Chapter 2

Electrostatics

Problem 2.1

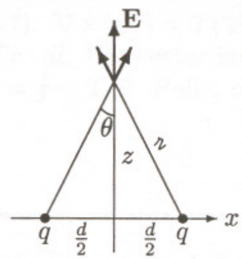
(a) $\boxed{\text{Zero.}}$

(b) $\boxed{F = \dfrac{1}{4\pi\epsilon_0}\dfrac{qQ}{r^2}}$, where r is the distance from center to each numeral. $\mathbf{F}$ points *toward* the missing q. *Explanation:* by superposition, this is equivalent to (a), with an extra $-q$ at 6 o'clock—since the force of all twelve is zero, the net force is that of $-q$ only.

(c) $\boxed{\text{Zero.}}$

(d) $\boxed{\dfrac{1}{4\pi\epsilon_0}\dfrac{qQ}{r^2}}$, pointing toward the missing q. Same reason as (b). Note, however, that if you explained (b) as a cancellation in pairs of opposite charges (1 o'clock against 7 o'clock; 2 against 8, etc.), with one unpaired q doing the job, then you'll need a *different* explanation for (d).

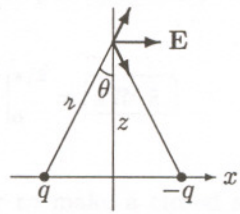
Problem 2.2



(a) "Horizontal" components cancel. Net vertical field is: $E_z = \frac{1}{4\pi\epsilon_0} 2 \frac{q}{\mathscr{r}^2}\cos\theta$.

Here $\mathscr{r}^2 = z^2 + \left(\frac{d}{2}\right)^2$; $\cos\theta = \frac{z}{\mathscr{r}}$, so $\boxed{\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{2qz}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\,\hat{\mathbf{z}}.}$

When $z \gg d$ you're so far away it just looks like a single charge $2q$; the field should reduce to $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\,\hat{\mathbf{z}}$. And it *does* (just set $d \to 0$ in the formula).



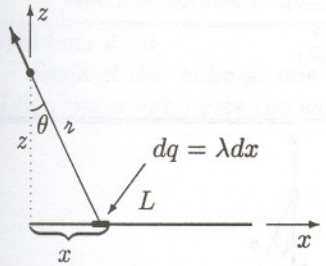
(b) This time the "vertical" components cancel, leaving

$\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2 \frac{q}{\mathscr{r}^2}\sin\theta\,\hat{\mathbf{x}}$, or

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qd}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\,\hat{\mathbf{x}}.}$$

From far away, $(z \gg d)$, the field goes like $\mathbf{E} \approx \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\,\hat{\mathbf{z}}$, which, as we shall see, is the field of a *dipole*. (If we set $d \to 0$, we get $\mathbf{E} = 0$, as is appropriate; to the extent that this configuration looks like a single point charge from far away, the net charge is zero, so $\mathbf{E} \to 0$.)

Problem 2.3



$$E_z = \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda\,dx}{\mathscr{r}^2}\cos\theta;\ (\mathscr{r}^2 = z^2 + x^2;\ \cos\theta = \tfrac{z}{\mathscr{r}})$$
$$= \frac{1}{4\pi\epsilon_0}\lambda z\int_0^L \frac{1}{(z^2+x^2)^{3/2}}dx$$
$$= \frac{1}{4\pi\epsilon_0}\lambda z\left[\frac{1}{z^2}\frac{x}{\sqrt{z^2+x^2}}\right]\Bigg|_0^L = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{z}\frac{L}{\sqrt{z^2+L^2}}.$$
$$E_x = -\frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda\,dx}{\mathscr{r}^2}\sin\theta = -\frac{1}{4\pi\epsilon_0}\lambda\int\frac{x\,dx}{(x^2+z^2)^{3/2}}$$
$$= -\frac{1}{4\pi\epsilon_0}\lambda\left[-\frac{1}{\sqrt{x^2+z^2}}\right]\Bigg|_0^L = -\frac{1}{4\pi\epsilon_0}\lambda\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+L^2}}\right].$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{z}\left[\left(-1+\frac{z}{\sqrt{z^2+L^2}}\right)\hat{\mathbf{x}} + \left(\frac{L}{\sqrt{z^2+L^2}}\right)\hat{\mathbf{z}}\right].}$$

For $z \gg L$ you expect it to look like a point charge $q = \lambda L$: $\mathbf{E} \to \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z^2}\hat{\mathbf{z}}$. It checks, for with $z \gg L$ the $\hat{\mathbf{x}}$ term $\to 0$, and the $\hat{\mathbf{z}}$ term $\to \frac{1}{4\pi\epsilon_0}\frac{\lambda}{z}\frac{L}{z}\hat{\mathbf{z}}$.

Problem 2.4

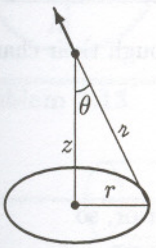
From Ex. 2.1, with $L \to \frac{a}{2}$ and $z \to \sqrt{z^2 + \left(\frac{a}{2}\right)^2}$ (distance from center of edge to P), field of *one* edge is:

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{\lambda a}{\sqrt{z^2+\frac{a^2}{4}}\sqrt{z^2+\frac{a^2}{4}+\frac{a^2}{4}}}.$$

There are 4 sides, and we want vertical components only, so multiply by $4\cos\theta = 4\frac{z}{\sqrt{z^2+\frac{a^2}{4}}}$:

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{4\lambda a z}{\left(z^2+\frac{a^2}{4}\right)\sqrt{z^2+\frac{a^2}{2}}}\hat{\mathbf{z}}.}$$

Problem 2.5



"Horizontal" components cancel, leaving: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{\int\frac{\lambda dl}{\mathscr{r}^2}\cos\theta\right\}\hat{\mathbf{z}}$.
Here, $\mathscr{r}^2 = r^2 + z^2$, $\cos\theta = \frac{z}{\mathscr{r}}$ (both constants), while $\int dl = 2\pi r$. So

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda(2\pi r)z}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}}.}$$

Problem 2.6

Break it into rings of radius r, and thickness dr, and use Prob. 2.5 to express the field of each ring. Total charge of a ring is $\sigma \cdot 2\pi r \cdot dr = \lambda \cdot 2\pi r$, so $\lambda = \sigma dr$ is the "line charge" of each ring.

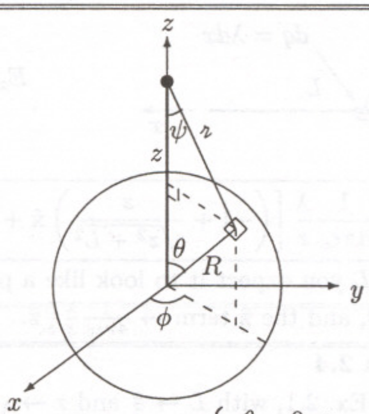
$$E_{\text{ring}} = \frac{1}{4\pi\epsilon_0}\frac{(\sigma dr)2\pi r z}{(r^2+z^2)^{3/2}};\quad E_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\int_0^R \frac{r}{(r^2+z^2)^{3/2}}dr.$$

$$\boxed{\mathbf{E}_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}.}$$

For $R \gg z$ the second term $\to 0$, so $\mathbf{E}_{\text{plane}} = \frac{1}{4\pi\epsilon_0} 2\pi\sigma\hat{\mathbf{z}} = \boxed{\dfrac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}.}$

For $z \gg R$, $\frac{1}{\sqrt{R^2+z^2}} = \frac{1}{z}\left(1 + \frac{R^2}{z^2}\right)^{-1/2} \approx \frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right)$, so $[\,] \approx \frac{1}{z} - \frac{1}{z} + \frac{1}{2}\frac{R^2}{z^3} = \frac{R^2}{2z^3}$,
and $E = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{2z^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}$, where $Q = \pi R^2 \sigma$. ✓

Problem 2.7



$\mathbf{E}$ is clearly in the z direction. From the diagram,

$dq = \sigma\, da = \sigma R^2 \sin\theta\, d\theta\, d\phi$,
$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta$,
$\cos\psi = \frac{z - R\cos\theta}{\mathscr{r}}$.

So

$$E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma R^2 \sin\theta\, d\theta\, d\phi (z - R\cos\theta)}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}. \qquad \int d\phi = 2\pi.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_0^\pi \frac{(z - R\cos\theta)\sin\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}\,d\theta. \quad \text{Let } u = \cos\theta;\ du = -\sin\theta\, d\theta; \left\{\begin{array}{l}\theta = 0 \Rightarrow u = +1 \\ \theta = \pi \Rightarrow u = -1\end{array}\right\}.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_{-1}^{1} \frac{z - Ru}{(R^2 + z^2 - 2Rzu)^{3/2}}\,du. \quad \text{Integral can be done by partial fractions—or look it up.}$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\left[\frac{1}{z^2}\frac{zu - R}{\sqrt{R^2 + z^2 - 2Rzu}}\right]_{-1}^{1} = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{z^2}\left\{\frac{(z-R)}{|z-R|} - \frac{(-z-R)}{|z+R|}\right\}.$$

For $z > R$ (outside the sphere), $E_z = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$, so $\boxed{\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{z^2}\hat{\mathbf{z}}.}$

For $z < R$ (inside), $E_z = 0$, so $\boxed{\mathbf{E} = 0.}$

Problem 2.8

According to Prob. 2.7, all shells *interior* to the point (i.e. at smaller r) contribute as though their charge were concentrated at the center, while all exterior shells contribute nothing. Therefore:

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{int}}}{r^2}\hat{\mathbf{r}},$$

where Q_{int} is the total charge interior to the point. *Outside* the sphere, *all* the charge is interior, so

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}.}$$

Inside the sphere, only that fraction of the total which is interior to the point counts:

$$Q_{\text{int}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}Q = \frac{r^3}{R^3}Q, \text{ so } \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{r^3}{R^3}Q\frac{1}{r^2}\hat{\mathbf{r}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}.}$$

Problem 2.9

(a) $\rho = \epsilon_0 \nabla\cdot\mathbf{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot kr^3\right) = \epsilon_0 \frac{1}{r^2}k(5r^4) = \boxed{5\epsilon_0 k r^2.}$

