

# Chapter 12

## Electrodynamics and Relativity

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### Problem 12.1

Let  $\mathbf{u}$  be the velocity of a particle in  $\mathcal{S}$ ,  $\bar{\mathbf{u}}$  its velocity in  $\bar{\mathcal{S}}$ , and  $\mathbf{v}$  the velocity of  $\bar{\mathcal{S}}$  with respect to  $\mathcal{S}$ . Galileo's velocity addition rule says that  $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{v}$ . For a free particle,  $\mathbf{u}$  is constant (that's Newton's first law in  $\mathcal{S}$ ).

- (a) If  $\mathbf{v}$  is constant, then  $\bar{\mathbf{u}} = \bar{\mathbf{u}} - \mathbf{v}$  is also constant, so Newton's first law holds in  $\bar{\mathcal{S}}$ , and hence  $\mathcal{S}$  is inertial.  
(b) If  $\bar{\mathcal{S}}$  is inertial, then  $\bar{\mathbf{u}}$  is also constant, so  $\mathbf{v} = \mathbf{u} - \bar{\mathbf{u}}$  is constant.
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### Problem 12.2

(a)  $m_A \mathbf{u}_A + m_B \mathbf{u}_B = m_C \mathbf{u}_C + m_D \mathbf{u}_D; \quad \mathbf{u}_i = \bar{\mathbf{u}}_i + \mathbf{v}.$

$$m_A(\bar{\mathbf{u}}_A + \mathbf{v}) + m_B(\bar{\mathbf{u}}_B + \mathbf{v}) = m_C(\bar{\mathbf{u}}_C + \mathbf{v}) + m_D(\bar{\mathbf{u}}_D + \mathbf{v}),$$

$$m_A \bar{\mathbf{u}}_A + m_B \bar{\mathbf{u}}_B + (m_A + m_B)\mathbf{v} = m_C \bar{\mathbf{u}}_C + m_D \bar{\mathbf{u}}_D + (m_C + m_D)\mathbf{v}.$$

Assuming mass is conserved,  $(m_A + m_B) = (m_C + m_D)$ , it follows that

$$m_A \bar{\mathbf{u}}_A + m_B \bar{\mathbf{u}}_B = m_C \bar{\mathbf{u}}_C + m_D \bar{\mathbf{u}}_D, \text{ so momentum is conserved in } \bar{\mathcal{S}}.$$

(b)  $\frac{1}{2}m_A u_A^2 + \frac{1}{2}m_B u_B^2 = \frac{1}{2}m_C u_C^2 + \frac{1}{2}m_D u_D^2 \Rightarrow$

$$\frac{1}{2}m_A(\bar{u}_A^2 + 2\bar{\mathbf{u}}_A \cdot \mathbf{v} + v^2) + \frac{1}{2}m_B(\bar{u}_B^2 + 2\bar{\mathbf{u}}_B \cdot \mathbf{v} + v^2) = \frac{1}{2}m_C(\bar{u}_C^2 + 2\bar{\mathbf{u}}_C \cdot \mathbf{v} + v^2) + \frac{1}{2}m_D(\bar{u}_D^2 + 2\bar{\mathbf{u}}_D \cdot \mathbf{v} + v^2)$$

$$\frac{1}{2}m_A \bar{u}_A^2 + \frac{1}{2}m_B \bar{u}_B^2 + 2\mathbf{v} \cdot (m_A \bar{\mathbf{u}}_A + m_B \bar{\mathbf{u}}_B) + \frac{1}{2}v^2(m_A + m_B)$$

$$= \frac{1}{2}m_C \bar{u}_C^2 + \frac{1}{2}m_D \bar{u}_D^2 + 2\mathbf{v} \cdot (m_C \bar{\mathbf{u}}_C + m_D \bar{\mathbf{u}}_D) + \frac{1}{2}v^2(m_C + m_D).$$

But the middle terms are equal by conservation of momentum, and the last terms are equal by conservation of mass, so  $\frac{1}{2}m_A \bar{u}_A^2 + \frac{1}{2}m_B \bar{u}_B^2 = \frac{1}{2}m_C \bar{u}_C^2 + \frac{1}{2}m_D \bar{u}_D^2$ . qed

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### Problem 12.3

(a)  $v_G = v_{AB} + v_{BC}; \quad v_E = \frac{v_{AB} + v_{BC}}{1 + v_{AB} v_{BC}/c^2} \approx v_G \left(1 - \frac{v_{AB} v_{BC}}{c^2}\right) \Rightarrow \frac{v_G - v_E}{v_G} = \frac{v_{AB} v_{BC}}{c^2}.$

In mi/h,  $c = (186,000 \text{ mi/s}) \times (3600 \text{ sec/hr}) = 6.7 \times 10^8 \text{ mi/hr.}$

$$\therefore \frac{v_G - v_E}{v_G} = \frac{(5)(60)}{(6.7 \times 10^8)^2} = 6.7 \times 10^{-16} \Rightarrow \boxed{6.7 \times 10^{-14}\% \text{ error, (pretty small!)}}$$

(b)  $\left(\frac{1}{2}c + \frac{3}{4}c\right) / \left(1 + \frac{1}{2} \cdot \frac{3}{4}\right) = \left(\frac{5}{4}c\right) / \left(\frac{11}{8}\right) = \boxed{\frac{10}{11}c} \text{ (still less than } c\text{).}$

(c) To simplify the notation, let  $\beta \equiv v_{AC}/c$ ,  $\beta_1 \equiv v_{AB}/c$ ,  $\beta_2 \equiv v_{BC}/c$ . Then Eq. 12.3 says:  $\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$ , or:

$$\beta^2 = \frac{\beta_1^2 + 2\beta_1 \beta_2 + \beta_2^2}{(1 + 2\beta_1 \beta_2 + \beta_1^2 \beta_2^2)} = \frac{1 + 2\beta_1 \beta_2 + \beta_1^2 \beta_2^2}{(1 + 2\beta_1 \beta_2 + \beta_1^2 \beta_2^2)} - \frac{(1 + \beta_1^2 \beta_2^2 - \beta_1^2 - \beta_2^2)}{(1 + 2\beta_1 \beta_2 + \beta_1^2 \beta_2^2)} = 1 - \frac{(1 - \beta_1^2)(1 - \beta_2^2)}{(1 + \beta_1 \beta_2)^2} = 1 - \Delta,$$

where  $\Delta \equiv (1 - \beta_1^2)(1 - \beta_2^2)/(1 + \beta_1\beta_2)^2$  is clearly a *positive* number. So  $\beta_2 < 1$ , and hence  $|v_{AC}| < c$ . qed

### Problem 12.4

(a) Velocity of bullet relative to ground:  $\frac{1}{2}c + \frac{1}{3}c = \frac{5}{6}c = \frac{10}{12}c$ .

Velocity of getaway car:  $\frac{3}{4}c = \frac{9}{12}c$ . Since  $v_b > v_g$ , bullet does reach target.

(b) Velocity of bullet relative to ground:  $\frac{\frac{1}{2}c + \frac{1}{3}c}{1 + \frac{1}{2} \cdot \frac{1}{3}} = \frac{\frac{5}{6}c}{\frac{7}{6}} = \frac{5}{7}c = \frac{20}{28}c$ .

Velocity of getaway car:  $\frac{3}{4}c = \frac{21}{28}c$ . Since  $v_g > v_b$ , bullet does not reach target.

### Problem 12.5

(a) Light from the 90th clock took  $\frac{90 \times 10^9 \text{ m}}{3 \times 10^8 \text{ m/s}} = 300 \text{ s} = 5 \text{ min}$  to reach me, so the time I see on the clock is 11:55 am.

(b) I observe 12 noon.

### Problem 12.6

$\left\{ \begin{array}{l} \text{light signal leaves } a \text{ at time } t'_a; \text{ arrives at earth at time } t_a = t'_a + d_a/c, \\ \text{light signal leaves } b \text{ at time } t'_b; \text{ arrives at earth at time } t_b = t'_b + d_b/c. \end{array} \right.$

$$\therefore \Delta t = t_b - t_a = t'_b - t'_a + \frac{(d_b - d_a)}{c} = \Delta t' + \frac{(-v\Delta t' \cos \theta)}{c} = \Delta t' \left[ 1 - \frac{v}{c} \cos \theta \right].$$

(Here  $d_a$  is the distance from  $a$  to earth, and  $d_b$  is the distance from  $b$  to earth.)

$$\Delta s = v\Delta t' \sin \theta = \frac{v \sin \theta \Delta t}{(1 - v/c \cos \theta)}; \quad u = \frac{v \sin \theta}{(1 - \frac{v}{c} \cos \theta)} \text{ is the apparent velocity.}$$

$$\begin{aligned} \frac{du}{d\theta} &= \frac{v[(1 - \frac{v}{c} \cos \theta)(\cos \theta) - \sin \theta(\frac{v}{c} \sin \theta)]}{(1 - \frac{v}{c} \cos \theta)^2} = 0 \Rightarrow (1 - \frac{v}{c} \cos \theta) \cos \theta = \frac{v}{c} \sin^2 \theta \\ &\Rightarrow \cos \theta = \frac{v}{c} (\sin^2 \theta + \cos^2 \theta) = \frac{v}{c} \end{aligned}$$

$\theta_{\max} = \cos^{-1}(v/c)$ . At this maximal angle,  $u = \frac{v\sqrt{1-v^2/c^2}}{1-v^2/c^2} = \frac{v}{\sqrt{1-v^2/c^2}}$ .

As  $v \rightarrow c$ ,  $u \rightarrow \infty$ , because the denominator  $\rightarrow 0$ , even though  $v < c$ .

### Problem 12.7

The student has not taken into account time dilation of the muon's "internal clock". In the *laboratory*, the muon lasts  $\gamma\tau = \frac{\tau}{\sqrt{1-v^2/c^2}}$ , where  $\tau$  is the "proper" lifetime,  $2 \times 10^{-6} \text{ s}$ . Thus

$$v = \frac{d}{\tau/\sqrt{1-v^2/c^2}} = \frac{d}{\tau} \sqrt{1-v^2/c^2}, \text{ where } d = 800 \text{ m.}$$

$$\left(\frac{\tau}{d}\right)^2 v^2 = 1 - \frac{v^2}{c^2}; \quad v^2 \left[ \left(\frac{\tau}{d}\right)^2 + \frac{1}{c^2} \right] = 1; \quad v^2 = \frac{1}{(\tau/d)^2 + (1/c)^2}.$$

$$\frac{v^2}{c^2} = \frac{1}{1 + (\tau c/d)^2}; \quad \frac{\tau c}{d} = \frac{(2 \times 10^{-6})(3 \times 10^8)}{800} = \frac{6}{8} = \frac{3}{4}; \quad \frac{v^2}{c^2} = \frac{1}{1 + 9/16} = \frac{16}{25}; \quad v = \frac{4}{5}c.$$

**Problem 12.8**

(a) Rocket clock runs slow; so earth clock reads  $\gamma t = \frac{1}{\sqrt{1-v^2/c^2}} \cdot 1 \text{ hr}$ . Here  $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-9/25}} = \frac{5}{4}$ .

$\therefore$  According to earth clocks signal was sent 1 hr and 15 min after take-off.

(b) By earth observer, rocket is now a distance  $(\frac{3}{5}c)(\frac{5}{4})(1 \text{ hr}) = \frac{3}{4}c \text{ hr}$  (three-quarters of a light hour) away. Light signal will therefore take  $\frac{3}{4} \text{ hr}$  to return to earth. Since it left 1 hr and 15 min after departure, light signal reaches earth 2 hrs after takeoff.

(c) Earth clocks run slow:  $t_{\text{rocket}} = \gamma \cdot (2 \text{ hrs}) = \frac{5}{4} \cdot (2 \text{ hrs}) = \boxed{2.5 \text{ hrs.}}$

**Problem 12.9**

$$L_c = 2L_v; \frac{L_c}{\gamma_c} = \frac{L_v}{\gamma_v}; \text{ so } \frac{2}{\gamma_c} = \frac{1}{\gamma_v} = \sqrt{1 - (\frac{1}{2})^2} = \sqrt{\frac{3}{4}}; \frac{1}{\gamma_c^2} = 1 - \frac{v^2}{c^2} = \frac{3}{16}; \frac{v^2}{c^2} = 1 - \frac{3}{16} = \frac{13}{16}; v = \frac{\sqrt{13}}{4}c.$$

**Problem 12.10**

Say length of mast (at rest) is  $l$ . To an observer on the boat, height of mast is  $l \sin \theta$ , horizontal projection is  $l \cos \theta$ . To observer on dock, the former is unaffected, but the latter is Lorentz contracted to  $\frac{1}{\gamma} l \cos \theta$ . Therefore:

$$\tan \bar{\theta} = \frac{l \sin \theta}{\frac{1}{\gamma} l \cos \theta} = \gamma \tan \theta, \text{ or } \boxed{\tan \bar{\theta} = \frac{\tan \theta}{\sqrt{1 - v^2/c^2}}}.$$

**Problem 12.11**

Naively, circumference/diameter =  $\frac{1}{\gamma}(2\pi R)/(2R) = \pi/\gamma = \pi\sqrt{1 - (\omega R/c)^2}$  — but this is nonsense. Point is: an accelerating object cannot remain rigid, in relativity. To decide what *actually* happens here, you need a specific model for the internal forces holding the disk together.

**Problem 12.12**

(iv)  $\Rightarrow t = \frac{\bar{t}}{\gamma} + \frac{vx}{c^2}$ . Put this into (i), and solve for  $x$ :

$$\bar{x} = \gamma x - \gamma v \left( \frac{\bar{t}}{\gamma} + \frac{vx}{c^2} \right) = \gamma x \left( 1 - \frac{v^2}{c^2} \right) - v\bar{t} = \gamma x \frac{1}{\gamma^2} - v\bar{t} = \frac{x}{\gamma} - v\bar{t}; \boxed{x = \gamma(\bar{x} + v\bar{t})}. \checkmark$$

Similarly, (i)  $\Rightarrow x = \frac{\bar{x}}{\gamma} + vt$ . Put this into (iv) and solve for  $t$ :

$$\bar{t} = \gamma t - \frac{\gamma v}{c^2} \left( \frac{\bar{x}}{\gamma} + vt \right) = \gamma t \left( 1 - \frac{v^2}{c^2} \right) - \frac{v}{c^2} \bar{x} = \frac{t}{\gamma} - \frac{v}{c^2} \bar{x}; \boxed{t = \gamma \left( \bar{t} + \frac{v}{c^2} \bar{x} \right)}. \checkmark$$

**Problem 12.13**

Let brother's accident occur at origin, time zero, in both frames. In system  $\mathcal{S}$  (Sophie's), the coordinates of Sophie's cry are  $x = 5 \times 10^5 \text{ m}$ ,  $t = 0$ . In system  $\bar{\mathcal{S}}$  (scientist's),  $\bar{t} = \gamma(t - \frac{v}{c^2}x) = -\gamma vx/c^2$ . Since this is *negative*, Sophie's cry occurred before the accident, in  $\bar{\mathcal{S}}$ .  $\gamma = \frac{1}{\sqrt{1-(12/13)^2}} = \frac{13}{\sqrt{169-144}} = \frac{13}{5}$ . So  $\bar{t} = -\left(\frac{13}{5}\right)\left(\frac{12}{13}c\right)(5 \times 10^5)/c^2 = -12 \times 10^5/3 \times 10^8 = -4 \times 10^{-3}$ .  $4 \times 10^{-3} \text{ s earlier.}$

**Problem 12.14**

(a) In  $\mathcal{S}$  it moves a distance  $dy$  in time  $dt$ . In  $\bar{\mathcal{S}}$ , meanwhile, it moves a distance  $d\bar{y} = dy$  in time  $d\bar{t} = \gamma(dt - \frac{v}{c^2}dx)$ .

$$\therefore \frac{d\bar{y}}{d\bar{t}} = \frac{dy}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{(dy/dt)}{\gamma \left( 1 - \frac{v}{c^2} \frac{dx}{dt} \right)}; \text{ or } \boxed{\bar{u}_y = \frac{u_y}{\gamma \left( 1 - \frac{vu_x}{c^2} \right)}; \bar{u}_z = \frac{u_z}{\gamma \left( 1 - \frac{vu_x}{c^2} \right)}}.$$

$$(b) \tan \bar{\theta} = -\frac{\bar{u}_y}{\bar{u}_x} = -\frac{u_y / [\gamma(1 - \frac{vu_x}{c^2})]}{(u_x - v) / (1 - \frac{vu_x}{c^2})} = \frac{1}{\gamma} \frac{(-u_y)}{(u_x - v)}.$$

In this case  $u_x = -c \cos \theta$ ;  $u_y = c \sin \theta \Rightarrow \tan \bar{\theta} = \frac{1}{\gamma} \left( \frac{-c \sin \theta}{-c \cos \theta - v} \right).$

$\tan \bar{\theta} = \frac{1}{\gamma} \left( \frac{\sin \theta}{\cos \theta + v/c} \right).$  [Compare  $\tan \bar{\theta} = \gamma \frac{\sin \theta}{\cos \theta}$  in Prob. 12.10. The point is that *velocities* are sensitive

not only to the transformation of *distances*, but also of *times*. That's why there is no *universal* rule for translating angles—you have to know whether it's an angle made by a *velocity* vector or a *position* vector.]

### Problem 12.15

Bullet relative to ground:  $\frac{5}{7}c$ , Outlaws relative to police:  $\frac{\frac{3}{4}c - \frac{1}{2}c}{1 - \frac{3}{4} \cdot \frac{1}{2}} = \frac{(1/4)c}{(5/8)} = \frac{2}{5}c$ .

Bullet relative to outlaws:  $\frac{\frac{5}{7}c - \frac{3}{4}c}{1 - \frac{5}{7} \cdot \frac{3}{4}} = \frac{-(1/28)c}{(13/28)} = -\frac{1}{13}c$ . [Velocity of  $A$  relative to  $B$  is minus the velocity of  $B$  relative to  $A$ , so all entries below the diagonal are trivial. Note that in every case  $v_{\text{bullet}} < v_{\text{outlaws}}$ , so no matter how you look at it, the bad guys get away.]

speed of → relative to ↓	Ground	Police	Outlaws	Bullet	Do they escape?
Ground	0	$\frac{1}{2}c$	$\frac{3}{4}c$	$\frac{5}{7}c$	Yes
Police	$-\frac{1}{2}c$	0	$\frac{2}{5}c$	$\frac{1}{3}c$	Yes
Outlaws	$-\frac{3}{4}c$	$-\frac{2}{5}c$	0	$-\frac{1}{13}c$	Yes
Bullet	$-\frac{5}{7}c$	$-\frac{1}{3}c$	$\frac{1}{13}c$	0	Yes

### Problem 12.16

(a) Moving clock runs slow, by a factor  $\gamma = \frac{1}{\sqrt{1-(4/5)^2}} = \frac{5}{3}$ . Since 18 years elapsed on the moving clock,  $\frac{5}{3} \times 18 = 30$  years elapsed on the stationary clock. 51 years old.

(b) By earth clock, it took 15 years to get there, at  $\frac{4}{5}c$ , so  $d = \frac{4}{5}c \times 15$  years = 12c years (12 light years).

(c)  $t = 15$  years,  $x = 12c$  years.

(d)  $\tilde{t} = 9$  years,  $\tilde{x} = 0$ . [She got *on* at the origin in  $\tilde{S}$ , and rode along with  $\tilde{S}$ , so she's *still* at the origin. If you doubt these values, use the Lorentz transformations, with  $x$  and  $t$  from (c).]

