Chapter 6

Magnetostatic Fields in Matter

Problem 6.1

$$\mathbf{N} = \mathbf{m}_2 \times \mathbf{B}_1;\ \mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1\right];\ \hat{\mathbf{r}} = \hat{\mathbf{y}}; \mathbf{m}_1 = m_1\hat{\mathbf{z}};\ \mathbf{m}_2 = m_2\hat{\mathbf{y}}.\quad \mathbf{B}_1 = -\frac{\mu_0}{4\pi}\frac{m_1}{r^3}\hat{\mathbf{z}}.$$

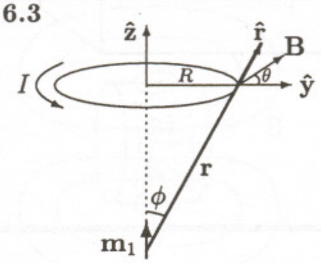
$\mathbf{N} = -\frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}(\hat{\mathbf{y}}\times\hat{\mathbf{z}}) = -\frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}\hat{\mathbf{x}}$. Here $m_1 = \pi a^2 I$, $m_2 = b^2 I$. So $\boxed{\mathbf{N} = -\frac{\mu_0}{4}\frac{(abI)^2}{r^3}\hat{\mathbf{x}}.}$ Final orientation : $\boxed{\text{downward}}$ $(-\hat{\mathbf{z}})$.

Problem 6.2

$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B}$; $d\mathbf{N} = \mathbf{r} \times d\mathbf{F} = I\,\mathbf{r} \times (d\mathbf{l} \times \mathbf{B})$. Now (Prob. 1.6): $\mathbf{r} \times (d\mathbf{l} \times \mathbf{B}) + d\mathbf{l} \times (\mathbf{B} \times \mathbf{r}) + \mathbf{B} \times (\mathbf{r} \times d\mathbf{l}) = 0$. But $d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] = d\mathbf{r} \times (\mathbf{r} \times \mathbf{B}) + \mathbf{r} \times (d\mathbf{r} \times \mathbf{B})$ (since $\mathbf{B}$ is constant), and $d\mathbf{r} = d\mathbf{l}$, so $d\mathbf{l} \times (\mathbf{B} \times \mathbf{r}) = \mathbf{r} \times (d\mathbf{l} \times \mathbf{B}) - d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})]$. Hence $2\mathbf{r} \times (d\mathbf{l} \times \mathbf{B}) = d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] - \mathbf{B} \times (\mathbf{r} \times d\mathbf{l})$. $d\mathbf{N} = \frac{1}{2}I\left\{d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] - \mathbf{B} \times (\mathbf{r} \times d\mathbf{l})\right\}$. $\therefore \mathbf{N} = \frac{1}{2}I\left\{\oint d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] - \mathbf{B} \times \oint(\mathbf{r} \times d\mathbf{l})\right\}$. But the first term is zero ($\oint d(\cdots) = 0$), and the second integral is $2\mathbf{a}$ (Eq. 1.107). So $\mathbf{N} = -I(\mathbf{B} \times \mathbf{a}) = \mathbf{m} \times \mathbf{B}$. qed

Problem 6.3

(a)



According to Eq. 6.2, $F = 2\pi IRB\cos\theta$. But $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1]}{r^3}$, and $B\cos\theta = \mathbf{B}\cdot\hat{\mathbf{y}}$, so $B\cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}) - (\mathbf{m}_1\cdot\hat{\mathbf{y}})\right]$. But $\mathbf{m}_1\cdot\hat{\mathbf{y}} = 0$ and $\hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = \sin\phi$, while $\mathbf{m}_1\cdot\hat{\mathbf{r}} = m_1\cos\theta$. $\therefore\ B\cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\sin\phi\cos\phi$.

$F = 2\pi IR\frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\sin\phi\cos\phi$. Now $\sin\phi = \frac{R}{r}$, $\cos\phi = \sqrt{r^2 - R^2}/r$, so $F = 3\frac{\mu_0}{2}m_1 IR^2\frac{\sqrt{r^2-R^2}}{r^5}$.

But $IR^2\pi = m_2$, so $F = \frac{3\mu_0}{2\pi}m_1 m_2\frac{\sqrt{r^2-R^2}}{r^5}$, while for a dipole, $R \ll r$, so $\boxed{F = \frac{3\mu_0}{2\pi}\frac{m_1 m_2}{r^4}.}$

(b) $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}_2\cdot\mathbf{B}) = (\mathbf{m}_2\cdot\boldsymbol{\nabla})\mathbf{B} = \left(m_2\frac{d}{dz}\right)\left[\frac{\mu_0}{4\pi}\frac{1}{z^3}\underbrace{(3(\mathbf{m}_1\cdot\hat{\mathbf{z}})\hat{\mathbf{z}} - \mathbf{m}_1)}_{2\mathbf{m}_1}\right] = \frac{\mu_0}{2\pi}m_1 m_2\hat{\mathbf{z}}\underbrace{\frac{d}{dz}\left(\frac{1}{z^3}\right)}_{-3\frac{1}{z^4}}$,

or, since $z = r$: $\boxed{\mathbf{F} = -\frac{3\mu_0}{2\pi}\frac{m_1 m_2}{r^4}\hat{\mathbf{z}}.}$

Problem 6.4

$$d\mathbf{F} = I\left\{(dy\,\hat{\mathbf{y}}) \times \mathbf{B}(0,y,0) + (dz\,\hat{\mathbf{z}}) \times \mathbf{B}(0,\epsilon,z) - (dy\,\hat{\mathbf{y}}) \times \mathbf{B}(0,y,\epsilon) - (dz\,\hat{\mathbf{z}}) \times \mathbf{B}(0,0,z)\right\}$$

$$= I\Big\{-(dy\,\hat{\mathbf{y}}) \times \underbrace{[\mathbf{B}(0,y,\epsilon) - \mathbf{B}(0,y,0)]}_{\approx\,\epsilon\frac{\partial\mathbf{B}}{\partial z}} + (dz\,\hat{\mathbf{z}}) \times \underbrace{[\mathbf{B}(0,\epsilon,z) - \mathbf{B}(0,0,z)]}_{\approx\,\epsilon\frac{\partial\mathbf{B}}{\partial y}}\Big\}$$

