How can the quantity of garland needed to decorate my Christmas tree be determined, striking a balance between meeting personal aesthetic preferences and minimizing waste?

Introduction

During Christmas, my family bought a Christmas tree to place at home. Unfortunately, it lacked decorations, so we decided to go to the store to buy some garland for the tree. However, we were uncertain of how much we needed to buy, so we guessed the amount we needed. In our attempt to minimize waste, we tried to buy as little garland as we thought we needed, which ended up needing more garland. We promptly returned to the store to buy more garland, but this time we ended up with way too much garland. Faced with these troubles, I decided that for my investigation, I would find a method to calculate the amount of garland required to wrap around a tree based on certain parameters. Then, I will manipulate the spacing of the garland to obtain a balance between meeting personal aesthetic preferences and minimizing waste.

Aim and Methodology

The goal of this paper is to devise a general formula which accounts for certain parameters of the tree as well as the spacing between successive rotations of the garland to obtain the amount of the garland I would need to buy. To do so, I must first mathematically model the Christmas tree and the garland that wraps around the tree. This involves making some assumptions and approximations, to simplify the model:

1. **The Christmas tree is "ideal"** – The Christmas tree is modelled as a *cone*, based on the assumption that an "ideal" tree would be radially symmetrical all around and that the slant of the tree with respect to its vertical axis is a straight line.

2. **The garland wraps uniformly** – This means that it wraps in a perfect spiral around the tree, without sagging. The garland should also wrap around the tree with equal spacings between subsequent rotations. Since the tree is approximated to be a cone, the garland wrapping around the tree can be modelled as a *circular conical spiral*.

The following parameters will be considered for the calculations:

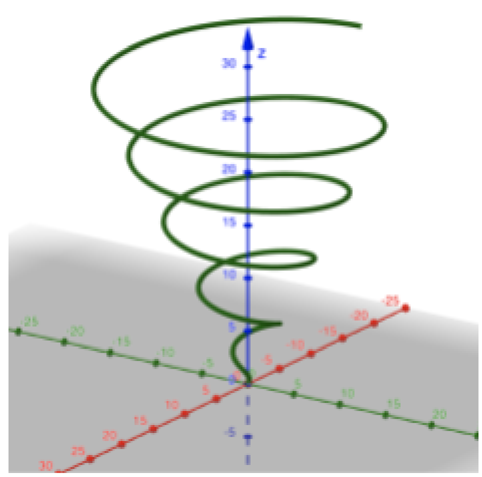

Figure 1: Circular Conical Spiral (modelled using Geogebra)

Parameter	*Description*
H	height of the tree
R	radius of the base of the tree
λ	slant distance (spacing) between successive rotations of the garland

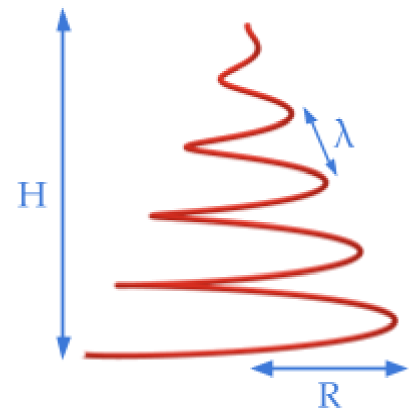

Figure 2: Parameters

Modelling the Garland

The garland can be modelled as a *circular conical spiral*, which has the following parametric function (Rejbrand, n.d.):

$$C(t) = \begin{pmatrix} at\cos t \\ at\sin t \\ bt \end{pmatrix}, \qquad \forall t \in [0, H] \tag{1}$$

where the constants $a, b \in \mathbb{R}$. t is the independent variable, and the bounds of the function are restricted between 0 and H because that is the range which represents the height of the tree, and hence only that part of the curve is useful. For a given value t, the parametric function outputs a point on the curve, and as we generate an infinite amount of points between 0 and H, a locus of points are generated, forming the shape of the space curve. The z-axis of the spiral coincides with the tree's axis of symmetry and $r(0)$ represents the tip of the tree.