(e) Lorentz transformations:  $\begin{cases} \tilde{x} = \gamma(x + vt) \\ \tilde{t} = \gamma(t + \frac{v}{c^2}x) \end{cases}$  (note that  $v$  is *negative*, since  $\tilde{S}$  is going to the *left*).

$$\therefore \tilde{x} = \frac{5}{3}(12c \text{ yrs} + \frac{4}{5}c \cdot 15 \text{ yrs}) = \frac{5}{3} \cdot 24c \text{ yrs} = \boxed{40c \text{ years.}}$$

$$\tilde{t} = \frac{5}{3}(15 \text{ yrs} + \frac{4}{5} \frac{c}{c^2} \cdot 12c \text{ yrs}) = \frac{5}{3}(15 + \frac{48}{5}) \text{ yrs} = (25 + 16) \text{ yrs} = \boxed{41 \text{ years.}}$$

(f) Set her clock ahead 32 years, from 9 to 41 ( $\tilde{t} \rightarrow \tilde{t}$ ). Return trip takes 9 years (moving time), so her clock will now read 50 years at her arrival. Note that this is  $\frac{5}{3} \cdot 30$  years—precisely what she would calculate if the *stay-at-home* had been the traveler, for 30 years of his own time.

(g) (i)  $\tilde{t} = 9$  yrs,  $x = 0$ . What is  $t$ ?  $t = \frac{v}{c^2}x + \frac{\tilde{t}}{\gamma} = \frac{3}{5} \cdot 9 = \frac{27}{5} = 5.4$  years, and he started at age 21, so he's 26.4 years old. (Younger than the traveler (!) because to the traveler it's the stay-at-home who's moving.)

(ii)  $\tilde{t} = 41$  yrs,  $x = 0$ . What is  $t$ ?  $t = \frac{\tilde{t}}{\gamma} = \frac{3}{5} \cdot 41 = \frac{123}{5} = 24.6$  years, and he started at 21, so he's 45.6 years old.

(h) It will take another  $5.4$  years of earth time for the return, so when she gets back, she will say her twin's age is  $45.6 + 5.4 = 51$  years—which is what we found in (a). But note that to make it work from traveler's point of view you *must* take into account the jump in *perceived* age of stay-at-home when she changes coordinates from  $\bar{\mathcal{S}}$  to  $\check{\mathcal{S}}$ .

### Problem 12.17

$$\begin{aligned} -\bar{a}^0\bar{b}^0 + \bar{a}^1\bar{b}^1 + \bar{a}^2\bar{b}^2 + \bar{a}^3\bar{b}^3 &= -\gamma^2(a^0 - \beta a^1)(b^0 - \beta b^1) + \gamma^2(a^1 - \beta a^0)(b^1 - \beta b^0) + a^2 b^2 + a^3 b^3 \\ &= -\gamma^2(a^0 b^0 - \beta a^1 b^1 - \beta a^1 b^0 + \beta^2 a^1 b^1 - a^1 b^1 + \beta a^0 b^0 + \beta a^0 b^1 - \beta^2 a^0 b^0) + a^2 b^2 + a^3 b^3 \\ &= -\gamma^2 a^0 b^0 (1 - \beta^2) + \gamma^2 a^1 b^1 (1 - \beta^2) + a^2 b^2 + a^3 b^3 \\ &= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \quad \text{qed} \quad [\text{Note: } \gamma^2(1 - \beta^2) = 1.] \end{aligned}$$

### Problem 12.18

$$(a) \begin{pmatrix} \bar{c}\bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad (\text{using the notation of Eq. 12.24, for best comparison}).$$

$$(b) \Lambda = \begin{pmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$(c) \text{Multiply the matrices: } \Lambda = \begin{pmatrix} \bar{\gamma} & 0 & -\bar{\gamma}\bar{\beta} & 0 \\ 0 & 1 & 0 & 0 \\ -\bar{\gamma}\bar{\beta} & 0 & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma\bar{\gamma} & -\gamma\bar{\gamma}\beta & -\bar{\gamma}\bar{\beta} & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ -\bar{\gamma}\gamma\bar{\beta} & \gamma\bar{\gamma}\beta\bar{\beta} & \bar{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Yes, the order *does* matter. In the other order, “bars” and “no-bars” would be switched, and this would give a *different matrix*.

### Problem 12.19

(a) Since  $\tanh \theta = \frac{\sinh \theta}{\cosh \theta}$ , and  $\cosh^2 \theta - \sinh^2 \theta = 1$ , we have:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \tanh^2 \theta}} = \frac{\cosh \theta}{\sqrt{\cosh^2 \theta - \sinh^2 \theta}} = \cosh \theta; \quad \gamma\beta = \cosh \theta \tanh \theta = \sinh \theta.$$

$$\therefore \Lambda = \begin{pmatrix} \cosh \theta & -\sinh \theta & 0 & 0 \\ -\sinh \theta & \cosh \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad \text{Compare: } R = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b)  $\bar{u} = \frac{u - v}{1 - \frac{uv}{c^2}} \Rightarrow \frac{\bar{u}}{c} = \frac{(u/c) - (v/c)}{1 - (\frac{u}{c})(\frac{v}{c})} \Rightarrow \tanh \bar{\phi} = \frac{\tanh \phi - \tanh \theta}{1 - \tanh \phi \tanh \theta}$ , where  $\tanh \phi = u/c$ ,  $\tanh \theta = v/c$ ;

$\tanh \bar{\phi} = \bar{u}/c$ . But a “trig” formula for hyperbolic functions (CRC Handbook, 18th Ed., p. 204) says:

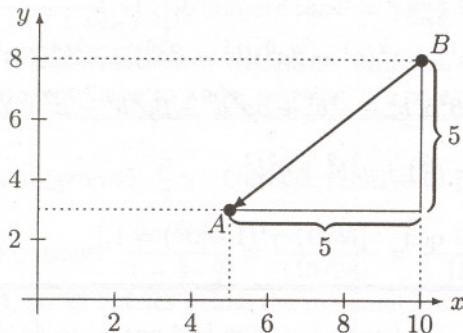
$$\frac{\tanh \phi - \tanh \theta}{1 - \tanh \phi \tanh \theta} = \tanh(\phi - \theta). \quad \therefore \tanh \bar{\phi} = \tanh(\phi - \theta), \text{ or: } \bar{\phi} = \phi - \theta.$$

**Problem 12.20**

(a) (i)  $I = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -(5-15)^2 + (10-5)^2 + (8-3)^2 + (0-0)^2 = -100 + 25 + 25 = -50.$

(ii)  No. (In such a system  $\Delta \bar{t} = 0$ , so  $I$  would have to be *positive*, which it *isn't*.)

(iii)  Yes.



$\bar{s}$  travels in the direction from  $B$  toward  $A$ , making the trip in time  $10/c$ .

$$\therefore \mathbf{v} = \frac{-5\hat{x} - 5\hat{y}}{10/c} = \left[ -\frac{c}{2}\hat{x} - \frac{c}{2}\hat{y} \right].$$

Note that  $\frac{v^2}{c^2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ , so  $v = \frac{1}{\sqrt{2}}c$ , safely less than  $c$ .

(b) (i)  $I = -(3-1)^2 + (5-2)^2 + 0 + 0 = -4 + 9 = 5.$

(ii)  Yes. By Lorentz transformation:  $\Delta(ct) = \gamma[\Delta(ct) - \beta(\Delta x)]$ . We want  $\Delta \bar{t} = 0$ , so  $\Delta(ct) = \beta(\Delta x)$ ; or  $\frac{v}{c} = \frac{\Delta(ct)}{\Delta x} = \frac{(3-1)}{(5-2)} = \frac{2}{3}$ . So  $v = \frac{2}{3}c$ , in the  $+x$  direction.

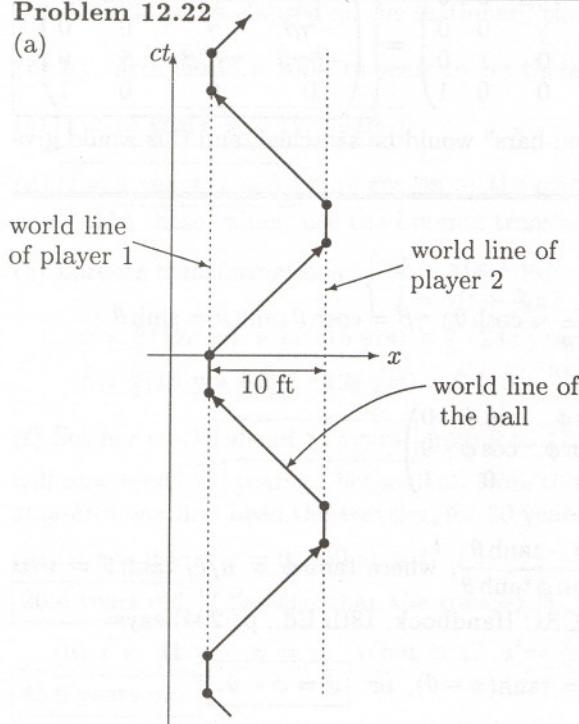
(iii)  No. (In such a system  $\Delta x = \Delta y = \Delta z = 0$  so  $I$  would be negative, which it *isn't*.)

**Problem 12.21**

Using Eq. 12.18 (iv):  $\Delta \bar{t} = \gamma(\Delta t - \frac{v}{c^2} \Delta x) = 0 \Rightarrow \Delta t = \frac{v}{c^2} \Delta x$ , or  $v = \frac{\Delta t}{\Delta x} c^2 = \frac{t_B - t_A}{x_B - x_A} c^2$ .

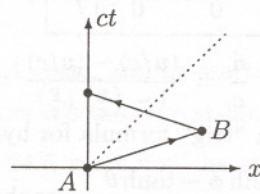
**Problem 12.22**

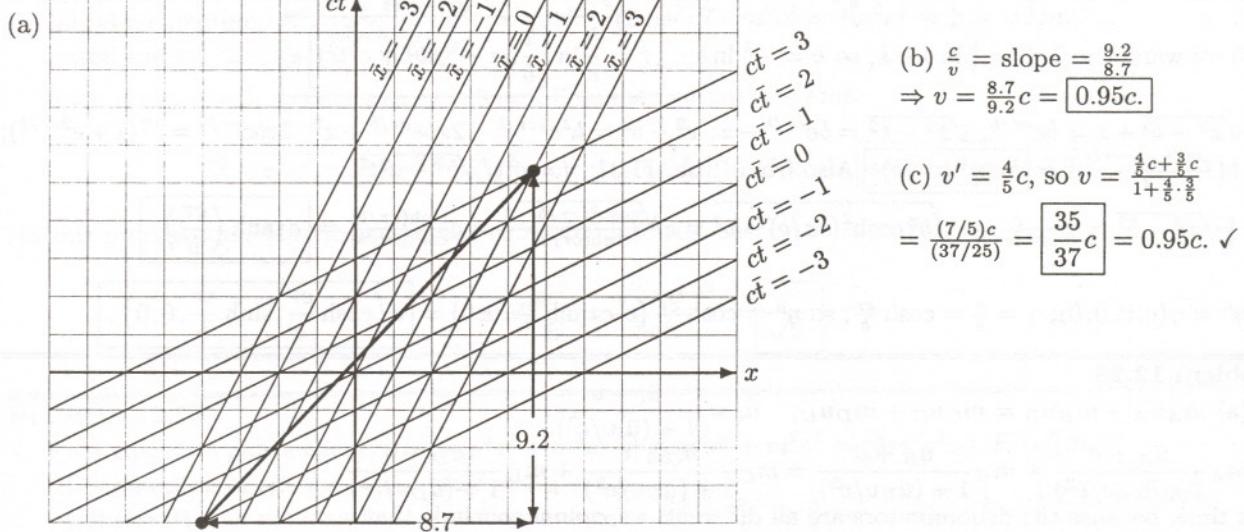
(a)



Truth is, you never *do* communicate with the other person *right now*—you communicate with the person he/she *will be* when the message gets there; and the response comes back to and older and wiser you.

(b)  No way. It is true that a moving observer might say she arrived at  $B$  before she left  $A$ , but for the *round trip* everyone must agree that she arrives back after she set out.



**Problem 12.23**

(b)  $\frac{c}{v} = \text{slope} = \frac{9.2}{8.7}$   
 $\Rightarrow v = \frac{8.7}{9.2}c = \boxed{0.95c}$ .

(c)  $v' = \frac{4}{5}c$ , so  $v = \frac{\frac{4}{5}c + \frac{3}{5}c}{1 + \frac{4}{5} \cdot \frac{3}{5}}$   
 $= \frac{(7/5)c}{(37/25)} = \boxed{\frac{35}{37}c} = 0.95c. \checkmark$

**Problem 12.24**

(a)  $(1 - \frac{u^2}{c^2})\eta^2 = u^2; u^2(1 + \frac{\eta^2}{c^2}) = \eta^2; u = \frac{1}{\sqrt{1 + \eta^2/c^2}} \eta.$

(b)  $\frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-\tanh^2 \theta}} = \frac{\cosh \theta}{\sqrt{\cosh^2 \theta - \sinh^2 \theta}} = \cosh \theta; \eta = \frac{1}{\sqrt{1-u^2/c^2}} u = \cosh \theta c \tanh \theta = \boxed{c \sinh \theta}.$

**Problem 12.25**

(a)  $u_x = u_y = u \cos 45^\circ = \frac{1}{\sqrt{2}} \frac{2}{\sqrt{5}} c = \boxed{\sqrt{\frac{2}{5}} c}.$

(b)  $\frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-4/5}} = \frac{\sqrt{5}}{\sqrt{5-4}} = \sqrt{5}; \eta = \frac{u}{\sqrt{1-u^2/c^2}} \Rightarrow \boxed{\eta_x = \eta_y = \sqrt{2} c}.$

(c)  $\eta^0 = \gamma c = \boxed{\sqrt{5} c}.$

(d) Eq. 12.45  $\Rightarrow \bar{u}_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{\sqrt{2/5} c - \sqrt{2/5} c}{1 - \frac{2}{5}} = \boxed{0}.$

$\bar{u}_y = \frac{1}{\gamma} \left( \frac{u_y}{1 - \frac{u_x v}{c^2}} \right) = \sqrt{1 - \frac{2}{5}} \frac{\sqrt{2/5} c}{1 - \frac{2}{5}} = \frac{\sqrt{2/5}}{\sqrt{3/5}} c = \boxed{\sqrt{\frac{2}{3}} c}.$

(e)  $\bar{\eta}_x = \gamma(\eta_x - \beta\eta^0) = \sqrt{1 - \frac{2}{5}} \left( \sqrt{2} c - \sqrt{\frac{2}{5}} \sqrt{5} c \right) = \boxed{0}. \quad \bar{\eta}_y = \eta_y = \sqrt{2} c.$

(f)  $\frac{1}{\sqrt{1-\bar{u}^2/c^2}} = \frac{1}{\sqrt{1-(2/3)}} = \sqrt{3}; \bar{\eta} = \sqrt{3} \bar{u} \Rightarrow \left\{ \begin{array}{l} \bar{\eta}_x = \sqrt{3} \bar{u}_x = 0. \checkmark \\ \bar{\eta}_y = \sqrt{3} \bar{u}_y = \sqrt{2} c. \checkmark \end{array} \right\}$

**Problem 12.26**

$\eta^\mu \eta_\mu = -(\eta^0)^2 + \eta^2 = \frac{1}{(1-u^2/c^2)} (-c^2 + u^2) = -c^2 \frac{(1-u^2/c^2)}{(1-u^2/c^2)} = \boxed{-c^2}.$

**Problem 12.27**

(a) From Prob. 11.31 we have  $\gamma = \frac{1}{b}\sqrt{b^2 + c^2t^2}$ .  $\therefore \tau = \int \frac{1}{\gamma} dt = b \int \frac{dt}{\sqrt{b^2 + c^2t^2}} = \frac{b}{c} \ln(ct + \sqrt{b^2 + c^2t^2}) + k$ ; at  $t = 0$  we want  $\tau = 0$ :  $0 = \frac{b}{c} \ln b + k$ , so  $k = -\frac{b}{c} \ln b$ ;  $\boxed{\tau = \frac{b}{c} \ln \left[ \frac{1}{b}(ct + \sqrt{b^2 + c^2t^2}) \right]}.$

(b)  $\sqrt{x^2 - b^2} + x = be^{c\tau/b}$ ;  $\sqrt{x^2 - b^2} = be^{c\tau/b} - x$ ;  $x^2 - b^2 = b^2 e^{2c\tau/b} - 2xbe^{c\tau/b} + x^2$ ;  $2xbe^{c\tau/b} = b^2(1 + e^{2c\tau/b})$ ;  $x = b \left( \frac{e^{c\tau/b} + e^{-c\tau/b}}{2} \right) = \boxed{b \cosh(c\tau/b)}$ . Also from Prob. 11.31:  $v = c^2t/\sqrt{b^2 + c^2t^2}$ .

$$v = \frac{c}{x} \sqrt{x^2 - b^2} = \frac{c}{b \cosh(c\tau/b)} \sqrt{b^2 \cosh^2(c\tau/b) - b^2} = c \frac{\sqrt{\cosh^2(c\tau/b) - 1}}{\cosh(c\tau/b)} = c \frac{\sinh(c\tau/b)}{\cosh(c\tau/b)} = \boxed{c \tanh\left(\frac{c\tau}{b}\right)}.$$

(c)  $\eta^\mu = \gamma(c, v, 0, 0)$ ;  $\gamma = \frac{x}{b} = \cosh \frac{c\tau}{b}$ , so  $\eta^\mu = \cosh \frac{c\tau}{b} (c, c \tanh \frac{c\tau}{b}, 0, 0) = \boxed{c \left( \cosh \frac{c\tau}{b}, \sinh \frac{c\tau}{b}, 0, 0 \right)}.$

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**Problem 12.28**

(a)  $m_A u_A + m_B u_B = m_C u_C + m_D u_D$ ;  $u_i = \frac{\bar{u}_i + v}{1 + (\bar{u}_i v / c^2)}$ .

$$m_A \frac{\bar{u}_A + v}{1 + (\bar{u}_A v / c^2)} + m_B \frac{\bar{u}_B + v}{1 + (\bar{u}_B v / c^2)} = m_C \frac{\bar{u}_C + v}{1 + (\bar{u}_C v / c^2)} + m_D \frac{\bar{u}_D + v}{1 + (\bar{u}_D v / c^2)}.$$

This time, because the denominators are all different, we *cannot* conclude that

$$m_A \bar{u}_A + m_B \bar{u}_B = m_C \bar{u}_C + m_D \bar{u}_D.$$

As an explicit counterexample, suppose all the masses are equal, and  $u_A = -u_B = v$ ;  $u_C = u_D = 0$ . This is a symmetric “completely inelastic” collision in  $\mathcal{S}$ , and momentum is clearly conserved ( $0 = 0$ ). But the Einstein velocity addition rule gives  $\bar{u}_A = 0$ ,  $\bar{u}_B = -2u/(1 + u^2/c^2)$ ,  $\bar{u}_C = \bar{u}_D = -u$ , so in  $\bar{\mathcal{S}}$  the (incorrectly defined) momentum is *not* conserved:

$$m \left( \frac{-2u}{1 + u^2/c^2} \right) \neq -2mu.$$

(b)  $m_A \eta_A + m_B \eta_B = m_C \eta_C + m_D \eta_D$ ;  $\eta_i = \gamma(\bar{\eta}_i + \beta \bar{\eta}_i^0)$ . (The inverse Lorentz transformation.)

$$m_A \gamma(\bar{\eta}_A + \beta \bar{\eta}_A^0) + m_B \gamma(\bar{\eta}_B + \beta \bar{\eta}_B^0) = m_C \gamma(\bar{\eta}_C + \beta \bar{\eta}_C^0) + m_D \gamma(\bar{\eta}_D + \beta \bar{\eta}_D^0)$$
. The gamma's cancel:

$$m_A \bar{\eta}_A + m_B \bar{\eta}_B + \beta(m_A \bar{\eta}_A^0 + m_B \bar{\eta}_B^0) = m_C \bar{\eta}_C + m_D \bar{\eta}_D + \beta(m_C \bar{\eta}_C^0 + m_D \bar{\eta}_D^0).$$

