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On Certain Lattice Theta Series

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Abstract

This work partially establishes the connection between polynomials of Riemann theta constants and lattice theta series of 24-dimensional unimodular lattices. Both the polynomials of Riemann theta constants and lattice theta series of 16-dimensional unimodular lattices were used as modular forms of weight-8 in the modular form approach in construction of superstring measures. It was shown that in case for 16-dimensional unimodular lattices there are very elegant relations between the two. We aimed to generalize the question for 24-dimensional unimodular lattices and were able to partially answer it.

1 Introduction

In this work we study the relation between 2 different ways of expressing certain modular forms: via polynomials of Riemann theta constants and via lattice theta series. The motivation of studying these specific modular forms comes from physics. The problem of finding NSR superstring measures is a long-standing problem in superstring theory. One of the approaches is to guess the answer based on known mathematical requirements it should satisfy. In case of low genera the problem can be reformulated in the terms of modular forms: one needs to construct the weight-8 modular forms of certain properties. We consider 2 different ways of constructing these modular forms:

- via polynomials of degree-16 in Riemann theta-constants (so-called Grushevsky ansatz [2])
- via lattice theta series of 16-dimensional unimodular lattices (so-called OPSMY ansatz [5])

It turned out [1] that there is a very natural connection between the two. The question arises: what could be said about the analogous constructions for weights other than 8?

In this work we try to address this question. We consider the weight-12 modular forms constructed similarly to original weight-8 Grushevsky functions (we refer to these new weight-12 expressions as *weight-12 Grushevsky functions*) and lattice theta series correponding to 24-dimensional unimodular lattices.

We were able to partially solve this problem: we have identified the underlying lattice for 3 out of 4 polynomials among weight-12 Grushevsky functions. One of these answers is rather trivial, the second one easily follows from known results, and the third one is nontrivial, was not considered in literature before, and is the main result of this work.

We start by providing all the necessary definitions for modular forms (section 2), Riemann theta constants (section 3), and lattice theta series (section 4). Then, in section 5, we formulate the known result in case of 16-dimensional lattices and proceed by proving our own results in the case of 24-dimensional lattices.

2 Modular forms

We begin with definitions of modular forms, modular group $\Gamma^{(g)}$ and its important subgroup $\Gamma^{(g)}(1,2)$. Both Grushevsky ansatz and OPSMY ansatz are interesting exactly because of their modular properties.

Definition 2.1 (Modular group)**.** Modular group $\Gamma^{(g)} = \mathrm{Sp}(2g,\mathbb{Z})$ is defined as

$$\Gamma^{(g)} = \mathrm{Sp}(2g,\mathbb{Z}) = \{M \in \mathbb{Z}^{2g\times 2g} : M^t\Omega M = \Omega\} \tag{1}$$

where Ω is defined to be

$$\Omega = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix} \tag{2}$$

Definition 2.2 (Siegel half-space)**.** Siegel half-space $\mathcal{H}^{(g)}$ is defined to be the set of all symmetric $\mathbb{C}^{g\times g}$ matrices with positive definite imaginary part.

Definition 2.3 (Action of $\Gamma^{(g)}$)**.** Modular group $\Gamma^{(g)} = \mathrm{Sp}(2g,\mathbb{Z})$ acts on Siegel half-space $\mathcal{H}^{(g)}$ the following way:

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma^{(g)} \tag{3}$$

$$\gamma : \tau \mapsto (A\tau + B)(C\tau + D)^{-1} \tag{4}$$

Definition 2.4 (Modular form)**.** A holomorphic function f on $\mathcal{H}^{(g)}$ is called a modular form of weight k with respect to subgroup $\Gamma' \leq \Gamma^{(g)}$ if

$$\forall \gamma \in \Gamma' \quad f(\gamma\tau) = \det(C\tau + D)^k f(\tau) \tag{5}$$

Definition 2.5 ($\Gamma^{(g)}(1,2)$)**.** The subgroup $\Gamma^{(g)}(1,2)$ consists of all the elements

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma^{(g)} \tag{6}$$

of the modular group $\Gamma^{(g)}$ s.t. all the diagonal elements of AB^t and CD^t are even.

3 Riemann theta constants

Now we are going to define the notion of Riemann theta constants: the basic blocks of Grushevsky functions. Then we define the set of functions, linear combinations of which form the Grushevsky ansatz for superstring measures: weight-8 Grushevsky functions. At the end of this section we define the generalized construction: weight-12 Grushevsky functions.

Definition 3.1 (Riemann theta constant with characteristic)**.** The Riemann theta constant with characteristic $e = \left[\begin{array}{c} \vec{\delta} \\ \vec{\varepsilon} \end{array}\right]$ is the function on Siegel half-space $\mathcal{H}^{(g)}$ defined as

$$\theta_e(\tau) = \theta\left[\begin{array}{c} \vec{\delta} \\ \vec{\varepsilon} \end{array}\right](\tau) = \sum_{\vec{n}\in\mathbb{Z}^g} \exp\left(\pi\big(\vec{n} + \frac{\vec{\delta}}{2}\big)^t \tau\big(\vec{n} + \frac{\vec{\delta}}{2}\big) + \pi i\big(\vec{n} + \frac{\vec{\delta}}{2}\big)^t \vec{\varepsilon}\right) \tag{7}$$

where $\vec{\delta}, \vec{\varepsilon}$ are binary vectors of length g (i.e. $\in \mathbb{Z}_2^g$).

Although those are called "constants", they are functions of an argument $\tau \in \mathcal{H}^{(g)}$. Usually theta function is considered to have 2 arguments: $\theta = \theta(z, \theta)$. And when we substitute $z = 0$ we get a function of 1 argument defined in (7).

Definition 3.2 (Γ_e)**.** For characteristic $e = \left[\begin{array}{c} \vec{\delta} \\ \vec{\varepsilon} \end{array}\right]$ we define the subgroup $\Gamma_e \leq \Gamma^{(g)}$ as

$$\Gamma_e = \gamma[e]\Gamma_g(1,2)\gamma[e]^{-1} \tag{8}$$

where $\gamma[e]$ is an element of $\Gamma^{(g)}$ which maps the Riemann theta constant with zero characteristic to characteristic e.

