Chapter 7

Electrodynamics

Problem 7.1

(a) Let Q be the charge on the inner shell. Then $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$ in the space between them, and $(V_a - V_b) = -\int_b^a \mathbf{E}\cdot d\mathbf{r} = -\frac{1}{4\pi\epsilon_0}Q\int_b^a \frac{1}{r^2}dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$.

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \sigma\frac{Q}{\epsilon_0} = \frac{\sigma}{\epsilon_0}\frac{4\pi\epsilon_0(V_a - V_b)}{(1/a - 1/b)} = \boxed{4\pi\sigma\frac{(V_a - V_b)}{(1/a - 1/b)}.}$$

(b) $R = \dfrac{V_a - V_b}{I} = \boxed{\dfrac{1}{4\pi\sigma}\left(\dfrac{1}{a}-\dfrac{1}{b}\right).}$

(c) For large b ($b \gg a$), the second term is negligible, and $R = 1/4\pi\sigma a$. Essentially all of the resistance is in the region right around the inner sphere. Successive shells, as you go out, contribute less and less, because the cross-sectional area ($4\pi r^2$) gets larger and larger. For the two submerged spheres, $R = \frac{2}{4\pi\sigma a} = \frac{1}{2\pi\sigma a}$ (one R as the current leaves the first, one R as it converges on the second). Therefore $I = V/R = \boxed{2\pi\sigma a V.}$

Problem 7.2

(a) $V = Q/C = IR$. Because positive I means the charge on the capacitor is *decreasing*,
$\dfrac{dQ}{dt} = -I = -\dfrac{1}{RC}Q$, so $Q(t) = Q_0 e^{-t/RC}$. But $Q_0 = Q(0) = CV_0$, so $\boxed{Q(t) = CV_0 e^{-t/RC}.}$

Hence $I(t) = -\dfrac{dQ}{dt} = CV_0\dfrac{1}{RC}e^{-t/RC} = \boxed{\dfrac{V_0}{R}e^{-t/RC}.}$

(b) $W = \boxed{\frac{1}{2}cV_0^2.}$ The energy delivered to the resistor is $\displaystyle\int_0^\infty P\,dt = \int_0^\infty I^2 R\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-2t/RC}dt =$ $\displaystyle\frac{V_0^2}{R}\left(-\frac{RC}{2}e^{-2t/RC}\right)\Big|_0^\infty = \frac{1}{2}CV_0^2.$ ✓

(c) $V_0 = Q/C + IR$. This time positive I means Q is *increasing*: $\dfrac{dQ}{dt} = I = \dfrac{1}{RC}(CV_0 - Q) \Rightarrow \dfrac{dQ}{Q - CV_0} =$ $-\dfrac{1}{RC}dt \Rightarrow \ln(Q - CV_0) = -\dfrac{1}{RC}t + \text{constant} \Rightarrow Q(t) = CV_0 + ke^{-t/RC}$. But $Q(0) = 0 \Rightarrow k = -CV_0$, so $\boxed{Q(t) = CV_0\left(1 - e^{-t/RC}\right).}$ $I(t) = \dfrac{dQ}{dt} = CV_0\left(\dfrac{1}{RC}e^{-t/RC}\right) = \boxed{\dfrac{V_0}{R}e^{-t/RC}.}$

(d) Energy from battery: $\int_0^\infty V_0 I\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-t/RC}dt = \frac{V_0^2}{R}\left(-RCe^{-t/RC}\right)\Big|_0^\infty = \frac{V_0^2}{R}RC = \boxed{CV_0^2.}$

Since $I(t)$ is the same as in (a), the energy delivered to the resistor is again $\boxed{\frac{1}{2}CV_0^2.}$ The final energy in the capacitor is also $\boxed{\frac{1}{2}CV_0^2,}$ so $\boxed{\text{half}}$ the energy from the battery goes to the capacitor, and the other half to the resistor.

Problem 7.3

(a) $I = \int \mathbf{J}\cdot d\mathbf{a}$, where the integral is taken over a surface enclosing the positively charged conductor. But $\mathbf{J} = \sigma\mathbf{E}$, and Gauss's law says $\int \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q$, so $I = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}Q$. But $Q = CV$, and $V = IR$, so $I = \frac{\sigma}{\epsilon_0}CIR$, or $\boxed{R = \frac{\epsilon_0}{\sigma C}.}$ qed

(b) $Q = CV = CIR \Rightarrow \frac{dQ}{dt} = -I = -\frac{1}{RC}Q \Rightarrow \boxed{Q(t) = Q_0 e^{-t/RC}}$, or, since $V = Q/C$, $V(t) = V_0 e^{-t/RC}$. The time constant is $\tau = RC = \boxed{\epsilon_0/\sigma.}$

Problem 7.4

$I = J(s)\,2\pi sL \;\Rightarrow\; J(s) = I/2\pi sL.\quad E = J/\sigma = I/2\pi s\sigma L = I/2\pi kL.$

$V = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = -\frac{I}{2\pi kL}(a-b).$ So $\boxed{R = \frac{b-a}{2\pi kL}.}$

Problem 7.5

$$I = \frac{\mathcal{E}}{r+R};\; P = I^2R = \frac{\mathcal{E}^2 R}{(r+R)^2};\; \frac{dP}{dR} = \mathcal{E}^2\left[\frac{1}{(r+R)^2} - \frac{2R}{(r+R)^3}\right] = 0 \Rightarrow r + R = 2R \Rightarrow \boxed{R = r.}$$

Problem 7.6

$\mathcal{E} = \oint \mathbf{E}\cdot d\mathbf{l} = \boxed{\text{zero}}$ for *all* electrostatic fields. It *looks* as though $\mathcal{E} = \oint \mathbf{E}\cdot d\mathbf{l} = (\sigma/\epsilon_0)h$, as would indeed be the case if the field were really just σ/ϵ_0 inside and zero outside. But in fact there is always a "fringing field" at the edges (Fig. 4.31), and this is evidently just right to kill off the contribution from the left end of the loop. The current is $\boxed{\text{zero.}}$

Problem 7.7

(a) $\mathcal{E} = -\frac{d\Phi}{dt} = -Bl\frac{dx}{dt} = -Blv$; $\mathcal{E} = IR \Rightarrow \boxed{I = \frac{Blv}{R}.}$ (Never mind the minus sign—it just tells you the *direction* of flow: $(\mathbf{v}\times\mathbf{B})$ is *upward*, in the bar, so *downward* through the resistor.)

(b) $F = IlB = \boxed{\frac{B^2l^2v}{R},}$ to the $\boxed{\text{left.}}$

(c) $F = ma = m\frac{dv}{dt} = -\frac{B^2l^2}{R}v \Rightarrow \frac{dv}{dt} = -\left(\frac{B^2l^2}{Rm}\right)v \Rightarrow \boxed{v = v_0 e^{-\frac{B^2l^2}{mR}t}.}$

(d) The energy goes into heat in the resistor. The power delivered to resistor is I^2R, so

$$\frac{dW}{dt} = I^2R = \frac{B^2l^2v^2}{R^2}R = \frac{B^2l^2}{R}v_0^2 e^{-2\alpha t},\text{ where } \alpha \equiv \frac{B^2l^2}{mR};\quad \frac{dW}{dt} = \alpha m v_0^2 e^{-2\alpha t}.$$

The total energy delivered to the resistor is $W = \alpha m v_0^2\int_0^\infty e^{-2\alpha t}dt = \alpha m v_0^2\left.\frac{e^{-2\alpha t}}{-2\alpha}\right|_0^\infty = \alpha m v_0^2\frac{1}{2\alpha} = \frac{1}{2}mv_0^2.$ ✓

Problem 7.8

(a) The field of long wire is $\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$, so $\Phi = \int \mathbf{B}\cdot d\mathbf{a} = \frac{\mu_0 I}{2\pi}\int_s^{s+a}\frac{1}{s}(a\,ds) = \boxed{\frac{\mu_0 I a}{2\pi}\ln\left(\frac{s+a}{s}\right)}.$

(b) $\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 I a}{2\pi}\frac{d}{dt}\ln\left(\frac{s+a}{s}\right)$, and $\frac{ds}{dt} = v$, so $-\frac{\mu_0 I a}{2\pi}\left(\frac{1}{s+a}\frac{ds}{dt} - \frac{1}{s}\frac{ds}{dt}\right) = \boxed{\frac{\mu_0 I a^2 v}{2\pi s(s+a)}}.$
The field points *out* of the page, so the force on a charge in the nearby side of the square is *to the right*. In the far side it's also to the right, but here the field is weaker, so the current flows $\boxed{\text{counterclockwise.}}$

(c) This time the flux is *constant*, so $\boxed{\mathcal{E} = 0.}$

Problem 7.9

Since $\nabla\cdot\mathbf{B} = 0$, Theorem 2(c) (Sect. 1.6.2) guarantees that $\int \mathbf{B}\cdot d\mathbf{a}$ is the same for *all* surfaces with a given boundary line.

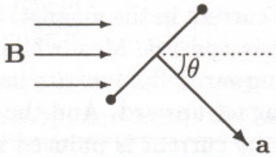
Problem 7.10

$\Phi = \mathbf{B}\cdot\mathbf{a} = Ba^2\cos\theta$

Here $\theta = \omega t$, so

$\mathcal{E} = -\frac{d\Phi}{dt} = -Ba^2(-\sin\omega t)\omega;$

$\boxed{\mathcal{E} = B\omega a^2\sin\omega t.}$



(view from above)

Problem 7.11

$\mathcal{E} = Blv = IR \Rightarrow I = \frac{Bl}{R}v \Rightarrow$ upward magnetic force $= IlB = \frac{B^2l^2}{R}v$. This opposes the gravitational force downward:

$$mg - \frac{B^2l^2}{R}v = m\frac{dv}{dt};\quad \frac{dv}{dt} = g - \alpha v,\ \text{where } \alpha \equiv \frac{B^2l^2}{mR}.\quad g - \alpha v_t = 0 \Rightarrow v_t = \frac{g}{\alpha} = \boxed{\frac{mgR}{B^2l^2}}.$$

$$\frac{dv}{g-\alpha v} = dt \Rightarrow -\frac{1}{\alpha}\ln(g-\alpha v) = t + \text{const.} \Rightarrow g - \alpha v = Ae^{-\alpha t};\ \text{at } t=0,\ v=0,\ \text{so } A = g.$$

$$\alpha v = g(1-e^{-\alpha t});\quad v = \frac{g}{\alpha}(1-e^{-\alpha t}) = \boxed{v_t(1-e^{-\alpha t}).}$$

At 90% of terminal velocity, $v/v_t = 0.9 = 1 - e^{-\alpha t} \Rightarrow e^{-\alpha t} = 1 - 0.9 = 0.1;\ \ln(0.1) = -\alpha t;\ \ln 10 = \alpha t;$
$t = \frac{1}{\alpha}\ln 10$, or $\boxed{t_{90\%} = \frac{v_t}{g}\ln 10.}$

Now the numbers: $m = 4\eta A l$, where η is the mass density of aluminum, A is the cross-sectional area, and l is the length of a side. $R = 4l/A\sigma$, where σ is the conductivity of aluminum. So

$$v_t = \frac{4\eta A l g 4l}{A\sigma B^2 l^2} = \frac{16\eta g}{\sigma B^2} = \frac{16 g\eta\rho}{B^2},\ \text{and}\ \left\{\begin{array}{l}\rho = 2.8\times10^{-8}\,\Omega\,\text{m}\\ g = 9.8\,\text{m/s}^2\\ \eta = 2.7\times10^3\,\text{kg/m}^3\\ B = 1\,\text{T}\end{array}\right\}.$$

So $v_t = \frac{(16)(9.8)(2.7\times10^3)(2.8\times10^{-8})}{1} = \boxed{1.2\ \text{cm/s};}\ t_{90\%} = \frac{1.2\times10^{-2}}{9.8}\ln(10) = \boxed{2.8\ \text{ms.}}$

$\boxed{\text{If the loop were cut, it would fall freely, with acceleration } g.}$

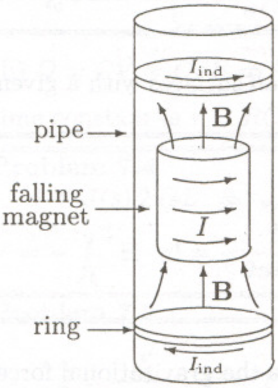
Problem 7.12

$$\Phi = \pi\left(\frac{a}{2}\right)^2 B = \frac{\pi a^2}{4} B_0 \cos(\omega t);\ \mathcal{E} = -\frac{d\Phi}{dt} = \frac{\pi a^2}{4} B_0 \omega \sin(\omega t).\quad I(t) = \frac{\mathcal{E}}{R} = \boxed{\frac{\pi a^2 \omega}{4R} B_0 \sin(\omega t).}$$

Problem 7.13

$$\Phi = \int B\,dx\,dy = kt^2 \int_0^a dx \int_0^a y^3\,dy = \frac{1}{4}kt^2 a^5.\quad \mathcal{E} = -\frac{d\Phi}{dt} = -\ \boxed{\tfrac{1}{2}kta^5.}$$

Problem 7.14



Suppose the current (I) in the magnet flows counterclockwise (viewed from above), as shown, so its field, near the ends, points *upward.* A ring of pipe *below* the magnet experiences an increasing upward flux, as the magnet approaches, and hence (by Lenz's law) a current (I_{ind}) will be induced in it such as to produce a *downward* flux. Thus I_{ind} must flow *clockwise*, which is *opposite* to the current in the magnet. Since opposite currents repel, the force on the magnet is *upward.* Meanwhile, a ring *above* the magnet experiences a *decreasing* (upward) flux, so *its* induced current is *parallel* to I, and it *attracts* the magnet upward. And the flux through rings *next to* the magnet is constant, so *no* current is induced in them. *Conclusion:* the delay is due to forces exerted on the magnet by induced eddy currents in the pipe.

