IB Math Analysis and Approaches (HL)
Internal Assessment

Determining the Optimal Quantity of Garland Needed to Decorate My Christmas Tree

Session: May 2024
Page Count: 7 pages
Word Count: 1220 words

Introduction

During Christmas, my family bought a Christmas tree to place at home. Unfortunately, we lacked decorations, so we decided to go to the store to buy some garland for the tree. However, uncertain of how much we needed to buy, we guessed the amount we needed. We tried to buy as little garland as we thought was necessary in order to minimize waste, but we ended up buying not enough garland. We then promptly returned to the store to buy more garland, but this time we ended up with way too much garland. Faced with these troubles, I decided that for my investigation, I would find a method to calculate the amount of garland required to wrap around my Christmas tree based on some chosen parameters, as well as find the optimal spacing for the garland in order to obtain a balance between meeting personal aesthetic preferences and minimizing waste.

Aim and Methodology

The goal of this paper is to devise a general formula which accounts for certain parameters of the tree as well as the spacing between successive rotations of the garland to obtain the amount of the garland I would need to buy. To do so, I must first mathematically model the Christmas tree and the garland that wraps around the tree. This involves making some assumptions and approximations, in order to simplify the model:

1. **The Christmas tree is "ideal"** – The Christmas tree is modelled as a *cone*, based on the assumption that an "ideal" tree would be radially symmetrical all around and that the slant of the tree is a straight line.

2. **The garland wraps uniformly** – This means that it wraps in a perfect spiral around the tree, without sagging. The garland should also wrap around the tree with equal spacing between subsequent rotations. Since the tree is approximated to be a cone, the garland wrapping around the tree can be modelled as a *circular conical spiral*.

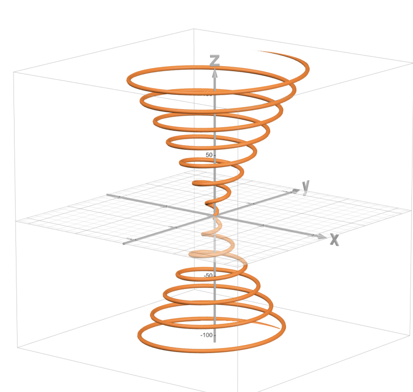


Figure 1: Unbounded Conical Spiral.

The following input parameters will be considered for the calculations (visualized in Figure 2):

Parameter	Description
H	height of the tree
R	radius of the based of the tree
λ	slant distance (spacing) between successive rotations of garland

Diagrams and graphs will be used throughout the paper to aid in the explanation of concepts, and unless otherwise mentioned, they are generated using the online graphing calculator *Desmos*.

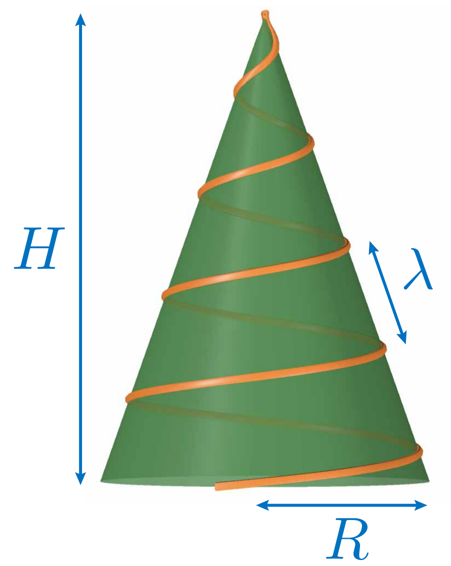


Figure 2: Input Parameters.

Modelling the Garland

The garland can be modelled as a circular conical spiral, and applying lower and upper bounds, it has the parametric function (Rejbrand, n.d.):

$$C(t) = \begin{pmatrix} at\cos t \\ at\sin t \\ bt \end{pmatrix}, \quad \forall t \in \left[0, \frac{H}{b}\right] \tag{1}$$

where the constants $a, b \in \mathbb{R}$. Lower and upper bounds are applied because the cone has height H, and we only want parts of the curve where $z(t)$ is between 0 and H. Thus, $0 \leq bt \leq H$, and dividing by b, we find that $0 \leq t \leq \frac{H}{b}$, which are the bounds applied to $C(t)$. t is the independent variable, and for any given value of t, the parametric function outputs a point on the curve, and as we generate an infinite amount of points from the lower bound to the upper bound, the locus of points formed generate the shape of the space curve. The z-axis of the spiral coincides with the tree's axis of radial symmetry and $C(0)$ represents the tip of the tree.