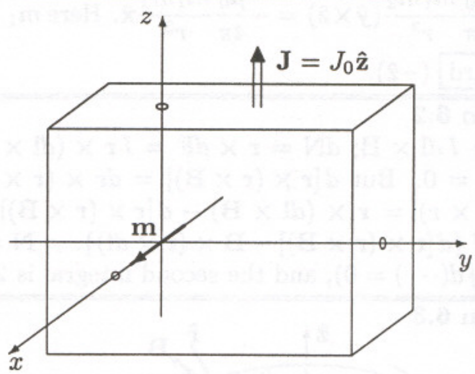
$$\Rightarrow I\epsilon^2\left\{\hat{\mathbf{z}} \times \frac{\partial\mathbf{B}}{\partial y} - \hat{\mathbf{y}} \times \frac{\partial\mathbf{B}}{\partial z}\right\}. \quad \left[\text{Note that } \int dy\,\frac{\partial\mathbf{B}}{\partial z}\Big|_{0,y,0} \approx \epsilon\,\frac{\partial\mathbf{B}}{\partial z}\Big|_{0,0,0} \text{ and } \int dz\,\frac{\partial\mathbf{B}}{\partial y}\Big|_{0,0,z} \approx \epsilon\,\frac{\partial\mathbf{B}}{\partial y}\Big|_{0,0,0}.\right]$$

$$\mathbf{F} = m\left\{\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & 1 \\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y} \end{vmatrix} - \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & 0 \\ \frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z} \end{vmatrix}\right\} = m\left\{\hat{\mathbf{y}}\frac{\partial B_x}{\partial y} - \hat{\mathbf{x}}\frac{\partial B_y}{\partial y} - \hat{\mathbf{x}}\frac{\partial B_z}{\partial z} - \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\right\}$$

$$= m\left[\hat{\mathbf{x}}\frac{\partial B_x}{\partial x} + \hat{\mathbf{y}}\frac{\partial B_x}{\partial y} + \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\right] \quad \left(\text{using } \boldsymbol{\nabla}\cdot\mathbf{B} = 0 \text{ to write } \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = -\frac{\partial B_x}{\partial x}\right).$$

But $\mathbf{m}\cdot\mathbf{B} = mB_x$ (since $\mathbf{m} = m\hat{\mathbf{x}}$, here), so $\boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}) = m\boldsymbol{\nabla}(B_x) = m\left(\frac{\partial B_x}{\partial x}\hat{\mathbf{x}} + \frac{\partial B_x}{\partial y}\hat{\mathbf{y}} + \frac{\partial B_x}{\partial z}\hat{\mathbf{z}}\right)$.
Therefore $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B})$. qed

Problem 6.5



(a) $\mathbf{B} = \mu_0 J_0 x\hat{\mathbf{y}}$ (Prob. 5.14).
$\mathbf{m}\cdot\mathbf{B} = 0$, so Eq. 6.3 says $\boxed{\mathbf{F} = 0.}$

(b) $\mathbf{m}\cdot\mathbf{B} = m_0\mu_0 J_0 x$, so $\boxed{\mathbf{F} = m_0\mu_0 J_0\hat{\mathbf{x}}.}$

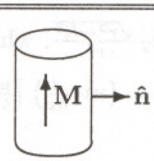
(c) Use product rule #4: $\boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E})$
$= \mathbf{p}\times(\boldsymbol{\nabla}\times\mathbf{E}) + \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{p}) + (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E} + (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{p}$.
But $\mathbf{p}$ does not depend on (x, y, z), so the second and fourth terms vanish, and $\boldsymbol{\nabla}\times\mathbf{E} = 0$, so the first term is zero. Hence $\boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E}) = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}$. qed

This argument does *not* apply to the magnetic analog, since $\boldsymbol{\nabla}\times\mathbf{B} \neq 0$. In fact, $\boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}) = (\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B} + \mu_0(\mathbf{m}\times\mathbf{J})$.
$(\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B}_a = m_0\frac{\partial}{\partial x}(\mathbf{B}) = m_0\mu_0 J_0\hat{\mathbf{y}}$, $(\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B}_b = m_0\frac{\partial}{\partial y}(\mu_0 J_0 x\hat{\mathbf{y}}) = 0$.

Problem 6.6

Aluminum, copper, copper chloride, and sodium all have an *odd* number of electrons, so we expect them to be paramagnetic. The rest (having an even number) should be diamagnetic.

Problem 6.7



$\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = 0$; $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = M\hat{\boldsymbol{\phi}}$.
The field is that of a surface current $\mathbf{K}_b = M\hat{\boldsymbol{\phi}}$, but that's just a solenoid, so the field $\boxed{\text{outside is zero,}}$ and inside $B = \mu_0 K_b = \mu_0 M$. Moreover, it points upward (in the drawing), so $\boxed{\mathbf{B} = \mu_0\mathbf{M}.}$

Problem 6.8

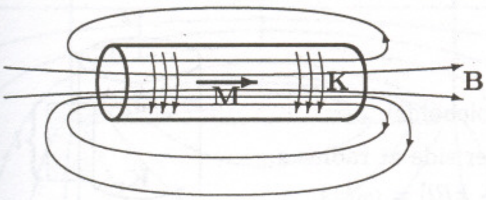
$$\nabla\times\mathbf{M} = \mathbf{J}_b = \frac{1}{s}\frac{\partial}{\partial s}(s\,ks^2)\hat{\mathbf{z}} = \frac{1}{s}(3ks^2)\hat{\mathbf{z}} = 3ks\hat{\mathbf{z}}, \quad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = ks^2(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{s}}) = -kR^2\hat{\mathbf{z}}.$$

So the bound current flows up the cylinder, and returns down the surface. [Incidentally, the *total* current should be zero ... *is* it? *Yes,* for $\int J_b\,da = \int_0^R (3ks)(2\pi s\,ds) = 2\pi kR^3$, while $\int K_b\,dl = (-kR^2)(2\pi R) = -2\pi kR^3$.] Since these currents have cylindrical symmetry, we can get the field by Ampère's law:

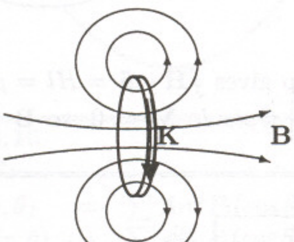
$$B\cdot 2\pi s = \mu_0 I_{\text{enc}} = \mu_0\int_0^s J_b\,da = 2\pi k\mu_0 s^3 \Rightarrow \boxed{\mathbf{B} = \mu_0 ks^2\hat{\boldsymbol{\phi}}} = \mu_0\mathbf{M}.$$

Outside the cylinder $I_{\text{enc}} = 0$, so $\boxed{\mathbf{B} = 0.}$

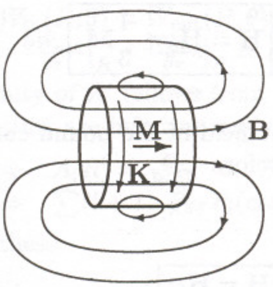
Problem 6.9



$\boxed{\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = M\hat{\boldsymbol{\phi}}.}$
(Essentially a long solenoid)



(Essentially a physical dipole)



(Intermediate case)

[The external fields are the same as in the electrical case; the *internal* fields (inside the bar) are completely different—in fact, opposite in direction.]

