IB Math Analysis and Approaches (HL)

Extended Essay

May 2024 Session

Mathematical Techniques and Applications of Camera Calibration

Research Question: What mathematical techniques can be employed to develop highly accurate camera models, and what are their real-world applications where these models prove valuable?

Word Count: 1670 words

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1 Introduction

Camera calibration, also known as camera resectioning, is the process of determining the intrinsic and extrinsic parameters of a camera. The knowledge of the accurate values of these values parameters are essential, as it enables us to create a mathematical model which describes how a camera projects 3D points from a scene onto the 2D image it captures. The intrinsic parameters deal with the camera's internal characteristics, while the extrinsic parameters describe its position and orientation in the world. The importance of a well-calibrated camera becomes very apparent in photogrammetric applications, where precise measurements of 3-dimensional physical objects are derived from photographic images.

Photogrammetry, as a comprehensive science in its own right, concerns itself with obtaining accurate measurements of 3-dimensional physical objects through photographic imagery. Photogrammetry was first employed by Prussian architect Albrecht Meydenbauer in the 1860s, who used photogrammetric techniques to create some of the most detailed topographic plans and elevations drawings[1]. Today, photogrammetric techniques are used in a multitude of applications spanning diverse fields, including but not limited to: computer vision, topographical mapping, medical imaging, and forensic analysis.

While camera calibration is essential in ensuring the accuracy of photogrammetric applications, it itself also relies on these very same photogrammetric techniques in order to estimate these parameters. This underscores the essential relationship between photogrammetry and camera calibration. In essence, the developments of photogrammetry and camera calibration are closely intertwined, and this shows the importance of understanding and accurately determining a camera's intrinsic and extrinsic parameters for various applications.

1.1 Problem Statement

While manufacturers of cameras often report parameters of cameras, such as the nominal focal length and pixel sizes of their camera sensor, these figures are typically approximations which can vary from camera to camera, particularly in consumer-grade cameras. As such, the use of these estimates by manufacturers are unsuitable in developing camera models for applications requiring high accuracy. Combined with the potential for manufacturing defects as well as unknown lens distortion coefficients further necessitates the need for a reliable method for determining the parameters of a camera.

Camera calibration emerges as the answer to these problems, allowing us to create very accurate estimates for the parameters of a camera. As such, it is important that we ac

[1] Albertz, "A Look Back; 140 Years of Photogrammetry," 1.

Importance of reserach question

In order to accurately determine the position of 3D points based on data from multiple 2D images, we must have knowledge of the parameters of the camera.

This process of calculating

2 Approach

2.1 Camera Model

A camera model is a projection model that approximates the function of a camera by describing a mathematical relationship between points in 3D space and its projection onto the sensor grid of the camera. In order to accurately model a camera, we must first understand the general workings of a camera.

Modern lens cameras are highly sophisticated, built with an array of complex mechanisms and a wide range of features. The complexity of cameras can be better understood by breaking down their components down into three main elements critical to image projection: the lens, the aperture, and the sensor grid (CCD).

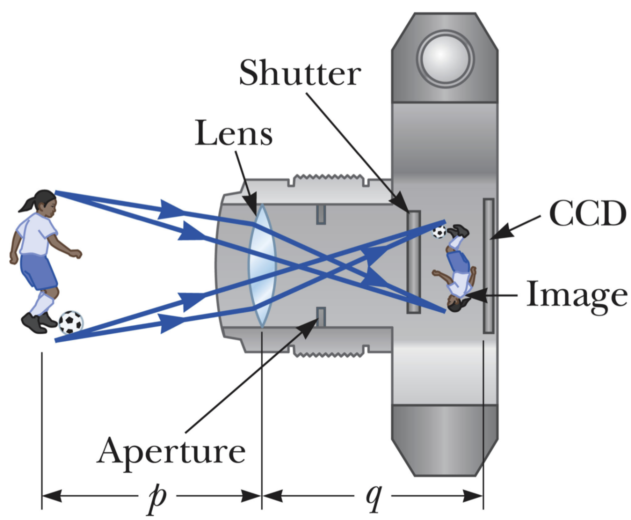

Figure 2.1: Lens camera. Adapted from Colton, "Warm-Up Exercise 30."

The lenses

However, we can simplify the model of the lens camera by collapsing the mechanisms of the camera into 3 main functional components that are important to the image projection: the lens, the aperture, and the sensor grid (CCD). This simplified model is visualized in

Figure 2.1. The lens focuses incoming light rays towards the aperture, before they project inverted onto the sensor grid. However, even this simplified model of a lens camera is too complex to model, as there is no simple mathematical equation which accurately describes the behavior of a lens. As such, we can further simplify our camera model by building upon the pinhole camera model, which is one of the simplest and most commonly used camera models in camera calibration.

