IB Math Analysis and Approaches (HL)

Extended Essay

May 2024 Session

The Mathematics Techniques of Camera Calibration

Research Question: What are the various strategies and mathematical techniques employed in camera calibration to develop precise and accurate camera models?

Word Count: 2187 words

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1 Introduction

Camera calibration, also known as camera resectioning, is the process of determining the intrinsic and extrinsic parameters of a camera. The intrinsic parameters deal with the camera's internal characteristics, while the extrinsic parameters describe its position and orientation in the world. The knowledge of the accurate values of these values parameters are essential, as it enables us to create a mathematical model which describes how a camera projects 3D points from a scene onto the 2D image it captures. The importance of a well-calibrated camera becomes very apparent in photogrammetric applications, where precise measurements of 3-dimensional physical objects are derived from photographic images.

Photogrammetry is the science of obtaining accurate measurements of 3-dimensional physical objects through photographic imagery. Photogrammetry was first employed by Prussian architect Albrecht Meydenbauer in the 1860s, who used photogrammetric techniques to create some of the most detailed topographic plans and elevations drawings[1]. Today, photogrammetric techniques are used in a multitude of applications spanning diverse fields, including but not limited to: 3D-model generation, computer vision, topographical mapping, medical imaging, and forensic analysis.

While camera calibration is essential in ensuring the accuracy of photogrammetric applications, it itself also relies on these very same photogrammetric techniques in order to estimate these parameters. In essence, the developments of photogrammetry and camera calibration are closely intertwined, underscoring the essential relationship between photogrammetry and camera calibration.

1.1 Problem Statement

While manufacturers of cameras often report parameters of cameras, such as the nominal focal length and pixel sizes of their camera sensor, these figures are typically approximations which can vary from camera to camera, particularly in consumer-grade cameras. As such, the use of these estimates by manufacturers are unsuitable in developing camera models for applications requiring high accuracy. Combined with the potential for manufacturing defects

[1] Albertz, "A Look Back; 140 Years of Photogrammetry," 1.

as well as unknown lens distortion coefficients further necessitates the need for a reliable method for determining the parameters of a camera.

Camera calibration emerges as the answer to these problems, allowing us to create very accurate models for the camera as well as generate estimates for its parameters. As such, it is important that we understand the mathematical techniques use in camera calibration, and why they find applications across many different real-world applications.

2 Approach

There are countless different approaches one could take to calibrate a camera,

A camera model is a projection model which approximates the function of a camera by describing a mathematical relationship between points in 3D space and its projection onto the sensor grid of the camera. In order to construct such a model, we must first understand the general workings of a camera.

however they all build upon techniques first described in multiple highly influential papers, most notably Tsai's "A Versatile Camera Calibration Technique for High-Accuracy 3D Machine Vision Metrology Using Off-the-shelf TV Cameras and Lenses" and Zhang's "A Flexible New Technique for Camera Calibration".

2.1 Camera Model

The modern lens camera is highly sophisticated, built with an array of complex mechanisms and a wide range of features such as zoom and autofocus. However, we only need to focus on its three principal elements critical to image projection: the lens, the aperture, and the sensor grid (CCD).

- **Lens** – Focuses incoming light rays and projects it onto the sensor grid. Modern cameras have compound lenses (lenses made up of several lens elements) in order to minimize undesired effects such as aberration, blurriness, and distortion.
- **Aperture** – Controls the amount of light that reaches the sensor. By adjusting the

aperture size, the exposure and depth of field can be modified.

- **Sensor Grid** – Captures the projected image

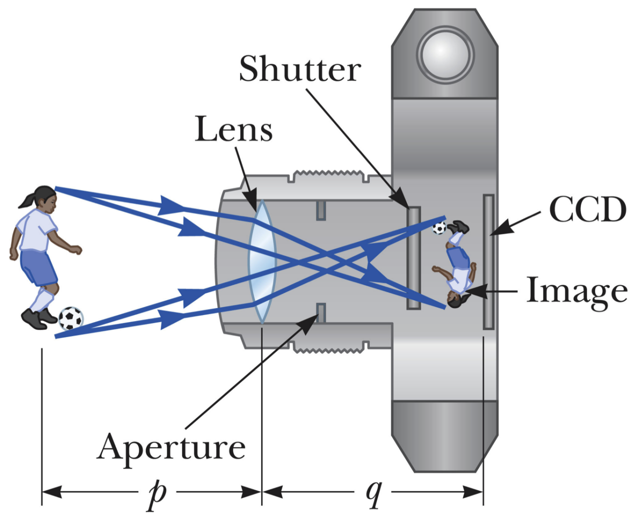

Figure 2.1: Lens camera. Adapted from Colton, "Warm-Up Exercise 30."

However, even this simplified model of a lens camera is still too complicated to describe, as it is impossible to encapsulate the complex behavior of a lens succinctly using one simple mathematical equation. As such, it is mathematically convenient to approximate the camera as a pinhole camera. In doing so, we ignore lens distortion, but it distills the behavior of a camera to its most fundamental and essential dynamics: the projection of points in 3D space onto the flat 2D sensor plane.

2.1.1 Pinhole Camera Model

A pinhole camera is a simple camera without a lens. It instead relies on the use of a tiny hole as the aperture of the camera, and light rays pass through the hole, projecting an inverted image onto the image plane. The pinhole camera model is based on the pinhole camera, however it goes further by making the assumptions that the aperture is infinitely small. This means that any incoming light ray can only travel straight through the pinhole, and that a point in space can only map to one single point on image plane.

If necessary, one could reintroduce distortion and shear terms in order to minimize the error, but this is often not needed for low to medium precision applications, as the distortion

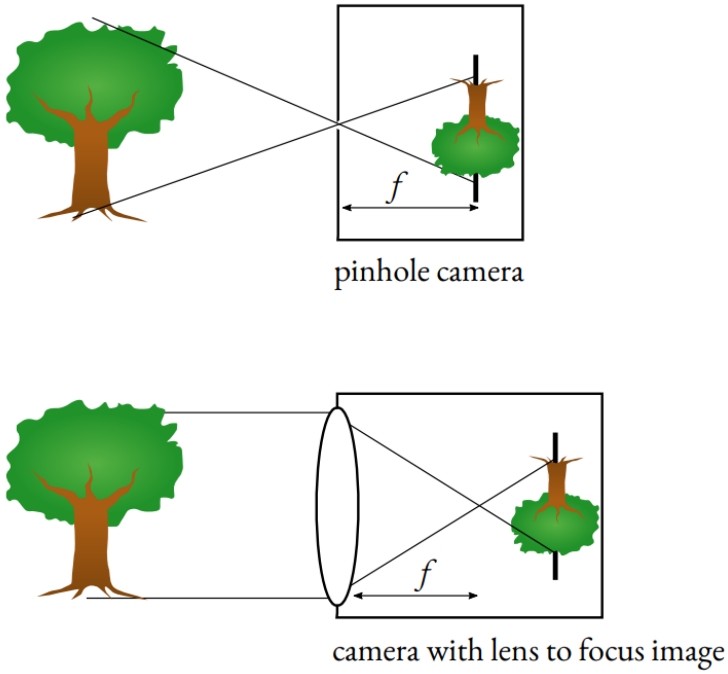

Figure 2.2: Difference between a pinhole camera and a lens camera. Adapted from Lê, "Camera Model: Intrinsic Parameters."

of modern lenses are already minimal. As such, its ease of use has led it to become one of the most frequently employed camera models in the field of camera calibration.

2.2 Calibration Object

The calibration object is an object with known dimensions

Calibration objects can be roughly separated into 3 categories, based on the dimension of the calibration object used[2]:

- **3D** –
- **2D** –
- **1D** –

[2] Zhang, "Camera Calibration."