Problem 7.15

In the quasistatic approximation, $\mathbf{B} = \begin{cases} \mu_0 n I\,\hat{\mathbf{z}}, & (s < a); \\ 0, & (s > a). \end{cases}$

Inside: for an "amperian loop" of radius $s < a$,

$$\Phi = B\pi s^2 = \mu_0 n I \pi s^2;\ \oint \mathbf{E}\cdot d\mathbf{l} = E\,2\pi s = -\frac{d\Phi}{dt} = -\mu_0 n \pi s^2 \frac{dI}{dt};\quad \boxed{\mathbf{E} = -\frac{\mu_0 n s}{2}\frac{dI}{dt}\,\hat{\boldsymbol{\phi}}.}$$

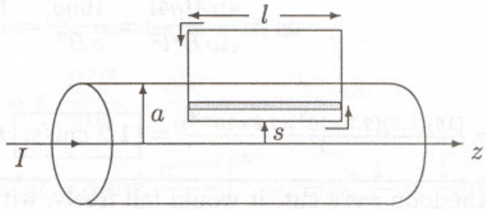
Outside: for an "amperian loop" of radius $s > a$:

$$\Phi = B\pi a^2 = \mu_0 n I \pi a^2;\ E\,2\pi s = -\mu_0 n \pi a^2 \frac{dI}{dt};\quad \boxed{\mathbf{E} = -\frac{\mu_0 n a^2}{2s}\frac{dI}{dt}\,\hat{\boldsymbol{\phi}}.}$$

Problem 7.16

(a) The magnetic field (in the quasistatic approximation) is "circumferential". This is analogous to the current in a solenoid, and hence the field is $\boxed{\text{longitudinal.}}$

(b) Use the "amperian loop" shown.
Outside, $\mathbf{B} = 0$, so here $\mathbf{E} = 0$ (like $\mathbf{B}$ outside a solenoid).
So $\oint \mathbf{E}\cdot d\mathbf{l} = El = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} = -\frac{d}{dt}\int_s^a \frac{\mu_0 I}{2\pi s'} l\,ds'$
$\therefore E = -\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln\left(\frac{a}{s}\right)$. But $\frac{dI}{dt} = -I_0\omega\sin\omega t$,



so $\boxed{\mathbf{E} = \frac{\mu_0 I_0 \omega}{2\pi}\sin(\omega t)\ln\left(\frac{a}{s}\right)\hat{\mathbf{z}}.}$

Problem 7.17

(a) The field inside the solenoid is $B = \mu_0 nI$. So $\Phi = \pi a^2 \mu_0 nI \Rightarrow \mathcal{E} = -\pi a^2 \mu_0 n (dI/dt)$.

In magnitude, then, $\mathcal{E} = \pi a^2 \mu_0 nk$. Now $\mathcal{E} = I_r R$, so $\boxed{I_{\text{resistor}} = \dfrac{\pi a^2 \mu_0 nk}{R}}$.

B is to the right and increasing, so the field of the loop is to the *left*, so the current is counterclockwise, or $\boxed{\text{to the right,}}$ through the resistor.

(b) $\Delta\Phi = 2\pi a^2 \mu_0 nI$; $I = \dfrac{dQ}{dt} = \dfrac{\mathcal{E}}{R} = -\dfrac{1}{R}\dfrac{d\Phi}{dt} \Rightarrow \Delta Q = \dfrac{1}{R}\Delta\Phi$, in magnitude. So $\boxed{\Delta Q = \dfrac{2\pi a^2 \mu_0 nI}{R}}$.

Problem 7.18

$\Phi = \int \mathbf{B}\cdot d\mathbf{a}$; $\mathbf{B} = \dfrac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$; $\Phi = \dfrac{\mu_0 I a}{2\pi}\displaystyle\int_a^{2a}\frac{ds}{s} = \frac{\mu_0 I a \ln 2}{2\pi}$; $\mathcal{E} = I_{\text{loop}} R = \dfrac{dQ}{dt} R = -\dfrac{d\Phi}{dt} = -\dfrac{\mu_0 a \ln 2}{2\pi}\dfrac{dI}{dt}$.

$$dQ = -\frac{\mu_0 a \ln 2}{2\pi R}\,dI \Rightarrow \boxed{Q = \frac{I\mu_0 a \ln 2}{2\pi R}}.$$

The field of the wire, at the square loop, is *out of the page*, and *decreasing*, so the field of the induced current must point out of page, within the loop, and hence the induced current flows $\boxed{\text{counterclockwise.}}$

Problem 7.19

In the quasistatic approximation, $\mathbf{B} = \begin{cases} \frac{\mu_0 NI}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{(inside toroid);} \\ 0, & \text{(outside toroid)} \end{cases}$

(Eq. 5.58). The flux around the toroid is therefore

$$\Phi = \frac{\mu_0 NI}{2\pi}\int_a^{a+w}\frac{1}{s}h\,ds = \frac{\mu_0 NIh}{2\pi}\ln\left(1+\frac{w}{a}\right) \approx \frac{\mu_0 Nhw}{2\pi a}I. \quad \frac{d\Phi}{dt} = \frac{\mu_0 Nhw}{2\pi a}\frac{dI}{dt} = \frac{\mu_0 Nhwk}{2\pi a}.$$

The electric field is the same as the *magnetic* field of a circular current (Eq. 5.38):

$$\mathbf{B} = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}}\hat{\mathbf{z}},$$

with (Eq. 7.18)

$$I \to -\frac{1}{\mu_0}\frac{d\Phi}{dt} = -\frac{Nhwk}{2\pi a}. \quad \text{So } \mathbf{E} = \frac{\mu_0}{2}\left(-\frac{Nhwk}{2\pi a}\right)\frac{a^2}{(a^2+z^2)^{3/2}}\hat{\mathbf{z}} = \boxed{-\frac{\mu_0}{4\pi}\frac{Nhwka}{(a^2+z^2)^{3/2}}\hat{\mathbf{z}}.}$$

Problem 7.20

(a) From Eq. 5.38, the field (on the axis) is $\mathbf{B} = \frac{\mu_0 I}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\hat{\mathbf{z}}$, so the flux through the little loop (area πa^2) is $\boxed{\Phi = \dfrac{\mu_0 \pi I a^2 b^2}{2(b^2+z^2)^{3/2}}}$.

(b) The field (Eq. 5.86) is $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$, where $m = I\pi a^2$. Integrating over the spherical "cap" (bounded by the big loop and centered at the little loop):

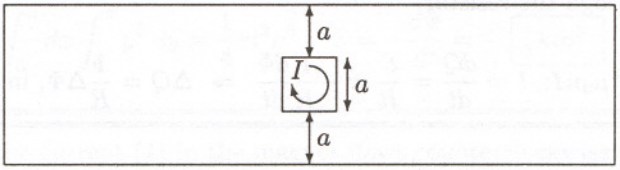
$$\Phi = \int \mathbf{B}\cdot d\mathbf{a} = \frac{\mu_0}{4\pi}\frac{I\pi a^2}{r^3}\int(2\cos\theta)(r^2\sin\theta\,d\theta\,d\phi) = \frac{\mu_0 I a^2}{2r}2\pi\int_0^{\bar{\theta}}\cos\theta\sin\theta\,d\theta$$

where $r = \sqrt{b^2 + z^2}$ and $\sin\bar{\theta} = b/r$. Evidently $\Phi = \frac{\mu_0 I \pi a^2}{r} \left.\frac{\sin^2\theta}{2}\right|_0^{\bar{\theta}} = \boxed{\dfrac{\mu_0 \pi I a^2 b^2}{2(b^2+z^2)^{3/2}}}$, the same as in (a)!!

(c) Dividing off I ($\Phi_1 = M_{12}I_2$, $\Phi_2 = M_{21}I_1$): $\boxed{M_{12} = M_{21} = \dfrac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}}.}$

Problem 7.21

$$\mathcal{E} = -\frac{d\Phi}{dt} = -M\frac{dI}{dt} = -Mk.$$



It's hard to calculate M using a current in the little loop, so, exploiting the equality of the mutual inductances, I'll find the flux through the *little* loop when a current I flows in the *big* loop: $\Phi = MI$. The field of *one* long wire is $B = \frac{\mu_0 I}{2\pi s} \Rightarrow \Phi_1 = \frac{\mu_0 I}{2\pi}\int_a^{2a} \frac{1}{s} a\, ds = \frac{\mu_0 I a}{2\pi}\ln 2$, so the *total* flux is

$$\Phi = 2\Phi_1 = \frac{\mu_0 I a \ln 2}{\pi} \Rightarrow M = \frac{\mu_0 a \ln 2}{\pi} \Rightarrow \boxed{\mathcal{E} = \frac{\mu_0 k a \ln 2}{\pi},} \text{ in magnitude.}$$

Direction: The net flux (through the big loop), due to I in the little loop, is *into the page.* (Why? Field lines point *in*, for the inside of the little loop, and *out* everywhere outside the little loop. The big loop encloses *all* of the former, and only *part* of the latter, so *net* flux is *inward.*) This flux is *increasing*, so the induced current in the big loop is such that *its* field points *out* of the page: it flows $\boxed{\text{counterclockwise.}}$

Problem 7.22

$B = \mu_0 n I \Rightarrow \Phi_1 = \mu_0 n I \pi R^2$ (flux through a single turn). In a length l there are nl such turns, so the total flux is $\Phi = \mu_0 n^2 \pi R^2 I l$. The self-inductance is given by $\Phi = LI$, so the self-inductance per unit length is $\boxed{\mathcal{L} = \mu_0 n^2 \pi R^2.}$

Problem 7.23

The field of one wire is $B_1 = \frac{\mu_0}{2\pi}\frac{I}{s}$, so $\Phi = 2 \cdot \frac{\mu_0 I}{2\pi} \cdot l \int\limits_{\epsilon}^{d-\epsilon} \frac{ds}{s} = \frac{\mu_0 I l}{\pi}\ln\left(\frac{d-\epsilon}{\epsilon}\right)$. The ϵ in the numerator is negligible (compared to d), but in the denominator we *cannot* let $\epsilon \to 0$, else the flux is *infinite.*

$\boxed{L = \dfrac{\mu_0 l}{\pi}\ln(d/\epsilon)}$. Evidently the size of the wire itself is critical in determining L.

Problem 7.24

(a) In the quasistatic approximation $\mathbf{B} = \dfrac{\mu_0}{2\pi s}\hat{\boldsymbol{\phi}}$. So $\Phi_1 = \dfrac{\mu_0 I}{2\pi}\displaystyle\int_a^b \frac{1}{s} h\, ds = \dfrac{\mu_0 I h}{2\pi}\ln(b/a)$.

This is the flux through *one* turn; the *total* flux is N times Φ_1: $\Phi = \dfrac{\mu_0 N h}{2\pi}\ln(b/a) I_0 \cos(\omega t)$. So

$\mathcal{E} = -\dfrac{d\Phi}{dt} = \dfrac{\mu_0 N h}{2\pi}\ln(b/a) I_0 \omega \sin(\omega t) = \dfrac{(4\pi \times 10^{-7})(10^3)(10^{-2})}{2\pi}\ln(2)(0.5)(2\pi\, 60)\sin(\omega t)$

$= \boxed{2.61 \times 10^{-4}\sin(\omega t)}$ (in volts), where $\omega = 2\pi\, 60 = 377/\text{s}$. $I_r = \dfrac{\mathcal{E}}{R} = \dfrac{2.61 \times 10^{-4}}{500}\sin(\omega t)$

$= \boxed{5.22 \times 10^{-7}\sin(\omega t)}$ (amperes).

(b) $\mathcal{E}_b = -L\dfrac{dI_r}{dt}$; where (Eq. 7.27) $L = \frac{\mu_0 N^2 h}{2\pi}\ln(b/a) = \frac{(4\pi\times10^{-7})(10^6)(10^{-2})}{2\pi}\ln(2) = 1.39 \times 10^{-3}$ (henries).

Therefore $\mathcal{E}_b = -(1.39 \times 10^{-3})(5.22 \times 10^{-7}\,\omega)\cos(\omega t) = \boxed{-2.74 \times 10^{-7}\cos(\omega t)}$ (volts).