Theorem 3.3 ([4])**.** *For every* $\gamma \in \Gamma(1,2)$

$$\theta\left[\begin{array}{c} D\vec{\delta} - C\vec{\varepsilon} \\ -B\vec{\delta} + A\vec{\varepsilon} \end{array}\right](\gamma\tau) = \zeta_\gamma \det(C\tau + D)^{1/2} \exp\left(\frac{\pi i}{4}\left(2\vec{\delta}^t B^t C\vec{\varepsilon} - \vec{\delta}^t B^t D\vec{\delta} - \vec{\varepsilon}^t A^t C\vec{\varepsilon}\right)\right)\theta\left[\begin{array}{c} \vec{\delta} \\ \vec{\varepsilon} \end{array}\right](\tau) \tag{9}$$

where ζ_γ *is the eighth root of unity* $\zeta_\gamma^8 = 1$ *depending only on* γ.

It is straightforward to see that by theorem 3.3 Riemann theta constant θ_0^{16} is a weight-8 modular form w.r.t. $\Gamma(1,2)$. More generally, Riemann theta constant θ_0^{8k} is a weight-$4k$ modular form w.r.t. $\Gamma(1,2)$ by same reasoning.

Definition 3.4 (Even/odd characteristic)**.** The characteristic $\left[\begin{array}{c} \vec{\delta} \\ \vec{\varepsilon} \end{array}\right]$ is called even if $\vec{\delta}\cdot\vec{\varepsilon}$ is even. Otherwise, the characteristic is called odd.

An interesting fact is that Riemann theta constants with odd characteristic are zero.

Lemma 3.5 ([4]). *$e \in \mathbb{Z}_2^g$ is odd. Then $\theta_e(\tau) \equiv 0$.*

Definition 3.6 (Original weight-8 Grushevsky functions).

$$\xi_0^{8,(g)}[e] = \theta_e^{16} \tag{10}$$

$$\xi_1^{8,(g)}[e] = \theta_e^8 \sum_{e_1}^{N_e} \theta_{e+e_1}^8 \tag{11}$$

$$\xi_2^{8,(g)}[e] = \theta_e^4 \sum_{e_1}^{N_e} \theta_{e+e_1}^4 \theta_{e+e_2}^4 \theta_{e+e_1+e_2}^4 \tag{12}$$

$$\dots$$

$$\xi_p^{8,(g)}[e] = \sum_{e_1,\dots,e_p}^{N_e} \left\{ \theta_e \cdot \Big(\prod_i^p \theta_{e+e_i} \Big) \cdot \Big(\prod_{i<j}^p \theta_{e+e_i+e_j} \Big) \cdot \dots \cdot \theta_{e+e_1+\dots+e_p} \right\}^{2^{4-p}} \tag{13}$$

where $p = 0..4$.

Grushevsky ansatz in genus g for superstring measures is defined as the linear combinations of these weight-8 Grushevsky functions. We will not provide the coefficients here, because they are not important for this work.

By summing up the θ_e for different characteristics e we obtain modular form w.r.t. somewhat large subgroup of $\Gamma^{(g)}$ (more precisely, w.r.t. subgroup Γ_e).

Theorem 3.7 ([3]). *$\xi_p^{8,(g)}[e]$ are weight-8 modular forms w.r.t. subgroup Γ_e.*

Grushevsky functions are constructed as polynomials of Riemann theta constants, thus keeping the total polynomial degree divisible by 8 kills ζ_γ. This is why we want to increase the polynomial degree to 24 after 18: to keep the modularity property. Another view on this move will be covered in section 4 when we discover the similar modularity property of lattice theta series.

Now, we can introduce the following definition:

Definition 3.8 (Weight-12 Grushevsky functions).

$$\xi_0^{12,(g)}[e] = \theta_e^{24} \tag{14}$$

$$\xi_1^{12,(g)}[e] = \theta_e^{12} \sum_{e_1}^{N_e} \theta_{e+e_1}^{12} \tag{15}$$

$$\dots$$

$$\xi_p^{12,(g)}[e] = \sum_{e_1,\dots,e_p}^{N_e} \left\{ \theta_e \cdot \Big(\prod_i^p \theta_{e+e_i} \Big) \cdot \Big(\prod_{i<j}^p \theta_{e+e_i+e_j} \Big) \cdot \dots \cdot \theta_{e+e_1+\dots+e_p} \right\}^{3\cdot 2^{3-p}} \tag{16}$$

where $p = 0..3$.

Grushevsky did not define or study weight-12 Grushevsky functions (because there is probably no application of them in superstring theory). They are original to the present work. We have selected this naming to reflect that the construction originates from Grushevsky ansatz for superstring measures. We consider those Grushevsky functions purely as mathematical objects and keep this naming throughout the article.

Theorem 3.9. *$\xi_p^{12,(g)}$ are weight-12 modular forms w.r.t. subgroup Γ_e.*

Proof. The proof is analogous to the proof of referenced theorem 3.7. ∎

4 Lattice theta series

In this section we will introduce the definition of lattice and the associated lattice theta series. Also we will talk about the gluing theory, which allows classification of lattices with fixed dimension.

Definition 4.1 (Lattice)**.** The set generated by all the integer linear combinations of a basis $v_1, .., v_n \in \mathbb{R}^n$ is called the lattice (generated by $v_1, .., v_n$). The number n is called the dimension of the lattice.

Definition 4.2 (Dual lattice)**.** Dual lattice Λ^* for lattice $\Lambda \subset \mathbb{R}^h$ is the set of all vectors $u \in \mathbb{R}^h$ s.t. $\forall v \in \Lambda \quad u \cdot v \in \mathbb{Z}$.

Definition 4.3 (Unimodular lattice)**.** Lattice Λ is called self-dual or unimodular if it coincides with it's dual Λ^*. An equivalent definition is that the Gram matrix MM^t (where M is the matrix formed with the basis of Λ as columns) has determinant 1.