A pinhole camera is a simple camera without a lens. Instead, it relies on the use of a tiny hole as the aperture of the camera, and light rays pass through the hole, projecting an inverted image onto the

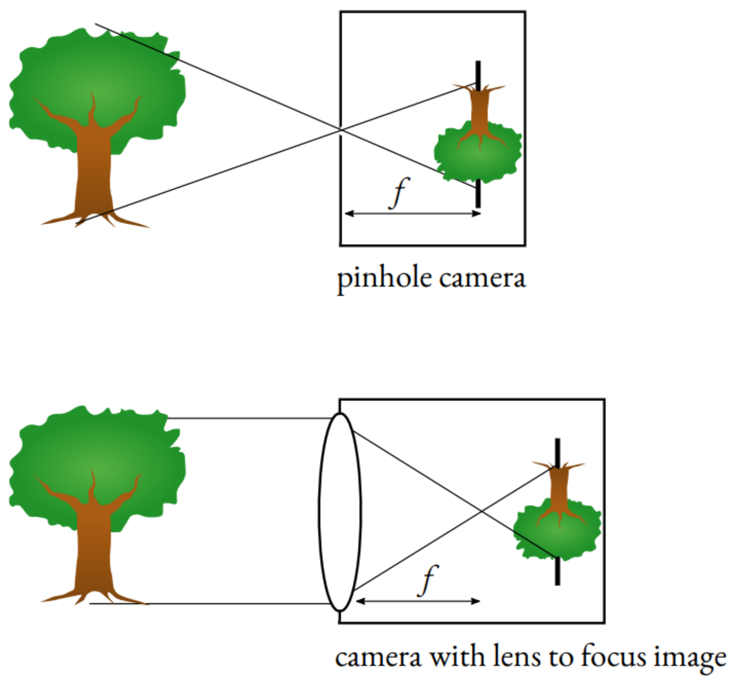

Figure 2.2: Difference between a pinhole camera and a lens camera. Adapted from Lê, "Camera Model: Intrinsic Parameters."

There are a few assumptions which are made by the pinhole camera model:

Extremely simple model for imaging geometry Doesn't strictly apply Mathematically convenient acceptable approximation.

- T

The pinhole camera model does not accurately describe the true workings of a camera, as some of the effects that the model fails to account for can be compensated the errors which results from these assumptions are sufficiently small to be neglected if a high quality camera is used. Additionally,

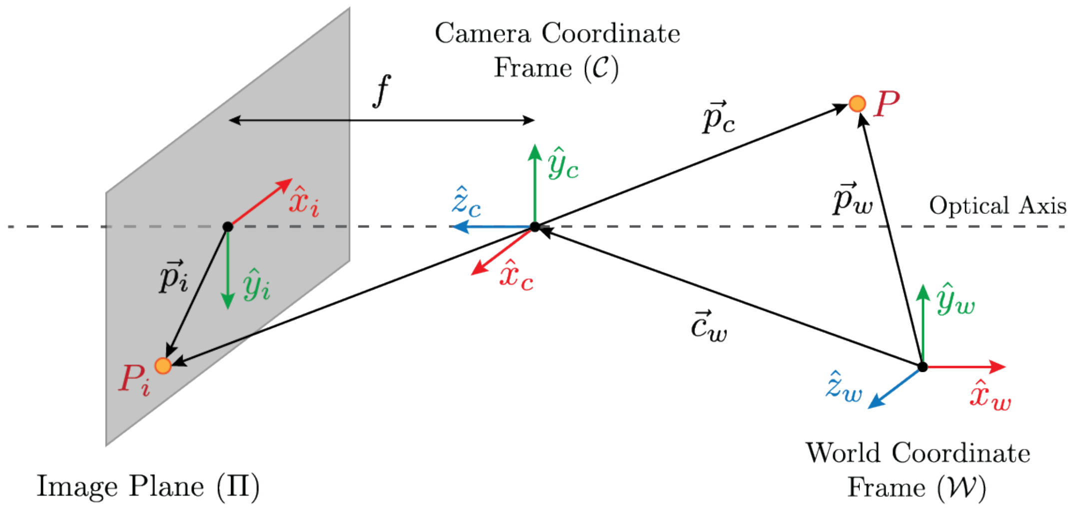

Figure 2.3: Pinhole camera model.

2.1.1 Geometry

For our camera model, we will establish 3 frames of

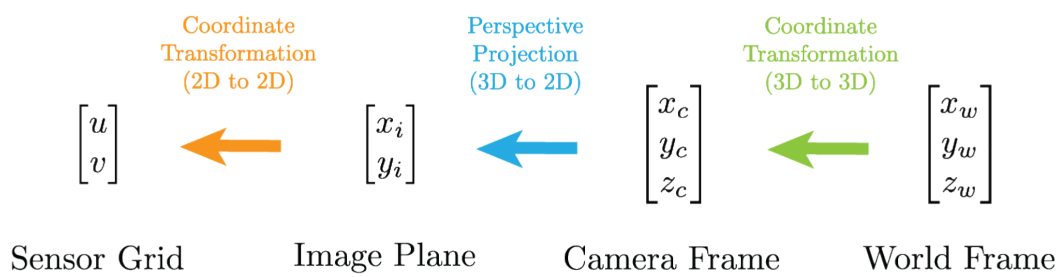

Figure 2.4: Coordinate remappings.

