1 Section 1.C Exercises

Exercises with solutions from Section 1.C of [LADR].

Exercise 1.C.1. For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$:

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$;

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$.

Solution. Let U denote the set in each part of this question.

(a) This is a subspace of $\mathbf{F}^3$. Clearly the zero vector belongs to U. Suppose that $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ belong to U and $\alpha \in \mathbf{F}$. Then

$$(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) = 0 + 0 = 0,$$
$$\alpha x_1 + 2(\alpha x_2) + 3(\alpha x_3) = \alpha(x_1 + 2x_2 + 3x_3) = \alpha 0 = 0.$$

So $x + y$ and αx belong to U and hence U is a subspace of V by (1.34).

(b) This is not a subspace of $\mathbf{F}^3$ since it does not contain the zero vector.

(c) This is not a subspace of $\mathbf{F}^3$ since it is not closed under addition; $(1, 1, 0)$ and $(0, 0, 1)$ belong to U but $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$ does not belong to U.

(d) This is a subspace of $\mathbf{F}^3$; observe that $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 - 5x_3 = 0\}$. Clearly the zero vector belongs to U. Suppose that $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ belong to U and $\alpha \in \mathbf{F}$. Then

$$(x_1 + y_1) - 5(x_3 + y_3) = (x_1 - 5x_3) + (y_1 - 5y_3) = 0 + 0 = 0,$$
$$\alpha x_1 - 5(\alpha x_3) = \alpha(x_1 - 5x_3) = \alpha 0 = 0.$$

So $x + y$ and αx belong to U and hence U is a subspace of V by (1.34).

Exercise 1.C.2. Verify all the assertions in Example 1.35.

Solution. (a) The assertion is that if $b \in \mathbf{F}$, then

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\} = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 - 5x_4 = b\}$$

is a subspace of $\mathbf{F}^4$ if and only if $b = 0$. If $b \neq 0$, then U is not a subspace of $\mathbf{F}^4$ because the zero vector does not belong to U. If $b = 0$, then similar reasoning to Exercise 1.C.1 (d) shows that U is a subspace of $\mathbf{F}^4$.

(b) The assertion is that the set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$, i.e.

$$U = \{f : [0, 1] \to \mathbf{R}, f \text{ continuous}\}$$

is a subspace of $\mathbf{R}^{[0,1]}$. Clearly the zero function $0 : [0, 1] \to \mathbf{R}, x \mapsto 0$ is continuous and hence belongs to U. Suppose that $f, g \in U$ and $\alpha \in \mathbf{R}$. Then it is well-known that $f + g$ and αf are continuous functions on $[0, 1]$, i.e. $f + g \in U$ and $\alpha f \in U$. Hence U is a subspace of $\mathbf{R}^{[0,1]}$ by (1.34).

(c) The assertion is that the set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$. The function $0 : \mathbf{R} \to \mathbf{R}, x \mapsto 0$ is certainly differentiable, and it is well-known that the sum of differentiable functions and constant multiples of differentiable functions are also differentiable. Hence the assertion holds by (1.34).

(d) The assertion is that the set U of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$. First, suppose that $b \neq 0$. Then U is not a subspace since the zero function $0 : (0, 3) \to \mathbf{R}, x \mapsto 0$ certainly does not satisfy $0'(2) = b$. Now suppose that $b = 0$; then the zero function satisfies $0'(2) = b$ and hence belongs to U. If $f, g \in U$ and $\alpha \in \mathbf{R}$, then

$$(f + g)'(2) = f'(2) + g'(2) = 0 + 0 = 0,$$

$$(\alpha f)'(2) = \alpha f'(2) = \alpha 0 = 0.$$

Hence $f + g$ and αf belong to U and so U is a subspace of $\mathbf{R}^{(0,3)}$ by (1.34).

(e) The assertion is that the set U of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^\infty$. The zero sequence $(0, 0, 0, \ldots)$ certainly has limit 0 and so belongs to U. Suppose that $x = (x_n)$ and $y = (y_n)$ belong to U and $\alpha \in \mathbf{C}$. Then

$$\lim(x_n + y_n) = \lim x_n + \lim y_n = 0 + 0 = 0,$$

$$\lim(\alpha x_n) = \alpha \lim x_n = \alpha 0 = 0.$$

Hence $x + y$ and αx belong to U and so U is a subspace of $\mathbf{C}^\infty$ by (1.34).