To ensure that the spiral sits on the surface of the cone (the Christmas tree), I want to find the appropriate values a and b by relating them to the parameters I have chosen. Based on methodology inspired by a blog post by Alan Stewart and Heighway, $\rho(t)$ is defined as the radial distance from the tree's axis of symmetry. Using Pythagorean's theorem:

$$\rho(t) = \sqrt{x(t)^2 + y(t)^2}$$

where $x(t)$ and $y(t)$ represent the x and y components of $C(t)$ respectively.

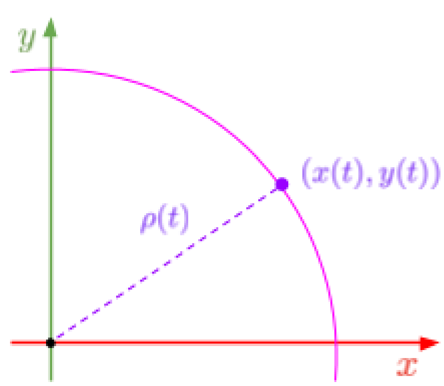

Figure 3: Radial distance from z-axis

➢ Substituting for $x(t)$ and $y(t)$

$$\begin{aligned}\rho(t) &= \sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t} \\ &= at\sqrt{\cos^2 t + \sin^2 t} \\ &= at \qquad (2)\end{aligned}$$

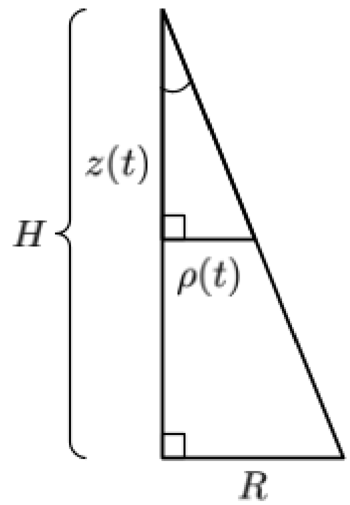

Figure 4: Proportional Relationship

From Figure 4, it can be seen that the triangle formed by $z(t)$ and $\rho(t)$ is similar to the triangle formed by the vertical cross-section of the cone, as a result of the shared interior angle. This allows us to establish the following proportional relationship:

$$\begin{aligned}\frac{R}{H} &= \frac{\rho(t)}{z(t)} = \frac{at}{bt} \\ \Rightarrow \frac{R}{H} &= \frac{a}{b} \qquad (3)\end{aligned}$$

In other words, for the spiral to lie on the surface of the cone, the ratio a over b must be proportional to R over H. In addition, this also tells us how modifying a and b changes the shape of the circular conical spiral. A larger value for a corresponds to a wider cone and vice versa, while a larger value for b corresponds to a larger vertical gap between consecutive spirals of the garland and vice versa.

Deriving an Equation for the Length of the Garland

With a function which models the garland, we can now calculate the length of the garland by calculating the *arc length* of $C(t)$. The arc length, L, is defined as the distance travelled along the path of the curve from one point to another ("8.1: Arc Length" 2017), and it can be evaluated by decomposing the curve $C(t)$ as an infinite sum of infinitesimally small line segments, dL. Thus, the length of the garland is represented by the definite integral:

$$L = \int_0^H \mathrm{d}L \qquad (4)$$

The lower and upper bounds on the integral are a result of the restrictions imposed on $C(t)$.

