IB Math Analysis and Approaches (HL)
Internal Assessment

Determining the Optimal Quantity of Garland Needed to Decorate My Christmas Tree

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Introduction

During Christmas, my family bought a Christmas tree to place at home. Unfortunately, we lacked decorations, so we decided to go to the store to buy some garland for the tree. However, uncertain of how much we needed to buy, we guessed the amount we needed. We tried to buy as little garland as we thought was necessary in order to minimize waste, but we ended up buying not enough garland. We then promptly returned to the store to buy more garland, but this time we ended up with way too much garland. When wrapping the garland around the tree, we also had a lot of trouble spacing successive rotations of the garland, and had to constantly undo and adjust our wrapping as we tried to make the way we wanted. Faced with these troubles, I decided that for my investigation, I would find a method to calculate the amount of garland required to wrap around my Christmas tree based on some chosen parameters, as well as find the optimal spacing for the garland in order to obtain a balance between meeting personal aesthetic preferences and minimizing waste.

Aim and Methodology

The goal of this paper is to devise a general formula which accounts for certain parameters of the tree as well as the spacing between successive rotations of the garland to obtain the amount of the garland I would need to buy. To do so, I must first mathematically model the Christmas tree and the garland that wraps around the tree. This involves making some assumptions and approximations, in order to simplify the model:

1. **The Christmas tree is "ideal"** – The Christmas tree is modelled as a *cone*, based on the assumption that an "ideal" tree would be radially symmetrical all around and that the slant of the tree is a straight line.

2. **The garland wraps uniformly** – This means that it wraps in a perfect spiral around the tree, without sagging. The garland should also wrap around the tree with equal spacing between subsequent rotations. Since the tree is approximated to be a cone, the garland wrapping around the tree can be modelled as a *circular conical spiral*.

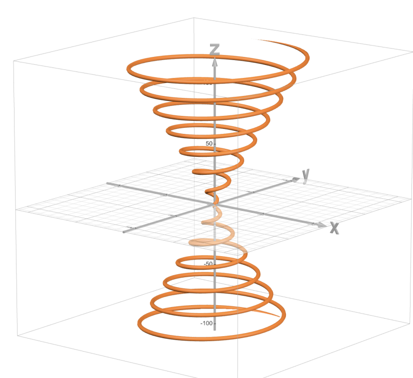


Figure 1: Unbounded Conical Spiral. (Generated using *Desmos*)

The following input parameters will be considered for the calculations (visualized in Figure 2):

Parameter	Description
H	height of the tree
R	radius of the based of the tree
λ	slant distance (spacing) between successive rotations of garland

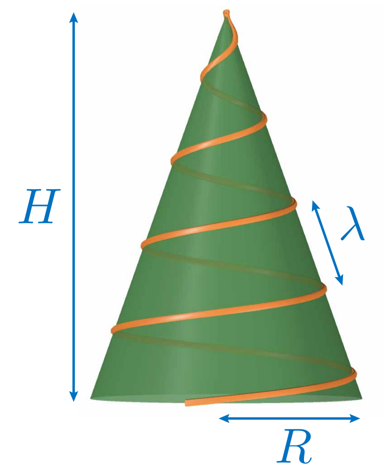


Figure 2: Input Parameters.

Diagrams and graphs will be used throughout the paper to aid in the explanation of concepts, and unless otherwise mentioned, all graphical visualizations are generated by myself using the online graphing calculator *Desmos* and labelled using *Adobe Illustrator*.

Modelling the Garland

The garland can be modelled as a *circular conical spiral*, and applying lower and upper bounds, it has the parametric function (Rejbrand, n.d.):

$$C(t) = \begin{pmatrix} at\cos t \\ at\sin t \\ bt \end{pmatrix}, \quad \forall t \in \left[0, \frac{H}{b}\right] \tag{1}$$

where the constants $a, b \in \mathbb{R}$. Lower and upper bounds are applied because the cone has height H, and we only want parts of the curve where $z(t)$ is between 0 and H. Thus, $0 \leq bt \leq H$, and dividing by b, we find that $0 \leq t \leq \frac{H}{b}$, which are the bounds applied to $C(t)$. t is the independent variable, and for any given value of t, the parametric function outputs a point on the curve, and as we generate an infinite amount of points from the lower bound to the upper bound, the locus of points formed generate the shape of the space curve. The z-axis of the spiral coincides with the tree's axis of radial symmetry and $C(0)$ represents the tip of the tree.

To ensure that the spiral sits on the surface of the cone (the Christmas tree), it is necessary to find appropriate values for a and b. Using methodology inspired by a blog post authored by Stewart and Heighway, we first define the radial distance $\rho(t)$ as the distance of a point on the curve to the tree's axis of radial symmetry, visualized in Figure 3. By the Pythagorean theorem:

$$\rho(t) = \sqrt{x(t)^2 + y(t)^2}$$

➢ Substituting for $x(t)$ and $y(t)$:

$$\begin{aligned}\rho(t) &= \sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t} \\ &= at\sqrt{\cos^2 t + \sin^2 t} \\ &= at \end{aligned} \tag{2}$$

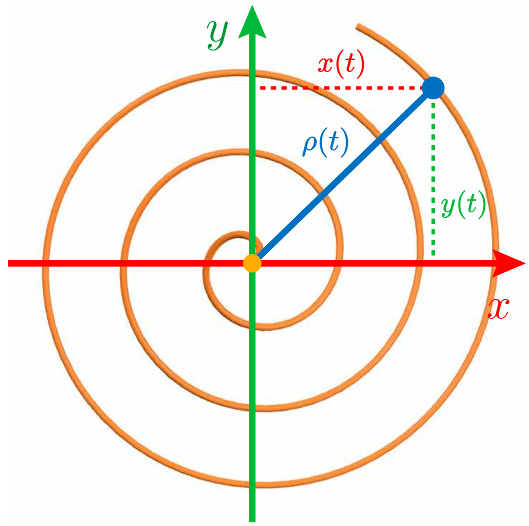


Figure 3: Radial distance of a point on the spiral to the z-axis (top-down view).

