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The Dipole Passmethod for Accelerator Toolbox

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Abstract

A new passmethod for a sector dipole in Accelerator Toolbox is documented in this note. This passmethod implements the 4th-order symplectic integration through the curvilinear coordinate system in the sector dipole to higher order correctly and includes the fringe field effect to the second order.

1 Limitations of the existing dipole passmethods in AT

In Accelerator Toolbox (AT) [1], every accelerator element is represented by the corresponding *passmethod* which describes how the 6-dimensional phase space coordinate vector of a particle is mapped from the entrance face of the element to its exit face. The passmethod for dipole magnets is very important because dipoles account for a considerable fraction of the circumference for any circular accelerator. There were three dipole passmethods in AT, *BendLinearPass*,*BndMPoleSymplectic4Pass* and *BndMPoleSymplectic4RadPass*. BendLinearPass uses a linear model that describes the dipole as a transfer matrix plus chromatic second order terms. BndMPoleSymplectic4Pass uses 4th-order symplectic integration to pass the phase space vectors through the dipole body. BndMPoleSymplectic4RadPass is different from BndMPoleSymplectic4Pass by only adding radiation energy loss in the integration. All three passmethods include fringe field effects to the first order (edge focusing). Clearly BndMPoleSymplectic4Pass is the main representation of dipoles in AT.

However, some limitations of this passmethod exist. First of all, as pointed out above, the second order effects of the fringe fields are not included. This has significant impact for machines like SPEAR3 which uses rectangular dipoles whose entrance and exit angles are nonzero. Secondly, this passmethod does not account for the intrinsic nonlinear effects of a gradient sector dipole. If one derives the second order transport map of such a dipole in AT with the passmethod, one finds that the only nonzero elements of the map are the ones associated with energy errors. For example, the only nonzero elements that affect the x coordinate are found to be T_{116},T_{126} and T_{166} [1] The geometric elements, such as T_{111},T_{112}, are all zeros, in contradiction to theoretic analysis [3, 4] and results of the MAD program [5].

[1]Note that we adopt the more universal convention with the 6th coordinate for δ when discussing the second order transport maps. This is different from AT, where the 5th coordinate is for δ. Also symmetry $T_{lmn} = T_{lnm}$ is assumed, where $l, m, n = 1, 2, \cdots, 6$ and $m \neq n$.

Causes of the second limitation have been discovered by looking into the source code of this passmethod. Symplectic integration is carried out as described in Ref. [6]. Every step through the dipole is divided into three thin kicks and four drift spaces with lengths and positions prescribed relative to the length of the step. The kicks are given by

$$\Delta p_x = L_k(-B_y/B\rho + h\delta - h^2 x), \tag{1}$$
$$\Delta p_y = L_k B_x/B\rho, \tag{2}$$
$$\Delta z = L_k h x, \tag{3}$$

where L_k is the field length the kick accounts for, $h = 1/\rho$ the curvature and magnetic fields B_x and B_y are based on the expansion form

$$(B_y + iB_x) = B\rho \sum_{n=0} (ia_n + b_n)(x + iy)^n, \tag{4}$$

where coefficients a_n and b_n are from *PolynomA*,*PolynomB* ($b_0 = h$, but it should be set to zero in PolynomB). Note that in AT, δ is the 5th coordinate (z the 6th) and consequently the z variable has an opposite sign to the MAD convention. We use the AT convention through out this note.

On the drift spaces between the thin kicks, the change of coordinates are

$$\Delta x = L_d p_x/(1 + \delta), \tag{5}$$
$$\Delta y = L_d p_y/(1 + \delta), \tag{6}$$
$$\Delta z = L_d(p_x^2 + p_y^2)/2(1 + \delta), \tag{7}$$

where L_d is the drift space length.

The above treatment is typical in tracking codes in the early days. However, it is a simplification that is accurate only to the first order. The loss of accuracy comes from two aspects, both associated with the omission of the curvilinear coordinate system that is used in the dipole body. First, the multipole field expansion used in the thin kick calculation is valid only on a straight geometry (i.e., Cartesian coordinates). Therefore, B_x and B_y don't represent magnetic fields seen by particles on the curved trajectory. By taking special care of $b_0 = h$ in Eq. (1), the linear transfer matrix is obtained correctly. But all higher order effects that come with the curved geometry are lost. Second, the above mapping of drift spaces is also valid only for a straight geometry, not the curvilinear coordinate system.

