New Methods of Measuring Emittance Using Beam Position Monitors

P. Li, B. G. Sun, Q. Luo, X. H. Wang, H. L. Xu, P. Lu

***Abstract*–Stripline position monitors can be used as a nonintercepting emittance monitor and it is independent of the beam distribution. Usually, difference/sum ratio method is used to pick up the second moment. Two new methods are put forward, which are a log-ratio method and a combined method. The analysis and comparing of these methods are introduced. Change the current of quadrupole magnets m times (m≥6) upstream from BPM to change the Transfer matrix, and then the emittance can be calculated.**

I. Introduction

The use of beam position monitors as nonintercepting emittance monitors has been proposed in 1983 by R. H. Miller etc al[1]. In general, the spatial distributions of electron beams from photoinjectors are unknown and are not well approximated by a Gaussian. However, it's ideal for beam position monitors to measuring the emittance because it requires no beam distribution assumptions. It was proved by S. J. Russell etc in 1993[2]. Usually, most laboratories use the difference/sum ratio (Δ/Σ) method to pick up the second moment $\sigma_x^2-\sigma_y^2$ from BPMs' signals [1]-[6]. However, we use other methods, which the log-ratio method and combined method are used to do it at Hefei Light Source (HLS).

In HLS, two stripline BPMs were installed at the end of 200MeV linear accelerator [7]. Some measurement research was made using the stripline BPMs.

II. *BPM Signal*

A beam position monitor consists of four electrodes placed around the beam pipe at 90° intervals, as shown in Fig. 1. The electrodes are located at top, bottom, left, right position. The distance between electrodes and BPM center is given by b and their azimuthal angle by β. And the chamber radius of BPM is given by a. Consider an infinitely long line current $I(r,\varphi)$ at radial location r and azimuthal angle φ. The image current density $j(r, \varphi, a, \theta)$ on a conducting circular cylindrical pipe of radius a at azimuthal angle θ is then [1]:

$$j(a,\theta,r,\varphi)=\frac{I(r,\varphi)}{2\pi a}\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\varphi)} \tag{1}$$
$$=1+2\sum_{n=1}^{\infty}(r/a)^n\cos[n(\theta-\varphi)]$$

Manuscript received October 26, 2007. This work was supported by the Natural Science Foundation of China (10675118) and the National "985 Project" (173123200402002).

P. Li, B. G. Sun, and Q. Luo are with the National Synchrotron Radiation Laboratory, University of Science and Technology of China, Hefei, Anhui 230029, P.R. China (e-mail: manlouh@mail.ustc.edu.cn; bgsun@ustc.edu.cn; luoqing@mail.ustc.edu.cn).

For a gaussian beam distribution $I_{beam}(r, \varphi)$, we assumed $(\tilde{x}, \tilde{y})$ is the beam centroid, and (σ_x, σ_y) are the rms half widths in the x and y directions. Here, $\tilde{x}=r\cos\varphi$, $\tilde{y}=r\sin\varphi$, and σ_x, $\sigma_y \ll a$. And we assume there is no coupling between x and y directions in the beam. For a total beam current I_{beam}, integrating (1) over r and φ with this gaussian distribution, we can get the induced image current j (a,θ) at radius a and azimuthal angle θ:

$$\begin{aligned}
j(a,\theta) &= \iint j(a,\theta,r,\varphi)drd\varphi \\
&=\frac{I_{beam}}{2\pi a}\{1+2[\frac{\tilde{x}}{a}\cos\theta+\frac{\tilde{y}}{a}\sin\theta] \\
&+2[(\frac{\sigma_x^2-\sigma_y^2}{a^2}+\frac{\tilde{x}^2-\tilde{y}^2}{a^2})\cos 2\theta+2\frac{\tilde{x}\tilde{y}}{a^2}\sin 2\theta] \\
&+2[\frac{3\sigma_x^2-3\sigma_y^2+\tilde{x}^2-3\tilde{y}^2}{a^2}\frac{\tilde{x}}{a}\cos 3\theta \\
&\quad+\frac{3\sigma_x^2-3\sigma_y^2+3\tilde{x}^2-\tilde{y}^2}{a^2}\frac{\tilde{y}}{a}\sin 3\theta] \\
&+2[\frac{3(\tilde{x}^2-\tilde{y}^2+\sigma_x^2-\sigma_y^2)^2-2\tilde{x}^4-2\tilde{y}^4}{a^4}\cos 4\theta \\
&\quad+\frac{4\tilde{x}\tilde{y}(\tilde{x}^2-\tilde{y}^2+3\sigma_x^2-3\sigma_y^2)}{a^4}\sin 4\theta] \\
&+\text{higher order terms}\}
\end{aligned} \tag{2}$$