Ratio of amplitudes: $\dfrac{2.74\times10^{-7}}{2.61\times10^{-4}} = \boxed{1.05\times10^{-3}} = \dfrac{\mu_0 N^2 h\omega}{2\pi R}\ln(b/a)$.

Problem 7.25

With I positive clockwise, $\mathcal{E} = -L\frac{dI}{dt} = Q/C$, where Q is the charge on the capacitor; $I = \frac{dQ}{dt}$, so $\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q = -\omega^2 Q$, where $\omega = \frac{1}{\sqrt{LC}}$. The general solution is $Q(t) = A\cos\omega t + B\sin\omega t$. At $t=0$, $Q = CV$, so $A = CV$; $I(t) = \frac{dQ}{dt} = -A\omega\sin\omega t + B\omega\sin\omega t$. At $t=0$, $I=0$, so $B=0$, and

$$I(t) = -CV\omega\sin\omega t = \boxed{-V\sqrt{\frac{C}{L}}\sin\left(\frac{t}{\sqrt{LC}}\right)}.$$

If you put in a resistor, the oscillation is "damped". This time $-L\frac{dI}{dt} = \frac{Q}{C} + IR$, so $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$. For an analysis of this case, see Purcell's *Electricity and Magnetism* (Ch. 8) or any book on oscillations and waves.

Problem 7.26

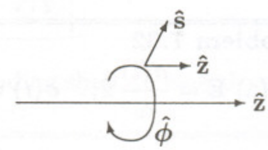
(a) $W = \frac{1}{2}LI^2$. $L = \mu_0 n^2\pi R^2 l$ (Prob. 7.22) $\boxed{W = \frac{1}{2}\mu_0 n^2\pi R^2 l I^2}$.

(b) $W = \frac{1}{2}\oint(\mathbf{A}\cdot\mathbf{I})dl$. $\mathbf{A} = (\mu_0 nI/2)R\,\hat{\boldsymbol{\phi}}$, at the surface (Eq. 5.70 or 5.71). So $W_1 = \frac{1}{2}\frac{\mu_0 nI}{2}RI\cdot 2\pi R$, for one turn. There are nl such turns in length l, so $W = \frac{1}{2}\mu_0 n^2\pi R^2 l I^2$. ✓

(c) $W = \frac{1}{2\mu_0}\int B^2\,d\tau$. $B = \mu_0 nI$, inside, and zero outside; $\int d\tau = \pi R^2 l$, so $W = \frac{1}{2\mu_0}\mu_0^2 n^2 I^2\pi R^2 l = \frac{1}{2}\mu_0 n^2\pi R^2 l I^2$. ✓

(d) $W = \frac{1}{2\mu_0}\left[\int B^2\,d\tau - \oint(\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}\right]$. This time $\int B^2\,d\tau = \mu_0^2 n^2 I^2\pi(R^2-a^2)l$. Meanwhile,
$\mathbf{A}\times\mathbf{B} = 0$ outside (at $s=b$). Inside, $\mathbf{A} = \frac{\mu_0 nI}{2}a\,\hat{\boldsymbol{\phi}}$ (at $s=a$), while $\mathbf{B} = \mu_0 nI\,\hat{\mathbf{z}}$.
$\mathbf{A}\times\mathbf{B} = \frac{1}{2}\mu_0^2 n^2 I^2 a\underbrace{(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}})}_{\hat{\mathbf{s}}}$

points *inward* ("out" of the volume)



$\oint(\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a} = \int(\frac{1}{2}\mu_0^2 n^2 I^2 a\hat{\mathbf{s}})\cdot[a\,d\phi\,dz(-\hat{\mathbf{s}})] = -\frac{1}{2}\mu_0^2 n^2 I^2 a^2 2\pi l$.
$W = \frac{1}{2\mu_0}\left[\mu_0^2 n^2 I^2\pi(R^2-a^2)l + \mu_0^2 n^2 I^2\pi a^2 l\right] = \frac{1}{2}\mu_0 n^2 I^2 R^2\pi l$. ✓

Problem 7.27

$$B = \frac{\mu_0 nI}{2\pi s};\ W = \frac{1}{2\mu_0}\int B^2\,d\tau = \frac{1}{2\mu_0}\frac{\mu_0^2 n^2 I^2}{4\pi^2}\int\frac{1}{s^2}hr\,d\phi\,ds = \frac{\mu_0 n^2 I^2}{8\pi^2}h2\pi\ln\left(\frac{b}{a}\right) = \boxed{\frac{1}{4\pi}\mu_0 n^2 I^2 h\ln(b/a)}.$$

$\boxed{L = \frac{\mu_0}{2\pi}n^2 h\ln(b/a)}$ (same as Eq. 7.27).

Problem 7.28

$$\oint\mathbf{B}\cdot d\mathbf{l} = B(2\pi s) = \mu_0 I_{\text{enc}} = \mu_0 I(s^2/R^2) \Rightarrow B = \frac{\mu_0 Is}{2\pi R^2}.$$

$$W = \frac{1}{2\mu_0}\int B^2\,d\tau = \frac{1}{2\mu_0}\frac{\mu_0^2 I^2}{4\pi^2 R^4}\int_0^R s^2(2\pi s)l\,ds = \frac{\mu_0 I^2 l}{4\pi R^4}\left(\frac{s^4}{4}\right)\Big|_0^R = \frac{\mu_0 l}{16\pi}I^2 = \frac{1}{2}LI^2.$$

So $L = \frac{\mu_0}{8\pi}l$, and $\mathcal{L} = L/l = \boxed{\mu_0/8\pi,}$ independent of R!

Problem 7.29

(a) Initial current: $I_0 = \mathcal{E}_0/R$. So $-L\dfrac{dI}{dt} = IR \Rightarrow \dfrac{dI}{dt} = -\dfrac{R}{L}I \Rightarrow I = I_0 e^{-Rt/L}$, or $\boxed{I(t) = \dfrac{\mathcal{E}_0}{R}e^{-Rt/L}}$.

(b) $P = I^2R = (\mathcal{E}_0/R)^2 e^{-2Rt/L}R = \dfrac{\mathcal{E}_0^2}{R}e^{-2Rt/L} = \dfrac{dW}{dt}$.

$$W = \frac{\mathcal{E}_0^2}{R}\int_0^\infty e^{-2Rt/L}\,dt = \frac{\mathcal{E}_0^2}{R}\left(-\frac{L}{2R}e^{-2Rt/L}\right)\Big|_0^\infty = \frac{\mathcal{E}_0^2}{R}(0 + L/2R) = \boxed{\frac{1}{2}L(\mathcal{E}_0/R)^2}$$

(c) $W_0 = \frac{1}{2}LI_0^2 = \frac{1}{2}\left(\mathcal{E}_0/R\right)^2$. ✓

Problem 7.30

(a) $\mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3}I_1[3(\mathbf{a}_1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{a}_1]$, since $\mathbf{m}_1 = I_1\mathbf{a}_1$. The flux through loop 2 is then

$$\Phi_2 = \mathbf{B}_1\cdot\mathbf{a}_2 = \frac{\mu_0}{4\pi}\frac{1}{r^3}I_1[3(\mathbf{a}_1\cdot\hat{\mathbf{r}})(\mathbf{a}_2\cdot\hat{\mathbf{r}}) - \mathbf{a}_1\cdot\mathbf{a}_2] = MI_1. \quad \boxed{M = \frac{\mu_0}{4\pi r^3}[3(\mathbf{a}_1\cdot\hat{\mathbf{r}})(\mathbf{a}_2\cdot\hat{\mathbf{r}}) - \mathbf{a}_1\cdot\mathbf{a}_2].}$$

(b) $\mathcal{E}_1 = -M\frac{dI_2}{dt}$, $\left.\frac{dW}{dt}\right|_1 = -\mathcal{E}_1 I_1 = MI_1\frac{dI_2}{dt}$. (This is the work done per unit time *against* the mutual emf in loop 1—hence the minus sign.) So (since I_1 is constant) $W_1 = MI_1I_2$, where I_2 is the final current in loop 2:

$$\boxed{W = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\mathbf{m}_2\cdot\hat{\mathbf{r}}) - \mathbf{m}_1\cdot\mathbf{m}_2].}$$

Notice that this is *opposite in sign* to Eq. 6.35. In Prob. 6.21 we assumed that the magnitudes of the dipole moments were *fixed*, and we did not worry about the energy necessary to sustain the currents themselves—only the energy required to move them into position and rotate them into their final orientations. But in *this* problem we are including it *all*, and it is a curious fact that this merely changes the sign of the answer. For commentary on this subtle issue see R. H. Young, *Am. J. Phys.* **66**, 1043 (1998), and the references cited there.

Problem 7.31

The displacement current density (Sect. 7.3.2) is $\mathbf{J}_d = \epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \frac{I}{A} = \frac{I}{\pi a^2}\,\hat{\mathbf{z}}$. Drawing an "amperian loop" at radius s,

$$\oint \mathbf{B}\cdot d\mathbf{l} = B\cdot 2\pi s = \mu_0 I_{d_{\text{enc}}} = \mu_0\frac{I}{\pi a^2}\cdot\pi s^2 = \mu_0 I\frac{s^2}{a^2} \Rightarrow B = \frac{\mu_0 I s^2}{2\pi s a^2}; \quad \boxed{\mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}.}$$

Problem 7.32

(a) $\mathbf{E} = \dfrac{\sigma(t)}{\epsilon_0}\,\hat{\mathbf{z}}$; $\quad\sigma(t) = \dfrac{Q(t)}{\pi a^2} = \dfrac{It}{\pi a^2}$; $\quad\boxed{\dfrac{It}{\pi\epsilon_0 a^2}\,\hat{\mathbf{z}}.}$

(b) $I_{d_{\text{enc}}} = J_d\pi s^2 = \epsilon_0\dfrac{dE}{dt}\pi s^2 = \boxed{I\dfrac{s^2}{a^2}.}\quad \oint\mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{d_{\text{enc}}} \Rightarrow B\,2\pi s = \mu_0 I\dfrac{s^2}{a^2} \Rightarrow \mathbf{B} = \boxed{\dfrac{\mu_0 I}{2\pi a^2}s\,\hat{\boldsymbol{\phi}}.}$

(c) A surface current flows radially outward over the left plate; let $I(s)$ be the total current crossing a circle of radius s. The charge density (at time t) is

$$\sigma(t) = \frac{[I - I(s)]t}{\pi s^2}.$$

Since we are told this is independent of s, it must be that $I - I(s) = \beta s^2$, for some constant β. But $I(a) = 0$, so $\beta a^2 = I$, or $\beta = I/a^2$. Therefore $I(s) = I(1 - s^2/a^2)$.

$$B\,2\pi s = \mu_0 I_{\text{enc}} = \mu_0[I - I(s)] = \mu_0\frac{s^2}{a^2} \Rightarrow \boxed{\mathbf{B} = \frac{\mu_0}{2\pi a^2}s\,\hat{\boldsymbol{\phi}}.}\ ✓$$

Problem 7.33

(a) $\mathbf{J}_d = \epsilon_0\dfrac{\mu_0 I_0\omega^2}{2\pi}\cos(\omega t)\ln(a/s)\,\hat{\mathbf{z}}$. But $I_0\cos(\omega t) = I$. So $\boxed{\mathbf{J}_d = \dfrac{\mu_0\epsilon_0}{2\pi}\omega^2 I\ln(a/s)\,\hat{\mathbf{z}}.}$

(b) $I_d = \displaystyle\int \mathbf{J}_d\cdot d\mathbf{a} = \frac{\mu_0\epsilon_0\omega^2 I}{2\pi}\int_0^a \ln(a/s)(2\pi s\,ds) = \mu_0\epsilon_0\omega^2 I\int_0^a(s\ln a - s\ln s)ds$

$$= \mu_0\epsilon_0\omega^2 I\left[(\ln a)\tfrac{s^2}{2} - \tfrac{s^2}{2}\ln s + \tfrac{s^2}{4}\right]\Big|_0^a = \mu_0\epsilon_0\omega^2 I\left[\cancel{\tfrac{a^2}{2}\ln a} - \cancel{\tfrac{a^2}{2}\ln a} + \tfrac{a^2}{4}\right] = \boxed{\frac{\mu_0\epsilon_0\omega^2 I a^2}{4}.}$$

(c) $\boxed{\frac{I_d}{I} = \frac{\mu_0\epsilon_0\omega^2 a^2}{4}.}$ Since $\mu_0\epsilon_0 = 1/c^2$, $I_d/I = (\omega a/2c)^2$. If $a = 10^{-3}$ m, and $\frac{I_d}{I} = \frac{1}{100}$, so that $\frac{\omega a}{2c} = \frac{1}{10}$, $\omega = \frac{2c}{10a} = \frac{3\times10^8\,\text{m/s}}{5\times10^{-3}\,\text{m}}$, or $\omega = 0.6\times10^{11}/\text{s} = \boxed{6\times10^{10}/\text{s};}$ $\nu = \frac{\omega}{2\pi} \approx 10^{10}$ Hz, or 10^4 megahertz. (This is the *microwave* region, *way* above radio frequencies.)