Definition 4.4 (Lattice theta series)**.** The genus-g lattice theta series associated with the lattice $\Lambda \subset \mathbb{R}^h$ is a function on the Siegel space $\mathcal{H}^{(g)}$ defined as

$$\vartheta_\Lambda^{(g)}(\tau) = \sum_{(\vec{p}_1,\ldots,\vec{p}_g)\in\Lambda^g} \exp\left(\pi i \sum_{k,l} (\vec{p_k} \cdot \vec{p_l})\tau_{kl}\right) \tag{17}$$

Lemma 4.5 ([5])**.** *If 8 divides the number of dimensions m of a unimodular lattice Λ, then the corresponding theta series $\vartheta_\Lambda^{(g)}$ for this lattice is a modular form of weight $m/2$ relative to $\Gamma^{(g)}(1,2)$.*

This lemma shows us that if we want to match against the generalized Grushevsky construction, which will consist of modular forms similarly to the original one, then we should consider the lattice theta series of m-dimensional lattices where $m = 8k$.

So the next step after the original $m = 16$ is the $m = 24$. And this gives another idea on why the generalized Grushevksy construction should contain weight-12 modular forms.

Theorem 4.6 ([6], Gluing theory)**.** *Suppose $L \subset \mathbb{R}^n$ is an integer lattice containing a sublattice*

$$L_1 \oplus L_2 \oplus .. \oplus L_k \tag{18}$$

$$\sum_i^k d_i = n \tag{19}$$

where every L_i is also an integer lattice and d_i is the dimension of L_i. Then L is generated by $L_1 \oplus L_2 \oplus .. \oplus L_k$ and some vectors $y^j, j = 1..J$ of form

$$y^j = y_1^j + y_2^j + .. + y_k^j \tag{20}$$

where each $y_i^j \in \mathbb{R}^n$ is from $L_i^* \subset \mathbb{R}^n$. Vectors $y^j = y_1^j + .. + y_n^j$ are called gluing vectors*.*

We will denote $L = \langle L_1 \oplus .. \oplus L_k, y^1, .., y^J\rangle$.

Definition 4.7 (A_n lattice)**.**

$$A_n = \{(x_1, .., x_{n+1}) \in \mathbb{Z}^{n+1} : \sum_i x_i = 0\} \tag{21}$$

Definition 4.8 (D_n, D_n^+ lattices)**.**

$$D_n = \{(x_1, .., x_n) \in \mathbb{Z}^n : \sum_i x_i \quad \text{is even}\} \tag{22}$$

There is also another important lattice, D_n^+, which is defined only for even n:

$$D_n^+ = D_n \cup \left(D_n + \left(\frac{1}{2}^n\right)\right) \tag{23}$$

Here and throughout the article we use the notation (a^n) to denote the vector $(a, a, .., a) \in \mathbb{R}^n$.

Definition 4.9 (E_7, E_8 lattices)**.** E_8 is another name for the lattice D_8^+. And the lattice E_7 is defined as the sublattice of E_8 orthogonal to the fixed minimal vector $v \in E_8$:

$$E_7 = \{x \in E_8 : x \cdot v = 0\} \tag{24}$$

(minimal vector of a lattice is any vector with smallest non-zero norm).

Theorem 4.10. *[6] There are 8 16-dimensional unimodular lattices*

Lattice theta series	*Lattice*	*Gluing vector*
ϑ_0	$\mathbb{Z}^{16}$	-
ϑ_1	$\mathbb{Z}^8 \oplus E_8$	-
ϑ_2	$\left(\mathbb{Z}^4 \oplus D_{12}\right)^+$	$\left(0^4, \frac{1}{2}^{12}\right)$
ϑ_3	$\left(\mathbb{Z}^2 \oplus E_7 \oplus E_7\right)^+$	$\left(\frac{1}{4}^6, -\frac{3}{4}^2, \frac{1}{4}^6, -\frac{3}{4}^2\right)$
ϑ_4	$(\mathbb{Z} \oplus A_{15})^+$	$\left(\frac{1}{4}^{12}, -\frac{3}{4}^4\right), \left(\frac{1}{2}^8, -\frac{1}{2}^8\right), \left(\frac{3}{4}^4, -\frac{1}{4}^{12}\right)$
ϑ_5	$D_8 \oplus D_8$	$\left(\frac{1}{2}^8, 0^7, 1\right)$
ϑ_6	$E_8 \oplus E_8$	-
ϑ_7	D_{16}^+	$\left(\frac{1}{2}^{16}\right)$

For every lattice Λ in 2-nd column the Λ^+ means the glued lattice in the sense of theorem 4.6.

OPSMY ansatz for superstring measures is defined as the linear combinations of these lattice theta series.

5 Weight-12 Grushevsky functions vs 24-dimensional lattice theta series

We start by recalling the known result for weight-8 modular forms. Then we consider our own case of weight-12 modular forms and formulate and prove the main results of this work: theorems 5.4 and 5.10.

Theorem 5.1 ([1])**.** *In every genus g there is an explicit connection between weight-8 Grushevsky functions and lattice theta series of unimodular 16-dimensional lattices, namely:*

$$\forall g \in \mathbb{Z}_{\geq 0} \quad \vartheta_p^{(g)} = 2^{-gp} \xi_p^{8,(g)}, \quad p = 0..4 \tag{25}$$

After the discussed theorem for $m = 16$, the next question is what lattices in dimension $m = 24$ correspond to the polynomials of Riemann theta constants given by our generalization: weight-12 Grushevsky functions. Let's recall it:

$$\xi_0^{12,(g)}[e] = \theta_e^{24}$$

$$\xi_1^{12,(g)}[e] = \theta_e^{12} \sum_{e_1}^{N_e} \theta_{e+e_1}^{12}$$

$$\ldots$$

$$\xi_p^{12,(g)}[e] = \sum_{e_1,\ldots,e_p}^{N_e} \left\{ \theta_e \cdot \left(\prod_i^p \theta_{e+e_i} \right) \cdot \left(\prod_{i<j}^p \theta_{e+e_i+e_j} \right) \cdot \ldots \cdot \theta_{e+e_1+\ldots+e_p} \right\}^{3 \cdot 2^{3-p}}$$

Clearly $p = 0..3$, so we have to deal with 4 polynomials.
The case $p = 0$ is a trivial one and is proved in [6].