There are countless different approaches one could take to calibrate a camera,

however they all build upon techniques first described in multiple highly influential papers, most notably Tsai's "A Versatile Camera Calibration Technique for High-Accuracy 3D Machine Vision Metrology Using Off-the-shelf TV Cameras and Lenses" and Zhang's "A Flexible New Technique for Camera Calibration".

2.2 Calibration Object

Calibration techniques can be roughly separated into 3 categories, based on the dimension of the calibration object used[2]:

3 Prerequisites

3.1 Homogenous Coordinates

While Euclidean space describes 2D and 3D space well, they are not sufficient in describing perspective projections.

Homogenous coordinates (also known as projective coordinates)

When (u,v)

In other words, with homogenous coordinates, we interpret our *Euclidean* space as an *affine* space

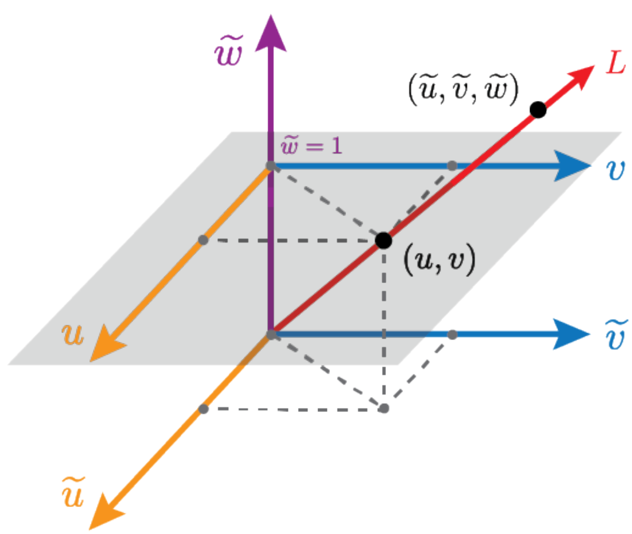

Figure 3.1: Homogenous coordinate system.

4 Coordinate Transformation

First, we

[2]Zhang, "Camera Calibration."

Now, we would like to find the extrinsic matrix, M_{ext}, which relates the positional vector $\vec{p}_w$ of point P in the world coordinate frame, to its positional vector $\vec{p}_c$ in the camera coordinate frame. Similar to what we did in section 5, we can express this in homogenous coordinates as follows:

$$\widetilde{p}_c = M_{ext}\,\widetilde{p}_w \tag{4.1}$$

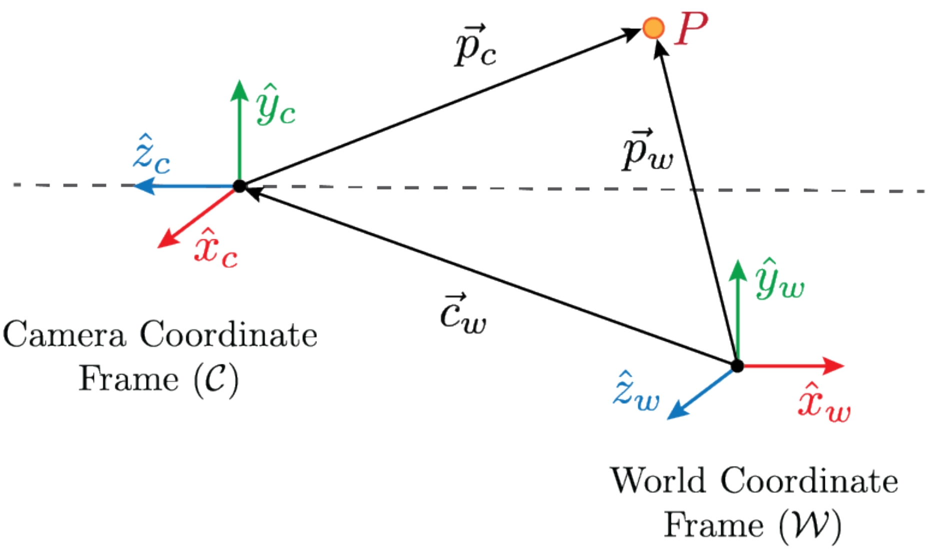

Figure 4.1: Coordinate transformation from the world coordinate frame to the camera frame.