Problem 7.34

Physically, this is the field of a point charge $-q$ at the origin, out to an expanding spherical shell of radius vt; outside this shell the field is zero. Evidently the shell carries the opposite charge, $+q$. *Mathematically*, using product rule #5 and Eq. 1.99:

$$\nabla\cdot\mathbf{E} = \theta(vt-r)\nabla\cdot\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}\right) - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}\cdot\nabla[\theta(vt-r)] = -\frac{q}{\epsilon_0}\delta^3(\mathbf{r})\theta(vt-r) - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\,\frac{\partial}{\partial r}\theta(vt-r).$$

But $\delta^3(\mathbf{r})\theta(vt-r) = \delta^3(\mathbf{r})\theta(t)$, and $\frac{\partial}{\partial r}\theta(vt-r) = -\delta(vt-r)$ (Prob. 1.45), so

$$\rho = \epsilon_0\nabla\cdot\mathbf{E} = \boxed{-q\delta^3(\mathbf{r})\theta(t) + \frac{q}{4\pi r^2}\delta(vt-r).}$$

(For $t<0$ the field and the charge density are zero everywhere.)

Clearly $\nabla\cdot\mathbf{B} = 0$, and $\nabla\times\mathbf{E} = 0$ (since $\mathbf{E}$ has only an r component, and it is independent of θ and ϕ). There remains only the Ampére/Maxwell law, $\nabla\times\mathbf{B} = 0 = \mu_0\mathbf{J} + \mu_0\epsilon_0\partial\mathbf{E}/\partial t$. Evidently

$$\mathbf{J} = -\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = -\epsilon_0\left\{-\frac{q}{4\pi\epsilon_0 r^2}\frac{\partial}{\partial t}\left[\theta(vt-r)\right]\right\}\hat{\mathbf{r}} = \boxed{\frac{q}{4\pi r^2}v\delta(vt-r)\,\hat{\mathbf{r}}.}$$

(The stationary charge at the origin does not contribute to $\mathbf{J}$, of course; for the expanding shell we have $\mathbf{J} = \rho\mathbf{v}$, as expected—Eq. 5.26.)

Problem 7.35

From $\nabla\cdot\mathbf{B} = \mu_0\rho_m$ it follows that the field of a point monopole is $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{\imath^2}\hat{\boldsymbol{\imath}}$. The force law has the form $\mathbf{F} \propto q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right)$ (see Prob. 5.21—the c^2 is needed on dimensional grounds). The proportionality constant must be 1 to reproduce "Coulomb's law" for point charges at rest. So $\boxed{\mathbf{F} = q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right).}$

Problem 7.36

Integrate the "generalized Faraday law" (Eq. 7.43iii), $\nabla\times\mathbf{E} = -\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}$, over the surface of the loop:

$$\int(\nabla\times\mathbf{E})\cdot d\mathbf{a} = \oint\mathbf{E}\cdot d\mathbf{l} = \mathcal{E} = -\mu_0\int\mathbf{J}_m\cdot d\mathbf{a} - \frac{d}{dt}\int\mathbf{B}\cdot d\mathbf{a} = -\mu_0 I_{m_{\text{enc}}} - \frac{d\Phi}{dt}.$$

But $\mathcal{E} = -L\dfrac{dI}{dt}$, so $\dfrac{dI}{dt} = \dfrac{\mu_0}{L}I_{m_{\text{enc}}} + \dfrac{1}{L}\dfrac{d\Phi}{dt}$, or $I = \dfrac{\mu_0}{L}\Delta Q_m + \dfrac{1}{L}\Delta\Phi$, where ΔQ_m is the total magnetic charge passing through the surface, and $\Delta\Phi$ is the change in flux through the surface. If we use the *flat* surface, then $\Delta Q_m = q_m$ and $\Delta\Phi = 0$ (when the monopole is far away, $\Phi = 0$; the flux builds up to $\mu_0 q_m/2$ just before it passes through the loop; then it abruptly drops to $-\mu_0 q_m/2$, and rises back up to zero as the monopole disappears into the distance). If we use a huge balloon-shaped surface, so that q_m remains *inside* it on the far side, then $\Delta Q_m = 0$, but Φ rises monotonically from 0 to $\mu_0 q_m$. In either case,

$$\boxed{I = \frac{\mu_0 q_m}{L}.}$$

Problem 7.37

$$E = \frac{V}{d} \Rightarrow J_c = \sigma E = \frac{1}{\rho}E = \frac{V}{\rho d}.\ J_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon E) = \epsilon\frac{\partial}{\partial t}\left[\frac{V_0\cos(2\pi\nu t)}{d}\right] = \frac{\epsilon V_0}{d}\left[-2\pi\nu\sin(2\pi\nu t)\right].$$

The ratio of the amplitudes is therefore:

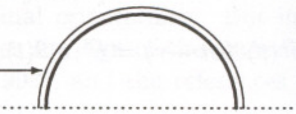
$$\frac{J_c}{J_d} = \frac{V_0}{\rho d}\frac{d}{2\pi\nu\epsilon V_0} = \frac{1}{2\pi\nu\epsilon\rho} = \left[2\pi(4\times 10^8)(81)(8.85\times 10^{-12})(0.23)\right]^{-1} = \boxed{2.41.}$$

Problem 7.38

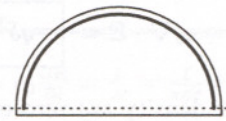
The potential and field in this configuration are identical to those in the upper half of Ex. 3.8. Therefore:

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a}$$

where the integral is over the hemispherical surface just outside the sphere.



But I can with impunity *close* this surface: (because $E = 0$ down there anyway—inside a conductor).



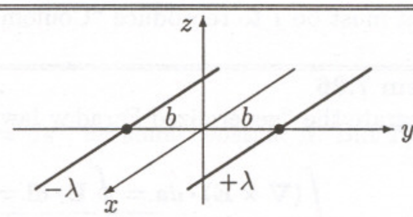
So $I = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}Q_{\text{enc}} = \frac{\sigma}{\epsilon_0}\int \sigma_e\,da$, where σ_e is the electric charge density on the surface of the hemisphere—to wit (Eq. 3.77) $\sigma_e = 3\epsilon_0 E_0\cos\theta$.

$$I = \frac{\sigma}{\epsilon_0}3\epsilon_0 E_0\int\cos\theta\,a^2\sin\theta\,d\theta\,d\phi = 3\sigma E_0 a^2 2\pi\underbrace{\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta}_{\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2} = \frac{1}{2}} = 3\sigma E_0\pi a^2.$$

But in this case $E_0 = V_0/d$, so $\boxed{I = \dfrac{3\sigma\pi V_0 a^2}{d}.}$

Problem 7.39

Begin with a different problem: two parallel wires carrying charges $+\lambda$ and $-\lambda$ as shown.



Field of one wire: $\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$; potential: $V = -\frac{\lambda}{2\pi\epsilon_0}\ln(s/a)$.

Potential of combination: $V = \frac{\lambda}{2\pi\epsilon_0}\ln(s_-/s_+)$,

or $V(y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{(y+b)^2+z^2}{(y-b)^2+z^2}\right\}$.

Find the locus of points of fixed V (i.e. equipotential surfaces):

$$e^{4\pi\epsilon_0 V/\lambda} \equiv \mu = \frac{(y+b)^2+z^2}{(y-b)^2+z^2} \Longrightarrow \mu(y^2 - 2yb + b^2 + z^2) = y^2 + 2yb + b^2 + z^2;$$

$$y^2(\mu-1) + b^2(\mu-1) + z^2(\mu-1) - 2yb(\mu+1) = 0 \Longrightarrow y^2 + z^2 + b^2 - 2yb\beta = 0 \quad \left(\beta \equiv \frac{\mu+1}{\mu-1}\right);$$

$$(y-b\beta)^2 + z^2 + b^2 - b^2\beta^2 = 0 \Longrightarrow (y-b\beta^2) + z^2 = b^2(\beta^2-1).$$

This is a *circle*, with center at $y_0 = b\beta = b\left(\frac{\mu+1}{\mu-1}\right)$ and radius $= b\sqrt{\beta^2-1} = b\sqrt{\frac{(\mu^2+2\mu+1)-(\mu^2-2\mu+1)}{(\mu-1)^2}} = \frac{2b\sqrt{\mu}}{\mu-1}$.

This suggests an image solution to the problem at hand. We want $y_0 = d$, radius $= a$, and $V = V_0$. These determine the parameters b, μ, and λ of the image solution:

$$\frac{d}{a} = \frac{y_0}{\text{radius}} = \frac{b\left(\frac{\mu+1}{\mu-1}\right)}{\frac{2b\sqrt{\mu}}{\mu-1}} = \frac{\mu+1}{2\sqrt{\mu}}. \quad \text{Call } \frac{d}{a} \equiv \alpha.$$

$$4\alpha^2\mu = (\mu+1)^2 = \mu^2 + 2\mu + 1 \Longrightarrow \mu^2 + (2-4\alpha^2)\mu + 1 = 0;$$

$$\mu = \frac{4\alpha^2 - 2 \pm \sqrt{4(1-2\alpha^2)^2 - 4}}{2} = 2\alpha^2 - 1 \pm \sqrt{1 - 4\alpha^2 + 4\alpha^4 - 1} = 2\alpha^2 - 1 \pm 2\alpha\sqrt{\alpha^2 - 1};$$

$$\frac{4\pi\epsilon_0 V_0}{\lambda} = \ln\mu \Longrightarrow \lambda = \frac{4\pi\epsilon_0 V_0}{\ln\left(2\alpha^2 - 1 \pm 2\alpha\sqrt{\alpha^2-1}\right)}.$$ That's the line charge in the image problem.

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \sigma\frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{\sigma}{\epsilon_0}\lambda l.$$

The current per unit length is $i = \dfrac{I}{l} = \dfrac{\sigma\lambda}{\epsilon_0} = \dfrac{4\pi\sigma V_0}{\ln\left(2\alpha^2 - 1 \pm 2\alpha\sqrt{\alpha^2-1}\right)}$. Which sign do we want? Suppose the cylinders are far apart, $d \gg a$, so that $\alpha \gg 1$.

$$(\;) = 2\alpha^2 - 1 \pm 2\alpha^2\sqrt{1 - 1/\alpha^2} = 2\alpha^2 - 1 \pm 2\alpha^2\left[1 - \frac{1}{2\alpha^2} - \frac{1}{8\alpha^4} + \cdots\right]$$
$$= 2\alpha^2(1\pm1) - (1\pm1) \mp \frac{1}{4\alpha^2} \pm \cdots = \begin{cases} 4\alpha^2 - 2 - 1/2\alpha^2 + \cdots \approx 4\alpha^2 & (+\text{ sign}), \\ -1/4\alpha^2 & (-\text{ sign}). \end{cases}$$

The current must surely *decrease* with increasing α, so evidently the + sign is correct:

$$\boxed{i = \frac{4\pi\sigma V_0}{\ln\left(2\alpha^2 - 1 + 2\alpha\sqrt{\alpha^2-1}\right)}, \quad \text{where } \alpha = \frac{d}{a}.}$$

Problem 7.40

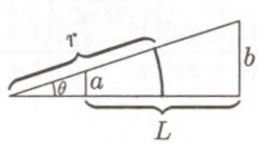
(a) The resistance of *one* disk (Ex. 7.1) is $dR = \frac{dz}{\sigma A} = \frac{\rho}{\pi r^2}\,dz$, where $r = \left(\frac{b-a}{L}\right)z + a$ is the radius of the disk. The total resistance is

$$R = \frac{\rho}{\pi}\int_0^L \frac{1}{\left[\left(\frac{b-a}{L}\right)z + a\right]^2}\,dz = \frac{\rho}{\pi}\left(\frac{L}{b-a}\right)\left\{\frac{-1}{\left[\left(\frac{b-a}{L}\right)z + a\right]}\right\}\Bigg|_0^L = \frac{\rho L}{\pi(b-a)}\left[-\frac{1}{(b-a+a)} + \frac{1}{a}\right]$$
$$= \frac{\rho L}{\pi(b-a)}\left(\frac{b-a}{ab}\right) = \boxed{\frac{\rho L}{\pi ab}}.$$

(b) In Ex. 7.1 the current was parallel to the axis; here it certainly is *not*. (Nor is it radial with respect to the apex of the cone, since the ends are *flat*. This is not an easy configuration to solve exactly.)