The radial distance is useful because any valid point of the curve should sit on the surface of the cone of height H and radius R, and thus the right triangle with base lengths equal to the radial distance $\rho(t)$ and vertical distance $z(t)$ as visualized in Figure 4 is similar to the right triangle formed by the vertical cross-section of the cone by angle-angle, due to the shared an interior angle. This allows us to establish the following proportional relationship:

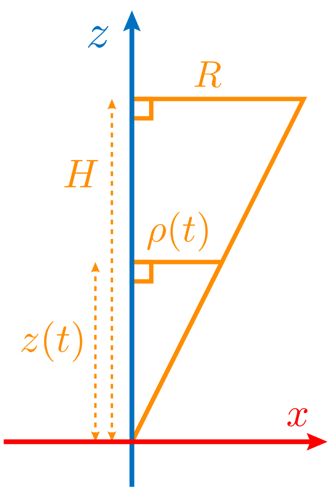
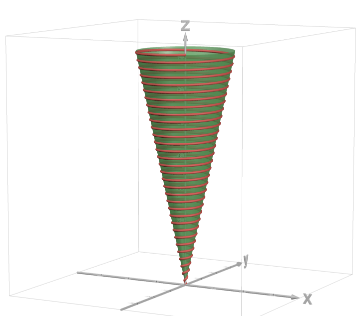


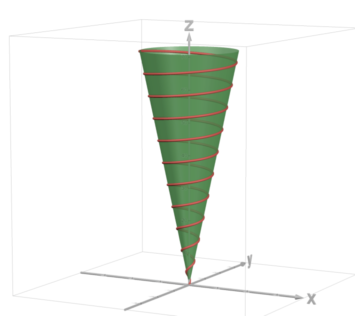
Figure 4: Similar Triangles.

$$\begin{aligned}\frac{R}{H} &= \frac{\rho(t)}{z(t)} = \frac{at}{bt} \\ \Rightarrow \frac{R}{H} &= \frac{a}{b}\end{aligned} \tag{3}$$

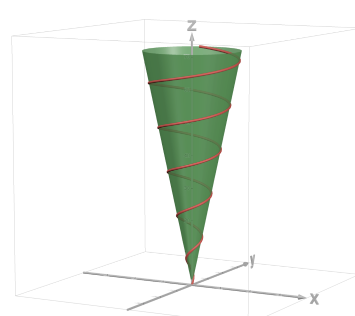
In other words, for the spiral to lie on the surface of the cone with radius R and height H, the ratio a over b must be proportional to R over H. This relationship is visualized in Figure 5, where we can see that larger values for a and b correspond to larger spacing between successive rotations of garland, while smaller values lead to smaller spacing between successive rotations of the garland. From this, we can establish a relationship between λ, a, and b.



(a) $a = 0.075$, $b = 0.360$



(b) $a = 0.225$, $b = 1.08$



(c) $a = 0.450$, $b = 2.16$

Figure 5: Spirals that have the same ratio of $\frac{a}{b}$ lie on the same cone

For every rotation of the garland, t increases by 2π because that is the period of the trigonometric functions sine and cosine. Thus, the change in radial distance, $\Delta\rho$, and change in vertical distance, Δz, after one full period would be:

$$\begin{aligned} \Delta\rho &= \rho(t+2\pi) - \rho(t) \\ &= a \cdot (t+2\pi) - at \\ &= 2\pi a \end{aligned} \qquad\qquad \begin{aligned} \Delta z &= z(t+2\pi) - z(t) \\ &= b \cdot (t+2\pi) - bt \\ &= 2\pi b \end{aligned}$$

Since the distance between sucessive rotation of the garland is equal to λ, by the Pythagorean theorem:

$$\lambda^2 = \Delta\rho^2 + \Delta z^2 = 4\pi^2 a^2 + 4\pi^2 b^2 \tag{4}$$

From equation 3, b can be isolated to get that $b = \frac{H}{R}a$, which can be substituted back into equation 4:

$$\Rightarrow \lambda^2 = 4\pi^2 a^2 + 4\pi^2 \cdot \frac{H^2}{R^2} a^2 = 4\pi^2 a^2 \left(1 + \frac{H^2}{R^2}\right)$$

➢ Isolating for a:

$$a = \sqrt{\frac{\lambda^2}{4\pi^2(1+\frac{H^2}{R^2})}} = \frac{\lambda}{2\pi\sqrt{1+\frac{H^2}{R^2}}} = \frac{\lambda R}{2\pi\sqrt{R^2+H^2}}$$

➢ Recognizing that $S = \sqrt{R^2+H^2}$ is the slant height of the cone:

$$\Rightarrow a = \frac{\lambda R}{2\pi S} \tag{5}$$

➢ Plugging this back in equation 3, we get:

$$b = \frac{\lambda H}{2\pi S} \tag{6}$$

Thus, we finally have that the parametric equation for the garland is:

$$C(t) = \frac{\lambda}{2\pi S}\begin{pmatrix} Rt\cos t \\ Rt\sin t \\ Ht \end{pmatrix}, \quad \forall t \in \left[0, \frac{2\pi S}{\lambda}\right] \tag{7}$$

Deriving an Equation for the Length of the Garland

With a function which models the garland, we can now calculate the length of the garland by calculating the *arc length* of $C(t)$. The arc length, L, is defined as the distance travelled along the path of a curve from one point to another ("8.1: Arc Length" 2017). The arc length of $C(t)$ can be evaluated by decomposing the curve into an infinite amount of infinitesimally small line segments, $\mathrm{d}L$, and adding their lengths. Thus, the length of the garland is represented by the definite integral:

$$L = \int_0^{\frac{2\pi S}{\lambda}} \mathrm{d}L \tag{8}$$

The lower and upper bounds on the integral are a result of the restrictions imposed on $C(t)$.

