IB Math Analysis and Approaches (HL)

Extended Essay

May 2024 Session

Mathematical Techniques and Applications of Camera Calibration

Research Question: What mathematical techniques can be employed to develop highly accurate camera models, and what are their real-world applications where these models prove valuable?

Word Count: 2136 words

Contents

1 Introduction

Camera calibration, also known as camera resectioning, is the process of determining the intrinsic and extrinsic parameters of a camera. The knowledge of the accurate values of these values parameters are essential, as it enables us to create a mathematical model which describes how a camera projects 3D points from a scene onto the 2D image it captures. The intrinsic parameters deal with the camera's internal characteristics, while the extrinsic parameters describe its position and orientation in the world. The importance of a well-calibrated camera becomes very apparent in photogrammetric applications, where precise measurements of 3-dimensional physical objects are derived from photographic images.

Photogrammetry, as a comprehensive science in its own right, concerns itself with obtaining accurate measurements of 3-dimensional physical objects through photographic imagery. Photogrammetry was first employed by Prussian architect Albrecht Meydenbauer in the 1860s, who used photogrammetric techniques to create some of the most detailed topographic plans and elevations drawings[1]. Today, photogrammetric techniques are used in a multitude of applications spanning diverse fields, including but not limited to: computer vision, topographical mapping, medical imaging, and forensic analysis.

While camera calibration is essential in ensuring the accuracy of photogrammetric applications, it itself also relies on these very same photogrammetric techniques in order to estimate these parameters. This underscores the essential relationship between photogrammetry and camera calibration. In essence, the developments of photogrammetry and camera calibration are closely intertwined, and this shows the importance of understanding and accurately determining a camera's intrinsic and extrinsic parameters for various applications.

1.1 Problem Statement

While manufacturers of cameras often report parameters of cameras, such as the nominal focal length and pixel sizes of their camera sensor, these figures are typically approximations which can vary from camera to camera, particularly in consumer-grade cameras. As such, the use of these estimates by manufacturers are unsuitable in developing camera models for

[1] Albertz, "A Look Back; 140 Years of Photogrammetry," 1.

applications requiring high accuracy. Combined with the potential for manufacturing defects as well as unknown lens distortion coefficients further necessitates the need for a reliable method for determining the parameters of a camera.

Camera calibration emerges as the answer to these problems, allowing us to create very accurate estimates for the parameters of a camera. As such, it is important that we understand the mathematical techniques that can be used to construct camera models, and how these models can be applied in different real-world applications.

2 Approach

There are countless different approaches one could take to calibrate a camera,

however they all build upon techniques first described in multiple highly influential papers, most notably Tsai's "A Versatile Camera Calibration Technique for High-Accuracy 3D Machine Vision Metrology Using Off-the-shelf TV Cameras and Lenses" and Zhang's "A Flexible New Technique for Camera Calibration".

2.1 Camera Model

A camera model is a projection model which approximates the function of a camera by describing a mathematical relationship between points in 3D space and its projection onto the sensor grid of the camera. In order to construct such a model, we must first understand the general workings of a camera.

The modern lens Camera is highly sophisticated, built with an array of complex mechanisms and a wide range of features. The complexity of cameras can be better understood simplifying the lens camera into three main elements critical to image projection: the lens, the aperture, and the sensor grid (CCD).

- Lens – Focuses incoming light rays and projects it onto the sensor grid. Modern cameras have compound lenses (lenses made up of several lens elements) in order to minimize undesired effects such as aberration, blurriness, and distortion.

- Aperture – Controls the amount of light that reaches the sensor. By adjusting the aperture size, the exposure and depth of field can be modified.
- Sensor Grid – Captures the projected image created by the lens and the aperture.

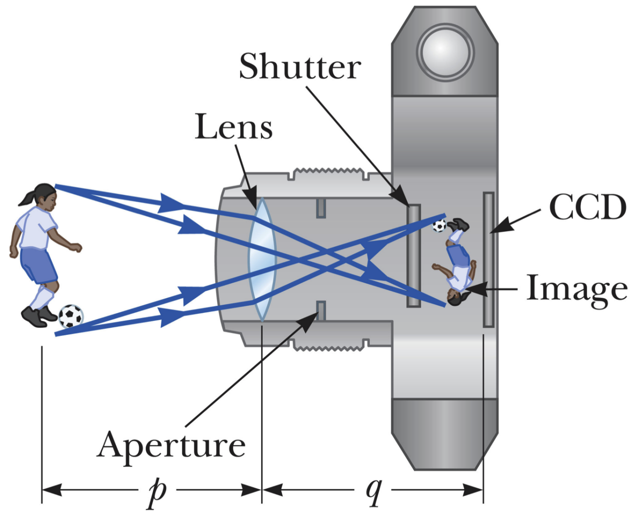

Figure 2.1: Lens camera. Adapted from Colton, "Warm-Up Exercise 30."

However, even a simplified model of a lens camera is still too complicated to describe, as it is impossible to encapsulate the complex behavior of a lens using a simple mathematical equation. As such, we need to sacrifice some precision by find a simpler model which sufficiently describes the behavior of a lens camera.

2.1.1 Pinhole Camera Model

A pinhole camera is a simple camera without a lens. Instead, it relies on the use of a tiny hole as the aperture of the camera, and light rays pass through the hole, projecting an inverted image onto the image plane. The pinhole camera model is based on the pinhole camera, however it goes further by making a few important assumptions for the purpose of simplification:

1. **The aperture is infinitely small** – This means that any incoming light ray can only travel straight through the pinhole, and that a point in space can only map to one single point on image plane. This allows us to establish a relationship between a 3D point and its 2D projection onto the image plane.

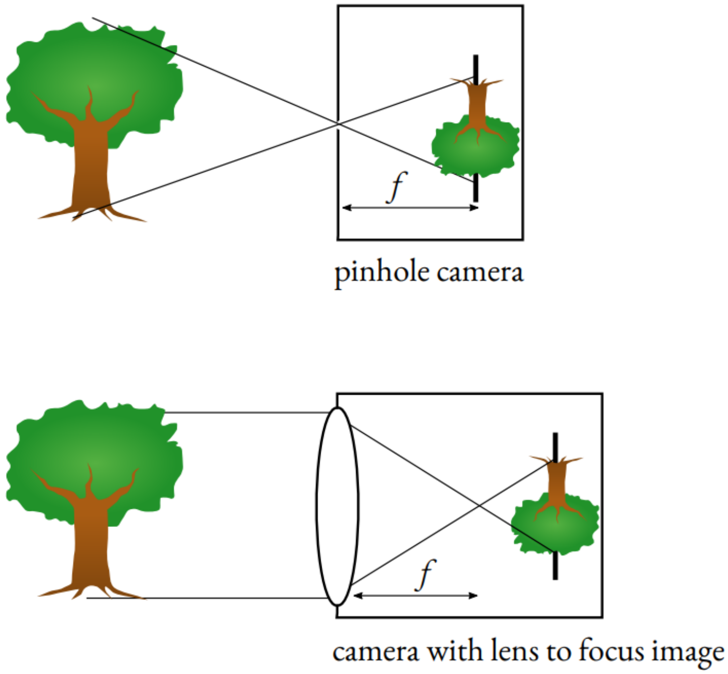

Figure 2.2: Difference between a pinhole camera and a lens camera. Adapted from Lê, "Camera Model: Intrinsic Parameters."

2. **Infinite depth of field** –

Extremely simple model for imaging geometry Doesn't strictly apply Mathematically convenient acceptable approximation.