(c) This time the flow *is* radial, and we can add the resistances of nested spherical shells: $dR = \dfrac{\rho}{A}\,dr$, where

$$A = \int_0^\theta r^2\sin\theta\,d\theta\,d\phi = 2\pi r^2(-\cos\theta)\Big|_0^\theta = 2\pi r^2(1-\cos\theta).$$



$$R = \frac{\rho}{2\pi(1-\cos\theta)}\int_{r_a}^{r_b}\frac{1}{r^2}\,dr = \frac{\rho}{2\pi(1-\cos\theta)}\left(\frac{r_b-r_a}{r_a r_b}\right). \text{ Now } \frac{a}{r_a}=\frac{b}{r_b}=\sin\theta.$$

$$= \frac{\rho(b-a)}{2\pi ab}\frac{\sin\theta}{(1-\cos\theta)}. \text{ But } \sin\theta = \frac{b-a}{\sqrt{L^2+(b-a)^2}} \text{ and } \cos\theta = \frac{L}{\sqrt{L^2+(b-a)^2}}.$$

$$= \boxed{\frac{\rho(b-a)^2}{2\pi ab}\frac{1}{\left[\sqrt{L^2+(b-a)^2}-L\right]}.}$$

[Note that if $b-a \ll L$, then $\sqrt{L^2+(b-a)^2} \cong L\left[1+\frac{1}{2}\frac{(b-a)^2}{L^2}\right]$, and $R \cong \frac{\rho(b-a)^2}{2\pi ab}\frac{1}{(b-a)^2/2L} = \frac{\rho L}{\pi ab}$, as in (a).]

Problem 7.41

From Prob. 3.23, $$\begin{cases} V_{\text{in}}(s,\phi) &= \displaystyle\sum_{k=1}^{\infty} s^k b_k \sin(k\phi), \quad (s<a); \\ V_{\text{out}}(s,\phi) &= \displaystyle\sum_{k=1}^{\infty} s^{-k} d_k \sin(k\phi), \quad (s>a). \end{cases}$$

(We don't need the cosine terms, because V is clearly an *odd* function of ϕ.) At $s=a$, $V_{\text{in}} = V_{\text{out}} = V_0\phi/2\pi$. Let's start with V_{in}, and use Fourier's trick to determine b_k:

$\displaystyle\sum_{k=1}^{\infty} a^k b_k \sin(k\phi) = \frac{V_0\phi}{2\pi} \Rightarrow \sum_{k=1}^{\infty} a^k b_k \int_{-\pi}^{\pi}\sin(k\phi)\sin(k'\phi)\,d\phi = \frac{V_0}{2\pi}\int_{-\pi}^{\pi}\phi\sin(k'\phi)\,d\phi.$ But

$\displaystyle\int_{-\pi}^{\pi}\sin(k\phi)\sin(k'\phi)\,d\phi = \pi\delta_{kk'}$, and

$\displaystyle\int_{-\pi}^{\pi}\phi\sin(k'\phi)\,d\phi = \left[\frac{1}{(k')^2}\sin(k'\phi) - \frac{\phi}{k'}\cos(k'\phi)\right]\Bigg|_{-\pi}^{\pi} = -\frac{2\pi}{k'}\cos(k'\phi) = -\frac{2\pi}{k'}(-1)^{k'}.$ So

$\displaystyle\pi a^k b_k = \frac{V_0}{2\pi}\left[-\frac{2\pi}{k}(-1)^k\right]$, or $\displaystyle b_k = -\frac{V_0}{\pi k}\left(-\frac{1}{a}\right)^k$, and hence $\displaystyle V_{\text{in}}(s,\phi) = -\frac{V_0}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{s}{a}\right)^k\sin(k\phi).$

Similarly, $\displaystyle V_{\text{out}}(s,\phi) = -\frac{V_0}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{a}{s}\right)^k\sin(k\phi)$. Both sums are of the form $\displaystyle S \equiv \sum_{k=1}^{\infty}\frac{1}{k}(-x)^k\sin(k\phi)$ (with $x = s/a$ for $r<a$ and $x = a/s$ for $r>a$). This series can be summed explicitly, using Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$): $\displaystyle S = \text{Im}\sum_{k=1}^{\infty}\frac{1}{k}(-x)^k e^{ik\phi} = \text{Im}\sum_{k=1}^{\infty}\frac{1}{k}\left(-xe^{i\phi}\right)^k.$

But $\displaystyle\ln(1+w) = w - \frac{1}{2}w^2 + \frac{1}{3}w^3 - \frac{1}{4}w^4\cdots = -\sum_{k=1}^{\infty}\frac{1}{k}(-w)^k$, so $S = -\text{Im}\left[\ln\left(1+xe^{i\phi}\right)\right]$.

Now $\ln\left(Re^{i\theta}\right) = \ln R + i\theta$, so $S = -\theta$, where

$$\tan\theta = \frac{\text{Im}\left(1+xe^{i\phi}\right)}{\text{Re}\left(1+xe^{i\phi}\right)} = \frac{\frac{1}{2i}\left[\left(1+xe^{i\phi}\right)-\left(1+xe^{-i\phi}\right)\right]}{\frac{1}{2}\left[\left(1+xe^{i\phi}\right)+\left(1+xe^{-i\phi}\right)\right]} = \frac{x\left(e^{i\phi}-e^{-i\phi}\right)}{i\left[2+x\left(e^{i\phi}+e^{-i\phi}\right)\right]} = \frac{x\sin\phi}{1+x\cos\phi}.$$

$$\boxed{\text{Conclusion: } \left\{ \begin{array}{lcl} V_{\text{in}}(s,\phi) & = & \dfrac{V_0}{\pi}\tan^{-1}\left(\dfrac{s\sin\phi}{a+s\cos\phi}\right), \quad (s<a); \\[2ex] V_{\text{out}}(s,\phi) & = & \dfrac{V_0}{\pi}\tan^{-1}\left(\dfrac{a\sin\phi}{s+a\cos\phi}\right), \quad (s>a). \end{array}\right.}$$

(b) From Eq. 2.36, $\sigma(\phi) = -\epsilon_0 \left\{ \left.\dfrac{\partial V_{\text{out}}}{\partial s}\right|_{s=a} - \left.\dfrac{\partial V_{\text{in}}}{\partial s}\right|_{s=a} \right\}$.

$$\begin{aligned} \frac{\partial V_{\text{out}}}{\partial s} &= \frac{V_0}{\pi}\left\{ \frac{1}{\left[1+\left(\frac{a\sin\phi}{s+a\cos\phi}\right)^2\right]} \frac{(-a\sin\phi)}{(s+a\cos\phi)^2} \right\} = -\frac{V_0}{\pi}\left[\frac{a\sin\phi}{(s+a\cos\phi)^2+(a\sin\phi)^2}\right] \\ &= -\frac{V_0}{\pi}\left(\frac{a\sin\phi}{s^2+2as\cos\phi+a^2}\right); \\ \frac{\partial V_{\text{in}}}{\partial s} &= \frac{V_0}{\pi}\left\{ \frac{1}{\left[1+\left(\frac{s\sin\phi}{a+s\cos\phi}\right)^2\right]} \frac{[(a+s\cos\phi)\sin\phi - s\sin\phi\cos\phi]}{(a+s\cos\phi)^2} \right\} = \frac{V_0}{\pi}\left[\frac{a\sin\phi}{(a+s\cos\phi)^2+(s\sin\phi)^2}\right] \\ &= \frac{V_0}{\pi}\left(\frac{a\sin\phi}{s^2+2as\cos\phi+a^2}\right). \end{aligned}$$

$$\left.\frac{\partial V_{\text{in}}}{\partial s}\right|_{s=a} = -\left.\frac{\partial V_{\text{out}}}{\partial s}\right|_{s=a} = \frac{V_0}{2\pi a}\left(\frac{\sin\phi}{1+\cos\phi}\right), \text{ so } \sigma(\phi) = \frac{\epsilon_0 V_0}{\pi a}\frac{\sin\phi}{(1+\cos\phi)} = \boxed{\frac{\epsilon_0 V_0}{\pi a}\tan(\phi/2).}$$

Problem 7.42

(a) Faraday's law says $\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, so $\mathbf{E}=0 \Rightarrow \frac{\partial\mathbf{B}}{\partial t}=0 \Rightarrow \mathbf{B}(\mathbf{r})$ is independent of t.

(b) Faraday's law in integral form (Eq. 7.18) says $\oint \mathbf{E}\cdot d\mathbf{l} = -d\Phi/dt$. In the wire itself $\mathbf{E}=0$, so Φ through the loop is constant.

(c) Ampère-Maxwell$\Rightarrow$ $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$, so $\mathbf{E}=0$, $\mathbf{B}=0 \Rightarrow \mathbf{J}=0$, and hence any current must be at the surface.

(d) From Eq. 5.68, a rotating shell produces a uniform magnetic field (inside): $\mathbf{B} = \frac{2}{3}\mu_0\sigma\omega a\hat{\mathbf{z}}$. So to *cancel* such a field, we need $\sigma\omega a = -\dfrac{3}{2}\dfrac{B_0}{\mu_0}$. Now $\mathbf{K} = \sigma\mathbf{v} = \sigma\omega a\sin\theta\,\hat{\boldsymbol{\phi}}$, so $\boxed{\mathbf{K} = -\dfrac{3B_0}{2\mu_0}\sin\theta\,\hat{\boldsymbol{\phi}}.}$

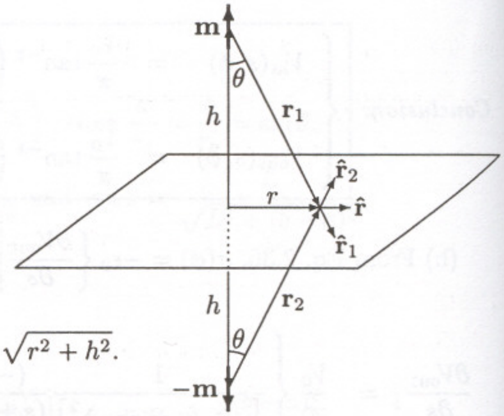
Problem 7.43

(a) To make the field *parallel* to the plane, we need image monopoles of the *same* sign (compare Figs. 2.13 and 2.14), so the image dipole points $\boxed{\text{down } (-z).}$

(b) From Prob. 6.3 (with $r \to 2z$):

$$F = \boxed{\frac{3\mu_0}{2\pi}\frac{m^2}{(2z)^4}.} \quad \frac{3\mu_0}{2\pi}\frac{m^2}{(2h)^4} = Mg \Rightarrow h = \boxed{\frac{1}{2}\left(\frac{3\mu_0 m^2}{2\pi Mg}\right)^{1/4}}.$$

(c) Using Eq. 5.87, and referring to the figure:



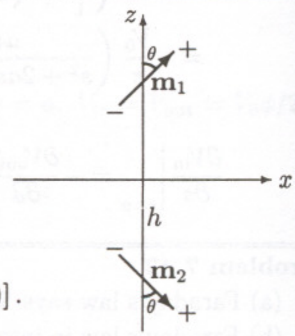
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{1}{(r_1)^3}\left\{[3(m\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_1)\,\hat{\mathbf{r}}_1 - m\hat{\mathbf{z}}] + [3(-m\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_2)\,\hat{\mathbf{r}}_2 + m\hat{\mathbf{z}}]\right\}$$
$$= \frac{3\mu_0 m}{4\pi(r_1)^3}\left[(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_1)\,\hat{\mathbf{r}}_1 - (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_2)\,\hat{\mathbf{r}}_2\right]. \quad \text{But } \hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_1 = -\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_2 = \cos\theta.$$
$$= -\frac{3\mu_0 m}{4\pi(r_1)^3}\cos\theta(\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2). \quad \text{But } \hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2 = 2\sin\theta\,\hat{\mathbf{r}}.$$
$$= -\frac{3\mu_0 m}{2\pi(r_1)^3}\sin\theta\cos\theta\,\hat{\mathbf{r}}. \quad \text{But } \sin\theta = \frac{r}{r_1},\ \cos\theta = \frac{h}{r_1},\ \text{and } r_1 = \sqrt{r^2+h^2}.$$
$$= -\frac{3\mu_0 m h}{2\pi}\frac{r}{(r^2+h^2)^{5/2}}\,\hat{\mathbf{r}}.$$

Now $\mathbf{B} = \mu_0(\mathbf{K}\times\hat{\mathbf{z}}) \Rightarrow \hat{\mathbf{z}}\times\mathbf{B} = \mu_0\,\hat{\mathbf{z}}\times(\mathbf{K}\times\hat{\mathbf{z}}) = \mu_0\left[\mathbf{K} - \hat{\mathbf{z}}(\mathbf{K}\cdot\hat{\mathbf{z}})\right] = \mu_0\mathbf{K}$. (I used the BAC-CAB rule, and noted that $\mathbf{K}\cdot\hat{\mathbf{z}} = 0$, because the surface current is in the xy plane.)

$$\mathbf{K} = \frac{1}{\mu_0}(\hat{\mathbf{z}}\times\mathbf{B}) = -\frac{3mh}{2\pi}\frac{r}{(r^2+h^2)^{5/2}}(\hat{\mathbf{z}}\times\hat{\mathbf{r}}) = -\frac{3mh}{2\pi}\frac{r}{(r^2+h^2)^{5/2}}\,\hat{\boldsymbol{\phi}}. \quad \text{qed}$$

Problem 7.44

Say the angle between the dipole ($\mathbf{m}_1$) and the z axis is θ (see diagram).