Problem 6.10

$K_b = M$, so the field inside a *complete* ring would be $\mu_0 M$. The field of a square loop, at the center, is given by Prob. 5.8: $B_{\text{sq}} = \sqrt{2}\,\mu_0 I/\pi R$. Here $I = Mw$, and $R = a/2$, so

$$B_{\text{sq}} = \frac{\sqrt{2}\,\mu_0 Mw}{\pi(a/2)} = \frac{2\sqrt{2}\,\mu_0 Mw}{\pi a}; \quad \text{net field in gap}: \ \boxed{\mathbf{B} = \mu_0\mathbf{M}\left(1 - \frac{2\sqrt{2}\,w}{\pi a}\right).}$$

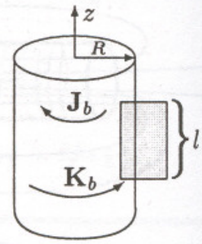
Problem 6.11

As in Sec. 4.2.3, we want the average of $\mathbf{B} = \mathbf{B}_{\text{out}} + \mathbf{B}_{\text{in}}$, where $\mathbf{B}_{\text{out}}$ is due to molecules *outside* a small sphere around point P, and $\mathbf{B}_{\text{in}}$ is due to molecules *inside* the sphere. The average of $\mathbf{B}_{\text{out}}$ is same as field at center (Prob. 5.57b), and for this it is OK to use Eq. 6.10, since the center is "far" from all the molecules in question:

$$\mathbf{A}_{\text{out}} = \frac{\mu_0}{4\pi} \int\limits_{\text{outside}} \frac{\mathbf{M} \times \hat{\boldsymbol{\imath}}}{\imath^2} d\tau$$

The average of $\mathbf{B}_{\text{in}}$ is $\frac{\mu_0}{4\pi}\left(\frac{2\mathbf{m}}{R^3}\right)$—Eq. 5.89—where $\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M}$. Thus the average $\mathbf{B}_{\text{in}}$ is $2\mu_0\mathbf{M}/3$. But what is *left out* of the integral $\mathbf{A}_{\text{out}}$ is the contribution of a uniformly magnetized sphere, to wit: $2\mu_0\mathbf{M}/3$ (Eq. 6.16), and this is precisely what $\mathbf{B}_{\text{in}}$ puts back in. So we'll get the correct macroscopic field using Eq. 6.10. qed

Problem 6.12



(a) $\mathbf{M} = ks\hat{\mathbf{z}}$; $\mathbf{J}_b = \nabla\times\mathbf{M} = -k\hat{\boldsymbol{\phi}}$; $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = kR\hat{\boldsymbol{\phi}}$.

$\mathbf{B}$ is in the z direction (this is essentially a superposition of solenoids). So

$\boxed{\mathbf{B} = 0 \text{ outside.}}$ Use the amperian loop shown (shaded)—inner side at radius s:

$\oint \mathbf{B}\cdot d\mathbf{l} = Bl = \mu_0 I_{\text{enc}} = \mu_0\left[\int J_b\, da + K_b l\right] = \mu_0\left[-kl(R-s) + kRl\right] = \mu_0 kls.$

$\therefore \boxed{\mathbf{B} = \mu_0 ks\hat{\mathbf{z}} \text{ inside.}}$

(b) By symmetry, $\mathbf{H}$ points in the z direction. That same amperian loop gives $\oint \mathbf{H}\cdot d\mathbf{l} = Hl = \mu_0 I_{f_{\text{enc}}} = 0$, since there *is* no free current here. So $\boxed{\mathbf{H} = 0}$, and hence $\boxed{\mathbf{B} = \mu_0\mathbf{M}.}$ *Outside* $\mathbf{M} = 0$, so $\mathbf{B} = 0$; *inside* $\mathbf{M} = ks\hat{\mathbf{z}}$, so $\mathbf{B} = \mu_0 ks\hat{\mathbf{z}}$.

Problem 6.13

(a) The field of a magnetized sphere is $\frac{2}{3}\mu_0\mathbf{M}$ (Eq. 6.16), so $\boxed{\mathbf{B} = \mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M},}$ with the sphere removed.

In the cavity, $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$, so $\mathbf{H} = \frac{1}{\mu_0}\left(\mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M}\right) = \mathbf{H}_0 + \mathbf{M} - \frac{2}{3}\mathbf{M} \Rightarrow \boxed{\mathbf{H} = \mathbf{H}_0 + \frac{1}{3}\mathbf{M}.}$

(b)

The field inside a long solenoid is $\mu_0 K$. Here $K = M$, so the field of the bound current on the inside surface of the cavity is $\mu_0 M$, pointing *down*. Therefore

$$\boxed{\mathbf{B} = \mathbf{B}_0 - \mu_0\mathbf{M};}$$

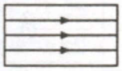
$$\mathbf{H} = \frac{1}{\mu_0}(\mathbf{B}_0 - \mu_0\mathbf{M}) = \frac{1}{\mu_0}\mathbf{B}_0 - \mathbf{M} \Rightarrow \boxed{\mathbf{H} = \mathbf{H}_0.}$$

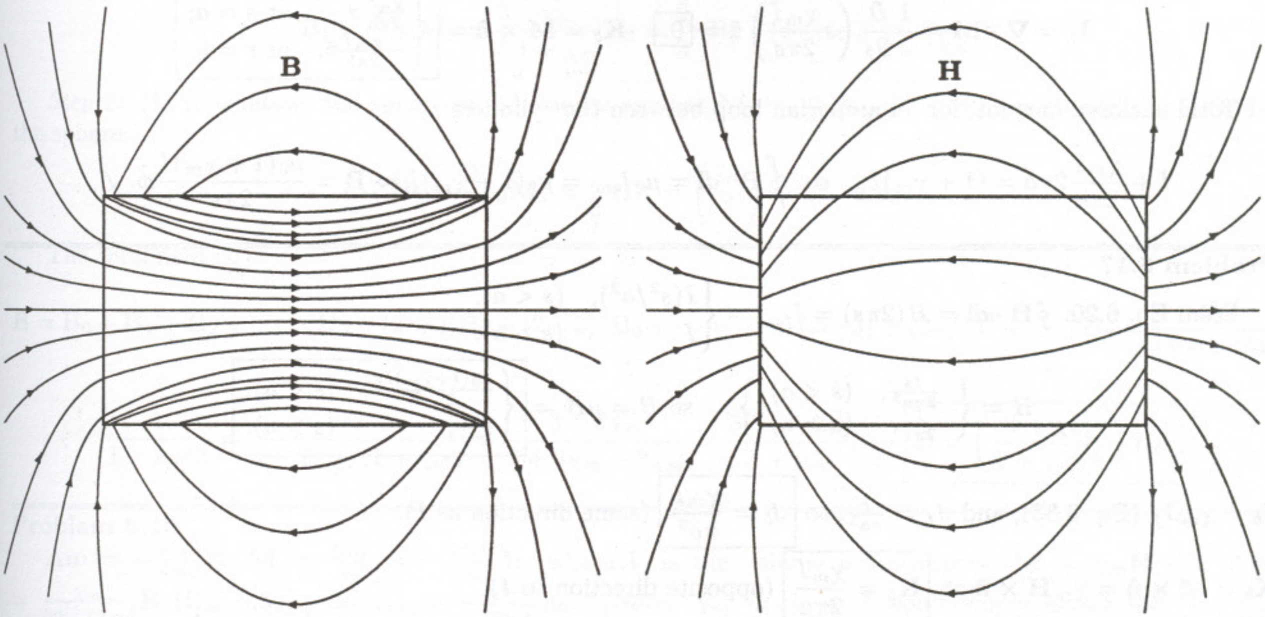
(c)

This time the bound currents are small, and far away from the center, so $\boxed{\mathbf{B} = \mathbf{B}_0,}$ while $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}_0 = \mathbf{H}_0 + \mathbf{M} \Rightarrow \boxed{\mathbf{H} = \mathbf{H}_0 + \mathbf{M}.}$

[*Comment*: In the wafer, $\mathbf{B}$ is the field in the medium; in the needle, $\mathbf{H}$ is the $\mathbf{H}$ in the medium; in the sphere (intermediate case) both $\mathbf{B}$ and $\mathbf{H}$ are modified.]