Theorem 5.2 ([6], $p = 0$)**.**

$$\forall g \in \mathbb{Z}_{\geq 0} \quad \xi_0^{12,(g)} = \vartheta_{\mathbb{Z}^{24}}^{(g)} \tag{26}$$

The case $p = 1$ is also known, but we provide the proof to demonstrate the approach we follow for $p = 2$.

Theorem 5.3 ($p = 1$)**.**

$$\forall g \in \mathbb{Z}_{\geq 0} \quad \xi_1^{12,(g)}[0] = 2^{-g} \cdot \vartheta^{(g)}_{\mathbb{Z}^{12} \oplus D_{12}^+} \tag{27}$$

Proof.

$$\begin{aligned}
\frac{\xi_1^{12,(g)}[0]}{\theta_0^{12}} = \sum_e \theta_e^{12} &= \sum_e \Big(\sum_{\vec{n} \in \mathbb{Z}^g} \exp\big[\pi i (\vec{n} + \frac{\vec{\delta}}{2})^t \tau (\vec{n} + \frac{\vec{\delta}}{2})) + (\vec{n} + \frac{\vec{\delta}}{2})^t \vec{\varepsilon}\big] \Big)^{12} \\
&= \sum_e \sum_{\substack{\vec{n}^a \in \mathbb{Z}^g \\ a=1..12}} \exp\big[\pi i \Big((\sum_{a=1}^{12} \vec{n}^a + \frac{\vec{\delta}}{2})^t \tau (\sum_{a=1}^{12} \vec{n}^a + \frac{\vec{\delta}}{2}) + (\sum_{a=1}^{12} \vec{n}^a + \frac{\vec{\delta}}{2})^t \vec{\varepsilon} \Big)\big] \\
&= \sum_e \sum_{\substack{\vec{n}^a \in \mathbb{Z}^g \\ a=1..12}} \exp\big[\pi i \Big((\sum_{a=1}^{12} \vec{n}^a + \frac{\vec{\delta}}{2})^t \tau (\sum_{a=1}^{12} \vec{n}^a + \frac{\vec{\delta}}{2}) + (\sum_{a=1}^{12} \vec{n}^a)^t \vec{\varepsilon} \Big)\big]
\end{aligned} \tag{28}$$

If any component of the vector $\sum_a \vec{n}^a$ is odd, then there exist 2^{g-1} vectors $\vec{\varepsilon} \in \mathbb{Z}^g$ where this component is equal to 1 and the same number of vectors where this component is equal to 0. Every such pair of vectors $\vec{\varepsilon}$ creates different in sign, but equal in absolute value terms $\big(\exp(\pi i(..))\big)$ which cancel out each other. So, we consider only vectors $\vec{n}^a$ s.t. the components of the vector $\sum_a \vec{n}^a$ are even. The term $(\sum_a \vec{n}^a)^t \vec{\varepsilon}$ is therefore always even and thus the exp[..] does not depend on it since $2\pi i$-periodicity of exp. So the sum over all $\vec{\varepsilon}$ (we had the sum over $e = \begin{bmatrix} \vec{\delta} \\ \vec{\varepsilon} \end{bmatrix}$) is replaced by the multiplier 2^g.

After dividing by 2^g we get the following series:

$$\sum_{\vec{\delta} \in \mathbb{Z}_2^g} \sum_{\substack{\vec{n}^a \in \mathbb{Z}^g \\ a=1..12 \\ \sum_{a=1}^{12} \vec{n}^a \vdots 2}} \exp\Big(\pi i \sum_{a=1}^{12} (\vec{n}^a + \frac{\vec{\delta}}{2})^t \tau (\vec{n}^a + \frac{\vec{\delta}}{2})\Big) \tag{29}$$

And it is easy to see now that the lattice theta series associated with lattice

$$\Lambda_1 \cup \Big(\Lambda_1 + (\frac{1}{2}^{12})\Big) \tag{30}$$

where

$$\Lambda_1 = \{(n^1, n^2, .., n^{12}) \in \mathbb{Z}^{12} | \sum_a n^a \vdots 2\} \tag{31}$$

coincides with the one given in (29). By definition the lattice $\Lambda_1 \cup \big(\Lambda_1 + (\frac{1}{2}^{12})\big)$ is exactly the lattice D_{12}^+. □

The following result for $p = 2$ is the main result of this work (alongside with theorem 5.10).

Theorem 5.4 ($p = 2$)**.**

$$\forall g \in \mathbb{Z}_{\geq 0} \quad \xi_2^{12,(g)}[0] = 2^{-2g} \cdot \vartheta^{(g)}_{\mathbb{Z}^6 \oplus \Lambda^+} \tag{32}$$

Λ^+ is the lattice obtained from $D_6^3 = D_6 \oplus D_6 \oplus D_6$ by gluing it with

$$\vec{\alpha} = \Big(\frac{1}{2}^6, 0^6, \frac{1}{2}^6\Big) \tag{33}$$

$$\vec{\beta} = \Big(0^6, \frac{1}{2}^6, \frac{1}{2}^6\Big) \tag{34}$$

$$\vec{\delta} = (1, 0^5, 1, 0^5, 1, 0^5) \tag{35}$$

Proof of the theorem will be separated into 4 lemmas.

Lemma 5.5. *Lattice corresponding to the theta series*

$$2^{2g}\cdot\frac{\xi_2^{12,(g)}[0]}{\theta_0^6} \tag{36}$$

is the lattice Λ^+*:*

$$\Lambda^+=\Lambda\cup(\Lambda+\vec{\alpha})\cup(\Lambda+\vec{\beta})\cup(\Lambda+\vec{\gamma}) \tag{37}$$

where

$$\vec{\alpha}=(\frac{1}{2}^6,0^6,\frac{1}{2}^6) \tag{38}$$

$$\vec{\beta}=(0^6,\frac{1}{2}^6,\frac{1}{2}^6) \tag{39}$$

$$\vec{\gamma}=(\frac{1}{2}^6,\frac{1}{2}^6,0^6) \tag{40}$$

.

