^4}\,{}^{(0)}T^{00} \tag{3.8}$$

Similarly expanding to 1PN order, we get:

$$\nabla^2\left[{}^{(2)}g_{ij}\right] = -\frac{8\pi G}{c^4}\delta_{ij}\,{}^{(0)}T^{00}, \tag{3.9}$$

$$\nabla^2\left[{}^{(3)}g_{0i}\right] = \frac{16\pi G}{c^4}\,{}^{(1)}T^{0i}, \tag{3.10}$$

$$\begin{aligned}\nabla^2\left[{}^{(4)}g_{00}\right] &= \partial_0^2\left[{}^{(2)}g_{00}\right] + g_{ij}\partial_i\partial_j\left[{}^{(2)}g_{00}\right] - \partial_i\left[{}^{(2)}g_{00}\right]\partial_i\left[{}^{(2)}g_{00}\right] \\ &\quad - \frac{8\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii} - 2\,{}^{(2)}g_{00}\,{}^{(0)}T^{00}\right\},\end{aligned} \tag{3.11}$$

The solution to 3.8 is with the boundary condition that it vanishes at infinity is:

$$g_{00} = -2\phi \tag{3.12}$$

$$\phi(t,\mathbf{x}) = -\frac{G}{c^4}\int \frac{T^{00}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{3.13}$$

Similarly, the 1PN order equations 3.10 and eq. 3.9 are solved:

$${}^{(2)}g_{ij} = -2\phi\delta_{ij}, \tag{3.14}$$

$${}^{(2)}g_{0i} = \varsigma_i \tag{3.15}$$

and

$$\varsigma(t,\mathbf{x}) = -\frac{4G}{c^4}\int \frac{T^{0i}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{3.16}$$

Now we solve 3.11,

$$\begin{aligned}\nabla^2\left[{}^{(4)}g_{00}\right] &= \partial_0^2(-2\phi) + 4\phi\delta_{ij}\partial_i\partial_j(\phi) - \partial_i(-2\phi)\partial_i(-2\phi) \\ &\quad - \frac{8\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii} + 4\phi\,{}^{(0)}T^{00}\right\},\end{aligned}$$

Using the vector identity:

$$\partial^i\phi\partial_i\phi = \frac{1}{2}\nabla^2(\phi^2) - \phi\nabla^2\phi$$

we get:

$$\begin{aligned}\nabla^2\left[{}^{(4)}g_{00}\right] &= -2\partial_0^2(\phi) - 4\left(\frac{1}{2}\nabla^2(\phi^2) - \phi\nabla^2\phi\right) + 4\phi\nabla^2\phi \\ &\quad - \frac{8\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii} + 4\phi\,{}^{(0)}T^{00}\right\}, \\ &= -4\left(\frac{1}{2}\nabla^2(\phi^2) - \phi\nabla^2\phi\right) + 4(\phi)\nabla^2(\phi) - 2\partial_0^2(\phi) - 8\phi\nabla^2(\phi) - \frac{8\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii}\right\} \\ &= -2\nabla^2(\phi^2) - 2\partial_0^2(\phi) - \frac{8\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii}\right\} \\ &= -2\nabla^2(\phi^2 + \psi)\end{aligned}$$

where,

$$\nabla^2\psi = \partial_0^2(\phi) + \frac{4\pi G}{c^4}\left\{ {}^{(2)}T^{00} + {}^{(2)}T^{ii} \right\} \tag{3.17}$$

which has the solution:

$$\psi = \int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left\{ \frac{1}{4\pi}\partial_0^2(\phi) + \frac{G}{c^4}\left[{}^{(2)}T^{00} + {}^{(2)}T^{ii} \right] \right\}$$

phi and ς^i are not independent as the gauge conditions constrain them. Moreover, their explicit expressions also show that they are related through conservation of the stress energy tensor.

ϕ, ψ and ε are instantaneous potentials as their value depends on stress energy tensor at the same time (retardation effects are small correctons of $0(\epsilon^2)$). We can re express the solutions in terms of retarded potentials to understand them better and use them to compute higher order potentials.

Expanding g_{00} to 1PN order we have:

$$\begin{aligned} g_{00} &= -1 - 2\phi - 2(\phi^2 + \psi) + O(\epsilon^6) \\ &= -1 - 2(\phi+\psi) - 2\phi^2 + O(\epsilon^6) \end{aligned}$$

Replacing ϕ^2 with $(\phi+\psi)^2$ because ψ is already of higher order and the additional terms will be beyond 1PN order. Introducing:

$$V = -c^2(\phi+\psi), \tag{3.18}$$

with these the solution for g_{00} to 1PN order:

$$\begin{aligned} g_{00} &= -1 + \frac{2V}{c^2} - \frac{2V^4}{c^4} + O\left(\frac{1}{c^6}\right) \\ &= -e^{-\frac{2V}{c^2}} + O\left(\frac{1}{c^6}\right) \end{aligned}$$

The potential satisfies the eq:

$$\nabla^2\phi = \frac{4\pi G}{c^4}\,{}^{(0)}T^{00} \tag{3.19}$$

while ψ is given by eq: 3.17, combining the two:

$$\nabla^2(\phi+\psi) = \partial_0^2(\phi) + \frac{4\pi G}{c^4}\left\{ {}^{(0)}T^{00} + {}^{(2)}T^{00} + {}^{(2)}T^{ii} \right\} \tag{3.20}$$