But  $m_i \eta_i^0 = p_i^0 = E_i/c$ , so if  $\boxed{\text{energy is conserved}}$  in  $\bar{\mathcal{S}}$  ( $\bar{E}_A + \bar{E}_B = \bar{E}_C + \bar{E}_D$ ), then so too is the momentum (correctly defined):

$$m_A \bar{\eta}_A + m_B \bar{\eta}_B = m_C \bar{\eta}_C + m_D \bar{\eta}_D. \quad \text{qed}$$


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**Problem 12.29**

$$\gamma mc^2 - mc^2 = nmc^2 \Rightarrow \gamma = n + 1 = \frac{1}{\sqrt{1-u^2/c^2}} \Rightarrow 1 - \frac{u^2}{c^2} = \frac{1}{(n+1)^2}.$$

$$\therefore \frac{u^2}{c^2} = 1 - \frac{1}{(n+1)^2} = \frac{n^2 + 2n + 1 - 1}{(n+1)^2} = \frac{n(n+2)}{(n+1)^2}; \boxed{u = \frac{\sqrt{n(n+2)}}{n+1} c}.$$


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**Problem 12.30**

$$E_T = E_1 + E_2 + \dots; \quad p_T = p_1 + p_2 + \dots; \quad \bar{p}_T = \gamma(p_T - \beta E_T/c) = 0 \Rightarrow \beta = v/c = p_T c/E_T.$$

$$v = c^2 p_T / E_T = \boxed{c^2(p_1 + p_2 + \dots) / (E_1 + E_2 + \dots)}.$$


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**Problem 12.31**

$$E_\mu = \frac{(m_\pi^2 + m_\mu^2)}{2m_\pi} c^2 = \gamma m_\mu c^2 \Rightarrow \gamma = \frac{(m_\pi^2 + m_\mu^2)}{2m_\pi m_\mu} = \frac{1}{\sqrt{1 - v^2/c^2}}; \quad 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2};$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} = 1 - \frac{4m_\pi^2 m_\mu^2}{(m_\pi^2 + m_\mu^2)^2} = \frac{m_\pi^4 + 2m_\pi^2 m_\mu^2 + m_\mu^4 - 4m_\pi^2 m_\mu^2}{(m_\pi^2 + m_\mu^2)^2} = \frac{(m_\pi^2 - m_\mu^2)^2}{(m_\pi^2 + m_\mu^2)^2}; \quad v = \boxed{\left( \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} \right) c}.$$

**Problem 12.32**

Initial momentum:  $E^2 - p^2 c^2 = m^2 c^4 \Rightarrow p^2 c^2 = (2mc^2)^2 - m^2 c^4 = 3m^2 c^4 \Rightarrow p = \sqrt{3} mc.$

Initial energy:  $2mc^2 + mc^2 = 3mc^2.$

Each is conserved, so final energy is  $3mc^2$ , final momentum is  $\sqrt{3} mc.$

$$E^2 - p^2 c^2 = (3mc^2)^2 - (\sqrt{3} mc)^2 c^2 = 6m^2 c^4 = M^2 c^4 \Rightarrow M = \sqrt{6} m \approx 2.5m.$$

(In this process some kinetic energy was converted into rest energy, so  $M > 2m$ .)

$$v = \frac{pc}{E} = \frac{\sqrt{3} mc c^2}{3mc^2} = \boxed{\frac{c}{\sqrt{3}}}.$$

**Problem 12.33**

First calculate pion's energy:  $E^2 = p^2 c^2 + m^2 c^4 = \frac{9}{16} m^2 c^4 + m^2 c^4 = \frac{25}{16} m^2 c^4 \Rightarrow E = \frac{5}{4} mc^2.$

Conservation of energy:  $\frac{5}{4} mc^2 = E_A + E_B$

Conservation of momentum:  $\frac{5}{4} mc^2 = p_A + p_B = \frac{E_A}{c} - \frac{E_B}{c} \Rightarrow \frac{3}{4} mc^2 = E_A - E_B \quad \left. \right\} 2E_A = 2mc^2.$

$$\Rightarrow E_A = mc^2; \quad E_B = \frac{1}{4} mc^2.$$

**Problem 12.34**

Classically,  $E = \frac{1}{2} mv^2$ . In a colliding beam experiment, the relative velocity (classically) is *twice* the velocity of either one, so the relative energy is  $4E$ .



Let  $\bar{S}$  be the system in which ① is at rest. Its speed  $v$ , relative to  $S$ , is just the speed of ① in  $S$ .

$$p^0 = \gamma(p^0 - \beta p^1) \Rightarrow \frac{\bar{E}}{c} = \gamma\left(\frac{E}{c} - \beta p\right), \text{ where } p \text{ is the momentum of ② in } S.$$

$$E = \gamma Mc^2, \text{ so } \gamma = \frac{E}{Mc^2}; \quad p = -\gamma Mv = -\gamma M\beta c; \quad \bar{E} = \gamma\left(\frac{E}{c} + \beta\gamma M\beta c\right)c = \gamma(E + \gamma Mc^2\beta^2).$$

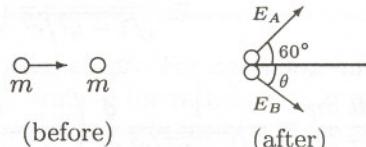
$$\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow 1 - \beta^2 = \frac{1}{\gamma^2} \Rightarrow \beta^2 = 1 - \frac{1}{\gamma^2} = \frac{\gamma^2 - 1}{\gamma^2}; \quad \bar{E} = \frac{E}{Mc^2} E + \left[\left(\frac{E}{Mc^2}\right)^2 - 1\right] Mc^2.$$

$$\bar{E} = \frac{E^2}{Mc^2} + \frac{E^2}{Mc^2} - Mc^2; \quad \bar{E} = \frac{2E^2}{Mc^2} - Mc^2.$$

$$\text{For } E = 30 \text{ GeV and } Mc^2 = 1 \text{ GeV, we have } \bar{E} = \frac{(2)(900)}{1} - 1 = 1800 - 1 = [1799 \text{ GeV}] = [60E].$$

**Problem 12.35**

One photon is impossible, because in the “center of momentum” frame (Prob. 12.30) we’d be left with a photon *at rest*, whereas photons *have* to travel at speed  $c$ .



$$\left\{ \begin{array}{l} \text{Cons. of energy: } \sqrt{p_0 c^2 + m^2 c^4} + mc^2 = E_A + E_B. \\ \text{Cons. of mom.: } \left\{ \begin{array}{l} \text{horizontal: } p_0 = \frac{E_A}{c} \cos 60^\circ + \frac{E_B}{c} \cos \theta \Rightarrow E_B \cos \theta = p_0 c - \frac{1}{2} E_A, \\ \text{vertical: } 0 = \frac{E_A}{c} \sin 60^\circ - \frac{E_B}{c} \sin \theta \Rightarrow E_B \sin \theta = \frac{\sqrt{3}}{2} E_A; \end{array} \right. \end{array} \right\} \text{square and add:}$$

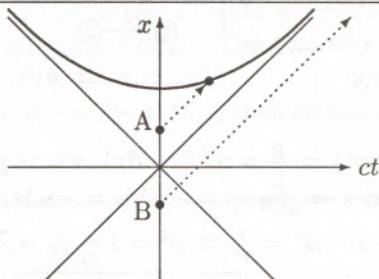
$$\begin{aligned}
 E_B^2(\cos^2 \theta + \sin^2 \theta) &= p_0 c^2 - p_0 c E_A + \frac{1}{4} E_A^2 + \frac{3}{4} E_A^2 \\
 \Rightarrow E_B^2 &= p_0 c^2 - p_0 c E_A + E_A^2 = \left[ \sqrt{p_0^2 c^2 + m^2 c^4} + mc^2 - E_A \right]^2 \\
 &= p_0 c^2 + m^2 c^4 + 2\sqrt{p_0^2 c^2 + m^2 c^4}(mc^2 - E_A) + m^2 c^4 - 2E_A mc^2 + E_A^2. \text{ Or:} \\
 -p_0 c E_A &= 2m^2 c^4 + 2mc^2 \sqrt{p_0^2 c^2 + m^2 c^4} - 2E_A \sqrt{p_0^2 c^2 + m^2 c^4} - 2E_A mc^2; \\
 E_A(m c^2 + \sqrt{p_0^2 c^2 + m^2 c^4} - p_0 c / 2) &= m^2 c^4 + mc^2 \sqrt{p_0^2 c^2 + m^2 c^4}; \\
 E_A = mc^2 \frac{(mc^2 + \sqrt{p_0^2 c^2 + m^2 c^4})}{(mc^2 + \sqrt{p_0^2 c^2 + m^2 c^4} - p_0 c / 2)} &\cdot \frac{(mc^2 - \sqrt{p_0^2 c^2 + m^2 c^4} - p_0 c / 2)}{(mc^2 - \sqrt{p_0^2 c^2 + m^2 c^4} - p_0 c / 2)} \\
 = mc^2 \frac{(\cancel{m} c^4 - p_0^2 c^2 - \cancel{m} c^4 - \frac{1}{2} p_0 m c^3 - \frac{p_0 c}{2} \sqrt{p_0^2 c^2 + m^2 c^4})}{(\cancel{m} c^4 - p_0 m c^3 + \frac{p_0^2 c^2}{4} - p_0^2 c^2 - \cancel{m} c^4)} &= \boxed{\frac{mc^2}{2} \frac{(mc + 2p_0 + \sqrt{p_0^2 + m^2 c^2})}{(mc + \frac{3}{4} p_0)}}.
 \end{aligned}$$

**Problem 12.36**

$$\begin{aligned}
 \mathbf{F} = \frac{d\mathbf{p}}{dt} &= \frac{d}{dt} \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} = m \left\{ \frac{\frac{d\mathbf{u}}{dt}}{\sqrt{1-u^2/c^2}} + \mathbf{u} \left( -\frac{1}{2} \right) \frac{-\frac{1}{c^2} 2\mathbf{u} \cdot \frac{d\mathbf{u}}{dt}}{(1-u^2/c^2)^{3/2}} \right\} \\
 &= \frac{m}{\sqrt{1-u^2/c^2}} \left\{ \mathbf{a} + \frac{\mathbf{u}(\mathbf{u} \cdot \mathbf{a})}{(c^2-u^2)} \right\}. \quad \text{qed}
 \end{aligned}$$

**Problem 12.37**

At constant force you go in "hyperbolic" motion. Photon A, which left the origin at  $t < 0$ , catches up with you, but photon B, which passes the origin at  $t > 0$ , never does.

**Problem 12.38**

$$\begin{aligned}
 (a) \alpha^0 &= \frac{d\eta_0}{d\tau} = \frac{d\eta_0}{dt} \frac{dt}{d\tau} = \left[ \frac{d}{dt} \left( \frac{c}{\sqrt{1-u^2/c^2}} \right) \right] \frac{1}{\sqrt{1-u^2/c^2}} \\
 &= \frac{c}{\sqrt{1-u^2/c^2}} \left( -\frac{1}{2} \right) \frac{(-\frac{1}{c^2}) 2\mathbf{u} \cdot \mathbf{a}}{(1-u^2/c^2)^{3/2}} = \boxed{\frac{1}{c} \frac{\mathbf{u} \cdot \mathbf{a}}{(1-u^2/c^2)^2}}.
 \end{aligned}$$

$$\begin{aligned}
 \alpha &= \frac{d\eta}{d\tau} = \frac{dt}{d\tau} \frac{d\eta}{dt} = \frac{1}{\sqrt{1-u^2/c^2}} \frac{d}{dt} \left( \frac{\mathbf{u}}{\sqrt{1-u^2/c^2}} \right) = \frac{1}{\sqrt{1-u^2/c^2}} \left\{ \frac{\mathbf{a}}{\sqrt{1-u^2/c^2}} + \mathbf{u} \left( -\frac{1}{2} \right) \frac{-\frac{1}{c^2} 2\mathbf{u} \cdot \mathbf{a}}{(1-u^2/c^2)^{3/2}} \right\} \\
 &= \boxed{\frac{1}{(1-u^2/c^2)} \left[ \mathbf{a} + \frac{\mathbf{u}(\mathbf{u} \cdot \mathbf{a})}{(c^2-u^2)} \right]}.
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \alpha_\mu \alpha^\mu &= -(\alpha^0)^2 + \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = -\frac{1}{c^2} \frac{(\mathbf{u} \cdot \mathbf{a})^2}{(1-u^2/c^2)^4} + \frac{1}{(1-u^2/c^2)^4} \left[ \mathbf{a} \left( 1 - \frac{u^2}{c^2} \right) + \frac{1}{c^2} \mathbf{u}(\mathbf{u} \cdot \mathbf{a}) \right]^2 \\
 &= \frac{1}{(1-u^2/c^2)^4} \left\{ -\frac{1}{c^2} (\mathbf{u} \cdot \mathbf{a})^2 + a^2 \left( 1 - \frac{u^2}{c^2} \right)^2 + \frac{2}{c^2} \left( 1 - \frac{u^2}{c^2} \right) (\mathbf{u} \cdot \mathbf{a})^2 + \frac{1}{c^4} u^2 (\mathbf{u} \cdot \mathbf{a})^2 \right\} \\
 &= \frac{1}{(1-u^2/c^2)^4} \left\{ a^2 \left( 1 - \frac{u^2}{c^2} \right)^2 + \frac{(\mathbf{u} \cdot \mathbf{a})^2}{c^2} \underbrace{\left( -1 + 2 - 2 \frac{u^2}{c^2} + \frac{u^2}{c^2} \right)}_{(1 - \frac{u^2}{c^2})} \right\} \\
 &= \boxed{\frac{1}{(1-u^2/c^2)^2} \left[ a^2 + \frac{(\mathbf{u} \cdot \mathbf{a})^2}{(c^2 - u^2)} \right]}.
 \end{aligned}$$

$$(c) \eta^\mu \eta_\mu = -c^2, \text{ so } \frac{d}{d\tau}(\eta^\mu \eta_\mu) = \alpha^\mu \eta_\mu + \eta^\mu \alpha_\mu = 2\alpha^\mu \eta_\mu = 0, \text{ so } \boxed{\alpha^\mu \eta_\mu = 0}.$$

$$(d) K^\mu = \frac{dp^\mu}{d\tau} = \frac{d}{d\tau}(m\eta^\mu) = \boxed{m\alpha^\mu}. \quad \boxed{K^\mu \eta_\mu = m\alpha^\mu \eta_\mu = 0}.$$

### Problem 12.39

$K_\mu K^\mu = -(K^0)^2 + \mathbf{K} \cdot \mathbf{K}$ . From Eq. 12.70,  $\mathbf{K} \cdot \mathbf{K} = \frac{F^2}{(1-u^2/c^2)}$ . From Eq. 12.71:

$$K^0 = \frac{1}{c} \frac{dE}{d\tau} = \frac{1}{c\sqrt{1-u^2/c^2}} \frac{d}{dt} \left( \frac{mc^2}{\sqrt{1-u^2/c^2}} \right) = \frac{mc}{\sqrt{1-u^2/c^2}} \left[ -\frac{1}{2} \frac{(-1/c^2)}{(1-u^2/c^2)^{3/2}} 2\mathbf{u} \cdot \mathbf{a} \right] = \frac{m}{c} \frac{(\mathbf{u} \cdot \mathbf{a})}{(1-u^2/c^2)^2}.$$

$$\text{But (Eq. 12.73): } \mathbf{u} \cdot \mathbf{F} = uF \cos \theta = \frac{m}{\sqrt{1-u^2/c^2}} \left[ (\mathbf{u} \cdot \mathbf{a}) + \frac{u^2(\mathbf{u} \cdot \mathbf{a})}{c^2(1-u^2/c^2)} \right] = \frac{m(\mathbf{u} \cdot \mathbf{a})}{(1-u^2/c^2)^{3/2}}, \text{ so}$$

$$K^0 = \frac{uF \cos \theta}{c\sqrt{1-u^2/c^2}}; \quad K_\mu K^\mu = \frac{F^2}{(1-u^2/c^2)} - \frac{u^2 F^2 \cos^2 \theta}{c^2(1-u^2/c^2)} = \left[ \frac{1 - (u^2/c^2) \cos^2 \theta}{(1-u^2/c^2)} \right] F^2. \quad \text{qed}$$

### Problem 12.40

$$\mathbf{F} = \frac{m}{\sqrt{1-u^2/c^2}} \left[ \mathbf{a} + \frac{\mathbf{u}(\mathbf{u} \cdot \mathbf{a})}{c^2 - u^2} \right] = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \Rightarrow \mathbf{a} + \frac{\mathbf{u}(\mathbf{u} \cdot \mathbf{a})}{(c^2 - u^2)} = \frac{q}{m} \sqrt{1-u^2/c^2} (\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

$$\text{Dot in } \mathbf{u}: (\mathbf{u} \cdot \mathbf{a}) + \frac{u^2(\mathbf{u} \cdot \mathbf{a})}{c^2(1-u^2/c^2)} = \frac{\mathbf{u} \cdot \mathbf{a}}{(1-u^2/c^2)} = \frac{q}{m} \sqrt{1-u^2/c^2} [\mathbf{u} \cdot \mathbf{E} + \underbrace{\mathbf{u} \cdot (\mathbf{u} \times \mathbf{B})}_{=0}];$$

$$\therefore \frac{\mathbf{u}(\mathbf{u} \cdot \mathbf{a})}{(c^2 - u^2)} = \frac{q}{m} \sqrt{1-u^2/c^2} \frac{\mathbf{u}(\mathbf{u} \cdot \mathbf{E})}{c^2}. \quad \text{So } \mathbf{a} = \frac{q}{m} \sqrt{1-u^2/c^2} [\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{c^2} \mathbf{u}(\mathbf{u} \cdot \mathbf{E})]. \quad \text{qed}$$

### Problem 12.41

One way to see it is to look back at the general formula for  $\mathbf{E}$  (Eq. 10.29). For a uniform infinite plane of charge, moving at constant velocity in the plane,  $\dot{\mathbf{j}} = 0$  and  $\dot{\rho} = 0$ , while  $\rho$  (or rather,  $\sigma$ ) is independent of  $t$  (so retardation does nothing). Therefore the field is exactly the same as it would be for a plane at rest (except that  $\sigma$  itself is altered by Lorentz contraction).

A more elegant argument exploits the fact that  $\mathbf{E}$  is a *vector* (whereas  $\mathbf{B}$  is a *pseudovector*). This means that any given component changes sign if the configuration is reflected in a plane perpendicular to that direction. But in Fig. 12.35(b), if we reflect in the  $x y$  plane the configuration is unaltered, so the  $z$  component of  $\mathbf{E}$  would

have to stay the *same*. Therefore it must in fact be zero. (By contrast, if you reflect in a plane perpendicular to the  $y$  direction the charges trade places, so it is perfectly appropriate that the  $y$  component of  $\mathbf{E}$  should reverse its sign.)

**Problem 12.42**

(a) Field is  $\sigma_0/\epsilon_0$ , and it points perpendicular to the positive plate, so:

$$\mathbf{E}_0 = \frac{\sigma_0}{\epsilon_0} (\cos 45^\circ \hat{x} + \sin 45^\circ \hat{y}) = \boxed{\frac{\sigma_0}{\sqrt{2}\epsilon_0} (-\hat{x} + \hat{y})}.$$

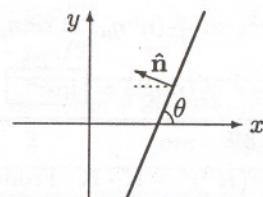
(b) From Eq. 12.108,  $E_x = E_{x_0} = -\frac{\sigma_0}{\sqrt{2}\epsilon_0}$ ;  $E_y = \gamma E_{y_0} = \gamma \frac{\sigma_0}{\sqrt{2}E_0}$ . So  $\boxed{\mathbf{E} = \frac{\sigma_0}{\sqrt{2}\epsilon_0} (-\hat{x} + \gamma \hat{y})}$ .