Proof.

$$\frac{\xi_2^{12,(g)}[0]}{\theta_0^6}=\sum_{e_1,e_2}\theta_{e_1}^6\theta_{e_2}^6\theta_{e_1+e_2}^6 \tag{41}$$

Now, let $e_i=\begin{bmatrix}\vec{\delta}_i\\ \vec{\varepsilon}_i\end{bmatrix}, i=1,2$ and consider

$$\begin{aligned}
\theta_{e_1}^6&=\Big(\sum_{\vec{n}_1\in\mathbb{Z}^g}\exp\big[\pi i\big((\vec{n}_1+\frac{\vec{\delta}_1}{2})^t\tau(\vec{n}_1+\frac{\vec{\delta}_1}{2})+(\vec{n}_1+\frac{\vec{\delta}_1}{2})^t\vec{\varepsilon}_1\big)\big]\Big)^6 &(42)\\
&=\sum_{\vec{n}_1^1\in\mathbb{Z}^g}(..)\cdot\sum_{\vec{n}_1^2\in\mathbb{Z}^g}(..)\cdot..\cdot\sum_{\vec{n}_1^6\in\mathbb{Z}^g}(..) &(43)\\
&=\sum_{\vec{n}_1^1\in\mathbb{Z}^g}\sum_{\vec{n}_1^2\in\mathbb{Z}^g}..\sum_{\vec{n}_1^6\in\mathbb{Z}^g}\exp\big[\pi i\big(\sum_{a=1}^6(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})^t\tau(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})+\sum_{a=1}^6(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})^t\vec{\varepsilon}_1\big)\big] &(44)
\end{aligned}$$

The term $\sum_{a=1}^6(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})^t\vec{\varepsilon}_1$ splits on $(\sum_{a=1}^6\vec{n}_1^a)^t\vec{\varepsilon}_1$ and $6\cdot\frac{\vec{\delta}_1^t}{2}\vec{\varepsilon}_1=3\vec{\delta}_1^t\vec{\varepsilon}_1$. Since the $2\pi i$-periodicity of exp, the latter is equivalent to $\vec{\delta}_1^t\vec{\varepsilon}_1$. The same holds for θ_{e_2} and $\theta_{e_1+e_2}$. Now

$$\begin{aligned}
\sum_{e_1,e_2}\theta_{e_1}^6\theta_{e_2}^6\theta_{e_1+e_2}^6=&\sum_{e_1,e_2}\sum_{\substack{\vec{n}_1^a\in\mathbb{Z}^g\\ a=1..6}}\sum_{\substack{\vec{n}_2^a\in\mathbb{Z}^g\\ a=1..6}}\sum_{\substack{\vec{n}_{12}^a\in\mathbb{Z}^g\\ a=1..6}}\exp\Big[\pi i\Big(\sum_{a=1}^6(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})^t\tau(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})\\
&+\sum_{a=1}^6(\vec{n}_2^a+\frac{\vec{\delta}_1}{2})^t\tau(\vec{n}_2^a+\frac{\vec{\delta}_1}{2})+\sum_{a=1}^6(\vec{n}_{12}^a+\frac{\vec{\delta}_1+\vec{\delta}_2}{2})^t\tau(\vec{n}_{12}^a+\frac{\vec{\delta}_1+\vec{\delta}_2}{2})\\
&+(\sum_{a=1}^6\vec{n}_1^a)^t\vec{\varepsilon}_1+(\sum_{a=1}^6\vec{n}_2^a)^t\vec{\varepsilon}_2+(\sum_{a=1}^6\vec{n}_{12}^a)^t(\vec{\varepsilon}_1+\vec{\varepsilon}_2)\\
&+\vec{\delta}_1^t\vec{\varepsilon}_1+\vec{\delta}_2^t\vec{\varepsilon}_2+(\vec{\delta}_1+\vec{\delta}_2)^t(\vec{\varepsilon}_1+\vec{\varepsilon}_2)\Big)\Big] \qquad (45)
\end{aligned}$$

If e is odd characteristic, then the corresponding Riemann theta constant θ_e is zero. So, we can assume that $\vec{\delta}_1^t\vec{\varepsilon}_1$, $\vec{\delta}_2^t\vec{\varepsilon}_2$, and $(\vec{\delta}_1+\vec{\delta}_2)^t(\vec{\varepsilon}_1+\vec{\varepsilon}_2)$ are even (otherwise the whole term exp(..) is zero). Again, since the $2\pi i$-periodicity of exp, we can neglect them:

$$\sum_{e_1,e_2}\theta^6_{e_1}\theta^6_{e_2}\theta^6_{e_1+e_2} = \sum_{e_1,e_2}\sum_{\substack{\vec{n}_1^a\in\mathbb{Z}^g\\ a=1..6}}\sum_{\substack{\vec{n}_2^a\in\mathbb{Z}^g\\ a=1..6}}\sum_{\substack{\vec{n}_{12}^a\in\mathbb{Z}^g\\ a=1..6}} \exp\Big[\pi i\Big(\sum_{a=1}^{6}(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})^t\tau(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})$$
$$+\sum_{a=1}^{6}(\vec{n}_2^a+\frac{\vec{\delta}_1}{2})^t\tau(\vec{n}_2^a+\frac{\vec{\delta}_1}{2})+\sum_{a=1}^{6}(\vec{n}_{12}^a+\frac{\vec{\delta}_1+\vec{\delta}_2}{2})^t\tau(\vec{n}_{12}^a+\frac{\vec{\delta}_1+\vec{\delta}_2}{2})$$
$$+(\sum_{a=1}^{6}\vec{n}_1^a+\vec{n}_{12}^a)^t\vec{\varepsilon}_1+(\sum_{a=1}^{6}\vec{n}_2^a+\vec{n}_{12}^a)^t\vec{\varepsilon}_2\Big)\Big] \tag{46}$$

Consider the vector $\sum_{a=1}^6 \vec{n}_1^a+\vec{n}_{12}^a$. If some component of it is odd, then there exist 2^{g-1} binary vectors $\vec{\varepsilon}_1\in\mathbb{Z}_2^g$ where the corresponding component is 1, and the same number of vectors where the corresponding component is 0. Those vectors produce terms that perfectly cancel each other.

So, every component of the vector $\sum_{a=1}^6 \vec{n}_1^a+\vec{n}_{12}^a$ must be even to give the non-zero result. And if so, by $2\pi i$-periodicity of exp, we can neglect the $(\sum_{a=1}^6 \vec{n}_1^a+\vec{n}_{12}^a)^t\vec{\varepsilon}_1$ inside the exp(..). The same applies for vector $\sum_{a=1}^6 \vec{n}_2^a+\vec{n}_{12}^a$.