Exercise 1.C.3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

Solution. Let U be the set in question. The zero function $0 : (-4, 4) \to \mathbf{R}, x \mapsto 0$ belongs to U since $0'(-1) = 0$ and $0(2) = 0$. Suppose that $f, g \in U$ and $\alpha \in \mathbf{R}$. Then

$$(f+g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f(2) + g(2)) = 3(f+g)(2),$$
$$(\alpha f)'(-1) = \alpha f'(-1) = \alpha(3f(2)) = 3(\alpha f(2)) = 3(\alpha f)(2).$$

Hence $f + g$ and αf belong to U and so U is a subspace of $\mathbf{R}^{(-4,4)}$ by (1.34).

Exercise 1.C.4. Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

Solution. Let U be the set in question and suppose that $b \neq 0$. Then U is not a subspace of $\mathbf{R}^{[0,1]}$ since the zero function $0 : [0, 1] \to \mathbf{R}, x \mapsto 0$ does not belong to U; $\int_0^1 0 = 0 \neq b$. Now suppose that $b = 0$, so that the zero function belongs to U. Suppose that $f, g \in U$ and $\alpha \in \mathbf{R}$. Then

$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0 + 0 = 0,$$
$$\int_0^1 (\alpha f) = \alpha \int_0^1 f = \alpha 0 = 0.$$

Hence $f + g$ and αf belong to U and so U is a subspace of $\mathbf{R}^{[0,1]}$ by (1.34).

Exercise 1.C.5. Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

Solution. The question is whether the subset

$$\mathbf{R}^2 = \{(x, y) : x, y \in \mathbf{R}\} \subseteq \{(z, w) : z, w \in \mathbf{C}\} = \mathbf{C}^2$$

is a subspace, where we are taking complex scalars in $\mathbf{C}^2$. This is not a subspace since it is not closed under scalar multiplication; for example, $(1, 0) \in \mathbf{R}^2$ but $i(1, 0) = (i, 0) \notin \mathbf{R}^2$.

Exercise 1.C.6. (a) Is $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?

(b) Is $\{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of $\mathbf{C}^3$?

Solution. (a) Let U be the set in question. Since for real numbers a and b we have $a^3 = b^3$ if and only if $a = b$, the set U can be written as

$$U = \{(a, a, c) \in \mathbf{R}^3 : a, c \in \mathbf{R}\}.$$

Clearly, $(0,0,0) \in U$. Suppose that $u = (a,a,c)$ and $v = (x,x,y)$ belong to U and $\lambda \in \mathbf{R}$. Then

$$u + v = (a,a,c) + (x,x,y) = (a+x, a+x, c+y) \in U,$$
$$\lambda u = \lambda(a,a,c) = (\lambda a, \lambda a, \lambda c) \in U.$$

Hence U is a subspace of $\mathbf{R}^3$ by (1.34).

(b) Let U be the set in question. Observe that

$$\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)^3 = \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)^3 = 1.$$

So if we let $u = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i, 1, 0\right)$ and $v = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i, 1, 0\right)$, then we have $u, v \in U$. However,

$$u + v = (-1, 2, 0) \notin U.$$

So U is not closed under addition and hence cannot be a subspace of $\mathbf{C}^3$.

Exercise 1.C.7. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbf{R}^2$.

Solution. Let $U = \{(a,b) : a, b \in \mathbf{Q}\} \subseteq \mathbf{R}^2$. Then U satisfies the required conditions since the sum of two rational numbers is a rational number and the additive inverse of a rational number is a rational number. However, U is not a subspace of $\mathbf{R}^2$ since it is not closed under scalar multiplication by arbitrary real numbers; for example, $(1,0) \in U$ but $\sqrt{2}(1,0) = \left(\sqrt{2}, 0\right) \notin U$.

Exercise 1.C.8. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

Solution. Let $U = \{(x,0) : x \in \mathbf{R}\} \cup \{(0,y) : y \in \mathbf{R}\}$, the union of the x- and y-axes. Then U is closed under scalar multiplication ($(\alpha x, 0)$ and $(0, \alpha y)$ belong to U for any $\alpha, x, y \in \mathbf{R}$) but U is not closed under addition; for example, $(1,0)$ and $(0,1)$ belong to U but $(1,0) + (0,1) = (1,1)$ does not. It follows that U is not a subspace of $\mathbf{R}^2$.