2 Symplectic integration for the dipole body

The symplectic integration may be derived following the Hamiltonian in the curvilinear coordinate system which can be found in Ref. [3]. By neglecting the difference between δ and p_t (which amounts for setting $\beta_s = 1$) and assuming the magnetic field is derived solely from the longitudinal component, $A_s(x, y, s)$, of the vector potential (i.e., $A_x = A_y = 0$), the Hamiltonian is written

$$H = 1 + \delta - (1 + hx)\frac{A_s}{B\rho} - (1 + hx)\sqrt{(1 + \delta)^2 - p_x^2 - p_y^2}. \tag{8}$$

For a sector dipole with mid-plane symmetry whose magnetic field is $B_x(y=0) = B_s(y=0) = 0$ and

$$B_y(x,0,s) \;=\; B_0 + B_1 x + B_2\frac{x^2}{2} + B_3\frac{x^3}{6} + \cdots,$$

the vector potential is $A_x = A_y = 0$ and [3]

$$\begin{aligned}
A_s \;=\; & -B_0\left(x - \frac{hx^2}{2(1+hx)}\right) - B_1\left(\frac{1}{2}(x^2-y^2) - \frac{h}{6}x^3 + \frac{h^2}{24}(4x^4-y^4) + \cdots\right) \\
& -B_2\left(\frac{1}{6}(x^3-3xy^2) - \frac{h}{24}(x^4-y^4) + \cdots\right) - B_3\left(\frac{1}{24}(x^4-6x^2y^2+y^4) + \cdots\right) \\
& + \cdots.
\end{aligned} \tag{9}$$

The Hamiltonian in Eq. (8) can be directly split into two parts for symplectic integration (see Appendix A). Instead, we will first expand the last term and keep terms up to third order to get

$$H \;=\; -(1+hx)\frac{A_s}{B\rho} - (1+\delta)hx + (1+hx)\frac{p_x^2+p_y^2}{2(1+\delta)}, \tag{10}$$

which can be split into two parts,

$$H \;=\; H_1 + H_2, \tag{11}$$

$$H_1 \;=\; (1+hx)\frac{p_x^2+p_y^2}{2(1+\delta)}, \tag{12}$$

$$H_2 \;=\; -(1+hx)\frac{A_s}{B\rho} - (1+\delta)hx. \tag{13}$$

Here the way to split the Hamiltonian is basically the same as BndMPoleSymplectic4Pass. The Hamiltonian H_1 is a generalized drift space in which a reference particle follows the curved reference trajectory. Its analytic solution can be found. Since H_1 is itself an approximation, we only need to keep the solution to the corresponding order, which is

$$x_2 \;=\; x + \frac{1+hx}{1+\delta}p_x L_d + \frac{hL_d^2}{4(1+\delta^2)}(p_x^2-p_y^2), \tag{14}$$

$$p_{x2} \;=\; p_x - \frac{hL_d}{2(1+\delta)}(p_x^2+p_y^2), \tag{15}$$

$$y_2 \;=\; y + \frac{1+hx}{1+\delta}p_y L_d + \frac{hL_d^2}{2(1+\delta)^2}p_x p_y, \tag{16}$$

$$z_2 \;=\; z + \frac{1+hx}{2(1+\delta)^2}(p_x^2+p_y^2)L_d, \tag{17}$$

where subscript "2" indicates values at the exit face of the drift element, L_d is the arc length of the reference trajectory in the drift space, all phase space coordinates on the right hand side are values at the entrance face, and p_y and δ are unchanged. The Hamiltonian H_1 to a higher order and its solution can also be obtained (see Appendix B).