3 Prerequisites

3.1 Notation

Vectors and Matrices. In this paper, lowercase letters are used to denote vectors, whereas capital letters are used for matrices. Depending on the context, vectors can also be attached with diacritics:

- $\vec{v}$ – a letter with an arrow above it denotes a positional vector or translational vector dealing with the transformations.
- $\widetilde{v}$ – a letter with a tilde above it denotes a vector represented in homogenous coordinates[3].

Transpose of Vectors and Matrices. The transpose of a vector or a matrix is an operation whereby the rows and columns of the vector or matrix are inverted, and this is denoted using the notation v^T or M^T. For example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\mathsf{T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

3.2 Homogenous Coordinates

While Euclidean space describes 2D and 3D space well, they are not sufficient in describing perspective projections, as it is unable to fully the capture the relationships inherent in projective projections and affine transformations, both of which are core concepts in this paper.

Homogenous coordinates forms the basis of projective geometry, because it unifies the treatment of common graphical transformations such as rotation and translations[4].

Given a point in with $\mathbb{R}^n$ coordinates $(a_1, a_2, \cdots, a_n)$

When

[3]See section 3.2.
[4]Bloomenthal and Rokne, “Homogeneous Coordinates,” 1.

Given the vector $[\,u, v\,]^{\mathsf{T}} \in \mathbb{R}^2$, we can express it in terms of homogenous coordinates:

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \equiv \begin{bmatrix} u\widetilde{w} \\ v\widetilde{w} \\ \widetilde{w} \end{bmatrix} \equiv \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{w} \end{bmatrix} \tag{3.1}$$

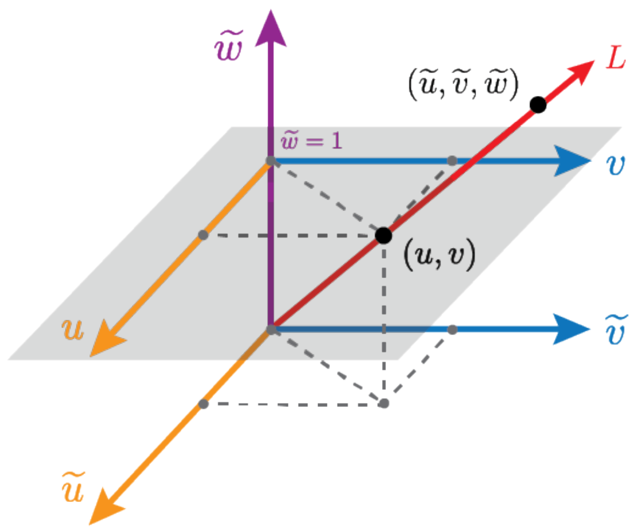

Figure 3.1: Homogenous coordinate system.

In other words, with homogenous coordinates, we interpret our *Euclidean* space as an *affine* space

4 Constructing the Pinhole Camera Model

4.1 General Strategy

4.2 Nomenclature

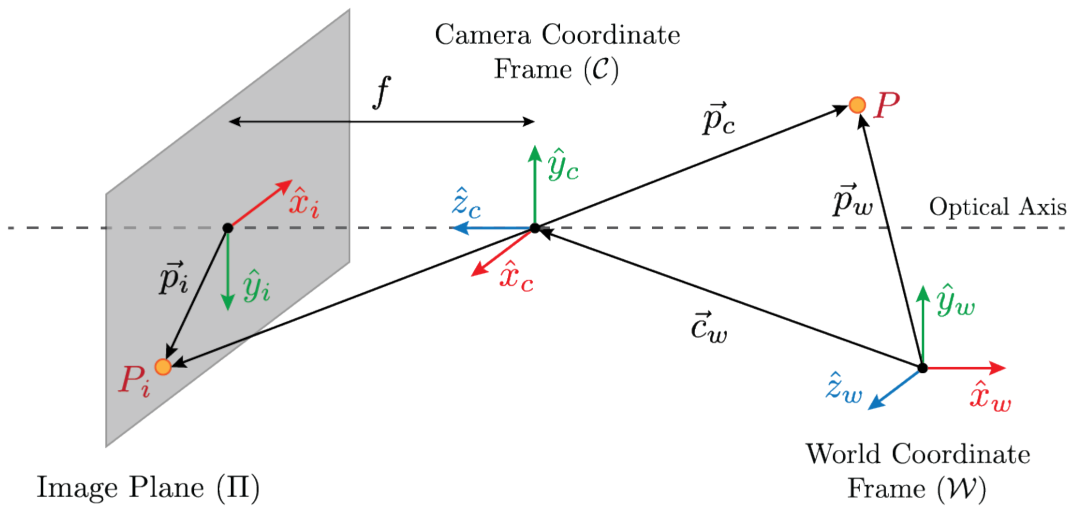

Figure 4.1: Pinhole camera model.

For our camera model, we will isntroduce 4 different coordinate systems:

- **The World Coordinate Frame $\mathcal{W}$**. Points are denoted as (x_w, y_w, z_w)
- **The Camera Coordinate Frame $\mathcal{C}$**. Points are denoted as (x_c, y_c, z_c).
- **The Image Plane Π**. Points are denoted as (x_i, y_i).
- **The Sensor Grid**. Points are denoted as (u, v).

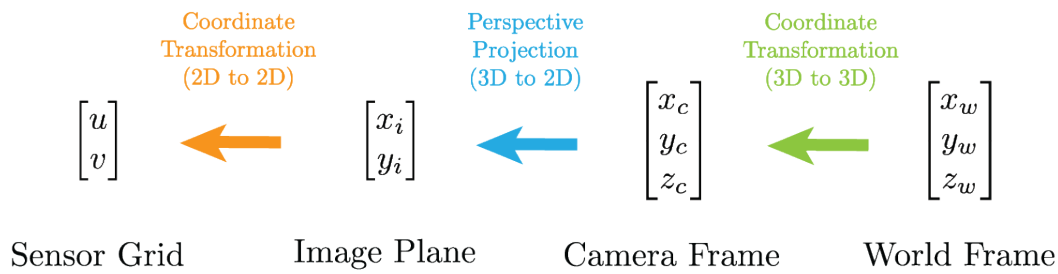

Figure 4.2: Coordinate transformations.

4.3 Intrinsic Parameters

First, we will focus on the projection of points in the 3D space onto the image plane. The goal is to construct a calibration matrix, K, which relates the position of the point P to its projection on the image plane. This can be expressed mathematically as follows:

$$\widetilde{p}_i = K\widetilde{p}_c \tag{4.1}$$

where $\vec{p}_i$ and $\vec{p}_c$ represents the position of the point P in the image plane Π and the camera frame $\mathcal{C}$ respectively.

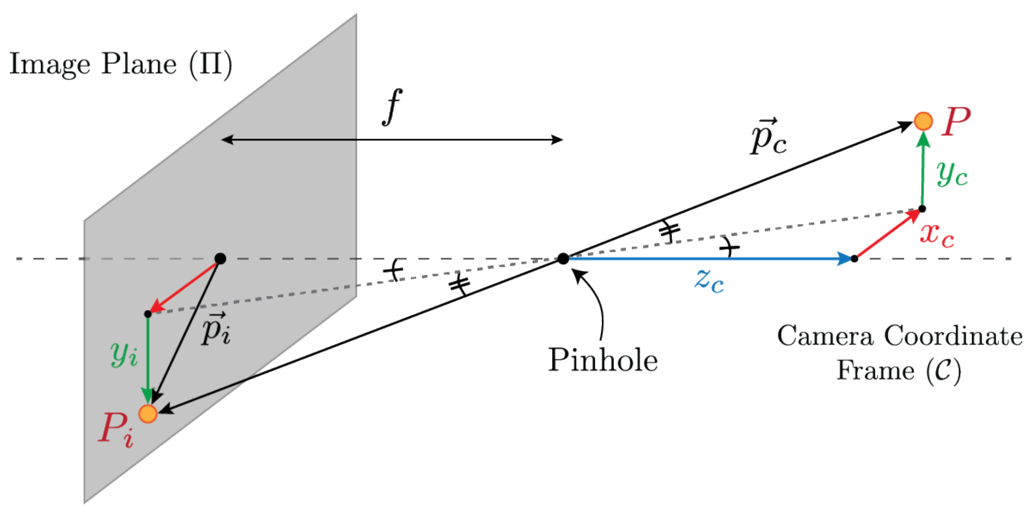

Figure 4.3: Perspective projection of the point P onto the image plane Π.