4.1 Extrinsic Matrix

For the extrinsic parameters of the camera, we have the position $\vec{c}_w$ of the camera in world coordinates and orientation R of the camera. The orientation, R, is a 3x3 rotational matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{4.2}$$

where:

- Row 1: Direction of $\hat{x}_c$ in world coordinate frame.
- Row 2: Direction of $\hat{y}_c$ in world coordinate frame.
- Row 3: Direction of $\hat{z}_c$ in world coordinate frame.

$$\vec{p}_c = R(\vec{p}_w - \vec{c}_w) \tag{4.3a}$$

$$= R\vec{p}_w - R\vec{c}_w \tag{4.3b}$$

$$\vec{p}_c = R\vec{p}_w + \vec{t} \tag{4.4}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{R} \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}_{\vec{t}} \tag{4.5}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_{ext}} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} \tag{4.6}$$

$$M_{ext} = \begin{bmatrix} R_{3\times3} & t_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix} \tag{4.7}$$

5 Perspective Projection

Next, we would like to find the intrinsic matrix, M_{int}, which relates the positional vector $\vec{p}_c$ of point P in the camera coordinate frame, to its positional vector $\vec{p}_i$ on the image plane. Using $\widetilde{p}_c$ and $\widetilde{p}_i$ to represent the homogenous coordinates of the vectors $\vec{p}_c$ and $\vec{p}_i$ respectively, we can express this mathematically as follows:

$$\widetilde{p}_i = M_{ext}\,\widetilde{p}_c \tag{5.1}$$

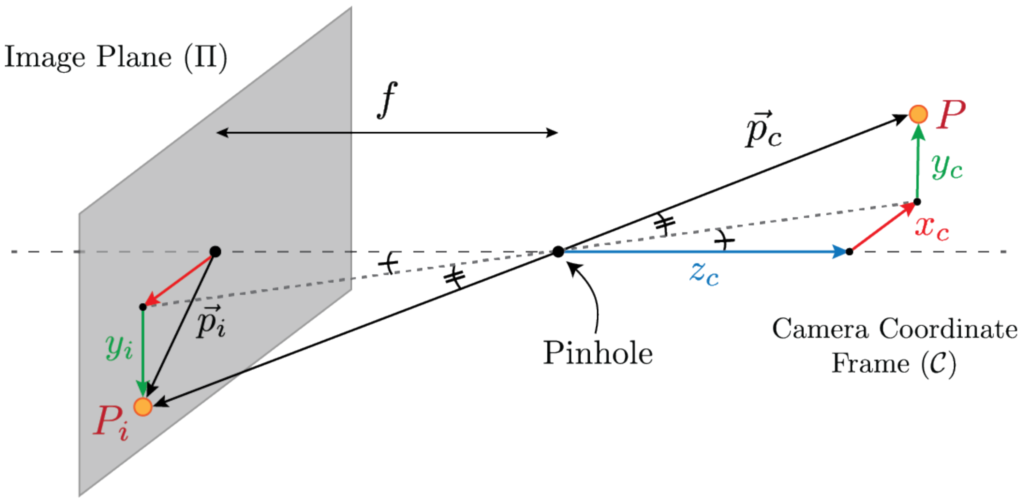

Figure 5.1: Perspective projection of the point onto the image plane Π.

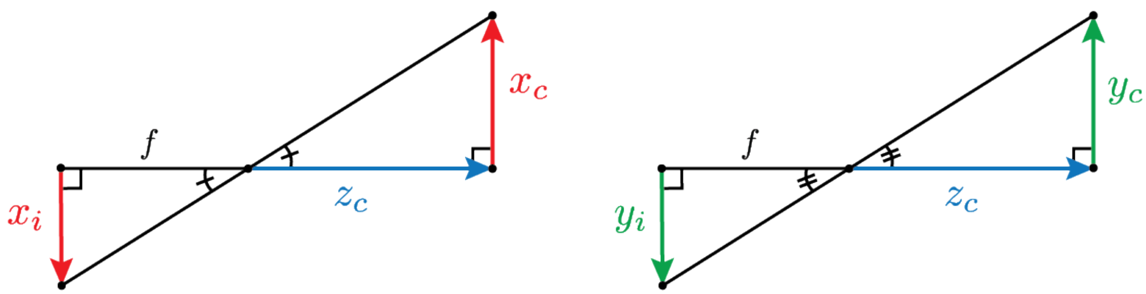

Figure 5.2: Similar triangles formed by perspective projection, which relate x_i to x_c and y_i to y_c

$$\frac{x_i}{f} = \frac{x_c}{z_c} \implies x_i = f\frac{x_c}{z_c} \tag{5.2a}$$

$$\frac{y_i}{f} = \frac{y_c}{z_c} \implies y_i = f\frac{y_c}{z_c} \tag{5.2b}$$

We can then relate the coordinates of the projection, (x_i, y_i), which are in real-world units, to its position (u, v) in pixels.