However, what is dL? To find what dL is, it is useful to think of it as a 3-dimensional vector. One important property of vectors is that they can be expressed as a sum of multiple vectors, and thus 3D vectors can be decomposed into their x, y, and z components. Using this line of thinking, it can similarly be said that $\mathrm{d}L$ can be decomposed into the infinitesimals $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$. The geometric intuition for this is visualized in Figure 5. Therefore, the relation between $\mathrm{d}L$ with $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ can be found by applying the 3D Pythagorean theorem:

$$\mathrm{d}L = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2}$$

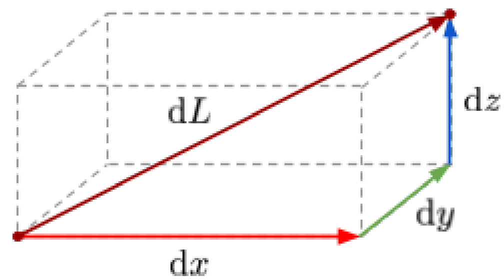

Figure 5: dL in terms of dx, dy, and dz

However, this is not of much use, as $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ are arbitrary. Thus, $\mathrm{d}t$ is introduced into the equation by multiplying the equation by $\mathrm{d}t/\mathrm{d}t$, and with some algebraic manipulation, $\mathrm{d}L$ can be expressed in terms of the derivatives of the x, y, and z components of the curves, which can be evaluated (Schlicker et al., n.d.):

$$\begin{aligned}\mathrm{d}L &= \sqrt{\frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2}{(\mathrm{d}t)^2}}\,\mathrm{d}t \\ &= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t\end{aligned}$$

➢ Substituting for $x(t)$, $y(t)$, and $z(t)$:

$$\begin{aligned}\mathrm{d}L &= \sqrt{\left(\frac{\mathrm{d}}{\mathrm{d}t}(at\cos t)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}(at\sin t)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}(bt)\right)^2}\,\mathrm{d}t \\ &= \sqrt{(a\cos t - at\sin t)^2 + (a\sin t + at\cos t)^2 + b^2}\,\mathrm{d}t\end{aligned}$$

➢ Expanding and simplifying:

$$\begin{aligned}\mathrm{d}L &= \sqrt{(a^2\cos^2 t - \cancel{2a^2t\sin t\cos t} + a^2t^2\sin^2 t) + (a^2\sin^2 t + \cancel{2a^2t\sin t\cos t} + a^2t^2\cos^2 t) + b^2}\,\mathrm{d}t \\ &= \sqrt{a^2\sin^2 t + a^2\cos^2 t + a^2t^2\sin^2 t + a^2t^2\cos^2 t + b^2}\,\mathrm{d}t \\ &= \sqrt{a^2\cancel{(\sin^2 t + \cos^2 t)} + a^2t^2\cancel{(\sin^2 t + \cos^2 t)} + b^2}\,\mathrm{d}t \\ &= \sqrt{a^2 + a^2t^2 + b^2}\,\mathrm{d}t \\ &= \sqrt{a^2 + b^2}\sqrt{1 + \frac{a^2}{a^2 + b^2}t^2}\,\mathrm{d}t \qquad (5)\end{aligned}$$

As such, an equation for $\mathrm{d}L$ has been derived. Substituting it for $\mathrm{d}L$ in ***Eqn. 4***, the integral is now expressed in terms of a and b:

$$L = \int_0^H \sqrt{a^2+b^2}\sqrt{1+\frac{a^2}{a^2+b^2}t^2}\,\mathrm{d}t \tag{6}$$

Although the constants a and b determine the shape of the curve, they do not represent any physical quantity which is of practical use to us. Therefore, they should be substituted with expressions based on the chosen input parameters H, R, and λ.

Substituting for Expressions with a and b

Let us first focus on the expression $\frac{a^2}{a^2+b^2}$. From ***Eqn. 3***, we know that a and b is proportional to H and R respectively. As such, similar triangles can be employed again to achieve a proportional relationship, as visualized in Figure 6.