While the pinhole camera model is an extremely basic model that does not accurately describe the true behavior of cameras, it is extremely simple. As such, for the sake of simplifying camera models, the pinhole camera model is often used. This model is one of the most basic and frequently employed camera models in the field of camera calibration.

2.2 Calibration Object

Calibration techniques can be roughly separated into 3 categories, based on the dimension of the calibration object used[2]:

- **3D** –
- **2D** –

[2]Zhang, "Camera Calibration."

- **1D** –

3 Prerequisites

Given the that this paper utilizes mathematical concepts beyond the scope of the IB Mathematics Analysis and Approaches HL curriculum, it is imperative that some notation and important ideas are introduced prior.

3.1 Notation

Vectors and Matrices. In this paper, lowercase letters are used to denote vectors, whereas capital letters are used for matrices. Depending on the context, vectors can also be attached with diacritics:

- $\vec{v}$ – a letter with an arrow above it denotes a positional vector or translational vector dealing with the transformation of points in space.
- $\widetilde{v}$ – a letter with a tilde above it denotes a vector represented in homogenous coordinates. This idea is explained in section 3.2.
- v – when explicitly stated, a letter without diacritics can also denote a vector if it does not fall into the categories listed above.

Transpose of Vectors and Matrices. The transpose of a vector or a matrix is an operation whereby the rows and columns of the vector or matrix are inverted, and this is denoted using the notation $v^\intercal$ or $M^\intercal$. For example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\intercal = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

L2 Norm. The L2 norm, or the magnitude of a vector, is the Euclidean distance of the tail of a vector to its tip.

3.2 Homogenous Coordinates

While Euclidean space describes 2D and 3D space well, they are not sufficient in describing perspective projections, as it is unable to fully the capture the relationships inherent in projective projections and affine transformations, both of which are core concepts in this paper.

Homogenous coordinates forms the basis of projective geometry, because it unifies the treatment of common graphical transformations such as rotation and translations[3].

Given a point in with $\mathbb{R}^n$ coordinates $(a_1, a_2, \cdots, a_n)$

When

Given the vector $[\,x, y\,]^\mathsf{T} \in \mathbb{R}^2$, we can express it in terms of homogenous coordinates:

$$\begin{bmatrix} x \\ y \end{bmatrix} \tag{3.1}$$

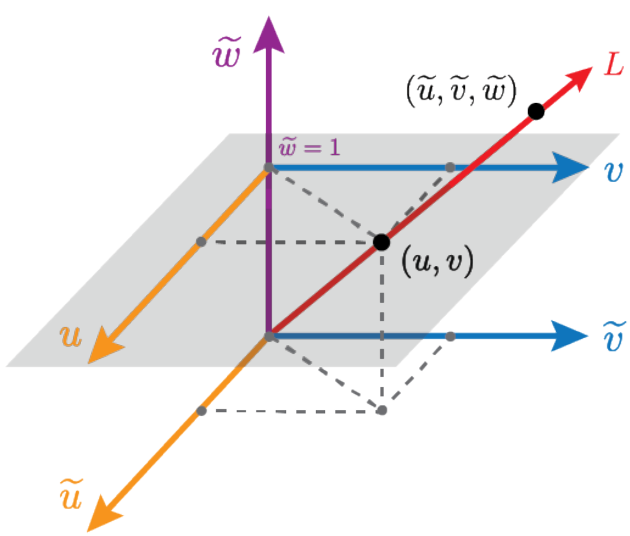

Figure 3.1: Homogenous coordinate system.

[3]Bloomenthal and Rokne, "Homogeneous Coordinates," 1.

In other words, with homogenous coordinates, we interpret our *Euclidean* space as an *affine* space

4 Constructing the Pinhole Camera Model

4.1 Nomenclature

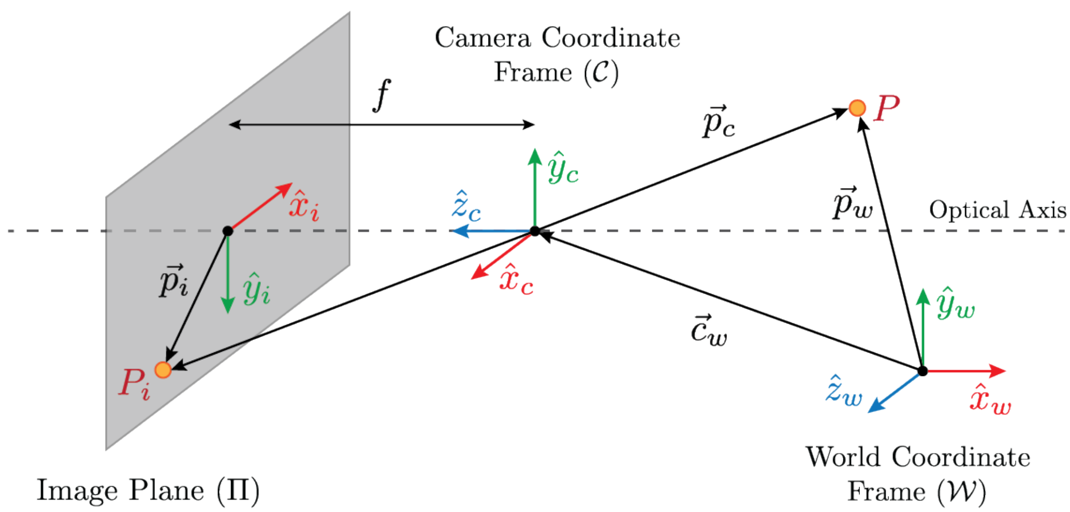

Figure 4.1: Pinhole camera model.

For our camera model, we will isntroduce 4 different coordinate systems:

- **The World Coordinate Frame $\mathcal{W}$**. Points are denoted as (x_w, y_w, z_w)
- **The Camera Coordinate Frame $\mathcal{C}$**. Points are denoted as (x_c, y_c, z_c).
- **The Image Plane Π**. Points are denoted as (x_i, y_i).
- **The Sensor Grid**. Points are denoted as (u, v).

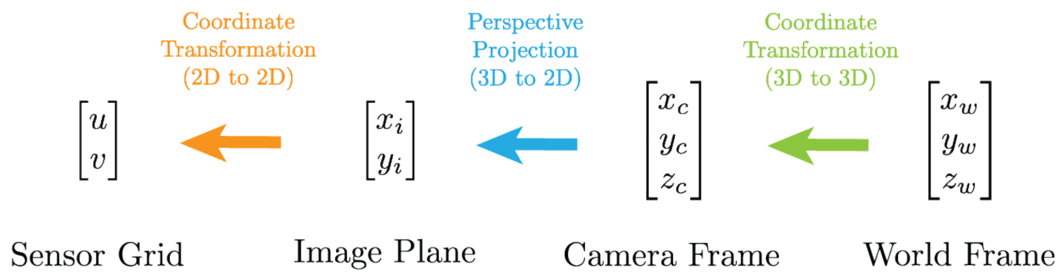

Figure 4.2: Coordinate remappings.

4.2 Intrinsic Parameters

First, we will focus on the projection of points in the 3D space onto the image plane. The goal is to construct a calibration matrix, K, which relates the position of the point P to its projection on the image plane. This can be expressed mathematically as follows:

$$\widetilde{p}_i = K\widetilde{p}_c \tag{4.1}$$

where $\vec{p}_i$ and $\vec{p}_c$ represents the position of P in the image plane Π and the camera frame $\mathcal{C}$ respectively.