To ensure that the spiral sits on the surface of the cone (the Christmas tree), it is necessary to find appropriate values for a and b. Using methodology inspired by a blog post authored by Stewart and Heighway, we first define the radial distance $\rho(t)$ as the distance of a point on the curve to the tree's axis of radial symmetry, visualized in Figure 3. By Pythagorean's theorem:

$$\rho(t) = \sqrt{x(t)^2 + y(t)^2}$$

➢ Substituting for $x(t)$ and $y(t)$:

$$\begin{aligned}\rho(t) &= \sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t}\\ &= at\sqrt{\cos^2 t + \sin^2 t}\\ &= at \qquad (2)\end{aligned}$$

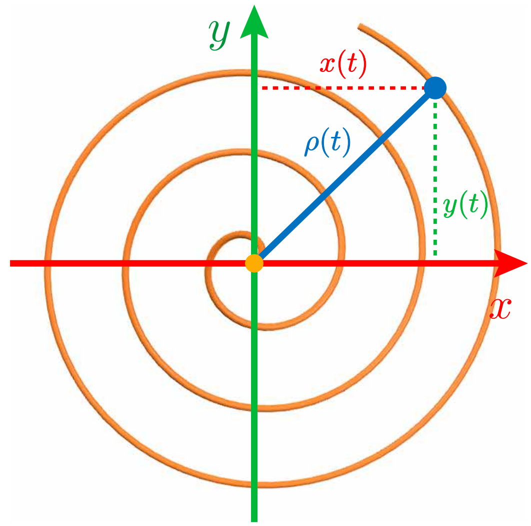


Figure 3: Radial distance of a point on the spiral to the z-axis (top-down view).

The radial distance is useful because the ratio of $\rho(t)$ over $z(t)$ for any given t will always be proportional to H over R for any given t. This because any valid point of the curve should sit on the surface of the cone of height H and radius R, and thus the radial distance $\rho(t)$ and vertical distance $z(t)$ of the point would form a right triangle as visualized in Figure 4, which would be similar to the triangle formed by the vertical cross-section of the cone by angle-angle, due to the shared an interior angle. This allows us to establish the following proportional relationship:

$$\begin{aligned}\frac{R}{H} &= \frac{\rho(t)}{z(t)} = \frac{at}{bt}\\ \Rightarrow \frac{R}{H} &= \frac{a}{b} \qquad (3)\end{aligned}$$

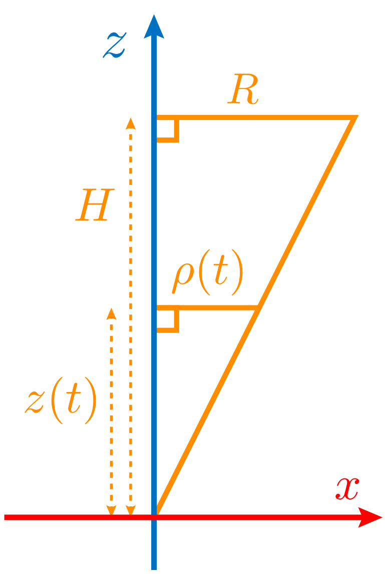
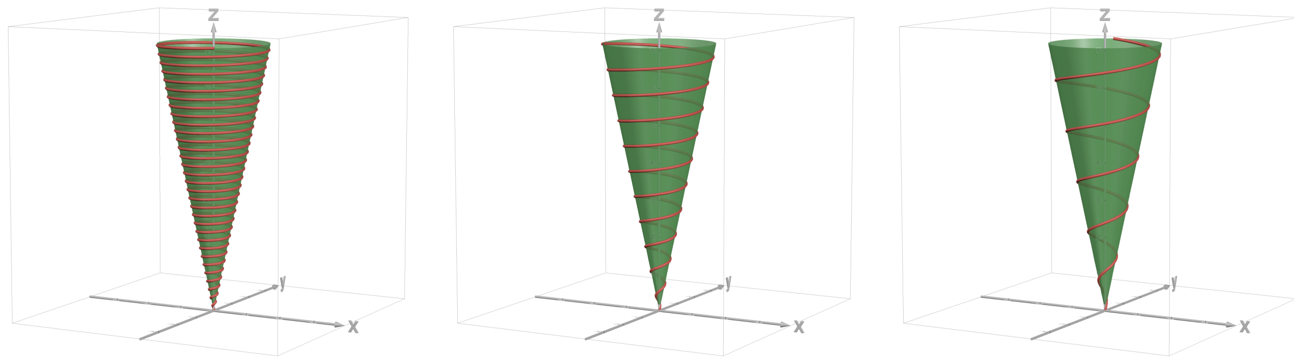


Figure 4: Similar Triangles.