(c) From Prob. 12.10:  $\tan \theta = \gamma$ , so  $\boxed{\theta = \tan^{-1} \gamma}$ .

(d) Let  $\hat{n}$  be a unit vector perpendicular to the plates in  $\mathcal{S}$ —evidently  $\hat{n} = -\sin \theta \hat{x} + \cos \theta \hat{y}$ ;  $|E| = \frac{\sigma_0}{\sqrt{2}\epsilon_0} \sqrt{1 + \gamma^2}$ .

So the angle  $\phi$  between  $\hat{n}$  and  $\mathbf{E}$  is:

$$\frac{\mathbf{E} \cdot \hat{n}}{|E|} = \cos \phi = \frac{1}{\sqrt{1 + \gamma^2}} (\sin \theta + \gamma \cos \theta) = \frac{\cos \theta}{\sqrt{1 + \gamma^2}} (\tan \theta + \gamma) = \frac{2\gamma}{\sqrt{1 + \gamma^2}} \cos \theta$$



But  $\gamma = \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{1-\cos^2 \theta}}{\cos \theta} = \sqrt{\frac{1}{\cos^2 \theta} - 1} \Rightarrow \frac{1}{\cos^2 \theta} = \gamma^2 + 1 \Rightarrow \cos \theta = \frac{1}{\sqrt{1+\gamma^2}}$ . So  $\boxed{\cos \phi = \left( \frac{2\gamma}{1 + \gamma^2} \right)}$ .

Evidently the field is **not** perpendicular to the plates in  $\mathcal{S}$ .

**Problem 12.43**

(a)  $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q(1-v^2/c^2)}{(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}$  (Eq. 12.92)  $\Rightarrow$

$$\begin{aligned} \int \mathbf{E} \cdot d\mathbf{a} &= \frac{q(1-v^2/c^2)}{4\pi\epsilon_0} \int \frac{R^2 \sin \theta d\theta d\phi}{R^2 (1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}} \\ &= \frac{q(1-v^2/c^2)}{4\pi\epsilon_0} 2\pi \int_0^\pi \frac{\sin \theta d\theta}{(1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}}. \quad \text{Let } u = \cos \theta, \text{ so } du = -\sin \theta d\theta, \sin^2 \theta = 1 - u^2. \\ &= \frac{q(1-v^2/c^2)}{2\epsilon_0} \int_{-1}^1 \frac{du}{[1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}u^2]^{3/2}} = \frac{q(1-v^2/c^2)}{2\epsilon_0} \left(\frac{c}{v}\right)^3 \int_{-1}^1 \frac{du}{\left(\frac{v^2}{c^2} - 1 + u^2\right)^{3/2}}. \end{aligned}$$

The integral is: 
$$\frac{u}{\left(\frac{c^2}{v^2} - 1\right) \sqrt{\frac{c^2}{v^2} - 1 + u^2}} \Big|_{-1}^{+1} = \frac{2}{\left(\frac{c^2}{v^2} - 1\right) \frac{c}{v}} = \left(\frac{v}{c}\right)^3 \frac{2}{(1-v^2/c^2)}.$$

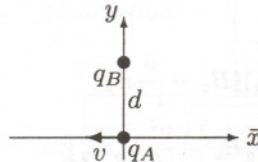
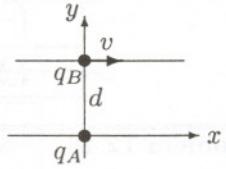
So  $\int \mathbf{E} \cdot d\mathbf{a} = \frac{q(1-v^2/c^2)}{2\epsilon_0} \left(\frac{c}{v}\right)^3 \left(\frac{v}{c}\right)^3 \frac{2}{(1-v^2/c^2)} = q. \checkmark$

(b) Using Eq. 12.111 and Eq. 12.92,  $\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0} \frac{1}{4\pi\epsilon_0} \frac{\mu_0}{4\pi} \frac{q^2 (1-v^2/c^2)^2 v \sin \theta}{R^4 (1 - \frac{v^2}{c^2} \sin^2 \theta)^3} (\underbrace{\hat{\mathbf{R}} \times \hat{\phi}}_{-\hat{\theta}});$

$$\mathbf{S} = \boxed{-\frac{q^2}{16\pi^2\epsilon_0} \frac{(1-v^2/c^2)^2 v \sin \theta}{R^4(1-\frac{v^2}{c^2} \sin^2 \theta)^3} \hat{\theta}}.$$

**Problem 12.44**

(a) Fields of  $A$  at  $B$ :  $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q_A}{d^2} \hat{\mathbf{y}}$ ;  $\mathbf{B} = 0$ . So force on  $q_B$  is  $\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{d^2} \hat{\mathbf{y}}}.$



(b) (i) From Eq. 12.68:  $\bar{\mathbf{F}} = \frac{\gamma}{4\pi\epsilon_0} \frac{q_A q_B}{d^2} \hat{\mathbf{y}}$ . (Note: here the particle is at rest in  $\bar{\mathcal{S}}$ .)

(ii) From Eq. 12.92, with  $\theta = 90^\circ$ :  $\bar{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \frac{q_A(1-v^2/c^2)}{(1-v^2/c^2)^{3/2}} \frac{1}{d^2} \hat{\mathbf{y}} = \frac{\gamma}{4\pi\epsilon_0} \frac{q_A}{d^2} \hat{\mathbf{y}}$   
(this also follows from Eq. 12.108).

$\bar{\mathbf{B}} \neq 0$ , but since  $v_B = 0$  in  $\bar{\mathcal{S}}$ , there is no magnetic force anyway, and  $\boxed{\bar{\mathbf{F}} = \frac{\gamma}{4\pi\epsilon_0} \frac{q_A q_B}{d^2} \hat{\mathbf{y}}}$  (as before).

**Problem 12.45**

Here  $\theta = 90^\circ$ ,  $\hat{\mathbf{r}} = \hat{\mathbf{y}}$ ,  $\hat{\phi} = \hat{\mathbf{z}}$ ,  $\mathbf{r} = r$ , so (using  $c^2 = 1/\mu_0\epsilon_0$ ):

$$\mathbf{E} = -\frac{q}{4\pi\epsilon_0} \frac{\gamma}{r^2} \hat{\mathbf{y}}, \quad \mathbf{B} = -\frac{q}{4\pi\epsilon_0} \frac{v}{c^2} \frac{\gamma}{r^2} \hat{\mathbf{z}}, \quad \text{where } \gamma = \frac{1}{\sqrt{1-v^2/c^2}}.$$

Note that  $(E^2 - B^2 c^2) = \left(\frac{q}{4\pi\epsilon_0 r}\right)^2 \gamma^2 (1 - \frac{v^2}{c^2}) = \left(\frac{q}{4\pi\epsilon_0 r}\right)^2$  is invariant, because it doesn't depend on  $v$ . We can use this as a check.

System A:  $v_A = v$ , so  $\mathbf{E} = -\frac{q}{4\pi\epsilon_0} \frac{\bar{\gamma}}{r^2} \hat{\mathbf{y}}$ ,  $\mathbf{B} = -\frac{q}{4\pi\epsilon_0} \frac{v}{c^2} \frac{\bar{\gamma}}{r^2} \hat{\mathbf{z}}$ , where  $\bar{\gamma} = \frac{1}{\sqrt{1-v^2/c^2}}$ .

$$\mathbf{F} = q[\mathbf{E} + (-v\hat{\mathbf{x}}) \times \mathbf{B}] = -\frac{q^2}{4\pi\epsilon_0} \frac{\bar{\gamma}}{r^2} [\hat{\mathbf{y}} - \frac{v^2}{c^2} (\hat{\mathbf{x}} \times \hat{\mathbf{z}})] = -\frac{q^2}{4\pi\epsilon_0} \frac{\bar{\gamma}}{r^2} \left(1 + \frac{v^2}{c^2}\right) \hat{\mathbf{y}}.$$

System B:  $v_B = \frac{v+v}{1+v^2/c^2} = \frac{2v}{(1+v^2/c^2)}$

$$\begin{aligned} \gamma_B &= \frac{1}{\sqrt{1 - \frac{4v^2/c^2}{(1+v^2/c^2)^2}}} = \frac{(1+v^2/c^2)}{\sqrt{1 - 2\frac{v^2}{c^2} + \frac{v^4}{c^4}}} = \frac{(1+v^2/c^2)}{(1-v^2/c^2)} = \bar{\gamma}^2 \left(1 + \frac{v^2}{c^2}\right); v_b \gamma_B = 2v \bar{\gamma}^2. \\ \therefore \mathbf{E} &= -\frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \bar{\gamma}^2 \left(1 + \frac{v^2}{c^2}\right) \hat{\mathbf{y}}; \mathbf{B} = -\frac{q}{4\pi\epsilon_0} \frac{2v}{c^2} \frac{\bar{\gamma}^2}{r^2} \hat{\mathbf{z}}. \end{aligned}$$

$$[Check: E^2 - B^2 c^2 = \left(\frac{q}{4\pi\epsilon_0 r}\right)^2 \bar{\gamma}^4 \left(1 + \frac{2v^2}{c^2} + \frac{v^4}{c^4} - \frac{4v^2}{c^2}\right) = \left(\frac{q}{4\pi\epsilon_0 r}\right)^2 \bar{\gamma}^4 \frac{1}{\bar{\gamma}^4} = \left(\frac{q}{4\pi\epsilon_0 r}\right)^2. \checkmark]$$

$$\mathbf{F} = q\mathbf{E} = -\frac{q^2}{4\pi\epsilon_0} \frac{\bar{\gamma}^2}{r^2} \left(1 + \frac{v^2}{c^2}\right) \hat{\mathbf{y}}. (+q \text{ at rest} \Rightarrow \text{no magnetic force}). [Check: Eq. 12.68 \Rightarrow F_A = \frac{1}{\bar{\gamma}} F_B. \checkmark]$$

System C:  $v_C = 0$ .  $\mathbf{E} = -\frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{\mathbf{y}}$ ;  $\mathbf{B} = 0$ ;  $\mathbf{F} = q\mathbf{E} = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{r^2} \hat{\mathbf{y}}$ .

[The relative velocity of  $B$  and  $C$  is  $2v/(1+v^2/c^2)$ , and the corresponding  $\gamma$  is  $\bar{\gamma}^2(1+v^2/c^2)$ . So Eq. 12.68  $\Rightarrow F_C = \frac{1}{\bar{\gamma}^2(1+v^2/c^2)} F_B$ .  $\checkmark$ ]

*Summary:*

$$\left. \begin{array}{|c|c|c|} \hline & \left( -\frac{q}{4\pi\epsilon_0 r^2} \right) \gamma \hat{\mathbf{y}} & \left( -\frac{q}{4\pi\epsilon_0 r^2} \right) \gamma^2 \left( 1 + \frac{v^2}{c^2} \right) \hat{\mathbf{y}} & \left( -\frac{q}{4\pi\epsilon_0 r^2} \right) \hat{\mathbf{y}} \\ \hline & \left( -\frac{q}{4\pi\epsilon_0 r^2} \right) \frac{v}{c^2} \gamma \hat{\mathbf{z}} & \left( -\frac{q}{4\pi\epsilon_0 r^2} \right) \frac{2v}{c^2} \gamma^2 \hat{\mathbf{z}} & 0 \\ \hline & \left( -\frac{q^2}{4\pi\epsilon_0 r^2} \right) \gamma \left( 1 + \frac{v^2}{c^2} \right) \hat{\mathbf{y}} & \left( -\frac{q^2}{4\pi\epsilon_0 r^2} \right) \gamma^2 \left( 1 + \frac{v^2}{c^2} \right) \hat{\mathbf{y}} & \left( -\frac{q}{4\pi\epsilon_0 r^2} \right) \hat{\mathbf{y}} \\ \hline \end{array} \right\} \gamma = \frac{1}{\sqrt{1-v^2/c^2}}.$$

### Problem 12.46

(a) From Eq. 12.108:

$$\begin{aligned} \bar{\mathbf{E}} \cdot \bar{\mathbf{B}} &= \bar{E}_x \bar{B}_x + \bar{E}_y \bar{B}_y + \bar{E}_z \bar{B}_z = E_x B_x + \gamma^2 (E_y - v B_z) (B_y + \frac{v}{c^2} E_z) + \gamma (E_z + v B_y) (B_z - \frac{v}{c^2} E_y) \\ &= E_x B_x + \gamma^2 \{ E_y B_y + \frac{v}{c^2} \cancel{E}_y E_z - v \cancel{B}_y B_z - \frac{v^2}{c^2} E_z B_z + E_z B_z - \frac{v}{c^2} \cancel{E}_y E_z + v \cancel{B}_y B_z - \frac{v^2}{c^2} E_y B_y \} \\ &= E_x B_x + \gamma^2 \left[ E_y B_y \left( 1 - \frac{v^2}{c^2} \right) + E_z B_z \left( 1 - \frac{v^2}{c^2} \right) \right] = E_x B_x + E_y B_y + E_z B_z = \mathbf{E} \cdot \mathbf{B}. \quad \text{qed} \end{aligned}$$

$$(b) \bar{E}^2 - c^2 \bar{B}^2 = [E_x^2 + \gamma^2 (E_y - v B_z)^2 + \gamma^2 (E_z + v B_y)^2] - c^2 [B_x^2 + \gamma^2 (B_y + \frac{v}{c^2} E_z)^2 + \gamma^2 (B_z - \frac{v}{c^2} E_y)^2]$$

$$= E_x^2 + \gamma^2 (E_y^2 - 2 E_y v B_z + v^2 B_z^2 + E_z^2 + 2 E_z v B_y + v^2 B_y^2 - c^2 B_y^2 - c^2 2 \frac{v}{c^2} \cancel{B}_y E_z)$$

$$- c^2 \frac{v^2}{c^4} E_z^2 - c^2 B_z^2 + c^2 2 \frac{v}{c^2} \cancel{B}_z E_y - c^2 \frac{v^2}{c^4} E_y^2 - c^2 B_x^2$$

$$= E_x^2 - c^2 B_x^2 + \gamma^2 \left[ E_y^2 \left( 1 - \frac{v^2}{c^2} \right) + E_z^2 \left( 1 - \frac{v^2}{c^2} \right) - c^2 B_y^2 \left( 1 - \frac{v^2}{c^2} \right) - c^2 B_z^2 \left( 1 - \frac{v^2}{c^2} \right) \right]$$

$$= (E_x^2 + E_y^2 + E_z^2) - c^2 (B_x^2 + B_y^2 + B_z^2) = E^2 - B^2 c^2. \quad \text{qed}$$

(c) No. For if  $\mathbf{B} = 0$  in one system, then  $(E^2 - c^2 B^2)$  is *positive*. Since it is invariant, it must be positive in *any* system. Therefore  $\mathbf{E} \neq 0$  in *all* systems.

### Problem 12.47

(a) Making the appropriate modifications in Eq. 9.48 (and picking  $\delta = 0$  for convenience),

$$\boxed{\mathbf{E}(x, y, z, t) = E_0 \cos(kx - \omega t) \hat{\mathbf{y}}, \quad \mathbf{B}(x, y, z, t) = \frac{E_0}{c} \cos(kx - \omega t) \hat{\mathbf{z}}, \quad \text{where } k \equiv \frac{\omega}{c}.}$$

(b) Using Eq. 12.108 to transform the fields:

$$\bar{E}_x = \bar{E}_z = 0, \quad \bar{E}_y = \gamma(E_y - v B_z) = \gamma E_0 \left[ \cos(kx - \omega t) - \frac{v}{c} \cos(kx - \omega t) \right] = \alpha E_0 \cos(kx - \omega t),$$

$$\bar{B}_x = \bar{B}_y = 0, \quad \bar{B}_z = \gamma(B_z - \frac{v}{c^2} E_y) = \gamma E_0 \left[ \frac{1}{c} \cos(kx - \omega t) - \frac{v}{c^2} \cos(kx - \omega t) \right] = \alpha \frac{E_0}{c} \cos(kx - \omega t),$$

where  $\alpha \equiv \gamma \left( 1 - \frac{v}{c} \right) = \sqrt{\frac{1-v/c}{1+v/c}}$

Now the inverse Lorentz transformations (Eq. 12.19)  $\Rightarrow x = \gamma(\bar{x} + vt)$  and  $t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)$ , so

$$kx - \omega t = \gamma \left[ k(\bar{x} + vt) - \omega \left( \bar{t} + \frac{v}{c^2}\bar{x} \right) \right] = \gamma \left[ \left( k - \frac{\omega v}{c^2} \right) \bar{x} - (\omega - kv)\bar{t} \right] = \bar{k}\bar{x} - \bar{\omega}\bar{t},$$

where (recalling that  $k = \omega/c$ ):  $\bar{k} \equiv \gamma \left( k - \frac{\omega v}{c^2} \right) = \gamma k(1 - v/c) = \alpha k$  and  $\bar{\omega} \equiv \gamma\omega(1 - v/c) = \alpha\omega$ .

*Conclusion:*

$$\bar{\mathbf{E}}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \bar{E}_0 \cos(\bar{k}\bar{x} - \bar{\omega}\bar{t}) \hat{y}, \quad \bar{\mathbf{B}}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \frac{\bar{E}_0}{c} \cos(\bar{k}\bar{x} - \bar{\omega}\bar{t}) \hat{z},$$

$$\text{where } \bar{E}_0 = \alpha E_0, \quad \bar{k} = \alpha k, \quad \bar{\omega} = \alpha\omega, \quad \text{and } \alpha \equiv \sqrt{\frac{1 - v/c}{1 + v/c}}.$$

(c)  $\bar{\omega} = \omega \sqrt{\frac{1 - v/c}{1 + v/c}}$ . This is the **Doppler shift** for light.  $\bar{\lambda} = \frac{2\pi}{\bar{k}} = \frac{2\pi}{\alpha k} = \frac{\lambda}{\alpha}$ . The velocity of the

wave in  $\bar{S}$  is  $\bar{v} = \frac{\bar{\omega}}{2\pi} \bar{\lambda} = \frac{\omega}{\lambda} = [c]$ . **Yup**, this is exactly what I expected (the velocity of a light wave is the same in any inertial system).

(d) Since intensity goes like  $E^2$ , the ratio is  $\frac{\bar{I}}{I} = \frac{\bar{E}_0^2}{E_0^2} = \alpha^2 = \frac{1 - v/c}{1 + v/c}$ .

Dear Al,

The amplitude, frequency, and intensity of the light wave will all **decrease to zero** as you run faster and faster. It'll get so faint you won't be able to see it, and so red-shifted even your night-vision goggles won't help. But it'll still be going  $3 \times 10^8$  m/s relative to you. Sorry about that.