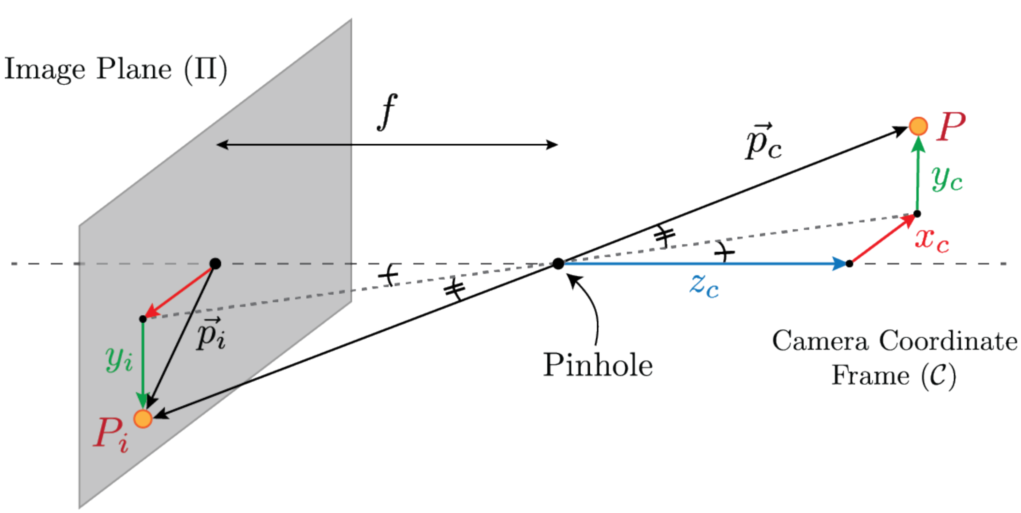

Figure 4.3: Perspective projection of the point P onto the image plane Π.

When a straight line is drawn from P to its projection P_i through the aperture, it intersects the optical axis. Deconstructing this intersection in the x and y direction, pairs of similar triangles are formed on the x and y plane. This is visualized in Figure 4.4.

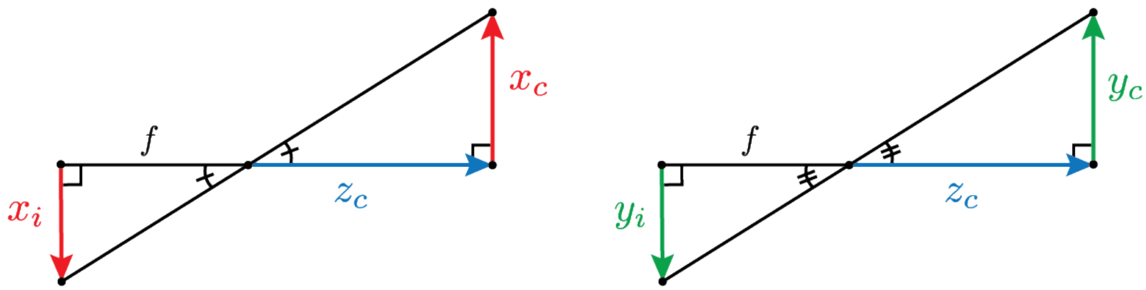

Figure 4.4: Similar triangles formed by perspective projection, which relate x_i to x_c and y_i to y_c

$$\frac{x_i}{f} = \frac{x_c}{z_c} \implies x_i = f\frac{x_c}{z_c} \tag{4.2a}$$

$$\frac{y_i}{f} = \frac{y_c}{z_c} \implies y_i = f\frac{y_c}{z_c} \tag{4.2b}$$

We can then relate the coordinates of the projection, (x_i, y_i), which are in real-world units, to its position (u, v) in pixels.

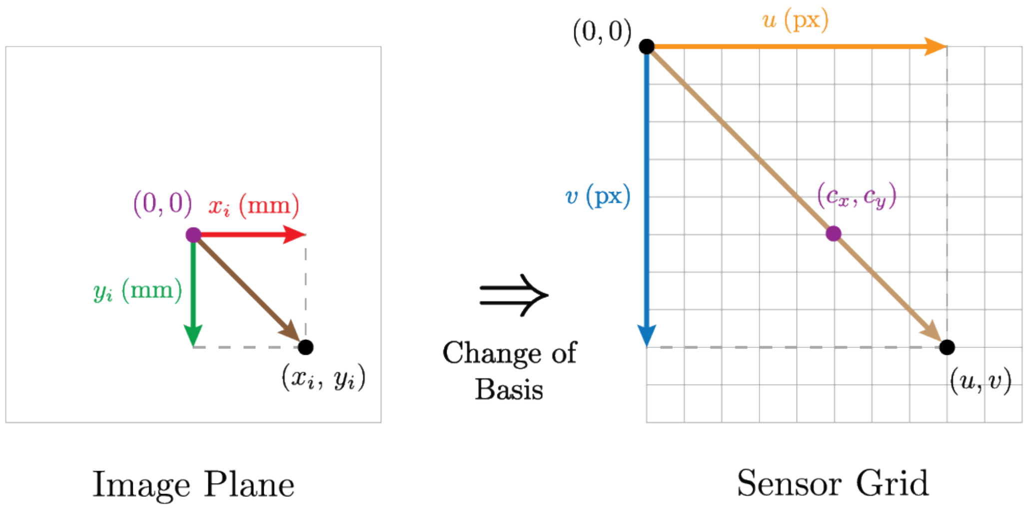

Figure 4.5: Conversion from image plane coordinates to sensor grid coordinates

Let m_x and m_y represent the pixel density of the image sensor in the x and y axes of the image sensor plane respectively.

$$u = m_x x_i + c_x$$

$$v = m_y y_i + c_y$$

Replacing x_i and y_i for the result we obtained from 4.2a and 4.2b, we get:

$$u = m_x f \frac{x_c}{z_c} + c_x$$

$$v = m_y f \frac{y_c}{z_c} + c_y$$

Since m_x, m_y, and f are all unknowns, we can combine the products $m_x f$ and $m_y f$ to f_x and f_y respectively. Under this new scheme, we define f_x and f_y as the horizontal and vertical focal lengths of camera.

$$u = f_x \frac{x_c}{z_c} + c_x \tag{4.3a}$$

$$v = f_y \frac{y_c}{z_c} + c_y \tag{4.3b}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \sim \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{w} \end{bmatrix} \equiv \begin{bmatrix} z_c u \\ z_c v \\ z_c \end{bmatrix} = \begin{bmatrix} f_x x_c + z_c c_x \\ f_y y_c + z_c c_y \\ z_c \end{bmatrix} = \underbrace{\begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}}_{K} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \tag{4.4}$$

$$K = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \tag{4.5}$$

Note that K that is an *upper triangular matrix*. It is a special kind of square matrix with all of its non-zero entries above the main diagonal. This is an important property which we will exploit when extracting the intrinsic matrix from the projection matrix in section 5.