Problem 6.14

M:  ; **B is the same as the field of a short solenoid;** $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$.



Problem 6.15

"Potentials":

$$\begin{cases} W_{\text{in}}(r,\theta) & = \sum A_l r^l P_l(\cos\theta), \quad (r < R); \\ W_{\text{out}}(r,\theta) & = \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta), \quad (r > R). \end{cases}$$

Boundary Conditions:

$$\begin{cases} \text{(i)} \quad W_{\text{in}}(R,\theta) = W_{\text{out}}(R,\theta), \\ \text{(ii)} \quad -\frac{\partial W_{\text{out}}}{\partial r}\Big|_R + \frac{\partial W_{\text{in}}}{\partial r}\Big|_R = M^{\perp} = M\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = M\cos\theta. \end{cases}$$

(The continuity of W follows from the gradient theorem: $W(\mathbf{b}) - W(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} \nabla W \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{H}\cdot d\mathbf{l}$; if the two points are infinitesimally separated, this last integral $\to 0$.)

$$\begin{cases} \text{(i)} \quad \Rightarrow \quad A_l R^l = \frac{B_l}{R^{l+1}} \;\Rightarrow\; B_l = R^{2l+1} A_l, \\ \text{(ii)} \quad \Rightarrow \quad \sum (l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) + \sum l A_l R^{l-2} P_l(\cos\theta) = M\cos\theta. \end{cases}$$

Combining these:

$$\sum (2l+1) R^{l-1} A_l P_l(\cos\theta) = M\cos\theta, \text{ so } A_l = 0\ (l \neq 1), \text{ and } 3A_1 = M \Rightarrow A_1 = \frac{M}{3}.$$

Thus $W_{\text{in}}(r,\theta) = \dfrac{M}{3} r\cos\theta = \dfrac{M}{3} z$, and hence $\mathbf{H}_{\text{in}} = -\nabla W_{\text{in}} = -\dfrac{M}{3}\hat{\mathbf{z}} = -\dfrac{1}{3}\mathbf{M}$, so

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0\left(-\frac{1}{3}\mathbf{M} + \mathbf{M}\right) = \boxed{\frac{2}{3}\mu_0\mathbf{M}.}\ \checkmark$$

Problem 6.16

$\oint \mathbf{H}\cdot d\mathbf{l} = I_{f_{enc}} = I$, so $\mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}}$. $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \boxed{\mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\boldsymbol{\phi}}.}$ $\mathbf{M} = \chi_m\mathbf{H} = \boxed{\frac{\chi_m I}{2\pi s}\hat{\boldsymbol{\phi}}.}$

$$\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\chi_m I}{2\pi s}\right)\hat{\mathbf{z}} = \boxed{0.}\quad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \boxed{\begin{cases}\frac{\chi_m I}{2\pi a}\hat{\mathbf{z}}, & \text{at } s=a;\\ -\frac{\chi_m I}{2\pi b}\hat{\mathbf{z}}, & \text{at } r=b.\end{cases}}$$

Total enclosed current, for an amperian loop between the cylinders:

$$I + \frac{\chi_m I}{2\pi a}2\pi a = (1+\chi_m)I, \quad \text{so } \oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} = \mu_0(1+\chi_m)I \Rightarrow \mathbf{B} = \frac{\mu_0(1+\chi_m)I}{2\pi s}\hat{\boldsymbol{\phi}}. \checkmark$$

Problem 6.17

From Eq. 6.20: $\oint \mathbf{H}\cdot d\mathbf{l} = H(2\pi s) = I_{f_{enc}} = \begin{cases} I(s^2/a^2), & (s<a);\\ I & (s>a).\end{cases}$

$$H = \left\{\begin{array}{ll}\frac{Is}{2\pi a^2}, & (s<a)\\ \frac{I}{2\pi s}, & (s>a)\end{array}\right\},\quad \text{so } B = \mu H = \boxed{\begin{cases}\frac{\mu_0(1+\chi_m)Is}{2\pi a^2}, & (s<a);\\ \frac{\mu_0 I}{2\pi s}, & (s>a).\end{cases}}$$

$\mathbf{J}_b = \chi_m\mathbf{J}_f$ (Eq. 6.33), and $J_f = \frac{I}{\pi a^2}$, so $\boxed{J_b = \frac{\chi_m I}{\pi a^2}}$ (same direction as I).

$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \chi_m\mathbf{H}\times\hat{\mathbf{n}} \Rightarrow \boxed{\mathbf{K}_b = \frac{\chi_m I}{2\pi a}}$ (opposite direction to I).

$I_b = J_b(\pi a^2) + K_b(2\pi a) = \chi_m I - \chi_m I = \boxed{0}$ (as it *should* be, of course).

Problem 6.18

By the method of Prob. 6.15:

For large r, we want $\mathbf{B}(r,\theta) \to \mathbf{B}_0 = B_0\,\hat{\mathbf{z}}$, so $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} \to \frac{1}{\mu_0}B_0\,\hat{\mathbf{z}}$, and hence $W \to -\frac{1}{\mu_0}B_0 z = -\frac{1}{\mu_0}B_0 r\cos\theta$.

"Potentials":

$$\begin{cases} W_{\text{in}}(r,\theta) &= \sum A_l r^l P_l(\cos\theta), & (r<R);\\ W_{\text{out}}(r,\theta) &= -\frac{1}{\mu_0}B_0 r\cos\theta + \sum \frac{B_l}{r^{l+1}}P_l(\cos\theta), & (r>R).\end{cases}$$

Boundary Conditions:

$$\begin{cases} \text{(i)} & W_{\text{in}}(R,\theta) = W_{\text{out}}(R,\theta),\\ \text{(ii)} & -\mu_0\frac{\partial W_{\text{out}}}{\partial r}\Big|_R + \mu\frac{\partial W_{\text{in}}}{\partial r}\Big|_R = 0.\end{cases}$$

(The latter follows from Eq. 6.26.)

$$\text{(ii)} \Rightarrow \mu_0\left[\frac{1}{\mu_0}B_0\cos\theta + \sum(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta)\right] + \mu\sum lA_l R^{l-1}P_l(\cos\theta) = 0.$$

For $l \neq 1$, (i) $\Rightarrow B_l = R^{2l+1}A_l$, so $[\mu_0(l+1)+\mu l]A_l R^{l-1} = 0$, and hence $A_l = 0$.