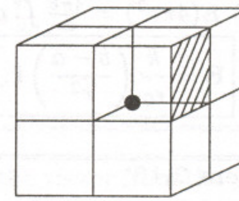
(b) *By Gauss's law:* $Q_{\text{enc}} = \epsilon_0 \oint \mathbf{E}\cdot d\mathbf{a} = \epsilon_0 (kR^3)(4\pi R^2) = \boxed{4\pi\epsilon_0 k R^5.}$

By direct integration: $Q_{\text{enc}} = \int \rho\, d\tau = \int_0^R (5\epsilon_0 k r^2)(4\pi r^2 dr) = 20\pi\epsilon_0 k \int_0^R r^4 dr = 4\pi\epsilon_0 k R^5.\checkmark$

Problem 2.10

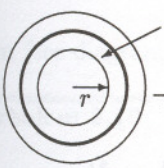
Think of this cube as one of 8 surrounding the charge. Each of the 24 squares which make up the surface of this larger cube gets the same flux as every other one, so:

$$\int\limits_{\substack{\text{one}\\\text{face}}} \mathbf{E}\cdot d\mathbf{a} = \frac{1}{24} \int\limits_{\substack{\text{whole}\\\text{large}\\\text{cube}}} \mathbf{E}\cdot d\mathbf{a}.$$



The latter is $\frac{1}{\epsilon_0}q$, by Gauss's law. Therefore $\boxed{\int\limits_{\substack{\text{one}\\\text{face}}} \mathbf{E}\cdot d\mathbf{a} = \frac{q}{24\epsilon_0}.}$

Problem 2.11

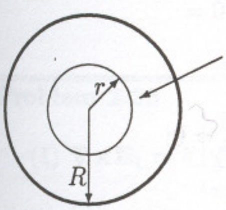


Gaussian surface: *Inside*: $\oint \mathbf{E}\cdot d\mathbf{a} = E(4\pi r^2) = \frac{1}{\epsilon_0}Q_{\text{enc}} = 0 \Rightarrow \boxed{\mathbf{E} = 0.}$

Gaussian surface: *Outside*: $E(4\pi r^2) = \frac{1}{\epsilon_0}(\sigma 4\pi R^2) \Rightarrow \boxed{\mathbf{E} = \frac{\sigma R^2}{\epsilon_0 r^2}\,\hat{\mathbf{r}}.}$

(As in Prob. 2.7.)

Problem 2.12

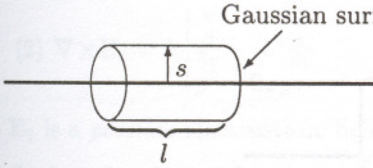


$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 4\pi r^2 = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3 \rho.$ So

$$\boxed{\mathbf{E} = \frac{1}{3\epsilon_0}\rho r\hat{\mathbf{r}}.}$$

Since $Q_{\text{tot}} = \frac{4}{3}\pi R^2 \rho$, $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\hat{\mathbf{r}}$ (as in Prob. 2.8).

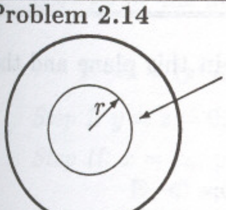
Problem 2.13



$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\lambda l.$ So

$\boxed{\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}}$ (same as Ex. 2.1).

Problem 2.14



$$\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 4\pi r^2 = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\int \rho\, d\tau = \frac{1}{\epsilon_0}\int (k\bar{r})(\bar{r}^2 \sin\theta\, d\bar{r}\, d\theta\, d\phi)$$
$$= \frac{1}{\epsilon_0}k\,4\pi \int_0^r \bar{r}^3 d\bar{r} = \frac{4\pi k}{\epsilon_0}\frac{r^4}{4} = \frac{\pi k}{\epsilon_0}r^4.$$

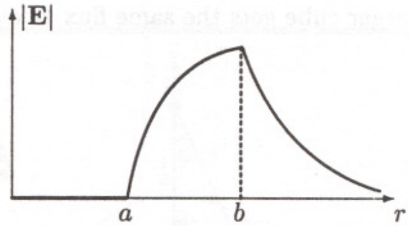
$$\therefore \boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\pi k r^2 \hat{\mathbf{r}}.}$$

Problem 2.15

(i) $Q_{\text{enc}} = 0$, so $\boxed{\mathbf{E} = 0.}$

(ii) $\oint \mathbf{E} \cdot d\mathbf{a} = E(4\pi r^2) = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \int \rho \, d\tau = \frac{1}{\epsilon_0} \int \frac{k}{\bar{r}^2} \bar{r}^2 \sin\theta \, d\bar{r} \, d\theta \, d \; phi$

$= \frac{4\pi k}{\epsilon_0} \int_a^r d\bar{r} = \frac{4\pi k}{\epsilon_0}(r - a) \therefore \boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{r-a}{r^2}\right)\hat{\mathbf{r}}.}$

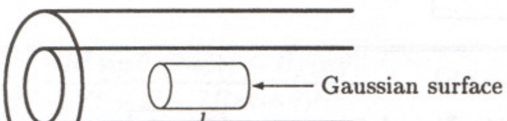


(iii) $E(4\pi r^2) = \frac{4\pi k}{\epsilon_0} \int_a^b d\bar{r} = \frac{4\pi k}{\epsilon_0}(b - a)$, so

$$\boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{b-a}{r^2}\right)\hat{\mathbf{r}}.}$$

Problem 2.16

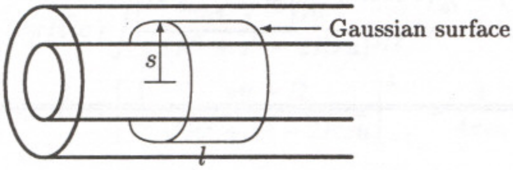
(i)



$\oint \mathbf{E} \cdot d\mathbf{a} = E \cdot 2\pi s \cdot l = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \rho \pi s^2 l;$

$$\boxed{\mathbf{E} = \frac{\rho s}{2\epsilon_0}\hat{\mathbf{s}}.}$$

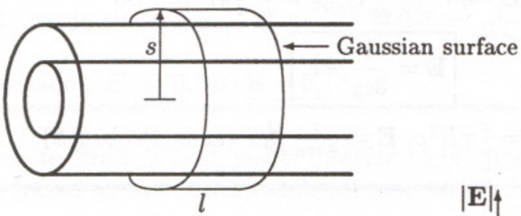
(ii)



$\oint \mathbf{E} \cdot d\mathbf{a} = E \cdot 2\pi s \cdot l = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \rho \pi a^2 l;$

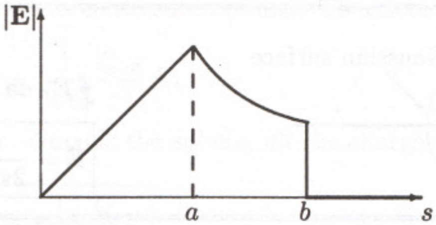
$$\boxed{\mathbf{E} = \frac{\rho a^2}{2\epsilon_0 s}\hat{\mathbf{s}}.}$$

(iii)



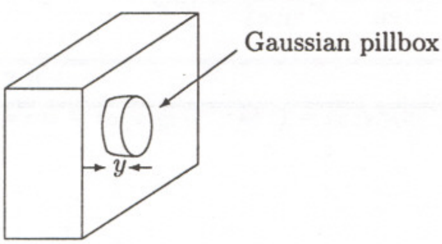
$\oint \mathbf{E} \cdot d\mathbf{a} = E \cdot 2\pi s \cdot l = \frac{1}{\epsilon_0} Q_{\text{enc}} = 0;$

$\boxed{\mathbf{E} = 0.}$



Problem 2.17

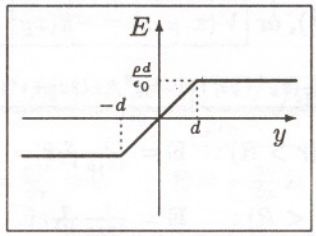
On the $x\,z$ plane $E = 0$ by symmetry. Set up a Gaussian "pillbox" with one face in this plane and the other at y.



$\int \mathbf{E} \cdot d\mathbf{a} = E \cdot A = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} A y \rho;$

$$\boxed{\mathbf{E} = \frac{\rho}{\epsilon_0} y \, \hat{\mathbf{y}}} \text{ (for } |y| < d).$$

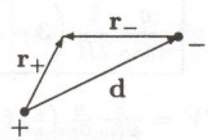
$Q_{\text{enc}} = \frac{1}{\epsilon_0} A d \rho \Rightarrow \boxed{\mathbf{E} = \frac{\rho}{\epsilon_0} d\,\hat{\mathbf{y}}}$ (for $y > d$).



Problem 2.18

From Prob. 2.12, the field inside the positive sphere is $\mathbf{E}_+ = \frac{\rho}{3\epsilon_0}\mathbf{r}_+$, where $\mathbf{r}_+$ is the vector from the positive center to the point in question. Likewise, the field of the negative sphere is $-\frac{\rho}{3\epsilon_0}\mathbf{r}_-$. So the *total* field is

$$\mathbf{E} = \frac{\rho}{3\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_-)$$



But (see diagram) $\mathbf{r}_+ - \mathbf{r}_- = \mathbf{d}$. So $\boxed{\mathbf{E} = \frac{\rho}{3\epsilon_0}\mathbf{d}.}$

Problem 2.19

$$\nabla\times\mathbf{E} = \frac{1}{4\pi\epsilon_0}\nabla\times\int\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho\,d\tau = \frac{1}{4\pi\epsilon_0}\int\left[\nabla\times\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\right]\rho\,d\tau \quad \text{(since } \rho \text{ depends on } \mathbf{r}', \text{ not } \mathbf{r})$$

$$= 0 \quad \left(\text{since } \nabla\times\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 0, \text{ from Prob. 1.62}\right).$$

Problem 2.20

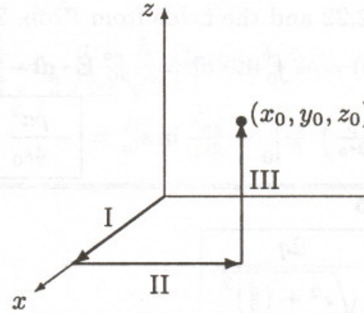
$$(1)\ \nabla\times\mathbf{E}_1 = k\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & 2yz & 3zx\end{vmatrix} = k\left[\hat{\mathbf{x}}(0-2y) + \hat{\mathbf{y}}(0-3z) + \hat{\mathbf{z}}(0-x)\right] \neq 0,$$

so $\mathbf{E}_1$ is an *impossible* electrostatic field.

$$(2)\ \nabla\times\mathbf{E}_2 = k\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2 & 2xy+z^2 & 2yz\end{vmatrix} = k\left[\hat{\mathbf{x}}(2z-2z) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(2y-2y)\right] = 0,$$

so $\mathbf{E}_2$ is a *possible* electrostatic field.