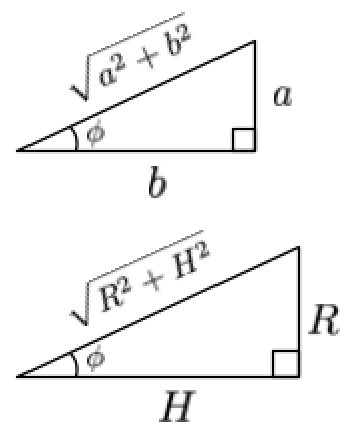

Figure 6: Similar Triangles

- ➢ Let the interior angle of triangle be ϕ. Recall that the sine of an acute angle of a right triangle is equal to the side opposite to the angle over the hypotenuse. Thus:

$$\sin\phi = \frac{a}{\sqrt{a^2+b^2}} = \frac{R}{\sqrt{R^2+H^2}}$$

- ➢ Squaring both sides, we find an expression for $\frac{a^2}{a^2+b^2}$ in terms of R and H:

$$\frac{a^2}{a^2+b^2} = \frac{R^2}{R^2+H^2} \tag{7}$$

Now, onto the other expression containing a and b, which is $\sqrt{a^2+b^2}$. For this expression, trigonometric ratios are to our advantage, because it is not a fraction. However, another property of right triangles can be used, which is that scaling one side of a right triangle also scales the other sides by the same factor. This can be shown by multiplying Pythagoras' theorem by a scale factor:

$$a^2+b^2=c^2 \Leftrightarrow ka^2+kb^2=kc^2$$

We previously established that the triangle formed by a and b is similar to the triangle formed by the vertical cross-section of the cone, and since the parameter specifying the spacing of the garland, λ, lies on the slant of the tree, λ must therefore equal to $k\sqrt{a^2+b^2}$, where $k \in \mathbb{R}$ is some scale factor.

For every rotation of the garland, t increases by 2π since that is the period of sine and cosine. Thus, for a given t value, the radial distance after one rotation would be $\rho(t + 2\pi)$ and the height after one rotation $z(t + 2\pi)$. Therefore:

$$\begin{aligned}\rho(t+2\pi) &= a[t+2\pi] \\ &= at + 2\pi a \\ &= \rho(t) + 2\pi a\end{aligned} \qquad\qquad \begin{aligned}z(t+2\pi) &= b[t+2\pi] \\ &= bt + 2\pi b \\ &= z(t) + 2\pi b\end{aligned}$$

This tells us that for every rotation, the radial distance ρ of the spiral increases by $2\pi a$ and the vertical change increases by $2\pi b$. Since the slant distance between consecutive rotations of the garland is λ, the following equation is reached using Pythagoras' theorem:

$$\begin{aligned}\lambda^2 &= \sqrt{(2\pi a)^2 + (2\pi b)^2} \\ &\Rightarrow \lambda = 2\pi\sqrt{a^2+b^2}\end{aligned}$$

➢ Isolating for $\sqrt{a^2 + b^2}$:

$$\sqrt{a^2+b^2} = \frac{\lambda}{2\pi} \tag{8}$$

Thus, we can now substitute $\frac{R^2}{R^2+H^2}$ for $\frac{a^2}{a^2+b^2}$ and $\frac{\lambda}{2\pi}$ for $\sqrt{a^2 + b^2}$ into ***Eqn. 6*** to finally obtain:

$$L = \int_0^H \frac{\lambda}{2\pi}\sqrt{1 + \frac{R^2}{R^2+H^2}t^2}\,\mathrm{d}t \tag{9}$$

Evaluating the Integral

Now, we evaluate the integral so that we can finally obtain a general solution for the length of the garland based on our chosen parameters.

➢ To begin, we can factor out $\frac{\lambda}{2\pi}$ from the integral since it is constant:

$$L = \frac{\lambda}{2\pi}\int_0^H \sqrt{1 + \frac{R^2}{R^2+H^2}t^2}\,\mathrm{d}t$$

➢ Factor $R^2 + H^2$ from the square root:

$$\Rightarrow L = \frac{\lambda}{2\pi\sqrt{R^2+H^2}} \int_0^H \sqrt{R^2+H^2+R^2t^2}\,\mathrm{d}t$$

➢ Recognizing that $S = \sqrt{R^2 + H^2}$ is the slant height of a cone, we substitute it in the equation,

$$\Rightarrow L = \frac{\lambda}{2\pi S} \int_0^H \sqrt{S^2+R^2t^2}\,\mathrm{d}t$$

Since the integral is of the form $\sqrt{c^2 + x^2}$, the integral can be evaluated using trigonometric substitution.