As such, we can express M_{int} as $[\,K \mid 0\,]$.

$$M_{int} = [\,K \mid 0\,] \tag{4.6}$$

4.3 Extrinsic Parameters

Next, we will focus on finding the position

Now, we would like to find the extrinsic matrix, M_{ext}, which relates the positional vector $\vec{p}_w$ of point P in the world coordinate frame, to its positional vector $\vec{p}_c$ in the camera coordinate frame. Similar to what we did in section 4.2, we can express this in homogenous coordinates as follows:

$$\widetilde{p}_c = M_{ext}\, \widetilde{p}_w \tag{4.7}$$

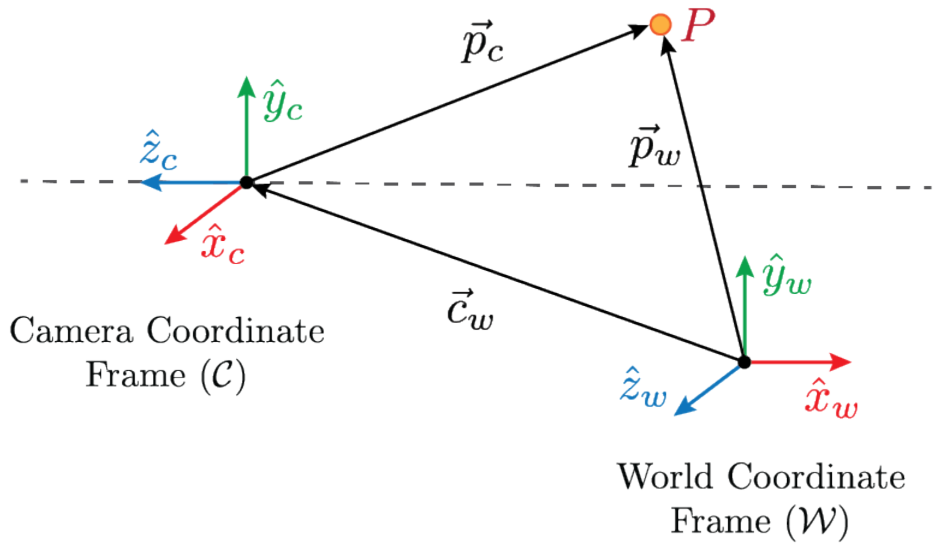

Figure 4.6: Coordinate transformation from the world coordinate frame to the camera frame.

For the extrinsic parameters of the camera, we have the position $\vec{c}_w$ of the camera in world coordinates and orientation R of the camera. The orientation, R, is a 3x3 rotational matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{4.8}$$

where:

- Row 1: Direction of $\hat{x}_c$ in world coordinate frame.
- Row 2: Direction of $\hat{y}_c$ in world coordinate frame.
- Row 3: Direction of $\hat{z}_c$ in world coordinate frame.

$$\vec{p}_c = R(\vec{p}_w - \vec{c}_w) \tag{4.9a}$$

$$= R\vec{p}_w - R\vec{c}_w \tag{4.9b}$$

$$\vec{p}_c = R\vec{p}_w + \vec{t} \tag{4.10}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{R} \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}_{\vec{t}} \tag{4.11}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_{ext}} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} \tag{4.12}$$

$$M_{ext} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \tag{4.13}$$

4.4 Putting It All Together

When we combine the equations $\widetilde{p}_c = M_{ext}\,\widetilde{p}_w$ (eq. 4.7) and $\widetilde{p}_i = M_{int}\,\widetilde{p}_c$ (eq. 4.1), we obtain

$$\widetilde{p}_i = M_{int}\,M_{ext}\,\widetilde{p}_w \tag{4.14}$$

To simplify our camera model, we can define a new matrix, $P \in \mathbb{R}^{3\times 4}$, which is equal to the product $M_{int}\,M_{ext}$. Since M_{ext} is a 4×4 matrix and M_{int} is a 3×4 matrix, their matrix product produces a 3×4 matrix.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \equiv \underbrace{\begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix}}_{\left[\, R \,|\, \vec{t} \,\right]} \tag{4.15}$$

Replacing P for $M_{int}\, M_{ext}$ in equation 4.14, we obtain

$$\widetilde{p}_i = P\, \widetilde{p}_w \tag{4.16}$$

The implications of this equation is very important, as it means that we can project the nth point $\left[x_w^{(n)}, y_w^{(n)}, z_w^{(n)}\right]^\mathsf{T}$ in the world coordinate frame $\mathcal{W}$ to its pixel coordinates $[u_n, v_n]^\mathsf{T}$ on the image plane Π simply by using the projection matrix. But now, we need to figure out a way to solve for the project matrix.

Given that we have equation 4.16 which relates

When expressing the pixel coordinate in homogenous coordinates, equation 4.16 becomes

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} \sim \begin{bmatrix} \widetilde{u}_n \\ \widetilde{v}_n \\ \widetilde{w}_n \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} x_w^{(n)} \\ y_w^{(n)} \\ z_w^{(n)} \\ 1 \end{bmatrix} \tag{4.17}$$

5 Camera Calibration

5.1 Overall Strategy

5.2 Solving for the Projection Matrix

$$\widetilde{u}_n = p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14}$$

$$\widetilde{v}_n = p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24}$$

$$\widetilde{w}_n = p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}$$

$$u_n = \frac{\widetilde{u}_n}{\widetilde{w}_n} = \frac{p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14}}{p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}}$$

$$v_n = \frac{\widetilde{v}_n}{\widetilde{w}_n} = \frac{p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24}}{p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}}$$

$$u_n(p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}) = p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14}$$

$$v_n(p_{31}x_w^{(n)} + p_{32}y_w^{(n)} + p_{33}z_w^{(n)} + p_{34}) = p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24}$$

$$0 = p_{11}x_w^{(n)} + p_{12}y_w^{(n)} + p_{13}z_w^{(n)} + p_{14} - p_{31}u_nx_w^{(n)} - p_{32}u_ny_w^{(n)} - p_{33}u_nz_w^{(n)} - p_{34}u_n \tag{5.1a}$$

$$0 = p_{21}x_w^{(n)} + p_{22}y_w^{(n)} + p_{23}z_w^{(n)} + p_{24} - p_{31}v_nx_w^{(n)} - p_{32}v_ny_w^{(n)} - p_{33}v_nz_w^{(n)} - p_{34}v_n \tag{5.1b}$$

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} x_w^{(1)} & y_w^{(1)} & z_w^{(1)} & 1 & 0 & 0 & 0 & 0 & -u_1 x_w^{(1)} & -u_1 y_w^{(1)} & -u_1 z_w^{(1)} & -u_1 \\ 0 & 0 & 0 & 0 & x_w^{(1)} & y_w^{(1)} & z_w^{(1)} & 1 & -v_1 x_w^{(1)} & -v_1 y_w^{(1)} & -v_1 z_w^{(1)} & -v_1 \\ & & \vdots & & & & & & & \vdots & & \\ x_w^{(n)} & y_w^{(n)} & z_w^{(n)} & 1 & 0 & 0 & 0 & 0 & -u_n x_w^{(n)} & -u_n y_w^{(n)} & -u_n z_w^{(n)} & -u_n \\ 0 & 0 & 0 & 0 & x_w^{(n)} & y_w^{(n)} & z_w^{(n)} & 1 & -v_n x_w^{(n)} & -v_n y_w^{(n)} & -v_n z_w^{(n)} & -v_n \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix}}_{p} \tag{5.2}
$$

homogenous linear system overdetermined

5.3 Constrained Least Squares Solution

We have now established a way to solve for the

Now, we need to solve for $Gp = 0$

$$
\underset{p}{\text{minimize}} \ \|Gp\|^2 \ \text{subject to} \ \|p\|^2 = 1 \tag{5.3}
$$

For a given arbitrary vector $v \in \mathbb{R}^n$, the magnitude is equal to $\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. As such, we can rewrite the square of the magnitude of v, $\|v\|^2$, as:

$$
\|v\|^2 = v_1^2 + v_2^2 + \cdots v_n^2 = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v^\mathsf{T} v
$$

Thus, in equation 5.3, we can replace $\|Gp\|^2$ with $p^\mathsf{T} A^\mathsf{T} Gp$ and $\|p\|^2$ for $p^\mathsf{T} p$ to obtain

$$\underset{p}{\text{minimize}} \;\; \left(p^\mathsf{T} G^\mathsf{T} Gp\right) \;\; \text{subject to} \;\; p^\mathsf{T} p = 1 \tag{5.4}$$