When a straight line is drawn from P to its projection P_i through the aperture, it intersects the optical axis. Deconstructing this intersection in the x and y direction, pairs of similar triangles are formed on the x and y plane.

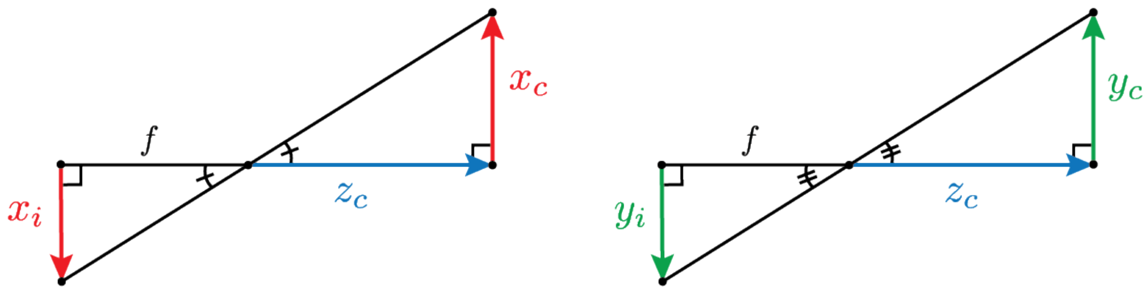

Figure 4.4: Similar triangles formed by perspective projection, which relate x_i to x_c and y_i to y_c.

$$\frac{x_i}{f} = \frac{x_c}{z_c} \implies x_i = f\frac{x_c}{z_c} \tag{4.2a}$$

$$\frac{y_i}{f} = \frac{y_c}{z_c} \implies y_i = f\frac{y_c}{z_c} \tag{4.2b}$$

We can then relate the coordinates of the projection, (x_i, y_i), which are in real-world units, to its position (u, v) in pixels.

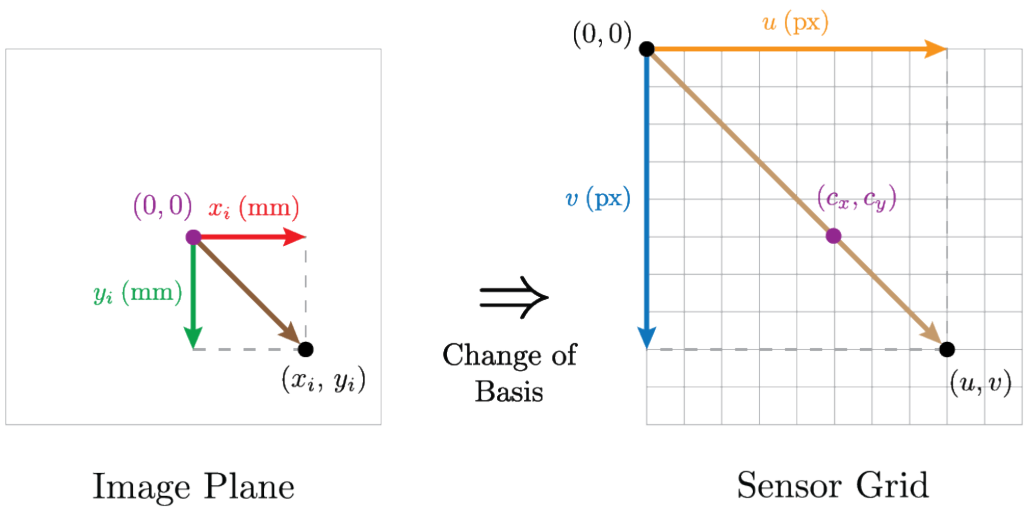

Figure 4.5: Conversion from image plane coordinates to sensor grid coordinates

Let m_x and m_y represent the pixel density of the image sensor in the x and y axes of the image sensor plane respectively.

$$u = m_x x_i + c_x$$

$$v = m_y y_i + c_y$$

Replacing x_i and y_i for the result we obtained from 4.2a and 4.2b, we get:

$$u = m_x f \frac{x_c}{z_c} + c_x$$

$$v = m_y f \frac{y_c}{z_c} + c_y$$

Since m_x, m_y, and f are all unknowns, we can combine the products $m_x f$ and $m_y f$ to f_x and f_y respectively. Under this new scheme, we define f_x and f_y as the horizontal and vertical focal lengths of camera.

$$u = f_x \frac{x_c}{z_c} + c_x$$

$$v = f_y \frac{y_c}{z_c} + c_y$$

Multiply both sides of the equations by z_c.

$$z_c u = f_x x_c + z_c c_x \tag{4.3a}$$

$$z_c v = f_y y_c + z_c c_y \tag{4.3b}$$

Doing so allows us to express the relationship as a matrix transformation using homogenous coordinates, by letting $\widetilde{w} = z_c$.

$$\begin{bmatrix} z_c u \\ z_c v \\ z_c \end{bmatrix} = \begin{bmatrix} f_x x_c + z_c c_x \\ f_y y_c + z_c c_y \\ z_c \end{bmatrix} = \underbrace{\begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}}_{K} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \tag{4.4}$$

$$K = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \tag{4.5}$$

In this case, K is what is known as the *calibration matrix*. It is a matrix transformation which maps a point represented in the camera coordinate frame to the coordinates of their projection onto the sensor plane. An important property worth noting is that K is an *upper triangular matrix*. It is a special kind of square matrix with all of its non-zero entries above the main diagonal. This is an important property which we will exploit when extracting the intrinsic matrix from the projection matrix in section 5.

4.4 Extrinsic Parameters

Next, we will focus on finding the position

Now, we would like to find the extrinsic matrix, M_{ext}, which relates the positional vector $\vec{p}_w$ of point P in the world coordinate frame, to its positional vector $\vec{p}_c$ in the camera coordinate frame. Similar to what we did in section 4.3, we can express this in homogenous coordinates as follows:

$$\widetilde{p}_c = M_{ext}\,\widetilde{p}_w \tag{4.6}$$

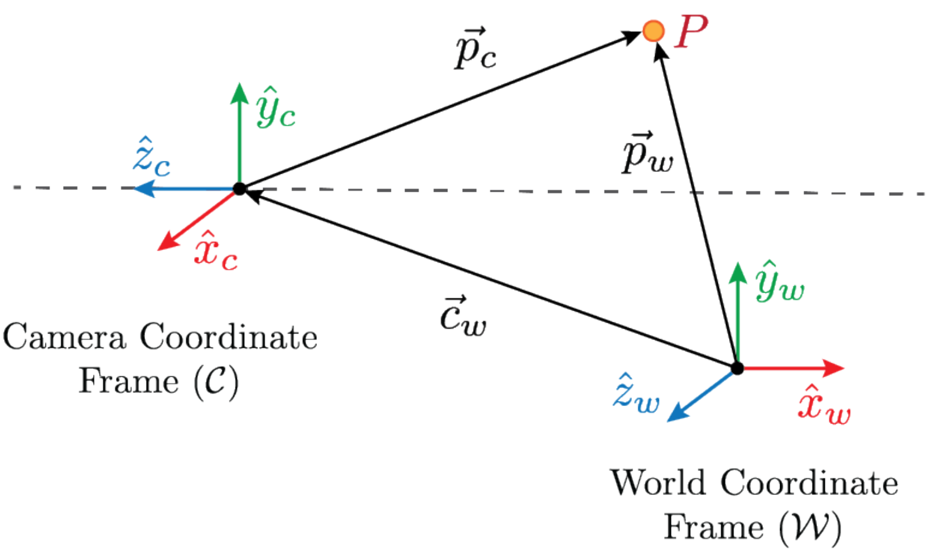

Figure 4.6: Coordinate transformation from the world coordinate frame to the camera frame.