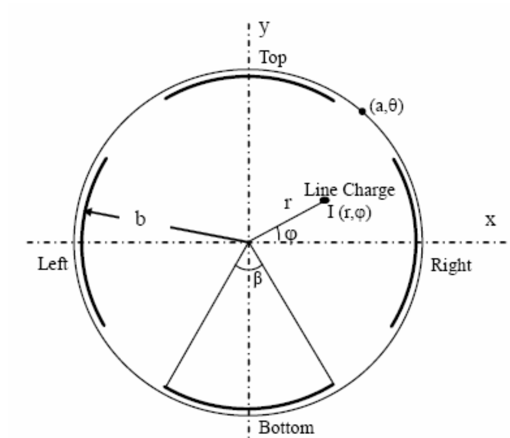

Fig.1. BPM electrode positions.

In fact, this gaussian restriction is not necessary, and a transverse emittance measurement using beam position monitors is independent of the beam distribution [2].

Consider BPM that consists of four pickup stripline electrodes, as shown in Fig.1, here $b\approx a$. Integrating the equation (2) over θ from ϕ-β/2 to ϕ+β/2 (ϕ=0^0, 90^0, 180^0, 270^0), we can get the result of signal of each stripline electrode.

To make it looks simple here, we introduce some symbols:

$$\begin{cases} z_{1x}=2\dfrac{\sin(\beta/2)}{\beta/2}\dfrac{\tilde{x}}{b}, z_{1y}=2\dfrac{\sin(\beta/2)}{\beta/2}\dfrac{\tilde{y}}{b}, \\ z_2=2\dfrac{\sin\beta}{\beta}\dfrac{\sigma_x^2-\sigma_y^2+\tilde{x}^2-\tilde{y}^2}{b^2}, \\ z_{3x}=2\dfrac{\sin(3\beta/2)}{3\beta/2}\dfrac{3\sigma_x^2-3\sigma_y^2+\tilde{x}^2-3\tilde{y}^2}{b^2}\dfrac{\tilde{x}}{b}, \\ z_{3y}=2\dfrac{\sin(3\beta/2)}{3\beta/2}\dfrac{3\sigma_x^2-3\sigma_y^2+3\tilde{x}^2-\tilde{y}^2}{b^2}\dfrac{\tilde{y}}{b}, \\ z_4=\dfrac{\sin(2\beta)}{\beta}\dfrac{3\left(\sigma_x^2-\sigma_y^2+\tilde{x}^2-\tilde{y}^2\right)^2-2\left(\tilde{x}^4+\tilde{y}^4\right)}{b^4}. \end{cases} \tag{3}$$

Table I gives the result of signal from each stripline for different modes up to the 4th order terms, which are normalized to $I_{beam}/2\pi b/\beta$.

TABLE I
SIGNALS FROM STRIPLINE PROBES NORMALIZED TO $I_{beam}/2\pi b/\beta$

	Right (ϕ=0)	Left (ϕ=180^0)	Top (ϕ=90^0)	Bottom (ϕ=270^0)
Monopole(n=0)	1	1	1	1
Dipole(n=1)	z_{1x}	$-z_{1x}$	z_{1y}	$-z_{1y}$
Quadrupole(n=2)	z_2	z_2	$-z_2$	$-z_2$
Sextupole (n=3)	z_{3x}	$-z_{3x}$	$-z_{3y}$	z_{3y}
Octupole(n=4)	z_4	z_4	z_4	z_4

According to Table I, we get the normalized signal from each stripline electrode:

$$\begin{cases} R=1+z_{1x}+z_2+z_{3x}+z_4+\cdots, \\ L=1-z_{1x}+z_2-z_{3x}+z_4\cdots, \\ T=1+z_{1y}-z_2-z_{3y}+z_4+\cdots, \\ B=1-z_{1y}-z_2+z_{3y}+z_4\cdots. \end{cases} \tag{4}$$

III. METHOD TO PICK UP $\sigma_x^2-\sigma_y^2$

A. Difference/sum ratio method

Since we got signal from each stripline electrode, we can pick up the second moment $\sigma_x^2-\sigma_y^2$ by some methods. Usually, most laboratories [1]-[6] use difference/sum ratio (Δ/Σ) method.