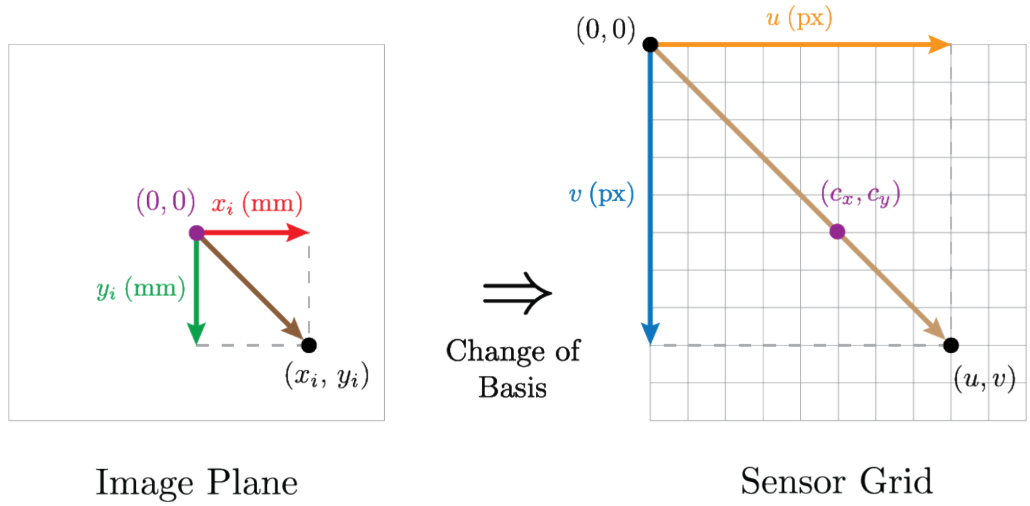

Figure 5.3: Conversion from image plane coordinates to sensor grid coordinates

Let m_x and m_y represent the pixel density of the image sensor in the x and y axes of the image sensor plane respectively.

$$u = m_x x_i + c_x$$

$$v = m_y y_i + c_y$$

Replacing x_i and y_i for the result we obtained from 5.2a and 5.2b, we get:

$$u = m_x f \frac{x_c}{z_c} + c_x$$

$$v = m_y f \frac{y_c}{z_c} + c_y$$

Since m_x, m_y, and f are all unknowns, we can combine the products $m_x f$ and $m_y f$ to f_x and f_y respectively. Under this new scheme, we define f_x and f_y as the horizontal and vertical focal lengths of camera.

$$u = f_x \frac{x_c}{z_c} + c_x \tag{5.3a}$$

$$v = f_y \frac{y_c}{z_c} + c_y \tag{5.3b}$$

5.1 Intrinsic Matrix

$$\begin{bmatrix} u \\ v \end{bmatrix} \sim \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{w} \end{bmatrix} \equiv \begin{bmatrix} z_c u \\ z_c v \\ z_c \end{bmatrix} = \begin{bmatrix} f_x x_c + z_c c_x \\ f_y y_c + z_c c_y \\ z_c \end{bmatrix} = \underbrace{\begin{bmatrix} f_x & 0 & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{M_{int}} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} \tag{5.4}$$

$$K = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \tag{5.5}$$

Note that K that is an *upper triangular matrix*. It is a special kind of square matrix with all of its non-zero entries above the main diagonal. This is an important property which we will exploit when extracting the intrinsic matrix from the projection matrix in section 6.

As such, we can express M_{int} as $[\,K \mid 0\,]$.

$$M_{int} = [\,K \mid 0\,] \tag{5.6}$$

6 Solving for the Projection Matrix

When we combine the equations $\widetilde{p_c} = M_{ext}\,\widetilde{p_w}$ (eq. 4.1) and $\widetilde{p_i} = M_{int}\,\widetilde{p_c}$ (eq. 5.1), we obtain

$$\widetilde{p_i} = M_{int}\,M_{ext}\,\widetilde{p_w} \tag{6.1}$$

To simplify our camera model, we can define a new matrix, $P \in \mathbb{R}^{3\times4}$, which is equal to the product $M_{int}\,M_{ext}$. Since M_{ext} is a 4×4 matrix and M_{int} is a 3×4 matrix, their matrix product produces a 3×4 matrix.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \equiv \underbrace{\begin{bmatrix} f_x & 0 & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{M_{int}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_{ext}} \tag{6.2}$$

Replacing P for $M_{int}\, M_{ext}$ in equation 6.1, we obtain

$$\widetilde{p}_i = P\,\widetilde{p}_w \tag{6.3}$$

The implications of this equation is very important, as it means that we can project the nth point $\left[x_w^{(n)}, y_w^{(n)}, z_w^{(n)}\right]^\mathsf{T}$ in the world coordinate frame $\mathcal{W}$ to its pixel coordinates $[u_n, v_n]^\mathsf{T}$ on the image plane Π simply by using the projection matrix. But now, we need to figure out a way to solve for the project matrix.