Exercise 1.C.9. A function $f : \mathbf{R} \to \mathbf{R}$ is called ***periodic*** if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution. This is not a subspace of $\mathbf{R}^{\mathbf{R}}$ since it is not closed under addition. Consider the periodic functions $\sin(x)$ and $\sin\left(\sqrt{2}x\right)$ and let $f(x) = \sin(x) + \sin\left(\sqrt{2}x\right)$. We will show that $f(x)$ cannot be periodic. Seeking a contradiction, suppose there is a positive real number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$, i.e.

$$\sin(x) + \sin\left(\sqrt{2}x\right) = \sin(x+p) + \sin\left(\sqrt{2}x + \sqrt{2}p\right) \text{ for all } x \in \mathbf{R}. \tag{1}$$

By differentiating this equation twice, we see that

$$\sin(x) + 2\sin\left(\sqrt{2}x\right) = \sin(x+p) + 2\sin\left(\sqrt{2}x + \sqrt{2}p\right) \text{ for all } x \in \mathbf{R}. \tag{2}$$

Subtracting equation (1) from equation (2) gives us

$$\sin\left(\sqrt{2}x\right) = \sin\left(\sqrt{2}x + \sqrt{2}p\right) \text{ for all } x \in \mathbf{R}, \tag{3}$$

which together with equation (1) implies that

$$\sin(x) = \sin(x+p) \text{ for all } x \in \mathbf{R}. \tag{4}$$

In particular, by taking $x = 0$ in equation (4) we must have $0 = \sin(p)$, which can be the case if and only if $p = n\pi$ for some positive integer n (p was assumed to be positive). Substituting this value of p and $x = 0$ into equation (3) gives $0 = \sin\left(n\sqrt{2}\pi\right)$, which can be the case if and only if $n\sqrt{2}\pi = m\pi$ for some integer m. This implies that $\sqrt{2} = \frac{m}{n}$, a rational number; but it is well-known that $\sqrt{2}$ is not a rational number, so we have found our contradiction. It follows that f cannot be periodic and hence the set of all periodic functions from $\mathbf{R}$ to $\mathbf{R}$ is not a subspace since it is not closed under addition.

Exercise 1.C.10. Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Solution. See Exercise 1.C.11.

Exercise 1.C.11. Prove that the intersection of every collection of subspaces of V is a subspace of V.

Solution. Let $\mathscr{U}$ be an arbitrary collection of subspaces of V. We will show that $U' = \bigcap_{U \in \mathscr{U}} U$ is a subspace of V. Since $0 \in U$ for each $U \in \mathscr{U}$, we have $0 \in U'$. Suppose that $x, y \in U', \alpha \in \mathbf{F}$ and $U \in \mathscr{U}$. Then $x, y \in U$, so $x + y \in U$ and $\alpha x \in U$ since U is a subspace of V. Since U was arbitrary, we have $x + y \in U'$ and $\alpha x \in U'$; it follows that U' is a subspace of V by (1.34).

Exercise 1.C.12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution. Suppose that U and W are subspaces of V. We want to show that $U \cup W$ is a subspace of V if and only if either $U \subseteq W$ or $W \subseteq U$. If either of U or W is contained in the other then either $U \cup W = U$ or $U \cup W = W$; in either case, $U \cup W$ is a subspace of V. Suppose that $U \cup W$ is a subspace of V and $U \nsubseteq W$; we need to show that $W \subseteq U$. Since $U \nsubseteq W$, there is a $u \in U$ such that $u \notin W$. Let $w \in W$ be given. By assumption, $U \cup W$ is a subspace of V, so $u + w \in U \cup W$. It cannot be the case that $u + w \in W$, since then $u + w - w = u \in W$, so it must be the case that $u + w \in U$. Then $u + w - u = w \in U$ and hence $W \subseteq U$ as desired.

Exercise 1.C.13. Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.
[*This exercise is surprisingly harder than the previous exercise, possibly because this exercise is not true if we replace* **F** *with a field containing only two elements.*]

Solution. Let U_1, U_2, and U_3 be subspaces of V. We want to show that $U = U_1 \cup U_2 \cup U_3$ is a subspace of V if and only if some U_i contains the other two. If some U_i contains the other two, say U_1 contains U_2 and U_3, then $U = U_1$ is a subspace of V. Now suppose that U is a subspace of V, and note that if any U_i is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $U = U_2 \cup U_3$ and we may apply Exercise 1.C.12 to say that either $U_2 \subseteq U_3$ or $U_3 \subseteq U_2$; in either case, one U_i contains the other two. Suppose therefore that no U_i is contained in the union of the other two, and seeking a contradiction suppose also that no U_i contains the other two. Hence

$$U_1 \nsubseteq (U_2 \cup U_3) \quad \text{and} \quad (U_2 \cup U_3) \nsubseteq U_1.$$