The solution of H_2 up to the octupole component, corresponding to terms explicitly given in Eq. (9), is

$$\begin{aligned}\Delta p_x/L_k &= -(K_1+h^2)x - K_2(x^2-y^2) - K_3(x^3-2xy^2) \\ &\quad -h\Big(K_1(x^2-\frac{1}{2}y^2) + K_2(x^3-\frac{4}{3}xy^2)\Big) \qquad (18)\\ \Delta p_y/L_k &= K_1 y + 2K_2 xy + K_3(2x^2y - y^3) + h\Big(K_1 xy + \frac{4}{3}K_2 x^2 y + \frac{1}{6}(hK_1 - 2K_2)y^3\Big), \qquad (19)\\ \Delta z/L_k &= hx, \qquad (20)\end{aligned}$$

where terms in the last brackets of Eqs. (18–19) are the curvature effect not accounted for by the existing AT passmethod. Higher order components are included in the implementation of the new passmethod. But their corresponding curvature effects are neglected.

Combining the solutions of H_1 and H_2 with the technique given in Ref. [6], we obtain a fourth-order symplectic integrator for the dipole body.

3 Fringe field effect to the second order

Second order transport maps for fringe fields have been derived by Brown [8] and Helm [7] and can be found in Ref. [2, 3, 4]. The new AT passmethod follows the formulae given in Ref. [3] except some misprints are corrected. These formulae are reproduced below. Let the entrance and exit angles be ψ_1 and ψ_2 respectively, and the curvature of the entrance and exit faces be $H_1 = 1/R_1$ and $H_2 = 1/R_2$ respectively. If we consider the finite extent of the fringe field (i.e., a soft fringe field model), the vertical focusing angle is defined to be

$$\bar{\psi}_i = \psi_i - hgI_i(1+\sin^2\psi_i)\sec\psi_i, \qquad i = 1,2 \qquad (21)$$

where g is the full gap of the dipole magnet and I_i are the fringe field integrals evaluated at the corresponding faces. The second order map at the entrance face is

$$\begin{aligned}T_{111} &= T_{234} = T_{414} = -T_{212} = -T_{313} = -\frac{h}{2}\tan^2\psi_1, \qquad (22)\\ T_{133} &= -T_{423} = \frac{h}{2}\sec^2\psi_1, \qquad (23)\\ T_{211} &= -T_{413} = \frac{h}{2R_1}\sec^3\psi_1 + K_1\tan\psi_1, \qquad (24)\\ T_{233} &= -\frac{h}{2R_1}\sec^3\psi_1 - K_1\tan\psi_1 + \frac{h^2}{2}\tan\psi_1(\tan^2\bar{\psi}_1 + \sec^2\psi_1), \qquad (25)\end{aligned}$$

and at the exit face

$$\begin{aligned}T_{111} &= T_{234} = T_{414} = -T_{212} = -T_{313} = \frac{h}{2}\tan^2\psi_2, \qquad (26)\\ T_{133} &= -T_{423} = -\frac{h}{2}\sec^2\psi_2, \qquad (27)\\ T_{211} &= \frac{h}{2R_2}\sec^3\psi_2 + K_1\tan\psi_2 - \frac{h^2}{2}\tan^3\psi_2, \qquad (28)\end{aligned}$$

$$T_{233} = -\frac{h}{2R_2}\sec^3\psi_2 - K_1\tan\psi_2 - \frac{h^2}{2}\tan\psi_2\tan^2\bar{\psi}_2, \tag{29}$$

$$T_{413} = -\frac{h}{2R_2}\sec^3\psi_2 - K_1\tan\psi_2 + \frac{h^2}{2}\tan\psi_2\sec^2\psi_2, \tag{30}$$

with all other elements being zero. The above formulae produce exactly the same result as the MAD program as described in the next section.

4 The new passmethod and its validation

The new passmethod is modified from the existing code and is renamed *BndMPoleSymplectic4E2Pass* with "E2" added to indicate that it includes edge effects to the second order. It now ignores the element field PolynomA and accepts two new optional fields, *H1* and *H2*, which represent the curvatures of the entrance and exit pole faces, respectively.

Results of the new AT passmethod are compared to that of the MAD program for the following cases to validate the code.

