1 Section 3.4 Exercises

Exercises with solutions from Section 3.4 of [UA].

Exercise 3.4.1. If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Solution. P is closed, so $P \cap K$ must be compact (Exercise 3.3.4 (a)). However, $P \cap K$ need not be perfect. For a counterexample, consider $P = [0, 1]$ and $K = \{0\}$.

Exercise 3.4.2. Does there exist a perfect set consisting of only rational numbers?

Solution. No. By Theorem 3.4.3, a non-empty perfect set must be uncountable, but any subset of $\mathbf{Q}$ is either finite or countably infinite.

Exercise 3.4.3. Review the portion of the proof given in Example 3.4.2 and follow these steps to complete the argument.

(a) Because $x \in C_1$, argue that there exists an $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| \leq 1/3$.

(b) Finish the proof by showing that for each $n \in \mathbf{N}$, there exists $x_n \in C \cap C_n$, different from x satisfying $|x - x_n| \leq 1/3^n$.

Solution. (a) We have $C_1 = [0, 1/3] \cup [2/3, 1]$. The endpoints of these intervals are never removed at any subsequent stage of the construction of the Cantor set, so they belong to C. Since $x \in C_1$, it must belong to one of these intervals, say the interval $[0, 1/3]$. If $0 \leq x < 1/3$, then take $x_1 = 1/3$, and if $x = 1/3$, then take $x_1 = 0$. We can make similar choices if $x \in [2/3, 1]$. In any case, we have chosen an $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| \leq 1/3$.

(b) Let $n \in \mathbf{N}$ be given. The set C_n consists of 2^n disjoint closed intervals each of length $1/3^n$. The endpoints of these intervals are never removed at any subsequent stage of the construction of the Cantor set, so they belong to C. Since $x \in C$, we have $x \in C_n$ and hence x must belong to one of the disjoint closed intervals, say $I = [a, b]$ where $b - a = 1/3^n$. If $a \leq x < b$, then let $x_n = b$, and if $x = b$ then let $x_n = a$. In either case, we have chosen an $x_n \in C \cap C_n$ such that $x \neq x_n$ and $|x - x_n| \leq b - a = 1/3^n$.

Thus x is the limit of a sequence (x_n) contained in C such that $x_n \neq x$ for all $n \in \mathbf{N}$. It follows that x is a limit point of C and hence that C contains no isolated points.

Exercise 3.4.4. Repeat the Cantor construction from Section 3.1 starting with the interval $[0, 1]$. This time, however, remove the open middle *fourth* from each component.

(a) Is the resulting set compact? Perfect?

(b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

Solution. We begin with $B_0 := [0,1]$ and remove the open middle fourth to obtain $B_1 = \left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]$. Notice that each interval has length $\frac{3}{8}$. Next we remove the open middle fourth from each of the two intervals of B_1 to obtain

$$B_2 = \left(\left[0, \tfrac{9}{64}\right] \cup \left[\tfrac{15}{64}, \tfrac{24}{64}\right]\right) \cup \left(\left[\tfrac{40}{64}, \tfrac{49}{64}\right] \cup \left[\tfrac{55}{64}, 1\right]\right).$$

Notice that each interval has length $\left(\frac{3}{8}\right)^2$. We continue in this fashion, obtaining sets B_n consisting of 2^n disjoint closed intervals each of length $\left(\frac{3}{8}\right)^n$, and define our Cantor-like set $B := \bigcap_{n=0}^{\infty} B_n$.

(a) The set B is compact and perfect; the arguments used for the Cantor set work equally well for B. Since each B_n is the finite union of closed sets, B_n is closed, and then since B is the intersection of closed sets, B is also closed. Clearly B is also bounded and hence B is compact. As in Exercise 3.4.3, given any $x \in B$, we can find a sequence of endpoints (x_n) such that $x_n \in B \cap B_n, x_n \neq x$, and $|x - x_n| \leq \left(\frac{3}{8}\right)^n$ for each $n \in \mathbf{N}$. It follows that x is a limit point of B and hence that B has no isolated points. Since B is also closed, we see that B is a perfect set.

(b) At the first stage, we remove an interval of length $\frac{1}{4}$. At the n^{th} stage $(n = 2, 3, 4 \ldots)$, we remove 2^{n-1} intervals each of length $\frac{1}{4}\left(\frac{3}{8}\right)^{n-1}$. Thus the length of B is

$$1 - \left(\tfrac{1}{4} + 2 \cdot \tfrac{1}{4} \cdot \tfrac{3}{8} + 2^2 \cdot \tfrac{1}{4} \cdot \left(\tfrac{3}{8}\right)^2 + \cdots\right) = 1 - \tfrac{1}{4}\left(1 + \tfrac{3}{4} + \left(\tfrac{3}{4}\right)^2 + \cdots\right) = 1 - \frac{\frac{1}{4}}{1 - \frac{3}{4}} = 0.$$