It follows that there exists a $u \in U_1$ such that $u \notin U_2 \cup U_3$ and a $v \in U_2 \cup U_3$ such that $v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\}$ and observe that no element of W belongs to U_1, for if $v + \lambda u \in U_1$ then $v + \lambda u - \lambda u = v \in U_1$; but $v \notin U_1$. Then since $W \subseteq U$ and $W \cap U_1 = \emptyset$, it must be the case that $W \subseteq U_2 \cup U_3$. W is infinite since $\mathbf{F}$ is infinite, so at least one of U_2 and U_3 must contain infinitely many members of W; say U_j. Then there exists $\lambda \neq \mu$ in $\mathbf{F}$ such that $v + \lambda u$ and $v + \mu u$ both belong to U_j, which implies that $(\lambda - \mu)u \in U_j$. Since $\lambda - \mu \neq 0$, this gives $u \in U_j$, which contradicts $u \notin U_2 \cup U_3$. We may conclude that one U_i contains the other two.

Exercise 1.C.14. Verify the assertion in Example 1.38.

Solution. Let

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}, \quad W = \{(x, x, x, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\},$$
$$E = \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}.$$

The assertion is that $U+W=E$. First, suppose that $u+w\in U+W$ for some $u=(x,x,y,y)\in U$ and $w=(a,a,a,b)\in W$. Then

$$u+w=(x,x,y,y)+(a,a,a,b)=(x+a,x+a,y+a,y+b),$$

which has the form of an element of E. It follows that $U+W\subseteq E$. Now suppose that $v=(x,x,y,z)\in E$. Then observe that

$$v=(x,x,y,z)=(x,x,y,y)+(0,0,0,z-y)$$

which has the form of an element of $U+W$. It follows that $E\subseteq U+W$ and we may conclude that $U+W=E$.

Exercise 1.C.15. Suppose U is a subspace of V. What is $U+U$?

Solution. We have $U+U=U$. For $u+v\in U+U$, we have $u+v\in U$ since U is a subspace of V; it follows that $U+U\subseteq U$. For $u\in U$, we have $u=u+0\in U+U$, so that $U\subseteq U+U$.

Exercise 1.C.16. Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U+W=W+U$?

Solution. The operation is commutative, since addition of vectors in V is commutative. If $u+w\in U+W$, then $u+w=w+u\in W+U$, so that $U+W\subseteq W+U$. Similarly, $W+U\subseteq U+W$.

Exercise 1.C.17. Is the operation of addition on the subspaces of V associative? In other words, if U_1,U_2,U_3 are subspaces of V, is

$$(U_1+U_2)+U_3=U_1+(U_2+U_3)?$$

Solution. The operation is associative, since addition of vectors in V is associative. If $(u_1+u_2)+u_3\in(U_1+U_2)+U_3$, then $(u_1+u_2)+u_3=u_1+(u_2+u_3)\in U_1+(U_2+U_3)$, so that $(U_1+U_2)+U_3\subseteq U_1+(U_2+U_3)$. Similarly, $U_1+(U_2+U_3)\subseteq(U_1+U_2)+U_3$.

Exercise 1.C.18. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution. The subspace $\{0\}$ is the additive identity for the operation. If U is a subspace of V then $u+0=u$ for any $u\in U$; it follows that $U+\{0\}=U$.

Since $\{0\}+\{0\}=\{0\}$, the subspace $\{0\}$ is its own additive inverse. We claim that no other subspace has an additive inverse, i.e. if U is a subspace of V with $U\neq\{0\}$, then there does not exist a subspace W satisfying $U+W=\{0\}$; indeed, simply observe that $U\subseteq U+W$ for any subspace W.

Exercise 1.C.19. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Solution. This is false. For a counterexample, consider the real vector space $\mathbf{R}$. Then

$$\{0\} + \mathbf{R} = \mathbf{R} + \mathbf{R} = \mathbf{R},$$

but $\{0\} \neq \mathbf{R}$.

Exercise 1.C.20. Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

Solution. Let $W = \{(0, a, 0, b) \in \mathbf{F}^4 : a, b \in \mathbf{F}\}$; it is easily verified that W is a subspace of $\mathbf{F}^4$. Suppose that $v = (v_1, v_2, v_3, v_4) \in U \cap W$. Since $v \in W$ we must have $v_1 = v_3 = 0$, then since $v \in U$ we must have $v_2 = v_1 = 0$ and $v_4 = v_3 = 0$, i.e. $v = 0$. It follows that $U \cap W = \{0\}$ and hence by (1.45) the sum $U + W$ is direct. Let $v = (x, y, z, t) \in \mathbf{F}^4$ be given. Then observe that

$$v = (x, y, z, t) = (x, x, z, z) + (0, y - x, 0, t - z) \in U \oplus W.$$

Hence $\mathbf{F}^4 = U \oplus W$.

