1 Section 1.A Exercises

Exercises with solutions from Section 1.A of [LADR].

Exercise 1.A.1. Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$1/(a+bi) = c+di.$$

Solution. Observe that

$$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}.$$

So the desired real numbers are $c = a/(a^2+b^2)$ and $d = -b/(a^2+b^2)$.

Exercise 1.A.2. Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution. Let $z = \frac{-1+\sqrt{3}i}{2}$, so that $2z = -1+\sqrt{3}i$. Then

$$\begin{aligned} &(2z)^2 = 4z^2 = (-1+\sqrt{3}i)^2 = 1 - 2\sqrt{3}i - 3 = -2 - 2\sqrt{3}i \\ \implies &(2z)^3 = (4z^2)(2z) = (-2-2\sqrt{3}i)(-1+\sqrt{3}i) = 2 - 2\sqrt{3}i + 2\sqrt{3}i + 6 = 8, \end{aligned}$$

i.e. $8z^3 = 8$. It follows that $z^3 = 1$.

Exercise 1.A.3. Find two distinct square roots of i.

Solution. Let $z_1 = \frac{1+i}{\sqrt{2}}$ and $z_2 = -z_1$ (z_1 and z_2 are distinct since $z_1 \neq 0$). Then

$$2z_1^2 = (1+i)^2 = 2i \implies z_1^2 = i,$$

and hence $z_2^2 = (-z_1)^2 = z_1^2 = i$.

Exercise 1.A.4. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Solution. Suppose $\alpha = x + yi$ and $\beta = u + vi$. Then

$$\alpha + \beta = (x+u) + (y+v)i = (u+x) + (v+y)i = \beta + \alpha,$$

where we have used commutativity of addition in $\mathbf{R}$.

Exercise 1.A.5. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution. Suppose that $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$. Then

$$\begin{aligned}(\alpha + \beta) + \lambda &= ((x + u) + (y + v)i) + \lambda = ((x + u) + s) + ((y + v) + t)i \\ &= (x + (u + s)) + (y + (v + t))i = \alpha + ((u + s) + (v + t)i) = \alpha + (\beta + \lambda),\end{aligned}$$

where we have used associativity of addition in $\mathbf{R}$.

Exercise 1.A.6. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution. Suppose that $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$. Then

$$\begin{aligned}(\alpha\beta)\lambda &= ((xu - yv) + (xv + yu)i)\lambda \\ &= ((xu - yv)s - (xv + yu)t) + ((xu - yv)t + (xv + yu)s)i \\ &= ((xu)s - (yv)s - (xv)t - (yu)t) + ((xu)t - (yv)t + (xv)s + (yu)s)i \\ &= (x(us) - x(vt) - y(ut) - y(vs)) + (x(ut) + x(vs) + y(us) - y(vt))i \\ &= (x(us - vt) - y(ut + vs)) + (x(ut + vs) + y(us - vt))i \\ &= \alpha((us - vt) + (ut + vs)i) \\ &= \alpha(\beta\lambda),\end{aligned}$$

where we have used several algebraic properties of $\mathbf{R}$.

Exercise 1.A.7. Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Solution. Suppose that $\alpha = x + yi$ and let $\beta = -x - yi$. Then

$$\alpha + \beta = (x - x) + (y - y)i = 0 + 0i = 0.$$

To see that β is unique, suppose β' also satisfies $\alpha + \beta' = 0$. Then

$$\beta = \beta + 0 = \beta + (\alpha + \beta') = (\alpha + \beta) + \beta' = 0 + \beta' = \beta',$$

where we have used associativity and commutativity of addition in $\mathbf{C}$.

Exercise 1.A.8. Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Solution. Suppose that $\alpha = x + yi$. Since $\alpha \neq 0$, it must be the case that x and y are not both zero. It follows that $x^2 + y^2 \neq 0$, so let $\beta = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$. Then

$$\alpha\beta = (x + yi)\left(\frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i\right) = \frac{x^2 + y^2}{x^2 + y^2} + \frac{xy - xy}{x^2 + y^2}i = 1 + 0i = 1.$$

To see that β is unique, suppose β' also satisfies $\alpha\beta' = 1$. Then

$$\beta = \beta 1 = \beta(\alpha\beta') = (\alpha\beta)\beta' = 1\beta' = \beta',$$

where we have used associativity and commutativity of multiplication in $\mathbf{C}$.