Again writing $\partial_0^2\phi = \partial_0^2(\phi+\psi)$, re arranging the above equation we get:

$$\Box V = -\frac{4\pi G}{c^4}\left[{}^{(0)}T^{00} + {}^{(2)}T^{00} + {}^{(2)}T^{ii} \right] \qquad = \frac{4\pi G}{c^4}\left[T^{00} + T^{ii}\right]$$

Defining active gravitational mass density:

$$\sigma = \frac{1}{c^2}[T^{00} + T^i_i]$$

with this, 1PN equation for g_{00} becomes:

$$\Box V = -4\pi G\sigma \tag{3.21}$$

Now V can be written as the retarded integral:

$$V(t,\mathbf{x}) = -\frac{G}{c^4}\int d^3x' \frac{\sigma(t - |\mathbf{x}-\mathbf{x}'|/c, x')}{|\mathbf{x}-\mathbf{x}'|} \tag{3.22}$$

The retarded potential can be expanded in terms of instantaneous potentials by expanding $\sigma(t - |\mathbf{x} - \mathbf{x}'|/c, x')$ for small retardation effects.

$$\sigma(t - |\mathbf{x} - \mathbf{x}'|/c, x') = \sigma(t, x') - \frac{\mathbf{x} - \mathbf{x}'}{c}\partial_t \sigma + \dots$$

Similarly, for g_{ij} and g_{0j} we can write those in terms of V. Replacing $varepsilon^i$ with V^i (retardation effects are already of higher order) in eq. 3.15 using the active mass current density,

$$\sigma_i \equiv \frac{1}{c}T^{0i}$$

Expressing in terms of σ_i, 1PN solution to the g_{0j} equation is:

$$V_i(t, \mathbf{x}) = -\frac{G}{c^4}\int d^3x' \frac{\sigma_i(t - |\mathbf{x} - \mathbf{x}'|/c, x')}{|\mathbf{x} - \mathbf{x}'|} \tag{3.23}$$

and in g_{ij} we can replace $-c\phi^2$ with V which is of higher order.

Summarizing the 1PN solution in terms of V and V_i:

$$g_{00} = -1 + \frac{2V}{c^2} - \frac{2V^4}{c^4} + O\left(\frac{1}{c^6}\right)$$
$$g_{0i} = -\frac{4}{c^3}V_i + O\left(\frac{1}{c^5}\right)$$
$$g_{0j} = \delta_{ij}\left(1 + \frac{2}{c^2}V\right) + O\left(\frac{1}{c^4}\right)$$

To 1PN order, energy momentum enters only in two combinations σ and σ_i.

At large distance r, from the source we can expand the potentials V and V_i using:

$$\frac{1}{\mathbf{x} - \mathbf{x}'} = \frac{1}{r} + \frac{\mathbf{x}.\mathbf{x}'}{r^3} + \dots \tag{3.24}$$

3.2.5 Difficulties of the PN expansion

We are trying to iteratively solve equations that have the form:

$$\Box h = S_{\mu\nu}(h),$$

where $S_{\mu\nu}$ is the source term that depends on the energy momentum tensor and on $h_{\mu\nu}$. We could do an expansion of the form:

$$h_{\mu\nu} = {}^{(0)}h_{\mu\nu} + {}^{(1)}h_{\mu\nu} + {}^{(2)}h_{\mu\nu} + \dots$$

At zeroth order, we simply set ${}^{(0)}h_{\mu\nu} = 0$, and then solve the equation of the form:

$$\nabla^2[{}^{(1)}h_{\mu\nu}] = (\text{matter sources})$$

This equation is then integrated by using Green's function. Then, at the next iteration we have an equation of the form:

$$\nabla^2[{}^{(2)}h_{\mu\nu}] = (\text{matter sources}) + (\text{terms that depend on } {}^{(1)}h_{\mu\nu}), \tag{3.25}$$

which again we will solve using Green's function and the Poisson integral. Beyond 1PN order, the resulting Poisson integrals are *necessarily divergent.*. Even if the source is compact, the second term in equation eq. 3.25 extends over all space, *raising an issue of convergence at infinity.* Moreover, when we expand the potentials to a higher order l the factors $(\mathbf{x}.\mathbf{x}')^l$, that come from expansion of $\frac{1}{\mathbf{x}-\mathbf{x}'}$ diverge at large x'.

The correct non divergent solution to the Poisson equation is not necessarily given by the Poisson integral (correct sol later). Another problem is that the expansion of the retarded potential diverges for small retardation. Our solutions are of the form:

$$h_{\mu\nu} = \frac{1}{r} F_{\mu\nu}(t - r/c)$$

For $r/c << 1$, the expansion:

$$\frac{1}{r} F_{\mu\nu}(t - r/c) = \frac{1}{r} F_{\mu\nu}(t) - \frac{1}{c} \dot{F}_{\mu\nu}(t) - \frac{r}{2c^2} \ddot{F}_{\mu\nu}(t) + \dots$$

blows up as $r \to \infty$. So, we cannot use PN expansion at large distances from the source.