To calculate the dimension of B, we magnify the set by a factor of $\frac{8}{3}$, so that B_0 becomes the closed interval $\left[0, \frac{8}{3}\right]$. Then when we remove the open middle fourth of this interval, we are left with two intervals of length 1:

$$B_1 = [0,1] \cup \left[\tfrac{5}{3}, \tfrac{8}{3}\right].$$

Thus we will obtain two copies of B. Then the dimension x of B is given by solving $2 = \left(\frac{8}{3}\right)^x$, which gives

$$x = \frac{\log(2)}{\log\left(\frac{8}{3}\right)} \approx 0.7067.$$

Exercise 3.4.5. Let A and B be nonempty subsets of $\mathbf{R}$. Show that if there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.

Solution. Observe that V^{c} is a closed set which contains A (since $U \cap V = \emptyset \implies A \cap V = \emptyset$). Since $\overline{A}$ is the smallest closed set containing A (Theorem 3.2.12), we must have $\overline{A} \subseteq V^{\mathsf{c}}$, which gives

$$\overline{A} \subseteq V^{\mathsf{c}} \implies \overline{A} \cap V = \emptyset \implies \overline{A} \cap B = \emptyset.$$

Similarly, $A \cap \overline{B} = \emptyset$. Thus A and B are separated.

Exercise 3.4.6. Prove Theorem 3.4.6.

Solution. Suppose we have non-empty subsets $A, B \subseteq \mathbf{R}$ such that every convergent sequence contained in one of the subsets has a limit which does not belong to the other subset. Since a limit point of A is the limit of a sequence contained in A and an element of A is the limit of a constant sequence contained in A, and by assumption these limits do not belong to B, we see that $\overline{A} \cap B = \emptyset$. Similarly, $A \cap \overline{B} = \emptyset$. Thus A and B are separated.

Conversely, suppose that A and B are separated. If $(x_n) \to x$ is a convergent sequence contained in A, then $x \in \overline{A}$. It follows that $x \notin B$ since $\overline{A} \cap B = \emptyset$. Similarly, the limit of any convergent sequence contained in B must not belong to A.

We have now shown that for non-empty subsets $A, B \subseteq \mathbf{R}$, A and B being separated is equivalent to the condition that every convergent sequence contained in one of the subsets has a limit which does not belong to the other subset.

Proving Theorem 3.4.6 is equivalent to showing that a subset $E \subseteq \mathbf{R}$ is disconnected if and only if there exist non-empty subsets $A, B \subseteq E$ such that $E = A \cup B$ and every convergent sequence contained in one of the subsets has a limit which does not belong to the other subset. By the previous discussion, such subsets are separated. So the theorem follows by the definition of disconnectedness.

Exercise 3.4.7. A set E is *totally disconnected* if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A, y \in B$, and $E = A \cup B$.

(a) Show that $\mathbf{Q}$ is totally disconnected.

(b) Is the set of irrational numbers totally disconnected?

Solution. (a) Suppose that $p < q$ are rational numbers. By the density of $\mathbf{I}$ in $\mathbf{R}$, there exists an irrational number y such that $p < y < q$. Define the sets

$$A = (-\infty, y) \cap \mathbf{Q} \quad \text{and} \quad B = (y, \infty) \cap \mathbf{Q}.$$

Then $p \in A, q \in B$, and since $y \notin \mathbf{Q}$, we have $A \cup B = \mathbf{Q}$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, we have $\overline{A} = (-\infty, y]$ and $\overline{B} = [y, \infty)$. It follows that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ and hence that A and B are separated. Thus $\mathbf{Q}$ is totally disconnected.

(b) **I** is also totally disconnected. To see this, reverse the roles of **Q** and **I** in the solution to part (a).

Exercise 3.4.8. Follow these steps to show that the Cantor set is totally disconnected in the sense described in Exercise 3.4.7.

Let $C = \bigcap_{n=0}^{\infty} C_n$, as defined in Section 3.1.

(a) Given $x, y \in C$, with $x < y$, set $\epsilon = y - x$. For each $n = 0, 1, 2, \ldots$, the set C_n consists of a finite number of closed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval of C_N.

(b) Show that C is totally disconnected.

Solution. (a) If I is an interval of length δ, then any $a, b \in I$ must satisfy $|a - b| \leq \delta$. In the construction of C, each C_n consists of 2^n disjoint closed intervals each of length 3^{-n}. Thus we can find an N large enough so that C_N consists of closed intervals each of length $3^{-N} < \epsilon = y - x$, i.e. whose length is smaller than the distance between x and y. Then x and y cannot possibly belong to the same interval of C_N.