For $l = 1$, (i) $\Rightarrow A_1 R = -\frac{1}{\mu_0}B_0 R + B_1/R^2$, and (ii) $\Rightarrow B_0 + 2\mu_0 B_1/R^3 + \mu A_1 = 0$, so $A_1 = -3B_0/(2\mu_0+\mu)$.

$$W_{\text{in}}(r,\theta) = -\frac{3B_0}{(2\mu_0+\mu)}r\cos\theta = -\frac{3B_0 z}{(2\mu_0+\mu)}.\quad \mathbf{H}_{\text{in}} = -\boldsymbol{\nabla}W_{\text{in}} = \frac{3B_0}{(2\mu_0+\mu)}\hat{\mathbf{z}} = \frac{3\mathbf{B}_0}{(2\mu_0+\mu)}.$$

$$\mathbf{B} = \mu\mathbf{H} = \frac{3\mu\mathbf{B}_0}{(2\mu_0+\mu)} = \boxed{\left(\frac{1+\chi_m}{1+\chi_m/3}\right)\mathbf{B}_0.}$$

By the method of Prob. 4.23:

Step 1: $\mathbf{B}_0$ magnetizes the sphere: $\mathbf{M}_0 = \chi_m \mathbf{H}_0 = \frac{\chi_m}{\mu_0(1+\chi_m)}\mathbf{B}_0$. This magnetization sets up a field within the sphere given by Eq. 6.16:

$$\mathbf{B}_1 = \frac{2}{3}\mu_0\mathbf{M}_0 = \frac{2}{3}\frac{\chi_m}{1+\chi_m}\mathbf{B}_0 = \frac{2}{3}\kappa\mathbf{B}_0 \quad (\text{where } \kappa \equiv \tfrac{\chi_m}{1+\chi_m}).$$

Step 2: $\mathbf{B}_1$ magnetizes the sphere an additional amount $\mathbf{M}_1 = \frac{\kappa}{\mu_0}\mathbf{B}_1$. This sets up an additional field in the sphere:

$$\mathbf{B}_2 = \frac{2}{3}\mu_0\mathbf{M}_1 = \frac{2}{3}\kappa\mathbf{B}_1 = \left(\frac{2\kappa}{3}\right)^2\mathbf{B}_0, \quad \text{etc.}$$

The *total* field is:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2 + \cdots = \mathbf{B}_0 + (2\kappa/3)\mathbf{B}_0 + (2\kappa/3)^2\mathbf{B}_0 + \cdots = \left[1 + (2\kappa/3) + (2\kappa/3)^2 + \cdots\right]\mathbf{B}_0 = \frac{\mathbf{B}_0}{(1-2\kappa/3)}.$$

$$\frac{1}{1-2\kappa/3} = \frac{3}{3-2\chi_m/(1+\chi_m)} = \frac{3+3\chi_m}{3+3\chi_m-2\chi_m} = \frac{3(1+\chi_m)}{3+\chi_m}, \text{ so } \boxed{\mathbf{B} = \left(\frac{1+\chi_m}{1+\chi_m/3}\right)\mathbf{B}_0.}$$

Problem 6.19

$\Delta\mathbf{m} = -\frac{e^2r^2}{4m_e}\mathbf{B}$; $\mathbf{M} = \frac{\Delta\mathbf{m}}{V} = -\frac{e^2r^2}{4m_eV}\mathbf{B}$, where V is the volume per electron. $\mathbf{M} = \chi_m\mathbf{H}$ (Eq. 6.29) $= \frac{\chi_m}{\mu_0(1+\chi_m)}\mathbf{B}$ (Eq. 6.30). So $\chi_m = -\frac{e^2r^2}{4m_eV}\mu_0$. [*Note*: $\chi_m \ll 1$, so I won't worry about the $(1+\chi_m)$ term; for the same reason we need not distinguish $\mathbf{B}$ from $\mathbf{B}_{\text{else}}$, as we did in deriving the Clausius-Mossotti equation in Prob. 4.38.] Let's say $V = \frac{4}{3}\pi r^3$. Then $\chi_m = -\frac{\mu_0}{4\pi}\left(\frac{3e^2}{4m_er}\right)$. I'll use 1 Å$= 10^{-10}$ m for r. Then $\chi_m = -(10^{-7})\left(\frac{3(1.6\times10^{-19})^2}{4(9.1\times10^{-31})(10^{-10})}\right) = \boxed{-2\times10^{-5},}$ which is not bad—Table 6.1 says $\chi_m = -1\times10^{-5}$. However, I used only *one electron* per atom (copper has 29) and a very crude value for r. Since the orbital radius is smaller for the inner electrons, they count for less ($\Delta m \sim r^2$). I have also neglected competing paramagnetic effects. But never mind ... this is in the right ball park.

Problem 6.20

Place the object in a region of zero magnetic field, and heat it above the Curie point—or simply drop it on a hard surface. If it's delicate (a watch, say), place it between the poles of an electromagnet, and magnetize it back and forth many times; each time you reverse the direction, reduce the field slightly.

Problem 6.21

(a) Identical to Prob. 4.7, only starting with Eqs. 6.1 and 6.3 instead of Eqs. 4.4 and 4.5.

(b) Identical to Prob. 4.8, but starting with Eq. 5.87 instead of 3.104.

(c) $U = -\frac{\mu_0}{4\pi}\frac{1}{r^3}[3\cos\theta_1\cos\theta_2 - \cos(\theta_2-\theta_1)]m_1m_2$. Or, using $\cos(\theta_2-\theta_1) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$,

$$\boxed{U = \frac{\mu_0}{4\pi}\frac{m_1m_2}{r^3}\left(\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2\right).}$$

Stable position occurs at minimum energy: $\frac{\partial U}{\partial\theta_1} = \frac{\partial U}{\partial\theta_2} = 0$

$$\begin{cases} \frac{\partial U}{\partial\theta_1} = \frac{\mu_0m_1m_2}{4\pi r^3}(\cos\theta_1\sin\theta_2 + 2\sin\theta_1\cos\theta_2) = 0 \Rightarrow 2\sin\theta_1\cos\theta_2 = -\cos\theta_1\sin\theta_2; \\ \frac{\partial U}{\partial\theta_2} = \frac{\mu_0m_1m_2}{4\pi r^3}(\sin\theta_1\cos\theta_2 + 2\cos\theta_1\sin\theta_2) = 0 \Rightarrow 2\sin\theta_1\cos\theta_2 = -4\cos\theta_1\sin\theta_2. \end{cases}$$

Thus $\sin\theta_1\cos\theta_2 = \sin\theta_2\cos\theta_1 = 0.$ $\begin{cases} \text{Either } \sin\theta_1 = \sin\theta_2 = 0: & \longrightarrow\overset{①}{}\longrightarrow \text{ or } \longrightarrow\overset{②}{}\longleftarrow \\ \text{or } \cos\theta_1 = \cos\theta_2 = 0: & \underset{③}{\uparrow\uparrow} \text{ or } \underset{④}{\uparrow\downarrow} \end{cases}$

Which of these is the *stable* minimum? Certainly not ② or ③—for these $\mathbf{m}_2$ is not parallel to $\mathbf{B}_1$, whereas we know $\mathbf{m}_2$ will line up along $\mathbf{B}_1$. It remains to compare ① (with $\theta_1 = \theta_2 = 0$) and ④ (with $\theta_1 = \pi/2, \theta_2 = -\pi/2$): $U_1 = \frac{\mu_0 m_1 m_2}{4\pi r^3}(-2)$; $U_2 = \frac{\mu_0 m_1 m_2}{4\pi r^3}(-1)$. U_1 is the lower energy, hence the more stable configuration.