So, we use the PN expansion only in the near zone and use a different formalism in the regions far from sources. This method is called matched asymptotic expansion.

3.2.6 The Relaxed Einstein Equations

First we recast Einstein equations in a form that will be convenient. We define a field $h^{\alpha\beta}$ as:

$$\mathsf{h}^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta} \tag{3.26}$$

This is an exact definition where we have given up the assumption of keeping $h^{\alpha\beta}$ small. In the limit that $\mathsf{h}_{\alpha\beta}$ is small, $-g = 1 + h$,

$$\begin{aligned} -\mathsf{h}^{\alpha\beta} &\approx \eta^{\alpha\beta} - \sqrt{1+h}(\eta^{\alpha\beta} - h^{\alpha\beta}) \\ &= h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h \end{aligned}$$

So, it reduces to the trace reversed perturbation, but with an overall negative sign. Harmonic gauge condition becomes:

$$\partial_\beta \mathsf{h}^{\alpha\beta} = 0?$$

In the harmonic gauge, Einstein equations take the form:

$$\Box \mathsf{h}^{\alpha\beta} = \frac{16\pi G}{c^4} \tau^{\alpha\beta} \tag{3.27}$$

where $\tau^{\alpha\beta}$ is defined as:

$$\tau^{\alpha\beta} \equiv (-g) T^{\alpha\beta} + \frac{c^4}{16\pi G} \Lambda^{\alpha\beta},$$

$T^{\alpha\beta}$ is the matter energy-momentum tensor. The tensor $\Lambda^{\alpha\beta}$ does not depend on the matter variables and is defined by

$$\Lambda^{\alpha\beta} = \frac{16\pi G}{c^4}(-g) t^{\alpha\beta}_{LL} + \left(\partial_\nu \mathrm{h}^{\alpha\mu} \partial_\mu \mathrm{h}^{\beta\nu} - \mathrm{h}^{\mu\nu} \partial_\mu \partial_\nu \mathrm{h}^{\alpha\beta}\right), \tag{3.28}$$

where $t^{\alpha\beta}_{LL}$ is the Landau-Lifshitz energy-momentum pseudotensor,

$$\begin{aligned} \frac{16\pi G}{c^4}(-g) t^{\alpha\beta}_{LL} =& g_{\lambda\mu} g^{\nu\rho} \partial_\nu \mathrm{h}^{\alpha\lambda} \partial_\rho \mathrm{h}^{\beta\mu} + \frac{1}{2} g_{\lambda\mu} g^{\alpha\beta} \partial_\rho \mathrm{h}^{\lambda\nu} \partial_\nu \mathrm{h}^{\rho\mu} \\ & - g_{\mu\nu} \left(g^{\lambda\alpha} \partial_\rho \mathrm{h}^{\beta\nu} + g^{\lambda\beta} \partial_\rho \mathrm{h}^{\alpha\nu}\right) \partial_\lambda h^{\rho\mu} \\ & + \frac{1}{8} \left(2 g^{\alpha\lambda} g^{\beta\mu} - g^{\alpha\beta} g^{\lambda\mu}\right) \left(2 g_{\nu\rho} g_{\sigma\tau} - g_{\rho\sigma} g_{\nu\tau}\right) \partial_\lambda \mathrm{h}^{\nu\tau} \partial_\mu \mathrm{h}^{\rho\sigma}. \end{aligned}$$

Since $t_{LL}^{\alpha\beta}$ depends on the metric $g_{\mu\nu}$, it is a highly non-linear function of $\mathrm{h}_{\mu\nu}$. Using the De Donder gauge condition, we see that the last term in eq. 3.28

$$\partial_\nu \mathrm{h}^{\alpha\mu}\partial_\mu \mathrm{h}^{\beta\nu} - \mathrm{h}^{\mu\nu}\partial_\mu\partial_\nu \mathrm{h}^{\alpha\beta} = \partial_\mu\partial_\nu \left(\mathrm{h}^{\alpha\mu}\mathrm{h}^{\beta\nu} - \mathrm{h}^{\mu\nu}\mathrm{h}^{\alpha\beta}\right).$$

Defining:

$$\chi^{\alpha\beta\mu\nu} = \frac{c^4}{16\pi G}\left(\mathrm{h}^{\alpha\mu}\mathrm{h}^{\beta\nu} - \mathrm{h}^{\mu\nu}\mathrm{h}^{\alpha\beta}\right).$$

Thus, we can also rewrite eq. 3.27 as

$$\Box h^{\alpha\beta} = +\frac{16\pi G}{c^4}\left[(-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta}\right) + \partial_\mu\partial_\nu\chi^{\alpha\beta\mu\nu}\right],$$

Since Einstein equations impose the covariant conservation of energy, eq. 3.27 along with the harmonic gauge condition is completely equivalent to the Einstein field equations. The gauge condition implies:

$$\partial_\beta \tau^{\alpha\beta} = 0.$$

Eq. 3.27 alone does not constrain the dynamics of matter variables. An arbitrary time-dependent $T^{\alpha\beta}$ would satisfy the eq. 3.27. This is why the ten components of eq. 3.27 are called relaxed Einstein equations, as the requirement that matter variables follow equations of motion has been relaxed. Eq. 3.27 has the solution of the form:

$$\mathsf{h}^{\alpha\beta} = -\frac{4G}{c^4}\int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\tau^{\alpha\beta}(t - |\mathbf{x}-\mathbf{x}'|/c, \mathbf{x}')$$

Then, we can impose the gauge condition on this solution. Comparing this solution with the one in linearized theory eq. **??**, here $\tau^{\alpha\beta}$ is itself a function of h and its derivatives. So, the above equation is just an *integro-differential* eq, which is practically unsolvable. Therefore, we have to employ approximation methods. In the near region, we can use PN expansion and retardation effects are small, while in the far region, we will have gravitational wavesand retardation effects will be significant. The source term $\tau^{\alpha\beta}$ extends over all space-time. Naively trying to solve or expand this integral, results in divergences.