The field of the image dipole ($\mathbf{m}_2$) is

$$\mathbf{B}(z) = \frac{\mu_0}{4\pi}\frac{1}{(h+z)^3}\left[3(\mathbf{m}_2\cdot\hat{\mathbf{z}})\,\hat{\mathbf{z}} - \mathbf{m}_2\right]$$

for points on the z axis (Eq. 5.87). The torque on $\mathbf{m}_1$ is (Eq. 6.1)

$$\mathbf{N} = \mathbf{m}_1\times\mathbf{B} = \frac{\mu_0}{4\pi(2h)^3}\left[3(\mathbf{m}_2\cdot\hat{\mathbf{z}})(\mathbf{m}_1\times\hat{\mathbf{z}}) - (\mathbf{m}_1\times\mathbf{m}_2)\right].$$

But $\mathbf{m}_1 = m(\sin\theta\,\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}})$, $\mathbf{m}_2 = m(\sin\theta\,\hat{\mathbf{x}} - \cos\theta\hat{\mathbf{z}})$, so $\mathbf{m}_2\cdot\hat{\mathbf{z}} = -m\cos\theta$, $\mathbf{m}_1\times\hat{\mathbf{z}} = -m\sin\theta\,\hat{\mathbf{y}}$, and $\mathbf{m}_1\times\mathbf{m}_2 = 2m^2\sin\theta\cos\theta\hat{\mathbf{y}}$.

$$\mathbf{N} = \frac{\mu_0}{4\pi(2h)^3}\left[3m^2\sin\theta\cos\theta\,\hat{\mathbf{y}} - 2m^2\sin\theta\cos\theta\,\hat{\mathbf{y}})\right] = \frac{\mu_0 m^2}{4\pi(2h)^3}\sin\theta\cos\theta\,\hat{\mathbf{y}}.$$

Evidently the torque is zero for $\theta = 0$, $\pi/2$, or π. But 0 and π are clearly unstable, since the nearby ends of the dipoles (minus, in the figure) dominate, and they repel. The stable configuration is $\theta = \pi/2$: $\boxed{\text{parallel to the surface}}$ (contrast Prob. 4.6).

In this orientation, $\mathbf{B}(z) = -\frac{\mu_0 m}{4\pi(h+z)^3}\,\hat{\mathbf{x}}$, and the force on $\mathbf{m}_1$ is (Eq. 6.3):

$$\mathbf{F} = \boldsymbol{\nabla}\left[-\frac{\mu_0 m^2}{4\pi(h+z)^3}\right]\Bigg|_{z=h} = \frac{3\mu_0 m^2}{4\pi(h+z)^4}\,\hat{\mathbf{z}}\Bigg|_{z=h} = \frac{3\mu_0 m^2}{4\pi(2h)^4}\,\hat{\mathbf{z}}.$$

At equilibrium this force upward balances the weight Mg:

$$\frac{3\mu_0 m^2}{4\pi(2h)^4} = Mg \Rightarrow h = \boxed{\frac{1}{2}\left(\frac{3\mu_0 m^2}{4\pi Mg}\right)^{1/4}}.$$

Incidentally, this is $(1/2)^{1/4} = 0.84$ times the height it would adopt in the orientation *perpendicular* to the plane (Prob. 7.43b).

Problem 7.45

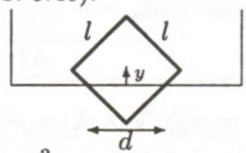
$\mathbf{f} = \mathbf{v} \times \mathbf{B}$; $\mathbf{v} = \omega a \sin\theta\,\hat{\boldsymbol{\phi}}$; $\mathbf{f} = \omega a B_0 \sin\theta(\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}})$. $\mathcal{E} = \int \mathbf{f}\cdot d\mathbf{l}$, and $d\mathbf{l} = a\,d\theta\,\hat{\boldsymbol{\theta}}$.

So $\mathcal{E} = \omega a^2 B_0 \int_0^{\pi/2} \sin\theta(\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}})\cdot\hat{\boldsymbol{\theta}}\,d\theta$. But $\hat{\boldsymbol{\theta}}\cdot(\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}) = \hat{\mathbf{z}}\cdot(\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}}) = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = \cos\theta$.

$$\mathcal{E} = \omega a^2 B_0 \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \omega a^2 B_0 \left[\frac{\sin^2\theta}{2}\right]\Big|_0^{\pi/2} = \boxed{\frac{1}{2}\omega a^2 B_0}$$ (same as the rotating disk in Ex. 7.4).

Problem 7.46

(a) In the "square" orientation (□), it falls at terminal velocity $\boxed{v_{\text{square}} = \dfrac{mgR}{B^2 l^2}}$ (Prob. 7.11). In the "diamond" orientation (◇), the magnetic force upward is $F = IBd$ (Prob. 5.40).



The flux is $\Phi = B\left[l^2 - (d/2)^2\right]$, and $d/2 = l/\sqrt{2} - y$,
so $\Phi = B\left[l^2 - (l/\sqrt{2} - y)^2\right]$.
$\mathcal{E} = -\frac{d\Phi}{dt} = -2B\left(l/\sqrt{2} - y\right)\frac{dy}{dt}$. But $\frac{dy}{dt} = -v$.
So $\mathcal{E} = 2Bv\left(l/\sqrt{2} - y\right) = IR \Rightarrow I = \frac{2Bv}{R}\left(l/\sqrt{2} - y\right)$; $F = 2\cdot\frac{2B^2 v}{R}\left(l/\sqrt{2} - y\right)^2 = mg$ (at terminal velocity).

$$\boxed{v_{\text{diamond}} = \frac{mgR}{4B^2\left(l/\sqrt{2} - y\right)^2}.}$$ (This works for negative y as well as positive, if you replace y by $|y|$.)

Thus $\dfrac{v_{\text{square}}}{v_{\text{diamond}}} = \left(\dfrac{mgR}{B^2 l^2}\right)\dfrac{4B^2\left(l/\sqrt{2} - y\right)^2}{mgR} = \boxed{\left(\sqrt{2} - 2y/l\right)^2.}$ At first $(y \sim l/\sqrt{2})$ the "diamond" falls faster; toward the halfway mark $(y \sim 0)$, the "square" falls twice as fast; then the diamond again takes over. The *total* time it takes for the square to fall is:

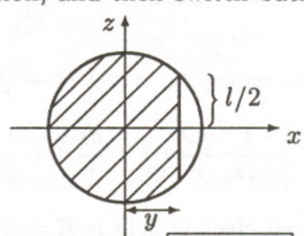
$$t_{\text{square}} = \frac{l}{v_{\text{square}}} = \boxed{\frac{B^2 l^3}{mgR}}$$

(assuming it always goes at the terminal velocity, which—as we found in Prob. 7.11—is close to the truth, if the field is strong). For the diamond, t is

$$-\int \frac{dy}{v_{\text{diamond}}} = -\frac{8B^2}{mgR}\int_{l/\sqrt{2}}^{0} -\left(l/\sqrt{2} - y\right)^2 dy = \frac{8B^2}{mgR}\left[\frac{1}{3}\left(l/\sqrt{2} - y\right)^3\right]\Bigg|_{l/\sqrt{2}}^{0} = \frac{8B^2}{mgR}\frac{1}{3}\frac{l^3}{2\sqrt{2}} = \boxed{\frac{2\sqrt{2}}{3}\frac{B^2 l^3}{mgR}.}$$

So $t_{\text{square}}/t_{\text{diamond}} = 3/2\sqrt{2} = 1.06$. The *"square" falls faster, overall.* If free to rotate, it would start out in the "diamond" orientation, switch to "square" for the middle portion, and then switch back to diamond, always trying to present the minimum *chord* at the field's edge.

(b) $F = IBl$; $\Phi = 2B\int_{-a}^{y}\sqrt{a^2 - x^2}\,dx$ (a = radius of circle).
$\mathcal{E} = -\frac{d\Phi}{dt} = -2B\sqrt{a^2 - y^2}\frac{dy}{dt} = 2Bv\sqrt{a^2 - y^2} = IR$.
$I = \frac{2Bv}{R}\sqrt{a^2 - y^2}$; $l/2 = \sqrt{a^2 - y^2}$. So $F = \frac{4B^2 v}{R}(a^2 - y^2) = mg$.



$$v_{\text{circle}} = \frac{mgR}{4B^2(a^2 - y^2)};$$

$$t_{\text{circle}} = \int_{+a}^{-a} -\frac{dy}{v} = \frac{4B^2}{mgR}\int_{-a}^{a}(a^2 - y^2)dy = \frac{4B^2}{mgR}\left(a^2 y - \frac{1}{3}y^3\right)\Big|_{-a}^{a} = \frac{4B^2}{mgR}\left(\frac{4}{3}a^3\right) = \boxed{\frac{16}{3}\frac{B^2 a^3}{mgR}.}$$

Problem 7.47

(a) In magnetostatics

$$\nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \ \Rightarrow\ \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'.$$

For Faraday electric fields (with $\rho = 0$), therefore,

$$\nabla \cdot \mathbf{E} = 0,\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ \Rightarrow\ \mathbf{E}(\mathbf{r}, t) = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{\mathbf{B}(\mathbf{r}', t) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

(with the substitution $\mathbf{J} \to -\frac{1}{\mu_0}\frac{\partial \mathbf{B}}{\partial t}$.)

(b) From Prob. 5.50a,

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi}\int \frac{\mathbf{B}(\mathbf{r}', t) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau', \text{ so } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}.$$

[*Check:* $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\frac{\partial \mathbf{B}}{\partial t}$, and we recover Faraday's law.]

(c) The Coulomb field is zero inside and $\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{\sigma 4\pi R^2}{r^2}\,\hat{\mathbf{r}} = \frac{\sigma R^2}{\epsilon_0 r^2}\,\hat{\mathbf{r}}$ outside. The Faraday field is $-\frac{\partial \mathbf{A}}{\partial t}$, where $\mathbf{A}$ is given (in the quasistatic approximation) by Eq. 5.67, with ω a function of time. Letting $\dot{\omega} \equiv d\omega/dt$,

$$\boxed{\mathbf{E}(r, \theta, \phi, t) = \begin{cases} \dfrac{\mu_0 R \dot{\omega} \sigma}{3} r \sin\theta\, \hat{\boldsymbol{\phi}} & (r < R), \\[2ex] \dfrac{\sigma R^2}{\epsilon_0 r^2}\,\hat{\mathbf{r}} + \dfrac{\mu_0 R^4 \dot{\omega} \sigma}{3}\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}} & (r > R). \end{cases}}$$

Problem 7.48

$qBR = mv$ (Eq. 5.3). If R is to stay fixed, then $qR\dfrac{dB}{dt} = m\dfrac{dv}{dt} = ma = F = qE$, or $E = R\dfrac{dB}{dt}$. But $\oint \mathbf{E} \cdot d\mathbf{l} = -\dfrac{d\Phi}{dt}$, so $E\, 2\pi R = -\dfrac{d\Phi}{dt}$, so $-\dfrac{1}{2\pi R}\dfrac{d\Phi}{dt} = R\dfrac{dB}{dt}$, or $B = -\dfrac{1}{2}\left(\dfrac{1}{\pi R^2}\Phi\right) +$ constant. If at time $t = 0$ the field is off, then the constant is zero, and $B(R) = \dfrac{1}{2}\left(\dfrac{1}{\pi R^2}\Phi\right)$ (in magnitude). Evidently the field at R must be *half* the *average* field over the cross-section of the orbit. qed

Problem 7.49

Initially, $\frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2} \Rightarrow T = \frac{1}{2}mv^2 = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r}$. After the magnetic field is on, the electron circles in a new orbit, of radius r_1 and velocity v_1:

$$\frac{mv_1^2}{r_1} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r_1^2} + qv_1 B \Rightarrow T_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r_1} + \frac{1}{2}qv_1 r_1 B.$$

But $r_1 = r + dr$, so $(r_1)^{-1} = r^{-1}\left(1 + \frac{dr}{r}\right)^{-1} \cong r^{-1}\left(1 - \frac{dr}{r}\right)$, while $v_1 = v + dv$, $B = dB$. To first order, then,

$$T_1 = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r}\left(1 - \frac{dr}{r}\right) + \frac{1}{2}q(vr)\, dB, \text{ and hence } dT = T_1 - T = \frac{qvr}{2}\, dB - \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\, dr.$$

Now, the induced electric field is $E = \frac{r}{2}\frac{dB}{dt}$ (Ex. 7.7), so $m\frac{dv}{dt} = qE = \frac{qr}{2}\frac{dB}{dt}$, or $m\, dv = \frac{qr}{2}dB$. The increase in kinetic energy is therefore $dT = d(\frac{1}{2}mv^2) = mv\, dv = \frac{qvr}{2}dB$. Comparing the two expressions, I conclude that $dr = 0$. qed

Problem 7.50

$\mathcal{E} = -\frac{d\Phi}{dt} = -\alpha$. So the current in R_1 and R_2 is $I = \frac{\alpha}{R_1 + R_2}$; by Lenz's law, it flows counterclockwise. Now the voltage across R_1 (which voltmeter #1 measures) is $V_1 = IR_1 = \boxed{\frac{\alpha R_1}{R_1 + R_2}}$ (V_b is the *higher* potential), and $V_2 = -IR_2 = \boxed{\frac{-\alpha R^2}{R_1 + R_2}}$ (V_b is *lower*).

