

# Browsing large graphs with XJS, a graph drawing tool in JavaScript

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**Abstract.** There has been progress in visualization of large graphs recently. Tools appeared that can render a huge graph in seconds. However, if we request that the node labels are readable, and the edges are routed around the nodes, then the problem remains difficult. Interacting with a large graph in an Internet browser with the same ease as browsing an online map is still a challenging task. In this paper we describe a few novel approaches to large graph visualization that we developed in open-source JavaScript software.

We give a new efficient edge routing algorithm, where the edges are routed around the nodes. The algorithm produces edge paths which are visually appealing and shortest in their homotopy class.

To facilitate graph visualization with WebGL, or any other platform supporting tiles, we propose a new simple and efficient tiling method. The method guarantees that in every view, except of the highest level, the number of visible entities per tile is not larger than a predefined bound.

The edge routing algorithm mentioned above is reused at the tiling stage to simplify the paths on the lower levels.

## Introduction

Our software is open source, it is represented by a set of NPM packages. It runs on the client desktop or on a phone, and renders the graph in an Internet browser. We target large but not huge graphs. The maximum number of vertices of the graphs we applied our tool at was 28k, and the maximum number of the edges was 237k.

The algorithms described below were discovered while we programmed our tool. We believe these algorithms can be useful to other developers as well. The findings seemed to us interesting enough to put them into a paper.

The paper has sections Introduction, Related Work, Edge routing in XJS, Tiling, and Future work.

Let us start with a short review of some relevant to us publications.

## 29 Related work

30 A popular graph drawing tool Graphviz [1] applies method Scalable Force-  
31 Directed Placement [2] for large graphs, with no support for tiling. Its edge  
32 routing for this method builds the whole visibility graph and routes edges on it.  
33 This can be very slow because the visibility graph can have  $O(n^2)$  edges, where  
34  $n$  is the number of the nodes in the graph. Interestingly, the funnel algorithm [3,  
35 4], the last step of our approach, is used in Graphviz for the edge routing in the  
36 Sugiyama layout. We are not aware of any tool that integrates Graphviz and  
37 uses tiling as well.

38 yWorks [5] has method "Organic edge routing" that produces edge routes  
39 around the nodes. We could find only a very general description of the method:  
40 "The algorithm is based on a force directed layout paradigm. Nodes act as re-  
41 pulsive forces on edges in order to guarantee a certain minimal distance between  
42 nodes and edges. Edges tend to contract themselves. Using simulated annealing,  
43 this finally leads to edge layouts that are calculated for each edge separately".  
44 It seems the algorithm runs in  $O(n + m)\log(n + m)$  time, where  $n$  is the number  
45 of the nodes and  $m$  is the number of the edges.

46 ReGraph [6] uses WebGL as the viewing platform. It can render a large graph  
47 using straight lines for the edges. The tool does not support tiling, but instead  
48 the user interactively opens the node that is a cluster of nodes.

49 "graph-tool.skewed" [7] does not implement its own layout algorithms or edge  
50 routing algorithms, but instead provides a nice wrapper around the algorithms  
51 from other layout tools.

52 Circos [8] visualizes large graphs in a circular layout. It does not support  
53 tiles.

54 Cosmograph [9] uses a GPU to calculate the layout of a graph and can  
55 handle a graph with a million nodes. It renders edges as straight lines. It does  
56 not support tiling.

57 The authors of [10] implemented GraphMaps, a tool for large graph visu-  
58 alization. The tool only runs on Windows. The edge were routed as polylines  
59 on a triangulation and were not optimized. The tool supported tiling, but the  
60 problem of the limiting number of visible entities was not solved.

61 In [11] an approach to visualize a huge graph is described. The method uses  
62 tiles and edge bundling following [12], which is applied at the last moment during  
63 the graph browsing. The latter calculation is done on the client side. The rest  
64 and the majority of the calculations runs on several servers.

## 65 Edge routing in XJS

66 We believe that short and smooth edges, that are not obstructed by the nodes,  
67 are easier to follow than longer edges with kinks. We believe that such edges  
68 help in understanding the graph structure. In addition, we were looking for a  
69 fast algorithm. This was our motivation to come up with the routing described  
70 here.