3.2.7 The Blanchet Damour Approach

For a self gravitating slowly moving source, two length scales are important, the size of the object or orbital radius of the binary, d and the length $\mathcal{R}$ boundary of the near zone. Near zone extends to $\mathcal{R} >> d$. PN approximation breaks down in the far zone $r >> \mathcal{R}$. Outside the the source, $r > d$, energy momentum tensor is zero, and the contribution to $\tau^{\alpha\beta}$ comes from the field itself. If the field inside the source is weak then at $r = d$, the spacetime wont differ much from flat spacetime and away from the source, it will start approaching Minkowski spacetime as r increases. Therefore, in the region $d < r < \infty$, we can use Post Minkowskian approximation. PN expansion is valid in the region $0 < r < \mathcal{R}$, so the two expansions overlap in the region $d < r < \mathcal{R}$. In the Blanchet Damour approach, we use PN expansion in the near region, post Minkowskian expansion outside the source and match these two up in the overlapping region.

3.2.8 Post Minkowskian Expansion

outside the source, we solve vacuum Einstein equations. If we are considering a weak source (will extend this to a strong source laterr), in the first approximation metric is the Minkowski metric $\eta^{\alpha\beta}$. At the distance r, we give expansions in terms of R_s/r where $R_s = 2Gm/c^2$, where m is the mass of system. R_s is proportional to G so we can also expans the metric in powers of G as:

$$\sqrt{-g}g^{\alpha\beta} = \eta^{\alpha\beta} + G\mathsf{h}_1^{\alpha\beta} + G^2\mathsf{h}_2^{\alpha\beta}$$

so, we have

$$\mathsf{h}^{\alpha\beta} = \sum_{n=1}^{\infty} G^n h_n^{\alpha\beta}$$

Plugging it in relaxed Einstein equations 3.27, and setting $T^{\alpha\beta} = 0$,

$$\Box \mathsf{h}^{\alpha\beta} = \Lambda^{\alpha\beta}[\mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3 ... \mathsf{h}_{n-1}] \tag{3.29}$$

We expand tensor $\Lambda^{\alpha\beta}$ in powers of $\mathsf{h}^{\alpha\beta}$ and can write

$$\Lambda^{\alpha\beta} = N^{\alpha\beta}[h, h] + M^{\alpha\beta}[h, h, h] + L^{\alpha\beta}[h, h, h, h] + O(h^5)$$

where the coefficients can be by expanding $\Lambda^{\alpha\beta}$ in powers of $\mathsf{h}^{\alpha\beta}$. Now we solve relaxed Einstein's equations order by order. Since $\Lambda^{\alpha\beta}$, is quadratic in $\mathsf{h}^{\alpha\beta}$, at first order we have:

$$\Box \mathsf{h}_1^{\alpha\beta} = 0 \tag{3.30}$$

and at higher orders we get:

$$\Box h_2^{\alpha\beta} = N^{\alpha\beta}[h_1, h_1] \tag{3.31}$$

$$\Box h_3^{\alpha\beta} = M^{\alpha\beta}[h_1, h_1, h_1] + N^{\alpha\beta}[h_1, h_2] + N^{\alpha\beta}[h_2, h_1] \tag{3.32}$$

and so on with the gauge conditions:

$$\partial_\beta \mathsf{h}_n^{\alpha\beta} = 0$$

We first find general solutions to 3.30. The most general solution to 3.30, outside the source is written in terms of retarded multipolar waves:

$$\mathsf{h}_1^{\alpha\beta} = \sum_{l=0}^{\text{inf}} \partial_L \left[\frac{1}{r} K_L^{\alpha\beta}(t - r/c)\right]$$

Where K_L are a symmetric traceless tensors and the term is bracket satisfies the wave equation because it is a function of retarded time, $\Box K_L^{\alpha\beta} u/r = 0$, where $u = t - r/c$. This solution is only acceptable in the region $r > d$, as it becomes singular at $r = 0$. We have not yet used the harmonic gauge condition. So, we have ten independent components of the tensor and using the gauge condition the solution takes a *canonical* form parametrized by two moments $I_L(u)$ and $J_L(u)$:

$$h_1^{\alpha\beta} = k_1^{\alpha\beta} + \partial^\alpha \phi_1^\beta + \partial^\beta \phi_1^\alpha - \eta^{\alpha\beta} \partial_\mu \phi_1^\mu$$