(b) Let $[a, b]$ be the closed interval of C_N which contains x and note that the open interval $\left(b, b + \frac{1}{3^N}\right)$ was either removed at the N^{th} stage of construction or is a subset of an open interval which was removed at a previous stage of construction. So if we set $t := b + \frac{1}{2 \cdot 3^N}$, then $t \notin C$. Since $y \notin [a, b]$ and $y > x$, we must have $y > t$. Define

$$A = (-\infty, t) \cap C \quad \text{and} \quad B = (t, \infty) \cap C.$$

Then $x \in A, y \in B$, and since $t \notin C$, we have $A \cup B = C$. If $(z_n) \to z$ is a convergent sequence contained in A, then the Order Limit Theorem implies that $z \leq t$ and hence that $z \notin B$. Similarly, the limit of any convergent sequence contained in B cannot belong to A. Thus A and B are separated by Theorem 3.4.6 (see Exercise 3.4.6). It follows that C is totally disconnected.

Exercise 3.4.9. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rational numbers, and for each $n \in \mathbf{N}$ set $\epsilon_n = 1/2^n$. Define $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ and let $F = O^{\mathsf{c}}$.

(a) Argue that F is a closed, nonempty set consisting only of irrational numbers.

(b) Does F contain any nonempty open intervals? Is F totally disconnected? (See Exercise 3.4.7 for the definition.)

(c) Is it possible to know whether F is perfect? If not, can we modify this construction to produce a nonempty perfect set of irrational numbers?

Solution. (a) O is an open set since it is a union of open intervals, so $F = O^c$ must be closed. To see that F is non-empty, suppose otherwise. Then $O = \mathbf{R}$, so the collection $\{V_{\epsilon_n}(r_n) : n \in \mathbf{N}\}$ is an open cover of the compact set $[0, 10]$. Thus there exist finitely many indices $n_1 < \cdots < n_K$ such that

$$[0, 10] \subseteq V_{\epsilon_{n_1}}(r_{n_1}) \cup \cdots \cup V_{\epsilon_{n_K}}(r_{n_K}).$$

However, the interval $[0, 10]$ has length 10, whereas the set $V_{\epsilon_{n_1}}(r_{n_1}) \cup \cdots \cup V_{\epsilon_{n_K}}(r_{n_K})$ has total length at most

$$\sum_{k=1}^{K} \frac{1}{2^{n_k - 1}} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} = 2,$$

since $\left|V_{\epsilon_{n_k}}(r_{n_k})\right| = 2\epsilon_{n_k} = 1/2^{n_k-1}$. So we have a set of length 10 contained inside a set of length 2, which is a contradiction. Thus F is non-empty. Clearly, $\mathbf{Q} \subseteq O$, so $F = O^c$ can contain only irrational numbers.

(b) F cannot contain any non-empty open intervals, since this would imply that F contains a rational number (indeed, infinitely many rational numbers), but by part (a) F contains only irrational numbers.

To see that F is totally disconnected, let us show that any subset of a totally disconnected set is also totally disconnected. Suppose we have sets $E \subseteq G \subseteq \mathbf{R}$ such that G is totally disconnected. Let $x, y \in E$ be given. Then since x and y belong to the totally disconnected set G, there exist separated sets A and B such that $x \in A, y \in B$, and $G = A \cup B$. Set $A' = A \cap E$ and $B' = B \cap E$ and note that $x \in A'$ and $y \in B'$. Furthermore, $A' \subseteq A$ and $B' \subseteq B$, so

$$\overline{A'} \subseteq \overline{A} \implies \overline{A'} \cap B' \subseteq \overline{A} \cap B' \subseteq \overline{A} \cap B = \emptyset.$$

Thus $\overline{A'} \cap B' = \emptyset$, and similarly $A' \cap \overline{B'} = \emptyset$, so that A' and B' are separated. Finally,

$$E = E \cap G = E \cap (A \cup B) = (A \cap E) \cup (B \cap E) = A' \cup B'.$$

It follows that E is totally disconnected.

Since F is a subset of $\mathbf{I}$, which we showed was totally disconnected in Exercise 3.4.7, by the previous paragraph we have that F is totally disconnected.

(c) There are enumerations of $\mathbf{Q}$ which, when used in this construction, will result in an F which is not perfect, i.e. an F with at least one isolated point. We will construct such an enumeration (r_n), which gives an F with $\sqrt{2}$ as an isolated point, via the following four step process. (Any irrational number would also work in place of $\sqrt{2}$.)

Step 1. We will first construct a strictly increasing sequence (p_n) of distinct rational numbers such that:

(1.1) $p_1 < p_2 < p_3 < \cdots < \sqrt{2}$;

(1.2) $\left(\sqrt{2} - \frac{1}{16}, \sqrt{2}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(p_n)$;

(1.3) $\sqrt{2} \notin \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(p_n)$.