For Δ/Σ method, we get from the equation (3) and (4)

$$\begin{aligned} Q_{\Delta/\Sigma}&=\frac{R+L-T-B}{R+L+T+B}=z_2+O\left(\frac{1}{b^6}\right) \\ &=2\frac{\sin\beta}{\beta}\left(\frac{\sigma_x^2-\sigma_y^2}{b^2}+\frac{\tilde{x}^2-\tilde{y}^2}{b^2}\right)+O\left(\frac{1}{b^6}\right). \end{aligned} \tag{5}$$

In the equation (5), there are the item $\sigma_x^2-\sigma_y^2$ and the item $\tilde{x}^2-\tilde{y}^2$. To get the second moment $\sigma_x^2-\sigma_y^2$, the item $\tilde{x}^2-\tilde{y}^2$ must be known.

From the equation (3) and (4), while omit 3[rd] order, we approximately get

$$\begin{cases} \dfrac{R-L}{R+L}\approx 4\dfrac{\sin(\beta/2)}{\beta}\dfrac{\tilde{x}}{b}, \\ \dfrac{T-B}{T+B}\approx 4\dfrac{\sin(\beta/2)}{\beta}\dfrac{\tilde{y}}{b}. \end{cases} \tag{6}$$

We can solve $\tilde{x}$, $\tilde{y}$ from the equation (6), and then take the result into the equation (5), we get the second moment $\sigma_x^2-\sigma_y^2$.

We suppose the first moment is eliminated perfectly, and it has no effect in the process of computing the second moment.

For the second moment, the sensitivity of Δ/Σ is:

$$S_{\Delta/\Sigma}=2\frac{\sin\beta}{\beta}\frac{1}{b^2} \tag{7}$$

B. Log-ratio method

When we use logarithmic detector after BPM, the signal we got is 20lgR, 20lgL, 20lgT and 20lgB at the output of data acquisition system. So, a log-ratio method is put forward.

According to

$$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5+\cdots\cdots(-1<x\le 1). \tag{8}$$

Then,

$$\begin{cases} 20\log R=\dfrac{20}{\ln 10}\{z_{1x}+z_2+z_{3x}-\dfrac{1}{2}z_{1x}^2-z_{1x}z_2+\dfrac{1}{3}z_{1x}^3+\cdots\}, \\ 20\log L=\dfrac{20}{\ln 10}\{-z_{1x}+z_2-z_{3x}-\dfrac{1}{2}z_{1x}^2+z_{1x}z_2-\dfrac{1}{3}z_{1x}^3+\cdots\}, \\ 20\log T=\dfrac{20}{\ln 10}\{z_{1y}-z_2-z_{3y}-\dfrac{1}{2}z_{1y}^2+z_{1y}z_2+\dfrac{1}{3}z_{1y}^3+\cdots\}, \\ 20\log B=\dfrac{20}{\ln 10}\{-z_{1y}-z_2+z_{3y}-\dfrac{1}{2}z_{1y}^2-z_{1y}z_2-\dfrac{1}{3}z_{1y}^3+\cdots\}. \end{cases} \tag{9}$$

For log-ratio method, we get from the equation (3) and (9)

$$\begin{aligned} Q_{\log-ratio}&=20\lg\frac{RL}{TB}=\frac{80}{\ln 10}\left(z_2-\frac{z_{1x}^2-z_{1y}^2}{4}\right)+O\left(\frac{1}{b^4}\right) \\ &=\frac{160}{\ln 10}\frac{\sin\beta}{\beta}\left[\frac{\sigma_x^2-\sigma_y^2}{b^2}+\left(1-\frac{\tan\beta/2}{\beta}\right)\frac{\tilde{x}^2-\tilde{y}^2}{b^2}\right]+O\left(\frac{1}{b^4}\right) \end{aligned} \tag{10}$$

In the same way, from the equation (3) and (4), while omit 3[rd] order, we approximately get

$$\begin{cases} 20\lg\dfrac{R}{L}\approx\dfrac{160}{\ln 10}\dfrac{\sin(\beta/2)}{\beta}\dfrac{\tilde{x}}{b}+O\left(\dfrac{1}{b^3}\right), \\ 20\lg\dfrac{T}{B}\approx\dfrac{160}{\ln 10}\dfrac{\sin(\beta/2)}{\beta}\dfrac{\tilde{y}}{b}. \end{cases} \tag{11}$$

We can solve $\tilde{x}$, $\tilde{y}$ from equation (11), and then take the result into equation (10), we got the second moment $\sigma_x^2-\sigma_y^2$.