➢ Let $t = \frac{S}{R}$ tan θ. Thus dt = $\frac{S}{R}$ sec^{2}θ dθ. Substituting them into the integral,

$$\begin{aligned} \Rightarrow L &= \frac{\lambda}{2\pi \cancel{S}} \int_0^{t=H} \sqrt{S^2+R^2\left(\frac{S}{R}\tan\theta\right)^2} \cdot \frac{\cancel{S}}{R}\sec^2\theta\,\mathrm{d}\theta \\ &= \frac{\lambda}{2\pi R} \int_0^{t=H} \sqrt{S^2+S^2\tan^2\theta} \cdot \sec^2\theta\,\mathrm{d}\theta \\ &= \frac{\lambda}{2\pi R} \int_0^{t=H} S\sec\theta \cdot \sec^2\theta\,\mathrm{d}\theta \\ &= \frac{\lambda S}{2\pi R} \int_0^{t=H} \sec^3\theta\,\mathrm{d}\theta \end{aligned} \tag{10}$$

Then, using integration by parts, the integral sec^3θ can be evaluated.

➢ Let u = sec θ and dv = sec^2θ dθ. Therefore du = sec θ dθ and v = tan θ.

$$\Rightarrow L = \frac{\lambda S}{2\pi R}\sec\theta\tan\theta\Big|_0^{t=H} - \frac{\lambda S}{2\pi R}\int_0^{t=H}\sec\theta\tan^2\theta\,\mathrm{d}\theta$$

➢ Using the trigonometric identity $\tan^2\theta = \sec^2\theta - 1$, $\sec\theta\ \tan^2\theta = \sec^3\theta - \sec\theta$,

$$\begin{aligned} \Rightarrow L &= \frac{\lambda S}{2\pi R}\sec\theta\tan\theta\Big|_0^{t=H} - \frac{\lambda S}{2\pi R}\int_0^{t=H}(\sec^3\theta - \sec\theta)\,\mathrm{d}\theta \\ &= \frac{\lambda S}{2\pi R}\sec\theta\tan\theta\Big|_0^{t=H} - \frac{\lambda S}{2\pi R}\int_0^{t=H}\sec^3\theta\,\mathrm{d}\theta + \frac{\lambda S}{2\pi R}\int_0^{t=H}\sec\theta\,\mathrm{d}\theta \end{aligned}$$

➢ Since $\frac{\lambda S}{2\pi R}\int_0^{t=H} \sec^3\theta\,\mathrm{d}\theta = L$ ***(Eqn. 10)***,

$$\Rightarrow L = \frac{\lambda S}{2\pi R}\sec\theta\tan\theta\bigg|_0^{t=H} - L + \frac{\lambda S}{2\pi R}\int_0^{t=H}\sec\theta\,\mathrm{d}\theta$$

$$\Rightarrow 2L = \frac{\lambda S}{2\pi R}\left[\sec\theta\tan\theta\,\Big|_0^{t=H} + \int_0^{t=H}\sec\theta\,\mathrm{d}\theta\right]$$

$$\Rightarrow L = \frac{\lambda S}{4\pi R}\left[\sec\theta\tan\theta\,\Big|_0^{t=H} + \int_0^{t=H}\sec\theta\,\mathrm{d}\theta\right]$$

➢ By $\int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + c$,

$$\Rightarrow L = \frac{\lambda S}{4\pi R}\left[\,\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\,\right]_0^{t=H} \tag{11}$$

➢ Since we substituted $t = \frac{S}{R}\tan\theta$, then $\tan\theta = \frac{R}{S}t$. Then, using the trigonometric identity $\sec^2\theta = 1 + \tan^2\theta$, we can find an equation for $\sec\theta$:

$$\begin{aligned}\sec\theta &= \sqrt{1+\tan^2\theta}\\ &= \sqrt{1+\frac{R^2}{S^2}t^2}\\ &= \frac{\sqrt{S^2+R^2t}}{S}\end{aligned}$$