For the extrinsic parameters of the camera, we have the position $\vec{c}_w$ of the camera in world coordinates and orientation R of the camera. The orientation, R, is a 3x3 rotational matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{4.7}$$

where:

- Row 1: Direction of $\hat{x}_c$ in world coordinate frame.
- Row 2: Direction of $\hat{y}_c$ in world coordinate frame.

- Row 3: Direction of $\hat{z}_c$ in world coordinate frame.

$$\vec{p}_c = R(\vec{p}_w - \vec{c}_w) \tag{4.8a}$$

$$= R\vec{p}_w - R\vec{c}_w \tag{4.8b}$$

$$\vec{p}_c = R\vec{p}_w + \vec{t} \tag{4.9}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{R} \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}_{\vec{t}} \tag{4.10}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_{ext}} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} \tag{4.11}$$

$$M_{ext} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \tag{4.12}$$

4.5 Putting It All Together

When we combine the equations $\widetilde{p}_c = M_{ext}\,\widetilde{p}_w$ (eq. 4.6) and $\widetilde{p}_i = M_{int}\,\widetilde{p}_c$ (eq. 4.1), we obtain

$$\widetilde{p}_i = M_{int}\,M_{ext}\,\widetilde{p}_w \tag{4.13}$$

To simplify our camera model, we can define a new matrix, $P \in \mathbb{R}^{3\times4}$, which is equal to the product $M_{int}\, M_{ext}$. Since M_{ext} is a 4×4 matrix and M_{int} is a 3×4 matrix, their matrix product produces a 3×4 matrix.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \equiv \underbrace{\begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix}}_{[\,R\,|\,\vec{t}\,]} \tag{4.14}$$

Replacing P for $M_{int}\, M_{ext}$ in equation 4.13, we obtain

$$\widetilde{p}_i = P\, \widetilde{p}_w \tag{4.15}$$

The implications of this equation is very important, as it means that we can project the nth point $\left[x_w^{(n)}, y_w^{(n)}, z_w^{(n)}\right]^\mathsf{T}$ in the world coordinate frame $\mathcal{W}$ to its pixel coordinates $[u_n, v_n]^\mathsf{T}$ on the image plane Π simply by using the projection matrix. But now, we need to figure out a way to solve for the project matrix.

Given that we have equation 4.15 which relates

When expressing the pixel coordinate in homogenous coordinates, equation 4.15 becomes

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} \sim \begin{bmatrix} \widetilde{u}_n \\ \widetilde{v}_n \\ \widetilde{w}_n \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w^{(n)} \\ y_w^{(n)} \\ z_w^{(n)} \\ 1 \end{bmatrix} \tag{4.16}$$

5 Camera Calibration

5.1 Overall Strategy

5.2 Solving for the Projection Matrix

$$\widetilde{u}_n = p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14}$$
$$\widetilde{v}_n = p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24}$$
$$\widetilde{w}_n = p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}$$

$$u_n = \frac{\widetilde{u}_n}{\widetilde{w}_n} = \frac{p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14}}{p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}}$$
$$v_n = \frac{\widetilde{v}_n}{\widetilde{w}_n} = \frac{p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24}}{p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}}$$

$$u_n(p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}) = p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14}$$
$$v_n(p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}) = p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24}$$

$$0 = p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14} - p_{31}u_nx_w^{(n)} - p_{32}u_ny_w^{(n)} - p_{33}u_nz_w^{(n)} - p_{34}u_n \tag{5.1a}$$
$$0 = p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24} - p_{31}v_nx_w^{(n)} - p_{32}v_ny_w^{(n)} - p_{33}v_nz_w^{(n)} - p_{34}v_n \tag{5.1b}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} x_w^{(1)} & y_w^{(1)} & z_w^{(1)} & 1 & 0 & 0 & 0 & 0 & -u_1x_w^{(1)} & -u_1y_w^{(1)} & -u_1z_w^{(1)} & -u_1 \\ 0 & 0 & 0 & 0 & x_w^{(1)} & y_w^{(1)} & z_w^{(1)} & 1 & -v_1x_w^{(1)} & -v_1y_w^{(1)} & -v_1z_w^{(1)} & -v_1 \\ & & \vdots & & & & & & & \vdots & & \\ x_w^{(n)} & y_w^{(n)} & z_w^{(n)} & 1 & 0 & 0 & 0 & 0 & -u_nx_w^{(n)} & -u_ny_w^{(n)} & -u_nz_w^{(n)} & -u_n \\ 0 & 0 & 0 & 0 & x_w^{(n)} & y_w^{(n)} & z_w^{(n)} & 1 & -v_nx_w^{(n)} & -v_ny_w^{(n)} & -v_nz_w^{(n)} & -v_n \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix}}_{p} \tag{5.2}$$

homogenous linear system overdetermined

5.3 Constrained Least Squares Solution

We have now established a way to solve for the

Now, we need to solve for $Gp = 0$

$$\underset{p}{\text{minimize}} \ \|Gp\|^2 \ \text{subject to} \ \|p\|^2 = 1 \tag{5.3}$$

For a given arbitrary vector $v \in \mathbb{R}^n$, the magnitude is equal to $\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. As such, we can rewrite the square of the magnitude of v, $\|v\|^2$, as:

$$\|v\|^2 = v_1^2 + v_2^2 + \cdots v_n^2 = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v^\mathsf{T} v$$

Thus, in equation 5.3, we can replace $\|Gp\|^2$ with $p^\mathsf{T} A^\mathsf{T} G p$ and $\|p\|^2$ for $p^\mathsf{T} p$ to obtain

$$\underset{p}{\text{minimize}} \ \left(p^\mathsf{T} G^\mathsf{T} G p\right) \ \text{ subject to } \ p^\mathsf{T} p = 1 \tag{5.4}$$

The Lagrangian[5] of equation 5.4 is

$$\mathcal{L}(p, \lambda) = p^\mathsf{T} G^\mathsf{T} G p - \lambda \left(p^\mathsf{T} p - 1\right) \tag{5.5}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. Since p is minimized when $\mathcal{L}$ is minimized, we need to look for the absolute minimum of $\mathcal{L}$, which are located at its critical points. To find these points, we want to look for values of p and λ where all partial derivatives of the Lagrangian are zero, i.e.

$$\frac{\partial \mathcal{L}}{\partial p} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

where ∂ is used to denote a partial derivative (see Appendix **??**). We will focus on the partial derivative of $\mathcal{L}$ with respect to p. Using product rule for partial derivatives, we obtain:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial p} &= \frac{\partial}{\partial p} \left[p^\mathsf{T} G^\mathsf{T} G p - \lambda \left(p^\mathsf{T} p - 1\right)\right] \overset{\text{set}}{=} 0 \\ &\Rightarrow 2 G^\mathsf{T} G p - 2\lambda p = 0 \\ &\Rightarrow G^\mathsf{T} G p = \lambda p \end{aligned} \tag{5.6}$$

which is an eigenvalue problem for $G^\mathsf{T} G$. Potential solutions for p are eigenvectors that satisfy equation 5.6,[6] with $\lambda \in \mathbb{R}$ as the eigenvalue. Since 5.4 is a minimization problem, the minimized eigenvector p is the one which has the smallest eigenvalue λ.[7]

which states that for a given matrix $M \in \mathbb{R}^{n \times n}$, determine the eigenvector $x \in \mathbb{R}^n, x \neq 0$ and the eigenvalue $\lambda \in \mathbb{C}$ such that:

[5] Ghojogh, Karray, and Crowley, "Eigenvalue and Generalized Eigenvalue Problems," 2.
[6] Nayar, *Linear Camera Model*.
[7] Ghojogh, Karray, and Crowley, "Eigenvalue and Generalized Eigenvalue Problems."