Problem 7.51

$$\mathcal{E} = vBh = -L\frac{dI}{dt};\ F = IhB = m\frac{dv}{dt};\ \frac{d^2v}{dt^2} = \frac{hB}{m}\frac{dI}{dt} = -\frac{hB}{m}\left(\frac{hB}{L}\right)v,\ \boxed{\frac{d^2v}{dt^2} = -\omega^2 v,}\ \text{with}\ \boxed{\omega = \frac{hB}{\sqrt{mL}}}.$$

Problem 7.52

A point on the upper loop: $\mathbf{r}_2 = (a\cos\phi_2, a\sin\phi_2, z)$; a point on the lower loop: $\mathbf{r}_1 = (b\cos\phi_1, b\sin\phi_1, 0)$.

$$\begin{aligned}
\mathscr{r}^2 = (\mathbf{r}_2 - \mathbf{r}_1)^2 &= (a\cos\phi_2 - b\cos\phi_1)^2 + (a\sin\phi_2 - b\sin\phi_1)^2 + z^2 \\
&= a^2\cos^2\phi_2 - 2ab\cos\phi_2\cos\phi_1 + b^2\cos^2\phi_1 + a^2\sin^2\phi_2 - 2ab\sin\phi_1\sin\phi_2 + b^2\sin^2\phi_1 + z^2 \\
&= a^2 + b^2 + z^2 - 2ab(\cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1) = a^2 + b^2 + z^2 - 2ab\cos(\phi_2 - \phi_1) \\
&= (a^2 + b^2 + z^2)[1 - 2\beta\cos(\phi_2 - \phi_1)] = \frac{ab}{\beta}[1 - 2\beta\cos(\phi_2 - \phi_1)].
\end{aligned}$$

$d\mathbf{l}_1 = b\,d\phi_1\,\hat{\boldsymbol{\phi}}_1 = b\,d\phi_1[-\sin\phi_1\,\hat{\mathbf{x}} + \cos\phi_1\,\hat{\mathbf{y}}]$; $d\mathbf{l}_2 = a\,d\phi_2\,\hat{\boldsymbol{\phi}}_2 = a\,d\phi_2[-\sin\phi_2\,\hat{\mathbf{x}} + \cos\phi_2\,\hat{\mathbf{y}}]$, so
$d\mathbf{l}_1 \cdot d\mathbf{l}_2 = ab\,d\phi_1\,d\phi_2[\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2] = ab\cos(\phi_2 - \phi_1)\,d\phi_1\,d\phi_2$.

$$M = \frac{\mu_0}{4\pi}\oint\oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}} = \frac{\mu_0}{4\pi}\frac{ab}{\sqrt{ab/\beta}}\int\int \frac{\cos(\phi_2 - \phi_1)}{\sqrt{1 - 2\beta\cos(\phi_2 - \phi_1)}}\,d\phi_2\,d\phi_1.$$

Both integrals run from 0 to 2π. Do the ϕ_2 integral first, letting $u \equiv \phi_2 - \phi_1$:

$$\int_{-\phi_1}^{2\pi - \phi_1} \frac{\cos u}{\sqrt{1 - 2\beta\cos u}}\,du = \int_0^{2\pi} \frac{\cos u}{\sqrt{1 - 2\beta\cos u}}\,du$$

(since the integral runs over a complete cycle of $\cos u$, we may as well change the limits to $0 \to 2\pi$). Then the ϕ_1 integral is just 2π, and

$$M = \frac{\mu_0}{4\pi}\sqrt{ab\beta}\,2\pi\int_0^{2\pi} \frac{\cos u}{\sqrt{1 - 2\beta\cos u}}\,du = \frac{\mu_0}{2}\sqrt{ab\beta}\int_0^{2\pi} \frac{\cos u}{\sqrt{1 - 2\beta\cos u}}\,du.$$

(a) If a is small, then $\beta \ll 1$, so (using the binomial theorem)

$$\frac{1}{\sqrt{1 - 2\beta\cos u}} \cong 1 + \beta\cos u,\ \text{and}\ \int_0^{2\pi} \frac{\cos u}{\sqrt{1 - 2\beta\cos u}}\,du \cong \int_0^{2\pi}\cos u\,du + \beta\int_0^{2\pi}\cos^2 u\,du = 0 + \beta\pi,$$

and hence $M = (\mu_0\pi/2)\sqrt{ab\beta^3}$. Moreover, $\beta \cong ab/(b^2 + z^2)$, so $M \cong \boxed{\frac{\mu_0\pi a^2 b^2}{2(b^2 + z^2)^{3/2}}}$ (same as in Prob. 7.20).

(b) More generally,

$$(1+\epsilon)^{-1/2} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots \Longrightarrow \frac{1}{\sqrt{1-2\beta\cos u}} = 1 + \beta\cos u + \frac{3}{2}\beta^2\cos^2 u + \frac{5}{2}\beta^3\cos^3 u + \cdots,$$

so

$$M = \frac{\mu_0}{2}\sqrt{ab\beta}\left\{\int_0^{2\pi}\cos u\,du + \beta\int_0^{2\pi}\cos^2 u\,du + \frac{3}{2}\beta^2\int_0^{2\pi}\cos^3 u\,du + \frac{5}{2}\beta^3\int_0^{2\pi}\cos^4 u\,du + \cdots\right\}$$

$$= \frac{\mu_0}{2}\sqrt{ab\beta}\left[0 + \beta(\pi) + \frac{3}{2}\beta^2(0) + \frac{5}{2}\beta^3(\frac{3}{4}\pi) + \cdots\right] = \boxed{\frac{\mu_0\pi}{2}\sqrt{ab\beta^3}\left(1 + \frac{15}{8}\beta^2 + (\;)\beta^4 + \cdots\right).} \quad \text{qed}$$

Problem 7.53

Let Φ be the flux of $\mathbf{B}$ through a *single* loop of either coil, so that $\Phi_1 = N_1\Phi$ and $\Phi_2 = N_2\Phi$. Then

$$\mathcal{E}_1 = -N_1\frac{d\Phi}{dt}, \quad \mathcal{E}_2 = -N_2\frac{d\Phi}{dt}, \text{ so } \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}. \text{ qed}$$

Problem 7.54

(a) Suppose current I_1 flows in coil 1, and I_2 in coil 2. Then (if Φ is the flux through *one* turn):

$$\Phi_1 = I_1L_1 + MI_2 = N_1\Phi; \quad \Phi_2 = I_2L_2 + MI_1 = N_2\Phi, \quad \text{or } \Phi = I_1\frac{L_1}{N_1} + I_2\frac{M}{N_1} = I_2\frac{L_2}{N_2} + I_1\frac{M}{N_2}.$$

In case $I_1 = 0$, we have $\frac{M}{N_1} = \frac{L_2}{N_2}$; if $I_2 = 0$, we have $\frac{L_1}{N_1} = \frac{M}{N_2}$. Dividing: $\frac{M}{L_1} = \frac{L_2}{M}$, or $L_1L_2 = M^2$. qed

(b) $-\mathcal{E}_1 = \frac{d\Phi_1}{dt} = L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt} = V_1\cos(\omega t)$; $-\mathcal{E}_2 = \frac{d\Phi_2}{dt} = L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} = -I_2R$. qed

(c) Multiply the first equation by L_2: $L_1L_2\frac{dI_1}{dt} + L_2\frac{dI_2}{dt}M = L_2V_1\cos\omega t$. Plug in $L_2\frac{dI_2}{dt} = -I_2R - M\frac{dI_1}{dt}$. $M^2\frac{dI_1}{dt} - MRI_2 - M^2\frac{dI_1}{dt} = L_2V_1\cos\omega t \Rightarrow \boxed{I_2(t) = -\frac{L_2V_1}{MR}\cos\omega t.}$ $L_1\frac{dI_1}{dt} + M\left(\frac{L_2V_1}{MR}\omega\sin\omega t\right) = V_1\cos\omega t$.

$$\frac{dI_1}{dt} = \frac{V_1}{L_1}\left(\cos\omega t - \frac{L_2}{R}\omega\sin\omega t\right) \Rightarrow \boxed{I_1(t) = \frac{V_1}{L_1}\left(\frac{1}{\omega}\sin\omega t + \frac{L_2}{R}\cos\omega t\right).}$$

(d) $\dfrac{V_{\text{out}}}{V_{\text{in}}} = \dfrac{I_2R}{V_1\cos\omega t} = \dfrac{-\frac{L_2V_1}{MR}\cos\omega t\,R}{V_1\cos\omega t} = -\dfrac{L_2}{M} = -\dfrac{N_2}{N_1}$. The ratio of the amplitudes is $\dfrac{N_2}{N_1}$. qed

(e) $P_{\text{in}} = V_{\text{in}}I_1 = (V_1\cos\omega t)(\frac{V_1}{L_1})\left(\frac{1}{\omega}\sin\omega t + \frac{L_2}{R}\cos\omega t\right) = \boxed{\frac{(V_1)^2}{L_1}\left(\frac{1}{\omega}\sin\omega t\cos\omega t + \frac{L_2}{R}\cos^2\omega t\right).}$

$P_{\text{out}} = V_{\text{out}}I_2 = (I_2)^2R = \boxed{\frac{(L_2V_1)^2}{M^2R}\cos^2\omega t.}$ Average of $\cos^2\omega t$ is $1/2$; average of $\sin\omega t\cos\omega t$ is zero.

So $\langle P_{\text{in}}\rangle = \frac{1}{2}(V_1)^2\left(\frac{L_2}{L_1R}\right)$; $\langle P_{\text{out}}\rangle = \frac{1}{2}(V_1)^2\left[\frac{(L_2)^2}{M^2R}\right] = \frac{1}{2}(V_1)^2\left[\frac{(L_2)^2}{L_1L_2R}\right]$; $\boxed{\langle P_{\text{in}}\rangle = \langle P_{\text{out}}\rangle = \frac{(V_1)^2L_2}{2L_1R}.}$

Problem 7.55

(a) The continuity equation says $\frac{\partial\rho}{\partial t} = -\,\boldsymbol{\nabla}\cdot\mathbf{J}$. Here the right side is independent of t, so we can integrate: $\rho(t) = (-\,\boldsymbol{\nabla}\cdot\mathbf{J})t+$ constant. The "constant" may be a function of $\mathbf{r}$—it's only constant with respect to t. So, putting in the $\mathbf{r}$ dependence explicitly, and noting that $\boldsymbol{\nabla}\cdot\mathbf{J} = -\dot{\rho}(\mathbf{r},0)$, $\rho(\mathbf{r},t) = \dot{\rho}(\mathbf{r},0)t + \rho(\mathbf{r},0)$. qed

(b) Suppose $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau$ and $\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau$. We want to show that $\nabla\cdot\mathbf{B} = 0$, $\nabla\times\mathbf{B} = \mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$; $\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho$, and $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, provided that $\mathbf{J}$ is independent of t.