Given that we have equation 6.3 which relates

When expressing the pixel coordinate in homogenous coordinates, equation 6.3 becomes

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} \sim \begin{bmatrix} \widetilde{u}_n \\ \widetilde{v}_n \\ \widetilde{w}_n \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w^{(n)} \\ y_w^{(n)} \\ z_w^{(n)} \\ 1 \end{bmatrix} \tag{6.4}$$

$$\widetilde{u}_n = p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14} \tag{6.5}$$

$$\widetilde{v}_n = p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24} \tag{6.6}$$

$$\widetilde{w}_n = p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34} \tag{6.7}$$

$$u_n = \frac{\widetilde{u}_n}{\widetilde{w}_n} = \frac{p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14}}{p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}} \tag{6.8}$$

$$v_n = \frac{\widetilde{v}_n}{\widetilde{w}_n} = \frac{p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24}}{p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}} \tag{6.9}$$

$$u_n(p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}) = p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14} \tag{6.10}$$

$$v_n(p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}) = p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24} \tag{6.11}$$

$$0 = p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14} - p_{31}u_n x_w^{(n)} - p_{32}u_n y_w^{(n)} - p_{33}u_n z_w^{(n)} - p_{34}u_n \tag{6.12a}$$

$$0 = p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24} - p_{31}v_n x_w^{(n)} - p_{32}v_n y_w^{(n)} - p_{33}v_n z_w^{(n)} - p_{34}v_n \tag{6.12b}$$

$$\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} = \underbrace{\begin{bmatrix} x_w^{(1)} & y_w^{(1)} & z_w^{(1)} & 1 & 0 & 0 & 0 & 0 & -u_1x_w^{(1)} & -u_1y_w^{(1)} & -u_1z_w^{(1)} & -u_1 \\ 0 & 0 & 0 & 0 & x_w^{(1)} & y_w^{(1)} & z_w^{(1)} & 1 & -v_1x_w^{(1)} & -v_1y_w^{(1)} & -v_1z_w^{(1)} & -v_1 \\ & & \vdots & & & & & & & \vdots & & \\ x_w^{(n)} & y_w^{(n)} & z_w^{(n)} & 1 & 0 & 0 & 0 & 0 & -u_nx_w^{(n)} & -u_ny_w^{(n)} & -u_nz_w^{(n)} & -u_n \\ 0 & 0 & 0 & 0 & x_w^{(n)} & y_w^{(n)} & z_w^{(n)} & 1 & -v_nx_w^{(n)} & -v_ny_w^{(n)} & -v_nz_w^{(n)} & -v_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix}p_{11}\\p_{12}\\p_{13}\\p_{14}\\p_{21}\\p_{22}\\p_{23}\\p_{24}\\p_{31}\\p_{32}\\p_{33}\\p_{34}\end{bmatrix}}_{p} \tag{6.13}$$

homogenous linear system overdetermined

6.1 Constrained Least Squares Solution

We have now established a way to solve for the

Now, we need to solve for $Ap = 0$

$$\underset{p}{\text{minimize}} \ \|Ap\|^2 \ \text{subject to} \ \|p\|^2 = 1 \tag{6.14}$$

For a given arbitrary vector $v \in \mathbb{R}^n$, the magnitude of the vector, $\|v\|$, is equal to $\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. As such, we can rewrite the square of the magnitude of vector, $\|v\|^2$, as:

$$\|v\|^2 = v_1^2 + v_2^2 + \cdots v_n^2 = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v^\mathsf{T} v$$

where $v^\intercal$ is the transpose of vector v. Thus, in equation 6.14, we can replace $\|Ap\|^2$ with $p^\intercal A^\intercal Ap$ and $\|p\|^2$ for $p^\intercal p$ to obtain

$$\underset{p}{\text{minimize}} \;\; \left(p^\intercal A^\intercal Ap\right) \;\; \text{subject to} \;\; p^\intercal p = 1 \tag{6.15}$$

The Lagrangian[3] of equation 6.15 is

$$\mathcal{L}(p, \lambda) = p^\intercal A^\intercal Ap - \lambda\left(p^\intercal p - 1\right) \tag{6.16}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. Since p is minimized when $\mathcal{L}$ is minimized, we need to look for the absolute minimum of $\mathcal{L}$, which are located at its stationary points. To find these points, we find where all the partial derivatives of the Lagrangian are zero, i.e.

$$\frac{\partial \mathcal{L}}{\partial p} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

where ∂ is used to denote a partial derivative (see Appendix B). We will focus on the partial derivative of $\mathcal{L}$ with respect to p. Using product rule for partial derivatives, we obtain:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial p} = \frac{\partial}{\partial p}\left[p^\intercal A^\intercal Ap - \lambda\left(p^\intercal p - 1\right)\right] &\overset{\text{set}}{=} 0 \\ \Rightarrow 2A^\intercal Ap - 2\lambda p &= 0 \\ \Rightarrow A^\intercal Ap &= \lambda p \end{aligned} \tag{6.17}$$

which is an eigenvalue problem for $A^\intercal A$. Potential solutions for p are eigenvectors that satisfy equation 6.17,[4] with $\lambda \in \mathbb{R}$ as the eigenvalue. Since 6.15 is a minimization problem, the minimized eigenvector p is the one which has the smallest eigenvalue λ.[5]

[3] Ghojogh, Karray, and Crowley, "Eigenvalue and Generalized Eigenvalue Problems," 2.
[4] Nayar, *Linear Camera Model*.
[5] Ghojogh, Karray, and Crowley, "Eigenvalue and Generalized Eigenvalue Problems."