However, what is $\mathrm{d}L$? To find what $\mathrm{d}L$ is, it is useful to think of it as a 3-dimensional vector. One important property of vectors is that they can be expressed as a sum of multiple vectors, and thus 3D vectors can be decomposed into their x, y, and z components. Using this line of thinking, it can similarly be said that $\mathrm{d}L$ can be decomposed into the infinitesimals $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$, as visualized in Figure 6. Therefore, applying the 3D Pythagorean theorem:

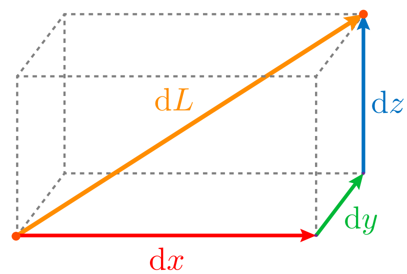


Figure 6: $\mathrm{d}L$ in terms of $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$

$$\mathrm{d}L = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2}$$

However, this is not of much use, as $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ are arbitrary. Thus, $\mathrm{d}t$ is introduced into the equation by multiplying the equation by $\frac{\mathrm{d}t}{\mathrm{d}t}$, and with some algebraic manipulation, $\mathrm{d}L$ can be expressed in terms of the derivatives of $x(t)$, $y(t)$, and $z(t)$ components of the curve, which can be easily evaluated (Schlicker, Keller, and Long, n.d.):

$$\mathrm{d}L = \sqrt{\frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2}{(\mathrm{d}t)^2}} \cdot \mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} \cdot \mathrm{d}t$$

- ➢ Substituting in $x(t)$, $y(t)$, and $z(t)$:

$$\begin{aligned}
\Rightarrow \mathrm{d}L &= \sqrt{\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{2\pi S}Rt\cos t\right)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{2\pi S}Rt\sin t\right)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{2\pi S}Ht\right)\right)^2} \cdot \mathrm{d}t \\
&= \frac{\lambda}{2\pi S}\sqrt{\left(\frac{\mathrm{d}}{\mathrm{d}t}(Rt\cos t)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}(Rt\sin t)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}(Ht)\right)^2} \cdot \mathrm{d}t \\
&= \frac{\lambda}{2\pi S}\sqrt{(R\cos t - Rt\sin t)^2 + (R\sin t - Rt\cos t)^2 + H^2} \cdot \mathrm{d}t
\end{aligned}$$

- ➢ Expanding and simplifying:

$$\begin{aligned}
\Rightarrow \mathrm{d}L &= \frac{\lambda}{2\pi S}\sqrt{(R^2\cos^2 t \cancel{-2R^2 t\sin t\cos t} + R^2t^2\sin^2 t) + (R^2\sin^2 t \cancel{+2R^2 t\sin t\cos t} + R^2t^2\cos^2 t) + H^2} \cdot \mathrm{d}t \\
&= \frac{\lambda}{2\pi S}\sqrt{R^2(\sin^2 t + \cos^2 t) + R^2t^2(\sin^2 t + \cos^2 t) + H^2} \cdot \mathrm{d}t \\
&= \frac{\lambda}{2\pi S}\sqrt{R^2 + H^2 + R^2t^2} \cdot \mathrm{d}t
\end{aligned}$$

$$= \frac{\lambda}{2\pi S}\sqrt{S^2 + R^2t^2} \cdot \mathrm{d}t \tag{9}$$

➢ Substituting this back into equation 8, we finally obtain:

$$L = \frac{\lambda}{2\pi S}\int_0^{\frac{2\pi S}{\lambda}} \sqrt{S^2 + R^2t^2} \cdot \mathrm{d}t \tag{10}$$

Evaluating the Integral

Now, we evaluate the integral so that we can finally obtain a general solution for the length of the garland based on our chosen parameters. Since the integral is of the form $\sqrt{c^2 + x^2}$, it can be evaluated using trigonometric substitution.

➢ Let $t = \frac{S}{R}\tan\theta$. Thus, $\mathrm{d}t = \frac{S}{R}\sec^2\theta\,\mathrm{d}\theta$. Substituting them into equation 10:

$$\begin{aligned}
\Rightarrow L &= \frac{\lambda}{2\pi \cancel{S}}\int_0^{t=\frac{2\pi S}{\lambda}} \sqrt{S^2 + R^2\left(\frac{S}{R}\tan\theta\right)^2} \cdot \frac{\cancel{S}}{R}\sec^2\theta\,\mathrm{d}\theta \\
&= \frac{\lambda}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} \sqrt{S^2 + S^2\tan^2\theta} \cdot \sec^2\theta\,\mathrm{d}\theta \\
&= \frac{\lambda}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} S\sec\theta \cdot \sec^2\theta\,\mathrm{d}\theta \\
&= \frac{\lambda S}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} \sec^3\theta\,\mathrm{d}\theta
\end{aligned} \tag{11}$$

Then, using integration by parts, the integral of $\sec^3\theta$ can be evaluated.