1. A gradient sector dipole with $L = 1.5048$ m, $\theta = \pi/17$,$K_1 = -0.3154$ m^{-2}.
2. The above gradient dipole with hard edge fringe fields, $\psi_1 = \psi_2 = \pi/34$.
3. The above gradient dipole with soft edge fringe fields, $\psi_1 = \psi_2 = \pi/34$, $I_1 = I_2 = 0.6$ and $g = 0.034$ m.
4. The above gradient dipole with soft edge fringe fields, plus curved pole profiles at both faces with $R_1 = R_2 = 10$ m.
5. The gradient dipole with soft edge fringe fields and pole curvature, plus additional body sextupole component with $K_2 = 4.0$ m^{-3} (MAD convention).

With NumIntSteps set to 20, the differences of corresponding elements of the transfer matrix and the second order transport map between the new AT passmethod and MAD are below 1.5×10^{-6} in SI units for cases 1–4. For case 5, NumIntSteps needs to be 50 to reach the same level.

The AT passmethods are also compared to MAD and Elegant [9] for the SPEAR3 achromatic lattice. Table 1 shows horizontal and vertical chromaticities and the second order momentum compaction factor for the existing passmethod BndMPoleSymplectic4Pass ("AT old"), the new passmethod BndMPoleSymplectic4E2Pass ("AT new"), MAD and Elegant. The field NumIntSteps is set to 20. Setting it to 50 doesn't change the result much. The new AT passmethod clearly agrees with MAD and Elegant while the old one does not do as well, especially for the vertical chromaticity.

The passmethod BndMPoleSymplectic4RadPass is also modified to include the changes described in the previous sections. The corresponding new passmethod is named *BndMPoleSymplectic4E2RadPass*. It is verified with the SPEAR3 dipole model.

Table 1: Comparison of chromaticities and the second order momentum compaction factor between AT, MAD and Elegant. NumIntSteps=20 for both AT passmethods.

Model	C_x	C_y	α_2
AT old	1.09730	1.214431	-0.0023962
AT new	1.12223	0.197202	-0.0024464
MAD	1.12367	0.196600	–
Elegant	1.12356	0.196566	-0.0024462

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Appendix A: another symplectic integrator

Following Ref. [6], the Hamiltonian Eq. (8) can be split into two parts as

$$
\begin{aligned}
H &= H_1 + H_2, && \text{(A-1)}\\
H_1 &= 1 + \delta - (1 + hx)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}, && \text{(A-2)}\\
H_2 &= -(1 + hx)\frac{A_s}{B\rho}, && \text{(A-3)}
\end{aligned}
$$

where H_1 represents a drift space in the curvilinear coordinates and H_2 represent a thin kick since the general momenta are absent from the latter.

The solution of H_1 is found to be

$$
\begin{aligned}
x_2 &= (\rho + x)\frac{\cos\phi}{\cos(\phi + hL_d)} - \rho, && \text{(A-4)}\\
p_{x2} &= \sqrt{(1+\delta)^2 - p_y^2}\sin(\phi + hL_d), && \text{(A-5)}\\
y_2 &= y + p_y\frac{(\rho + x)}{\sqrt{(1+\delta)^2 - p_y^2}}(\cos\phi\tan(\phi + hL_d) - \sin\phi), && \text{(A-6)}\\
z_2 &= z + (1+\delta)\frac{(\rho + x)}{\sqrt{(1+\delta)^2 - p_y^2}}(\cos\phi\tan(\phi + hL_d) - \sin\phi) - L_d, && \text{(A-7)}
\end{aligned}
$$

where subscript "2" indicates values at the exit face of the drift element, L_d is the arc length of the reference trajectory in the drift space, all phase space coordinates on the right hand side are values at the entrance face, p_y and δ are unchanged, and

$$\phi = \tan^{-1}\frac{p_x}{\sqrt{(1+\delta)^2 - p_y^2}}.$$

The solution of H_2 up to the octupole component, corresponding to terms explicitly given in Eq. (9), is

$$\begin{aligned}\Delta p_x/L_k = & -h-(K_1+h^2)x-K_2(x^2-y^2)-K_3(x^3-2xy^2) \\ & -h\Big(K_1(x^2-\frac{1}{2}y^2)+K_2(x^3-\frac{4}{3}xy^2)\Big)\end{aligned} \quad \text{(A-8)}$$

$$\Delta p_y/L_k = K_1y+2K_2xy+K_3(2x^2y-y^3)+h\Big(K_1xy+\frac{4}{3}K_2x^2y+\frac{1}{6}(hK_1-2K_2)y^3\Big) \quad \text{(A-9)}$$

where terms in the last brackets are the curvature effect not included in the existing AT passmethod.