In other words, for the spiral to lie on the surface of the cone with radius R and height H, the ratio a over b must be proportional to R over H. This relationship is visualized in Figure 5, where we can see that larger values for a and b correspond to larger spacing between successive rotations of garland, while smaller values lead to smaller spacing between successive rotations of the garland. From this, we can establish a relationship between λ, a, and b.



(a) $a = 0.075$, $b = 0.360$ **(b)** $a = 0.225$, $b = 1.08$ **(c)** $a = 0.450$, $b = 2.16$

Figure 5: Spirals that have the same ratio of $\frac{a}{b}$ lie on the same cone

For every rotation of the garland, t increases by 2π because that is the period of the trigonometric functions sine and cosine. Thus, the change in radial distance, $\Delta\rho$, and change in vertical distance, Δz, after one full period would be:

$$\begin{aligned}\Delta\rho &= \rho(t+2\pi) - \rho(t) \\ &= a\cdot(t+2\pi) - at \\ &= 2\pi a\end{aligned} \qquad \begin{aligned}\Delta z &= z(t+2\pi) - z(t) \\ &= b\cdot(t+2\pi) - bt \\ &= 2\pi b\end{aligned}$$

While λ represents the distance between consecutive rotations of garland, we can also think of it as the change in position of the spiral along of the slant length of the cone per rotation. Thus, by the Pythagorean theorem:

$$\lambda^2 = 4\pi^2 a^2 + 4\pi^2 b^2 \tag{4}$$

With this, we now have 2 equations with a and b, and we can represent a and b in terms of our chosen parameters. From equation 3, we can isolate b to get that $b = \frac{H}{R}a$, which can be substituted back into equation 4:

$$\Rightarrow \lambda^2 = 4\pi^2 a^2 + \frac{4\pi^2 a^2 H^2}{R^2} = 4\pi^2 a^2 \left(1 + \frac{H^2}{R^2}\right)$$

➢ Isolating for a:

$$a = \sqrt{\frac{\lambda^2}{4\pi^2(1+\frac{H^2}{R^2})}} = \frac{\lambda}{2\pi\sqrt{1+\frac{H^2}{R^2}}} = \frac{\lambda R}{2\pi\sqrt{R^2+H^2}}$$

➢ Recognizing that $S = \sqrt{R^2+H^2}$ is the slant height of the cone:

$$a = \frac{\lambda R}{2\pi S} \tag{5}$$

➢ Plugging this back in equation 3, we get:

$$b = \frac{\lambda H}{2\pi S} \tag{6}$$

Thus, we finally have that the parametric equation for the garland is:

$$C(t) = \frac{\lambda}{2\pi S}\begin{pmatrix} Rt\cos t \\ Rt\sin t \\ Ht \end{pmatrix}, \quad \forall t \in \left[0, \frac{2\pi S}{\lambda}\right] \tag{7}$$

Deriving an Equation for the Length of the Garland

With a function which models the garland, we can now calculate the length of the garland by calculating the *arc length* of $C(t)$. The arc length, L, is defined as the distance travelled along the path of a curve from one point to another ("8.1: Arc Length" 2017). The arc length of $C(t)$ can be evaluated by decomposing the curve into an infinite amount of infinitesimally small line segments, $\mathrm{d}L$, and adding their lengths. Thus, the length of the garland is represented by the definite integral:

$$L = \int_0^{\frac{2\pi S}{\lambda}} \mathrm{d}L \tag{8}$$

The lower and upper bounds on the integral are a result of the restrictions imposed on $C(t)$.