6 Extracting Parameters

Once we have solved for the projection for the projection matrix P, we can then extract the intrinsic and extrinsic parameters. We know that

$$\begin{aligned} P &= K \left[\, R \,|\, \vec{t} \,\right] \\ &= K \,[\, R \,|\, -R\vec{c_w} \,] \\ &= [\, KR \,|\, -KR\vec{c_w} \,] \end{aligned} \tag{6.1}$$

$$P = [\, Q \,|\, -Q\vec{c_w} \,] \tag{6.2}$$

$$Q = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{R}$$

Since K is in the form of an *upper right triangular matrix* and R is an *orthonormal matrix*, we can find unique solutions for K and R using a method called *RQ decomposition*.

6.1 RQ Decomposition

RQ decomposition is a technique which allows us to uniquely decompose a matrix A into a product $A = RQ$,

Since

6.2 Extracting the Translation Vector

$$-Q\vec{c_w} = \begin{bmatrix} p_{14} \\ p_{24} \\ p_{34} \end{bmatrix}$$

$$\Rightarrow \vec{c_w} = -Q^{-1} \begin{bmatrix} p_{14} \\ p_{24} \\ p_{34} \end{bmatrix} \tag{6.3}$$

6.3 Extracting Orientation as Angles

When constructing the extrinsic matrix in section 4.4, we defined the rotation matrix as the

We can represent the rotation in terms of *Tait-Bryan Angles*

$$R \equiv R_z(\gamma)R_y(\beta)R_x(\alpha) \tag{6.4}$$

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{6.5a}$$

$$R_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & -\cos(\beta) \end{bmatrix} \tag{6.5b}$$

$$R_z(\gamma) = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.5c}$$

$$
\begin{aligned}
R &= \begin{bmatrix} 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & -\cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\beta)\,\cos(\gamma) & \sin(\alpha)\,\sin(\beta)\,\cos(\gamma) - \cos(\alpha)\,\sin(\gamma) & \cos(\alpha)\,\sin(\beta)\,\cos(\gamma) + \sin(\alpha)\,\cos(\gamma) \\ \cos(\beta)\,\sin(\gamma) & \sin(\alpha)\,\sin(\beta)\,\sin(\gamma) + \cos(\alpha)\,\cos(\gamma) & \cos(\alpha)\,\sin(\beta)\,\sin(\gamma) - \sin(\alpha)\,\cos(\gamma) \\ -\sin(\beta) & \sin(\alpha)\,\cos(\beta) & \cos(\alpha)\,\cos(\beta) \end{bmatrix}
\end{aligned} \tag{6.6}
$$

We have that

$$
\begin{aligned}
r_{31} &= -\sin(\beta) \\
\Rightarrow \beta &= \sin^{-1}(-r_{31})
\end{aligned} \tag{6.7}
$$

$$
r_{32} = \sin(\alpha)\cos(\beta)
$$

$$
\begin{aligned}
\Rightarrow \alpha &= \sin^{-1}\left(\frac{r_{32}}{\cos(\beta)}\right) = \sin^{-1}\left(\frac{r_{32}}{\cos(\sin^{-1}(-r_{31}))}\right) \\
&= \sin^{-1}\left(\frac{r_{32}}{\sqrt{1-r_{31}^2}}\right)
\end{aligned} \tag{6.8}
$$

$$
r_{21} = \cos(\beta)\sin(\gamma)
$$

$$
\begin{aligned}
\Rightarrow \gamma &= \sin^{-1}\left(\frac{r_{21}}{\cos(\beta)}\right) = \sin^{-1}\left(\frac{r_{21}}{\cos(\sin^{-1}(-r_{31}))}\right) \\
&= \sin^{-1}\left(\frac{r_{21}}{\sqrt{1-r_{31}^2}}\right)
\end{aligned} \tag{6.9}
$$

7 Experimental Validation

In an attempt to show that the model works, I created the program

Figure 7.1: Photograph 1. The photo editing software *GIMP* was used for edge detection, and the coordinates of the calibration points were selected manually.

Figure 7.2: Graph produced by `Matplotlib` which displays the results of the trial

$$P = \begin{bmatrix} -2.5844 \times 10^{-3} & 1.7334 \times 10^{-3} & -4.6719 \times 10^{-4} & 6.0581 \times 10^{-1} \\ 4.8240 \times 10^{-4} & 4.4097 \times 10^{-4} & -3.1337 \times 10^{-3} & 7.9559 \times 10^{-1} \\ -3.3990 \times 10^{-7} & -3.1311 \times 10^{-7} & -2.8179 \times 10^{-7} & 4.1340 \times 10^{-4} \end{bmatrix}$$

		Canon EOS D80	**IPhone X**	**Nikon D100**
Focal Lengths	f_x	8404.1 px	3281.5 px	8144.4 px
	f_y	8387.9 px	3279.9 px	8142.6 px
Principal Point	c_x	3151.6 px	2043.0 px	1541.8 px
	c_y	1972.8 px	1453.1 px	1027.9 px
Tait-Bryan Angles	α	−81.86°	−60.21°	−70.83°
	β	44.27°	38.72°	46.44°
	γ	4.97°	21.64°	13.89°
Translation	t_x	494.8 mm	329.0 mm	840.3 mm
	t_y	537.6 mm	321.4 mm	766.0 mm
	t_z	128.3 mm	208.6 mm	317.2 mm
Reproj. Errors	μ_{max}	11.08 px	5.58 px	11.70 px
	μ_{avg}	3.56 px	2.55 px	2.81 px

Table 7.1: Intrinsic and Extrinsic Parameters calculated by `calicam`.

7.1 Validating Estimated Focal Length

Given that specification of cameras are readily available online, we can actually evaluate the accuracy of our calculated focal lengths. Assuming that the pixels are square, we estimate the focal lengths of our cameras to be the average of the horizontal and vertical focal lengths. Then, based on the manufacturer reported size of each individual pixel (known as the *pixel pitch*), we can convert our estimated focal length from pixels to millimeters.

	Calculated Focal Length[8]	**Reported Focal Length**	**% Error**
Canon EOS D80	$(8496\,\text{px})(3.73\,\mu\text{m/px}) \approx \boxed{31.7\,\text{mm}}$	32 mm	0.94 %
IPhone X	$(3280\,\text{px})(1.22\,\mu\text{m/px}) \approx \boxed{4.00\,\text{mm}}$	4 mm	–
Nikon D100	$(8143\,\text{px})(7.82\,\mu\text{m/px}) \approx \boxed{63.7\,\text{mm}}$	55 mm	15.6 %

Table 7.2: Comparison of Calculated vs. Reported Focal Length.

Considering that most

With exception to the **Nikon D100**, the

I suspect that the reason for the error is that I used a zoom lens for the Nikon D100, and the

[9]

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[8] Pixel pitches were retrieved from digicamdb.com.
[9] WayneF, "Answer to 'Are Lenses Marked with the True Focal Length?'"

