This is Exercise 1.2.4 from [UA].

1 An infinite partition of $\mathbb{N}$ into infinite sets

We will construct a countably infinite collection of sets $\{A_i : i \in \mathbb{N}\}$ such that the following three properties hold.

(1) Each A_i is countably infinite.

(2) $A_i \cap A_j = \varnothing$ for $i < j$.

(3) $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

Let p_i be the ith prime ($p_1 = 2, p_2 = 3, p_3 = 5$, and so on), and define

$$\begin{aligned}
A_1 &= \{n \in \mathbb{N} : n \text{ is divisible by 2}\} \cup \{1\} \\
A_2 &= \{n \in \mathbb{N} : n \text{ is divisible by 3 but not by 2}\} \\
A_3 &= \{n \in \mathbb{N} : n \text{ is divisible by 5 but not by 3 or 2}\} \\
&\cdots \\
A_i &= \{n \in \mathbb{N} : n \text{ is divisible by } p_i \text{ but not by } p_{i-1}, \ldots, 3, \text{ or } 2\} \\
&\cdots
\end{aligned}$$

We claim that these sets satisfy properties (1), (2), and (3).

(1) Each A_i is infinite since $p_i^k \in A_i$ for each $k \in \mathbb{N}$, and countably infinite since $A_i \subseteq \mathbb{N}$.

(2) For $i < j$, one has $n \in A_i$ only if p_i divides n; but in order to have $n \in A_j$ it is necessary that p_i does *not* divide n. It follows that $n \notin A_j$, so that $A_i \cap A_j = \varnothing$.

(3) It is clear that $\bigcup_{i=1}^{\infty} A_i \subseteq \mathbb{N}$. For the reverse inclusion, suppose $n \in \mathbb{N}$. If $n = 1$, then $n \in A_1$. If $n > 1$, then let j be the index of the smallest prime appearing in the unique prime factorization of n. It follows that $n \in A_j$, so that $n \in \bigcup_{i=1}^{\infty} A_i$.

Finally, properties (1) and (2) together imply that the A_i's are non-empty and distinct, so that the collection $\{A_i : i \in \mathbb{N}\}$ is indeed countably infinite.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edn.