However, what is $\mathrm{d}L$? To find what $\mathrm{d}L$ is, it is useful to think of it as a 3-dimensional vector. One important property of vectors is that they can be expressed as a sum of multiple vectors, and thus 3D vectors can be decomposed into their x, y, and z components. Using this line of thinking, it can similarly be said that $\mathrm{d}L$ can be decomposed into the infinitesimals $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$, as visualized in Figure 6. Therefore, applying the 3D Pythagorean theorem:

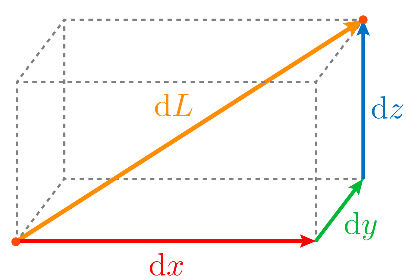


Figure 6: $\mathrm{d}L$ in terms of $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$

$$\mathrm{d}L = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2}$$

However, this is not of much use, as $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ are arbitrary. Thus, $\mathrm{d}t$ is introduced into the equation by multiplying the equation by $\frac{\mathrm{d}t}{\mathrm{d}t}$, and with some algebraic manipulation, $\mathrm{d}L$ can be expressed in terms of the derivatives of $x(t)$, $y(t)$, and $z(t)$ components of the curve, which can be easily evaluated (Schlicker, Keller, and Long, n.d.):

$$\mathrm{d}L = \sqrt{\frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2}{(\mathrm{d}t)^2}} \cdot \mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} \cdot \mathrm{d}t$$

➢ Substituting in $x(t)$, $y(t)$, and $z(t)$:

$$\begin{aligned}
\Rightarrow \mathrm{d}L &= \sqrt{\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{2\pi S}Rt\cos t\right)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{2\pi S}Rt\sin t\right)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{2\pi S}Ht\right)\right)^2} \cdot \mathrm{d}t \\
&= \frac{\lambda}{2\pi S}\sqrt{\left(\frac{\mathrm{d}}{\mathrm{d}t}(Rt\cos t)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}(Rt\sin t)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}(Ht)\right)^2} \cdot \mathrm{d}t \\
&= \frac{\lambda}{2\pi S}\sqrt{(R\cos t - Rt\sin t)^2 + (R\sin t - Rt\cos t)^2 + H^2} \cdot \mathrm{d}t
\end{aligned}$$

➢ Expanding and simplifying:

$$\begin{aligned}
\Rightarrow \mathrm{d}L &= \frac{\lambda}{2\pi S}\sqrt{(R^2\cos^2 t - \cancel{2R^2 t\sin t\cos t} + R^2t^2\sin^2 t) + (R^2\sin^2 t + \cancel{2R^2 t\sin t\cos t} + R^2t^2\cos^2 t) + H^2} \cdot \mathrm{d}t \\
&= \frac{\lambda}{2\pi S}\sqrt{R^2(\cancel{\sin^2 t + \cos^2 t}) + R^2t^2(\cancel{\sin^2 t + \cos^2 t}) + H^2} \cdot \mathrm{d}t \\
&= \frac{\lambda}{2\pi S}\sqrt{R^2 + H^2 + R^2t^2} \cdot \mathrm{d}t \\
&= \frac{\lambda}{2\pi S}\sqrt{S^2 + R^2t^2} \cdot \mathrm{d}t \qquad (9)
\end{aligned}$$

➢ Substituting this back into equation 8, we finally obtain:

$$L = \frac{\lambda}{2\pi S}\int_0^{\frac{2\pi S}{\lambda}} \sqrt{S^2 + R^2t^2} \cdot \mathrm{d}t \qquad (10)$$

Evaluating the Integral

Now, we evaluate the integral so that we can finally obtain a general solution for the length of the garland based on our chosen parameters. Since the integral is of the form $\sqrt{c^2 + x^2}$, it can be evaluated using trigonometric substitution.

➢ Let $t = \frac{S}{R}\tan\theta$. Thus, $\mathrm{d}t = \frac{S}{R}\sec^2\theta\,\mathrm{d}\theta$. Substituting them into equation 10:

$$\begin{aligned}
\Rightarrow L &= \frac{\lambda}{2\pi \cancel{S}}\int_0^{t=\frac{2\pi S}{\lambda}} \sqrt{S^2 + R^2\left(\frac{S}{R}\tan\theta\right)^2} \cdot \frac{\cancel{S}}{R}\sec^2\theta\,\mathrm{d}\theta \\
&= \frac{\lambda}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} \sqrt{S^2 + S^2\tan^2\theta} \cdot \sec^2\theta\,\mathrm{d}\theta \\
&= \frac{\lambda}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} S\sec\theta \cdot \sec^2\theta\,\mathrm{d}\theta \\
&= \frac{\lambda S}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} \sec^3\theta\,\mathrm{d}\theta \qquad (11)
\end{aligned}$$

Then, using integration by parts, the integral of $\sec^3\theta$ can be evaluated.