➢ Let $u = \sec\theta$ and $\mathrm{d}v = \sec^2\theta\,\mathrm{d}\theta$. Therefore, $\mathrm{d}u = \sec\theta\tan\theta$ and $v = \tan\theta$.

$$\Rightarrow L = \frac{\lambda S}{2\pi R}\left[\sec\theta\tan\theta\Big|_0^{t=\frac{2\pi S}{\lambda}} - \int_0^{t=\frac{2\pi S}{\lambda}} \sec\theta\tan^2\theta\,\mathrm{d}\theta\right]$$

➢ Using the identity $\tan^2\theta = \sec^2\theta - 1$:

$$\begin{aligned}
\Rightarrow L &= \frac{\lambda S}{2\pi R}\left[\sec\theta\tan\theta\Big|_0^{t=\frac{2\pi S}{\lambda}} - \int_0^{t=\frac{2\pi S}{\lambda}} (\sec^3\theta - \sec\theta)\,\mathrm{d}\theta\right] \\
&= \frac{\lambda S}{2\pi R}\left[\sec\theta\tan\theta\Big|_0^{t=\frac{2\pi S}{\lambda}} + \int_0^{t=\frac{2\pi S}{\lambda}} \sec\theta\,\mathrm{d}\theta\right] - \frac{\lambda S}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} \sec^3\theta\,\mathrm{d}\theta
\end{aligned}$$

➢ By $\int \sec\theta\,\mathrm{d}\theta = \ln(\sec\theta + \tan\theta) + c$:

$$\Rightarrow L = \frac{\lambda S}{2\pi R}\left[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\Big|_0^{t=\frac{2\pi S}{\lambda}}\right. - \frac{\lambda S}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} \sec^3\theta\,\mathrm{d}\theta$$

Note that when integrating, the argument of logarithms are typically wrapped in absolute value because indefinite integrals are usually evaluated for all real values, but logarithms are restricted to arguments greater than 0. However, since only positive answers for L are desired, as well as my chosen parameters being positive, the arguments of the natural log should never be negative, and thus the absolute value is not necessary.

➢ Back to the integral, since $\frac{\lambda S}{2\pi R}\int_0^{t=\frac{2\pi S}{\lambda}} \sec^3\theta \, \mathrm{d}\theta = L$ (equation 11):

$$\Rightarrow L = \frac{\lambda S}{2\pi R}\left[\sec\theta\tan\theta + \ln(\sec\theta+\tan\theta)\right|_0^{t=\frac{2\pi S}{\lambda}} - L$$
$$\Rightarrow 2L = \frac{\lambda S}{2\pi R}\left[\sec\theta\tan\theta + \ln(\sec\theta+\tan\theta)\right|_0^{t=\frac{2\pi S}{\lambda}}$$
$$\Rightarrow L = \frac{\lambda S}{4\pi R}\left[\sec\theta\tan\theta + \ln(\sec\theta+\tan\theta)\right|_0^{t=\frac{2\pi S}{\lambda}} \tag{12}$$

➢ Now, we want to reverse the substitution and bring the equation back in terms of our original variable, t. Recall that we substituted $t = \frac{S}{R}\tan\theta$, so $\tan\theta = \frac{R}{S}t$. Then, using the identity $\sec^2\theta = 1+\tan^2\theta$, we can find what $\sec\theta$ is equal to:

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{R^2}{S^2}t^2} = \frac{\sqrt{S^2+R^2t^2}}{S}$$

➢ Reversing the substitution in equation 12:

$$L = \frac{\lambda S}{4\pi R}\left[\frac{\sqrt{S^2+R^2t^2}}{S}\cdot\frac{R}{S}t + \ln\left(\frac{\sqrt{S^2+R^2t^2}}{S} + \frac{R}{S}t\right)\right|_0^{\frac{2\pi S}{\lambda}} \tag{13}$$

➢ Using **FTC Part 2** to evaluate the integral between the lower and upper bound:

$$\Rightarrow L = \frac{\lambda S}{4\pi R}\left[\frac{\sqrt{S^2+R^2\left(\frac{2\pi S}{\lambda}\right)^2}}{S}\cdot\frac{R}{S}\left(\frac{2\pi S}{\lambda}\right) + \ln\left(\frac{\sqrt{S^2+R^2\left(\frac{2\pi S}{\lambda}\right)^2}}{S} + \frac{R}{S}\left(\frac{2\pi S}{\lambda}\right)\right) - \right.$$
$$\left. -\cancel{\frac{\sqrt{S^2+R^2(0)^2}}{S}\cdot\frac{R}{S}(0)} - \ln\left(\frac{\sqrt{S^2+R^2(0)^2}}{S} + \cancel{\frac{R}{S}(0)}\right)\right]$$
$$= \frac{S}{2}\sqrt{1+\frac{4\pi^2R^2}{\lambda}} + \frac{\lambda S}{4\pi R}\left[\ln\left(\sqrt{1+\frac{4\pi^2R^2}{\lambda}} + \frac{2\pi R}{\lambda}\right) - \ln(1)\right]$$
$$= \frac{S}{2\lambda}\sqrt{\lambda^2+4\pi^2R^2} + \frac{\lambda S}{4\pi R}\ln\left(\frac{\sqrt{\lambda^2+4\pi^2R^2}+2\pi R}{\lambda}\right)$$

➢ After simplifying and reversing the substitution $S = \sqrt{R^2+H^2}$, we arrive at the final form of the general solution for the length of the garland:

$$\boxed{L(\lambda,R,H) = \frac{\sqrt{R^2+H^2}}{2\lambda}\sqrt{\lambda^2+4\pi^2R^2} + \frac{\lambda\sqrt{R^2+H^2}}{4\pi R}\left[\ln\left(\sqrt{\lambda^2+4\pi^2R^2}+2\pi R\right) - \ln\lambda\right]} \tag{14}$$

where $\lambda, R, H \in \mathbb{R}^+$.