Let's go by the indicated path:



$\mathbf{E}\cdot d\mathbf{l} = (y^2\,dx + (2xy + z^2)dy + 2yz\,dz)k$

Step I: $y = z = 0$; $dy = dz = 0$. $\mathbf{E}\cdot d\mathbf{l} = ky^2\,dx = 0$.

Step II: $x = x_0$, $y: 0 \to y_0$, $z = 0$. $dx = dz = 0$.

$\mathbf{E}\cdot d\mathbf{l} = k(2xy + z^2)dy = 2kx_0y\,dy$.

$\int_{II}\mathbf{E}\cdot d\mathbf{l} = 2kx_0\int_0^{y_0} y\,dy = kx_0y_0^2$.

Step III: $x = x_0$, $y = y_0$, $z: 0 \to z_0$; $dx = dy = 0$.

$\mathbf{E}\cdot d\mathbf{l} = 2kyz\,dz = 2ky_0z\,dz.$

$\int_{III}\mathbf{E}\cdot d\mathbf{l} = 2y_0k\int_0^{z_0} z\,dz = ky_0z_0^2.$

$$V(x_0,y_0,z_0) = -\int_0^{(x_0,y_0,z_0)}\mathbf{E}\cdot d\mathbf{l} = -k(x_0y_0^2 + y_0z_0^2), \text{ or } \boxed{V(x,y,z) = -k(xy^2+yz^2).}$$

Check: $-\nabla V = k[\frac{\partial}{\partial x}(xy^2+yz^2)\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}(xy^2+yz^2)\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}(xy^2+yz^2)\,\hat{\mathbf{z}}] = k[y^2\,\hat{\mathbf{x}} + (2xy+z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}] = \mathbf{E}.$ ✓

Problem 2.21

$$V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l}. \quad \begin{cases} \text{Outside the sphere } (r>R): & \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \\ \text{Inside the sphere } (r<R): & \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}. \end{cases}$$

So for $r>R$: $V(r) = -\int_\infty^r\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar{r}^2}\right)d\bar{r} = \frac{1}{4\pi\epsilon_0}q\left(\frac{1}{\bar{r}}\right)\Big|_\infty^r = \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{r},}$

and for $r<R$: $V(r) = -\int_\infty^R\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar{r}^2}\right)d\bar{r} - \int_R^r\left(\frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\bar{r}\right)d\bar{r} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2-R^2}{2}\right)\right]$

$$= \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3-\frac{r^2}{R^2}\right).}$$

When $r>R$, $\nabla V = \frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\mathbf{r}} = -\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$, so $\mathbf{E} = -\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}.$ ✓

When $r<R$, $\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\frac{\partial}{\partial r}\left(3-\frac{r^2}{R^2}\right)\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(-\frac{2r}{R^2}\right)\hat{\mathbf{r}} = -\frac{q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{\mathbf{r}}$; so $\mathbf{E} = -\nabla V = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}.$ ✓

Problem 2.22

$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}\hat{\mathbf{s}}$ (Prob. 2.13). In this case we cannot set the reference point at ∞, since the charge itself extends to ∞. Let's set it at $s=a$. Then

$$V(s) = -\int_a^s\left(\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{\bar{s}}\right)d\bar{s} = \boxed{-\frac{1}{4\pi\epsilon_0}2\lambda\ln\left(\frac{s}{a}\right).}$$

(In this form it is clear why $a=\infty$ would be no good—likewise the other "natural" point, $a=0$.)

$\nabla V = -\frac{1}{4\pi\epsilon_0}2\lambda\frac{\partial}{\partial s}\left(\ln\left(\frac{s}{a}\right)\right)\hat{\mathbf{s}} = -\frac{1}{4\pi\epsilon_0}2\lambda\frac{1}{s}\hat{\mathbf{s}} = -\mathbf{E}.$ ✓

Problem 2.23

$$V(0) = -\int_\infty^0\mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b\left(\frac{k}{\epsilon_0}\frac{(b-a)}{r^2}\right)dr - \int_b^a\left(\frac{k}{\epsilon_0}\frac{(r-a)}{r^2}\right)dr - \int_a^0(0)dr = \frac{k}{\epsilon_0}\frac{(b-a)}{b} - \frac{k}{\epsilon_0}\left(\ln\left(\frac{a}{b}\right) + a\left(\frac{1}{a}-\frac{1}{b}\right)\right)$$

$$= \frac{k}{\epsilon_0}\left\{1 - \frac{a}{b} - \ln\left(\frac{a}{b}\right) - 1 + \frac{a}{b}\right\} = \boxed{\frac{k}{\epsilon_0}\ln\left(\frac{b}{a}\right).}$$

Problem 2.24

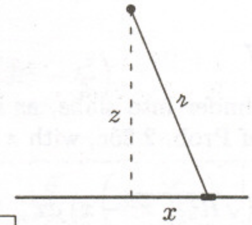
Using Eq. 2.22 and the fields from Prob. 2.16:

$$V(b) - V(0) = -\int_0^b\mathbf{E}\cdot d\mathbf{l} = -\int_0^a\mathbf{E}\cdot d\mathbf{l} - \int_a^b\mathbf{E}\cdot d\mathbf{l} = -\frac{\rho}{2\epsilon_0}\int_0^a s\,ds - \frac{\rho a^2}{2\epsilon_0}\int_a^b\frac{1}{s}ds$$

$$= -\left(\frac{\rho}{2\epsilon_0}\right)\frac{s^2}{2}\Big|_0^a + \frac{\rho a^2}{2\epsilon_0}\ln s\Big|_a^b = \boxed{-\frac{\rho a^2}{4\epsilon_0}\left(1+2\ln\left(\frac{b}{a}\right)\right).}$$

Problem 2.25

(a) $\boxed{V = \frac{1}{4\pi\epsilon_0}\frac{2q}{\sqrt{z^2+\left(\frac{d}{2}\right)^2}}.}$



(b) $V = \frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\frac{\lambda\,dx}{\sqrt{z^2+x^2}} = \frac{\lambda}{4\pi\epsilon_0}\ln(x+\sqrt{z^2+x^2})\Big|_{-L}^{L}$

$$= \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{L+\sqrt{z^2+L^2}}{-L+\sqrt{z^2+L^2}}\right]} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{L+\sqrt{z^2+L^2}}{z}\right).$$

(c) $V = \frac{1}{4\pi\epsilon_0}\int_0^R\frac{\sigma\,2\pi r\,dr}{\sqrt{r^2+z^2}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma\left(\sqrt{r^2+z^2}\right)\Big|_0^R = \boxed{\frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2}-z\right).}$

In each case, by symmetry $\frac{\partial V}{\partial y} = \frac{\partial V}{\partial x} = 0$. $\therefore \mathbf{E} = -\frac{\partial V}{\partial z}\hat{\mathbf{z}}$.

(a) $\mathbf{E} = -\frac{1}{4\pi\epsilon_0}2q\left(-\frac{1}{2}\right)\frac{2z}{\left(z^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}\hat{\mathbf{z}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{2qz}{\left(z^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}\hat{\mathbf{z}}}$ (agrees with Prob. 2.2a).

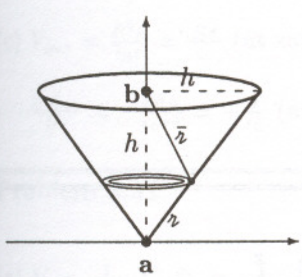
(b) $\mathbf{E} = -\frac{\lambda}{4\pi\epsilon_0}\left\{\frac{1}{(L+\sqrt{z^2+L^2})}\frac{1}{2}\frac{1}{\sqrt{z^2+L^2}}2z - \frac{1}{(-L+\sqrt{z^2+L^2})}\frac{1}{2}\frac{1}{\sqrt{z^2+L^2}}2z\right\}\hat{\mathbf{z}}$

$= -\frac{\lambda}{4\pi\epsilon_0}\frac{z}{\sqrt{z^2+L^2}}\left\{\frac{-L+\sqrt{z^2+L^2}-L-\sqrt{z^2+L^2}}{(z^2+L^2)-L^2}\right\}\hat{\mathbf{z}} = \boxed{\frac{2L\lambda}{4\pi\epsilon_0}\frac{1}{z\sqrt{z^2+L^2}}\hat{\mathbf{z}}}$ (agrees with Ex. 2.1).

(c) $\mathbf{E} = -\frac{\sigma}{2\epsilon_0}\left\{\frac{1}{2}\frac{1}{\sqrt{R^2+z^2}}2z-1\right\}\hat{\mathbf{z}} = \boxed{\frac{\sigma}{2\epsilon_0}\left[1-\frac{z}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}}$ (agrees with Prob. 2.6).

If the right-hand charge in (a) is $-q$, then $\boxed{V=0}$, which, naively, suggests $\mathbf{E} = -\nabla V = 0$, in contradiction with the answer to Prob. 2.2b. The point is that we only know V *on the z axis*, and from this we cannot hope to compute $E_x = -\frac{\partial V}{\partial x}$ or $E_y = -\frac{\partial V}{\partial y}$. That was OK in part (a), because we knew from symmetry that $E_x = E_y = 0$. But now $\mathbf{E}$ points in the x direction, so knowing V on the z axis is insufficient to determine $\mathbf{E}$.