7 Extracting Parameters

Once we have solved for the projection for the projection matrix P, we can then extract the intrinsic and extrinsic parameters. We know that

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = \underbrace{\begin{bmatrix} f_x & 0 & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{M_{int}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_{ext}} \quad \text{(?? revisited)}$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = KR \tag{7.1}$$

$$\begin{bmatrix} p_{14} \\ p_{24} \\ p_{34} \end{bmatrix} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = K\vec{t} \tag{7.2}$$

Since K is in the form of an *upper right triangular matrix* and R is an *orthonormal matrix*, we can find unique solutions for K and R using a method called *RQ decomposition.*

7.1 RQ Decomposition

RQ decomposition is a technique which allows us to uniquely decompose a matrix A into a product $A = RQ$,

Since

7.2 Extracting Orientation as Angles

When constructing the extrinsic matrix in section 4, we defined In our rotation matrix R, we represent the

$$R \equiv R_z(\gamma)R_y(\beta)R_x(\alpha) \tag{7.3}$$

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{7.4a}$$

$$R_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & -\cos(\beta) \end{bmatrix} \tag{7.4b}$$

$$R_z(\gamma) = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.4c}$$

$$\begin{aligned} R &= \begin{bmatrix} 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & -\cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\beta)\cos(\gamma) & \sin(\alpha)\sin(\beta)\cos(\gamma) - \cos(\alpha)\sin(\gamma) & \cos(\alpha)\sin(\beta)\cos(\gamma) + \sin(\alpha)\cos(\gamma) \\ \cos(\beta)\sin(\gamma) & \sin(\alpha)\sin(\beta)\sin(\gamma) + \cos(\alpha)\cos(\gamma) & \cos(\alpha)\sin(\beta)\sin(\gamma) - \sin(\alpha)\cos(\gamma) \\ -\sin(\beta) & \sin(\alpha)\cos(\beta) & \cos(\alpha)\cos(\beta) \end{bmatrix} \end{aligned} \tag{7.5}$$

We have that

$$\begin{aligned} r_{31} &= -\sin(\beta) \\ \Rightarrow \beta &= \sin^{-1}(-r_{31}) \end{aligned} \tag{7.6}$$

$$r_{21} = \cos(\beta)\sin(\gamma)$$

$$\begin{aligned} \Rightarrow \gamma = \sin^{-1}\left(\frac{r_{21}}{\cos(\beta)}\right) &= \sin^{-1}\left(\frac{r_{21}}{\cos(\sin^{-1}(-r_{31}))}\right) \\ &= \sin^{-1}\left(\frac{r_{21}}{\sqrt{1 - r_{31}^2}}\right) \end{aligned} \tag{7.7}$$

$$r_{32} = \sin(\alpha)\cos(\beta)$$

$$\begin{aligned} \Rightarrow \alpha = \sin^{-1}\left(\frac{r_{32}}{\cos(\beta)}\right) &= \sin^{-1}\left(\frac{r_{32}}{\cos(\sin^{-1}(-r_{31}))}\right) \\ &= \sin^{-1}\left(\frac{r_{32}}{\sqrt{1-r_{31}^2}}\right) \end{aligned} \tag{7.8}$$