Validation

In order to verify that my equation is correct, I decided to test a set of 9 data points with different values for λ, R, and H. This was done by first inputting the values into the formula and evaluating using a calculator, and then comparing the result to the output generated by *GeoGebra*'s `Length` function, which is capable of calculating the arc length of an inputted function given a starting t-value and ending t-value. The results are summarized in the table below.

Parameters			**Output**	
λ	R	H	$L(\lambda, R, H)$	GeoGebra
10	90	30	2686.72443	2686.72443
20	60	60	809.01467	809.01467
30	30	90	320.94376	320.94376
20	60	30	639.58225	639.58225
30	30	60	226.94151	226.94151
10	90	90	3604.61907	3604.61907
30	30	30	143.53041	143.53041
10	90	60	3063.33716	3063.33716
20	60	90	1031.29539	1031.29539

Table 6.1: Comparison between the output of my formula and *GeoGebra*.

From the table, we can see that the output from my formula is accurate, as it matches the output generated by GeoGebra to 5 decimals points. While this does not prove that my general formula is accurate for all values of λ, R, and H, having 9 points where the outputs are exactly the same to 5 decimals is enough to convince me that my formula is exactly correct.

Finding An Optimal Solution

Now that I have derived the general formula for the length of the garland and convinced myself that it is correct, I can now devise a way to meet my objective of meeting personal aesthetic preferences while minimizing waste. This is because the height H and radius R of the tree is fixed, and as such the spacing between successive rotations of the garland λ is the only parameter which could theoretically be optimized.

The first thing I did was to check whether $L(\lambda, R, H)$ had any minima. This is because if there were minima for the multivariate equation, whether global or local, this would mean that for certain values of R and H, there would exist optimal solution(s) for λ which result in shorter lengths of garland than other values of λ in their neighborhood, which would mean less garland used and thus less waste. One way that I could check for minima is to evaluate the partial derivatives of $L(\lambda, R, H)$ with respect to each of the 3 parameters λ, R, and H, and set them equal to zero. i.e. $\frac{\partial L}{\partial \lambda} = 0$, $\frac{\partial L}{\partial R} = 0$, and $\frac{\partial L}{\partial H} = 0$. This would give me 3 equations, and I could solve the system of equations to obtain values of λ, R, and H which would correspond to the critical points that could potentially be minima. However, given the complexity of the partial derivatives, it is probably very difficult or outright impossible to obtain a solution analytically. I would instead need to find a way to evaluate this numerically, and so I turned to *Wolfram Alpha*, which promptly told me that the equation in fact had no global or local minima at all. Thus, there are never any situations where certain values of λ are objectively more optimal in that it uses less garland than other λ values in its neighborhood, and as such I will have turn to more subjective means to define what I mean by "optimal" solutions.

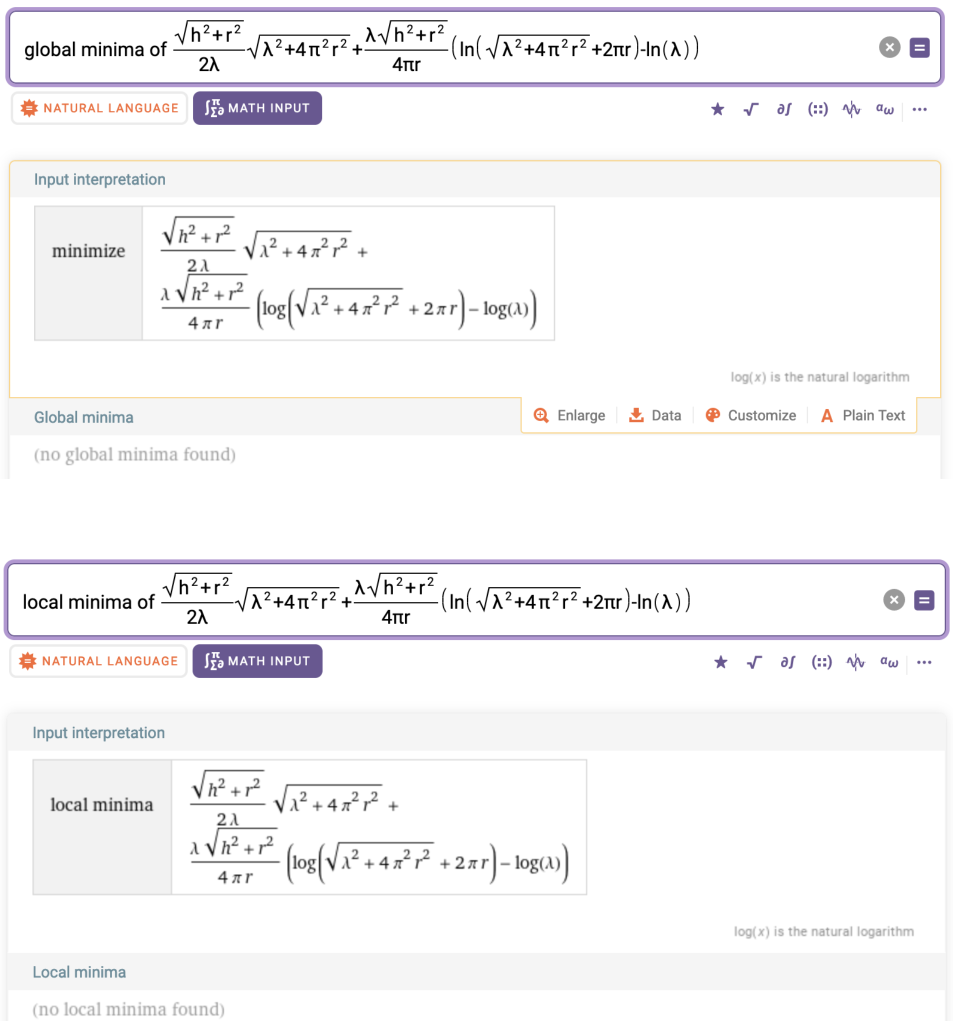


Figure 7: The equation has no global or local minima, according to *Wolfram Alpha*.