Sincerely,

David

### Problem 12.48

$$\bar{t}^{02} = \Lambda_\lambda^0 \Lambda_\sigma^2 t^{\lambda\sigma} = \Lambda_0^0 \Lambda_2^2 t^{02} + \Lambda_1^0 \Lambda_2^2 t^{12} = \gamma t^{02} + (-\gamma\beta)t^{12} = \gamma(t^{02} - \beta t^{12}).$$

$$\bar{t}^{03} = \Lambda_\lambda^0 \Lambda_\sigma^3 t^{\lambda\sigma} = \Lambda_0^0 \Lambda_3^3 t^{03} + \Lambda_1^0 \Lambda_3^3 t^{13} = \gamma t^{03} + (-\gamma\beta)t^{13} = \gamma(t^{03} - \beta t^{13}) = \gamma(t^{03} + \beta t^{31}).$$

$$\bar{t}^{23} = \Lambda_\lambda^2 \Lambda_\sigma^3 t^{\lambda\sigma} = \Lambda_2^2 \Lambda_3^3 t^{23} = t^{23}.$$

$$\bar{t}^{31} = \Lambda_\lambda^3 \Lambda_\sigma^1 t^{\lambda\sigma} = \Lambda_3^3 \Lambda_1^1 t^{30} + \Lambda_3^3 \Lambda_1^1 t^{31} = (-\gamma\beta)t^{30} + \gamma t^{31} = \gamma(t^{31} + \beta t^{03}).$$

$$\bar{t}^{12} = \Lambda_\lambda^1 \Lambda_\sigma^2 t^{\lambda\sigma} = \Lambda_1^1 \Lambda_2^2 t^{02} + \Lambda_1^1 \Lambda_2^2 t^{12} = (-\gamma\beta)t^{02} + \gamma t^{12} = \gamma(t^{12} - \beta t^{02}).$$

### Problem 12.49

Suppose  $t^{\nu\mu} = \pm t^{\mu\nu}$  (+ for symmetric, - for antisymmetric).

$$\bar{t}^{\kappa\lambda} = \Lambda_\mu^\kappa \Lambda_\nu^\lambda t^{\mu\nu}$$

$$\bar{t}^{\lambda\kappa} = \Lambda_\mu^\lambda \Lambda_\nu^\kappa t^{\mu\nu} = \Lambda_\nu^\lambda \Lambda_\mu^\kappa t^{\nu\mu} \quad [\text{Because } \mu \text{ and } \nu \text{ are both summed from } 0 \rightarrow 3,$$

it doesn't matter which we call  $\mu$  and which we call  $\nu$ .]

$$= \Lambda_\mu^\kappa \Lambda_\mu^\lambda (\pm t^{\mu\nu}) \quad [\text{I used the symmetry of } t^{\mu\nu}, \text{ and wrote the } \Lambda\text{'s in the other order.}]$$

$$= \pm \bar{t}^{\kappa\lambda}. \quad \text{qed}$$

**Problem 12.50**

$$\begin{aligned}
 F^{\mu\nu}F_{\mu\nu} &= F^{00}F^{00} - F^{01}F^{01} - F^{02}F^{02} - F^{03}F^{03} - F^{10}F^{10} - F^{20}F^{20} - F^{30}F^{30} \\
 &\quad + F^{11}F^{11} + F^{12}F^{12} + F^{13}F^{13} + F^{21}F^{21} + F^{22}F^{22} + F^{23}F^{23} + F^{31}F^{31} + F^{32}F^{32} + F^{33}F^{33} \\
 &= -(E_x/c)^2 - (E_y/c)^2 - (E_z/c)^2 - (E_x/c)^2 - (E_y/c)^2 - (E_z/c)^2 + B_z^2 + B_y^2 + B_z^2 + B_x^2 + B_y^2 \\
 &= 2B^2 - 2E^2/c^2 = \boxed{2\left(B^2 - \frac{E^2}{c^2}\right)},
 \end{aligned}$$

which, apart from the constant factor  $-\frac{2}{c^2}$ , is the invariant we found in Prob. 12.46(b).

$$\boxed{G^{\mu\nu}G_{\mu\nu} = 2(E^2/c^2 - B^2)} \text{ (the same invariant).}$$

$$\begin{aligned}
 F^{\mu\nu}G_{\mu\nu} &= -2(F^{01}G^{01} + F^{02}G^{02} + F^{03}G^{03}) + 2(F^{12}G^{12} + F^{13}G^{13} + F^{23}G^{23}) \\
 &= -2\left(\frac{1}{c}E_xB_x + \frac{1}{c}E_yB_y + \frac{1}{c}E_zB_z\right)2[B_z(-E_z/c) + (-B_y)(E_y/c) + B_x(-E_x/c)] \\
 &= -\frac{2}{c}(\mathbf{E} \cdot \mathbf{B}) - \frac{2}{c}(\mathbf{E} \cdot \mathbf{B}) = \boxed{-\frac{4}{c}(\mathbf{E} \cdot \mathbf{B})},
 \end{aligned}$$

which, apart from the factor  $-4/c$ , is the invariant of Prob. 12.46(a). [These are, incidentally, the *only* fundamental invariants you can construct from  $\mathbf{E}$  and  $\mathbf{B}$ .]

**Problem 12.51**

$$\left. \begin{array}{l} \mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{x} \hat{x} = \frac{\mu_0}{2\pi} \frac{\lambda c^2}{x} \hat{x} \\ \mathbf{B} = \frac{\mu_0}{4\pi} \frac{2\lambda v}{x} \hat{y} = \frac{\mu_0}{2\pi} \frac{\lambda v}{x} \hat{y} \end{array} \right\} F^{\mu\nu} = \frac{\mu_0\lambda}{2\pi x} \begin{pmatrix} 0 & c & 0 & 0 \\ -c & 0 & 0 & -v \\ 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 \end{pmatrix}.$$

**Problem 12.52**

$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$ . Differentiate:  $\partial_\mu \partial_\nu F^{\mu\nu} = \mu_0 \partial_\mu J^\mu$ .

But  $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$  (the combination is *symmetric*) while  $F^{\nu\mu} = -F^{\mu\nu}$  (*antisymmetric*).

$\therefore \partial_\mu \partial_\nu F^{\mu\nu} = 0$ . [Why? Well, these indices are both summed from  $0 \rightarrow 3$ , so it doesn't matter which we call  $\mu$ , which  $\nu$ :  $\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu F^{\nu\mu} = \partial_\mu \partial_\nu (-F^{\mu\nu}) = -\partial_\mu \partial_\nu F^{\mu\nu}$ . But if a quantity is equal to minus itself, it must be zero.] Conclusion:  $\partial_\mu J^\mu = 0$ . qed

**Problem 12.53**

We know that  $\partial_\nu G^{\mu\nu} = 0$  is equivalent to the two homogeneous Maxwell equations,  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ . All we have to show, then, is that  $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$  is *also* equivalent to them. Now this equation stands for 64 separate equations ( $\mu = 0 \rightarrow 3$ ,  $\nu = 0 \rightarrow 3$ ,  $\lambda = 0 \rightarrow 3$ , and  $4 \times 4 \times 4 = 64$ ). But many of them are redundant, or trivial.

Suppose two indices are the same (say,  $\mu = \nu$ ). Then  $\partial_\lambda F_{\mu\mu} + \partial_\mu F_{\mu\lambda} + \partial_\mu F_{\lambda\mu} = 0$ . But  $F_{\mu\mu} = 0$  and  $F_{\mu\lambda} = -F_{\lambda\mu}$ , so this is trivial:  $0 = 0$ . To get anything significant, then,  $\mu$ ,  $\nu$ ,  $\lambda$  must all be *different*. They could be *all spatial* ( $\mu, \nu, \lambda = 1, 2, 3 = x, y, z$  — or some permutation thereof), or *one temporal and two spatial* ( $\mu = 0, \nu, \lambda = 1, 2$  or  $2, 3$ , or  $1, 3$  — or some permutation). Let's examine these two cases separately.

*All spatial*: say,  $\mu = 1, \nu = 2, \lambda = 3$  (other permutations yield the same equation, or minus it).

$$\partial_3 F_{12} + \partial_1 F_{23} + \partial_2 F_{31} = 0 \Rightarrow \frac{\partial}{\partial z}(B_z) + \frac{\partial}{\partial x}(B_x) + \frac{\partial}{\partial y}(B_y) = 0 \Rightarrow \nabla \cdot \mathbf{B} = 0.$$

*One temporal:* say,  $\mu = 0, \nu = 1, \lambda = 2$  (other permutations of these indices yield the same result, or minus it).

$$\partial_2 F_{01} + \partial_0 F_{12} + \partial_1 F_{20} = 0 \Rightarrow \frac{\partial}{\partial y} \left( -\frac{E_x}{c} \right) + \frac{\partial}{\partial (ct)} (B_z) + \frac{\partial}{\partial x} \left( \frac{E_y}{c} \right) = 0,$$

or  $-\frac{\partial B_z}{\partial t} + \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) = 0$ , which is the  $z$  component of  $-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}$ . (If  $\mu = 0, \nu = 1, \lambda = 2$ , we get the  $y$  component; for  $\nu = 2, \lambda = 3$  we get the  $x$  component.)

*Conclusion:*  $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$  is equivalent to  $\nabla \cdot \mathbf{B} = 0$  and  $\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$ , and hence to  $\partial_\nu G^{\mu\nu} = 0$ . qed

### Problem 12.54

$K^0 = q\eta_\nu F^{0\nu} = q(\eta_1 F^{01} + \eta_2 F^{02} + \eta_3 F^{03}) = q(\eta \cdot \mathbf{E})/c = \boxed{\frac{q}{c} \gamma \mathbf{u} \cdot \mathbf{E}}$ . Now from Eq. 12.71 we know that  $K^0 = \frac{1}{c} \frac{dW}{d\tau}$ , where  $W$  is the energy of the particle. Since  $d\tau = \frac{1}{\gamma} dt$ , we have:

$$\frac{1}{c} \gamma \frac{dW}{dt} = \frac{q}{c} \gamma (\mathbf{u} \cdot \mathbf{E}) \Rightarrow \boxed{\frac{dW}{dt} = q(\mathbf{u} \cdot \mathbf{E})}.$$

This says the power delivered to the particle is force ( $q\mathbf{E}$ ) times velocity ( $\mathbf{u}$ ) — which is as it should be.

### Problem 12.55

$$\overline{\partial^0 \phi} = \frac{\partial}{\partial \bar{x}_0} \phi = -\frac{1}{c} \frac{\partial}{\partial \bar{t}} \phi = -\frac{1}{c} \left( \frac{\partial \phi}{\partial t} \frac{\partial t}{\partial \bar{t}} + \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \bar{t}} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \bar{t}} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \bar{t}} \right).$$

From Eq. 12.19, we have:  $\frac{\partial t}{\partial \bar{t}} = \gamma, \quad \frac{\partial x}{\partial \bar{t}} = \gamma v, \quad \frac{\partial y}{\partial \bar{t}} = \frac{\partial z}{\partial \bar{t}} = 0$ .

$$\text{So } \overline{\partial^0 \phi} = -\frac{1}{c} \gamma \left( \frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x} \right) \text{ or (since } ct = x^0 = -x_0\text{): } \overline{\partial^0 \phi} = \gamma \left( \frac{\partial \phi}{\partial x_0} - \frac{v}{c} \frac{\partial \phi}{\partial x^1} \right) = \gamma [(\partial^0 \phi) - \beta(\partial^1 \phi)].$$

$$\overline{\partial^1 \phi} = \frac{\partial}{\partial \bar{x}} \phi = \frac{\partial \phi}{\partial t} \frac{\partial t}{\partial \bar{x}} + \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \bar{x}} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \bar{x}} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \bar{x}} = \gamma \frac{v}{c^2} \frac{\partial \phi}{\partial t} + \gamma \frac{\partial \phi}{\partial x} = \gamma \left( \frac{\partial \phi}{\partial x_1} - \frac{v}{c} \frac{\partial \phi}{\partial x_0} \right) = \gamma [(\partial^1 \phi) - \beta(\partial^0 \phi)].$$

$$\overline{\partial^2 \phi} = \frac{\partial}{\partial \bar{y}} \phi = \frac{\partial \phi}{\partial t} \frac{\partial t}{\partial \bar{y}} + \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \bar{y}} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \bar{y}} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \bar{y}} = \frac{\partial \phi}{\partial y} = \partial^2 \phi.$$

$$\overline{\partial^3 \phi} = \frac{\partial}{\partial \bar{z}} \phi = \frac{\partial \phi}{\partial t} \frac{\partial t}{\partial \bar{z}} + \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \bar{z}} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \bar{z}} = \frac{\partial \phi}{\partial z} = \partial^3 \phi.$$

*Conclusion:*  $\partial^\mu \phi$  transforms in the same way as  $a^\mu$  (Eq. 12.27)—and hence is a contravariant 4-vector. qed

### Problem 12.56

According to Prob. 12.53,  $\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$  is equivalent to Eq. 12.129. Using Eq. 12.132, we find (in the notation of Prob. 12.55):

$$\begin{aligned} \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} &= \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} \\ &= \partial_\lambda (\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\mu (\partial_\nu A_\lambda - \partial_\lambda A_\nu) + \partial_\nu (\partial_\lambda A_\mu - \partial_\mu A_\lambda) \\ &= (\partial_\lambda \partial_\mu A_\nu - \partial_\mu \partial_\lambda A_\nu) + (\partial_\mu \partial_\nu A_\lambda - \partial_\nu \partial_\mu A_\lambda) + (\partial_\nu \partial_\lambda A_\mu - \partial_\lambda \partial_\nu A_\mu) = 0. \quad \text{qed} \end{aligned}$$

[Note that  $\partial_\lambda \partial_\mu A_\nu = \frac{\partial^2 A_\nu}{\partial x^\lambda \partial x^\mu} = \frac{\partial^2 A_\nu}{\partial x^\mu \partial x^\lambda} = \partial_\mu \partial_\lambda A_\nu$ , by equality of cross-derivatives.]

**Problem 12.57**

Step 1: rotate from  $xy$  to  $XY$ , using Eq. 1.29:

$$X = \cos \phi x + \sin \phi y$$

$$Y = -\sin \phi x + \cos \phi y$$

Step 2: Lorentz-transform from  $XY$  to  $\bar{X}\bar{Y}$ , using Eq. 12.18:

$$\bar{X} = \gamma(X - vt) = \gamma[\cos \phi x + \sin \phi y - \beta ct]$$

$$\bar{Y} = Y = -\sin \phi x + \cos \phi y$$

$$\bar{Z} = Z = z$$

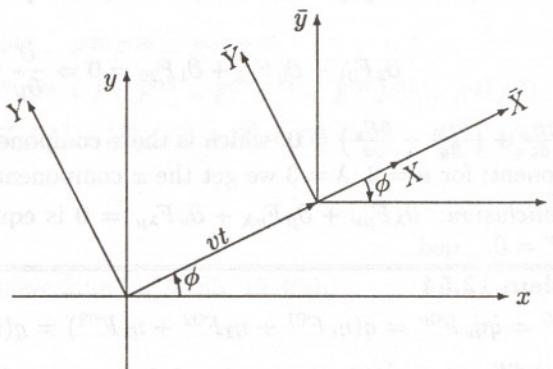
$$\bar{ct} = \gamma(ct - \beta X) = \gamma[ct - \beta(\cos \phi x + \sin \phi y)]$$

Step 3: Rotate from  $\bar{X}\bar{Y}$  to  $\bar{x}\bar{y}$ , using Eq. 1.29 with negative  $\phi$ :

$$\begin{aligned} \bar{x} &= \cos \phi \bar{X} - \sin \phi \bar{Y} = \gamma \cos \phi [\cos \phi x + \sin \phi y - \beta ct] - \sin \phi [-\sin \phi x + \cos \phi y] \\ &= (\gamma \cos^2 \phi + \sin^2 \phi)x + (\gamma - 1) \sin \phi \cos \phi y - \gamma \beta \cos \phi (ct) \end{aligned}$$

$$\begin{aligned} \bar{y} &= \sin \phi \bar{X} + \cos \phi \bar{Y} = \gamma \sin \phi (\cos \phi x + \sin \phi y - \beta ct) + \cos \phi (-\sin \phi x + \cos \phi y) \\ &= (\gamma - 1) \sin \phi \cos \phi x + (\gamma \sin^2 \phi + \cos^2 \phi)y - \gamma \beta \sin \phi (ct) \end{aligned}$$

In matrix form: 
$$\begin{pmatrix} \bar{ct} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma \beta \cos \phi & -\gamma \beta \sin \phi & 0 \\ -\gamma \beta \cos \phi & (\gamma \cos^2 \phi + \sin^2 \phi) & (\gamma - 1) \sin \phi \cos \phi & 0 \\ -\gamma \beta \sin \phi & (\gamma - 1) \sin \phi \cos \phi & (\gamma \sin^2 \phi + \cos^2 \phi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}.$$

**Problem 12.58**

In center-of-momentum system, threshold occurs when incident energy is just sufficient to cover the rest energy of the resulting particles, with none “wasted” as kinetic energy. Thus, in lab system, we want the outgoing  $K$  and  $\Sigma$  to have the same velocity, at threshold:

$$\begin{array}{c} \pi \rightarrow p \\ \text{Before} \end{array} \quad \begin{array}{c} K \Sigma \rightarrow \\ \text{After} \end{array}$$

$$\begin{array}{ccccc} \pi \rightarrow & & p & \leftarrow & \text{before (CM)} \\ & & & & \\ & \infty & \Sigma & \infty & \text{after (CM)} \end{array}$$

Initial momentum:  $p_\pi$ ; initial energy of  $\pi$ :  $E^2 - p^2 c^2 = m^2 c^4 \Rightarrow E_\pi^2 = m_\pi^2 c^4 + p_\pi^2 c^2$ .

Total initial energy:  $m_p c^2 + \sqrt{m_\pi^2 c^4 + p_\pi^2 c^2}$ . These are also the final energy and momentum:  $E^2 - p^2 c^2 = (m_K + m_\Sigma)^2 c^4$ .

$$(m_p c^2 + \sqrt{m_\pi^2 c^4 + p_\pi^2 c^2})^2 - p_\pi^2 c^2 = (m_K + m_\Sigma)^2 c^4$$

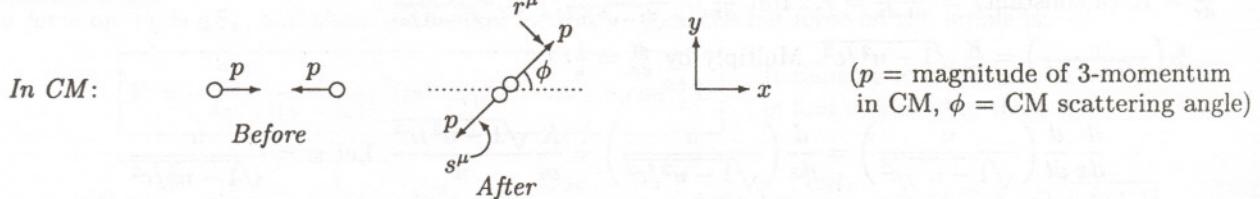
$$m_p^2 c^4 + \frac{2m_p c^2}{c^4} \sqrt{m_\pi^2 c^2 + p_\pi^2 c^2} c + m_\pi^2 c^4 + p_\pi^2 c^2 - p_\pi^2 c^2 = (m_K + m_\Sigma)^2 c^4$$

$$\frac{2m_p}{c} \sqrt{m_\pi^2 c^2 + p_\pi^2} = (m_K + m_\Sigma)^2 - m_p^2 - m_\pi^2$$

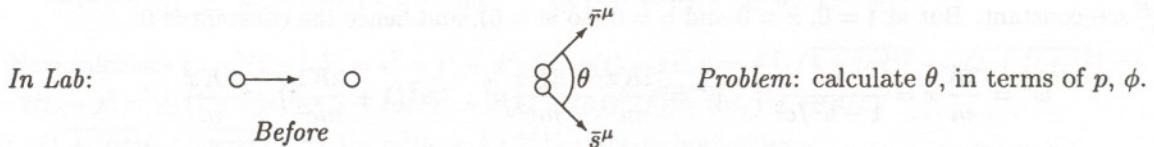
$$(m_\pi^2 c^2 + p_\pi^2) \frac{4m_p^2}{c^2} = (m_K + m_\Sigma)^4 - 2(m_p^2 + m_\pi^2)(m_K + m_\Sigma)^2 + m_p^4 + m_\pi^4 + 2m_p^2 m_\pi^2$$

$$\frac{4m_p^2}{c^2} p_\pi^2 = (m_K + m_\Sigma)^4 - 2(m_p^2 + m_\pi^2)(m_K + m_\Sigma)^2 + (m_p^2 - m_\pi^2)^2$$

$$\begin{aligned} p_\pi &= \frac{c}{2m_p} \sqrt{(m_K + m_\Sigma)^4 - 2(m_p^2 + m_\pi^2)(m_K + m_\Sigma)^2 + (m_p^2 - m_\pi^2)^2} \\ &= \frac{1}{(2m_p c^2)c} \sqrt{(m_K c^2 + m_\Sigma c^2)^4 - 2[(m_p c^2)^2 + (m_\pi c^2)^2](m_K c^2 + m_\Sigma c^2)^2 + [(m_p c^2)^2 - (m_\pi c^2)^2]^2} \\ &= \frac{1}{2c(900)} \sqrt{(1700)^4 - 2[(900)^2 + (150)^2](1700)^2 + [(900)^2 - (150)^2]^2} \\ &= \frac{1}{1800c} \sqrt{(8.35 \times 10^{12}) - (4.81 \times 10^{12}) + (0.62 \times 10^{12})} = \frac{1}{1800c} (2.04 \times 10^6) = 1133 \text{ MeV}/c. \end{aligned}$$

**Problem 12.59**

Outgoing 4-momenta:  $r^\mu = (\frac{E}{c}, p \cos \phi, p \sin \phi, 0)$ ;  $s^\mu = (\frac{E}{c}, -p \cos \phi, -p \sin \phi, 0)$ .