The Lagrangian[4] of equation 5.4 is

$$\mathcal{L}(p, \lambda) = p^\mathsf{T} G^\mathsf{T} Gp - \lambda \left(p^\mathsf{T} p - 1\right) \tag{5.5}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. Since p is minimized when $\mathcal{L}$ is minimized, we need to look for the absolute minimum of $\mathcal{L}$, which are located at its critical points. To find these points, we want to look for values of p and λ where all partial derivatives of the Lagrangian are zero, i.e.

$$\frac{\partial \mathcal{L}}{\partial p} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

where ∂ is used to denote a partial derivative (see Appendix **??**). We will focus on the partial derivative of $\mathcal{L}$ with respect to p. Using product rule for partial derivatives, we obtain:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial p} = \frac{\partial}{\partial p} \left[p^\mathsf{T} G^\mathsf{T} Gp - \lambda \left(p^\mathsf{T} p - 1\right)\right] \overset{\text{set}}{=} 0 \\ \Rightarrow 2G^\mathsf{T} Gp - 2\lambda p = 0 \\ \Rightarrow G^\mathsf{T} Gp = \lambda p \end{aligned} \tag{5.6}$$

which is an eigenvalue problem for $G^\mathsf{T} G$. Potential solutions for p are eigenvectors that satisfy equation 5.6,[5] with $\lambda \in \mathbb{R}$ as the eigenvalue. Since 5.4 is a minimization problem, the minimized eigenvector p is the one which has the smallest eigenvalue λ.[6]

which states that for a given matrix $M \in \mathbb{R}^{n \times n}$, determine the eigenvector $x \in \mathbb{R}^n, x \neq 0$ and the eigenvalue $\lambda \in \mathbb{C}$ such that:

[4] Ghojogh, Karray, and Crowley, "Eigenvalue and Generalized Eigenvalue Problems," 2.
[5] Nayar, *Linear Camera Model*.
[6] Ghojogh, Karray, and Crowley, "Eigenvalue and Generalized Eigenvalue Problems."

6 Extracting Parameters

Once we have solved for the projection for the projection matrix P, we can then extract the intrinsic and extrinsic parameters. We know that

$$\begin{aligned} P &= K \left[R \,|\, \vec{t}\, \right] \\ &= K \left[R \,|\, -R\vec{c_w} \right] \\ &= \left[KR \,|\, -KR\vec{c_w} \right] \end{aligned} \tag{6.1}$$

$$P = \left[Q \,|\, -Q\vec{c_w} \right] \tag{6.2}$$

$$Q = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{R}$$

Since K is in the form of an *upper right triangular matrix* and R is an *orthonormal matrix*, we can find unique solutions for K and R using a method called *RQ decomposition.*

6.1 RQ Decomposition

RQ decomposition is a technique which allows us to uniquely decompose a matrix A into a product $A = RQ$,

Since

6.2 Extracting the Translation Vector

$$
\begin{aligned}
-Q\vec{c_w} &= \begin{bmatrix} p_{14} \\ p_{24} \\ p_{34} \end{bmatrix} \\
\Rightarrow \vec{c_w} &= -Q^{-1} \begin{bmatrix} p_{14} \\ p_{24} \\ p_{34} \end{bmatrix}
\end{aligned} \tag{6.3}
$$

6.3 Extracting Orientation as Angles

When constructing the extrinsic matrix in section 4.3, we defined the rotation matrix as the

We can represent the rotation in terms of *Tait-Bryan Angles*

$$R \equiv R_z(\gamma)R_y(\beta)R_x(\alpha) \tag{6.4}$$

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{6.5a}$$

$$R_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & -\cos(\beta) \end{bmatrix} \tag{6.5b}$$

$$R_z(\gamma) = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.5c}$$

$$R = \begin{bmatrix} 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & -\cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\beta)\ \cos(\gamma) & \sin(\alpha)\ \sin(\beta)\ \cos(\gamma) - \cos(\alpha)\ \sin(\gamma) & \cos(\alpha)\ \sin(\beta)\ \cos(\gamma) + \sin(\alpha)\ \cos(\gamma) \\ \cos(\beta)\ \sin(\gamma) & \sin(\alpha)\ \sin(\beta)\ \sin(\gamma) + \cos(\alpha)\ \cos(\gamma) & \cos(\alpha)\ \sin(\beta)\ \sin(\gamma) - \sin(\alpha)\ \cos(\gamma) \\ -\sin(\beta) & \sin(\alpha)\ \cos(\beta) & \cos(\alpha)\ \cos(\beta) \end{bmatrix} \tag{6.6}$$

We have that

$$r_{31} = -\sin(\beta)$$

$$\Rightarrow \beta = \sin^{-1}(-r_{31}) \tag{6.7}$$

$$r_{32} = \sin(\alpha)\cos(\beta)$$

$$\begin{aligned} \Rightarrow \alpha &= \sin^{-1}\left(\frac{r_{32}}{\cos(\beta)}\right) = \sin^{-1}\left(\frac{r_{32}}{\cos(\sin^{-1}(-r_{31}))}\right) \\ &= \sin^{-1}\left(\frac{r_{32}}{\sqrt{1-r_{31}^2}}\right) \end{aligned} \tag{6.8}$$

$$r_{21} = \cos(\beta)\sin(\gamma)$$

$$\begin{aligned} \Rightarrow \gamma &= \sin^{-1}\left(\frac{r_{21}}{\cos(\beta)}\right) = \sin^{-1}\left(\frac{r_{21}}{\cos(\sin^{-1}(-r_{31}))}\right) \\ &= \sin^{-1}\left(\frac{r_{21}}{\sqrt{1-r_{31}^2}}\right) \end{aligned} \tag{6.9}$$

7 Experimental Validation

In an attempt to show that the model works, I created the program

Figure 7.1: Photograph 1. The photo editing software *GIMP* was used for edge detection, and the coordinates of the calibration points were selected manually.

Figure 7.2: Graph produced by `Matplotlib` which displays the results of the trial

$$P = \begin{bmatrix} -2.5844 \times 10^{-3} & 1.7334 \times 10^{-3} & -4.6719 \times 10^{-4} & 6.0581 \times 10^{-1} \\ 4.8240 \times 10^{-4} & 4.4097 \times 10^{-4} & -3.1337 \times 10^{-3} & 7.9559 \times 10^{-1} \\ -3.3990 \times 10^{-7} & -3.1311 \times 10^{-7} & -2.8179 \times 10^{-7} & 4.1340 \times 10^{-4} \end{bmatrix}$$

The reprojection error, μ, was calculated to be $\pm 3.02\,\mathrm{px}$.

8 Applications

Acknowledgements

I am very grateful to my supervisor Mr. Hoteit for his continual guidance and invaluable pieces of advice during the process of writing this extended essay. Additionally, I would like to express my appreciation to Mr. Auclair for dedicating his time to instruct me on camera operation and familiarizing me with camera settings. I am also indebted to Mr. Matthewson for his guidance with the manufacturing of my calibration object, and to Leon, for his help in operating the laser cutter. Last but not least, I want to thank Aditya for aiding me in the process of creating some of the diagrams used in my paper.