Given that there are no globally optimal solutions, I decided that I should instead focus on my particular Christmas tree, which has the dimensions $R = 15\,\text{in}$ and $H = 72\,\text{in}$. I realized it was important to first understand the dynamics of the function, and so I graphed the function in *Desmos*, with λ as the independent variable and L as the dependent variable, as visualized in Figure 8. Firstly, I noted that the function was decreasing over the entire domain, meaning that as λ increases, the length of garland required L decreases. This is reasonable, given that larger spacing between successive rotations of the garland would mean that garland would have to go around the tree a lesser amount of times before it reaches to the tip of the tree, and thus mean that a shorter length of garland is necessary.

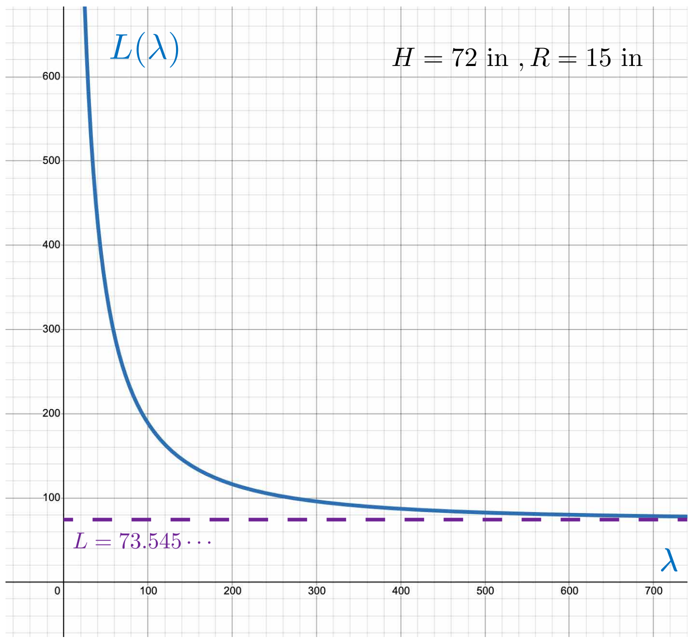


Figure 8: Graph of L vs. λ with $H = 72\,\text{in}$ and $R = 15\,\text{in}$

Another interesting thing to note is that the function is asymptotic, because from the graph we can see that as $\lambda \to +\infty$, L converges towards $L = 73.5459\,\text{in}$, which is the slant height of the cone ($\sqrt{15^2 + 72^2} = 73.5459\cdots$). In fact, I prove in Appendix A that:

$$\lim_{\lambda \to +\infty} L(\lambda, R, H) = \sqrt{R^2 + H^2} \qquad \forall R, H \in \mathbb{R}^+ \tag{15}$$

which is the slant height of the tree. This makes sense, given that the shortest length possible for the garland would be a straight line from the tip of the tree to the bottom of the tree, which would be equal to the slant height. Finally, we see that as λ decreases, the amount garland needed exponentially increases. Thus, while I want to space the garland such that the tree look nice according to my personal preferences, it is also important not to choose a spacing too small, as it would require a lot more garland, and not only is that wasteful and unnecessary spending, it is harmful to the environment as I will be using plastic tinsel garland.

Since the function does not have minima, I will have to subject to additional requirements in order to arrive at "optimal" solution(s). One thing I quickly realized was that the function was continuous, but in reality garland is typically sold in standard unit lengths; for example, the garland which my family bought is sold in 6 ft lengths (or 72 in). This means that the amount of garland that I buy can only be a positive multiple of unit lengths of garland, which can mathematically represented thus:

$$L_G = nG, \quad n \in \mathbb{Z}^+ \tag{16}$$

where L_G denotes the total length of garland required, G represents the individual unit lengths of the garland, and $n \in \mathbb{Z}^+$ is the number of lengths of garland that I need to buy. As such, if I wanted to minimize wasted and have as little excess garland as possible, I would want the theoretical minimum amount of garland, L, to be roughly equal to some multiple of G, i.e. $L(\lambda, R, H) = nG$. and find solutions for λ My initial approach was to equate $L(\lambda, R, H)$ to nG and isolate for λ. I could then input positive integers into n to get all possible values for λ which result in lengths of the garland that are multiples of G. However, I quickly realized that this was not viable, as the complexity of the equation from the square roots and the logarithms means that it is very difficult or most likely impossible to isolate for λ. As such, this problem will have to be solved numerically.

Given that we know the value of $G = 72\,\text{in}$, we can calculate the amount of garland necessary, n, by dividing L by G, and rounding up the result to a whole number. This can be mathematically represented using the ceiling function:

$$n = \left\lceil \frac{L(\lambda, R, H)}{G} \right\rceil \tag{17}$$

Plugging this for n in equation 16, we get that:

$$L_G(\lambda, R, H) = G \left\lceil \frac{L(\lambda, R, H)}{G} \right\rceil \tag{18}$$

Using this, I can graph L_G and compare it to L.