81 The edge routing starts, as in [13], by building a spanner graph, an approxi-  
82 mation of the full visibility graph, and then finding the edge routes as shortest  
83 paths on the spanner. The spanner, see Fig 2, is built on a variation of a Yao  
84 graph, which was introduced independently by Flinchbaugh, and Jones [14], and  
85 Yao [15]. This graph is built with a help of a set of cones with the apices at  
86 the vertices. Each cone of the set has the same angle, usually in the form of  $\frac{2\pi}{k}$ ,  
87 where  $k$  is a natural number,  $k = 12$  in our settings. The family of cones with  
88 the apex at a specific vertex partition the plane, as illustrated in Fig. 1. For each  
89 cone at most one edge is created connecting the cone apex with a vertex inside  
90 the cone. This way the spanner has at most  $kn$  edges, where  $n$  is the number of  
91 the vertices. We cover each node by a polygon with a relatively small number of  
92 corners, at most 8. Polygon corners play role of the vertices of the spanner. As a  
93 result, the spanner has  $O(N)$  edges, where  $N$  is the number of the graph nodes.

94  
95  
96 The approach of [13] applies local optimizations to shorten an edge path.  
97 Namely, it tries to shortcut one vertex at a time from the path, as illustrated in  
98 Fig 3. To smoothen a path, it fits Bezier segments into the polyline corners by  
99 using a binary search to find a large fitting segment, see Fig 4.  
100 We noticed that when the shortcutting of polyline corners fails, the resulting  
101 path might remain not visually appealing, as shown in Fig. 3.

102 We replace the shortcutting with a more precise, but still efficient optimiza-  
103 tion described below.  
104  
105

## 106 Path optimization

107 We finalize edge routes by the “funnel” algorithm [3, 4], routing a path inside a  
108 simple polygon, that is a polygon without holes.

109 An application of the ‘path in a simple polygon’ optimization to edge routing  
110 is not a new idea: the novelty of our work is in how we find the polygon and  
111 how we use it. The authors of Graphvis used the ‘funnel’ algorithm [16], but  
112 only for hierarchical layouts, where a simple polygon,  $\mathcal{P}$ , containing the path is  
113 available. They write: “If  $\mathcal{P}$  does not contain holes ... we can apply a standard  
114 “funnel” algorithm ... for finding Euclidean shortest paths in a simple polygon”.  
115 In general case, for a non-layered layout, they build the visibility graph which is  
116 very expensive for a large graph.

117 Here we find the polygon  $\mathcal{P}$  for any layout. We drop the requirement that  
118  $\mathcal{P}$  is simple. Indeed, to run the “funnel” algorithm one only needs a “sleeve”: a  
119 sequence of triangles leading from the start to the end of the path, where each  
120 triangle shares a side with its successor. Let us show how to build polygon  $\mathcal{P}$ ,  
121 create a sleeve, and produce an optimized path.

122 We call obstacles,  $\mathcal{O}$ , the set of polygons covering the original nodes, see  
123 Fig. 2. Before routing edges, we calculate a Constrained Delaunay Triangula-  
124 tion [17] on  $\mathcal{O}$ . Let us call this triangulation  $\mathcal{T}$ .

125 For each edge of the graph we proceed with the following steps.

133 We route a path, called  $\mathcal{L}$ , on the spanner, as illustrated by Fig. 5. Let  $\mathcal{S}$  and  
134  $\mathcal{E}$  be the obstacles containing correspondingly  $\mathcal{L}$ ’s start and end point. To obtain



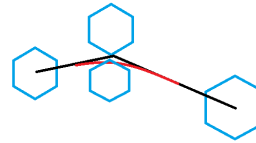
**Fig. 1.** Yao graph



**Fig. 2.** Spanner graph is built using the idea of Yao graphs. The dashed curves are the original node boundaries. Each original curve is surrounded by a polygon with some offset to allow the polyline paths smoothing without intersecting the former. The edge marked by the circles is created because the top vertex is inside the cone, and it is the closest among such vertices to the cone apex. The apex of the cone is the lower vertex of the edge. XJS uses cone angle  $\frac{\pi}{6}$ , so the edges of the spanner can deviate from the optimal direction by this angle. Therefore, the shortest paths on the spanner have length that is at most the optimal shortest length multiplied by  $\frac{1}{\cos(\frac{\pi}{6})} \simeq 1.155$ .



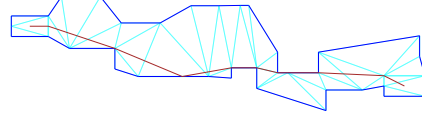
**Fig. 3.** Unsuccessful shortcut



**Fig. 4.** Fitting a Bezier segment into a polyline corner



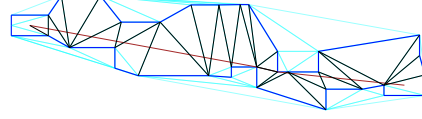
126 **Fig. 5.** Path  $\mathcal{L}$  with  $\mathcal{T}$ , a fragment.