Conclusion: They line up parallel, along the line joining them: $\longrightarrow\ \longrightarrow$

(d) They'd line up the same way: $\longrightarrow \longrightarrow \longrightarrow \longrightarrow \longrightarrow \longrightarrow$

Problem 6.22

$$\mathbf{F} = I\oint d\mathbf{l}\times\mathbf{B} = I\left(\oint d\mathbf{l}\right)\times\mathbf{B}_0 + I\oint d\mathbf{l}\times[(\mathbf{r}\cdot\boldsymbol{\nabla}_0)\mathbf{B}_0] - I\left(\oint d\mathbf{l}\right)\times[(\mathbf{r}_0\cdot\boldsymbol{\nabla}_0)\mathbf{B}_0] = I\oint d\mathbf{l}\times[(\mathbf{r}\cdot\boldsymbol{\nabla}_0)\mathbf{B}_0]$$

(because $\oint d\mathbf{l} = 0$). Now

$$(d\mathbf{l}\times\mathbf{B}_0)_i = \sum_{j,k}\epsilon_{ijk}\,dl_j(B_0)_k, \quad \text{and } (\mathbf{r}\cdot\boldsymbol{\nabla}_0) = \sum_l r_l(\nabla_0)_l, \text{ so}$$

$$\begin{aligned}
F_i &= I\sum_{j,k,l}\epsilon_{ijk}\left[\oint r_l\,dl_j\right][(\nabla_0)_l(B_0)_k] & \left\{\text{Lemma 1}: \oint r_l\,dl_j = \sum_m \epsilon_{ljm}a_m \text{ (proof below).}\right\} \\
&= I\sum_{j,k,l,m}\epsilon_{ijk}\epsilon_{ljm}a_m(\nabla_0)_l(B_0)_k & \left\{\text{Lemma 2}: \sum_j \epsilon_{ijk}\epsilon_{ljm} = \delta_{il}\delta_{km} - \delta_{im}\delta_{kl} \text{ (proof below).}\right\} \\
&= I\sum_{k,l,m}(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})\,a_m(\nabla_0)_l(B_0)_k = I\sum_k[a_k(\nabla_0)_i(B_0)_k - a_i(\nabla_0)_k(B_0)_k] \\
&= I\,[(\nabla_0)_i(\mathbf{a}\cdot\mathbf{B}_0) - a_i(\boldsymbol{\nabla}_0\cdot\mathbf{B}_0)].
\end{aligned}$$

But $\boldsymbol{\nabla}_0\cdot\mathbf{B}_0 = 0$ (Eq. 5.48), and $\mathbf{m} = I\mathbf{a}$ (Eq. 5.84), so $\mathbf{F} = \boldsymbol{\nabla}_0(\mathbf{m}\cdot\mathbf{B}_0)$ (the subscript just reminds us to take the derivatives at the point where $\mathbf{m}$ is located). qed

Proof of Lemma 1:
Eq. 1.108 says $\oint(\mathbf{c}\cdot\mathbf{r})\,d\mathbf{l} = \mathbf{a}\times\mathbf{c} = -\mathbf{c}\times\mathbf{a}$. The jth component is $\sum_p\oint c_p r_p\,dl_j = -\sum_{p,m}\epsilon_{jpm}c_p a_m$. Pick $c_p = \delta_{pl}$ (i.e. 1 for the lth component, zero for the others). Then $\oint r_l\,dl_j = -\sum_m\epsilon_{jlm}a_m = \sum_m\epsilon_{ljm}a_m$. qed

Proof of Lemma 2:
$\epsilon_{ijk}\epsilon_{ljm} = 0$ unless ijk and ljm are both permutations of 123. In particular, i must either be l or m, and k must be the other, so

$$\sum_j \epsilon_{ijk}\epsilon_{ljm} = A\delta_{il}\delta_{km} + B\delta_{im}\delta_{kl}.$$

To determine the constant A, pick $i = l = 1$, $k = m = 3$; the only contribution comes from $j = 2$:

$$\epsilon_{123}\epsilon_{123} = 1 = A\delta_{11}\delta_{33} + B\delta_{13}\delta_{31} = A \Rightarrow A = 1.$$

To determine B, pick $i = m = 1$, $k = l = 3$:

$$\epsilon_{123}\epsilon_{321} = -1 = A\delta_{13}\delta_{31} + B\delta_{11}\delta_{33} = B \Rightarrow B = -1.$$

So

$$\sum_j \epsilon_{ijk}\epsilon_{ljm} = \delta_{il}\delta_{km} - \delta_{im}\delta_{kl}. \quad \text{qed}$$

Problem 6.23

(a) The *electric* field inside a uniformly *polarized* sphere, $\mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}$ (Eq. 4.14) translates to $\mathbf{H} = -\frac{1}{3\mu_0}(\mu_0\mathbf{M}) = -\frac{1}{3}\mathbf{M}$. But $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$. So the *magnetic* field inside a uniformly *magnetized* sphere is $\mathbf{B} = \mu_0(-\frac{1}{3}\mathbf{M}+\mathbf{M}) = \boxed{\frac{2}{3}\mu_0\mathbf{M}}$ (same as Eq. 6.16).

(b) The *electric* field inside a sphere of linear *dielectric* in an otherwise uniform *electric* field is $\mathbf{E} = \frac{1}{1+\chi_e/3}\mathbf{E}_0$ (Eq. 4.49). Now χ_e translates to χ_m, for then Eq. 4.30 ($\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$) goes to $\mu_0\mathbf{M} = \mu_0\chi_m\mathbf{H}$, or $\mathbf{M} = \chi_m\mathbf{H}$ (Eq. 6.29). So Eq. 4.49 $\Rightarrow \mathbf{H} = \frac{1}{1+\chi_m/3}\mathbf{H}_0$. But $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H}$, and $\mathbf{B}_0 = \mu_0\mathbf{H}_0$ (Eqs. 6.31 and 6.32), so the *magnetic* field inside a sphere of linear *magnetic* material in an otherwise uniform *magnetic* field is
$\frac{\mathbf{B}}{\mu_0(1+\chi_m)} = \frac{1}{(1+\chi_m/3)}\frac{\mathbf{B}_0}{\mu_0}$, or $\boxed{\mathbf{B} = \left(\frac{1+\chi_m}{1+\chi_m/3}\right)\mathbf{B}_0}$ (as in Prob. 6.18).