Exercise 1.A.9. Show that $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Solution. Suppose that $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$. Then

$$\begin{aligned}\lambda(\alpha+\beta) &= (s(x+u) - t(y+v)) + (s(y+v) + t(x+u))i \\ &= (sx + su - ty - tv) + (sy + sv + tx + tu)i \\ &= [(sx - ty) + (sy + tx)i] + [(su - tv) + (sv + tu)i] \\ &= \lambda\alpha + \lambda\beta,\end{aligned}$$

where we have used distributivity in $\mathbf{R}$.

Exercise 1.A.10. Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution. Take $x = (\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2})$.

Exercise 1.A.11. Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Solution. Suppose there was such a λ. Then

$$\lambda(2 - 3i) = 12 - 5i \implies \lambda = \frac{12 - 5i}{2 - 3i} = 3 + 2i.$$

However,

$$(3 + 2i)(-6 + 7i) = -32 + 9i \neq -32 - 9i.$$

Exercise 1.A.12. Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

Solution. Suppose $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$, and $z = (z_1, \ldots, z_n)$. Then

$$\begin{aligned}(x+y)+z &= (x_1 + y_1, \ldots, x_n + y_n) + z \\ &= ((x_1 + y_1) + z_1, \ldots, (x_n + y_n) + z_n) \\ &= (x_1 + (y_1 + z_1), \ldots, x_n + (y_n + z_n)) \\ &= x + (y_1 + z_1, \ldots, y_n + z_n) \\ &= x + (y + z),\end{aligned}$$

where we have used associativity of addition in $\mathbf{F}$.

Exercise 1.A.13. Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Solution. Suppose that $x = (x_1, \ldots, x_n)$. Then

$$\begin{aligned}(ab)x &= ((ab)x_1, \ldots, (ab)x_n) \\ &= (a(bx_1), \ldots, a(bx_n)) \\ &= a(bx_1, \ldots, bx_n) \\ &= a(bx),\end{aligned}$$

where we have used associativity of multiplication in $\mathbf{F}$.

Exercise 1.A.14. Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Solution. Suppose that $x = (x_1, \ldots, x_n)$. Then

$$1x = (1x_1, \ldots, 1x_n) = (x_1, \ldots, x_n) = x,$$

where we have used that $1x_j = x_j$ for any $x_j \in \mathbf{F}$.

Exercise 1.A.15. Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.

Solution. Suppose that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$\begin{aligned}\lambda(x + y) &= \lambda(x_1 + y_1, \ldots, x_n + y_n) \\ &= (\lambda(x_1 + y_1), \ldots, \lambda(x_n + y_n)) \\ &= (\lambda x_1 + \lambda y_1, \ldots, \lambda x_n + \lambda y_n) \\ &= (\lambda x_1, \ldots, \lambda x_n) + (\lambda y_1, \ldots, \lambda y_n) \\ &= \lambda(x_1, \ldots, x_n) + \lambda(y_1, \ldots, y_n) \\ &= \lambda x + \lambda y,\end{aligned}$$

where we have used distributivity in $\mathbf{F}$.

Exercise 1.A.16. Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Solution. Suppose that $x = (x_1, \ldots, x_n)$. Then

$$\begin{aligned}(a + b)x &= (a + b)(x_1, \ldots, x_n) \\ &= ((a + b)x_1, \ldots, (a + b)x_n) \\ &= (ax_1 + bx_1, \ldots, ax_n + bx_n) \\ &= (ax_1, \ldots, ax_n) + (bx_1, \ldots, bx_n) \\ &= a(x_1, \ldots x_n) + b(x_1, \ldots, x_n) \\ &= ax + bx,\end{aligned}$$

where we have used distributivity in $\mathbf{F}$.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edn.