Lorentz transformation:  $\bar{r}_x = \gamma(r_x - \beta r^0)$ ;  $\bar{r}_y = r_y$ ;  $\bar{s}_x = \gamma(s_x - \beta s^0)$ ;  $\bar{s}_y = s_y$ .

Now  $E = \gamma mc^2$ ;  $p = -\gamma mv$  ( $v$  here is to the left);  $E^2 - p^2 c^2 = m^2 c^4$ , so  $\beta = -\frac{pc}{E}$ .

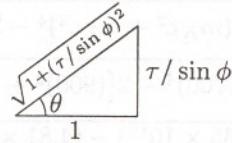
$$\therefore \bar{r}_x = \gamma(p \cos \phi + \frac{pc}{E} \frac{E}{c}) = \gamma p(1 + \cos \phi); \bar{r}_y = p \sin \phi; \bar{s}_x = \gamma p(1 - \cos \phi); \bar{s}_y = -p \sin \phi.$$

$$\begin{aligned} \cos \theta &= \frac{\bar{r} \cdot \bar{s}}{\bar{r} \bar{s}} = \frac{\gamma^2 p^2 (1 - \cos^2 \phi) - p^2 \sin^2 \phi}{\sqrt{[\gamma^2 p^2 (1 + \cos \phi)^2 + p^2 \sin^2 \phi] [\gamma^2 p^2 (1 - \cos \phi)^2 + p^2 \sin^2 \phi]}} \\ &= \frac{(\gamma^2 - 1) \sin^2 \phi}{\sqrt{[\gamma^2 (1 + \cos \phi)^2 + \sin^2 \phi] [\gamma^2 (1 - \cos \phi)^2 + \sin^2 \phi]}} \\ &= \frac{(\gamma^2 - 1)}{\sqrt{\left[\gamma^2 \left(\frac{1+\cos \phi}{\sin \phi}\right)^2 + 1\right] \left[\gamma^2 \left(\frac{1-\cos \phi}{\sin \phi}\right)^2 + 1\right]}} = \frac{(\gamma^2 - 1)}{\sqrt{(\gamma^2 \cot^2 \frac{\phi}{2} + 1) (\gamma^2 \tan^2 \frac{\phi}{2} + 1)}} \end{aligned}$$

$$\begin{aligned}
 \cos \theta &= \frac{\omega}{\sqrt{(1 + \cot^2 \frac{\phi}{2} + \omega \cot^2 \frac{\phi}{2})(1 + \tan^2 \frac{\phi}{2} + \omega \tan^2 \frac{\phi}{2})}} \quad (\text{where } \omega \equiv \gamma^2 - 1) \\
 &= \frac{\omega}{\sqrt{(\csc^2 \frac{\phi}{2} + \omega \cot^2 \frac{\phi}{2})(\sec^2 \frac{\phi}{2} + \omega \tan^2 \frac{\phi}{2})}} = \frac{\omega \sin \frac{\phi}{2} \cos \frac{\phi}{2}}{\sqrt{(1 + \omega \cos^2 \frac{\phi}{2})(1 + \omega \sin^2 \frac{\phi}{2})}} \\
 &= \frac{\frac{1}{2}\omega \sin \phi}{\sqrt{[1 + \frac{1}{2}\omega(1 + \cos \phi)][1 + \frac{1}{2}\omega(1 - \cos \phi)]}} = \frac{\sin \phi}{\sqrt{[(\frac{2}{\omega} + 1) + \cos \phi][( \frac{2}{\omega} + 1) - \cos \phi]}} \\
 &= \frac{\sin \phi}{\sqrt{(\frac{2}{\omega} + 1)^2 - \cos^2 \phi}} = \frac{\sin \phi}{\sqrt{\frac{4}{\omega^2} + \frac{4}{\omega} + \sin^2 \phi}} = \frac{1}{\sqrt{1 + (\tau / \sin \phi)^2}}, \text{ where } \tau^2 = \frac{4}{\omega^2} + \frac{4}{\omega}.
 \end{aligned}$$

$$\sin \theta = \frac{\tau}{\sin \phi}. \quad \tau^2 = \frac{4}{\omega^2}(1 + \omega) = \frac{4}{(\gamma^2 - 1)^2} \gamma^2, \text{ so } \tan \theta = \frac{2\gamma}{(\gamma^2 - 1) \sin \phi}.$$

$$\text{Or, since } (\gamma^2 - 1) = \gamma^2 \left(1 - \frac{1}{\gamma^2}\right) = \gamma^2 \frac{v^2}{c^2}, \quad \tan \theta = \frac{2c^2}{\gamma v^2 \sin \phi}.$$



### Problem 12.60

$\frac{dp}{d\tau} = K$  (a constant)  $\Rightarrow \frac{dp}{dt} \frac{dt}{d\tau} = K$ . But  $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}}$ ;  $p = \frac{mu}{\sqrt{1-u^2/c^2}}$ .

$\therefore \frac{d}{dt} \left( \frac{u}{\sqrt{1-u^2/c^2}} \right) = \frac{K}{m} \sqrt{1-u^2/c^2}$ . Multiply by  $\frac{dt}{dx} = \frac{1}{u}$ :

$$\begin{aligned}
 \frac{dt}{dx} \frac{d}{dt} \left( \frac{u}{\sqrt{1-u^2/c^2}} \right) &= \frac{d}{dx} \left( \frac{u}{\sqrt{1-u^2/c^2}} \right) = \frac{K}{m} \frac{\sqrt{1-u^2/c^2}}{u}. \text{ Let } w = \frac{u}{\sqrt{1-u^2/c^2}}. \\
 \frac{dw}{dx} = \frac{K}{m} \frac{1}{w} ; \quad w \frac{dw}{dx} = \frac{1}{2} \frac{d}{dx} w^2 = \frac{k}{m} ; \quad \frac{d(w^2)}{dx} = \frac{2K}{m} \Rightarrow d(w^2) &= \frac{2K}{m} (dx).
 \end{aligned}$$

$\therefore w^2 = \frac{2K}{m} x + \text{constant}$ . But at  $t = 0$ ,  $x = 0$  and  $u = 0$  (so  $w = 0$ ), and hence the constant is 0.

$$w^2 = \frac{2K}{m} x = \frac{u^2}{1-u^2/c^2}; \quad u^2 = \frac{2Kx}{m} - \frac{2Kx}{mc^2} u^2; \quad u^2 \left(1 + \frac{2Kx}{mc^2}\right) = \frac{2Kx}{m}.$$

$$u^2 = \frac{2Kx/m}{1 + \frac{2Kx}{mc^2}} = \frac{c^2}{1 + (\frac{mc^2}{2Kx})}; \quad \frac{dx}{dt} = \frac{c}{\sqrt{1 + (\frac{mc^2}{2Kx})}}; \quad ct = \int \sqrt{1 + (\frac{mc^2}{2Kx})} dx.$$

$$\text{Let } \frac{mc^2}{2K} \equiv a^2; \quad ct = \int \frac{\sqrt{x+a^2}}{\sqrt{x}} dx. \quad \text{Let } x \equiv y^2; dx = 2y dy; \sqrt{x} = y.$$

$$ct = \int \frac{\sqrt{y^2 + a^2}}{y} 2y dy = 2 \int \sqrt{y^2 + a^2} dy = \left[ y \sqrt{y^2 + a^2} + a^2 \ln(y + \sqrt{y^2 + a^2}) \right] + \text{constant.}$$

At  $t = 0$ ,  $x = 0 \Rightarrow y = 0$ , so  $0 = a^2 \ln a + \text{constant} \Rightarrow \text{constant} = -a^2 \ln a$ .

$$\therefore ct = y \sqrt{y^2 + a^2} + a^2 \ln(y/a + \sqrt{(y/a)^2 + 1}) = a^2 \left[ \left( \frac{y}{a} \right) \sqrt{\left( \frac{y}{a} \right)^2 + 1} + \ln \left( \frac{y}{a} + \sqrt{\left( \frac{y}{a} \right)^2 + 1} \right) \right].$$

$$\text{Let: } z \equiv y/a = \sqrt{x} \sqrt{\frac{2K}{mc^2}} = \sqrt{\frac{2Kx}{mc^2}}. \quad \text{Then } \frac{2Kt}{mc} = z \sqrt{1+z^2} + \ln(z + \sqrt{1+z^2}).$$

**Problem 12.61**

(a)  $x(t) = \frac{c}{\alpha} [\sqrt{1 + (\alpha t)^2} - 1]$ , where  $\alpha = \frac{F}{mc}$ . The force of  $+q$  on  $-q$  will be the mirror image of the force of  $-q$  on  $+q$  (in the  $x$  axis), so the net force is in the  $x$  direction (the net magnetic force is zero). All we need is the  $x$  component of  $\mathbf{E}$ .

The field at  $+q$  due to  $-q$  is: (Eq. 10.65)

$$\mathbf{E} = -\frac{q}{4\pi\epsilon_0} \frac{\mathbf{z}}{(\mathbf{r} \cdot \mathbf{u})^3} [\mathbf{u}(c^2 - v^2) + \mathbf{u}(\mathbf{z} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{z} \cdot \mathbf{u})].$$

$$\mathbf{u} = c\mathbf{z} - \mathbf{v} \Rightarrow u_x = c\frac{l}{z} - v = \frac{1}{z}(cl - vr); \mathbf{z} \cdot \mathbf{u} = c\mathbf{z} - \mathbf{z} \cdot \mathbf{v} = (cr - lv); \mathbf{z} \cdot \mathbf{a} = la. \text{ So:}$$

$$\begin{aligned} E_x &= -\frac{q}{4\pi\epsilon_0} \frac{\mathbf{z}}{(cr - vl)^3} \left[ \frac{1}{z}(cl - vr)(c^2 - v^2) + \underbrace{\frac{1}{z}(cl - lv)la - a(cr - lv)}_{\frac{1}{z}ca(l^2 - r^2)} \right] \\ &= -\frac{q}{4\pi\epsilon_0} \frac{1}{(cr - vl)^3} [(cl - vr)(c^2 - v^2) - cad^2]. \end{aligned}$$

The force on  $+q$  is  $qE_x$ , and there is an equal force on  $-q$ , so the net force on the dipole is:

$$\mathbf{F} = -\frac{2q^2}{4\pi\epsilon_0} \frac{1}{(cr - lv)^3} [(cl - vr)(c^2 - v^2) - cad^2] \hat{x}.$$

It remains to determine  $r$ ,  $l$ ,  $v$ , and  $a$ , and plug these in.

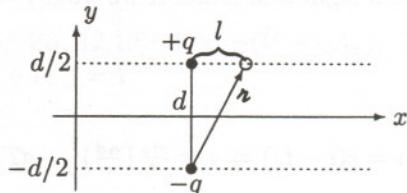
$$\begin{aligned} v(t) &= \frac{dx}{dt} = \frac{c}{\alpha} \frac{1}{2} \frac{1}{\sqrt{1 + (\alpha t)^2}} 2\alpha^2 t = \frac{c\alpha t}{\sqrt{1 - (\alpha t)^2}}; v = v(t_r) = \frac{c\alpha t_r}{T}, \text{ where } T \equiv \sqrt{1 + (\alpha t_r)^2}. \\ a(t_r) &= \frac{dv}{dt_r} = \frac{c\alpha}{T} + c\alpha t_r \left( -\frac{1}{2} \right) \frac{2\alpha^2 t_r}{T^3} = \frac{c\alpha}{T^3} [1 + (\alpha t_r)^2 - (\alpha t_r)^2] = \frac{c\alpha}{T^3}. \end{aligned}$$

$$\begin{aligned} \text{Now calculate } t_r: c^2(t - t_r)^2 &= r^2 = l^2 + d^2; l = x(t) - x(t_r) = \frac{c}{\alpha} [\sqrt{1 + (\alpha t)^2} - \sqrt{1 + (\alpha t_r)^2}], \text{ so} \\ \cancel{t^2} - 2tt_r + \cancel{t_r^2} &= \frac{1}{\alpha^2} [1 + (\cancel{\alpha t})^2 + 1 + (\cancel{\alpha t_r})^2 - 2\sqrt{1 + (\alpha t)^2}\sqrt{1 + (\alpha t_r)^2}] + (d/c)^2 \\ (\star) \sqrt{1 + (\alpha t)^2}\sqrt{1 + (\alpha t_r)^2} &= 1 + \alpha^2 tt_r + \frac{1}{2} \left( \frac{\alpha d}{c} \right)^2. \text{ Square both sides:} \end{aligned}$$

$$\begin{aligned} \cancel{t^2} + (\alpha t)^2 + (\alpha t_r)^2 + \cancel{\alpha^4 t^2 t_r^2} &= \cancel{t^2} + \cancel{\alpha^2 t^2 t_r^2} + \frac{1}{4} \left( \frac{\alpha d}{c} \right)^4 + 2\alpha^2 tt_r + \left( \frac{\alpha d}{c} \right)^2 + \alpha^2 tt_r \left( \frac{\alpha d}{c} \right)^2 \\ t^2 + t_r^2 - 2tt_r - tt_r \left( \frac{\alpha d}{c} \right)^2 - \left( \frac{d}{c} \right)^2 - \frac{\alpha^2}{4} \left( \frac{d}{c} \right)^4 &= 0. \end{aligned}$$

At this point we could solve for  $t_r$  in terms of  $t$ , but since  $v$  and  $a$  are already expressed in terms of  $t_r$ , it is simpler to solve for  $t$  (in terms of  $t_r$ ), and express everything in terms of  $t_r$ :

$$\begin{aligned} t^2 - tt_r \left[ 2 + \left( \frac{\alpha d}{c} \right)^2 \right] + \left[ t_r^2 - \left( \frac{d}{c} \right)^2 - \frac{\alpha^2}{4} \left( \frac{d}{c} \right)^4 \right] &= 0 \implies \\ t &= \frac{1}{2} \left\{ t_r \left[ 2 + \left( \frac{\alpha d}{c} \right)^2 \right] \pm \sqrt{t_r^2 \left[ 4 + 4 \left( \frac{\alpha d}{c} \right)^2 + \left( \frac{\alpha d}{c} \right)^4 \right] - 4t_r^2 + 4 \left( \frac{d}{c} \right)^2 + \alpha^2 \left( \frac{d}{c} \right)^4} \right\} \\ &= t_r \left[ 1 + \frac{1}{2} \left( \frac{\alpha d}{c} \right)^2 \right] \pm \sqrt{\left[ 1 + (\alpha t_r)^2 \right] \left( \frac{d}{c} \right)^2 \left[ 1 + \left( \frac{\alpha d}{2c} \right)^2 \right]} \end{aligned}$$



Which sign? For small  $\alpha$  we want  $t \approx t_r + d/c$ , so we need the + sign:

$$t = t_r \left[ 1 + \frac{1}{2} \left( \frac{\alpha d}{c} \right)^2 \right] + \frac{d}{c} TD, \text{ where } D \equiv \sqrt{1 + \left( \frac{\alpha d}{2c} \right)^2}.$$

So  $\tau = c(t - t_r) \Rightarrow \tau = \frac{ct_r}{2} \left( \frac{\alpha d}{c} \right)^2 + dTD$ . Now go back to Eq. (★) and solve for  $\sqrt{1 + (\alpha t)^2}$ :

$$\begin{aligned} \sqrt{1 + (\alpha t)^2} &= \frac{1}{T} \left\{ 1 + \frac{1}{2} \left( \frac{\alpha d}{c} \right)^2 + \alpha^2 t_r \left[ t_r \left( 1 + \frac{1}{2} \left( \frac{\alpha d}{c} \right)^2 \right) + \frac{d}{c} TD \right] \right\} \\ &= \frac{1}{T} \left\{ \underbrace{\left[ 1 + (\alpha t_r)^2 \right]}_{T^2} \left[ 1 + \frac{1}{2} \left( \frac{\alpha d}{c} \right)^2 \right] + \frac{\alpha^2 t_r d}{c} TD \right\} = \left[ 1 + \frac{1}{2} \left( \frac{\alpha d}{c} \right)^2 \right] T + \frac{\alpha^2 t_r d}{c} D. \end{aligned}$$

$$l = \frac{c}{\alpha} \left[ \sqrt{1 + (\alpha t^2)} - \sqrt{1 + (\alpha t_r)^2} \right] = \frac{c}{\alpha} \left\{ \left[ X + \frac{1}{2} \left( \frac{\alpha d}{c} \right)^2 \right] T + \frac{\alpha^2 t_r d}{c} D - X' \right\} = \alpha d \left( \frac{d}{2c} T + t_r D \right).$$

Putting all this in, the numerator in square brackets in  $\mathbf{F}$  becomes:

$$\begin{aligned} [ ] &= \left\{ \alpha d \left( \frac{d}{2c} T + t_r D \right) - \frac{c \alpha t_r}{T} \left[ \frac{ct_r}{2} \left( \frac{\alpha d}{c} \right)^2 + dTD \right] \right\} \left( c^2 - \frac{c^2 \alpha^2 t_r^2}{T^2} \right) - c \frac{c \alpha}{T^3} d^2 \\ &= \alpha d \left[ \frac{d}{2c} T + t_r D - \frac{d(\alpha t_r)^2}{2cT} - t_r D \right] \frac{c^2}{T^2} [1 + (\cancel{\alpha t_r})^2 - (\cancel{\alpha t_r})^2] - \frac{c^2 \alpha d^2}{T^3} \\ &= \frac{c^2 \alpha d^2}{T^3} \left[ \frac{1}{2} T^2 - \frac{1}{2} (\alpha t_r)^2 - 1 \right] = \frac{c^2 \alpha d^2}{2T^3} [1 + (\alpha t_r)^2 - (\alpha t_r)^2 - 2] = -\frac{c^2 \alpha d^2}{2T^3}. \end{aligned}$$

$$\therefore \mathbf{F} = \frac{q^2}{4\pi\epsilon_0} \frac{c^2 \alpha d^2}{[(\alpha r - lv)T]^3} \hat{x}. \quad \text{It remains to compute the denominator:}$$

$$\begin{aligned} (\alpha r - lv)T &= \left\{ c \left[ \frac{ct_r}{2} \left( \frac{\alpha d}{c} \right)^2 + dTD \right] - \alpha d \left( \frac{d}{2c} T + t_r D \right) \frac{c \alpha t_r}{T} \right\} T \\ &= \left[ \frac{1}{2} \cancel{\alpha^2 t_r d^2} + cdTD - \frac{1}{2} \cancel{\alpha^2 t_r d^2} - \frac{cd(\alpha t_r)^2}{T} D \right] T = cdD \underbrace{[T^2 - (\alpha t_r)^2]}_{1 + (\cancel{\alpha t_r})^2 - (\cancel{\alpha t_r})^2} = dcD. \end{aligned}$$

$$\therefore \mathbf{F} = \frac{q^2}{4\pi\epsilon_0} \frac{c^2 d^2 \alpha}{c^3 d^3 D^3} \hat{x} = \boxed{\frac{q^2}{4\pi\epsilon_0} \frac{\alpha}{cd[1 + (\alpha d/2c)^2]^{3/2}} \hat{x}} \quad \left( \alpha = \frac{F}{mc} \right).$$