127 **Fig. 6.** Polygon  $\mathcal{P}$  containing  $\mathcal{L}$ .



128 **Fig. 7.** New triangulation of  $\mathcal{P}$ .



129 **Fig. 8.** The optimized path together  
130 with the sleeve diagonals.

135  $\mathcal{P}$ , let us consider  $\mathcal{U}$ , the set of all triangles  $t \in \mathcal{T}$  such that either  $t \subset \mathcal{S} \cup \mathcal{E}$ , or  $t$   
136 intersects  $\mathcal{L}$  and is not inside of any obstacle in  $\mathcal{O} \setminus \{S, E\}$ . The union of  $\mathcal{U}$  gives  
137 us  $\mathcal{P}$ . The boundary of  $\mathcal{P}$  comprizes all sides  $e$  of the triangles from  $\mathcal{U}$  such that  
138  $e$  belongs to exactly one triangle from  $\mathcal{U}$ , see Fig. 6.

139 To create the sleeve [3, 4], we need to have a triangulation of  $\mathcal{P}$  such that every  
140 edge of the triangulation is either a boundary edge of  $\mathcal{P}$ , or a diagonal of  $\mathcal{P}$ .  
141 Because  $\mathcal{U}$  might not have this property, as in Fig. 6, we create a new Constrained  
142 Delaunay Triangulation of  $\mathcal{P}$ , where the set of constrained edges is the boundary  
143 of  $\mathcal{P}$ , see Fig. 7.

144 We trace path  $\mathcal{L}$  through the new triangulation and obtain the sleeve. Finally,  
145 we apply the funnel algorithm on the sleeve and obtain the path which is the  
146 shortest in the homotopy class of  $\mathcal{L}$ , as illustrated in Fig. 8.

147 The discussion [18] of the algorithm helped us in the implementation.

148 Polygon  $\mathcal{P}$  is not necessarily simple, as shown in Fig. 9. In this example the  
149 path that we calculate with the funnel algorithm is not the shortest path inside  
150 of  $\mathcal{P}$ .

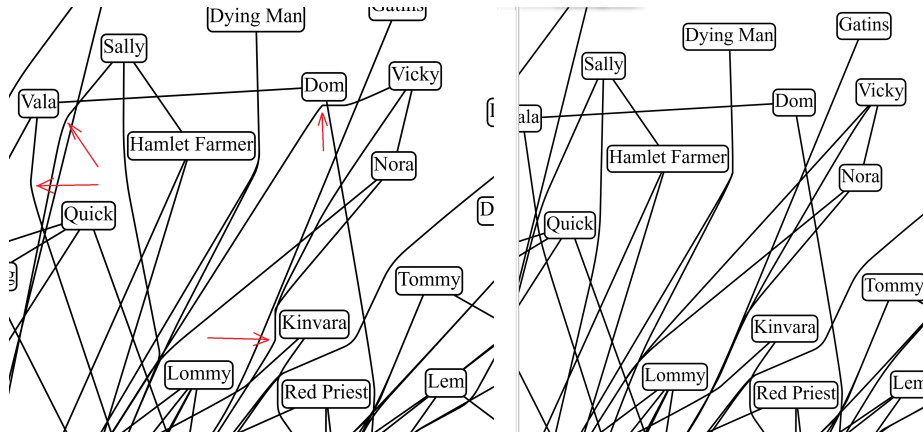
## 151 Performance and quality comparison

155 In Fig. 10 we compare the paths generated by the old and the new method. We  
156 can see that the paths produced by the new method have no kinks. We also  
157 know that these paths are the shortest in their 'channels'. Arguably, the new  
158 method produces better paths.

159 Our performance experiments are summarized in Table. 1. We see that the  
160 older approach outperforms the new one on the smaller graphs; those with the  
161 number of nodes under 2000. The new method is faster on the rest of the graphs.  
162 We still prefer to use the new method independently of the graph size since the  
163 slowdown is insignificant, but the quality of the paths is better.



131 **Fig. 9.**  $\mathcal{P}$  is not simple. The dotted path is shorter than the dashed one that  
 132 was found by the routing.