➢ Let $u = \sec\theta$ and $\mathrm{d}v = \sec^2\theta\,\mathrm{d}\theta$. Therefore, $\mathrm{d}u = \sec\theta\tan\theta$ and $v = \tan\theta$.

$$\Rightarrow L = \frac{\lambda S}{2\pi R}\left[\sec\theta\tan\theta\,\Big|_0^{t=\frac{2\pi S}{\lambda}} - \int_0^{t=\frac{2\pi S}{\lambda}} \sec\theta\tan^2\theta\,\mathrm{d}\theta\right]$$

➢ Using the identity $\tan^2\theta = \sec^2\theta - 1$:

$$\begin{aligned}\Rightarrow L &= \frac{\lambda S}{2\pi R}\left[\sec\theta\tan\theta\,\Big|_0^{t=\frac{2\pi S}{\lambda}} - \int_0^{t=\frac{2\pi S}{\lambda}} (\sec^3\theta - \sec\theta)\,\mathrm{d}\theta\right] \\ &= \frac{\lambda S}{2\pi R}\left[\sec\theta\tan\theta\,\Big|_0^{t=\frac{2\pi S}{\lambda}} + \int_0^{t=\frac{2\pi S}{\lambda}} \sec\theta\,\mathrm{d}\theta\right] - \frac{\lambda S}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} \sec^3\theta\,\mathrm{d}\theta\end{aligned}$$

➢ By $\int\sec\theta\,\mathrm{d}\theta = \ln(\sec\theta + \tan\theta) + c$:

$$\Rightarrow L = \frac{\lambda S}{2\pi R}\Big[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\Big]_0^{t=\frac{2\pi S}{\lambda}} - \frac{\lambda S}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} \sec^3\theta\,\mathrm{d}\theta$$

➢ Since $\frac{\lambda S}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} \sec^3\theta\,\mathrm{d}\theta = L$ (equation 11):

$$\begin{aligned}\Rightarrow L &= \frac{\lambda S}{2\pi R}\Big[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\Big]_0^{t=\frac{2\pi S}{\lambda}} - L \\ \Rightarrow 2L &= \frac{\lambda S}{2\pi R}\Big[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\Big]_0^{t=\frac{2\pi S}{\lambda}} \\ \Rightarrow L &= \frac{\lambda S}{4\pi R}\Big[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\Big]_0^{t=\frac{2\pi S}{\lambda}} \qquad (12)\end{aligned}$$

➢ Now, we want to reverse the substitution and bring the equation back in terms of our original variable, t. Recall that we substituted $t = \frac{S}{R}\tan theta$, so $\tan\theta = \frac{R}{S}t$. Then, using the identity $\sec^2\theta = 1 + \tan^2\theta$, we can find what $\sec\theta$ is equal to:

$$\sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + \frac{R^2}{S^2}t^2} = \frac{\sqrt{S^2 + R^2t^2}}{S}$$

➢ Reversing the substitution in equation 12:

$$L = \frac{\lambda S}{4\pi R}\left[\frac{\sqrt{S^2 + R^2t^2}}{S}\cdot\frac{R}{S}t + \ln\left(\frac{\sqrt{S^2 + R^2t^2}}{S} + \frac{R}{S}t\right)\right]_0^{t=\frac{2\pi S}{\lambda}}$$

References

"8.1: Arc Length." 2017. Mathematics LibreTexts, April 27, 2017, 7:19 p.m. (Z). Accessed February 26, 2024. https://math.libretexts.org/Bookshelves/Calculus/Map%3A_Calculus__Early_Transcendentals_(Stewart)/08%3A_Further_Applications_of_Integration/8.01%3A_Arc_Length.

Rejbrand, Andreas. n.d. "Conical Helix." The Rejbrand Encyclopædia of Curves and Surfaces. Accessed February 23, 2024. https://trecs.se/conicalHelix.php.

Schlicker, Steve, Mitchel T. Keller, and Nicholas Long. n.d. *Arc Length and Curvature.* Accessed February 26, 2024. https://activecalculus.org/multi/S-9-8-Arc-Length-Curvature.html.