Problem 2.26



$$V(\mathbf{a}) = \frac{1}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi r}{\eta}\right)d\eta = \frac{2\pi\sigma}{4\pi\epsilon_0}\frac{1}{\sqrt{2}}(\sqrt{2}h) = \frac{\sigma h}{2\epsilon_0}.$$

(where $r = \eta/\sqrt{2}$)

$$V(\mathbf{b}) = \frac{1}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi r}{\mathscr{r}}\right)d\eta, \quad \text{where } \mathscr{r} = \sqrt{h^2+\eta^2-\sqrt{2}h\eta}.$$

$$= \frac{2\pi\sigma}{4\pi\epsilon_0}\frac{1}{\sqrt{2}}\int_0^{\sqrt{2}h}\frac{\eta}{\sqrt{h^2+\eta^2-\sqrt{2}h\eta}}d\eta$$

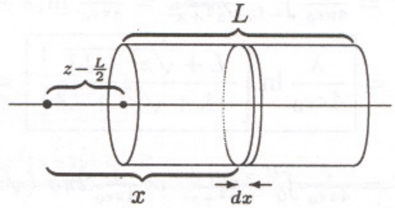
$$= \frac{\sigma}{2\sqrt{2}\epsilon_0}\left[\sqrt{h^2+\eta^2-\sqrt{2}h\eta}+\frac{h}{\sqrt{2}}\ln(2\sqrt{h^2+\eta^2-\sqrt{2}h\eta}+2\eta-\sqrt{2}h)\right]_0^{\sqrt{2}h}$$

$$= \frac{\sigma}{2\sqrt{2}\epsilon_0}\left[h+\frac{h}{\sqrt{2}}\ln(2h+2\sqrt{2}h-\sqrt{2}h)-h-\frac{h}{\sqrt{2}}\ln(2h-\sqrt{2}h)\right] = \frac{\sigma}{2\sqrt{2}\epsilon_0}\frac{h}{\sqrt{2}}\left[\ln(2h+\sqrt{2}h)-\ln(2h-\sqrt{2}h)\right]$$

$$= \frac{\sigma h}{4\epsilon_0}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) = \frac{\sigma h}{4\epsilon_0}\ln\left(\frac{(2+\sqrt{2})^2}{2}\right) = \frac{\sigma h}{2\epsilon_0}\ln(1+\sqrt{2}).$$

$$\therefore \boxed{V(\mathbf{a})-V(\mathbf{b}) = \frac{\sigma h}{2\epsilon_0}\left[1-\ln(1+\sqrt{2})\right].}$$

Problem 2.27



Cut the cylinder into slabs, as shown in the figure, and use result of Prob. 2.25c, with $z \to x$ and $\sigma \to \rho\, dx$:

$$V = \frac{\rho}{2\epsilon_0} \int\limits_{z-L/2}^{z+L/2} \left(\sqrt{R^2+x^2} - x\right) dx$$

$$= \frac{\rho}{2\epsilon_0}\frac{1}{2}\left[x\sqrt{R^2+x^2} + R^2 \ln\left(x+\sqrt{R^2+x^2}\right) - x^2\right]\Big|_{z-L/2}^{z+L/2}$$

$$= \boxed{\frac{\rho}{4\epsilon_0}\left\{\left(z+\tfrac{L}{2}\right)\sqrt{R^2+\left(z+\tfrac{L}{2}\right)^2} - \left(z-\tfrac{L}{2}\right)\sqrt{R^2+\left(z-\tfrac{L}{2}\right)^2} + R^2 \ln\left[\frac{z+\frac{L}{2}+\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}{z-\frac{L}{2}+\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}\right] - 2zL\right\}.}$$

(*Note:* $-\left(z+\frac{L}{2}\right)^2 + \left(z-\frac{L}{2}\right)^2 = -z^2 - zL - \frac{L^2}{4} + z^2 - zL + \frac{L^2}{4} = -2zL.$)

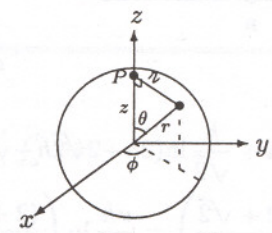
$$\mathbf{E} = -\nabla V = -\hat{\mathbf{z}}\frac{\partial V}{\partial z} = -\frac{\hat{\mathbf{z}}\rho}{4\epsilon_0}\left\{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2} + \frac{\left(z+\frac{L}{2}\right)^2}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}} - \sqrt{R^2+\left(z-\frac{L}{2}\right)^2} - \frac{\left(z-\frac{L}{2}\right)^2}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}\right.$$

$$\left. + R^2 \underbrace{\left[\frac{1+\frac{z+\frac{L}{2}}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}}{z+\frac{L}{2}+\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}} - \frac{1+\frac{z-\frac{L}{2}}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}}{z-\frac{L}{2}+\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}\right]}_{\frac{1}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}} - \frac{1}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}} - 2L\right\}$$

$$\mathbf{E} = -\frac{\hat{\mathbf{z}}\rho}{4\epsilon_0}\left\{2\sqrt{R^2+\left(z+\frac{L}{2}\right)^2} - 2\sqrt{R^2+\left(z-\frac{L}{2}\right)^2} - 2L\right\}$$

$$= \boxed{\frac{\rho}{2\epsilon_0}\left[L - \sqrt{R^2+\left(z+\frac{L}{2}\right)^2} + \sqrt{R^2+\left(z-\frac{L}{2}\right)^2}\right]\hat{\mathbf{z}}.}$$

Problem 2.28



Orient axes so P is on z axis.

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\eta}\, d\tau. \quad \left\{\begin{array}{l}\text{Here } \rho \text{ is constant, } d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi, \\ \eta = \sqrt{z^2+r^2-2rz\cos\theta}.\end{array}\right.$$

$$V = \frac{\rho}{4\pi\epsilon_0}\int \frac{r^2 \sin\theta\, dr\, d\theta\, d\phi}{\sqrt{z^2+r^2-2rz\cos\theta}}\ ; \ \int_0^{2\pi} d\phi = 2\pi.$$

$$\int_0^{\pi} \frac{\sin\theta}{\sqrt{z^2+r^2-2rz\cos\theta}}\, d\theta = \frac{1}{rz}\left(\sqrt{r^2+z^2-2rz\cos\theta}\right)\Big|_0^{\pi} = \frac{1}{rz}\left(\sqrt{r^2+z^2+2rz} - \sqrt{r^2+z^2-2rz}\right)$$

$$= \frac{1}{rz}\left(r+z-|r-z|\right) = \left\{\begin{array}{l} 2/z\ , \text{ if } r < z, \\ 2/r\ , \text{ if } r > z. \end{array}\right\}$$

$$\therefore V = \frac{\rho}{4\pi\epsilon_0}\cdot 2\pi\cdot 2\left\{\int_0^z \frac{1}{z}r^2dr + \int_z^R \frac{1}{r}r^2dr\right\} = \frac{\rho}{\epsilon_0}\left\{\frac{1}{z}\frac{z^3}{3} + \frac{R^2-z^2}{2}\right\} = \frac{\rho}{2\epsilon_0}\left(R^2 - \frac{z^2}{3}\right).$$

But $\rho = \frac{q}{\frac{4}{3}\pi R^3}$, so $V(z) = \frac{1}{2\epsilon_0}\frac{3q}{4\pi R^3}\left(R^2 - \frac{z^2}{3}\right) = \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{z^2}{R^2}\right)$; $\boxed{V(r) = \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right).}$ ✓

Problem 2.29

$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\nabla^2\int\left(\frac{\rho}{\mathscr{r}}\right)d\tau = \frac{1}{4\pi\epsilon_0}\int\rho(\mathbf{r}')\left(\nabla^2\frac{1}{\mathscr{r}}\right)d\tau$ (since ρ is a function of $\mathbf{r}'$, not $\mathbf{r}$)

$= \frac{1}{4\pi\epsilon_0}\int\rho(\mathbf{r}')[-4\pi\delta^3(\mathbf{r}-\mathbf{r}')]\,d\tau = -\frac{1}{\epsilon_0}\rho(\mathbf{r})$. ✓

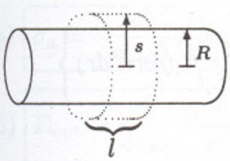
Problem 2.30.

(a) Ex. 2.4: $\mathbf{E}_{\text{above}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$; $\mathbf{E}_{\text{below}} = -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$ ($\hat{\mathbf{n}}$ always pointing *up*); $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$. ✓

Ex. 2.5: At each surface, $E = 0$ one side and $E = \frac{\sigma}{\epsilon_0}$ other side, so $\Delta E = \frac{\sigma}{\epsilon_0}$. ✓

Prob. 2.11: $\mathbf{E}_{\text{out}} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{\mathbf{r}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{r}}$; $\mathbf{E}_{\text{in}} = 0$; so $\Delta\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{r}}$. ✓

(b)



Outside: $\oint\mathbf{E}\cdot d\mathbf{a} = E(2\pi s)l = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}(2\pi R)l \Rightarrow \mathbf{E} = \frac{\sigma}{\epsilon_0}\frac{R}{s}\hat{\mathbf{s}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{s}}$ (at surface).

Inside: $Q_{\text{enc}} = 0$, so $\mathbf{E} = 0$. $\therefore\ \Delta\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{s}}$. ✓

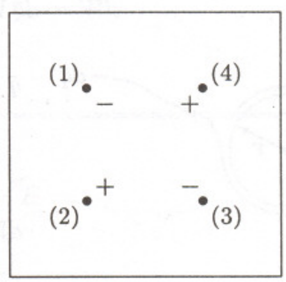
(c) $V_{\text{out}} = \frac{R^2\sigma}{\epsilon_0 r} = \frac{R\sigma}{\epsilon_0}$ (at surface); $V_{\text{in}} = \frac{R\sigma}{\epsilon_0}$; so $V_{\text{out}} = V_{\text{in}}$. ✓

$\frac{\partial V_{\text{out}}}{\partial r} = -\frac{R^2\sigma}{\epsilon_0 r^2} = -\frac{\sigma}{\epsilon_0}$ (at surface); $\frac{\partial V_{\text{in}}}{\partial r} = 0$; so $\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r} = -\frac{\sigma}{\epsilon_0}$. ✓

Problem 2.31

(a) $V = \frac{1}{4\pi\epsilon_0}\sum\frac{q_i}{r_{ij}} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q}{a} + \frac{q}{\sqrt{2}a} + \frac{-q}{a}\right\} = \frac{q}{4\pi\epsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right)$.

$\therefore W_4 = qV = \boxed{\frac{q^2}{4\pi\epsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right).}$



(b) $W_1 = 0$, $W_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{-q^2}{a}\right)$; $W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{\sqrt{2}\,a} - \frac{q^2}{a}\right)$; $W_4 =$ (see (a)).