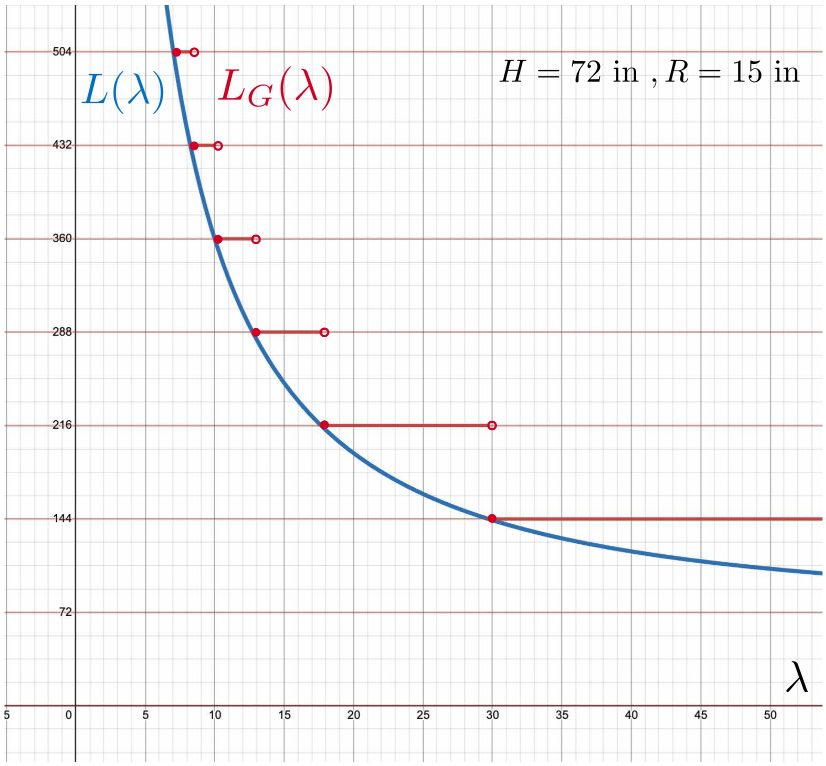


Figure 9: A comparison between L and L_G.

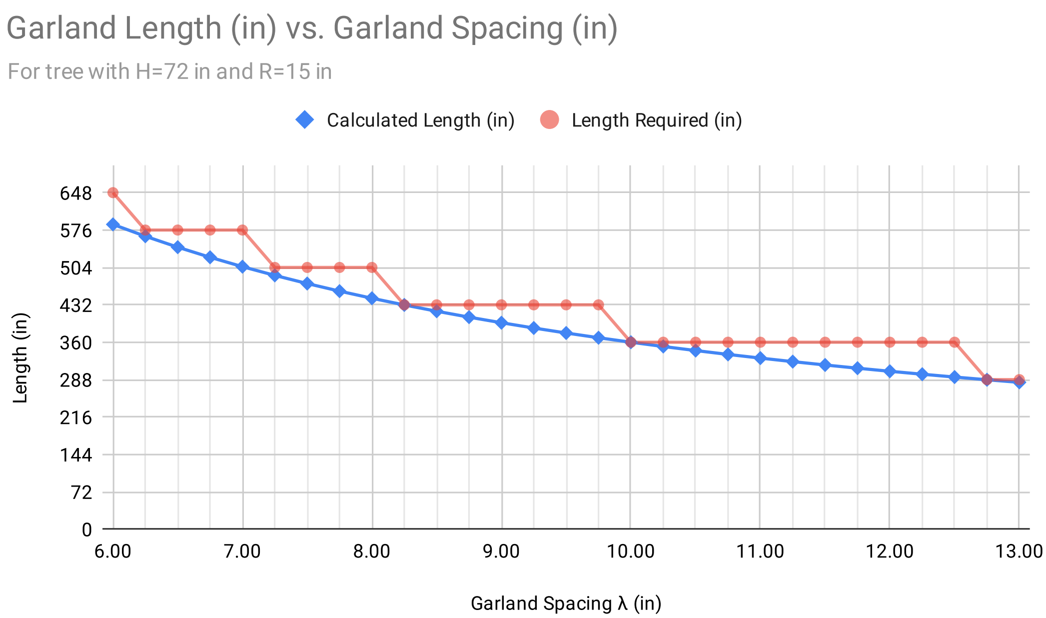
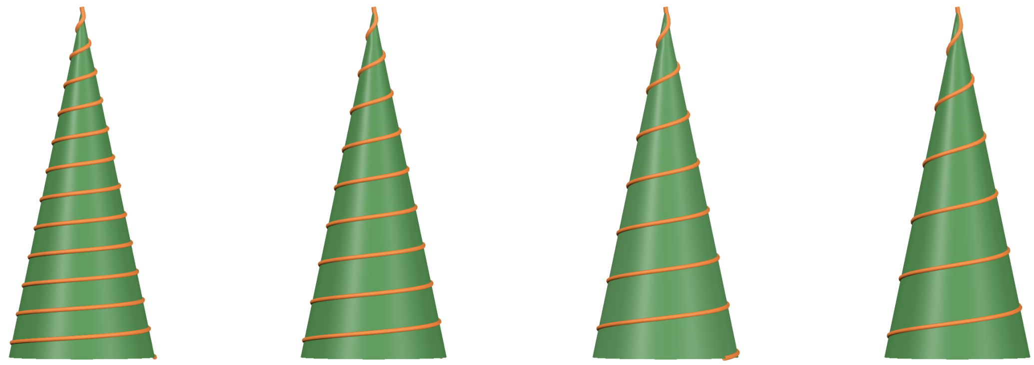


Figure 10: Graph showing the length of garland L vs. the garland spacing λ. (Generated using *Google Sheets*)

The fact that garland is sold in unit lengths means that there will almost always be some lengths of garland which will be excess unless the theoretical garland length L calculated by my general formula is exactly equal to L_G.