$k_1^{\alpha\beta}$ depends on STF multipole moments $I_L(u)$ and $J_L(u)$, which are arbitrary functions of spacetime and satisfy the harmonic gauge condition. It is given by:

$$k_1^{00} = -\frac{4}{c^2} \sum_{\ell \geqslant 0} \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{1}{r} \mathrm{I}_L(u)\right)$$

$$k_1^{0i} = \frac{4}{c^3} \sum_{\ell \geqslant 1} \frac{(-)^\ell}{\ell!} \left\{\partial_{L-1} \left(\frac{1}{r} \mathrm{I}_{iL-1}^{(1)}(u)\right) + \frac{\ell}{\ell+1} \epsilon_{iab} \partial_{aL-1} \left(\frac{1}{r}\ \mathrm{J}_{bL-1}(u)\right)\right\}$$

$$k_1^{ij} = -\frac{4}{c^4} \sum_{\ell \geqslant 2} \frac{(-)^\ell}{\ell!} \left\{\partial_{L-2} \left(\frac{1}{r} \mathrm{I}_{ijL-2}^{(2)}(u)\right) + \frac{2\ell}{\ell+1} \partial_{aL-2} \left(\frac{1}{r} \epsilon_{ab(i} \mathrm{J}_{j)bL-2}^{(1)}(u)\right)\right\}$$

where $I^{(1)}$ denotes the first derivative of I wrt u. The STF multipole moments explicitly are $I^L(u) = I, I_i, I_i j, \dots$ and $j^L(u) = J, J_i, J_i j, \dots$ and are called mass type and current type. They are *arbitrary* except for conservation of monopole moments gives the mass of the source $I \equiv M$ constant, total linear momentum $P_i \equiv I_i^{(1)} = 0$ and total angular momentum J_i constant and these follow from the gauge condition. Center of mass of the system is I_i and we can set it to zero by choosing origin of our coordinates system to coincide with it. These terms include the source's contribution to the waves, at the linearized level. They are arbitrary and not yet specified in terms of the stress energy tensor. The vector ϕ_1^α gives some arbitrary linear gauge transformations and they are given in terms of moments W_L, X_L, Y_L, Z_L multipole moments. This vector is given by:

$$\begin{aligned}
\varphi^0_{(1)} &= \frac{4}{c^3} \sum_{\ell \geqslant 0} \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{1}{r}\, \mathrm{W}_L(u) \right) \\
\varphi^i_{(1)} &= -\frac{4}{c^4} \sum_{\ell \geqslant 0} \frac{(-)^\ell}{\ell!} \partial_{iL} \left(\frac{1}{r} \mathrm{X}_L(u) \right) \\
&\quad - \frac{4}{c^4} \sum_{\ell \geqslant 1} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-1} \left(\frac{1}{r} \mathrm{Y}_{iL-1}(u) \right) + \frac{\ell}{\ell+1} \epsilon_{iab} \partial_{aL-1} \left(\frac{1}{r} \mathrm{Z}_{bL-1}(u) \right) \right\}
\end{aligned}$$

(These moments W_L, X_L, Y_L, Z_L play a physical role at the non linear level, ignoring these right now later results in a metric depending only on I_L and J_L. The gauge vector is important because we want to construct a full linear solution to the Einstein equation, discarding it will give back the result we got in linear theory.

Iteration of the solution Multipolar post-Minkowskian expansion We have found the general solution of the first order equation, and we want to use it to find the solution to eq. 3.31 and then use these to determine the next order solution and so on. The general problem is, how to integrate the equation 3.29 when the source term Λ_n is found by the previous iterations. We cannot use Green's functions because that requires knowing Λ_n in all of space, while we are working only in the region outside the source. The multipole expansion is valid only for $d/r < 1$.

Blanchet and Damour found the appropriate function that solves the equation. First, we do not need the full expansion but we are interested in computing the PN expansion to a given order and only a few multipoles contribute there. So, we iterate a truncated multipole expansion of $\mathsf{h}_1^{\alpha\beta}$. The solutio needs to have the same structure as the source term, irregular at $r = 0$ and satisfies the equation at $r > d$. So, we use a trick where we first regularize the source term by multiplying it with a factor r^B, where B is a complex number. $\Lambda_n^{\alpha\beta}$ is expanded to a multipolar order ℓ_{max}. If we have the source terms that go like $1/r^k$, maximal order of divergence is $k_{\max}$, the real part of B is larger such that the source is regular at $r \to 0$, we can use the retarded integral operator:

$$I_n^{\alpha\beta}(B) \equiv \Box^{-1}(r^B \Lambda_n^{\alpha\beta}) \tag{3.33}$$

where $\Box^{-1}$ denotes the convolution with the green's function:

$$\Box^{-1f}(t, \mathbf{x}) \equiv -\frac{1}{4\pi} \int_{R^3} \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} f(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}'$$

where we have also imposed the condition that in some remote past $r \to \inf$, the field becomes stationary. This way we have imposed the no incoming radiation condition (more on it later). $I^{\alpha\beta}$ can be expanded when $B \to 0$ in the form of a Laurent series:

$$I_n^{\alpha\beta}(B) = \sum_{p=p_0}^{\infty} B^p l_{n,p}^{\alpha\beta},$$

where $p_0 \in Z$ and for $p_0 < 0$, we have poles. Applying $\Box$ to both sides and using eq. 3.33, we have:

$$r^B \Lambda_n^{\alpha\beta} = \sum_{p=p_0}^{\infty} B^p \Box l_{n,p}^{\alpha\beta}$$