$W_{\text{tot}} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\left\{-1 + \frac{1}{\sqrt{2}} - 1 - 2 + \frac{1}{\sqrt{2}}\right\} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{2q^2}{a}\left(-2 + \frac{1}{\sqrt{2}}\right).}$

Problem 2.32

(a) $W = \frac{1}{2}\int \rho V\,d\tau$. From Prob. 2.21 (or Prob. 2.28): $V = \frac{\rho}{2\epsilon_0}\left(R^2 - \frac{r^2}{3}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$

$$W = \frac{1}{2}\rho\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\int_0^R\left(3 - \frac{r^2}{R^2}\right)4\pi r^2 dr = \frac{q\rho}{4\epsilon_0 R}\left[3\frac{r^3}{3} - \frac{1}{R^2}\frac{r^5}{5}\right]\Big|_0^R = \frac{q\rho}{4\epsilon_0 R}\left(R^3 - \frac{R^3}{5}\right)$$

$$= \frac{q\rho}{5\epsilon_0}R^2 = \frac{qR^2}{5\epsilon_0}\frac{q}{\frac{4}{3}\pi R^3} = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right).}$$

(b) $W = \frac{\epsilon_0}{2}\int E^2 d\tau$. *Outside* $(r > R)$ $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$; *Inside* $(r < R)$ $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$.

$$\therefore W = \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}q^2\left\{\int_R^\infty \frac{1}{r^4}(r^2 4\pi\,dr) + \int_0^R\left(\frac{r}{R^3}\right)^2(4\pi r^2 dr)\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left\{\left(-\frac{1}{r}\right)\Big|_R^\infty + \frac{1}{R^6}\left(\frac{r^5}{5}\right)\Big|_0^R\right\} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left(\frac{1}{R} + \frac{1}{5R}\right) = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{q^2}{R}.\checkmark$$

(c) $W = \frac{\epsilon_0}{2}\left\{\oint_{\mathcal{S}} V\mathbf{E}\cdot d\mathbf{a} + \int_{\mathcal{V}} E^2 d\tau\right\}$, where $\mathcal{V}$ is large enough to enclose all the charge, but otherwise arbitrary. Let's use a sphere of radius $a > R$. Here $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$.

$$W = \frac{\epsilon_0}{2}\left\{\int_{r=a}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right)\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)r^2\sin\theta\,d\theta\,d\phi + \int_0^R E^2 d\tau + \int_R^a\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)^2(4\pi r^2 dr)\right\}$$

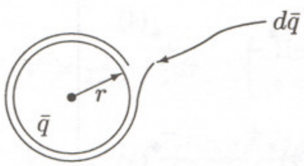
$$= \frac{\epsilon_0}{2}\left\{\frac{q^2}{(4\pi\epsilon_0)^2}\frac{1}{a}4\pi + \frac{q^2}{(4\pi\epsilon_0)^2}\frac{4\pi}{5R} + \frac{1}{(4\pi\epsilon_0)^2}4\pi q^2\left(-\frac{1}{r}\right)\Big|_R^a\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left\{\frac{1}{a} + \frac{1}{5R} - \frac{1}{a} + \frac{1}{R}\right\} = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{q^2}{R}.\checkmark$$

As $a \to \infty$, the contribution from the surface integral $\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\right)$ goes to zero, while the volume integral $\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\left(\frac{6a}{5R} - 1\right)\right)$ picks up the slack.

Problem 2.33

$$dW = d\bar{q}\,V = d\bar{q}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{\bar{q}}{r}, \quad (\bar{q} = \text{charge on sphere of radius } r).$$



$$\bar{q} = \frac{4}{3}\pi r^3\rho = q\frac{r^3}{R^3} \quad (q = \text{total charge on sphere}).$$

$$d\bar{q} = 4\pi r^2 dr\,\rho = \frac{4\pi r^2}{\frac{4}{3}\pi R^3}q\,dr = \frac{3q}{R^3}r^2 dr.$$

$$dW = \frac{1}{4\pi\epsilon_0}\left(\frac{qr^3}{R^3}\right)\frac{1}{r}\left(\frac{3q}{R^3}r^2 dr\right) = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{R^6}r^4 dr$$

$$W = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{R^6}\int_0^R r^4 dr = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{R^6}\frac{R^5}{5} = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right).\checkmark$$

Problem 2.34

(a) $W = \frac{\epsilon_0}{2}\int E^2\,d\tau$. $\quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ $(a < r < b)$, zero elsewhere.

$W = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int_a^b \left(\frac{1}{r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0}\int_a^b \frac{1}{r^2} = \boxed{\frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)}.$

(b) $W_1 = \frac{1}{8\pi\epsilon_0}\frac{q^2}{a}$, $W_2 = \frac{1}{8\pi\epsilon_0}\frac{q^2}{b}$, $\quad \mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$ $(r > a)$, $\mathbf{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{-q}{r^2}\,\hat{\mathbf{r}}$ $(r > b)$. So

$\mathbf{E}_1 \cdot \mathbf{E}_2 = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{-q^2}{r^4}$, $(r > b)$, and hence $\int \mathbf{E}_1 \cdot \mathbf{E}_2\,d\tau = -\left(\frac{1}{4\pi\epsilon_0}\right)^2 q^2 \int_b^\infty \frac{1}{r^4} 4\pi r^2 dr = -\frac{q^2}{4\pi\epsilon_0 b}$.

$W_{\text{tot}} = W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2\,d\tau = \frac{1}{8\pi\epsilon_0} q^2 \left(\frac{1}{a} + \frac{1}{b} - \frac{2}{b}\right) = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right).\checkmark$

Problem 2.35

(a) $\boxed{\sigma_R = \frac{q}{4\pi R^2};\ \sigma_a = \frac{-q}{4\pi a^2};\ \sigma_b = \frac{q}{4\pi b^2}.}$

(b) $V(0) = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_b^a (0)dr - \int_a^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_R^0 (0)dr = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{q}{b} + \frac{q}{R} - \frac{q}{a}\right).}$

(c) $\boxed{\sigma_b \to 0}$ (the charge "drains off"); $V(0) = -\int_\infty^a (0)dr - \int_a^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_R^0 (0)dr = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{q}{R} - \frac{q}{a}\right).}$

Problem 2.36

(a) $\boxed{\sigma_a = -\frac{q_a}{4\pi a^2};}\ \boxed{\sigma_b = -\frac{q_b}{4\pi b^2};}\ \boxed{\sigma_R = \frac{q_a + q_b}{4\pi R^2}.}$

(b) $\boxed{\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\,\hat{\mathbf{r}},}$ where $\mathbf{r}$ = vector from center of large sphere.

(c) $\boxed{\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\hat{\mathbf{r}}_a, \quad \mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\hat{\mathbf{r}}_b,}$ where $\mathbf{r}_a$ $(\mathbf{r}_b)$ is the vector from center of cavity a (b).

(d) $\boxed{\text{Zero.}}$

(e) σ_R changes (but not σ_a or σ_b); $\mathbf{E}_{\text{outside}}$ changes (but not $\mathbf{E}_a$ or $\mathbf{E}_b$); force on q_a and q_b still zero.

Problem 2.37

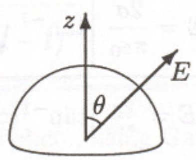
Between the plates, $E = 0$; outside the plates $E = \sigma/\epsilon_0 = Q/\epsilon_0 A$. So

$$P = \frac{\epsilon_0}{2}E^2 = \frac{\epsilon_0}{2}\frac{Q^2}{\epsilon_0^2 A^2} = \boxed{\frac{Q^2}{2\epsilon_0 A^2}.}$$

Problem 2.38

Inside, $\mathbf{E} = 0$; outside, $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$; so

$\mathbf{E}_{\text{ave}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\,\hat{\mathbf{r}}$; $\ f_z = \sigma(E_{\text{ave}})_z$; $\sigma = \frac{Q}{4\pi R^2}$.



$F_z = \int f_z da = \int \left(\frac{Q}{4\pi R^2}\right)\frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\right)\cos\theta\, R^2 \sin\theta\, d\theta\, d\phi$

$= \frac{1}{2\epsilon_0}\left(\frac{Q}{4\pi R}\right)^2 2\pi \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \frac{1}{\pi\epsilon_0}\left(\frac{Q}{4R}\right)^2 \left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{\pi/2} = \frac{1}{2\pi\epsilon_0}\left(\frac{Q}{4R}\right)^2 = \boxed{\frac{Q^2}{32\pi R^2\epsilon_0}.}$

Problem 2.39

Say the charge on the inner cylinder is Q, for a length L. The field is given by Gauss's law: $\int \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s \cdot L = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}Q \Rightarrow \mathbf{E} = \frac{Q}{2\pi\epsilon_0 L}\frac{1}{s}\,\hat{\mathbf{s}}$. Potential difference between the cylinders is

$$V(b) - V(a) = -\int_a^b \mathbf{E}\cdot d\mathbf{l} = -\frac{Q}{2\pi\epsilon_0 L}\int_a^b \frac{1}{s}ds = -\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right).$$

As set up here, a is at the higher potential, so $V = V(a) - V(b) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$.

$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$, so capacitance *per unit length* is $\boxed{\dfrac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}}$.

Problem 2.40

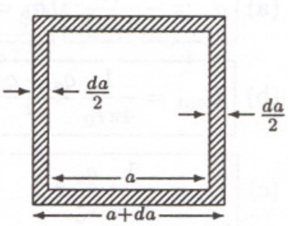
(a) W = (force)×(distance) = (pressure)×(area)×(distance) = $\boxed{\dfrac{\epsilon_0}{2}E^2 A\epsilon.}$

(b) W = (energy per unit volume)×(decrease in volume) = $\left(\epsilon_0\frac{E^2}{2}\right)(A\epsilon)$. Same as (a), confirming that the energy lost is equal to the work done.