Thus, the outermost sum over all characteristics $e_1=\begin{bmatrix}\vec{\delta}_1\\ \vec{\varepsilon}_1\end{bmatrix}, e_2=\begin{bmatrix}\vec{\delta}_2\\ \vec{\varepsilon}_2\end{bmatrix}$ is reduced to the sum over all binary vectors $\vec{\delta}_1,\vec{\delta}_2\in\mathbb{Z}_2^g$ with coefficient 2^{2g} and we get:

$$\sum_{e_1,e_2}\theta^6_{e_1}\theta^6_{e_2}\theta^6_{e_1+e_2} = 2^{2g}\sum_{\vec{\delta}_1,\vec{\delta}_2\in\mathbb{Z}_2^g}\sum_{\substack{\vec{n}_1^a\in\mathbb{Z}^g\\ a=1..6}}\sum_{\substack{\vec{n}_2^a\in\mathbb{Z}^g\\ a=1..6}}\sum_{\substack{\vec{n}_{12}^a\in\mathbb{Z}^g\\ a=1..6\\ \forall k\quad (\sum_{a=1}^6 \vec{n}_1^a+\vec{n}_{12}^a)_k \vdots 2\\ \forall k\quad (\sum_{a=1}^6 \vec{n}_2^a+\vec{n}_{12}^a)_k \vdots 2}} \exp\Big[\pi i\Big(\sum_{a=1}^{6}(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})^t\tau(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})$$
$$+\sum_{a=1}^{6}(\vec{n}_2^a+\frac{\vec{\delta}_1}{2})^t\tau(\vec{n}_2^a+\frac{\vec{\delta}_1}{2})+\sum_{a=1}^{6}(\vec{n}_{12}^a+\frac{\vec{\delta}_1+\vec{\delta}_2}{2})^t\tau(\vec{n}_{12}^a+\frac{\vec{\delta}_1+\vec{\delta}_2}{2})\Big)\Big] \tag{47}$$

Now consider the lattice

$$\Lambda=\{(n_1^1,..,n_1^6,n_2^1,..,n_2^6,n_{12}^1,..,n_{12}^6)\in\mathbb{Z}^{12}\big|\sum_{a=1}^{6}n_1^a+n_{12}^a,\sum_{a=1}^{6}n_2^a+n_{12}^a\quad\text{are even}\} \tag{48}$$

(here every n_j^i represents one of the g components of the vector $\vec{n}_j^i\in\mathbb{Z}^g$).

The corresponding lattice theta series is:

$$\vartheta_\Lambda^{(g)}(\tau)=\sum_{(\vec{p}_1,..,\vec{p}_g)\in\Lambda^g}\exp(\pi i\sum_{k,l}(\vec{p_k}\cdot\vec{p_l})\tau_{kl}) \tag{49}$$

Let's consider the sum inside the exp(..):

$$\sum_{k,l}(\vec{p_k}\cdot\vec{p_l})\tau_{kl}=\sum_{k,l}\tau_{kl}\Big(\sum_{a=1}^{6}n_1^a(k)\cdot n_1^a(l)+\sum_{a=1}^{6}n_2^a(k)\cdot n_2^a(l)+\sum_{a=1}^{6}n_{12}^a(k)\cdot n_{12}^a(l)\Big)$$
$$=\sum_{a=1}^{6}\sum_{k,l}n_1^a(k)\tau_{kl}n_1^a(l)+\sum_{a=1}^{6}\sum_{k,l}n_2^a(k)\tau_{kl}n_2^a(l)+\sum_{a=1}^{6}\sum_{k,l}n_{12}^a(k)\tau_{kl}n_{12}^a(l)$$
$$=\sum_{a=1}^{6}(\vec{n}_1^a)^t\tau\vec{n}_1^a+(\vec{n}_2^a)^t\tau\vec{n}_2^a+(\vec{n}_{12}^a)^t\tau\vec{n}_{12}^a \tag{50}$$

So the $\vartheta^{(g)}_\Lambda$ is almost what we want:

$$\vartheta^{(g)}_\Lambda(\tau) = \sum_{\substack{\vec{n}_1^a,\vec{n}_2^a,\vec{n}_{12}^a\in\mathbb{Z}^g \\ a=1..6 \\ \forall k \quad (\sum_{a=1}^6 \vec{n}_1^a+\vec{n}_{12}^a)_k \vdots 2 \\ \forall k \quad (\sum_{a=1}^6 \vec{n}_2^a+\vec{n}_{12}^a)_k \vdots 2}} \exp\Big[\pi i\Big(\sum_{a=1}^6(\vec{n}_1^a)^t\tau\vec{n}_1^a + \sum_{a=1}^6(\vec{n}_2^a)^t\tau\vec{n}_2^a + \sum_{a=1}^6(\vec{n}_{12}^a)^t\tau\vec{n}_{12}^a\Big)\Big] \tag{51}$$

Now, by gluing Λ with vectors $\vec{\alpha},\vec{\beta},\vec{\gamma}$ (defined below) we obtain the lattice Λ^+:

$$\Lambda^+ = \langle\Lambda,\vec{\alpha},\vec{\beta},\vec{\gamma}\rangle \tag{52}$$

where

$$\vec{\alpha} = (\frac{1}{2}^6, 0^6, \frac{1}{2}^6) \tag{53}$$

$$\vec{\beta} = (0^6, \frac{1}{2}^6, \frac{1}{2}^6) \tag{54}$$

$$\vec{\gamma} = (\frac{1}{2}^6, \frac{1}{2}^6, 0^6) \tag{55}$$

.