This sequence will be placed in the final enumeration (r_n) as $r_{4n} = p_n$, so that

$$r_4 = p_1, r_8 = p_2, r_{12} = p_3, \ldots$$

Step 2. Mirroring Step 1, we will construct a strictly decreasing sequence (q_n) of distinct rational numbers such that:

(2.1) $\sqrt{2} < \cdots < q_3 < q_2 < q_1$;

(2.2) $\left(\sqrt{2}, \sqrt{2} + \frac{1}{16}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(q_n)$;

(2.3) $\sqrt{2} \notin \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(q_n)$.

This sequence will be placed in the final enumeration (r_n) as $r_{4n-2} = q_n$, so that

$$r_2 = q_1, r_6 = q_2, r_{10} = q_3, \ldots$$

Step 3. There are infinitely many rational numbers which belong to neither of the sequences (p_n) nor (q_n) from Steps 1 and 2. We will construct a sequence (a_n) which enumerates these remaining rational numbers in such a way that $\sqrt{2}$ will not be excluded from F in the final construction, i.e. a sequence (a_n) such that:

(3.1) $a_m \neq a_n$ for $m \neq n$;

(3.2) for each rational $r \in (\{p_1, p_2, \ldots\} \cup \{q_1, q_2, \ldots\})^{\mathsf{c}}$, there exists an $n \in \mathbf{N}$ such that $a_n = r$.

(3.3) $\sqrt{2} \notin \bigcup_{n=1}^{\infty} V_{\epsilon_{2n-1}}(a_n)$.

This sequence will be placed in the final enumeration (r_n) as $r_{2n-1} = a_n$, so that

$$r_1 = a_1, r_3 = a_2, r_5 = a_3, \ldots$$

Step 4. We will combine the sequences $(p_n), (q_n)$, and (a_n) to obtain an enumeration (r_n) of $\mathbf{Q}$ given by

$$a_1, q_1, a_2, p_1, a_3, q_2, a_4, p_2, \ldots$$

Letting $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ and $F = O^{c}$, we will have

$$\left(\sqrt{2} - \tfrac{1}{16}, \sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{2} + \tfrac{1}{16}\right) \subseteq O \quad \text{and} \quad \sqrt{2} \notin O,$$

so that $\left(\sqrt{2} - \frac{1}{16}, \sqrt{2} + \frac{1}{16}\right) \cap F = \left\{\sqrt{2}\right\}$. Thus $\sqrt{2}$ will be an isolated point of F.

Step 1.

For each $n \in \mathbf{N}$, let p_n be a rational number satisfying

$$\sqrt{2} - \frac{1}{2^{4n}} - \frac{1}{2^{4n+4}} < p_n < \sqrt{2} - \frac{1}{2^{4n}};$$

the existence of such a rational number is guaranteed by the density of $\mathbf{Q}$ in $\mathbf{R}$. Observe that for each $n \in \mathbf{N}$ we have

$$p_n < \sqrt{2} - \frac{1}{2^{4n}} < \sqrt{2} - \frac{1}{2^{4n+4}} - \frac{1}{2^{4n+8}} < p_{n+1} \quad \text{and} \quad p_n < \sqrt{2}.$$

Thus the sequence (p_n) satisfies condition (1.1).

For any $n \in \mathbf{N}$ we have

$$p_n < p_{n+1} < \sqrt{2} - \frac{1}{2^{4n+4}} < p_n + \frac{1}{2^{4n}} < \sqrt{2},$$

so that $p_{n+1} \in (p_n, p_n + 2^{-4n}) \subseteq V_{\epsilon_{4n}}(p_n)$, i.e. the centre of $V_{\epsilon_{4n+4}}(p_{n+1})$ is contained in $V_{\epsilon_{4n}}(p_n)$. Thus, for any $N \in \mathbf{N}$, the union $\bigcup_{n=1}^{N} V_{\epsilon_{4n}}(p_n)$ must be an open interval:

$$\bigcup_{n=1}^{N} V_{\epsilon_{4n}}(p_n) = \left(p_1 - \tfrac{1}{16}, B\right),$$

where $B = \max\{p_n + \frac{1}{2^{4n}} : 1 \leq n \leq N\}$ (the exact value of B is not particularly important, but note that it must be strictly less than $\sqrt{2}$). Since

$$p_1 < \sqrt{2} - \frac{1}{16} < p_1 + \frac{1}{16},$$

we have $\sqrt{2} - \frac{1}{16} \in \bigcup_{n=1}^{N} V_{\epsilon_{4n}}(p_n)$ for any $N \in \mathbf{N}$. Let $y \in \mathbf{R}$ be such that $\sqrt{2} - \frac{1}{16} < y < \sqrt{2}$. Since (p_n) is increasing and converges to $\sqrt{2}$, we can find an $N \in \mathbf{N}$ such that $y < p_N < \sqrt{2}$. Then since $\sqrt{2} - \frac{1}{16}$ and p_N both belong to the open interval $\bigcup_{n=1}^{N} V_{\epsilon_{4n}}(p_n)$ and y lies between these two values, we must have $y \in \bigcup_{n=1}^{N} V_{\epsilon_{4n}}(p_n)$ and thus $y \in \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(p_n)$. Hence the sequence (p_n) satisfies condition (1.2).