Writing $r^B = e^{Blogr}$ and doing an expansion:

$$e^{Blogr}\Lambda_n^{\alpha\beta} = \sum_{p=p_0}^{\infty} B^p \Box l_{n,p}^{\alpha\beta}$$
$$\sum_{n=0}^{\infty} \frac{(Blogr)^n}{n!}\Lambda_n^{\alpha\beta} = \sum_{p=p_0}^{\infty} B^p \Box l_{n,p}^{\alpha\beta}$$

Equating it for same powers of B, we find that for $p_0 \leq p \leq -1$, $\Box l_{n,p}^{\alpha\beta} = 0$ and for $p \geq 0$,

$$\Box l_{n,p}^{\alpha\beta} = \frac{(logr)^p}{p!}\Lambda_n^{\alpha\beta}$$

for $p = 0$, let $u_n^{\alpha\beta} \equiv l_{n,p=0}^{\alpha\beta}$, we have $\Box u_n^{\alpha\beta} = \Lambda_n^{\alpha\beta}$. So, the solution is given by the coefficient of B^0 in the laurent expansion. This is called the finite part at B= 0 of the retarded integral and denoted as:

$$u_n^{\alpha\beta} = FP_{B=0}\Box^{-1}[r^B\Lambda_n^{\alpha\beta}] \tag{3.34}$$

This is one particular solution of the eq. 3.29. The solution in eq. 3.34 will not satisfy harmonic gauge condition, so we look for solution of the form:

$$\mathsf{h}_n^{\alpha\beta} = u_n^{\alpha\beta} + v_n^{\alpha\beta},$$

where $v_n^{\alpha\beta}$ is chosen such that $partial_\alpha v_n^{\alpha\beta} = -\partial_\alpha u_n^{\alpha\beta}$. Therefore, Mutipolar post Minkowskian expansion provides a well defined algorithm for computing the Minkowskian corrections to an arbitrary order. Now we return to finding the solution in the near region.

3.2.9 PN expansion in the near region

In the near region, we already found the solution at 1PN order in terms of $g^{\alpha\beta}$. We first express that in terms $\mathsf{h}^{\alpha\beta}$, using the relaxed Einstein equations. We find:

$$\mathsf{h}^{00} = -4V/c^2 + O(1/c^4)$$
$$\mathsf{h}^{0i} = O(1/c^3)$$
$$\mathsf{h}^{ij} = O(1/c^4)$$

Now, plugging this solution into eq. 3.27:

$$\Box h^{00} = \frac{16\pi G}{c^4}\left(1 + \frac{4V}{c^2}\right)T^{00} - \frac{14}{c^4}\partial_k V\partial_k V + O\left(\frac{1}{c^6}\right), \tag{3.35}$$
$$\Box h^{0i} = \frac{16\pi G}{c^4}T^{0i} + O\left(\frac{1}{c^5}\right), \tag{3.36}$$
$$\Box h^{ij} = \frac{16\pi G}{c^4}T^{ij} + \frac{4}{c^4}\left\{\partial_i V\partial_j V - \frac{1}{2}\delta_{ij}\partial_k V\partial_k V\right\} + O\left(\frac{1}{c^6}\right). \tag{3.37}$$

The solution to these equations:

$$h^{00} = -\frac{4}{c^2}V + \frac{4}{c^4}\left(W - 2V^2\right) + O\left(\frac{1}{c^6}\right), \tag{3.38}$$
$$h^{0i} = -\frac{4}{c^3}V_i + O\left(\frac{1}{c^5}\right), \tag{3.39}$$
$$h^{ij} = -\frac{4}{c^4}W_{ij} + O\left(\frac{1}{c^6}\right). \tag{3.40}$$

where W_{ij} is:

$$W_{ij}(t,\mathbf{x}) = G\int d^3x'\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_{ij} + \frac{1}{4\pi G}(\partial_i V\partial_j V - \frac{1}{2}\delta_{ij}\partial_k V\partial_k V)\right]\Big|_{\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c}.$$

Therefore by iterating over the solution, we have obtained equations at $O(1/c^6)$ similar to the equations that we had for $g_{\mu\nu}$ at 1PN order.

Multipolar PN Expansion

Multipolar PN expansion combines the PN expansion with multipole expansion. To 1PN order, we need the multipole expansion of the potentials V and V_i, which are written as:

$$V(t,\mathbf{x}) = G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{1}{r}F_L\left(t-\frac{r}{c}\right)\right]$$
$$V_i(t,\mathbf{x}) = G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{1}{r}G_{iL}\left(t-\frac{r}{c}\right)\right]$$

F_L and G_{iL} can be expressed in terms of the source terms.