(c) The average *electric* field over a sphere, due to charges within, is $\mathbf{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}$. Let's pretend the charges are all due to the frozen-in polarization of some medium (whatever ρ might be, we can solve $\boldsymbol{\nabla}\cdot\mathbf{P} = -\rho$ to find the appropriate $\mathbf{P}$). In this case there are *no* free charges, and $\mathbf{p} = \int \mathbf{P}\,d\tau$, so $\mathbf{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{1}{R^3}\int \mathbf{P}\,d\tau$, which translates to

$$\mathbf{H}_{\text{ave}} = -\frac{1}{4\pi\mu_0}\frac{1}{R^3}\int \mu_0\mathbf{M}\,d\tau = -\frac{1}{4\pi R^3}\mathbf{m}.$$

But $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$, so $\mathbf{B}_{\text{ave}} = -\frac{\mu_0}{4\pi}\frac{\mathbf{m}}{R^3} + \mu_0\mathbf{M}_{\text{ave}}$, and $\mathbf{M}_{\text{ave}} = \frac{\mathbf{m}}{\frac{4}{3}\pi R^3}$, so $\boxed{\mathbf{B}_{\text{ave}} = \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3}}$, in agreement with Eq. 5.89. (We must assume for this argument that all the currents are *bound*, but again it doesn't really matter, since we can model any current configuration by an appropriate frozen-in magnetization. See G. H. Goedecke, *Am. J. Phys.* **66**, 1010 (1998).)

Problem 6.24

$$\begin{array}{llll} \text{Eq. 2.15}: & \mathbf{E} & = & \rho\left\{\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\hat{\boldsymbol{\gamma}}}{\gamma^2}\,d\tau'\right\} \quad \text{(for uniform charge density);} \\ \text{Eq. 4.9}: & V & = & \mathbf{P}\cdot\left\{\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\hat{\boldsymbol{\gamma}}}{\gamma^2}\,d\tau'\right\} \quad \text{(for uniform polarization);} \\ \text{Eq. 6.11}: & \mathbf{A} & = & \mu_0\epsilon_0\mathbf{M}\times\left\{\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\hat{\boldsymbol{\gamma}}}{\gamma^2}\,d\tau'\right\} \quad \text{(for uniform magnetization).} \end{array}$$

For a uniformly charged sphere (radius R): $\left\{\begin{array}{lll} \mathbf{E}_{\text{in}} & = & \rho\left(\frac{1}{3\epsilon_0}\mathbf{r}\right) \quad \text{(Prob. 2.12),} \\ \mathbf{E}_{\text{out}} & = & \rho\left(\frac{1}{3\epsilon_0}\frac{R^3}{r^2}\hat{\mathbf{r}}\right) \quad \text{(Ex. 2.2).} \end{array}\right.$

So the scalar potential of a uniformly polarized sphere is: $\left\{\begin{array}{lll} V_{\text{in}} & = & \frac{1}{3\epsilon_0}(\mathbf{P}\cdot\mathbf{r}), \\ V_{\text{out}} & = & \frac{1}{3\epsilon_0}\frac{R^3}{r^2}(\mathbf{P}\cdot\hat{\mathbf{r}}), \end{array}\right.$

and the vector potential of a uniformly magnetized sphere is: $\left\{\begin{array}{lll} \mathbf{A}_{\text{in}} & = & \frac{\mu_0}{3}(\mathbf{M}\times\mathbf{r}), \\ \mathbf{A}_{\text{out}} & = & \frac{\mu_0}{3}\frac{R^3}{r^2}(\mathbf{M}\times\hat{\mathbf{r}}), \end{array}\right.$

(confirming the results of Ex. 4.2 and of Exs. 6.1 and 5.11).

Problem 6.25

(a) $\mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{2m}{z^3}\hat{\mathbf{z}}$ (Eq. 5.86, with $\theta = 0$). So $\mathbf{m}_2\cdot\mathbf{B}_1 = -\frac{\mu_0}{2\pi}\frac{m^2}{z^3}$. $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B})$ (Eq. 6.3) $\Rightarrow$ $\mathbf{F} = \frac{\partial}{\partial z}\left[-\frac{\mu_0}{2\pi}\frac{m^2}{z^3}\right]\hat{\mathbf{z}} = \frac{3\mu_0 m^2}{2\pi z^4}\hat{\mathbf{z}}$. This is the magnetic force *upward* (on the upper magnet); it balances the gravitational force downward ($-m_d g\hat{\mathbf{z}}$):

$$\frac{3\mu_0 m^2}{2\pi z^4} - m_d g = 0 \Rightarrow \boxed{z = \left[\frac{3\mu_0 m^2}{2\pi m_d g}\right]^{1/4}}.$$

(b) The middle magnet is repelled *upward* by lower magnet and *downward* by upper magnet:

$$\frac{3\mu_0 m^2}{2\pi x^4} - \frac{3\mu_0 m^2}{2\pi y^4} - m_d g = 0.$$

The top magnet is repelled *upward* by middle magnet, and attracted *downward* by lower magnet:

$$\frac{3\mu_0 m^2}{2\pi y^4} - \frac{3\mu_0 m^2}{2\pi (x+y)^4} - m_d g = 0.$$

Subtracting: $\frac{3\mu_0 m^2}{2\pi}\left[\frac{1}{x^4} - \frac{1}{y^4} - \frac{1}{y^4} + \frac{1}{(x+y)^4}\right] - m_d g + m_d g = 0$, or $\frac{1}{x^4} - \frac{2}{y^4} + \frac{1}{(x+y)^4} = 0$, so: $2 = \frac{1}{(x/y)^4} + \frac{1}{(x/y+1)^4}$.
Let $\alpha \equiv x/y$; then $2 = \frac{1}{\alpha^4} + \frac{1}{(\alpha+1)^4}$. Mathematica gives the numerical solution $\alpha = \boxed{x/y = 0.850115\ldots}$

Problem 6.26

At the interface, the perpendicular component of $\mathbf{B}$ is continuous (Eq. 6.26), and the parallel component of $\mathbf{H}$ is continuous (Eq. 6.25 with $\mathbf{K}_f = 0$). So $B_1^{\perp} = B_2^{\perp}$, $\mathbf{H}_1^{\parallel} = \mathbf{H}_2^{\parallel}$. But $\mathbf{B} = \mu\mathbf{H}$ (Eq. 6.31), so $\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel}$. Now $\tan\theta_1 = B_1^{\parallel}/B_1^{\perp}$, and $\tan\theta_2 = B_2^{\parallel}/B_2^{\perp}$, so

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{B_2^{\parallel}}{B_2^{\perp}}\frac{B_1^{\perp}}{B_1^{\parallel}} = \frac{B_2^{\parallel}}{B_1^{\parallel}} = \frac{\mu_2}{\mu_1}$$

(the same form, though for different reasons, as Eq. 4.68).