Let's calculate the theta series for Λ^+. If $\vec{p_k}\in(\Lambda+\vec{\alpha})$ and $\vec{p_l}\in\Lambda$, then

$$\begin{aligned}(\vec{p_k}\cdot\vec{p_l}) &= \sum_{a=1}^6(n_1^a(k)+\frac{1}{2})\cdot(n_1^a(l)+\frac{1}{2}) \\ &+\sum_{a=1}^6 n_2^a(k)\cdot n_2^a(l) \\ &+\sum_{a=1}^6(n_{12}^a(k)+\frac{1}{2})\cdot(n_{12}^a(l)+\frac{1}{2})\end{aligned} \tag{56}$$

If $\vec{p_k}\in(\Lambda+\vec{\alpha})$ and $\vec{p_l}\in(\Lambda+\vec{\beta})$, then

$$\begin{aligned}(\vec{p_k}\cdot\vec{p_l}) &= \sum_{a=1}^6(n_1^a(k)+\frac{1}{2})\cdot(n_1^a(l)+\frac{1}{2}) \\ &+\sum_{a=1}^6(n_2^a(k)+\frac{1}{2})\cdot(n_2^a(l)+\frac{1}{2}) \\ &+\sum_{a=1}^6(n_{12}^a(k)+\frac{1}{2}+\frac{1}{2})\cdot(n_{12}^a(l)+\frac{1}{2}+\frac{1}{2})\end{aligned} \tag{57}$$

And the same for all other cases. The point is, if we fix the vectors $(\vec{p}_1,..,\vec{p}_g)\in(\Lambda^+)^g$ and construct the matrix where they represent rows, then we can consider $\vec{n}_1^1,..,\vec{n}_1^6,\vec{n}_2^1,..,\vec{n}_2^6,\vec{n}_{12}^1,..,\vec{n}_{12}^6$ as columns of this matrix (see the definition of Λ). For example, when $\vec{p_k}\in(\Lambda+\vec{\alpha})$, we get the k-th component of each vector $\vec{n}_1^1,..,\vec{n}_1^6,\vec{n}_{12}^1,..,\vec{n}_{12}^6$ shifted by 1/2 (at the same time k-th component of $\vec{n}_2^1,..,\vec{n}_2^6$ is not shifted).

Now, we can view the $\sum_{k,l}(\vec{p_k}\cdot\vec{p_l})\tau_{kl}$, as the sum "over rows" of our matrix. The corresponding sum "over columns" could be written as:

$$\begin{aligned}\sum_{\vec{\delta}_1,\vec{\delta}_2}\Big(\sum_{a=1}^6(\vec{n}_1^a+\frac{\vec{\delta}_1}{2})^t\tau(\vec{n}_1^a+\frac{\vec{\delta}_1}{2}) + \sum_{a=1}^6(\vec{n}_2^a+\frac{\vec{\delta}_2}{2})^t\tau(\vec{n}_2^a+\frac{\vec{\delta}_2}{2}) \\ +\sum_{a=1}^6(\vec{n}_1^a+\frac{\vec{\delta}_1+\vec{\delta}_2}{2})^t\tau(\vec{n}_1^a+\frac{\vec{\delta}_1+\vec{\delta}_2}{2})\Big)\end{aligned} \tag{58}$$

where if we denote k-th row of our matrix by $\vec{p_k}$, then $(\vec{\delta}_1)_k = 0, (\vec{\delta}_2)_k = 0$ means $\vec{p_k} \in \Lambda$; $(\vec{\delta}_1)_k = 1, (\vec{\delta}_2)_k = 0$ means $\vec{p_k} \in (\Lambda + \vec{\alpha})$; $(\vec{\delta}_1)_k = 0, (\vec{\delta}_2)_k = 1$ means $\vec{p_k} \in (\Lambda + \vec{\beta})$; $(\vec{\delta}_1)_k = 1, (\vec{\delta}_2)_k = 1$ means $\vec{p_k} \in (\Lambda + \vec{\gamma})$. $\Big((\vec{\delta}_1 + \vec{\delta}_2)$ is the sum modulo $2\Big)$. $\square$

Lemma 5.6. *The lattice $\Lambda^+ = \langle \Lambda, \alpha, \beta, \gamma \rangle$ is the lattice*

$$\Lambda^+ = \Lambda \cup (\Lambda + \vec{\alpha}) \cup (\Lambda + \vec{\beta}) \cup (\Lambda + \vec{\gamma}) \tag{59}$$

where

$$\Lambda = \{(n_1^1, .., n_1^6, n_2^1, .., n_2^6, n_{12}^1, .., n_{12}^6) \in \mathbb{Z}^{12} | \sum_{a=1}^{6} n_1^a + n_{12}^a, \sum_{a=1}^{6} n_2^a + n_{12}^a \quad \textit{are even}\} \tag{60}$$

$$\vec{\alpha} = (\frac{1}{2}^6, 0^6, \frac{1}{2}^6) \tag{61}$$

$$\vec{\beta} = (0^6, \frac{1}{2}^6, \frac{1}{2}^6) \tag{62}$$

$$\vec{\gamma} = (\frac{1}{2}^6, \frac{1}{2}^6, 0^6) \tag{63}$$

$$\tag{64}$$

Proof. Every vector in $\langle \Lambda, \vec{\alpha}, \vec{\beta}, \vec{\delta} \rangle$ can be expressed as

$$x\vec{v} + a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma} \tag{65}$$

where $x, a, b, c \in \mathbb{Z}$. If all or neither of the a, b, c are odd, then the resulting vector lies in α. If a, b are odd and c is even, then the resulting vector lies in $\Lambda + \gamma$. The cases where b, c are odd, a is even and a, c are odd, b is even are analogous. If a is odd, and b, c are even, then the resulting vector lies in $\Lambda + \alpha$. The same for case when only b or c is odd. $\square$

Lemma 5.7. *The lattice Λ^+ is equal to the lattice obtained by gluing $D_6^3 = D_6 \oplus D_6 \oplus D_6$ with vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\delta}$, where*

$$\vec{\delta} = (1, 0^5, 1, 0^5, 1, 0^5) \tag{66}$$

Proof. The lattice Λ could be viewed as the sublattice of $\mathbb{Z}^{18}$ containing all the vectors of type $(e, e, e) \in \mathbb{Z}^{18}$ and $(o, o, o) \in \mathbb{Z}^{18}$ where each $e \in \mathbb{Z}^6$ denotes the vector with even sum of components and each $o \in \mathbb{Z}^6$ denotes the vector with odd sum of components.

The sublattice of Λ which contains (e, e, e) is precisely D_6^3. Let's show that D_6^3 glued with $\vec{\alpha}, \vec{\beta}, \vec{\delta}$ is exactly Λ^+.