Multipole Expansion to an Arbitrary order

We want to find the PN solution at all order. We can expand $\mathsf{h}_{\mu\nu}$ in the form:

$$\mathsf{h}^{\mu\nu} = \sum_{n=2}^{\infty}\frac{1}{c^n}{}^{(n)}\mathsf{h}^{\mu\nu}$$

where $1/c^n$ has been extracted to make the c dependence explicit. Similarly, we can expand the effective stress momentum tensor:

$$\tau^{\mu\nu} = \sum_{n=-2}^{\infty}\frac{1}{c^n}{}^{(n)}\tau^{\mu\nu}$$

Inserting into the relaxed Einstein equation and equating terms with same power of c, we get a recursive set of equations:

$$\nabla^2[{}^{(n)}\mathsf{h}^{\mu\nu}] = 16\pi G[{}^{(n-4)}\tau^{\mu\nu}] + \partial_t^2{}^{(n-2)}\mathsf{h}^{\mu\nu} \tag{3.41}$$

Recall that we cannot write solutions in the near zone using Green's functions. A particular solution to the above set of equations is found using a variant of the technique discussed in the last section. Given a function consider:

$$\left[\Delta^{-1}\left(r^B f\right)\right](\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}|\mathbf{x}'|^B f(\mathbf{x}')$$

If B is large and negative, the integral is regular as $|\mathbf{x}'| \to \infty$. Doing the same Laurent expansion, the coefficient u of B^0 is denoted by $\mathrm{FP}_{B=0}$,

$$u = \mathrm{FP}_{B=0}\left\{\Delta^{-1}\left[r^B f\right]\right\}$$

u satisfied $\nabla^2 u = f$ and is a well defined inversion of the Laplacian. When the integral converges $\mathrm{FP}_{B=0}\left\{\Delta^{-1}\left[r^B f\right]\right\}$ s the same as $\nabla^{-1}f$ If we now expand to n-th order in the PN expansion, we denote that by an overbar as:

$$\bar{\mathsf{h}}^{\mu\nu} = \sum_{m=2}^{n}\frac{1}{c^m}{}^{(m)}\mathsf{h}^{\mu\nu}$$

The particular solution we have found can be written as:

$$\bar{\mathsf{h}}^{\mu\nu} = \frac{16\pi G}{c^4}\mathcal{FP}\Box^{-1}_{\text{ret}}\bar{\tau}^{\mu\nu} \tag{3.42}$$

We add this to the general solution of the homogenous wave equation which has the form:

$$h^{\alpha\beta}_{\text{hom}} = \frac{16\pi G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{R^{\alpha\beta}_L\left(t-\frac{r}{c}\right)-R^{\alpha\beta}_L\left(t+\frac{r}{c}\right)}{2r}\right]$$

where $R^{\alpha\beta}_L$ are arbitrary functions of retarded and advanced times. This antisymmetric combination removes any outgoing radiation. Antisymmetric combination ensures that the solution is regular at $r=0$. Under time reversal, the equation is odd and therefore, it describes the radiation reaction.

3.2.10 Matching the Solution

In the external region $d<r<\infty$, for $d/r<1$, we found the solution in the form of a Minkowskian expansion. Multipole expansion is applicable so we write the solutions in terms of multpole moments. All higher order terms are determined through iteration in the form of a multipole expansion. In the region $0<r<\mathcal{R}$, where $\mathcal{R}$ is where near region ends, we found the solution in terms of a Post Newtonian expansion. Since we are considering slowly moving sources with $v\ll c$, we have $\mathcal{R}\gg d$, and the region where PN approximation is valid overlaps with the region where post-Minkowskian multipole expansion is valid. In the post-Minkowskian region solution is parametrized by multipole moments which are yet to be determined. In the PN solution, we have energy momentum tensor of the source. So, comparing these solutions in the overlapping region, we can fix the multipole moments in terms of the source terms. In the overlap region $d<r<\mathcal{R}$, $d/r<1$, so we can do a multipolar PN expansion in powers of d/r. Also, post Minkowskian expansion can be done in the same way as PN expansion in powers of v/c. When expanded (Blanchet and Damour 1986 eq. 5.5), we get:

$$h^{00}_n = O\left(\frac{1}{c^{2n}}\right),\quad h^{0i}_n = O\left(\frac{1}{c^{2n+1}}\right),\quad h^{ij}_n = O\left(\frac{1}{c^{2n}}\right) \tag{3.43}$$

So, to a given order in the PN expansion, we take a finite number of iterations of the post Minkowskian solution. For example, at 2PN order, we want to compute the $O\left(\frac{1}{c^4}\right)$ correction, so we compute g_{00} to $O\left(\frac{1}{c^6}\right)$, g_{0i} to $O\left(\frac{1}{c^5}\right)$ and g_{ij} to $O\left(\frac{1}{c^4}\right)$. So, eq 3.43 shows that we need to compute h_n to order $n=3$, so we do two iterations of the solution h_1. Comparing the PN expansion with re expanded post Minkowskian expansion, fixes the multipole moments in terms of the energy momentum tensors. They were computed to an arbitrary order in the PN expansion by Blanchet and Damour (2006 review). I_L and J_L are given by:

$$\begin{aligned}
\mathrm{I}_L(u) =&\mathcal{FP}\int d^3x\int_{-1}^{1}dz\left\{\delta_l(z)\hat{x}_L\Sigma-\frac{4(2l+1)\delta_{l+1}(z)}{c^2(l+1)(2l+3)}\hat{x}_{iL}\Sigma^{(1)}_i\right.\\
&\left.+\frac{2(2l+1)\delta_{l+2}(z)}{c^4(l+1)(l+2)(2l+5)}\hat{x}_{ijL}\Sigma^{(2)}_{ij}\right\}(u+z|\mathbf{x}|/c,\mathbf{x}),\\
\mathrm{J}_L(u) =&\mathcal{FP}\int d^3x\int_{-1}^{1}dz\epsilon_{ab\langle i_l}\left\{\delta_l(z)\hat{x}_{L-1\rangle a}\Sigma_b\right.\\
&\left.-\frac{(2l+1)\delta_{l+1}(z)}{c^2(l+2)(2l+3)}\hat{x}_{L-1\rangle ac}\Sigma^{(1)}_{bc}\right\}(u+z|\mathbf{x}|/c,\mathbf{x}).
\end{aligned}$$

where

$$\begin{aligned}
\Sigma &\equiv \frac{\bar{\tau}^{00}+\bar{\tau}^{ii}}{c^2}\\
\Sigma_i &\equiv \frac{\bar{\tau}^{0i}}{c}\\
\Sigma_{ij} &\equiv \bar{\tau}^{ij}
\end{aligned}$$

where $\bar{\tau}^{ii} \equiv \delta_{ij}\bar{\tau}^{ij}, \tau^{\mu\nu}$ is the effective stress energy tensor. The bar over a quantity denotes its PN expansion up to the required order The function $\delta_l(z)$ is given by 4.7.

Despite all complications of the non-linear theory, *the full nonlinear result for $\mathsf{h}_1^{\mu\nu}$, to all orders in the PN expansion, is obtained from the result of linearized theory simply replacing $T^{\mu\nu}$ with $\tau^{\mu\nu}$, and inserting the $\mathcal{FP}$ prescription.*

4 Appendix

4.0.1 Multipole Expansion and STF notation

For the Poisson equation $\nabla^2\phi = -4\pi\rho$, the solution is written in terms of the Green's function as:

$$\phi(\mathbf{x}) = \int d^3y \frac{1}{|\mathbf{x}-\mathbf{y}|}$$

A way of writing multipole expansion is:

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{y}|} &= \frac{1}{|\mathbf{x}|} - y^i\partial_i\frac{1}{|\mathbf{x}|} + \frac{1}{2}y^iy^j\partial_i\partial_j\frac{1}{|\mathbf{x}|} + \dots \\ &= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}y^{i_1}\dots y^{i_l}\partial_{i_1}\dots\partial_{i_l}\frac{1}{|\mathbf{x}|}.\end{aligned}$$

This expansion satisfies the Poisson equation. Now removing traces from the terms in expansion we get:

$$\phi(\mathbf{x}) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}Q_{i_1\dots i_l}\partial_{i_1}\dots\partial_{i_l}\frac{1}{|\mathbf{x}|}, \tag{4.1}$$

where

$$Q_{i_1\dots i_l} = \int d^3y\, y^{\langle i_1}\dots y^{i_l\rangle}\rho(\mathbf{y}), \tag{4.2}$$

where $y^{\langle i_1}\dots y^{i_l\rangle}$ denotes that we are taking the symmetric trace free parts of the tensors $y^{i_1}\dots y^{i_l}$. Now, we introduc multi index notation in which a tensor with l indices $i_1i_2...i_l$ is labeled by a letter L,

$$F_l \equiv F_{i_1i_2\dots i_l} \tag{4.3}$$

Similarly,

$$G_{iL} \equiv G_{ii_1i_2\dots i_l}$$

If we have repeated L indices, then a summation over all indices $i_1i_2...i_l$ is understood,

$$F_LG_L = \sum_{i_1i_2\dots i_l} F_{i_1i_2\dots i_l}G_{i_1i_2\dots i_l}$$

we use hat to indicate a tensor that is symmetric and trace free $\hat{K}_L$ or equivalently write it as $K_{<L>}$. A symmetric trace free tensor has $2l+1$ independent components, therefore it is an irreducible representation of the rotation group SO(3).

Solution to the relativistic wave equation $\Box\phi = -4\pi\rho$ can be written as:

$$\phi(t,\mathbf{x}) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{F_L\left(t-\frac{r}{c}\right)}{r}\right], \tag{4.4}$$

where F_L is an arbitrary function satisfying

$$\Box \left[\frac{F_L\left(t - \frac{r}{c}\right)}{r} \right] = 0, \tag{4.5}$$

Because the set of tensors F_L is a complate representation of the rotations group, this solution is the most general solution. Comparing this solution to the solution found using the Green's function:

$$\phi(t, \mathbf{x}) = \int d^3x' \, \frac{1}{|\mathbf{x} - \mathbf{x}'|} \, \rho \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right), \tag{3.187}$$

we can find the relativistic multipoles:

$$F_L(u) = \int d^3y \, y^{\langle L \rangle} \int_{-1}^{1} dz \, \delta_l(z) \rho \left(u + z\frac{|\mathbf{y}|}{c}, \mathbf{y} \right). \tag{4.6}$$

δ_l is:

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1} l!} (1 - z^2)^l \tag{4.7}$$

and it satisfies:

$$\int_{-1}^{1} dz \delta_l(z) = 1$$

and as $l \to \infty$, it approaches the Dirac delta function $\delta(z)$. (blanchet and damour 1989 appendix B)

5 Future Plans

Provide concluding remarks, summary, discussion, etc here.

6 Bibliography

Cite all your sources.