Problem 6.27

In view of Eq. 6.33, there is a *bound* dipole at the center: $\mathbf{m}_b = \chi_m \mathbf{m}$. So the *net* dipole moment at the center is $\mathbf{m}_{\text{center}} = \mathbf{m} + \mathbf{m}_b = (1+\chi_m)\mathbf{m} = \frac{\mu}{\mu_0}\mathbf{m}$. This produces a field given by Eq. 5.87:

$$\mathbf{B}_{\substack{\text{center}\\\text{dipole}}} = \frac{\mu}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right].$$

This accounts for the ***first*** term in the field. The remainder must be due to the bound surface current ($\mathbf{K}_b$) at $r = R$ (since there can be no volume bound current, according to Eq. 6.33). Let us make an educated guess (based either on the answer provided or on the analogous electrical Prob. 4.34) that the field due to the surface bound current is (for interior points) of the form $\mathbf{B}_{\substack{\text{surface}\\\text{current}}} = A\mathbf{m}$ (i.e. a constant, proportional to $\mathbf{m}$). In that case the magnetization will be:

$$\mathbf{M} = \chi_m \mathbf{H} = \frac{\chi_m}{\mu}\mathbf{B} = \frac{\chi_m}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] + \frac{\chi_m}{\mu}A\mathbf{m}.$$

This will produce bound currents $\mathbf{J}_b = \nabla\times\mathbf{M} = 0$, as it should, for $0 < r < R$ (no need to *calculate* this curl—the second term is *constant*, and the first is essentially the field of a dipole, which we know is curl-less, except at $r = 0$), and

$$\mathbf{K}_b = \mathbf{M}(R)\times\hat{\mathbf{r}} = \frac{\chi_m}{4\pi R^3}(-\mathbf{m}\times\hat{\mathbf{r}}) + \frac{\chi_m A}{\mu}(\mathbf{m}\times\hat{\mathbf{r}}) = \chi_m m\left(-\frac{1}{4\pi R^3} + \frac{A}{\mu}\right)\sin\theta\,\hat{\boldsymbol{\phi}}.$$

But this is exactly the surface current produced by a spinning sphere: $\mathbf{K} = \sigma\mathbf{v} = \sigma\omega R\sin\theta\,\hat{\boldsymbol{\phi}}$, with $(\sigma\omega R) \leftrightarrow \chi_m m\left(\frac{A}{\mu} - \frac{1}{4\pi R^3}\right)$. So the field it produces (for points inside) is (Eq. 5.68):

$$\mathbf{B}_{\substack{\text{surface}\\\text{current}}} = \frac{2}{3}\mu_0(\sigma\omega R) = \frac{2}{3}\mu_0\chi_m\mathbf{m}\left(\frac{A}{\mu} - \frac{1}{4\pi R^3}\right).$$

Everything is consistent, therefore, provided $A = \frac{2}{3}\mu_0\chi_m\left(\frac{A}{\mu} - \frac{1}{4\pi R^3}\right)$, or $A\left(1 - \frac{2\mu_0}{3\mu}\chi_m\right) = -\frac{2}{3}\frac{\mu_0\chi_m}{4\pi R^3}$. But $\chi_m = \left(\frac{\mu}{\mu_0}\right) - 1$, so $A\left(1 - \frac{2}{3} + \frac{2}{3}\frac{\mu_0}{\mu}\right) = -\frac{2}{3}\frac{(\mu-\mu_0)}{4\pi R^3}$, or $A\left(1 + \frac{2\mu_0}{\mu}\right) = 2\frac{(\mu_0-\mu)}{4\pi R^3}$; $A = \frac{\mu}{4\pi}\frac{2(\mu_0-\mu)}{R^3(2\mu_0+\mu)}$, and hence

$$\mathbf{B} = \frac{\mu}{4\pi}\left\{\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] + \frac{2(\mu_0-\mu)\mathbf{m}}{R^3(2\mu_0+\mu)}\right\}. \quad \text{qed}$$

The *exterior* field is that of the central dipole plus that of the surface current, which, according to Prob. 5.36, is *also* a perfect dipole field, of dipole moment

$$\mathbf{m}_{\substack{\text{surface}\\\text{current}}} = \frac{4}{3}\pi R^3(\sigma\omega R) = \frac{4}{3}\pi R^3\left(\frac{3}{2\mu_0}\mathbf{B}_{\substack{\text{surface}\\\text{current}}}\right) = \frac{2\pi R^3}{\mu_0}\frac{\mu}{4\pi}\frac{2(\mu_0-\mu)\mathbf{m}}{R^3(2\mu_0+\mu)} = \frac{\mu(\mu_0-\mu)\mathbf{m}}{\mu_0(2\mu_0+\mu)}.$$

So the *total* dipole moment is:

$$\mathbf{m}_{\text{tot}} = \frac{\mu}{\mu_0}\mathbf{m} + \frac{\mu}{\mu_0}\mathbf{m}\frac{(\mu_0-\mu)}{(2\mu_0+\mu)} = \frac{3\mu\mathbf{m}}{(2\mu_0+\mu)},$$

and hence the field (for $r > R$) is

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\left(\frac{3\mu}{2\mu_0+\mu}\right)\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right].}$$

Problem 6.28

The problem is that the field inside a *cavity* is not the same as the field in the material itself.

(a) *Ampére type.* The field deep inside the magnet is that of a long solenoid, $\mathbf{B}_0 \approx \mu_0\mathbf{M}$. From Prob. 6.13:

$$\begin{cases} \text{Sphere:} & \mathbf{B} = \mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M} = \frac{1}{3}\mu_0\mathbf{M}; \\ \text{Needle:} & \mathbf{B} = \mathbf{B}_0 - \mu_0\mathbf{M} = 0; \\ \text{Wafer:} & \mathbf{B} = \mu_0\mathbf{M}. \end{cases}$$

(b) *Gilbert type.* This is analogous to the *electric* case. The field at the center is approximately that midway between two distant point charges, $\mathbf{B}_0 \approx 0$. From Prob. 4.16 (with $\mathbf{E} \to \mathbf{B}$, $1/\epsilon_0 \to \mu_0$, $\mathbf{P} \to \mathbf{M}$):

$$\begin{cases} \text{Sphere:} & \mathbf{B} = \mathbf{B}_0 + \frac{\mu_0}{3}\mathbf{M} = \frac{1}{3}\mu_0\mathbf{M}; \\ \text{Needle:} & \mathbf{B} = \mathbf{B}_0 = 0; \\ \text{Wafer:} & \mathbf{B} = \mathbf{B}_0 + \mu_0\mathbf{M} = \mu_0\mathbf{M}. \end{cases}$$

In the cavities, then, the fields are the *same* for the two models, and this will be no test at all. $\boxed{\text{Yes.}}$ Fund it with \$1 M from the Office of Alternative Medicine.