152 **Fig. 10.** Comparing the old, on the left, and the new, on the right, paths. The  
 153 arrows on the left fragment point to the kinks that were removed by the new  
 154 method.

graph	nodes	edges	old method's time	new time
social network [19]	407	2639	1.0	1.4
b103 [20]	944	2438	1.6	2.0
b100 [21]	1463	5806	5.6	5.785
composers [22]	3405	13832	510.5	20.3
p2p-Gnutella04 [23]	10876	39994	375.4	304.2
facebook_combined [24]	4039	88234	132.2	123.7
lastfm_asia_edges [25]	7626	27807	43.3	54.7
deezer_europe_edges [25]	28283	92753	1596.9	1402.6
ca-HepPh [26]	12008	237010	521.2	495.0

**Table 1.** Performance comparison with time in seconds.

## 1 Tiling

We had two goals when working on tiling. The first goal was to make exploring the graph in our tool similar to using online maps. The second goal was efficiency. The algorithm works in three phases. The first phase builds the levels starting from the lowest level and proceeding to higher and more detailed levels, with smaller tiles, until no more tile subdivision is required. The second phase filters out the entities from the layers to satisfy the capacity quota. Finally, the third phase simplifies the edge routes to utilize the space freed by the filtered out entities.

A tile, in our settings, is a pair  $(rect, tiledata)$ , where  $rect$  is the rectangle of the tile and  $tiledata$  is a set of *tile elements* visible in  $rect$ . A *tile element* could be a node, an edge label, an edge arrowhead, or an *edge clip*. An edge clip is a pair  $(e, p)$ , where  $e$  is an edge and  $p$  is a continuous piece of the edge curve  $c_e$ . Sometimes we need several edge clips to trace an edge through a tile.

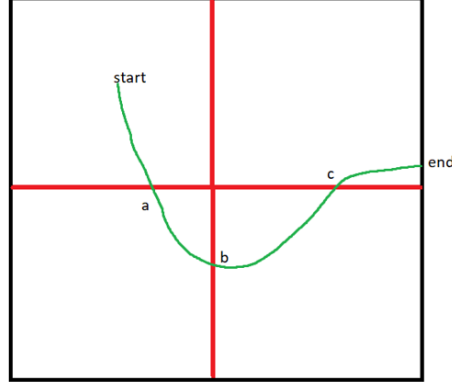
The initial tile, the only tile on level 0, is represented by pair  $(0, 0)$ . For  $z = 1$ , there are four tiles:  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 0)$ , and  $(1, 1)$ . Each tile  $(i, j)$  can be subdivided into four sub-tiles for level  $z + 1$ :  $(2i, 2j)$ ,  $(2i, 2j + 1)$ ,  $(2i + 1, 2j)$ , and  $(2i + 1, 2j + 1)$ .

Each  $z$ -level is represented by a map  $L_z$ , so  $L_z(i, j)$  gives us a specific tile. Empty tiles correspond to undefined  $L_z(i, j)$ .

We use edge clips to represent the edge intersections with the tiles and provide the renderer with the minimal geometry that is sufficient to render a tile. To achieve this we require property  $\mathcal{F}$ :

a) For each tile  $t$ , for each curve clip  $(e, p) \in t.tiledata$ , we have:  $p \subset t.rect$  and  $p$  might cross the boundary of the  $t.rect$  only at endpoints of  $p$ .

b) For each edge  $e$  we have : the union of all  $p$  for all  $(e, p) \in t.tiledata$  is equal to  $c_e \cap t.rect$ .



221 **Fig. 11.** Intersect curve  $[start, end]$  with the midlines. Sort the intersections pa-  
 222 rameters together with start, and end into array  $u = [start, a, b, c, end]$ . Split the  
 223 curve to sub-curves  $[start, a]$ ,  $[a, b]$ ,  $[b, c]$ ,  $[c, end]$ .

## 202 First phase of tiling

203 The first phase starts with  $L_0 = \{(0, 0) \rightarrow (rect, tiledata)\}$ : and *tiledata* com-  
 204 prising curve clips  $(e, c_e)$ , for all edges  $e$  of the graph, all graph nodes, all edge  
 205 labels, and all edge arrowheads. We ensure property  $\mathcal{F}$  by setting *rect* to a  
 206 padded bounding box of the graph, so each edge curve does not intersect the  
 207 boundary of *rect*.

208 Let us assume that  $L_z$  is already constructed and  $\mathcal{F}$  holds for its tiles. To  
 209 build level  $L_{z+1}$  we divide each tile  $t = L_z(i, j)$  into four sub-tiles of equal size.  
 210 For each node, arrowhead, or edge label of  $t.tiledata$ , if the bounding box of the  
 211 element intersects the sub-tile's rectangle then we add the element to the sub-tile  
 212 *tiledata*.