The sensitivity of log-ratio is:

$$S_{\log-ratio}=\frac{160}{\ln 10}\frac{\sin\beta}{\beta}\frac{1}{b^2}. \tag{12}$$

Comparing the log-ratio method with the Δ/Σ method, one can find

$$\frac{S_{\log-ratio}}{S_{\Delta/\Sigma}}=\frac{80}{\ln 10}=34.74. \tag{13}$$

So, log-ratio method has higher sensitivity than Δ/Σ method.

C. Combined method

Above the Δ/Σ method and the log-ratio method, because the item $\tilde{x}^2 - \tilde{y}^2$ is picking up using the approximate equations (5) and (11), the beam position will affect the second moment $\sigma_x^2 - \sigma_y^2$.

In the equation (10), because the item $1-\frac{\tan\beta/2}{\beta}$<0.5, it is little that the beam position affect the second moment $\sigma_x^2 - \sigma_y^2$ for the log-ratio method.

To decrease this effect farther, a new combined method is put forward.

According to the equation (5) and (10), we get

$$\sigma_x^2 - \sigma_y^2 = \frac{\beta}{\tan\beta/2}\frac{Q_{\log-ratio}}{S_{\log-ratio}} - \left(\frac{\beta}{\tan\beta/2} - 1\right)\frac{Q_{\Delta/\Sigma}}{S_{\Delta/\Sigma}}. \qquad (14)$$

Because this method combines the Δ/Σ method and the log-ratio method. According to the equation (14), the second moment $\sigma_x^2 - \sigma_y^2$ could be directly got. Theoretically, the combined method has more precision.

When $\beta = 60^0$, then

$$\sigma_x^2 - \sigma_y^2 = 1.8138\frac{Q_{\log-ratio}}{S_{\log-ratio}} - 0.8138\frac{Q_{\Delta/\Sigma}}{S_{\Delta/\Sigma}}. \qquad (15)$$

D. Numerical Results for different methods

To compare the difference among the three methods, some numerical calculations are made.

Fig.2 and Fig.3 show the three methods' nonlinear response. The horizon axis is ratio of the second moment and distance between electrodes and BPM center $(\sigma_x^2 - \sigma_y^2)/b^2$ which we set σ_x in simulation, and the vertical axis is the pick up results from different methods: Δ/Σ method (black), log-ratio method (red), and the combined method (blue). Obviously, the Δ/Σ method has the worst nonlinear response, and the combined method has the best linear response.

Fig.4 shows the beam position effect on the pick up second moment for different methods. Here, κ is the ratio of the pick up second moment and the setting second moment. Obviously, the combined method's result is closest to the value which we set in simulation for large range.

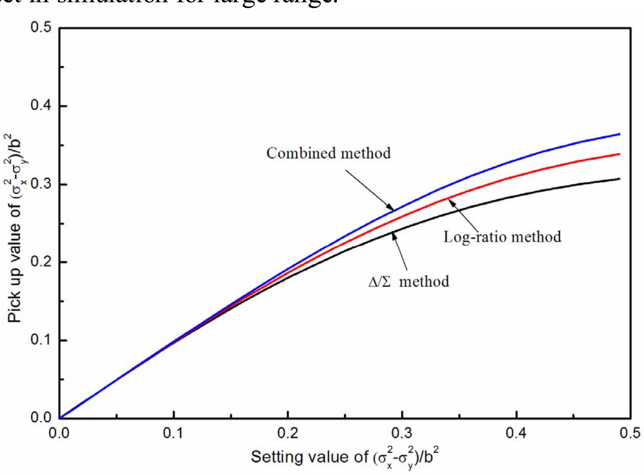

Fig .2. Different method's nonlinear response when $\tilde{x}/b$=0.1, $\tilde{y}$=0.