Energy must come from the “reservoir” of energy stored in the electromagnetic fields.

$$(b) F = mc\alpha = \frac{1}{2} \frac{q^2}{4\pi\epsilon_0} \frac{\alpha}{cd[1 + (\alpha d/2c)^2]^{3/2}} \Rightarrow \left[ 1 + \left( \frac{\alpha d}{2c} \right)^2 \right]^{3/2} = \frac{q^2}{8\pi\epsilon_0 mc^2 d} = \frac{\mu_0 q^2}{8\pi m d}.$$

(force on one end only)

$$\therefore \alpha = \frac{2c}{d} \sqrt{\left( \frac{\mu_0 q^2}{8\pi m d} \right)^{2/3} - 1}, \text{ so } \boxed{F = \frac{2mc^2}{d} \sqrt{\left( \frac{\mu_0 q^2}{8\pi m d} \right)^{2/3} - 1}}.$$

**Problem 12.62**

(a)  $A^\mu = (V/c, A_x, A_y, A_z)$  is a 4-vector (like  $x^\mu = (ct, x, y, z)$ ), so (using Eq. 12.19):  $V = \gamma(\bar{V} + v\bar{A}_x)$ . But  $\bar{V} = 0$ , and

$$\bar{A}_x = \frac{\mu_0}{4\pi} \frac{(\mathbf{m} \times \bar{\mathbf{r}})_x}{\bar{r}^3}.$$

Now  $(\mathbf{m} \times \bar{\mathbf{r}})_x = m_y \bar{z} - m_z \bar{y} = m_y z - m_z y$ . So

$$V = \gamma v \frac{\mu_0}{4\pi} \frac{(m_y z - m_z y)}{\bar{r}^3}.$$

Now  $\bar{x} = \gamma(x - vt) = \gamma R_x$ ,  $\bar{y} = y = R_y$ ,  $\bar{z} = z = R_z$ , where  $\mathbf{R}$  is the vector (in  $S$ ) from the (instantaneous) location of the dipole to the point of observation. Thus

$$\bar{r}^2 = \gamma^2 R_x^2 + R_y^2 + R_z^2 = \gamma^2(R_x^2 + R_y^2 + R_z^2) + (1 - \gamma^2)(R_y^2 + R_z^2) = \gamma^2(R^2 - \frac{v^2}{c^2} R^2 \sin^2 \theta)$$

(where  $\theta$  is the angle between  $\mathbf{R}$  and the  $x$  axis, so that  $R_y^2 + R_z^2 = R^2 \sin^2 \theta$ ).

$$\therefore V = \frac{\mu_0}{4\pi} \frac{v\gamma(m_y R_z - m_z R_y)}{\gamma^3 R^3 (1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}}; \quad \text{but } \mathbf{v} \cdot (\mathbf{m} \times \mathbf{R}) = v(\mathbf{m} \times \mathbf{R})_x = v(m_y R_z - m_z R_y), \quad \text{so}$$

$$V = \frac{\mu_0}{4\pi} \frac{\mathbf{v} \cdot (\mathbf{m} \times \mathbf{R})(1 - \frac{v^2}{c^2})}{R^3 (1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}},$$

$$\text{or, using } \mu_0 = \frac{1}{\epsilon_0 c^2} \text{ and } \mathbf{v} \cdot (\mathbf{m} \times \mathbf{R}) = \mathbf{R} \cdot (\mathbf{v} \times \mathbf{m}): \quad V = \frac{1}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}} \cdot (\mathbf{v} \times \mathbf{m})(1 - \frac{v^2}{c^2})}{c^2 R^2 (1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}}.$$

(b) In the nonrelativistic limit ( $v^2 \ll c^2$ ):

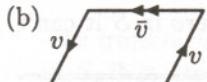
$$V = \frac{1}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}} \cdot (\mathbf{v} \times \mathbf{m})}{c^2 R^2} = \frac{1}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}} \cdot \mathbf{p}}{R^2}, \quad \text{with } \mathbf{p} = \frac{\mathbf{v} \times \mathbf{m}}{c^2},$$

which is the potential of an *electric* dipole.

**Problem 12.63**

(a)  $\mathbf{B} = -\frac{\mu_0}{2} K \hat{\mathbf{y}}$  (Eq. 5.56);  $\mathbf{N} = \mathbf{m} \times \mathbf{B}$  (Eq. 6.1), so  $\mathbf{N} = -\frac{\mu_0}{2} m K (\hat{\mathbf{z}} \times \hat{\mathbf{y}})$ .

$$\mathbf{N} = \frac{\mu_0}{2} m K \hat{\mathbf{x}} = \frac{\mu_0}{2} (\lambda v l^2)(\sigma v) \hat{\mathbf{x}} = \frac{\mu_0}{2} \lambda \sigma v^2 l^2 \hat{\mathbf{x}}.$$



Charge density on the front side:  $\lambda_0$  ( $\lambda = \gamma \lambda_0$ );  
Charge density on the back side:  $\bar{\lambda} = \bar{\gamma} \lambda_0$ , where  $\bar{v} = \frac{2v}{1+v^2/c^2} \Rightarrow$

$$\bar{\gamma} = \frac{1}{\sqrt{1 - \frac{4v^2/c^2}{(1+v^2/c^2)^2}}} = \frac{(1 + v^2/c^2)}{\sqrt{1 + 2\frac{v^2}{c^2} + \frac{v^4}{c^4} - 4\frac{v^2}{c^2}}} = \frac{1 + v^2/c^2}{\sqrt{1 - 2\frac{v^2}{c^2} + \frac{v^4}{c^4}}} = \frac{(1 + v^2/c^2)}{(1 - v^2/c^2)} = \gamma^2 \left(1 + \frac{v^2}{c^2}\right).$$

Length of front and back sides in this frame:  $l/\gamma$ . So the net charge on the back side is:

$$q_+ = \bar{\lambda} \frac{l}{\gamma} = \gamma^2 \left(1 + \frac{v^2}{c^2}\right) \frac{\lambda}{\gamma} \frac{l}{\gamma} = \left(1 + \frac{v^2}{c^2}\right) \lambda l.$$

Net charge on front side is:

$$q_- = \lambda_0 \frac{l}{\gamma} = \frac{\lambda l}{\gamma \gamma} = \frac{1}{\gamma^2} \lambda l.$$

So the dipole moment (note: charges on *sides* are equal):

$$\mathbf{p} = (q_+) \frac{l}{2} \hat{\mathbf{y}} - (q_-) \frac{l}{2} \hat{\mathbf{y}} = \left[ \left( 1 + \frac{v^2}{c^2} \right) \lambda l \frac{l}{2} - \frac{1}{\gamma^2} \lambda l \frac{l}{2} \right] \hat{\mathbf{y}} = \frac{\lambda l^2}{2} \left( 1 + \frac{v^2}{c^2} - 1 + \frac{v^2}{\gamma^2 c^2} \right) \hat{\mathbf{y}} = \boxed{\frac{\lambda l^2 v^2}{c^2} \hat{\mathbf{y}}}.$$

$$\mathbf{E} = \frac{\sigma_0}{2\epsilon_0} \hat{\mathbf{z}}, \text{ where } \sigma = \gamma \sigma_0, \text{ so } \mathbf{N} = \mathbf{p} \times \mathbf{E} = \frac{\lambda l^2 v^2}{c^2} \frac{\sigma}{2\epsilon_0 \gamma} (\hat{\mathbf{y}} \times \hat{\mathbf{z}}) = \boxed{\frac{1}{\gamma} \frac{\mu_0}{2} \lambda \sigma l^2 v^2 \hat{\mathbf{x}}}.$$

So apart from the relativistic factor of  $\gamma$  the torque is the same in both systems—but in  $\mathcal{S}$  it is the torque exerted by a *magnetic* field on a *magnetic* dipole, whereas in  $\bar{\mathcal{S}}$  it is the torque exerted by an *electric* field on an *electric* dipole.

### Problem 12.64

Choose axes so that  $\mathbf{E}$  points in the  $z$  direction and  $\mathbf{B}$  in the  $yz$  plane:  $\mathbf{E} = (0, 0, E)$ ;  $\mathbf{B} = (0, B \cos \phi, B \sin \phi)$ . Go to a frame moving at speed  $v$  in the  $x$  direction:

$$\bar{\mathbf{E}} = (0, -\gamma v B \sin \phi, \gamma(E + v B \cos \phi)); \quad \bar{\mathbf{B}} = (0, \gamma(B \cos \phi + \frac{v}{c^2} E), \gamma B \sin \phi).$$

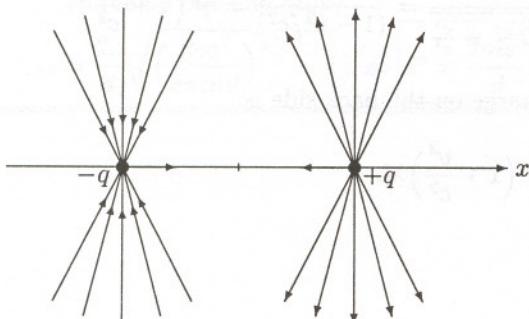
(I used Eq. 12.108.) Parallel provided  $\frac{-\gamma v B \sin \phi}{\gamma(B \cos \phi + \frac{v}{c^2} E)} = \frac{\gamma(E + v B \cos \phi)}{\gamma B \sin \phi}$ , or

$$\begin{aligned} -v B^2 \sin^2 \phi &= (B \cos \phi + \frac{v}{c^2} E)(E + v B \cos \phi) = EB \cos \phi + v B^2 \cos^2 \phi + \frac{v}{c^2} E^2 + \frac{v^2}{c^2} EB \cos \phi, \\ 0 &= v B^2 + \frac{v}{c^2} E^2 + EB \cos \phi \left(1 + \frac{v^2}{c^2}\right); \quad \frac{v}{1 + v^2/c^2} = -\frac{EB \cos \phi}{B^2 + E^2/c^2}. \end{aligned}$$

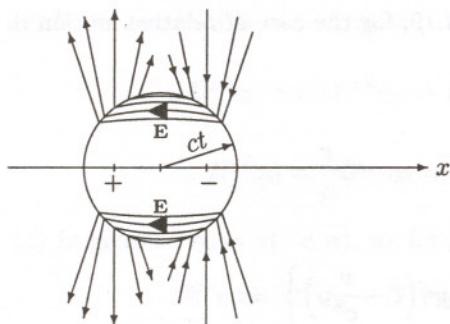
Now  $\mathbf{E} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & E \\ 0 & B \cos \phi & B \sin \phi \end{vmatrix} = -EB \cos \phi \hat{\mathbf{x}}$ . So  $\frac{\mathbf{v}}{1 + v^2/c^2} = \frac{\mathbf{E} \times \mathbf{B}}{B^2 + E^2/c^2}$ . qed

No, there can be no frame in which  $\mathbf{E} \perp \mathbf{B}$ , for  $(\mathbf{E} \cdot \mathbf{B})$  is invariant, and since it is not zero in  $\mathcal{S}$  it can't be zero in  $\bar{\mathcal{S}}$ .

### Problem 12.65



*Just before:*  
Field lines emanate  
from *present position*  
of particle.



*Just after:* Field lines *outside* sphere of radius  $ct$  emanate from position particle *would* have reached, had it kept going on its original "flight plan". *Inside* the sphere  $E = 0$ . On the surface the lines connect up (since they cannot simply terminate in empty space), as suggested in the figure.

This produces a dense cluster of tangentially-directed field lines, which expand with the spherical shell. This is a pictorial way of understanding the generation of *electromagnetic radiation*.

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### Problem 12.66

Equation 12.68 assumes the particle is (instantaneously) at rest in  $\mathcal{S}$ . Here the particle is at rest in  $\bar{\mathcal{S}}$ . So  $F_{\perp} = \frac{1}{\gamma} \bar{F}_{\perp}$ ,  $F_{\parallel} = \bar{F}_{\parallel}$ . Using  $\bar{\mathbf{F}} = q\bar{\mathbf{E}}$ , then,

$$F_x = \bar{F}_x = q\bar{E}_x, \quad F_y = \frac{1}{\gamma}\bar{F}_y = \frac{1}{\gamma}q\bar{E}_y, \quad F_z = \frac{1}{\gamma}\bar{F}_z = \frac{1}{\gamma}q\bar{E}_z.$$

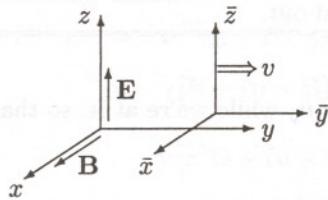
Invoking Eq. 12.108:

$$F_x = qE_x, \quad F_y = \frac{1}{\gamma}q\gamma(E_y - vB_z) = q(E_y - vB_z), \quad F_z = \frac{1}{\gamma}q\gamma(E_z + vB_y) = q(E_z + vB_y).$$

But  $\mathbf{v} \times \mathbf{B} = -vB_z \hat{x} + vB_y \hat{z}$ , so  $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ . qed

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### Problem 12.67



Rewrite Eq. 12.108 with  $x \rightarrow y$ ,  $y \rightarrow z$ ,  $z \rightarrow x$ :

$\bar{E}_y = E_y$	$\bar{E}_z = \gamma(E_z - vB_x)$	$\bar{E}_x = \gamma(E_x + vB_z)$
$\bar{B}_y = B_y$	$\bar{B}_z = \gamma\left(B_z + \frac{v}{c^2}E_x\right)$	$\bar{B}_x = \gamma\left(B_x - \frac{v}{c^2}E_z\right)$

This gives the fields in system  $\bar{\mathcal{S}}$  moving in the  $y$  direction at speed  $v$ .

Now  $\mathbf{E} = (0, 0, E_0)$ ;  $\mathbf{B} = (B_0, 0, 0)$ , so  $\bar{E}_y = 0$ ,  $\bar{E}_z = \gamma(E_0 - vB_0)$ ,  $\bar{E}_x = 0$ .

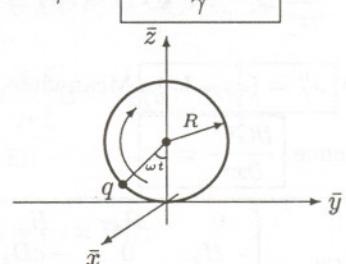
If we want  $\bar{\mathbf{E}} = 0$ , we must pick  $v$  so that  $E_0 - vB_0 = 0$ ; i.e.  $v = E_0/B_0$ .

(The condition  $E_0/B_0 < c$  guarantees that there is no problem *getting* to such a system.)

With this,  $\bar{B}_y = 0$ ,  $\bar{B}_z = 0$ ,  $\bar{B}_x = \gamma(B_0 - \frac{v}{c^2}E_0) = \gamma B_0(1 - \frac{v^2}{c^2}) = \gamma B_0 \frac{1}{\gamma^2} = \frac{1}{\gamma} B_0$ ;  $\bar{\mathbf{B}} = \frac{1}{\gamma} B_0 \hat{x}$ .

*The trajectory in  $\bar{\mathcal{S}}$ :* Since the particle started out at rest at the origin in  $\mathcal{S}$ , it started out with velocity  $-v\hat{y}$  in  $\bar{\mathcal{S}}$ . According to Eq. 12.72 it will move in a circle of radius  $R$ , given by

$$p = qBR, \text{ or } \gamma mv = q\left(\frac{1}{\gamma}B_0\right)R \Rightarrow \boxed{R = \frac{m\gamma^2 v}{qB_0}}.$$



The actual trajectory is given by  $\bar{x} = 0$ ;  $\bar{y} = -R \sin \omega \bar{t}$ ;  $\bar{z} = R(1 - \cos \omega \bar{t})$ ; where  $\omega = \frac{v}{R}$ .

*The trajectory in  $\mathcal{S}$ :* The Lorentz transformations Eqs. 12.18 and 12.19, for the case of relative motion in the  $y$  direction, read:

$$\begin{aligned}\bar{x} &= x \\ \bar{y} &= \gamma(y - vt) \\ \bar{z} &= z \\ \bar{t} &= \gamma\left(t - \frac{v}{c^2}y\right)\end{aligned}\quad \begin{aligned}x &= \bar{x} \\ y &= \gamma(\bar{y} + v\bar{t}) \\ z &= \bar{z} \\ t &= \gamma\left(\bar{t} + \frac{v}{c^2}\bar{y}\right)\end{aligned}$$

So the trajectory in  $\mathcal{S}$  is given by:

$$\begin{aligned}x = 0; \quad y &= \gamma(-R \sin \omega \bar{t} + v\bar{t}) = \gamma \left\{ -R \sin \left[ \omega \gamma \left( t - \frac{v}{c^2}y \right) \right] + v\gamma \left( t - \frac{v}{c^2}y \right) \right\}, \text{ or} \\ y \left( 1 + \gamma^2 \frac{v^2}{c^2} \right) &= \gamma^2 vt - \gamma R \sin \left[ \omega \gamma \left( t - \frac{v}{c^2}y \right) \right] \\ \gamma^2 y \left( 1 - \frac{v^2}{c^2} + \frac{v^2}{c^2} \right) &= \gamma^2 y \\ z &= R(1 - \cos^2 \omega \bar{t}) = R \left[ 1 - \cos \omega \gamma \left( t - \frac{v}{c^2}y \right) \right].\end{aligned}$$

So:  $x = 0; y = vt - \frac{R}{\gamma} \sin \left[ \omega \gamma \left( t - \frac{v}{c^2}y \right) \right]; z = R - R \cos \left[ \omega \gamma \left( t - \frac{v}{c^2}y \right) \right].$

We can get rid of the trigonometric terms by the usual trick:

$$\begin{aligned}\gamma(y - vt) &= -R \sin \left[ \omega \gamma \left( t - \frac{v}{c^2}y \right) \right] \\ z - R &= -R \cos \left[ \omega \gamma \left( t - \frac{v}{c^2}y \right) \right]\end{aligned}\quad \Rightarrow \boxed{\gamma^2(y - vt)^2 + (z - R)^2 = R^2}.$$

Absent the  $\gamma^2$ , this would be the cycloid we found back in Ch. 5 (Eq. 5.9). The  $\gamma^2$  makes it, as it were, an *elliptical cycloid* — same picture as p. 206, but with the horizontal axis stretched out.

### Problem 12.68

- (a)  $D = \epsilon_0 E + P$  suggests  $E \rightarrow \frac{1}{\epsilon_0} D$   
 $H = \frac{1}{\mu_0} B - M$  suggests  $B \rightarrow \mu_0 H$  } but it's a little cleaner if we divide by  $\mu_0$  while we're at it, so that

$E \rightarrow \frac{1}{\mu_0 \epsilon_0} D = c^2 D, B \rightarrow H$ . Then: 
$$D^{\mu\nu} = \begin{Bmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{Bmatrix}$$

Then (following the derivation on p. 539):

$$\frac{\partial}{\partial x^\nu} D^{0\nu} = c \nabla \cdot D = c\rho_f = J_f^0; \quad \frac{\partial}{\partial x^\nu} D^{1\nu} = \frac{1}{c} \frac{\partial}{\partial t} (-cD_x) + (\nabla \times H)_x = (J_f)_x; \quad \text{so} \quad \boxed{\frac{\partial D^{\mu\nu}}{\partial x^\nu} = J_f^\mu},$$

where  $\boxed{J_f^\mu = (c\rho_f, \mathbf{J}_f)}$ . Meanwhile, the homogeneous Maxwell equations ( $\nabla \cdot B = 0, E = -\frac{\partial B}{\partial t}$ ) are unchanged,

and hence  $\boxed{\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}$ .