213 The edge clip treatment is more involved. Let  $(e, p)$  be a curve clip belonging  
 214 to tile  $t$ . We find all intersections of curve  $p$  with the horizontal midline and the  
 215 vertical midline of  $t.rect$ . Each intersection can be represented as  $p[t_j]$ . We sort  
 216 sequence  $u = [start, \dots, t_j, \dots, end]$ , where  $[start, end]$  is the parameter domain  
 217 of  $p$ , in ascending order, and remove the duplicates.

218 Next we create curve clips  $(e, l_k) = (e, trim(p, u_k, u_{k+1}))$ , as shown in Fig 11.  
 219 We assign each curve clip  $(e, l_k)$  to the sub-tile with the rectangle containing the  
 220 bounding box of  $l_k$ .

224 Because, by the induction assumption property  $\mathcal{F}$  is true on  $L_z$ , and by  
 225 construction, each new curve clip can cross the boundary of the sub-tile only at  
 226 the clip endpoints. We also cover all the intersections of  $p$  with the sub-tiles with  
 227 the new edge clips, so the property  $\mathcal{F}$  holds for  $L_{z+1}$ .

228 Two parameters control the algorithm: tile capacity,  $\mathcal{C}$ , and the minimal size  
 229 of a tile:  $(\mathcal{W}, \mathcal{H})$ . If for each  $(i, j)$  the number of elements in  $L_z(i, j).tiledata$   
 230 is not greater than  $\mathcal{C}$ , and if  $w \leq \mathcal{W}$  and  $h \leq \mathcal{H}$ , where  $(w, h)$  is the current



231 tile size, then we try to build the next level  $L_{z+1}$ . Otherwise, the second phase  
 232 starts.

233 In our setting  $\mathcal{C} = 500$ , and  $(\mathcal{W}, \mathcal{H}) = 3(w, h)$ , where  $w$  is the average width  
 234 and  $h$  is the average height of the nodes of the graph. For efficiency, we do not  
 235 create a new curve in an edge clip but keep two parameters indicating the clipped  
 236 segment start and end. gnutella. One of the reasons was the memory limit on a  
 237 process in those browsers.

## 238 Second phase of tiling

239 The second phase of tiling filters out the entities from the lower layers. We do not  
 240 change the highest, the most detailed layer. We sort the nodes of the graph into  
 241 array  $N$  by PageRank [27]. For each layer  $L$ , except of the highest, we proceed  
 as follows.

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```

1: procedure FILTER( $L$ )
2:    $r \leftarrow \text{removeEntities}(L)$ 
3:   for all  $n$  in  $N$  do
4:     if ! $\text{addNodeToLayer}(n, r, N)$  then break
5:     end if
6:   end for
7: end procedure

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242  
 243 Here  $\text{removeEntities}(L)$  empties all the tiles of layer  $L$ , but returns map  $r$  allow-  
 244 ing to restore the tiles. Function  $\text{addNodeToLayer}(n)$  tries to add the node to  
 245  $L$ , it also tries to add the tile elements for self edges of  $n$ , and the tile elements  
 246 for the edges connecting  $n$  with the nodes ranked at least as high as  $n$ . These  
 247 nodes are the nodes already added to  $L$ .

248 This procedure guarantees that each tile of  $L$  has no more than  $\mathcal{C}$  elements.

## 249 Third phase of tiling

250 In the third phase we use a fact that some nodes are not present on the layer. For  
 251 all layers, except of the highest, we reroute the edges but only around the nodes  
 252 that are present in the layer. We do not calculate edge routes from scratch, but  
 253 use the existing routes and only apply the "funnel" heuristic in larger channels.  
 254 This gives us simpler edge routes but still has the visual stability during the  
 255 layer change while browsing.

## 256 Node labels visibility

257 Cite some papers on label positioning.

## 258 2 Future work

- 259 – Find a tiling method that guarantees that each tile has no more than  $\mathcal{C}$  ele-  
 260 ments on every level. One approach could be to use a more aggressive edge  
 261 bundling to reduce the number of edge clips in the tiles. A simple idea is  
 262 finding the repeated segments in the edge curves that naturally appear while  
 263 routing through the same graph with the same algorithm, and reuse the re-  
 264 peated segment to save memory.
- 265 – Our tile calculation is memory intensive and take a long time for larger  
 266 graphs. The largest graph from the Table 1 that we were able to load with  
 267 Chrome, and Edge using the tiling procedure was p2p-Gnutella04 [23]. One  
 268 of the reasons was the memory limit on a process in those browsers, another  
 269 was the long running-time of the tiling procedure. A possible measure would  
 270 be saving the tiles to the disk and loading them on demand.
- 271 – For the user convenience we would like to run the layout, routing, and tiling  
 272 in a worker thread to avoid blocking the main thread.

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