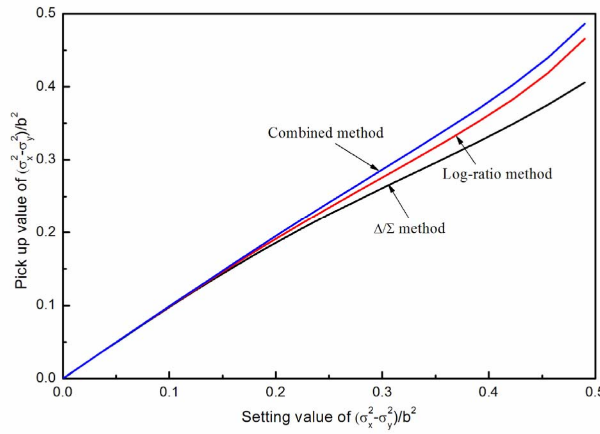

Fig. 3. Different method's nonlinear response when $\tilde{x}$=0, $\tilde{y}/b$=0.1.

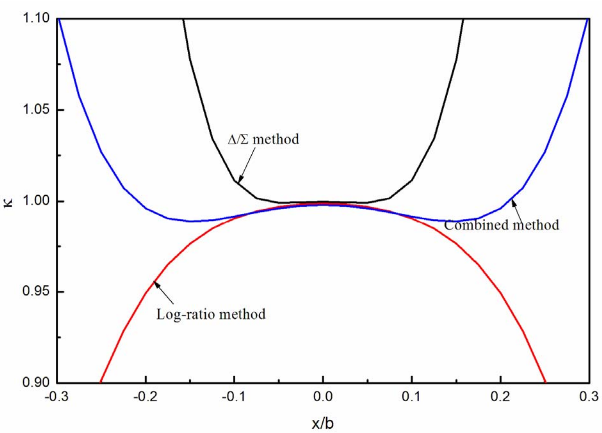

Fig. 4. The beam position effect on the pick up second moment for different methods when σ_x/b=0.1, σ_y=0.

IV. Emittance Measurement

Since we have picked up the second moment $\sigma_x^2 - \sigma_y^2$ with the three methods above, we can start to measure emittance [1].

$$\begin{aligned} Q_b &= \langle \sigma_x^2 - \sigma_y^2 \rangle_b = \sigma_{11b} - \sigma_{33b} \\ &= R_{11}^2\sigma_{11f} + 2R_{11}R_{12}\sigma_{12f} + R_{12}^2\sigma_{22f} \\ &\quad - R_{33}^2\sigma_{33f} - 2R_{33}R_{34}\sigma_{34f} - R_{34}^2\sigma_{44f}. \end{aligned} \qquad (16)$$

Where, the b subscript refers to the BPM location and the f subscript to a point upstream from BPM.

The constants R_{jk} are from the transfer matrix between the upstream point and the BPM:

$$M_f^b = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 \\ 0 & 0 & R_{33} & R_{34} \\ 0 & 0 & R_{43} & R_{44} \end{bmatrix}. \qquad (17)$$

Changing this transfer matrix m times (m≥6), and measuring σ_{11b}-σ_{33b} for each change, we can solve the vector $<\sigma_{11f}, \sigma_{12f}, \sigma_{22f}, \sigma_{33f}, \sigma_{34f}, \sigma_{44f}>^T$ in the least square method. In turn, it's easy to calculate the emittance in the x and y directions using the following formula:

$$\varepsilon_x = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}, \quad \varepsilon_y = \sqrt{\sigma_{33}\sigma_{44} - \sigma_{34}^2}.$$
(18)

In reference [4], S. J. Russell points out that how accurate these estimates will be depends upon the stability of the matrix equation. The stability of any matrix equation is inherent and cannot be improved using clever data processing techniques. S. J. Russell found a stable implementation of Miller's measurement: a triplet quadrupoles followed by a BPM.

V. EXPERIMENT

In Hefei Light Source 200MeV Linac, stripline BPM and its signal processing system have been developed [7]. The stripline BPM structure is 60^0 angle width and 183.8mm long. The BPM signal processing system consists of a front end module, a logarithmic detector module, a signal acquisition module and a software module. The signal processing system has a working frequency of 2856MHz, a processing bandwith of 10MHz, a detector dynamic range of 40dB.

We picked up the beam centroid position and the second moment $\sigma_x^2 – \sigma_y^2$ with the methods above successfully. The experiment results of beam's second moment $\sigma_x^2 – \sigma_y^2$ are shown in Fig.5. Here, QB7 and QB8 are the quadrupole magnets at upstream of BPM.