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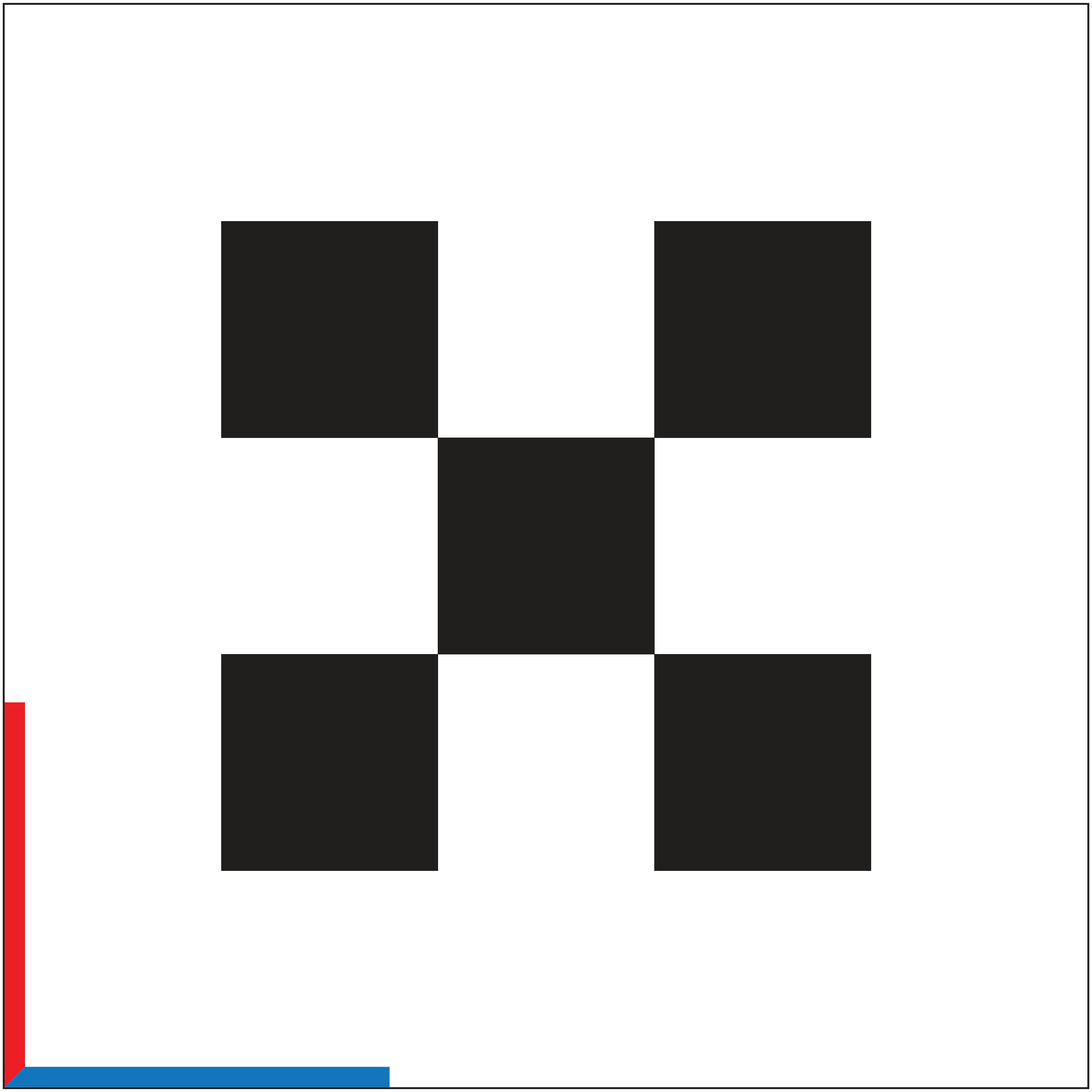
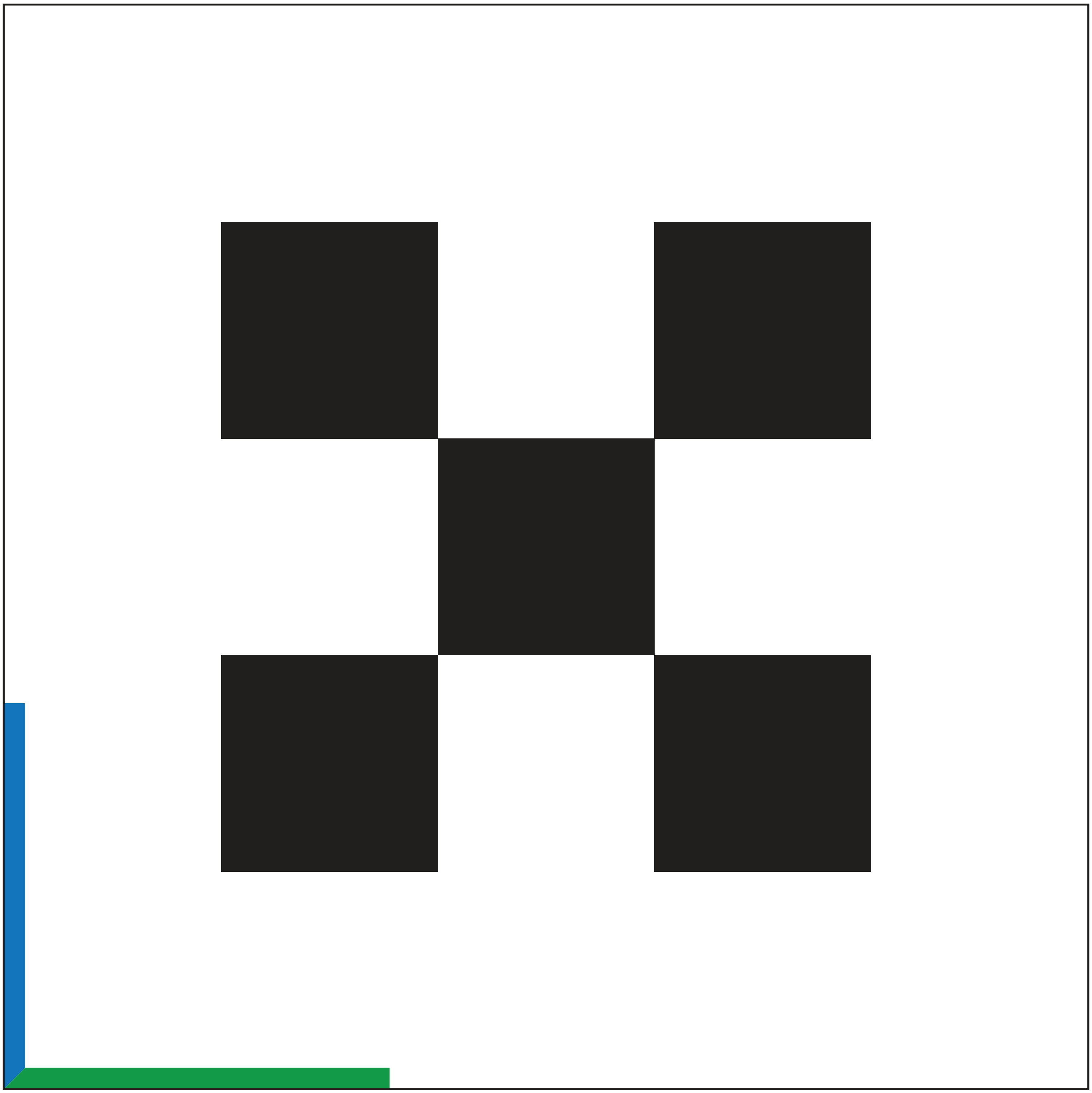
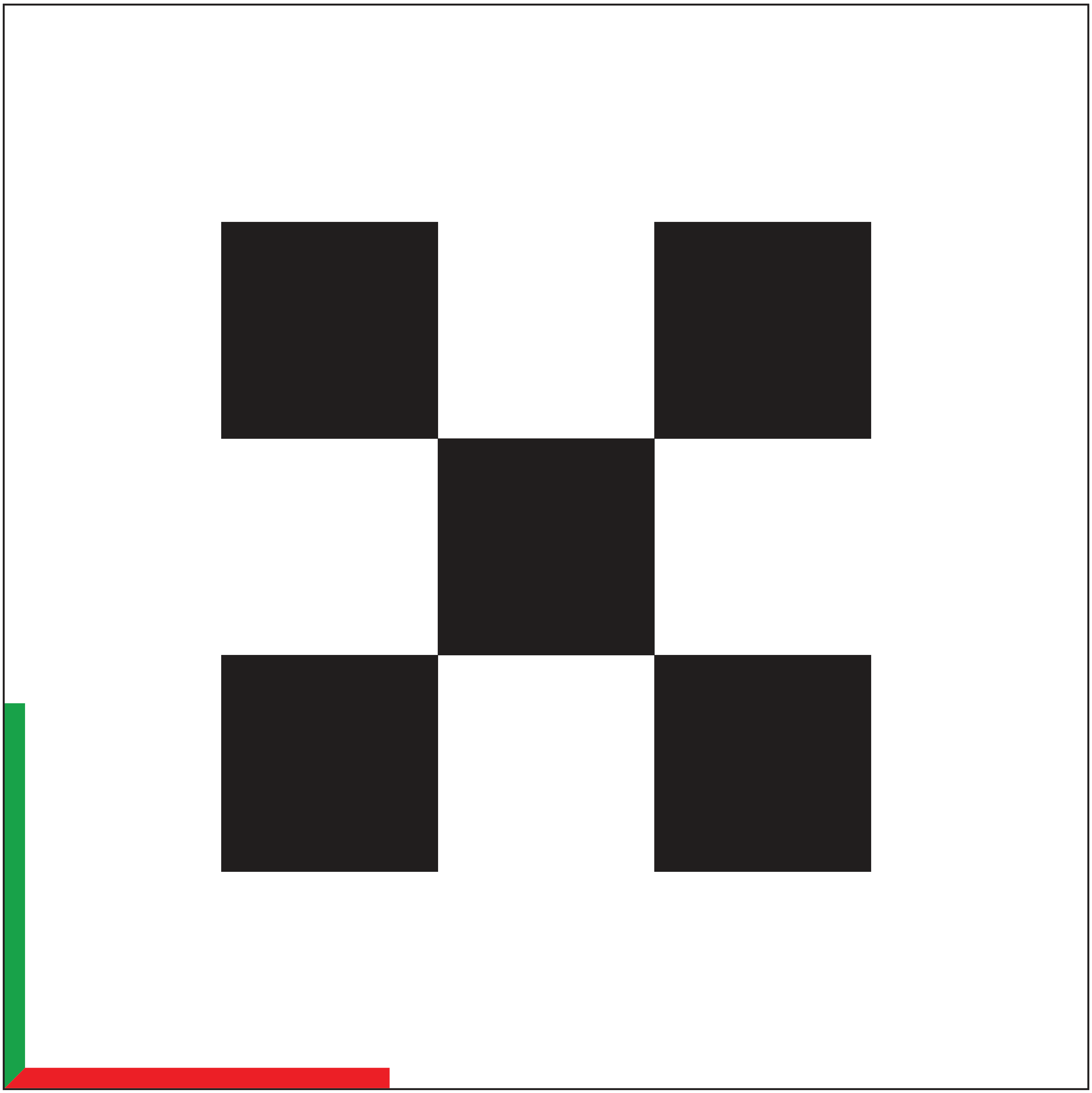
Appendix A Calibration Object Details