Finally, as noted above we have $p_n + 2^{-4n} < \sqrt{2}$ for all $n \in \mathbf{N}$, so the sequence (p_n) also satisfies condition (1.3).

Step 2.

The construction of the sequence (q_n) is analogous to the construction given in Step 1; for each $n \in \mathbf{N}$, let q_n be a rational number satisfying

$$\sqrt{2} + \frac{1}{2^{4n-2}} < q_n < \sqrt{2} + \frac{1}{2^{4n-2}} + \frac{1}{2^{4n+2}};$$

the existence of such a rational number is guaranteed by the density of $\mathbf{Q}$ in $\mathbf{R}$. Similar logic to that given in Step 1 shows that the sequence (q_n) satisfies condition (2.1), and furthermore that $\left(\sqrt{2}, \sqrt{2} + \frac{1}{4}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(q_n)$, which gives us condition (2.2). Condition (2.3) follows since $\sqrt{2} < q_n - \frac{1}{2^{4n-2}}$ for all $n \in \mathbf{N}$.

Step 3.

Since the sequences (p_n) and (q_n) constructed in Steps 1 and 2 are entirely contained inside the interval $[p_1, q_1]$, it is clear that there are infinitely many rational numbers left to enumerate. That is, letting

$$E = \mathbf{Q} \cap (\{p_1, p_2, \ldots\} \cup \{q_1, q_2, \ldots\})^{\mathsf{c}},$$

we have that E is countably infinite. However, enumerating E carelessly might exclude $\sqrt{2}$ from F in Step 4, since there are rational numbers in E arbitrarily close to $\sqrt{2}$; placing one of these rational numbers "too early" in the sequence (r_n) will include $\sqrt{2}$ in some $V_{\epsilon_n}(r_n)$. To surmount this problem, we will first partition E as follows. Let

$$A_n = \begin{cases} \left\{x \in \mathbf{R} : \epsilon_1 < \left|x - \sqrt{2}\right|\right\} & \text{if } n = 1, \\ \left\{x \in \mathbf{R} : \epsilon_{2n-1} < \left|x - \sqrt{2}\right| < \epsilon_{2n-3}\right\} & \text{if } n \geq 2. \end{cases}$$

Equivalently,

$$A_n = \begin{cases} \left(-\infty, \sqrt{2} - \epsilon_1\right) \cup \left(\sqrt{2} + \epsilon_1, \infty\right) & \text{if } n = 1, \\ \left(\sqrt{2} - \epsilon_{2n-3}, \sqrt{2} - \epsilon_{2n-1}\right) \cup \left(\sqrt{2} + \epsilon_{2n-1}, \sqrt{2} + \epsilon_{2n-3}\right) & \text{if } n \geq 2. \end{cases}$$

Now set $E_n = E \cap A_n$ for each $n \in \mathbf{N}$.

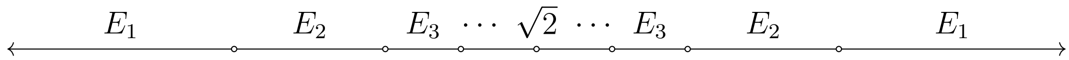

Figure 1: Partition of E

We have $\bigcup_{n=1}^{\infty} E_n = E$ since the only real numbers not contained in $\bigcup_{n=1}^{\infty} A_n$ are $\sqrt{2}$ and those of the form $\sqrt{2} \pm \epsilon_{2n-1}$ for some $n \in \mathbf{N}$, none of which are rational. Then since the collection $\{E_n : n \in \mathbf{N}\}$ is evidently pairwise disjoint, we have a partition of E.

Since $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \sqrt{2}$ and $\sqrt{2} \notin \overline{A_n}$ for any $n \in \mathbf{N}$, we see that there can be only finitely many terms of the sequences (p_n) and (q_n) contained in each A_n; it follows that each E_n is countably infinite. We can then enumerate each E_n:

$$E_n = \{e_{1,n}, e_{2,n}, e_{3,n}, \ldots\}.$$

These enumerations can be combined to form an enumeration (a_n) of E using the same diagonal method as that used in the proof that a countable union of countable sets is itself countable (see, for example, Exercise 1.5.3 (c)). To be precise, consider the following "infinite arrays".

$$\begin{array}{cccccc} e_{1,1} & e_{1,2} & e_{1,3} & e_{1,4} & e_{1,5} & \cdots \\ e_{2,1} & e_{2,2} & e_{2,3} & e_{2,4} & \ddots & \\ e_{3,1} & e_{3,2} & e_{3,3} & \ddots & & \\ e_{4,1} & e_{4,2} & \ddots & & & \\ e_{5,1} & \ddots & & & & \\ \vdots & & & & & \end{array} \qquad \begin{array}{cccccc} a_1 & a_3 & a_6 & a_{10} & a_{15} & \cdots \\ a_2 & a_5 & a_9 & a_{14} & \ddots & \\ a_4 & a_8 & a_{13} & \ddots & & \\ a_7 & a_{12} & \ddots & & & \\ a_{11} & \ddots & & & & \\ \vdots & & & & & \end{array}$$