Problem 2.41

From Prob. 2.4, the field at height z above the center of a square loop (side a) is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{4\lambda a z}{\left(z^2 + \frac{a^2}{4}\right)\sqrt{z^2 + \frac{a^2}{2}}}\,\hat{\mathbf{z}}.$$



Here $\lambda \to \sigma\frac{da}{2}$ (see figure), and we integrate over a from 0 to $\bar{a}$:

$$E = \frac{1}{4\pi\epsilon_0}2\sigma z\int_0^{\bar{a}}\frac{a\,da}{\left(z^2+\frac{a^2}{4}\right)\sqrt{z^2+\frac{a^2}{2}}}\text{. Let } u = \frac{a^2}{4}\text{, so } a\,da = 2\,du.$$

$$= \frac{1}{4\pi\epsilon_0}4\sigma z\int_0^{\bar{a}^2/4}\frac{du}{(u+z^2)\sqrt{2u+z^2}} = \frac{\sigma z}{\pi\epsilon_0}\left[\frac{2}{z}\tan^{-1}\left(\frac{\sqrt{2u+z^2}}{z}\right)\right]_0^{\bar{a}^2/4}$$

$$= \frac{2\sigma}{\pi\epsilon_0}\left\{\tan^{-1}\left(\frac{\sqrt{\frac{\bar{a}^2}{2}+z^2}}{z}\right) - \tan^{-1}(1)\right\};$$

$$\boxed{\mathbf{E} = \frac{2\sigma}{\pi\epsilon_0}\left[\tan^{-1}\sqrt{1+\frac{a^2}{2z^2}} - \frac{\pi}{4}\right]\hat{\mathbf{z}}.}$$

$a\to\infty$ (infinite plane): $E = \frac{2\sigma}{\pi\epsilon_0}\left[\tan^{-1}(\infty) - \frac{\pi}{4}\right] = \frac{2\sigma}{\pi\epsilon_0}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\sigma}{2\epsilon_0}$. ✓

$z \gg a$ (point charge): Let $f(x) = \tan^{-1}\sqrt{1+x} - \frac{\pi}{4}$, and expand as a Taylor series:

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \cdots$$

Here $f(0) = \tan^{-1}(1) - \frac{\pi}{4} = \frac{\pi}{4} - \frac{\pi}{4} = 0$; $f'(x) = \frac{1}{1+(1+x)}\frac{1}{2}\frac{1}{\sqrt{1+x}} = \frac{1}{2(2+x)\sqrt{1+x}}$, so $f'(0) = \frac{1}{4}$, so

$$f(x) = \frac{1}{4}x + (\ )x^2 + (\ )x^3 + \cdots$$

Thus (since $\frac{a^2}{2z^2} = x \ll 1$), $E \approx \frac{2\sigma}{\pi\epsilon_0}\left(\frac{1}{4}\frac{a^2}{2z^2}\right) = \frac{1}{4\pi\epsilon_0}\frac{\sigma a^2}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$. ✓

Problem 2.42

$$\rho = \epsilon_0\, \nabla\cdot \mathbf{E} = \epsilon_0 \left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{A}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{B\sin\theta\cos\phi}{r}\right)\right\}$$
$$= \epsilon_0\left[\frac{1}{r^2}A + \frac{1}{r\sin\theta}\frac{B\sin\theta}{r}(-\sin\phi)\right] = \boxed{\frac{\epsilon_0}{r^2}(A - B\sin\phi).}$$

Problem 2.43

From Prob. 2.12, the field inside a uniformly charged sphere is: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}$. So the force per unit volume is $\mathbf{f} = \rho\mathbf{E} = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right)\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)\mathbf{r} = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\mathbf{r}$, and the force in the z direction on $d\tau$ is:

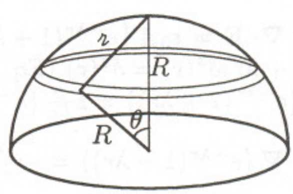
$$dF_z = f_z\, d\tau = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 r\cos\theta(r^2\sin\theta\, dr\, d\theta\, d\phi).$$

The total force on the "northern" hemisphere is:

$$F_z = \int f_z\, d\tau = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \int_0^R r^3 dr \int_0^{\pi/2}\cos\theta\sin\theta\, d\theta \int_0^{2\pi} d\phi$$
$$= \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\left(\frac{R^4}{4}\right)\left(\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2}\right)(2\pi) = \boxed{\frac{3Q^2}{64\pi\epsilon_0 R^2}.}$$

Problem 2.44

$$V_{\text{center}} = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma}{\mathscr{r}}da = \frac{1}{4\pi\epsilon_0}\frac{\sigma}{R}\int da = \frac{1}{4\pi\epsilon_0}\frac{\sigma}{R}(2\pi R^2) = \frac{\sigma R}{2\epsilon_0}$$



$$V_{\text{pole}} = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma}{\mathscr{r}}da\text{ , with}\begin{cases} da = 2\pi R^2\sin\theta\, d\theta, \\ \mathscr{r}^2 = R^2 + R^2 - 2R^2\cos\theta = 2R^2(1-\cos\theta).\end{cases}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi R^2)}{R\sqrt{2}}\int_0^{\pi/2}\frac{\sin\theta\, d\theta}{\sqrt{1-\cos\theta}} = \frac{\sigma R}{2\sqrt{2}\epsilon_0}\left.(2\sqrt{1-\cos\theta})\right|_0^{\pi/2}$$
$$= \frac{\sigma R}{\sqrt{2}\epsilon_0}(1-0) = \frac{\sigma R}{\sqrt{2}\epsilon_0}. \qquad \therefore V_{\text{pole}} - V_{\text{center}} = \boxed{\frac{\sigma R}{2\epsilon_0}(\sqrt{2}-1).}$$

Problem 2.45

First let's determine the electric field inside and outside the sphere, using Gauss's law:

$$\epsilon_0\oint\mathbf{E}\cdot d\mathbf{a} = \epsilon_0 4\pi r^2 E = Q_{\text{enc}} = \int\rho\, d\tau = \int(k\bar{r})\bar{r}^2\sin\theta\, d\bar{r}\, d\theta\, d\phi = 4\pi k\int_0^r \bar{r}^3 d\bar{r} = \begin{cases}\pi k r^4 & (r < R), \\ \pi k R^4 & (r > R).\end{cases}$$

So $\mathbf{E} = \frac{k}{4\epsilon_0} r^2\,\hat{\mathbf{r}}\ (r < R)$; $\quad \mathbf{E} = \frac{kR^4}{4\epsilon_0 r^2}\,\hat{\mathbf{r}}\ (r > R)$.

Method I:

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau \text{ (Eq. 2.45)} = \frac{\epsilon_0}{2}\int_0^R \left(\frac{kr^2}{4\epsilon_0}\right)^2 4\pi r^2 dr + \frac{\epsilon_0}{2}\int_R^\infty \left(\frac{kR^4}{4\epsilon_0 r^2}\right)^2 4\pi r^2 dr$$
$$= 4\pi\frac{\epsilon_0}{2}\left(\frac{k}{4\epsilon_0}\right)^2\left\{\int_0^R r^6 dr + R^8\int_R^\infty \frac{1}{r^2}dr\right\} = \frac{\pi k^2}{8\epsilon_0}\left\{\frac{R^7}{7} + R^8\left(-\frac{1}{r}\right)\Big|_R^\infty\right\} = \frac{\pi k^2}{8\epsilon_0}\left(\frac{R^7}{7} + R^7\right)$$
$$= \boxed{\frac{\pi k^2 R^7}{7\epsilon_0}.}$$

Method II:

$$W = \frac{1}{2}\int \rho V\, d\tau \quad \text{(Eq. 2.43)}.$$

For $r < R$, $V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^R \left(\frac{kR^4}{4\epsilon_0 r^2}\right) dr - \int_R^r \left(\frac{kr^2}{4\epsilon_0}\right) dr = -\frac{k}{4\epsilon_0}\left\{R^4\left(-\frac{1}{r}\right)\Big|_\infty^R + \frac{r^3}{3}\Big|_R^r\right\}$

$$= -\frac{k}{4\epsilon_0}\left(-R^3 + \frac{r^3}{3} - \frac{R^3}{3}\right) = \frac{k}{3\epsilon_0}\left(R^3 - \frac{r^3}{4}\right).$$
$$\therefore W = \frac{1}{2}\int_0^R (kr)\left[\frac{k}{3\epsilon_0}\left(R^3 - \frac{r^3}{4}\right)\right] 4\pi r^2 dr = \frac{2\pi k^2}{3\epsilon_0}\int_0^R \left(R^3 r^3 - \frac{1}{4}r^6\right) dr$$
$$= \frac{2\pi k^2}{3\epsilon_0}\left\{R^3\frac{R^4}{4} - \frac{1}{4}\frac{R^7}{7}\right\} = \frac{\pi k^2 R^7}{2\cdot 3\epsilon_0}\left(\frac{6}{7}\right) = \frac{\pi k^2 R^7}{7\epsilon_0}. \checkmark$$

Problem 2.46

$$\mathbf{E} = -\nabla V = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)\hat{\mathbf{r}} = -A\left\{\frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}}{r^2}\right\}\hat{\mathbf{r}} = \boxed{Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2}.}$$

$\rho = \epsilon_0\,\nabla\cdot\mathbf{E} = \epsilon_0 A\left\{e^{-\lambda r}(1+\lambda r)\,\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) + \frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+\lambda r)\right)\right\}$. But $\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r})$ (Eq. 1.99), and $e^{-\lambda r}(1+\lambda r)\delta^3(\mathbf{r}) = \delta^3(\mathbf{r})$ (Eq. 1.88). Meanwhile,
$\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = \hat{\mathbf{r}}\frac{\partial}{\partial r}\left(e^{-\lambda r}(1+\lambda r)\right) = \hat{\mathbf{r}}\left\{-\lambda e^{-\lambda r}(1+\lambda r) + e^{-\lambda r}\lambda\right\} = \hat{\mathbf{r}}(-\lambda^2 r e^{-\lambda r})$.