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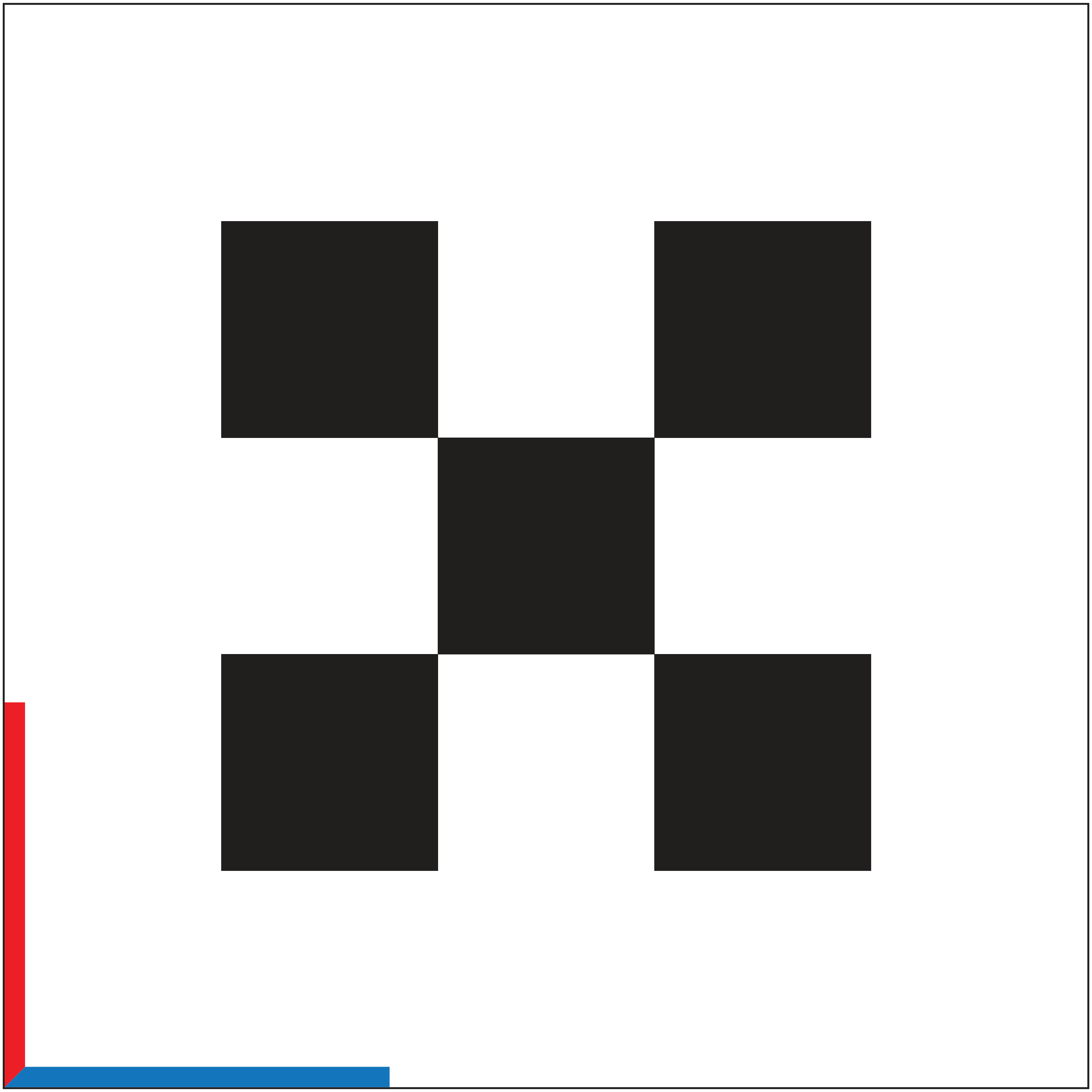
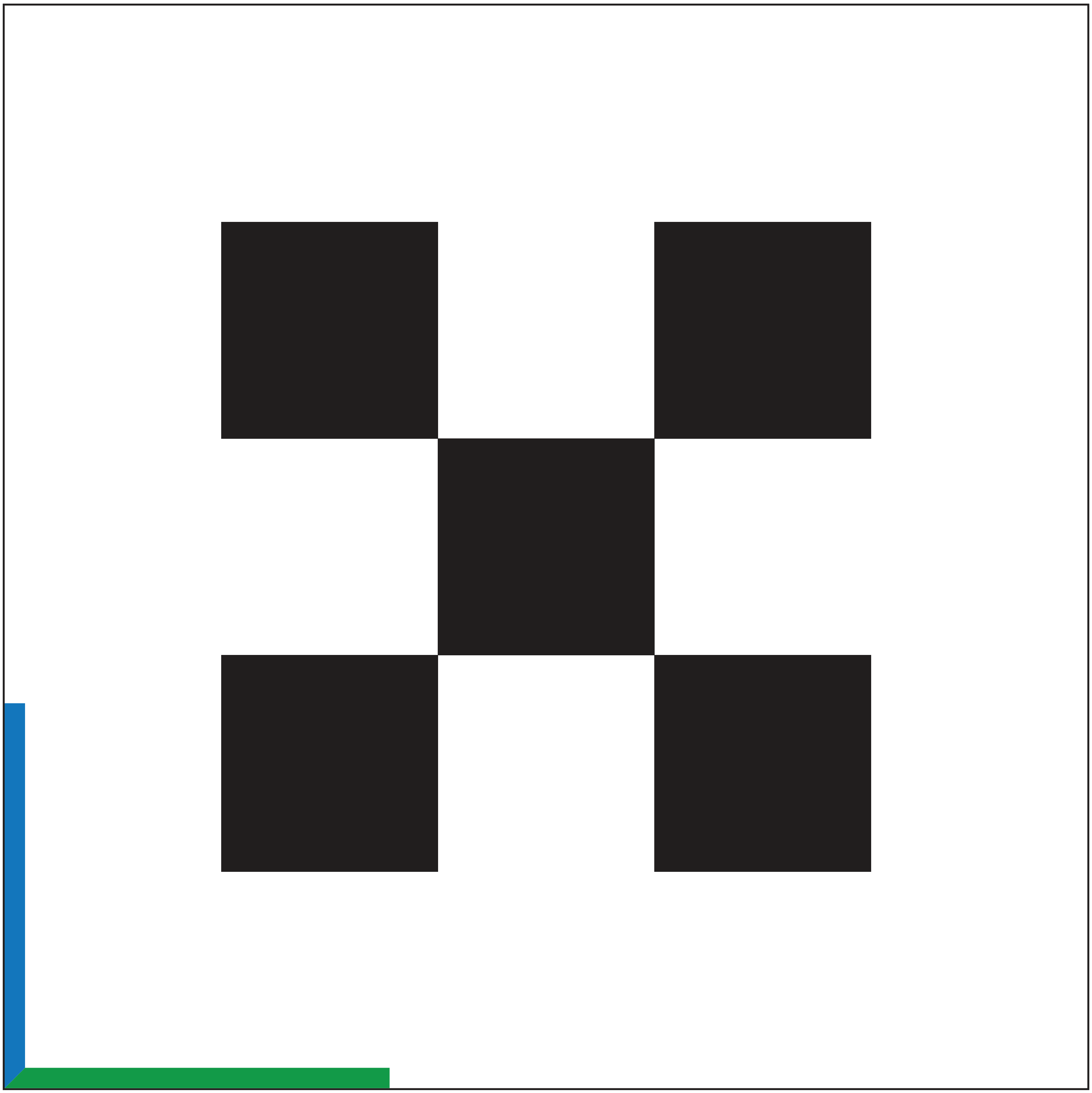
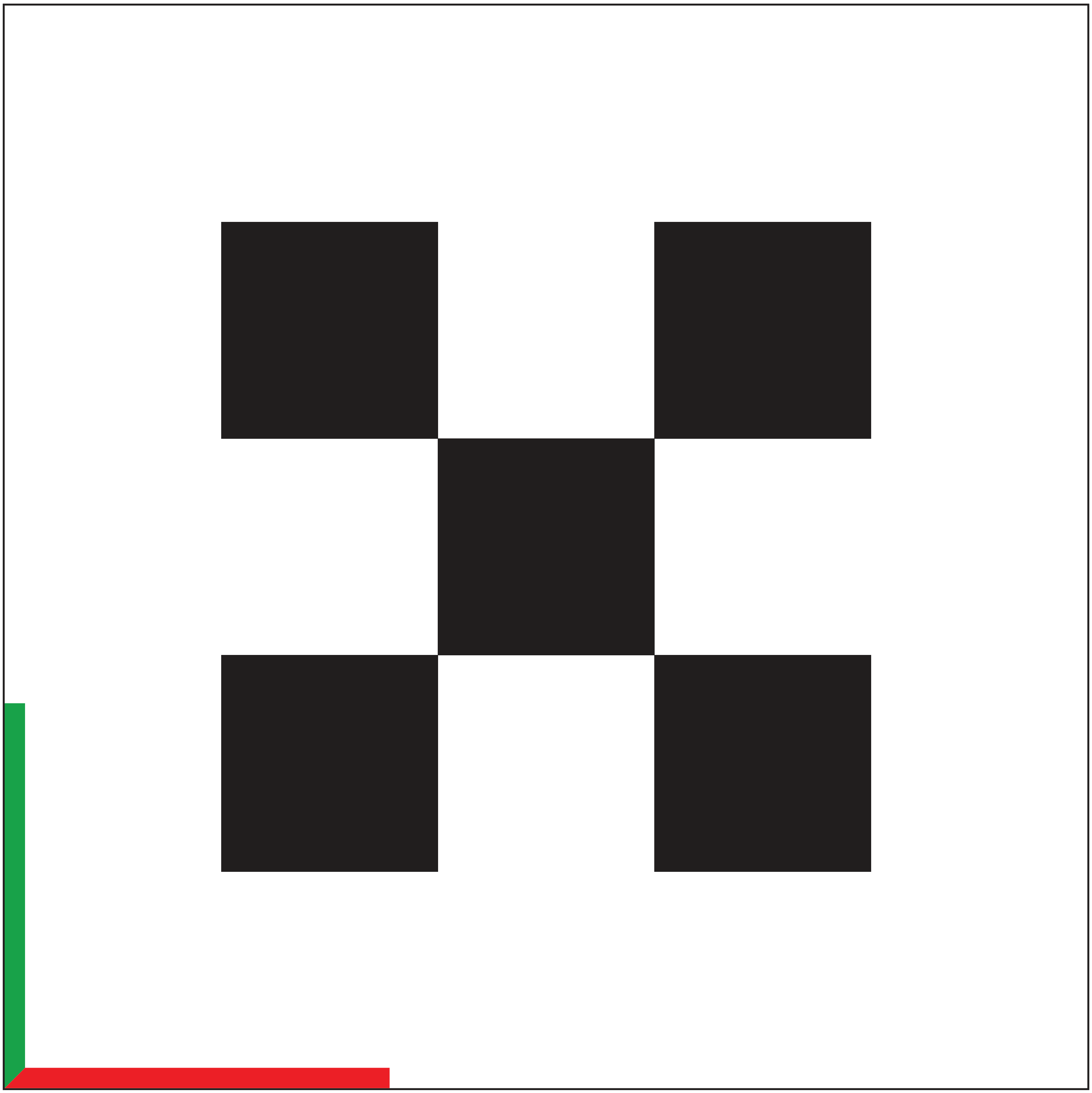
Appendix A Calibration Object Details