The enumeration of E_n is the n^{th} column of the left-hand array. The enumeration of E is obtained by letting a_N in the right-hand array be the element $e_{m,n}$ in the corresponding position of the left-hand array, so that

$$a_1 = e_{1,1}, a_2 = e_{2,1}, a_3 = e_{1,2}, a_4 = e_{3,1}, \ldots$$

This mapping is bijective because the collection $\{E_n : n \in \mathbf{N}\}$ is a partition of E. Thus the sequence (a_n) satisfies conditions (3.1) and (3.2).

To show that the sequence (a_n) satisfies condition (3.3), we need to show that for all $n \in \mathbf{N}$, $\sqrt{2} \notin V_{\epsilon_{2n-1}}(a_n)$. Let $n \in \mathbf{N}$ be given. Then a_n belongs to some column of the right-hand array above, say the N^{th} column. From the definition of our enumeration (a_n), we have $a_n = e_{m,N}$ for some $m \in \mathbf{N}$. It follows that $a_n \in E_N$ and hence that $\left|a_n - \sqrt{2}\right| > \epsilon_{2N-1}$, which gives $\sqrt{2} \notin V_{\epsilon_{2N-1}}(a_n)$.

If we examine the right-hand array, we see that the element at the top of the N^{th} column is $a_{N(N+1)/2}$ (the N^{th} triangular number), and furthermore that $n \geq N(N+1)/2$. Then

$$2n-1 \geq N(N+1)-1 \geq 2N-1 \implies \epsilon_{2n-1} \leq \epsilon_{2N-1} \implies V_{\epsilon_{2n-1}}(a_n) \subseteq V_{\epsilon_{2N-1}}(a_n).$$

Combining this with $\sqrt{2} \notin V_{\epsilon_{2N-1}}(a_n)$, we see that $\sqrt{2} \notin V_{\epsilon_{2n-1}}(a_n)$. Thus the sequence (a_n) satisfies condition (3.3).

Step 4.

We can now form our final enumeration (r_n), by setting

$$r_{2n-1} = a_n, \quad r_{4n-2} = q_n, \quad \text{and} \quad r_{4n} = p_n,$$

so that (r_n) is the sequence

$$a_1, q_1, a_2, p_1, a_3, q_2, a_4, p_2, \ldots$$

Let $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ and $F = O^{\mathsf{c}}$. By condition (1.2), we have

$$\left(\sqrt{2} - \tfrac{1}{16}, \sqrt{2}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(p_n) = \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(r_{4n}) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n) = O,$$

and by condition (2.2), we have

$$\left(\sqrt{2}, \sqrt{2} + \tfrac{1}{16}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(q_n) = \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(r_{4n-2}) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n) = O.$$

Thus $\left(\sqrt{2} - \frac{1}{16}, \sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{2} + \frac{1}{16}\right) \subseteq O$. Furthermore, since

$$\begin{aligned} O &= \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n) \\ &= \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(r_{4n}) \cup \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(r_{4n-2}) \cup \bigcup_{n=1}^{\infty} V_{\epsilon_{2n-1}}(r_{2n-1}) \\ &= \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(p_n) \cup \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(q_n) \cup \bigcup_{n=1}^{\infty} V_{\epsilon_{2n-1}}(a_n), \end{aligned}$$

conditions (1.3), (2.3), and (3.3) imply that $\sqrt{2} \notin O$. It follows that

$$\left(\sqrt{2} - \tfrac{1}{16}, \sqrt{2} + \tfrac{1}{16}\right) \cap F = \left\{\sqrt{2}\right\}.$$

Then $\sqrt{2}$ is an isolated point of F and thus F is not a perfect set.

Regarding the second half of the question, it is possible to modify the construction to produce a non-empty perfect set consisting of only irrational numbers. To do this, we start with any enumeration (r_n) of $\mathbf{Q}$ and inductively define a sequence of non-negative real numbers (ϵ_n) in such a way that if we set

$$O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n) \quad \text{and} \quad F = O^{\mathsf{c}},$$

then F will be a non-empty perfect of irrational numbers. Intuitively, we will inductively construct O as a union of disjoint open intervals, with no pair of these intervals sharing an endpoint. (In what follows, we adopt the convention that $V_\epsilon(x) = \emptyset$ if $\epsilon = 0$.)