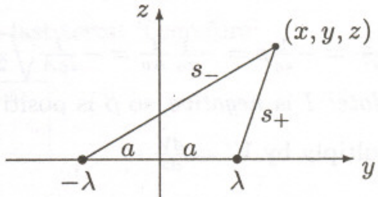
So $\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = -\frac{\lambda^2}{r}e^{-\lambda r}$, and $\boxed{\rho = \epsilon_0 A\left[4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right].}$

$$Q = \int \rho\, d\tau = \epsilon_0 A\left\{4\pi\int\delta^3(\mathbf{r})\,d\tau - \lambda^2\int\frac{e^{-\lambda r}}{r}4\pi r^2 dr\right\} = \epsilon_0 A\left(4\pi - \lambda^2 4\pi\int_0^\infty re^{-\lambda r}dr\right).$$

But $\int_0^\infty re^{-\lambda r}dr = \frac{1}{\lambda^2}$, so $Q = 4\pi\epsilon_0 A\left(1 - \frac{\lambda^2}{\lambda^2}\right) = \boxed{\text{zero.}}$

Problem 2.47

(a) Potential of $+\lambda$ is $V_+ = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_+}{a}\right)$, where s_+ is distance from λ_+ (Prob. 2.22).
Potential of $-\lambda$ is $V_- = +\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{a}\right)$, where s_- is distance from λ_-.



$\therefore$ Total $\boxed{V = \dfrac{\lambda}{2\pi\epsilon_0} \ln\left(\dfrac{s_-}{s_+}\right).}$

Now $s_+ = \sqrt{(y-a)^2 + z^2}$, and $s_- = \sqrt{(y+a)^2 + z^2}$, so

$$V(x,y,z) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\sqrt{(y+a)^2+z^2}}{\sqrt{(y-a)^2+z^2}}\right) = \boxed{\frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right].}$$

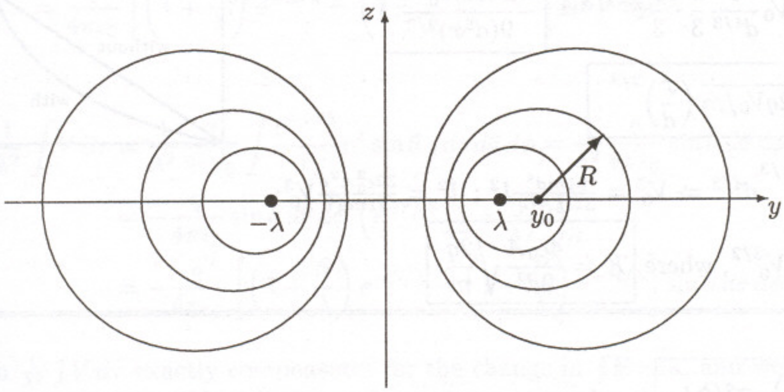
(b) Equipotentials are given by $\frac{(y+a)^2+z^2}{(y-a)^2+z^2} = e^{(4\pi\epsilon_0 V_0/\lambda)} = k =$ constant. That is:
$y^2 + 2ay + a^2 + z^2 = k(y^2 - 2ay + a^2 + z^2) \Rightarrow y^2(k-1) + z^2(k-1) + a^2(k-1) - 2ay(k+1) = 0$, or
$y^2 + z^2 + a^2 - 2ay\left(\frac{k+1}{k-1}\right) = 0$. The equation for a *circle*, with center at $(y_0, 0)$ and radius R, is
$(y - y_0)^2 + z^2 = R^2$, or $y^2 + z^2 + (y_0^2 - R^2) - 2yy_0 = 0$.
Evidently the equipotentials *are* circles, with $y_0 = a\left(\frac{k+1}{k-1}\right)$ and

$a^2 = y_0^2 - R^2 \Rightarrow R^2 = y_0^2 - a^2 = a^2\left(\frac{k+1}{k-1}\right)^2 - a^2 = a^2\frac{(k^2+2k+1-k^2+2k-1)}{(k-1)^2} = a^2\frac{4k}{(k-1)^2}$, or

$R = \frac{2a\sqrt{k}}{|k-1|}$; or, in terms of V_0:

$$y_0 = a\frac{e^{4\pi\epsilon_0 V_0/\lambda} + 1}{e^{4\pi\epsilon_0 V_0/\lambda} - 1} = a\frac{e^{2\pi\epsilon_0 V_0/\lambda} + e^{-2\pi\epsilon_0 V_0/\lambda}}{e^{2\pi\epsilon_0 V_0/\lambda} - e^{-2\pi\epsilon_0 V_0/\lambda}} = \boxed{a\coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right).}$$

$$R = 2a\frac{e^{2\pi\epsilon_0 V_0/\lambda}}{e^{4\pi\epsilon_0 V_0/\lambda} - 1} = a\frac{2}{(e^{2\pi\epsilon_0 V_0/\lambda} - e^{-2\pi\epsilon_0 V_0/\lambda})} = \frac{a}{\sinh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)} = \boxed{a\,\mathrm{csch}\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right).}$$



Problem 2.48

(a) $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ (Eq. 2.24), so $\boxed{\dfrac{d^2V}{dx^2} = -\dfrac{1}{\epsilon_0}\rho.}$

(b) $qV = \frac{1}{2}mv^2 \rightarrow \boxed{v = \sqrt{\dfrac{2qV}{m}}.}$

(c) $dq = A\rho\, dx$; $\frac{dq}{dt} = a\rho\frac{dx}{dt} = \boxed{A\rho v = I}$ (constant). (*Note*: ρ, hence also I, is *negative*.)

(d) $\frac{d^2V}{dx^2} = -\frac{1}{\epsilon_0}\rho = -\frac{1}{\epsilon_0}\frac{I}{Av} = -\frac{I}{\epsilon_0 A}\sqrt{\frac{m}{2qV}} \Rightarrow \boxed{\frac{d^2V}{dx^2} = \beta V^{-1/2}}$, where $\beta = -\frac{I}{\epsilon_0 A}\sqrt{\frac{m}{2q}}$.

(*Note:* I is *negative*, so β is positive; q is *positive*.)

(e) Multiply by $V' = \frac{dV}{dx}$:

$$V'\frac{dV'}{dx} = \beta V^{-1/2}\frac{dV}{dx} \Rightarrow \int V'\,dV' = \beta\int V^{-1/2}\,dV \Rightarrow \frac{1}{2}V'^2 = 2\beta V^{1/2} + \text{constant}.$$

But $V(0) = V'(0) = 0$ (cathode is at potential zero, and field at cathode is zero), so the constant is zero, and

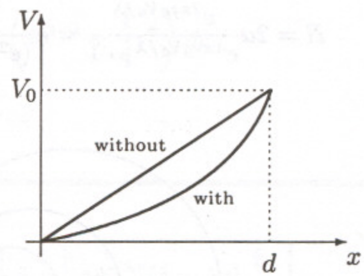
$$V'^2 = 4\beta V^{1/2} \Rightarrow \frac{dV}{dx} = 2\sqrt{\beta}\,V^{1/4} \Rightarrow V^{-1/4}dV = 2\sqrt{\beta}\,dx;$$
$$\int V^{-1/4}\,dV = 2\sqrt{\beta}\int dx \Rightarrow \frac{4}{3}V^{3/4} = 2\sqrt{\beta}\,x + \text{constant}.$$

But $V(0) = 0$, so this constant is also zero.

$$V^{3/4} = \frac{3}{2}\sqrt{\beta}\,x, \text{ so } V(x) = \left(\frac{3}{2}\sqrt{\beta}\right)^{4/3}x^{4/3}, \text{ or } V(x) = \left(\frac{9}{4}\beta\right)^{2/3}x^{4/3} = \left(\frac{81I^2m}{32\epsilon_0^2A^2q}\right)^{1/3}x^{4/3}.$$

Interms of V_0 (instead of I): $\boxed{V(x) = V_0\left(\frac{x}{d}\right)^{4/3}}$ (see graph).

Without space-charge, V would increase linearly: $V(x) = V_0\left(\frac{x}{d}\right)$.



$$\rho = -\epsilon_0\frac{d^2V}{dx^2} = -\epsilon_0 V_0\frac{1}{d^{4/3}}\frac{4}{3}\cdot\frac{1}{3}x^{-2/3} = \boxed{-\frac{4\epsilon_0 V_0}{9(d^2x)^{2/3}}}.$$

$$v = \sqrt{\frac{2q}{m}}\sqrt{V} = \boxed{\sqrt{2qV_0/m}\left(\frac{x}{d}\right)^{2/3}}.$$

(f) $V(d) = V_0 = \left(\frac{81I^2m}{32\epsilon_0^2A^2q}\right)^{1/3}d^{4/3} \Rightarrow V_0^3 = \frac{81md^4}{32\epsilon_0^2A^2q}I^2$; $I^2 = \frac{32\epsilon_0^2A^2q}{81md^4}V_0^3$;

$I = \frac{4\sqrt{2}\,\epsilon_0 A\sqrt{q}}{9\sqrt{m}\,d^2}V_0^{3/2} = KV_0^{3/2}$, where $\boxed{K = \frac{4\epsilon_0 A}{9d^2}\sqrt{\frac{2q}{m}}}$.

Problem 2.49

(a) $\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{-\mathscr{r}/\lambda}d\tau.}$

(b) $\boxed{\text{Yes.}}$ The field of a point charge at the origin is radial and symmetric, so $\nabla\times\mathbf{E} = 0$, and hence this is also true (by superposition) for any *collection* of charges.

(c)
$$V = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\frac{1}{4\pi\epsilon_0}q\int_\infty^r\frac{1}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}dr$$
$$= \frac{1}{4\pi\epsilon_0}q\int_r^\infty\frac{1}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}dr = \frac{q}{4\pi\epsilon_0}\left\{\int_r^\infty\frac{1}{r^2}e^{-r/\lambda}dr + \frac{1}{\lambda}\int_r^\infty\frac{1}{r}e^{-r/\lambda}dr\right\}.$$

Now $\int \frac{1}{r^2} e^{-r/\lambda} dr = -\frac{e^{-r/\lambda}}{r} - \frac{1}{\lambda}\int \frac{e^{-r/\lambda}}{r} dr \longleftarrow$ exactly right to kill the last term. Therefore

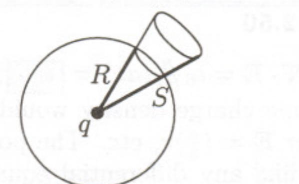
$$V(r) = \frac{q}{4\pi\epsilon_0}\left\{ -\frac{e^{-r/\lambda}}{r}\bigg|_r^{\infty} \right\} = \boxed{\frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}.}$$

(d)
$$\oint_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\cancel{4\pi}\epsilon_0} q \frac{1}{\cancel{R^2}}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\,\cancel{4\pi}\,\cancel{R^2} = \frac{q}{\epsilon_0}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}.$$

$$\int_{\mathcal{V}} V\,d\tau = \frac{q}{\cancel{4\pi}\epsilon_0}\int_0^R \frac{e^{-r/\lambda}}{r} r^2\,\cancel{4\pi}\,dr = \frac{q}{\epsilon_0}\int_0^R r e^{-r/\lambda} dr = \frac{q}{\epsilon_0}\left[\frac{e^{-r/\lambda}}{(1/\lambda)^2}\left(-\frac{r}{\lambda}-1\right)\right]_0^R$$
$$= \lambda^2 \frac{q}{\epsilon_0}\left\{ -e^{-R/\lambda}\left(1+\frac{R}{\lambda}\right)+1\right\}.$$

$$\therefore \oint_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{a} + \frac{1}{\lambda^2}\int_{\mathcal{V}} V\,d\tau = \frac{q}{\epsilon_0}\left\{\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}+1\right\} = \frac{q}{\epsilon_0}. \quad \text{qed}$$

(e) Does the result in (d) hold for a *non*spherical surface? Suppose we make a "dent" in the sphere—pushing a patch (area $R^2 \sin\theta\, d\theta\, d\phi$) from radius R out to radius S (area $S^2 \sin\theta\, d\theta\, d\phi$).