A.1 Panels

A.2 Grid Pattern







Appendix B Source Code

Project Structure

```
calicam
├── calicam
│   ├── __init__.py
│   ├── extract.py
│   ├── parser.py
│   ├── projection.py
│   └── vecs.py
└── run.py
```

run.py

```python
#!/usr/bin/env python3
import os
import sys
import numpy as np
import matplotlib.pyplot as plt
from argparse import ArgumentParser, RawDescriptionHelpFormatter

import calicam


def main():
    np.set_printoptions(precision=3, suppress=True)

    MAX_HELP_POSITION = 40

    parser = ArgumentParser(
        prog="calicam",
        description=(
            "Generates projection matrix and calculates intrinsic and extrinsic parameters.\n"
            "CSV inputs are in the format: x,y,z,u,v where 3D point = (x, y, z) and 2D point = (u,v)"),
        formatter_class=lambda prog: RawDescriptionHelpFormatter(prog, max_help_position=MAX_HELP_POSITION),
    )

    parser.add_argument("path", metavar="PATH", help="path to csv file with calibration points")
    parser.add_argument(
        "-d", "--data", metavar="DATA_PATH", help="path to csv file with model verification data")
    parser.add_argument(
        "-g", "--graph", nargs="?", const="", metavar="BKGD_IMG", help="generate graph")
    parser.add_argument("-t", "--title", metavar="TITLE", help="title of graph (ignored if `-g` is not passed)")
    parser.add_argument("-s", "--show", action="store_true", help="show graph (only necessary if `-o` is passed)")
    parser.add_argument("-o", "--out", metavar="GRAPH_PATH", help="graph output location ")
    parser.add_argument("--noprint", action="store_true", help="don't print output to terminal")

    args = parser.parse_args()

    output = []

    try:
        # GENERATE MODEL
        csv_path: str = args.path

        cali_world_coords, cali_image_coords = calicam.parse_data_from_csv(csv_path)
        proj_matrix, cali_matrix, rot_matrix, (tx, ty, tz) = calicam.calibrate_camera(
```

```
            cali_world_coords, cali_image_coords)

        (cx, cy), (fx, fy) = calicam.extract_intrinsics(cali_matrix)
        a, b, g = calicam.extract_orientation_zyx(rot_matrix)

        log_path: str = args.path

        output.append("\n" + "\n\n".join((
            f"Projection Matrix: \n{proj_matrix}",
            f"Calibration Matrix: \n{cali_matrix}",
            f"Rotation Matrix: \n{rot_matrix}",
            f"Focal Lengths: \n\tf_x = {fx:.2f} px \n\tf_y = {fy:.2f} px",
            f"Principal Point: \n\tc_x = {cx:.2f} px \n\tc_y = {cy:.2f} px",
            f"Translation: \n\tt_x = {tx:.2f} \n\tt_y = {ty:.2f} \n\tt_z = {tz:.2f}",
            f"Orientation: \n\t\u03B1 = {a:.2f}° \n\t\u03B2 = {b:.2f}° \n\t\u03B3 = {g:.2f}°",
        )))

        # MODEL VALIDATION
        data_path: str | None = args.data

        if data_path is not None:
            assert os.path.isfile(data_path), f"{data_path} does not exist."
            assert data_path.endswith(".csv"), f'{data_path} does not end with the extension ".csv".'

            data_world_coords, data_image_coords = calicam.parse_data_from_csv(data_path)

            predicted_coords = [
                calicam.project_point(proj_matrix, world_coord) for world_coord in data_world_coords
            ]
            reproj_errs = [
                calicam.euclidean(actual_coord, reproj_coord)
                for actual_coord, reproj_coord in zip(data_image_coords, predicted_coords)
            ]

            max_err = max(reproj_errs)
            avg_err = sum(reproj_errs) / len(reproj_errs)

            output.append(
                f"\nReprojection Errors: \n\t\u03BC_max = {max_err:.3f} px \n\t\u03BC_avg = {avg_err:.3f} px")

        # OUTPUT
        if not args.noprint:
            print(*output, sep="\n")

        # GRAPH
        image_path: str | None = args.graph

        if image_path is not None:
            ax: plt.Axes
            _, ax = plt.subplots(figsize=(8, 10))

            plt.gca().invert_yaxis()

            # image was provided
            if image_path != "":
                assert os.path.isfile(image_path), f"{image_path} does not exist."
                assert args.data, "Path to data csv file must be provide using -d flag to produce graph."

                img = plt.imread(image_path,)
                ax.imshow(img, cmap='gray')
                ax.autoscale(False)

            # graph data points and model points only if -d flag is specfied
            if data_path is not None:
                cmap = plt.cm.brg
                discrete_cmap = list(cmap(np.linspace(0, 1, len(data_image_coords))))
```

```
                # data points
                ax.scatter(
                    *zip(*data_image_coords),
                    label="Ground Truth",
                    s=50,
                    color=discrete_cmap,
                    marker="o",
                    alpha=0.6,
                )

                # predicted points
                ax.scatter(
                    *zip(*predicted_coords),
                    label="Prediction",
                    s=100,
                    color=discrete_cmap,
                    marker="x",
                )

            # calibration points
            ax.scatter(
                *zip(*cali_image_coords),
                label="Calibration Points",
                s=60,
                marker="D",
                color="darkorange",
            )

            # principle point
            ax.scatter(cx, cy, label="Principal Point", s=120, marker="p", color="magenta")

            graph_title: str = args.title or image_path

            plt.gca().update({"title": graph_title, "xlabel": "$u$ (px)", "ylabel": "$v$ (px)"})
            plt.legend()

            out_path: str | None = args.out

            if out_path:
                plt.savefig(out_path)

            # show graph if -s flag was specified or if a save location was not specified
            if args.show or not out_path:
                plt.show()

    except AssertionError as e:
        parser.error(str(e))  # pass error to argparse

    except KeyboardInterrupt:
        print(f"\nKeyboardInterrupt")
        sys.exit(1)

    sys.exit(0)


if __name__ == "__main__":
    main()
```