Suppose that after N steps we have chosen $\epsilon_1, \ldots, \epsilon_N$ such that:

(IH1) $\{r_1, \ldots, r_N\} \subseteq \bigcup_{n=1}^{N} V_{\epsilon_n}(r_n)$;

(IH2) for all $1 \leq n \leq N$, either $\epsilon_n = 0$ or ϵ_n is positive, irrational, and satisfies $\epsilon_n \leq \frac{\sqrt{2}}{2^n}$;

(IH3) $\overline{V_{\epsilon_m}(r_m)} \cap \overline{V_{\epsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $1 \leq m < n \leq N$.

Let $U = \bigcup_{n=1}^{N} V_{\epsilon_n}(r_n)$. There are two cases.

Case 1. This is the easier case. If $r_{N+1} \in U$, then set $\epsilon_{N+1} = 0$, so that $V_{\epsilon_{N+1}}(r_{N+1}) = \emptyset$. (IH1) combined with $r_{N+1} \in U$ gives us

$$\{r_1, \ldots, r_N, r_{N+1}\} \subseteq U = \bigcup_{n=1}^{N} V_{\epsilon_n}(r_n) = \bigcup_{n=1}^{N+1} V_{\epsilon_n}(r_n),$$

where the last equality follows from $V_{\epsilon_{N+1}}(r_{N+1}) = \emptyset$.

Combining (IH2) with $\epsilon_{N+1} = 0$, we see that for all $1 \leq n \leq N+1$, either $\epsilon_n = 0$ or ϵ_n is positive, irrational, and satisfies $\epsilon_n \leq \frac{\sqrt{2}}{2^n}$.

Similarly, combining (IH3) with $V_{\epsilon_{N+1}}(r_{N+1}) = \emptyset$, we have $\overline{V_{\epsilon_m}(r_m)} \cap \overline{V_{\epsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $1 \leq m < n \leq N+1$.

Case 2. This is the harder case. If $r_{N+1} \notin U$ then let $\epsilon_{n_1}, \dots, \epsilon_{n_J}$ be those ϵ's from $\epsilon_1, \dots, \epsilon_N$ which are non-zero; there must be at least one such ϵ_{n_j} by (IH1) and each ϵ_{n_j} must be positive and irrational by (IH2). Observe that

$$U = \bigcup_{n=1}^{N} V_{\epsilon_n}(r_n) = \bigcup_{j=1}^{J} V_{\epsilon_{n_j}}(r_{n_j}),$$

where each $V_{\epsilon_{n_j}}(r_{n_j})$ is a proper open interval. For each $1 \leq j \leq J$, note that since $r_{N+1} \notin U$, we must have $r_{N+1} \notin V_{\epsilon_{n_j}}(r_{n_j})$. Both of the endpoints of $V_{\epsilon_{n_j}}(r_{n_j})$ are the sum of a rational number and an irrational number and hence are irrational; since r_{N+1} is rational, we see that $r_{N+1} \notin [r_{n_j} - \epsilon_{n_j}, r_{n_j} + \epsilon_{n_j}]$. Given this, if we let d be the minimum of the distances from r_{N+1} to the endpoints of each $V_{\epsilon_{n_j}}$, i.e.

$$d = \min\left\{ \left| r_{n_j} - \epsilon_{n_j} - r_{N+1} \right|, \left| r_{n_j} + \epsilon_{n_j} - r_{N+1} \right| : 1 \leq j \leq J \right\},$$

then d must be positive. Furthermore, d must be irrational since it is the sum of a rational number and an irrational number, and for each $1 \leq j \leq J$, we have

$$\left[r_{N+1} - \tfrac{d}{2}, r_{N+1} + \tfrac{d}{2} \right] \cap [r_{n_j} - \epsilon_{n_j}, r_{n_j} + \epsilon_{n_j}] = \emptyset. \tag{1}$$

Set $\epsilon_{N+1} = \min\left\{ \frac{\sqrt{2}}{2^{N+1}}, \frac{d}{2} \right\}$. Then ϵ_{N+1} is positive, so $r_{N+1} \in V_{\epsilon_{N+1}}(r_{N+1})$. Combining this with (IH1) gives us

$$\{r_1, \dots, r_N, r_{N+1}\} \subseteq \bigcup_{n=1}^{N+1} V_{\epsilon_n}(r_n).$$

As noted before, d is positive and irrational, so ϵ_{N+1} is positive, irrational, and satisfies $\epsilon_{N+1} \leq \frac{\sqrt{2}}{2^{N+1}}$; combining this with (IH2) shows that for all $1 \leq n \leq N+1$, either $\epsilon_n = 0$ or ϵ_n is positive, irrational, and satisfies $\epsilon_n \leq \frac{\sqrt{2}}{2^n}$.