A.1 Panels

A.2 Grid Pattern







Appendix B Source Code

Project Structure

```
calicam
├── calicam
│   ├── __init__.py
│   ├── extract.py
│   ├── parser.py
│   ├── projection.py
│   └── vecs.py
└── run.py
```

run.py

```python
#!/usr/bin/env python3
import os
import sys
import numpy as np
import matplotlib.pyplot as plt
from argparse import ArgumentParser, RawDescriptionHelpFormatter

import calicam


def main():
    np.set_printoptions(precision=3, suppress=True)

    parser = ArgumentParser(
        prog="calicam",
        description=(
            "Generates projection matrix and calculates intrinsic and extrinsic parameters.\n"
            "CSV inputs are in the format: x,y,z,u,v where 3D point = (x, y, z) and 2D point = (u,v)"),
        formatter_class=RawDescriptionHelpFormatter,
    )

    parser.add_argument("path", help="path to csv file with calibration points")
    parser.add_argument(
        "-d", "--data", metavar="DATA_PATH", help="path to csv file with model verification data")
    parser.add_argument(
        "-g", "--graph", nargs="?", const="", metavar="BACKGROUND_IMAGE", help="generate graph")
    parser.add_argument("-t", "--title", metavar="TITLE", help="title of graph")
    parser.add_argument("-s", "--show", action="store_true", help="show graph")
    parser.add_argument("-o", "--out", metavar="GRAPH_OUPUT_PATH", help="graph output location")
```

```python
    parser.add_argument("-l", "--log", metavar="LOG_OUTPUT_PATH", help="log parameters")
    parser.add_argument("--noprint", action="store_true", help="don't print output")

    args = parser.parse_args()

    output = []

    try:
        # GENERATE MODEL
        csv_path: str = args.path

        cali_world_coords, cali_image_coords = calicam.parse_data_from_csv(csv_path)
        proj_matrix, cali_matrix, rot_matrix, (tx, ty, tz) = calicam.calibrate_camera(
            cali_world_coords, cali_image_coords)

        (cx, cy), (fx, fy) = calicam.extract_intrinsics(cali_matrix)
        a, b, g = calicam.extract_orientation_zyx(rot_matrix)

        log_path: str = args.path

        output.append("\n" + "\n\n".join((
            f"Projection Matrix: \n{proj_matrix}",
            f"Calibration Matrix: \n{cali_matrix}",
            f"Rotation Matrix: \n{rot_matrix}",
            f"Focal Lengths: \n\tf_x = {fx:.2f} px \n\tf_y = {fy:.2f} px",
            f"Principal Point: \n\tc_x = {cx:.2f} px \n\tc_y = {cy:.2f} px",
            f"Translation: \n\tt_x = {tx:.2f} \n\tt_y = {ty:.2f} \n\tt_z = {tz:.2f}",
            f"Orientation: \n\t\u03B1 = {a:.2f}° \n\t\u03B2 = {b:.2f}° \n\t\u03B3 = {g:.2f}°",
        )))

        # MODEL VALIDATION
        data_path: str | None = args.data

        if data_path is not None:
            assert os.path.isfile(data_path), f"{data_path} does not exist."
            assert data_path.endswith(".csv"), f'{data_path} does not end with the extension ".csv".'

            data_world_coords, data_image_coords = calicam.parse_data_from_csv(data_path)

            predicted_coords = [
                calicam.project_point(proj_matrix, world_coord) for world_coord in data_world_coords
            ]
            reproj_errs = [
                calicam.euclidean(actual_coord, reproj_coord)
                for actual_coord, reproj_coord in zip(data_image_coords, predicted_coords)
```

```
        ]

        max_err = max(reproj_errs)
        avg_err = sum(reproj_errs) / len(reproj_errs)

        output.append(
            f"\nReprojection Errors: \n\t\u03BC_max = {max_err:.3f} px \n\t\u03BC_avg = {avg_err:.3f} px")

    # OUTPUT
    if not args.noprint:
        print(*output, sep="\n")

    log_path: str = args.log
    if log_path is not None:
        assert os.path.isfile(data_path), f"{data_path} does not exist."
        with open(log_path, "w") as f:
            print(*output, sep="\n", file=f)

    # GRAPH
    image_path: str | None = args.graph

    if image_path is not None:
        ax: plt.Axes
        _, ax = plt.subplots(figsize=(8, 10))

        plt.gca().invert_yaxis()

        # image was provided
        if image_path != "":
            assert os.path.isfile(image_path), f"{image_path} does not exist."
            assert args.data, "Path to data csv file must be provide using -d flag to produce graph."

            img = plt.imread(image_path,)
            ax.imshow(img, cmap='gray')
            ax.autoscale(False)

        # graph data points and model points only if -d flag is specfied
        if data_path is not None:
            cmap = plt.cm.brg
            discrete_cmap = list(cmap(np.linspace(0, 1, len(data_image_coords))))

            # data points
            ax.scatter(
                *zip(*data_image_coords),
```

```
                    label="Ground Truth",
                    s=50,
                    color=discrete_cmap,
                    marker="o",
                    alpha=0.6,
                )

                # predicted points
                ax.scatter(
                    *zip(*predicted_coords),
                    label="Prediction",
                    s=100,
                    color=discrete_cmap,
                    marker="x",
                )

            # calibration points
            ax.scatter(
                *zip(*cali_image_coords),
                label="Calibration Points",
                s=60,
                marker="D",
                color="darkorange",
            )

            # principle point
            ax.scatter(cx, cy, label="Principal Point", s=120, marker="p", color="magenta")

            graph_title: str = args.title or image_path

            plt.gca().update({"title": graph_title, "xlabel": "$u$ (px)", "ylabel": "$v$ (px)"})
            plt.legend()

            out_path: str | None = args.out

            if out_path:
                plt.savefig(out_path)

            # show graph if -s flag was specified or if a save location was not specified
            if args.show or not out_path:
                plt.show()

    except AssertionError as e:
        parser.error(str(e))  # pass error to argparse