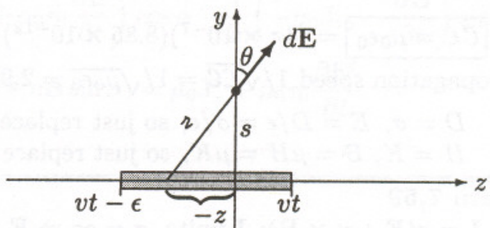
We know from Ch. 2 that Coulomb's law $\left(\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau\right)$ satisfies $\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho$ and $\nabla\times\mathbf{E} = 0$. Since $\mathbf{B}$ is constant (in time), the $\nabla\cdot\mathbf{E}$ and $\nabla\times\mathbf{E}$ equations are satisfied. From Chapter 5 (specifically, Eqs. 5.45-5.48) we know that the Biot-Savart law satisfies $\nabla\cdot\mathbf{B} = 0$. It remains only to check $\nabla\times\mathbf{B}$. The argument in Sect. 5.3.2 carries through until the equation following Eq. 5.52, where I invoked $\nabla'\cdot\mathbf{J} = 0$. In its place we now put $\nabla'\cdot\mathbf{J} = -\dot{\rho}$:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} - \frac{\mu_0}{4\pi}\int \underbrace{(\mathbf{J}\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}}_{(-\mathbf{J}tp\nabla')\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\ \text{(Eq. 5.52)}}\,d\tau \quad \text{(Eqs. 5.49-5.51)}$$

Integration by parts yields two terms, one of which becomes a surface integral, and goes to zero. The other is $\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}\nabla'\cdot\mathbf{J} = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}(-\dot{\rho})$. So:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} - \frac{\mu_0}{4\pi}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}(-\dot{\rho})d\tau = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left\{\frac{1}{4\pi\epsilon_0}\int\frac{\rho\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^3}\,d\tau\right\} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}. \quad \text{qed}$$

Problem 7.56



(a) $dE_z = \dfrac{1}{4\pi\epsilon_0}\dfrac{(-\lambda)dz}{\mathscr{r}^2}\sin\theta$

$\sin\theta = \dfrac{-z}{\mathscr{r}}$; $\mathscr{r} = \sqrt{z^2+s^2}$

$$E_z = \frac{\lambda}{4\pi\epsilon_0}\int\frac{z\,dz}{(z^2+s^2)^{3/2}} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{-1}{\sqrt{z^2+s^2}}\right]\Bigg|_{vt-\epsilon}^{vt}$$

$$\boxed{E_z = \frac{\lambda}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{(vt-\epsilon)^2+s^2}} - \frac{1}{\sqrt{(vt)^2+s^2}}\right\}.}$$

(b)

$$\Phi_E = \frac{\lambda}{4\pi\epsilon_0}\int_0^a\left\{\frac{1}{\sqrt{(vt-\epsilon)^2+s^2}} - \frac{1}{\sqrt{(vt)^2+s^2}}\right\}2\pi s\,ds = \frac{\lambda}{2\epsilon_0}\left[\sqrt{(vt-\epsilon)^2+s^2} - \sqrt{(vt)^2+s^2}\right]\Bigg|_0^a$$

$$= \boxed{\frac{\lambda}{2\epsilon_0}\left[\sqrt{(vt-\epsilon)^2+a^2} - \sqrt{(vt)^2+a^2} - (\epsilon - vt) + (vt)\right].}$$

(c) $$I_d = \epsilon_0\frac{d\Phi_E}{dt} = \boxed{\frac{\lambda}{2}\left\{\frac{v(vt-\epsilon)}{\sqrt{(vt-\epsilon)^2+a^2}} - \frac{v(vt)}{\sqrt{(vt)^2+a^2}} + 2v\right\}.}$$

As $\epsilon \to 0$, $vt < \epsilon$ also $\to 0$, so $I_d \to \frac{\lambda}{2}(2v) = \lambda v = I$. With an infinitesimal gap we attribute the magnetic field to *displacement* current, instead of real current, but we get the same answer. qed

Problem 7.57

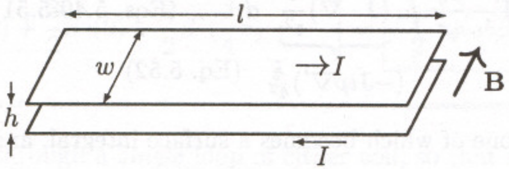
(a) $\nabla^2 V = \dfrac{1}{s}\dfrac{\partial}{\partial s}\left(s\dfrac{\partial(zf)}{\partial s}\right) + \dfrac{\partial^2(zf)}{\partial z^2} = \dfrac{z}{s}\dfrac{d}{ds}\left(s\dfrac{df}{ds}\right) = 0 \Rightarrow \dfrac{d}{ds}\left(s\dfrac{df}{ds}\right) = 0 \Rightarrow s\dfrac{df}{ds} = A$ (a constant) $\Rightarrow$ $A\dfrac{ds}{s} = df \Rightarrow f = A\ln(s/s_0)$ (s_0 another constant). But (ii) $\Rightarrow f(b) = 0$, so $\ln(b/s_0) = 0$, so $s_0 = b$, and

$V(s,z) = Az\ln(s/b)$. But (i) $\Rightarrow Az\ln(a/b) = -(I\rho z)/(\pi a^2)$, so $A = -\dfrac{I\rho}{\pi a^2}\dfrac{1}{\ln(a/b)}$; $\boxed{V(s,z) = -\dfrac{I\rho z\ln(s/b)}{\pi a^2\ln(a/b)}.}$

(b) $\mathbf{E} = -\boldsymbol{\nabla}V = -\dfrac{\partial V}{\partial s}\,\hat{\mathbf{s}} - \dfrac{\partial V}{\partial z}\,\hat{\mathbf{z}} = \dfrac{I\rho z}{\pi a^2}\dfrac{1}{s\ln(a/b)}\,\hat{\mathbf{s}} + \dfrac{I\rho}{\pi a^2}\dfrac{\ln(s/b)}{\ln(a/b)}\,\hat{\mathbf{z}} = \boxed{\dfrac{I\rho}{\pi a^2\ln(a/b)}\left(\dfrac{z}{s}\,\hat{\mathbf{s}} + \ln\left(\dfrac{s}{b}\right)\hat{\mathbf{z}}\right).}$

(c) $\sigma(z) = \epsilon_0\left[E_s(a^+) - E_s(a^-)\right] = \epsilon_0\left[\dfrac{I\rho}{\pi a^2\ln(a/b)}\left(\dfrac{z}{a}\right) - 0\right] = \boxed{\dfrac{\epsilon_0 I\rho z}{\pi a^3\ln(a/b)}.}$

Problem 7.58



(a) Parallel-plate capacitor: $E = \dfrac{1}{\epsilon_0}\sigma$; $V = Eh = \dfrac{1}{\epsilon_0}\dfrac{Q}{wl}h \Rightarrow C = \dfrac{Q}{V} = \dfrac{\epsilon_0 wl}{h} \Rightarrow \boxed{\mathcal{C} = \dfrac{\epsilon_0 w}{h}.}$

(b) $B = \mu_0 K = \mu_0\dfrac{I}{w}$; $\Phi = Bhl = \dfrac{\mu_0 I}{w}hl = LI \Rightarrow L = \dfrac{\mu_0 h}{w}l \Rightarrow \boxed{\mathcal{L} = \dfrac{\mu_0 h}{w}.}$

(c) $\boxed{\mathcal{CL} = \mu_0\epsilon_0} = (4\pi\times 10^{-7})(8.85\times 10^{-12}) = \boxed{1.112\times 10^{-17}\,\mathrm{s^2/m^2}.}$

(Propagation speed $1/\sqrt{\mathcal{LC}} = 1/\sqrt{\mu_0\epsilon_0} = 2.999\times 10^8\,\mathrm{m/s} = c$.)

(d) $D = \sigma$, $E = D/\epsilon = \sigma/\epsilon$, so just replace ϵ_0 by ϵ; $H = K$, $B = \mu H = \mu K$, so just replace μ_0 by μ. $\Big\}$ $\boxed{\mathcal{LC} = \epsilon\mu;}$ $\boxed{v = 1/\sqrt{\epsilon\mu}.}$

Problem 7.59

(a) $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B})$; $\mathbf{J}$ finite, $\sigma = \infty \Rightarrow \mathbf{E} + (\mathbf{v}\times\mathbf{B}) = 0$. Take the curl: $\boldsymbol{\nabla}\times\mathbf{E} + \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) = 0$. But Faraday's law says $\boldsymbol{\nabla}\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$. So $\dfrac{\partial\mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B})$. qed

(b) $\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \Rightarrow \oint\mathbf{B}\cdot d\mathbf{a} = 0$ for any closed surface. Apply this at time $(t + dt)$ to the surface consisting of $\mathcal{S}$, $\mathcal{S}'$, and $\mathcal{R}$:

$$\int_{\mathcal{S}'}\mathbf{B}(t+dt)\cdot d\mathbf{a} + \int_{\mathcal{R}}\mathbf{B}(t+dt)\cdot d\mathbf{a} - \int_{\mathcal{S}}\mathbf{B}(t+dt)\cdot d\mathbf{a} = 0$$

(the sign change in the third term comes from switching *outward* $d\mathbf{a}$ to *inward* $d\mathbf{a}$).

$$d\Phi = \int_{\mathcal{S}'}\mathbf{B}(t+dt)\cdot d\mathbf{a} - \int_{\mathcal{S}}\mathbf{B}(t)\cdot d\mathbf{a} = \int_{\mathcal{S}}\underbrace{\left[\mathbf{B}(t+dt) - \mathbf{B}(t)\right]}_{\frac{\partial\mathbf{B}}{\partial t}dt\text{ (for infinitesimal }dt)}\cdot d\mathbf{a} - \int_{\mathcal{R}}\mathbf{B}(t+dt)\cdot d\mathbf{a}$$

$$d\Phi = \left\{\int_{\mathcal{S}}\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a}\right\}dt - \int_{\mathcal{R}}\mathbf{B}(t+dt)\cdot\left[(d\mathbf{l}\times\mathbf{v})\,dt\right] \quad \text{(Figure 7.13).}$$

Since the second term is already first order in dt, we can replace $\mathbf{B}(t+dt)$ by $\mathbf{B}(t)$ (the distinction would be second order):

$$d\Phi = dt\int_{\mathcal{S}}\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a} - dt\oint_{\mathcal{C}}\underbrace{\mathbf{B}\cdot(d\mathbf{l}\times\mathbf{v})}_{(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}} = dt\left\{\int_{\mathcal{S}}\left(\frac{\partial\mathbf{B}}{\partial t}\right)\cdot d\mathbf{a} - \int_{\mathcal{S}}\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{a}\right\}.$$

$$\frac{d\Phi}{dt} = \int_S \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{B})\right]\cdot d\mathbf{a} = 0. \quad \text{qed}$$

Problem 7.60

(a)

$$\begin{aligned}
\nabla\cdot\mathbf{E}' &= (\nabla\cdot\mathbf{E})\cos\alpha + c(\nabla\cdot\mathbf{B})\sin\alpha = \frac{1}{\epsilon_0}\rho_e\cos\alpha + c\mu_0\rho_m\sin\alpha \\
&= \frac{1}{\epsilon_0}(\rho_e\cos\alpha + c\mu_0\epsilon_0\rho_m\sin\alpha) = \frac{1}{\epsilon_0}(\rho_e\cos\alpha + \frac{1}{c}\rho_m\sin\alpha) = \frac{1}{\epsilon_0}\rho_e'. \checkmark \\
\nabla\cdot\mathbf{B}' &= (\nabla\cdot\mathbf{B})\cos\alpha - \frac{1}{c}(\nabla\cdot\mathbf{E})\sin\alpha = \mu_0\rho_m\cos\alpha - \frac{1}{c\epsilon_0}\rho_e\sin\alpha \\
&= \mu_0(\rho_m\cos\alpha - \frac{1}{c\mu_0\epsilon_0}\rho_e\sin\alpha) = \mu_0(\rho_m\cos\alpha - c\rho_e\sin\alpha) = \mu_0\rho_m'. \checkmark \\
\nabla\times\mathbf{E}' &= (\nabla\times\mathbf{E})\cos\alpha + c(\nabla\times\mathbf{B})\sin\alpha = \left(-\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}\right)\cos\alpha + c\left(\mu_0\mathbf{J}_e + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\sin\alpha \\
&= -\mu_0(\mathbf{J}_m\cos\alpha - c\mathbf{J}_e\sin\alpha) - \frac{\partial}{\partial t}\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right) = -\mu_0\mathbf{J}_m' - \frac{\partial\mathbf{B}'}{\partial t}. \checkmark \\
\nabla\times\mathbf{B}' &= (\nabla\times\mathbf{B})\cos\alpha - \frac{1}{c}(\nabla\times\mathbf{E})\sin\alpha = \left(\mu_0\mathbf{J}_e + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\cos\alpha - \frac{1}{c}\left(-\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}\right)\sin\alpha \\
&= \mu_0(\mathbf{J}_e\cos\alpha + \frac{1}{c}\mathbf{J}_m\sin\alpha) + \mu_0\epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha) = \mu_0\mathbf{J}_e' + \mu_0\epsilon_0\frac{\partial\mathbf{E}'}{\partial t}. \checkmark
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{F}' &= q_e'(\mathbf{E}' + \mathbf{v}\times\mathbf{B}') + q_m'(\mathbf{B}' - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}') \\
&= \left(q_e\cos\alpha + \frac{1}{c}q_m\sin\alpha\right)\left[(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha) + \mathbf{v}\times\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right)\right] \\
&\quad + (q_m\cos\alpha - cq_e\sin\alpha)\left[\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right) - \frac{1}{c^2}\mathbf{v}\times(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha)\right] \\
&= q_e\Big[(\mathbf{E}\cos^2\alpha + c\mathbf{B}\sin\alpha\cos\alpha - c\mathbf{B}\sin\alpha\cos\alpha + \mathbf{E}\sin^2\alpha) \\
&\quad + \mathbf{v}\times\left(\mathbf{B}\cos^2\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha + \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha + \mathbf{B}\sin^2\alpha\right)\Big] \\
&\quad + q_m\Big[\left(\frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha + \mathbf{B}\sin^2\alpha + \mathbf{B}\cos^2\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha\right) \\
&\quad + \mathbf{v}\times\left(\frac{1}{c}\mathbf{B}\sin\alpha\cos\alpha - \frac{1}{c^2}\mathbf{E}\sin^2\alpha - \frac{1}{c^2}\mathbf{E}\cos^2\alpha - \frac{1}{c}\mathbf{B}\sin\alpha\cos\alpha\right)\Big] \\
&= q_e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right) = \mathbf{F}. \quad \text{qed}
\end{aligned}$$