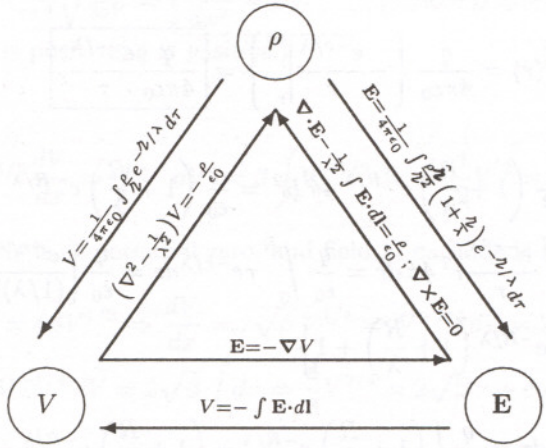
$$\Delta \oint \mathbf{E}\cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{S^2}\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda}(S^2\sin\theta\,d\theta\,d\phi) - \frac{1}{R^2}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}(R^2\sin\theta\,d\theta\,d\phi)\right\}$$
$$= \frac{q}{4\pi\epsilon_0}\left[\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]\sin\theta\,d\theta\,d\phi.$$

$$\Delta\frac{1}{\lambda^2}\int V\,d\tau = \frac{1}{\lambda^2}\frac{q}{4\pi\epsilon_0}\int \frac{e^{-r/\lambda}}{r} r^2 \sin\theta\,,dr\,d\theta\,d\phi = \frac{1}{\lambda^2}\frac{q}{4\pi\epsilon_0}\sin\theta\,d\theta\,d\phi\int_R^S r e^{-r/\lambda} dr$$
$$= -\frac{q}{4\pi\epsilon_0}\sin\theta\,d\theta\,d\phi\left(e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)\right)\bigg|_R^S$$
$$= -\frac{q}{4\pi\epsilon_0}\left[\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]\sin\theta\,d\theta\,d\phi.$$

So the change in $\frac{1}{\lambda^2}\int V\,d\tau$ exactly compensates for the change in $\oint \mathbf{E}\cdot d\mathbf{a}$, and we get $\frac{1}{\epsilon_0}q$ for the total using the dented sphere, just as we did with the perfect sphere. Any closed surface can be built up by successive distortions of the sphere, so the result holds for all shapes. By superposition, if there are many charges inside, the total is $\frac{1}{\epsilon_0}Q_{\text{enc}}$. Charges *outside* do not contribute (in the argument above we found that  for this volume $\oint \mathbf{E}\cdot d\mathbf{a} + \frac{1}{\lambda^2}\int V\,d\tau = 0$—and, again, the sum is not changed by distortions of the surface, as long as q remains outside). So the new "Gauss's Law" holds for *any* charge configuration.

(f) In differential form, "Gauss's law" reads: $\boxed{\boldsymbol{\nabla}\cdot\mathbf{E} + \frac{1}{\lambda^2}V = \frac{1}{\epsilon_0}\rho,}$ or, putting it all in terms of $\mathbf{E}$:

$\boldsymbol{\nabla}\cdot\mathbf{E} - \frac{1}{\lambda^2}\int \mathbf{E}\cdot d\mathbf{l} = \frac{1}{\epsilon_0}\rho$. Since $\mathbf{E} = -\boldsymbol{\nabla}V$, this also yields "Poisson's equation": $-\nabla^2 V + \frac{1}{\lambda^2}V = \frac{1}{\epsilon_0}\rho$.



Problem 2.50

$\rho = \epsilon_0 \, \nabla \cdot \mathbf{E} = \epsilon_0 \frac{\partial}{\partial x}(ax) = \boxed{\epsilon_0 a}$ (constant everywhere).

The same charge density would be compatible (as far as Gauss's law is concerned) with $\mathbf{E} = ay\hat{\mathbf{y}}$, for instance, or $\mathbf{E} = (\frac{a}{3})\mathbf{r}$, etc. The point is that Gauss's law (and $\nabla\times\mathbf{E} = 0$) by themselves *do not determine the field*—like any differential equations, they must be supplemented by appropriate ***boundary conditions.*** Ordinarily, these are so "obvious" that we impose them almost subconsciously ("E must go to zero far from the source charges")—or we appeal to symmetry to resolve the ambiguity ("the field must be the same—in magnitude—on both sides of an infinite plane of surface charge"). But in this case there are *no* natural boundary conditions, and no persuasive symmetry conditions, to fix the answer. The question "What is the electric field produced by a uniform charge density filling all of space?" is simply ***ill-posed***: it does not give us sufficient information to determine the answer. (Incidentally, it won't help to appeal to Coulomb's law $\left(\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \rho \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau\right)$—the integral is hopelessly indefinite, in this case.)

Problem 2.51

Compare Newton's law of universal gravitation to Coulomb's law:

$$\mathbf{F} = -G\frac{m_1 m_2}{r^2}\,\hat{\mathbf{r}}; \qquad \mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\,\hat{\mathbf{r}}.$$

Evidently $\frac{1}{4\pi\epsilon_0} \to G$ and $q \to m$. The gravitational energy of a sphere (translating Prob. 2.32) is therefore

$$\boxed{W_{\text{grav}} = \frac{3}{5}G\frac{M^2}{R}.}$$

Now, $G = 6.67 \times 10^{-11}$ N m^2/kg^2, and for the sun $M = 1.99 \times 10^{30}$ kg, $R = 6.96 \times 10^8$ m, so the sun's gravitational energy is $W = 2.28 \times 10^{41}$ J. At the current rate, this energy would be dissipated in a time

$$t = \frac{W}{P} = \frac{2.28 \times 10^{41}}{3.86 \times 10^{26}} = 5.90 \times 10^{14}\,\text{s} = \boxed{1.87 \times 10^7 \text{ years.}}$$

Problem 2.52

First eliminate z, using the formula for the ellipsoid:

$$\sigma(x,y) = \frac{Q}{4\pi ab}\frac{1}{\sqrt{c^2(x^2/a^4) + c^2(y^2/b^4) + 1 - (x^2/a^2) - (y^2/b^2)}}.$$

Now (for parts (a) and (b)) set $c \to 0$, "squashing" the ellipsoid down to an ellipse in the xy plane:

$$\sigma(x,y) = \frac{Q}{2\pi ab}\frac{1}{\sqrt{1 - (x/a)^2 - (y/b)^2}}.$$

(I multiplied by 2 to count both surfaces.)

(a) For the circular disk, set $a = b = R$ and let $r \equiv \sqrt{x^2+y^2}$. $\boxed{\sigma(r) = \frac{Q}{2\pi R}\frac{1}{\sqrt{R^2 - r^2}}.}$

(b) For the ribbon, let $Q/b \equiv \Lambda$, and then take the limit $b \to \infty$: $\boxed{\sigma(x) = \frac{\Lambda}{2\pi}\frac{1}{\sqrt{a^2 - x^2}}.}$

(c) Let $b = c$, $r \equiv \sqrt{y^2+z^2}$, making an ellipsoid of revolution:

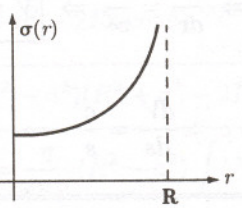
$$\frac{x^2}{a^2} + \frac{r^2}{c^2} = 1, \quad \text{with } \sigma = \frac{Q}{4\pi ac^2}\frac{1}{\sqrt{x^2/a^4 + r^2/c^4}}.$$

The charge on a ring of width dx is

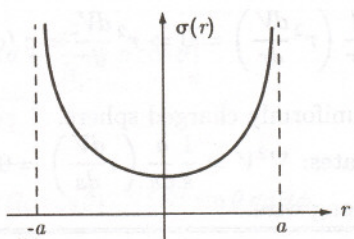
$$dq = \sigma 2\pi r\, ds, \quad \text{where } ds = \sqrt{dx^2 + dr^2} = dx\sqrt{1 + (dr/dx)^2}.$$

Now $\frac{2x\,dx}{a^2} + \frac{2r\,dr}{c^2} = 0 \Rightarrow \frac{dr}{dx} = -\frac{c^2x}{a^2r}$, so $ds = dx\sqrt{1 + \frac{c^4x^2}{a^4r^2}} = dx\frac{c^2}{r}\sqrt{x^2/a^4 + r^2/c^4}$. Thus

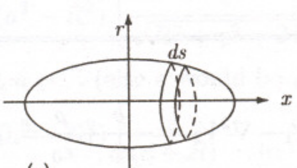
$$\lambda(x) = \frac{dq}{dx} = 2\pi r\frac{Q}{4\pi ac^2}\frac{1}{\sqrt{x^2/a^4 + r^2/c^4}}\frac{c^2}{r}\sqrt{x^2/a^4 + r^2/c^4} = \boxed{\frac{Q}{2a}.} \quad \textit{(Constant!)}$$



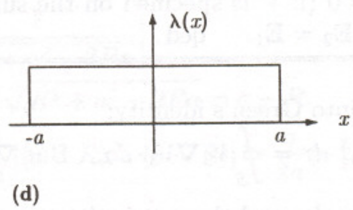
(a)



(b)



(c)



(d)