calicam/parser.py

```
import csv

from .vecs import *
```

```


def parse_data_from_csv(path: str) -> tuple[list[Vec3f], list[Vec2f]]:
    """
    Parses a csv and returns a list of 3D scene points and their corresonding
    2D image mappings.
    CSV format: x,y,z,u,v
    where 3D point = (x, y, z) and 2D point (u,v)
    """
    with open(path, "r") as f:
        reader = csv.reader(f, delimiter=",")

        world_coords = []
        image_coords = []
        for lno, line in enumerate(reader, start=1):
            assert len(line) == 5, f"Data on line {lno} in {path} is invalid."

            x, y, z, u, v = (float(s) for s in line)

            world_coords.append((x, y, z))
            image_coords.append((u, v))

    return world_coords, image_coords
```

calicam/projection.py

```
import numpy as np
import scipy.sparse.linalg
from nptyping import Shape, Double

from .vecs import *

ProjMatrix = np.ndarray[Shape["3, 4"], Double]


def generate_estimation_matrix(world_coords: list[Vec3f], image_coords: list[Vec2f]) -> np.ndarray:
    """
    Generates an estimation matrix from list of 3D world coords and
    their corresponding pixel coord mappings
    """
    rows = []
    for (x, y, z), (u, v) in zip(world_coords, image_coords):
        rows.append([x, y, z, 1.0, 0.0, 0.0, 0.0, 0.0, -u * x, -u * y, -u * z, -u])
        rows.append([0.0, 0.0, 0.0, 0.0, x, y, z, 1.0, -v * x, -v * y, -v * z, -v])
    return np.array(rows)


def generate_proj_matrix(world_coords: list[Vec3f], image_coords: list[Vec2f]) -> tuple[ProjMatrix, float]:
    """
    Takes 3D calibration points their corresponding pixel coord mappings and
    returns the projection matrix as a 3x4 matrix
    """
    assert len(world_coords) == len(image_coords), \
        f"The number of world coordinates ({world_coords}) and image coordinates ({image_coords}) do not
        ↪ match."

    assert len(world_coords) >= 6, \
        f"Need at least 6 calibration points, but only {len(world_coords)} were provided."

    G = generate_estimation_matrix(world_coords, image_coords)
    M = G.T @ G

    eigval, p = scipy.sparse.linalg.eigs(M, k=1, which="SM")  # solve for minimum p using eigenvalue problem
```

```
    proj_matrix = p.real.reshape(3, 4)  # take only real part of p and convert into 3x4 matrix

    return proj_matrix, eigval


def project_point(projection_matrix: ProjMatrix, world_coords: Vec3f) -> Vec2f:
    """
    Calculate pixel coordinate from 3D world coord using projection matrix
    """
    im_point = projection_matrix @ to_homogenous(world_coords)
    return to_inhomogenous(im_point)  # turn into inhomogenous coords
```

calicam/extract.py

```
import numpy as np
import scipy.linalg
from math import sqrt
from nptyping import  Shape, Double

from .projection import generate_proj_matrix, ProjMatrix
from .vecs import *

CalMatrix = np.ndarray[Shape["3, 3"], Double]
RotMatrix = np.ndarray[Shape["3, 3"], Double]


def calibrate_camera(world_coords: list[Vec3f],
                     image_coords: list[Vec2f]) -> tuple[ProjMatrix, CalMatrix, RotMatrix, Vec3f]:
    """
    Decomposes the projection matrix into the calibration matrix,
    rotation matrix, and translation matrix.
    """
    proj_matrix, _ = generate_proj_matrix(world_coords, image_coords)

    K, R = scipy.linalg.rq(proj_matrix[:, :3])  # rq decomposition

    # enforce positive diagonal on K
    D = np.diag(np.sign(np.diag(K)))
    K = K @ D
    R = D @ R

    # scale projection matrix and calibration matrix to reflect real world scaling
    scale_factor = 1 / K[2][2]
    proj_matrix *= scale_factor
    K *= scale_factor

    # extract translation vector from P
    t = tuple(-np.linalg.inv(proj_matrix[:, :3]) @ proj_matrix[:, 3])

    return proj_matrix, K, R, t


def extract_intrinsics(K: CalMatrix) -> tuple[Vec2f, Vec2f]:
    """
    Extract principle point and focal lengths from calibration matrix
    """
    focal_lengths = (K[0][0], K[1][1])
    principal_point = (K[0][2], K[1][2])
    return principal_point, focal_lengths


def extract_orientation_zyx(R: RotMatrix) -> Vec3f:
    """
```

```
    Extract tait-bryan angles (zyx) from rotation matrix
    """
    return (
        np.degrees(np.arcsin(R[2][1] / sqrt(1 - (R[2][0])**2))),  # alpha (x rotation)
        np.degrees(np.arcsin(-R[2][0])),  # beta (y rotation)
        np.degrees(np.arcsin(R[1][0] / sqrt(1 - (R[2][0])**2))),  # gamma (z rotation)
    )
```

calicam/vecs.py

```
from math import sqrt

Vecf = tuple[float, ...]

Vec2f = tuple[float, float]
Vec3f = tuple[float, float, float]


def to_homogenous(vec: Vecf) -> Vecf:
    return (*vec, 1.0)


def to_inhomogenous(vec: Vecf) -> Vecf:
    return tuple(map(lambda v_i: v_i / vec[-1], vec[:-1]))


def euclidean(a: Vecf, b: Vecf) -> float:
    return sqrt(sum((b_i - a_i)**2 for a_i, b_i in zip(a, b)))
```