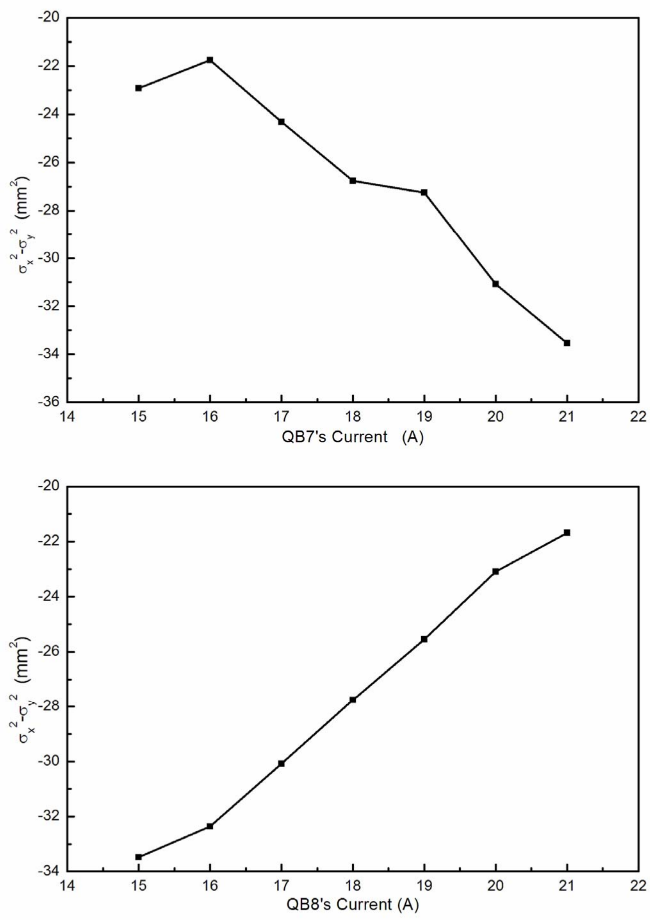

Fig.5. The second moment $\sigma_x^2 – \sigma_y^2$ vs quodrupole magnet's current.

In our experiment, due to the distance from the quadrupole magnets of QB7 and QB8 to BPM is too large, about 15m, the error is amplified strongly. And its matrix is close to singular, the matrix equation is very unstable. We failed to calculated emittance although we picked up the second moment successfully. Now we are developing a new BPM structure whose stripline length is 26.2mm (1/4 wavelength) and width angel is 45^0. This new BPM would be installed at a right position in next spring.

VI. CONCLUSION

Log-ratio method is an effective method to pick up the second moment of beam. It's better than Δ/Σ method on both sensitivity and nonlinear response aspects. Furthermore, we find a clever method—the combined method which needs no increasing of circuit's complex but has the best result.

It's better to change three or more quadrupole magnets' current to measure beam emittance. So that we can got the stable results.

REFERENCES

[1] R. H. Miller, J. E. Clendenin, et al, "Nonintercepting Emittance Monitor", *Proc. 12th Int. Conf. on High Energy Accelerators*, Fermilab, 1983, pp. 602-605.

[2] S. J. Russell and B. E. Caristen, "Measuring Emittance Using Beam Position Monitors", *Proceeding of the 1993 Particle Accelerator Conference*, 1993, pp. 2537-2539.

[3] A. Jansson, D.J. Williams, "A new optimised quadrupole pick-up design using magnetic coupling", *Nuclear Instrument and Methods in Physics Research A* 479, 2002, pp.233-242.

[4] R. Assmann, B. Dehning, J. Matheson, "Use of Movable Beam Position Monitors for Beam Size Measurements", *Proceeding of EPAC 2000*, Vienna, Austria, 2000, pp. 1693-1695.

[5] S. J. Russell, "Unstable Matrix Equations and Their Relationship to Measuring the Emittance of an Electron Beam Using Beam Position Monitors", *Nuclear Instrument and Methods in Physics Research A* 430, 1999, pp.498-506.

[6] S. J. Russell, "Emittance Measurements of the Sub-Picosecond Accelerator Electron Beam Using Beam Position Monitors", *Review of Scientific Instruments*, Vol.70, No.2, pp.1362-1366, Feb. 1999.

[7] J. H. Li, B. G. Sun, et al, "Research and Measurement of BPM System at HLS LINAC", *High Energy Physics and Nuclear Physics*, vol.31, no.1, pp.96-100, Jan. 2007.