```

```python
    except KeyboardInterrupt:
        print(f"\nKeyboardInterrupt")
        sys.exit(1)

    sys.exit(0)


if __name__ == "__main__":
    main()
```

calicam/parser.py

```python
import csv

from .vecs import *


def parse_data_from_csv(path: str) -> tuple[list[Vec3f], list[Vec2f]]:
    """
    Parses a csv and returns a list of 3D scene points and their corresonding
    2D image mappings.
    CSV format: x,y,z,u,v
    where 3D point = (x, y, z) and 2D point (u,v)
    """
    with open(path, "r") as f:
        reader = csv.reader(f, delimiter=",")

        world_coords = []
        image_coords = []
        for lno, line in enumerate(reader, start=1):
            assert len(line) == 5, f"Data on line {lno} in {path} is invalid."

            x, y, z, u, v = (float(s) for s in line)

            world_coords.append((x, y, z))
            image_coords.append((u, v))

    return world_coords, image_coords
```

calicam/projection.py

```python
import numpy as np
import scipy.sparse.linalg
from nptyping import Shape, Double

from .vecs import *

ProjMatrix = np.ndarray[Shape["3, 4"], Double]


def generate_estimation_matrix(world_coords: list[Vec3f], image_coords: list[Vec2f]) -> np.ndarray:
    """
    Generates an estimation matrix from list of 3D world coords and
    their corresponding pixel coord mappings
    """
    rows = []
    for (x, y, z), (u, v) in zip(world_coords, image_coords):
        rows.append([x, y, z, 1.0, 0.0, 0.0, 0.0, 0.0, -u * x, -u * y, -u * z, -u])
        rows.append([0.0, 0.0, 0.0, 0.0, x, y, z, 1.0, -v * x, -v * y, -v * z, -v])
    return np.array(rows)


def generate_proj_matrix(world_coords: list[Vec3f], image_coords: list[Vec2f]) -> tuple[ProjMatrix, float]:
    """
    Takes 3D calibration points their corresponding pixel coord mappings and
    returns the projection matrix as a 3x4 matrix
    """
    assert len(world_coords) == len(image_coords), \
        f"The number of world coordinates ({world_coords}) and image coordinates ({image_coords}) do not match."

    assert len(world_coords) >= 6, \
        f"Need at least 6 calibration points, but only {len(world_coords)} were provided."

    G = generate_estimation_matrix(world_coords, image_coords)
    M = G.T @ G

    eigval, p = scipy.sparse.linalg.eigs(M, k=1, which="SM")  # solve for minimum p using eigenvalue problem
    proj_matrix = p.real.reshape(3, 4)  # take only real part of p and convert into 3x4 matrix

    return proj_matrix, eigval


def project_point(projection_matrix: ProjMatrix, world_coords: Vec3f) -> Vec2f:
```

```
    """
    Calculate pixel coordinate from 3D world coord using projection matrix
    """
    im_point = projection_matrix @ to_homogenous(world_coords)
    return to_inhomogenous(im_point)  # turn into inhomogenous coords
```

calicam/extract.py

```
import numpy as np
import scipy.linalg
from math import sqrt
from nptyping import  Shape, Double

from .projection import generate_proj_matrix, ProjMatrix
from .vecs import *

CalMatrix = np.ndarray[Shape["3, 3"], Double]
RotMatrix = np.ndarray[Shape["3, 3"], Double]


def calibrate_camera(world_coords: list[Vec3f],
                     image_coords: list[Vec2f]) -> tuple[ProjMatrix, CalMatrix, RotMatrix, Vec3f]:
    """
    Decomposes the projection matrix into the calibration matrix,
    rotation matrix, and translation matrix.
    """
    proj_matrix, _ = generate_proj_matrix(world_coords, image_coords)

    K, R = scipy.linalg.rq(proj_matrix[:, :3])  # rq decomposition

    # enforce positive diagonal on K
    D = np.diag(np.sign(np.diag(K)))
    K = K @ D
    R = D @ R

    # scale projection matrix and calibration matrix to reflect real world scaling
    scale_factor = 1 / K[2][2]
    proj_matrix *= scale_factor
    K *= scale_factor

    # extract translation vector from P
    t = tuple(-np.linalg.inv(proj_matrix[:, :3]) @ proj_matrix[:, 3])
```

```python

    return proj_matrix, K, R, t


def extract_intrinsics(K: CalMatrix) -> tuple[Vec2f, Vec2f]:
    """
    Extract principle point and focal lengths from calibration matrix
    """
    focal_lengths = (K[0][0], K[1][1])
    principal_point = (K[0][2], K[1][2])
    return principal_point, focal_lengths


def extract_orientation_zyx(R: RotMatrix) -> Vec3f:
    """
    Extract tait-bryan angles (zyx) from rotation matrix
    """
    return (
        np.degrees(np.arcsin(R[2][1] / sqrt(1 - (R[2][0])**2))),  # alpha (x rotation)
        np.degrees(np.arcsin(-R[2][0])),  # beta (y rotation)
        np.degrees(np.arcsin(R[1][0] / sqrt(1 - (R[2][0])**2))),  # gamma (z rotation)
    )
```

calicam/vecs.py

```python
from math import sqrt

Vecf = tuple[float, ...]

Vec2f = tuple[float, float]
Vec3f = tuple[float, float, float]


def to_homogenous(vec: Vecf) -> Vecf:
    return (*vec, 1.0)


def to_inhomogenous(vec: Vecf) -> Vecf:
    return tuple(map(lambda v_i: v_i / vec[-1], vec[:-1]))


def euclidean(a: Vecf, b: Vecf) -> float:
```

```
    return sqrt(sum((b_i - a_i)**2 for a_i, b_i in zip(a, b)))
```