(b) 
$$H^{\mu\nu} = \begin{Bmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & -cD_z & cD_y \\ -H_y & cD_z & 0 & -cD_x \\ -H_z & -cD_y & cD_x & 0 \end{Bmatrix}$$

(c) If the material is at rest,  $\eta_\nu = (-c, 0, 0, 0)$ , and the sum over  $\nu$  collapses to a single term:

$$D^{\mu 0} \eta_0 = c^2 \epsilon F^{\mu 0} \eta_0 \Rightarrow D^{\mu 0} = c^2 \epsilon F^{\mu 0} \Rightarrow -c\mathbf{D} = -c^2 \epsilon \frac{\mathbf{E}}{c} \Rightarrow \mathbf{D} = \epsilon \mathbf{E} \text{ (Eq. 4.32), } \checkmark$$

$$H^{\mu 0} \eta_0 = \frac{1}{\mu} G^{\mu 0} \eta_0 \Rightarrow H^{\mu 0} = \frac{1}{\mu} G^{\mu 0} \Rightarrow -\mathbf{H} = -\frac{1}{\mu} \mathbf{B} \Rightarrow \mathbf{H} = \frac{1}{\mu} \mathbf{B} \text{ (Eq. 6.31). } \checkmark$$

(d) In general,  $\eta_\nu = \gamma(-c, \mathbf{u})$ , so, for  $\mu = 0$ :

$$D^{0\nu} \eta_\nu = D^{01} \eta_1 + D^{02} \eta_2 + D^{03} \eta_3 = cD_x(\gamma u_x) + cD_y(\gamma u_y) + cD_z(\gamma u_z) = \gamma c(\mathbf{D} \cdot \mathbf{u}),$$

$$F^{0\nu} \eta_\nu = F^{01} \eta_1 + F^{02} \eta_2 + F^{03} \eta_3 = \frac{E_x}{c}(\gamma u_x) + \frac{E_y}{c}(\gamma u_y) + \frac{E_z}{c}(\gamma u_z) = \frac{\gamma}{c}(\mathbf{E} \cdot \mathbf{u}), \text{ so}$$

$$D^{0\nu} \eta_\nu = c^2 \epsilon F^{0\nu} \eta_\nu \Rightarrow \gamma c(\mathbf{D} \cdot \mathbf{u}) = c^2 \epsilon \left( \frac{\gamma}{c} \right) (\mathbf{E} \cdot \mathbf{u}) \Rightarrow \mathbf{D} \cdot \mathbf{u} = \epsilon(\mathbf{E} \cdot \mathbf{u}). \quad [1]$$

$$H^{0\nu} \eta_\nu = H^{01} \eta_1 + H^{02} \eta_2 + H^{03} \eta_3 = H_x(\gamma u_x) + H_y(\gamma u_y) + H_z(\gamma u_z) = \gamma(\mathbf{H} \cdot \mathbf{u}),$$

$$G^{0\nu} \eta_\nu = G^{01} \eta_1 + G^{02} \eta_2 + G^{03} \eta_3 = B_x(\gamma u_x) + B_y(\gamma u_y) + B_z(\gamma u_z) = \gamma(\mathbf{B} \cdot \mathbf{u}), \text{ so}$$

$$H^{0\nu} \eta_\nu = \frac{1}{\mu} G^{0\nu} \eta_\nu \Rightarrow \gamma(\mathbf{H} \cdot \mathbf{u}) = \frac{1}{\mu}(\gamma)(\mathbf{B} \cdot \mathbf{u}) \Rightarrow \mathbf{H} \cdot \mathbf{u} = \frac{1}{\mu}(\mathbf{B} \cdot \mathbf{u}). \quad [2]$$

Similarly, for  $\mu = 1$ :

$$\begin{aligned} D^{1\nu} \eta_\nu &= D^{10} \eta_0 + D^{12} \eta_2 + D^{13} \eta_3 = (-cD_x)(-\gamma c) + H_z(\gamma u_y) + (-H_y)(\gamma u_z) = \gamma(c^2 D_x + u_y H_z - u_z H_y) \\ &= \gamma [c^2 \mathbf{D} + (\mathbf{u} \times \mathbf{H})]_x, \end{aligned}$$

$$\begin{aligned} F^{1\nu} \eta_\nu &= F^{10} \eta_0 + F^{12} \eta_2 + F^{13} \eta_3 = \frac{-E_x}{c}(-\gamma c) + B_z(\gamma u_y) + (-B_y)(\gamma u_z) = \gamma(E_x + u_y B_z - u_z B_y) \\ &= \gamma [\mathbf{E} + (\mathbf{u} \times \mathbf{B})]_x, \text{ so } D^{1\nu} \eta_\nu = c^2 \epsilon F^{1\nu} \eta_\nu \Rightarrow \end{aligned}$$

$$\gamma [c^2 \mathbf{D} + (\mathbf{u} \times \mathbf{H})]_x = c^2 \epsilon(\gamma) [\mathbf{E} + (\mathbf{u} \times \mathbf{B})]_x \Rightarrow \mathbf{D} + \frac{1}{c^2} (\mathbf{u} \times \mathbf{H}) = \epsilon [\mathbf{E} + (\mathbf{u} \times \mathbf{B})]. \quad [3]$$

$$\begin{aligned} H^{1\nu} \eta_\nu &= H^{10} \eta_0 + H^{12} \eta_2 + H^{13} \eta_3 = (-H_x)(-\gamma c) + (-cD_z)(\gamma u_y) + (cD_y)(\gamma u_z) \\ &= \gamma c(H_x - u_y D_z + u_z D_y) = \gamma c[\mathbf{H} - (\mathbf{u} \times \mathbf{D})]_x, \end{aligned}$$

$$\begin{aligned} G^{1\nu} \eta_\nu &= G^{10} \eta_0 + G^{12} \eta_2 + G^{13} \eta_3 = (-B_x)(-\gamma c) + \left( -\frac{E_z}{c} \right) (\gamma u_y) + \left( \frac{E_y}{c} \right) (\gamma u_z) \\ &= \frac{\gamma}{c}(c^2 B_x - u_y E_z + u_z E_y) = \frac{\gamma}{c} [c^2 \mathbf{B} - (\mathbf{u} \times \mathbf{E})]_x, \text{ so } H^{1\nu} \eta_\nu = \frac{1}{\mu} G^{1\nu} \eta_\nu \Rightarrow \end{aligned}$$

$$\gamma c[\mathbf{H} - (\mathbf{u} \times \mathbf{D})]_x = \frac{1}{\mu} \frac{\gamma}{c} [c^2 \mathbf{B} - (\mathbf{u} \times \mathbf{E})]_x \Rightarrow \mathbf{H} - (\mathbf{u} \times \mathbf{D}) = \frac{1}{\mu} \left[ \mathbf{B} - \frac{1}{c^2} (\mathbf{u} \times \mathbf{E}) \right]. \quad [4]$$

Use Eq. [4] as an expression for  $\mathbf{H}$ , plug this into Eq. [3], and solve for  $\mathbf{D}$ :

$$\mathbf{D} + \frac{1}{c^2} \mathbf{u} \times \left\{ (\mathbf{u} \times \mathbf{D}) + \frac{1}{\mu} \left[ \mathbf{B} - \frac{1}{c^2} (\mathbf{u} \times \mathbf{E}) \right] \right\} = \epsilon [\mathbf{E} + (\mathbf{u} \times \mathbf{B})];$$

$$\mathbf{D} + \frac{1}{c^2} [(\mathbf{u} \cdot \mathbf{D})\mathbf{u} - u^2 \mathbf{D}] = \epsilon [\mathbf{E} + (\mathbf{u} \times \mathbf{B})] - \frac{1}{\mu c^2} (\mathbf{u} \times \mathbf{B}) + \frac{1}{\mu c^4} [\mathbf{u} \times (\mathbf{u} \times \mathbf{E})].$$

Using Eq. [1] to rewrite  $\mathbf{u} \cdot \mathbf{D}$ :

$$\begin{aligned}\mathbf{D} \left(1 - \frac{u^2}{c^2}\right) &= -\frac{\epsilon}{c^2}(\mathbf{E} \cdot \mathbf{u})\mathbf{u} + \epsilon[\mathbf{E} + (\mathbf{u} \times \mathbf{B})] - \frac{1}{\mu c^2}(\mathbf{u} \times \mathbf{B}) + \frac{1}{\mu c^4}[(\mathbf{E} \cdot \mathbf{u})\mathbf{u} - u^2\mathbf{E}] \\ &= \epsilon \left\{ \left[1 - \frac{u^2}{\epsilon \mu c^4}\right] \mathbf{E} - \frac{1}{c^2} \left[1 - \frac{1}{\epsilon \mu c^2}\right] (\mathbf{E} \cdot \mathbf{u})\mathbf{u} + (\mathbf{u} \times \mathbf{B}) \left[1 - \frac{1}{\epsilon \mu c^2}\right] \right\}.\end{aligned}$$

Let  $\gamma \equiv \frac{1}{\sqrt{1 - u^2/c^2}}$ ,  $v \equiv \frac{1}{\sqrt{\epsilon \mu}}$ . Then

$$\boxed{\mathbf{D} = \gamma^2 \epsilon \left\{ \left(1 - \frac{u^2 v^2}{c^4}\right) \mathbf{E} + \left(1 - \frac{v^2}{c^2}\right) \left[ (\mathbf{u} \times \mathbf{B}) - \frac{1}{c^2}(\mathbf{E} \cdot \mathbf{u})\mathbf{u} \right] \right\}}.$$

Now use Eq. [3] as an expression for  $\mathbf{D}$ , plug this into Eq. [4], and solve for  $\mathbf{H}$ :

$$\begin{aligned}\mathbf{H} - \mathbf{u} \times \left\{ -\frac{1}{c^2}(\mathbf{u} \times \mathbf{H}) + \epsilon[\mathbf{E} + (\mathbf{u} \times \mathbf{B})] \right\} &= \frac{1}{\mu} \left[ \mathbf{B} - \frac{1}{c^2}(\mathbf{u} \times \mathbf{E}) \right]; \\ \mathbf{H} + \frac{1}{c^2}[(\mathbf{u} \cdot \mathbf{H})\mathbf{u} - u^2\mathbf{H}] &= \frac{1}{\mu} \left[ \mathbf{B} - \frac{1}{c^2}(\mathbf{u} \times \mathbf{E}) \right] + \epsilon(\mathbf{u} \times \mathbf{E}) + \epsilon[\mathbf{u} \times (\mathbf{u} \times \mathbf{B})].\end{aligned}$$

Using Eq. [2] to rewrite  $\mathbf{u} \cdot \mathbf{H}$ :

$$\begin{aligned}\mathbf{H} \left(1 - \frac{u^2}{c^2}\right) &= -\frac{1}{\mu c^2}(\mathbf{B} \cdot \mathbf{u})\mathbf{u} + \frac{1}{\mu} \left[ \mathbf{B} - \frac{1}{c^2}(\mathbf{u} \times \mathbf{E}) \right] + \epsilon(\mathbf{u} \times \mathbf{E}) + \epsilon[(\mathbf{B} \cdot \mathbf{u})\mathbf{u} - u^2\mathbf{B}] \\ &= \frac{1}{\mu} \left\{ \left[1 - \mu \epsilon u^2\right] \mathbf{B} + \left(\epsilon \mu - \frac{1}{c^2}\right) [(\mathbf{u} \times \mathbf{E}) + (\mathbf{B} \cdot \mathbf{u})\mathbf{u}] \right\}.\end{aligned}$$

$$\boxed{\mathbf{H} = \frac{\gamma^2}{\mu} \left\{ \left(1 - \frac{u^2}{v^2}\right) \mathbf{B} + \left(\frac{1}{v^2} - \frac{1}{c^2}\right) [(\mathbf{u} \times \mathbf{E}) + (\mathbf{B} \cdot \mathbf{u})\mathbf{u}] \right\}}.$$

### Problem 12.69

We know that (proper) power transforms as the zeroth component of a 4-vector:  $K^0 = \frac{1}{c} \frac{dW}{d\tau}$ . The Larmor formula says that for  $v = 0$ ,  $\frac{dW}{dt} = \frac{\mu_0 q^2 a^2}{6\pi c}$  (Eq. 11.70). Can we think of a 4-vector whose zeroth component reduces to this when the velocity is zero?

Well,  $a^2$  smells like  $(\alpha^\nu \alpha_\nu)$ , but how do we get a 4-vector in here? How about  $\eta^\mu$ , whose zeroth component is just  $c$ , when  $v = 0$ ? Try, then:

$$K^\mu = \frac{\mu_0 q^2}{6\pi c^3} (\alpha^\nu \alpha_\nu) \eta^\mu.$$

This has the right transformation properties, but we must check that it does reduce to the Larmor formula when  $v \rightarrow 0$ :

$$\frac{dW}{dt} = \frac{1}{\gamma} \frac{dW}{d\tau} = \frac{1}{\gamma} c K^0 = \frac{1}{\gamma} c \frac{\mu_0 q^2}{6\pi c^3} (\alpha^\nu \alpha_\nu) \eta^0, \text{ but } \eta^0 = c\gamma, \text{ so } \boxed{\frac{dW}{dt} = \frac{\mu_0 q^2}{6\pi c} (\alpha^\nu \alpha_\nu).}$$

[Incidentally, this tells us that the power itself (as opposed to *proper* power) is a *scalar*. If this had been obvious from the start, we could simply have looked for a Lorentz *scalar* that generalizes the Larmor formula.]

In Prob. 12.38(b) we calculated  $(\alpha^\nu \alpha_\nu)$  in terms of the *ordinary* velocity and acceleration:

$$\begin{aligned}\alpha^\nu \alpha_\nu &= \gamma^4 \left[ a^2 + \frac{(\mathbf{v} \cdot \mathbf{a})^2}{(c^2 - v^2)} \right] = \gamma^6 \left[ a^2 \gamma^{-2} + \frac{1}{c^2} (\mathbf{v} \cdot \mathbf{a})^2 \right] \\ &= \gamma^6 \left[ a^2 \left( 1 - \frac{v^2}{c^2} \right) + \frac{1}{c^2} (\mathbf{v} \cdot \mathbf{a})^2 \right] = \gamma^6 \left\{ a^2 - \frac{1}{c^2} [v^2 a^2 - (\mathbf{v} \cdot \mathbf{a})^2] \right\}.\end{aligned}$$

Now  $\mathbf{v} \cdot \mathbf{a} = va \cos \theta$ , where  $\theta$  is the angle between  $\mathbf{v}$  and  $\mathbf{a}$ , so:

$$v^2 a^2 - (\mathbf{v} \cdot \mathbf{a})^2 = v^2 a^2 (1 - \cos^2 \theta) = v^2 a^2 \sin^2 \theta = |\mathbf{v} \times \mathbf{a}|^2.$$

$$\alpha^\nu \alpha_\nu = \gamma^6 \left( a^2 - \left| \frac{\mathbf{v} \times \mathbf{a}}{c} \right|^2 \right).$$

$$\boxed{\frac{dW}{dt} = \frac{\mu_0 q^2}{6\pi c} \gamma^6 \left( a^2 - \left| \frac{\mathbf{v} \times \mathbf{a}}{c} \right|^2 \right)}, \text{ which is Liénard's formula (Eq. 11.73).}$$

### Problem 12.70

(a) It's inconsistent with the constraint  $\eta_\mu K^\mu = 0$  (Prob. 12.38(d)).

(b) We want to find a 4-vector  $b^\mu$  with the property that  $(\frac{d\alpha^\mu}{d\tau} + b^\mu) \eta_\mu = 0$ . How about  $b^\mu = \kappa (\frac{d\alpha^\nu}{d\tau} \eta_\nu) \eta^\mu$ ? Then  $(\frac{d\alpha^\mu}{d\tau} + b^\mu) \eta_\mu = \frac{d\alpha^\mu}{d\tau} \eta_\mu + \kappa \frac{d\alpha^\nu}{d\tau} \eta_\nu (\eta^\mu \eta_\mu)$ . But  $\eta^\mu \eta_\mu = -c^2$ , so this becomes  $(\frac{d\alpha^\mu}{d\tau} \eta_\mu) - c^2 \kappa (\frac{d\alpha^\nu}{d\tau} \eta_\nu)$ , which is zero, if we pick  $\kappa = 1/c^2$ . This suggests  $K_{\text{rad}}^\mu = \frac{\mu_0 q^2}{6\pi c} \left( \frac{d\alpha^\mu}{d\tau} + \frac{1}{c^2} \frac{d\alpha^\nu}{d\tau} \eta_\nu \eta^\mu \right)$ . Note that  $\eta^\mu = (c, \mathbf{v})\gamma$ , so the spatial components of  $b^\mu$  vanish in the nonrelativistic limit  $v \ll c$ , and hence this still reduces to the Abraham-Lorentz formula. [Incidentally,  $\alpha^\nu \eta_\nu = 0 \Rightarrow \frac{d}{d\tau} (\alpha^\nu \eta_\nu) = 0 \Rightarrow \frac{d\alpha^\nu}{d\tau} \eta_\nu + \alpha^\nu \frac{d\eta_\nu}{d\tau} = 0$ , so  $\frac{d\alpha^\nu}{d\tau} \eta_\nu = -\alpha^\nu \alpha_\nu$ , and hence  $b^\mu$  can just as well be written  $-\frac{1}{c^2} (\alpha^\nu \alpha_\nu) \eta^\mu$ .]

### Problem 12.71

Define the electric current 4-vector as before:  $J_e^\mu = (c\rho_e, \mathbf{J}_e)$ , and the magnetic current the same way:  $J_m^\mu = (c\rho_m, \mathbf{J}_m)$ . The fundamental laws are then

$$\boxed{\partial_\nu F^{\mu\nu} = \mu_0 J_e^\mu, \quad \partial_\nu G^{\mu\nu} = \frac{\mu_0}{c} J_m^\mu, \quad K^\mu = \left( q_e F^{\mu\nu} + \frac{q_m}{c} G^{\mu\nu} \right) \eta_\nu.}$$

The first of these reproduces  $\nabla \cdot \mathbf{E} = (1/\epsilon_0)\rho_e$  and  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_e + \mu_0 \epsilon_0 \partial \mathbf{E} / \partial t$ , just as before (p. 539); the second yields  $\nabla \cdot \mathbf{B} = (\mu_0/c)(c\rho_m) = \mu_0 \rho_m$  and  $-(1/c)(\partial \mathbf{B} / \partial t + \nabla \times \mathbf{E}) = (\mu_0/c)\mathbf{J}_m$ , or  $\nabla \times \mathbf{E} = -\mu_0 \mathbf{J}_m - \partial \mathbf{B} / \partial t$  (generalizing page 540). These are Maxwell's equations with magnetic charge (Eq. 7.43). The third (following the argument on p. 540) says

$$\begin{aligned}K^1 &= \frac{q_e}{\sqrt{1-u^2/c^2}} [\mathbf{E} + (\mathbf{u} \times \mathbf{B})]_x + \frac{q_m}{c} \left[ \frac{-c}{\sqrt{1-u^2/c^2}} (-B_x) + \frac{u_y}{\sqrt{1-u^2/c^2}} \left( -\frac{E_z}{c} \right) + \frac{u_z}{\sqrt{1-u^2/c^2}} \left( \frac{E_y}{c} \right) \right], \\ \mathbf{K} &= \frac{1}{\sqrt{1-u^2/c^2}} \left\{ q_e [\mathbf{E} + (\mathbf{u} \times \mathbf{B})] + q_m \left[ \mathbf{B} - \frac{1}{c^2} (\mathbf{u} \times \mathbf{E}) \right] \right\}, \text{ or} \\ \mathbf{F} &= q_e [\mathbf{E} + (\mathbf{u} \times \mathbf{B})] + q_m \left[ \mathbf{B} - \frac{1}{c^2} (\mathbf{u} \times \mathbf{E}) \right],\end{aligned}$$

which is the generalized Lorentz force law (Eq. 7.69).