(a) 6 in Spacing. **(b)** 8 in Spacing. **(c)** 10 in Spacing. **(d)** 12 in Spacing.

Figure 11: Comparison between different spacing for the garland.

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A Evaluating the Limit of $L(\lambda, R, H)$

$$
\begin{aligned}
&\lim_{\lambda\to+\infty}\left(\frac{\sqrt{R^2+H^2}}{2\lambda}\sqrt{\lambda^2+4\pi^2R^2}+\frac{\lambda\sqrt{R^2+H^2}}{4\pi R}\left[\ln\left(\sqrt{\lambda^2+4\pi^2R^2}+2\pi R\right)-\ln\lambda\right]\right)\\
&=\lim_{\lambda\to+\infty}\left(\frac{\sqrt{R^2+H^2}}{2\cancel{\lambda}}\cdot\cancel{\lambda}\cancelto{1}{\sqrt{1+\frac{4\pi^2R^2}{\lambda^2}}}\right)+\lim_{\lambda\to+\infty}\left(\frac{\lambda\sqrt{R^2+H^2}}{4\pi R}\cdot\ln\left(\frac{\sqrt{\lambda^2+4\pi^2R^2}+2\pi R}{\lambda}\right)\right)\\
&=\frac{\sqrt{R^2+H^2}}{2}+\lim_{\lambda\to+\infty}\frac{\ln\left(\frac{\sqrt{\lambda^2+4\pi^2R^2}+2\pi R}{\lambda}\right)}{\frac{4\pi R}{\lambda\sqrt{R^2+H^2}}}
\end{aligned}
$$

➢ Since the limit of the second term evaluates to $\frac{0}{0}$, by l'Hôpital's rule:

$$
\begin{aligned}
&=\frac{\sqrt{R^2+H^2}}{2}+\lim_{\lambda\to+\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\ln\left(\frac{\sqrt{\lambda^2+4\pi^2R^2}+2\pi R}{\lambda}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{4\pi R}{\lambda\sqrt{R^2+H^2}}\right)}\\
&=\frac{\sqrt{R^2+H^2}}{2}+\lim_{\lambda\to+\infty}\frac{\frac{-1}{\lambda}+\frac{\lambda}{2\pi R\sqrt{\lambda^2+4\pi^2R^2}+4\pi^2R^2+\lambda^2}}{\frac{-4\pi R}{\lambda^2\sqrt{R^2+H^2}}}\\
&=\frac{\sqrt{R^2+H^2}}{2}+\lim_{\lambda\to+\infty}\left(\frac{-\lambda^2\sqrt{R^2+H^2}}{4\pi R}\cdot\frac{\cancel{\lambda^2}-2\pi R\sqrt{\lambda^2+4\pi^2R^2}-4\pi^2R^2\cancel{-\lambda^2}}{2\pi R\lambda\sqrt{\lambda^2+4\pi^2R^2}+4\pi^2R^2\lambda+\lambda^3}\right)\\
&=\frac{\sqrt{R^2+H^2}}{2}+\lim_{\lambda\to+\infty}\left(\frac{\lambda^2\sqrt{R^2+H^2}}{4\pi R}\cdot\frac{2\pi R\lambda\sqrt{1+\frac{4\pi^2R^2}{\lambda^2}}+4\pi^2R^2}{2\pi R\lambda^2\sqrt{1+\frac{4\pi^2R^2}{\lambda^2}}+4\pi^2R^2\lambda+\lambda^3}\right)
\end{aligned}
$$

➢ Multiplying the numerator and denominator by $\frac{1}{\lambda^3}$:

$$
\begin{aligned}
&=\frac{\sqrt{R^2+H^2}}{2}+\lim_{\lambda\to+\infty}\left(\frac{\sqrt{R^2+H^2}}{4\pi R}\cdot\frac{2\pi R\sqrt{1+\frac{4\pi^2R^2}{\lambda^2}}+\frac{4\pi^2R^2}{\lambda}}{\frac{2\pi R\sqrt{1+\frac{4\pi^2R^2}{\lambda^2}}}{\lambda}+\frac{4\pi^2R^2}{\lambda^2}+1}\right)\\
&=\frac{\sqrt{R^2+H^2}}{2}+\lim_{\lambda\to+\infty}\left(\frac{\sqrt{R^2+H^2}}{2}\cdot\frac{\cancelto{1}{\sqrt{1+\frac{4\pi^2R^2}{\lambda^2}}}+\cancelto{0}{\frac{2\pi R}{\lambda}}}{\cancelto{0}{\frac{2\pi R\sqrt{1+\frac{4\pi^2R^2}{\lambda^2}}}{\lambda}}+\cancelto{0}{\frac{4\pi^2R^2}{\lambda^2}}+1}\right)\\
&=\frac{\sqrt{R^2+H^2}}{2}+\frac{\sqrt{R^2+H^2}}{2}\\
&=\boxed{\sqrt{R^2+H^2}}
\end{aligned}
$$

Thus, the limit of $L(\lambda, R.H)$ as $\lambda\to+\infty$ will always be equal to the slant height of a cone with height H and base radius R.