8 Experimental Validation

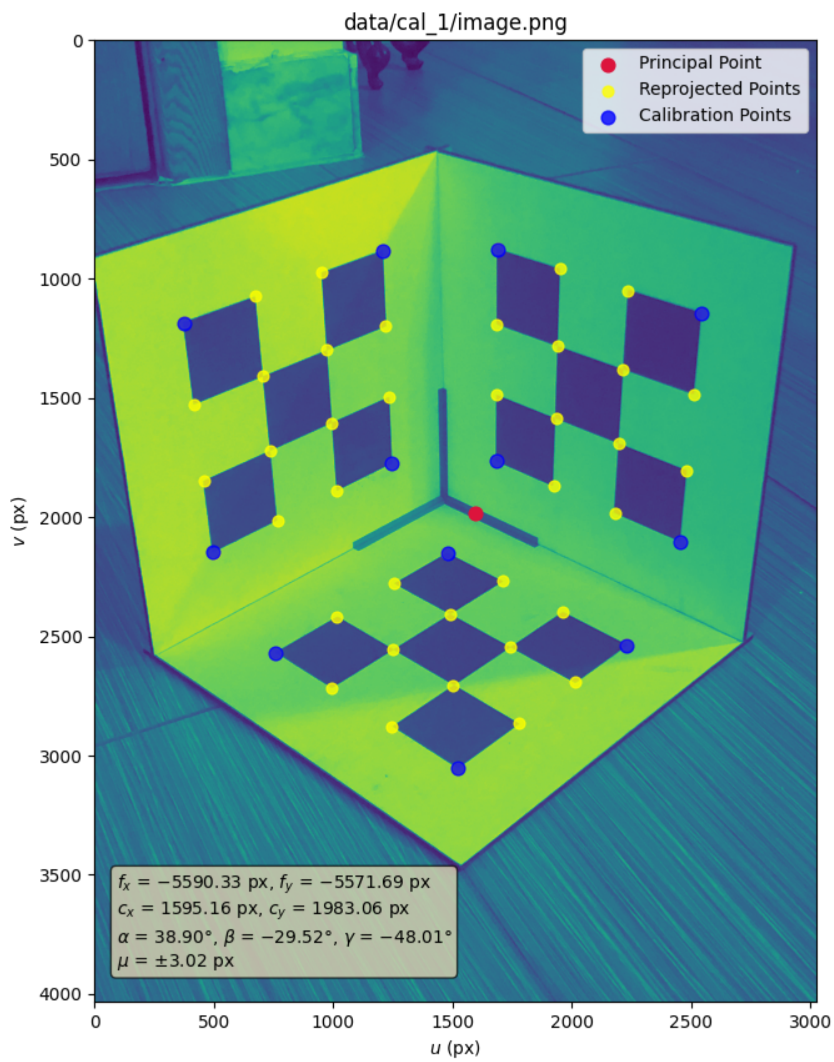

Figure 8.1: Graph displaying the

9 Applications

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Bibliography

Albertz, Joerg. "A Look Back; 140 Years of Photogrammetry." *Journal of the American Society for Photogrammetry and Remote Sensing* 73, no. 5 (May 2007): 504–506. Accessed September 9, 2023. https://www.asprs.org/wp-content/uploads/pers/2007journal/may/lookback.pdf.

Colton, John. "Physics 123 Lecture 30 Warm-up Questions." BYU Physics and Astronomy, November 5, 2012. Accessed October 17, 2023. https://physics.byu.edu/faculty/colton/docs/phy123-fall12/jitt30a.html.

Ghojogh, Benyamin, Fakhri Karray, and Mark Crowley. "Eigenvalue and Generalized Eigenvalue Problems: Tutorial." Comment: 8 pages, Tutorial paper. v2, v3: Added additional information. May 20, 2023. Accessed October 21, 2023. http://arxiv.org/abs/1903.11240. arXiv: 1903.11240 [cs, stat].

Lê, Hoàng-Ân. "Camera Model: Intrinsic Parameters." July 30, 2018. Accessed October 14, 2023. https://lhoangan.github.io/camera-params/.

Nayar, Shree, ed. *Linear Camera Model.* April 18, 2021. Accessed August 23, 2023. https://www.youtube.com/watch?v=qByYk6JggQU.

Zhang, Zhengyou. "Camera Calibration," May 2007. Accessed October 10, 2023. https://people.cs.rutgers.edu/elgammal/classes/cs534/lectures/CameraCalibration-book-chapter.pdf.

Appendix A Data

Table A.1: Data for the first image.

	World Coord (mm)			Pixel Coord (px)	
	x	y	z	u	v
XY Plane	40	40	0	1479	2151
	80	40	0	1250	2280
	120	40	0	1019	2419
	160	40	0	759	2568
	40	80	0	1710	2270
	80	80	0	1419	2407
	120	80	0	1252	2556
	160	80	0	994	2716
	40	120	0	1963	2400
	80	120	0	1742	2546
	120	120	0	1506	2706
	160	120	0	1250	2878
	40	160	0	2231	2537
	80	160	0	2014	2695
	120	160	0	1779	2868
	160	160	0	1535	3464
XZ Plane	40	0	40	1245	1774
	40	0	80	1233	1498
	40	0	120	1221	1201
	40	0	160	1207	883
	80	0	40	1015	1889
	80	0	80	995	1604
	80	0	120	973	1301
	80	0	160	952	975
	120	0	40	763	2013
	120	0	80	736	1720
	120	0	120	704	1410
	120	0	160	672	1076
	160	0	40	494	2148
	160	0	80	458	1848
	160	0	120	416	1527

	160	0	160	374	1186
YZ Plane	0	40	40	1683	1765
	0	80	40	1918	1868
	0	120	40	2176	1982
	0	160	40	2452	2102
	0	40	80	1684	1489
	0	80	80	1929	1587
	0	120	80	2194	1690
	0	160	80	2478	1804
	0	40	120	1685	1192
	0	80	120	1939	1283
	0	120	120	2213	1380
	0	160	120	2510	1487
	0	40	160	1689	878
	0	80	160	1948	960
	0	120	160	2236	1049
	0	160	160	2541	1144

Appendix B Partial Derivatives

Appendix C Lagrangian Method

Appendix D Eigenvalue and Eigenvector Problem

which states that for a given matrix $M \in \mathbb{R}^{n\times n}$, determine the eigenvector $x \in \mathbb{R}^n, x \neq 0$ and the eigenvalue $\lambda \in \mathbb{C}$ such that:

$$Mx = \lambda x$$