Let $1 \leq n \leq N$ be given. If $\epsilon_n = 0$, then the identity $\overline{V_{\epsilon_n}(r_n)} \cap \overline{V_{\epsilon_{N+1}}(r_{N+1})} = \emptyset$ is clear, since $V_{\epsilon_n}(r_n) = \emptyset$. If $\epsilon_n \neq 0$, then $n = n_j$ for some $1 \leq j \leq J$. In this case, we have

$$\overline{V_{\epsilon_n}(r_n)} = \overline{V_{\epsilon_{n_j}}(r_{n_j})} = [r_{n_j} - \epsilon_{n_j}, r_{n_j} + \epsilon_{n_j}],$$

$$\overline{V_{\epsilon_{N+1}}(r_{N+1})} = [r_{N+1} - \epsilon_{N+1}, r_{N+1} + \epsilon_{N+1}] \subseteq \left[r_{N+1} - \tfrac{d}{2}, r_{N+1} + \tfrac{d}{2} \right].$$

Then by equation (1), we see that $\overline{V_{\epsilon_n}(r_n)} \cap \overline{V_{\epsilon_{N+1}}(r_{N+1})} = \emptyset$. Combining this with (IH3), we see that $\overline{V_{\epsilon_m}(r_m)} \cap \overline{V_{\epsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $1 \leq m < n \leq N+1$.

This completes the induction step. For the base case, simply let $\epsilon_1 = \frac{\sqrt{2}}{2}$. Thus we obtain a sequence (ϵ_n) which satisfies (IH1), (IH2), and (IH3) for all $N \in \mathbf{N}$. In other words, the sequence (ϵ_n) has the following properties:

(A1) $\mathbf{Q} \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$;

(A2) for all $n \in \mathbf{N}$, either $\epsilon_n = 0$ or ϵ_n is positive, irrational, and satisfies $\epsilon_n \leq \frac{\sqrt{2}}{2^n}$;

(A3) $\overline{V_{\epsilon_m}(r_m)} \cap \overline{V_{\epsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $1 \leq m < n$.

Set $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ and $F = O^{\mathsf{c}}$. As in part (a), F is closed and, by (A1), consists solely of irrational numbers. By (A2), we have $\epsilon_n \leq \frac{\sqrt{2}}{2^n}$ for each $n \in \mathbf{N}$; a similar argument as in part (a) shows that O cannot be the entire real line and thus F is non-empty. We claim that F is perfect. To see this, suppose by way of contradiction that $x \in F$ is isolated. Then there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \cap F = \{x\}$. This implies that the intervals $(x - \delta, x)$ and $(x, x + \delta)$ are contained in O. We claim that if an interval such as $(x - \delta, x)$ is to be contained in O, then it must be entirely contained inside a single $V_{\epsilon_n}(r_n)$. To see this, suppose by way of contradiction that $a, b \in (x - \delta, x)$ are such that $a < b, a \in V_{\epsilon_m}(r_m)$, and $b \in V_{\epsilon_n}(r_n)$, with $m \neq n$. Then by (A3), it must be the case that

$$a < r_m + \epsilon_m < r_n - \epsilon_n < b.$$

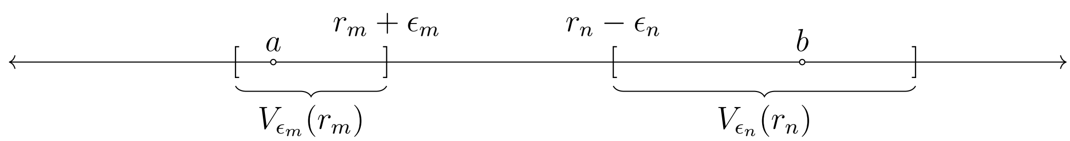

Figure 2: $V_{\epsilon_m}(r_m)$ and $V_{\epsilon_n}(r_n)$

Thus $r_m + \epsilon_m \in (a, b) \subseteq (x - \delta, x) \subseteq O$. There then exists a $k \in \mathbf{N}$ such that $r_m + \epsilon_m$ belongs to $V_{\epsilon_k}(r_k)$. If $k = m$, this says that an open interval contains one of its endpoints, which is a contradiction, and if $k \neq m$ then this violates (A3).

Thus if an interval such as $(x - \delta, x)$ is to be contained in O, it must be entirely contained inside a single $V_{\epsilon_n}(r_n)$. Then since $(x - \delta, x)$ and $(x, x + \delta)$ are disjoint, there exist $m, n \in \mathbf{N}$ with $m \neq n$ such that

$$(x - \delta, x) \subseteq V_{\epsilon_m}(r_m) \quad \text{and} \quad (x, x + \delta) \subseteq V_{\epsilon_n}(r_n).$$

This implies that

$$[x-\delta, x] \subseteq \overline{V_{\epsilon_m}(r_m)} \quad \text{and} \quad [x, x+\delta] \subseteq \overline{V_{\epsilon_n}(r_n)},$$

which in turn gives $x \in \overline{V_{\epsilon_m}(r_m)} \cap \overline{V_{\epsilon_n}(r_n)}$, contradicting (A3). We may conclude that F is a perfect set.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.