1 Riemann integral is continuous

This easy theorem supports the claim that "integrating functions makes them nicer", since f is required only to be Riemann-integrable and not necessarily continuous on its domain.

Theorem 1. Suppose we have some interval I, some $t_0 \in I$ and some Riemann-integrable function $f : I \to \mathbb{R}$. Define $g : I \to \mathbb{R}$ by

$$g(x) = \int_{t_0}^{x} f(t)\,\mathrm{d}t.$$

Then g is continuous.

Proof. Since f is Riemann-integrable, it is bounded by some $M > 0$. Let $y \in I$ be given and fix some $\varepsilon > 0$. Then provided we take $x \in I$ such that $|x - y| < \frac{\varepsilon}{M}$, we will have

$$\begin{aligned}
|g(x) - g(y)| &= \left| \int_{t_0}^{x} f(t)\,\mathrm{d}t - \int_{t_0}^{y} f(t)\,\mathrm{d}t \right| \\
&= \left| \int_{y}^{x} f(t)\,\mathrm{d}t \right| \\
&\leq \int_{y}^{x} |f(t)|\,\mathrm{d}t \\
&\leq M\,|x - y| \\
&< \varepsilon,
\end{aligned}$$

as desired. $\square$