➢ Thus:

$$\begin{aligned}\Rightarrow L &= \frac{\lambda S}{4\pi R}\left[\frac{\sqrt{S^2+R^2t^2}}{S}\cdot\frac{R}{S}t + \ln\left|\frac{\sqrt{S^2+R^2t^2}}{S} + \frac{R}{S}t\right|\right]_0^H\\ &= \frac{\lambda S}{4\pi R}\left[\frac{Rt\sqrt{S^2+R^2t^2}}{S^2} + \ln\left|\sqrt{S^2+R^2t^2}+Rt\right| - \ln(S)\right]_0^H\end{aligned}$$

➢ Finally, we arrive at the final solution by using **FTC Part 2**:

$$L(\lambda,R,H) = \frac{\lambda S}{4\pi R}\left[\frac{RH}{S^2}\sqrt{S^2+R^2H^2} + \ln\left(\sqrt{S^2+R^2H^2}+RH\right) - \ln(S)\right] \tag{12}$$

where $S = \sqrt{R^2+H^2}$

Finding the Correct Spacing of Garland

Now that I have derived the general solution for the length of the garland based on my chosen input parameters, I can now input the approximate parameters height $H = 72\ in$ and $R = 15\ in$ of my Christmas tree to find the specific solutions for my particular Christmas tree. However, I have yet to decide on the spacing of the garland, because my next objective is to find values of λ which meet my personal aesthetic preferences as well as minimize waste.

Minimizing Waste

Let G be the length of one piece of garland. First, we want to first narrow down possible solutions which minimize waste. When buying garland, it always comes in standard lengths. For example, the one which my family bought is $6\ ft$ long ($72\ in$). Thus, we want solutions where the total length of garland equal to multiples of the length of one length of garland so that there is no excess garland. Hence:

$$kG = \frac{\lambda S}{4\pi R}\left[\frac{RH}{S^2}\sqrt{S^2+R^2H^2}+\ln\left(\sqrt{S^2+R^2H^2}+RH\right)-\ln(S)\right], \quad k \in \mathbb{Z}^+ \quad (13)$$

- ➢ Now, isolating for λ, we obtain:

$$\lambda = kG \cdot \frac{4\pi RS}{RH\sqrt{R^2H^2+S^2}+S^2\left(\ln\left(\sqrt{R^2H^2+S^2}+RH\right)-\ln S\right)} \quad (14)$$

- ➢ Replacing for $R = 15\ in$, $H = 72\ in$, $G = 72\ in$, and $S = \sqrt{15^2 + 72^2} = 74.5\ in$,

$$\lambda = k(72) \cdot \frac{4\pi(72)(73.5)}{(15)(72)\sqrt{(15)^2(72)^2+(73.5)^2}+(73.5)^2\left(\ln\left(\sqrt{(15)^2(72)^2+(73.5)^2}+(15)(72)\right)-\ln 73.5\right)}$$

- ➢ Simplifying, we finally obtain that the spacing between consecutive rotation of garland should equal:

$$\lambda = k(0.841), \qquad k \in \mathbb{Z}^+ \quad (15)$$

This is a linear relationship, and this is visualized in Figure 7:

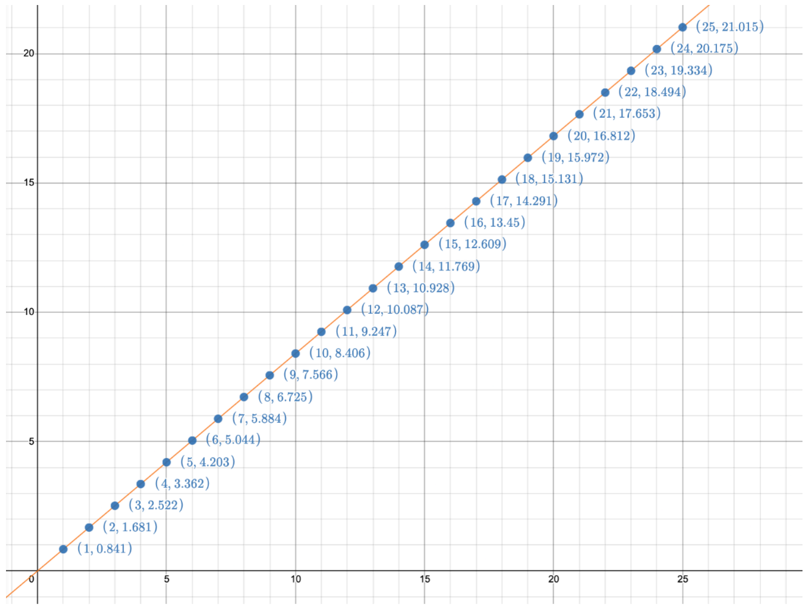

Figure 7: Graph showing the values for λ (in) for k from 1 to 25. (generated using *Desmos)*

Now that I have found different possible values for λ, I can finally calculate the total length of garland used, which is $L = kG,\ k \in \mathbb{Z}^{+}$ (***Eqn. 13***). The relationship between k and L is visualized in Figure 8.

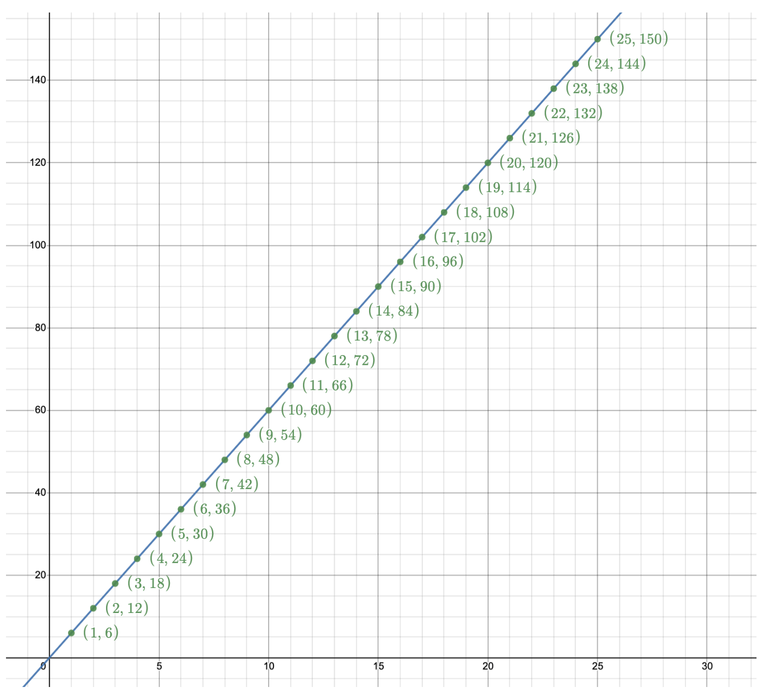

Figure 8: Graph showing the values of L (ft) for given values of k from 1 to 25. (generated using *Desmos)*

Here is the table of values for k values from 1 to 25.

k	$\lambda\ (in)$	$L\ (ft)$	k	$\lambda\ (in)$	$L\ (ft)$
1	0.84	6	14	11.8	84
2	1.68	12	15	12.6	90
3	2.52	18	16	13.4	96
4	3.36	24	17	14.3	102
5	4.20	30	18	15.1	108
6	5.04	36	19	16.0	114
7	5.88	42	20	16.8	120
8	6.72	48	21	17.7	126
9	7.57	54	22	18.5	132
10	8.41	60	23	19.3	138
11	9.25	66	24	20.0	144
12	10.1	72	25	21.0	150
13	10.9	78			

Factoring in Personal Aesthetics

optimal distance ~ 1 foot

will narrow down and find the best one

model around tree using geogebra