The 4th-order symplectic integrator for a general sector dipole based on the above scheme has been implemented. The advantage of this integrator is that it is accurate to high orders. One issue of this integrator is that a zero vector,$[0,0,0,0,0,0]'$, will not be exactly zero when it is passed to the exit. For example, it becomes $[0.5,0.8,0,0,0,0.3]'\times 10^{-6}$ for a sector dipole with length 1.5048 m, bending angle $\pi/17$ and defocusing gradient $K_1 = -0.3154$ m^{-2} if the number of integration steps (*NumIntSteps*) is set to 10. However, this is usually a minor inconvenience. In particle tracking the "kick" errors to x and x' affect only the closed orbit, not the dynamics (transfer matrix and higher order maps). The error to z may accumulate and cause artificial phase error in tracking. It also cause errors in momentum compaction factor calculation unless care is taken to subtract the overall shift of z out. However, the kick errors go down as $1/N^4$ as N=NumIntSteps goes up. In the example above, the error becomes $[0.5,0.8,0,0,0,0.3]'\times 10^{-10}$ if $N = 100$.

The passmethod that implements this scheme is named *BndMPoleSymplectic4E2APass* and its radiation counterpart is named *BndMPoleSymplectic4E2ARadPass*. To deal with the numerical "kick" error, especially the accumulation of phase error (z variable), one way is to increase the number of integration steps (the NumIntSteps field). Another remedy is provided by introducing a new optional field, *SyncError*, which is a 6-dimensional row vector obtained by tracking the zero vector through the element when this field is absent and turning the resulting vector into a row vector. The error vector passed to the code this way is subtracted from the exiting phase space vectors of particles being tracked. For the example discussed in the end of section 2, the SyncError field may be set by

BEND.SyncError = $[0.5, 0.8, 0, 0, 0, 0.3]$*1E-6,

when NumIntSteps is 10. It is advised not to use this trick on BndMPoleSymplectic4E2ARadPass since the zero vector is not supposed to be mapped to a zero vector when the particles lose energy in the dipole due to radiation.

Appendix B: Hamiltonian H_1 to the fourth order

If we expand the Hamiltonian in Eq. (8) to the fourth order and split it in the same manner, we get

$$H_1 = (1+hx)\frac{p_x^2+p_y^2}{2(1+\delta)}+\frac{p_x^2p_y^2}{4(1+\delta)^3}, \quad \text{(B-1)}$$

(B-2)

while H_2 remains the same as Eq. (13). The solution of Eq. (B-1) to the corresponding order is

$$\begin{aligned} x_2 &= x + \frac{1+hx}{1+\delta} p_x L_d + \frac{hL_d^2}{4(1+\delta^2)}(p_x^2 - p_y^2) + \frac{1}{2(1+\delta^2)} x p_y^2 + \frac{h^2 L_d^2}{4(1+\delta^2)} x(p_x^2 - p_y^2) \\ &\quad + \frac{L_d}{2(1+\delta)^3}(1 - \frac{1}{3}h^2 L_d^2) p_x p_y^2, \qquad \text{(B-3)} \\ p_{x2} &= p_x - \frac{hL_d}{2(1+\delta)}(p_x^2 + p_y^2) + \frac{h^2 L_d^2}{4(1+\delta)^2} p_x (p_x^2 + p_y^2), \qquad \text{(B-4)} \\ y_2 &= y + \frac{1+hx}{1+\delta} p_y L_d + \frac{(1+hx)hL_d^2}{2(1+\delta)^2} p_x p_y + \frac{L_d}{2(1+\delta)^2} p_x^2 p_y + \frac{h^2 L_d^3}{12(1+\delta)^3}(p_x^2 - p_y^2) p_y, \\ &\qquad \text{(B-5)} \\ z_2 &= z + \frac{1+hx}{2(1+\delta)^2}(p_x^2 + p_y^2) L_d, \qquad \text{(B-6)} \end{aligned}$$

with p_y and δ unchanged.

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