Firstly, consider the lattice D_6^3 glued with $\vec{\delta}$. It is easy to see that we obtain the lattice Λ: every vector of type (e, e, e) is by definition in D_6^3, and every vector of type (o, o, o) could be obtained from the suitable vector of type (e, e, e) by adding $\vec{\delta}$.

Now, glue the result with $\vec{\alpha}, \vec{\beta}$. It only remains to show that $\vec{\gamma}$ is also generated:

$$\vec{\alpha} + \vec{\beta} - \vec{\gamma} = (0^6, 0^6, 1^6)$$

and since the $(0^6, 0^6, 1^6)$ is inside the lattice Λ (as vector of type (e, e, e)) then the vector $\vec{\gamma}$ is inside the lattice Λ as well. $\square$

Lemma 5.8. *Λ^+ is unimodular.*

Proof. Let's construct the generating matrix M for the lattice Λ and see that $\det MM^t = 1$.

It is known [6] that vectors

$$
\begin{aligned}
&(-1,-1,0,0,0,0)\\
&(1,-1,0,0,0,0)\\
&(0,1,-1,0,0,0)\\
&(0,0,1,-1,0,0)\\
&(0,0,0,1,-1,0)\\
&(0,0,0,0,1,-1)
\end{aligned} \tag{67}
$$

form the generating basis of D_6. Consider the basis of D_6^3:

$$
\begin{aligned}
e_1^1 &= (-1,-1,0,0,0,0,0^6,0^6)\\
e_2^1 &= (1,-1,0,0,0,0,0^6,0^6)\\
e_3^1 &= (0,1,-1,0,0,0,0^6,0^6)\\
e_4^1 &= (0,0,1,-1,0,0,0^6,0^6)\\
e_5^1 &= (0,0,0,1,-1,0,0^6,0^6)\\
e_6^1 &= (0,0,0,0,1,-1,0^6,0^6)\\
e_1^2 &= (0^6,-1,-1,0,0,0,0,0^6)\\
e_2^2 &= (0^6,1,-1,0,0,0,0,0^6)\\
e_3^2 &= (0^6,0,1,-1,0,0,0,0^6)\\
e_4^2 &= (0^6,0,0,1,-1,0,0,0^6)\\
e_5^2 &= (0^6,0,0,0,1,-1,0,0^6)\\
e_6^2 &= (0^6,0,0,0,0,1,-1,0^6)\\
e_1^3 &= (0^6,0^6,-1,-1,0,0,0,0)\\
e_2^3 &= (0^6,0^6,1,-1,0,0,0,0)\\
e_3^3 &= (0^6,0^6,0,1,-1,0,0,0)\\
e_4^3 &= (0^6,0^6,0,0,1,-1,0,0)\\
e_5^3 &= (0^6,0^6,0,0,0,1,-1,0)\\
e_6^3 &= (0^6,0^6,0,0,0,0,1,-1)
\end{aligned} \tag{68}
$$

Replace the vector e_2^1 with $\vec{\delta}$, e_5^2 with $\vec{\beta}$ and e_5^3 with $\vec{\alpha}$. It is easy to see that we still generate D_6^3:

$$e_2^1 = e_1^1 + 2\vec{\delta} + e_1^2 - e_2^2 + e_1^3 - e_2^3 \tag{69}$$

$$e_5^2 = -5e_1^1 - 4\vec{\delta} - 4e_3^1 - 3e_4^1 - 2e_5^1 - e_6^1 - 2e_1^2 + 2e_2^2 - 5e_1^3 - 0 - 4e_3^3 - 3e_4^3 - 2e_5^3 - 2e_6^3 \tag{70}$$

$$e_5^3 = 5e_1^1 + 4e_2^1 + 4e_3^1 + 3e_4^1 + 2e_5^1 + 1e_6^1 - 1e_1^2 - 4e_2^2 - 4e_3^2 - 3e_4^2 - 2e_5^2 - 2e_6^2 + 2e_1^3 - 2e_2^3 + 2e_6^3 \tag{71}$$

Thus, the matrix M^t constructed with this basis as rows:

$$M^t = \begin{bmatrix}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix} \tag{72}$$

is the generating matrix for Λ^+. One can verify that $\det MM^t = 1$.

□

Theorem 5.9 ([6], Classification of unimodular lattices)**.** *All the unimodular lattices in dimension $m < 24$ which does not contain vectors of unit norm are listed in [6, Section 16.4, Table 16.7].*

Theorem 5.10. *Λ^+ is the 3rd out of 4 18-dimensional unimodular lattices provided in classification table in [6, Section 16.4, Table 16.7].*

Thus

$$\forall g \in \mathbb{Z}_{\geq 0} \quad \xi_2^{12,(g)}[0] = 2^{-2g}\, \vartheta^{(g)}_{Z^6 \oplus U_{18,3}} \tag{73}$$

where $U_{18,3}$ is the third 18-dimensional lattice in Conway's classification, and $\vartheta^{(g)}_{\mathbb{Z}^6 \oplus U_{18,3}}$ is the respective theta series for the direct sum of lattices.

Proof. It is easy to see that our lattice Λ^+ does not contain vectors of unit norm. All the vectors of form $v + \vec{\alpha}, v + \vec{\beta}, v + \vec{\gamma}$ where $v \in \Lambda$ has norm which is at least $3/2 + 3/2 = 3$. Vectors of type $(e, e, e) \in \Lambda$ has norm at least 2, and vectors of type $(o, o, o) \in \Lambda$ has norm at least 3.

And since the unimodularity of Λ^+ we can conclude that our lattice is presented in table in [6, Section 16.4, Table 16.7]. There are 4 lattices of dimension 18 there. They are defined by sublattices: $A_{17} \oplus A_1$, $D_{10} \oplus E_7 \oplus A_1$, D_6^3, and A_9^2. Clearly, our lattice is the one containing the sublattice D_6^3.

Formula (73) follows from theorem 5.4 and the above. □

The problem of finding the underlying lattice for the last case of $p = 3$ (polynomial $\xi_3^{12,(g)}$) requires another approach because powers of θ are no longer even (see the proof of theorem 5.4). So, this question remains open.

6 References

References

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