Exercise 1.C.21. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Solution. Let $W = \{(0, 0, a, b, c) \in \mathbf{F}^5 : a, b, c \in \mathbf{F}\}$; it is easily verified that W is a subspace of $\mathbf{F}^5$. Suppose that $v = (v_1, v_2, v_3, v_4, v_5) \in U \cap W$. Since $v \in U$ we must have $v = (v_1, v_2, v_1 + v_2, v_1 - v_2, 2v_1)$, then since $v \in W$ we must have $v_1 = v_2 = 0$ and and hence $v = 0$. It follows that $U \cap W = \{0\}$ and hence by (1.45) the sum $U + W$ is direct. Let $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbf{F}^5$ be given. Then observe that

$$\begin{aligned} v = (v_1, v_2, v_3, v_4, v_5) &= (v_1, v_2, v_1 + v_2, v_1 - v_2, 2v_1) \\ &\quad + (0, 0, v_3 - (v_1 + v_2), v_4 - (v_1 - v_2), v_5 - 2v_1) \in U \oplus W. \end{aligned}$$

Hence $\mathbf{F}^5 = U \oplus W$.

Exercise 1.C.22. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution. Let

$$W_1 = \{(0, 0, a, 0, 0) \in \mathbf{F}^5 : a \in \mathbf{F}\}, \quad W_2 = \{(0, 0, 0, b, 0) \in \mathbf{F}^5 : b \in \mathbf{F}\},$$
$$W_3 = \{(0, 0, 0, 0, c) \in \mathbf{F}^5 : c \in \mathbf{F}\}.$$

Suppose that $u = (x, y, x + y, x - y, 2x) \in U, w_1 = (0, 0, a, 0, 0) \in W_1, w_2 = (0, 0, 0, b, 0) \in W_2$, and $w_3 = (0, 0, 0, 0, c) \in W_3$ are such that $u + w_1 + w_2 + w_3 = 0$, i.e.

$$(x, y, x + y + a, x - y + b, 2x + c) = (0, 0, 0, 0, 0).$$

Then $x = y = 0$, which implies that $a = b = c = 0$. It follows that $u + w_1 + w_2 + w_3 = 0$ if and only if $u = w_1 = w_2 = w_3 = 0$, and so by (1.44) $U + W_1 + W_2 + W_3$ is a direct sum. Let $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbf{F}^5$ be given. Then observe that

$$\begin{aligned} v = (v_1, v_2, v_3, v_4, v_5) &= (v_1, v_2, v_1 + v_2, v_1 - v_2, 2v_1) + (0, 0, v_3 - (v_1 + v_2), 0, 0) \\ &\quad + (0, 0, 0, v_4 - (v_1 - v_2), 0) + (0, 0, 0, 0, v_5 - 2v_1) \in U \oplus W_1 \oplus W_2 \oplus W_3. \end{aligned}$$

Hence $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Exercise 1.C.23. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$

then $U_1 = U_2$.

Solution. This is false. For a counterexample, consider $V = \mathbf{R}^2$,

$$W = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, U_1 = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, U_2 = \{(y, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}.$$

It is easily verified that $W \cap U_1 = W \cap U_2 = \{0\}$, so that $W + U_1$ and $W + U_2$ are both direct sums, and that $W \oplus U_1 = W \oplus U_2 = \mathbf{R}^2$. However, $U_1 \neq U_2$, since $(1, 1) \in U_2$ but $(1, 1) \notin U_1$.

Exercise 1.C.24. A function $f : \mathbf{R} \to \mathbf{R}$ is called ***even*** if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f : \mathbf{R} \to \mathbf{R}$ is called ***odd*** if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$. Let U_{e} denote the set of real-valued even functions on $\mathbf{R}$ and let U_{o} denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^{\mathbf{R}} = U_{\mathrm{e}} \oplus U_{\mathrm{o}}$.

Solution. Suppose that $f \in U_{\mathrm{e}} \cap U_{\mathrm{o}}$. Then $f(x) = -f(x)$ for all $x \in \mathbf{R}$, which implies that $f(x) = 0$ for all $x \in \mathbf{R}$, i.e. $f = 0$. So $U_{\mathrm{e}} \cap U_{\mathrm{o}} = \{0\}$ and hence by (1.45) the sum $U_{\mathrm{e}} + U_{\mathrm{o}}$ is direct. Let $f : \mathbf{R} \to \mathbf{R}$ be given and define $f_{\mathrm{e}} : \mathbf{R} \to \mathbf{R}$, $f_{\mathrm{o}} : \mathbf{R} \to \mathbf{R}$ by

$$f_{\mathrm{e}}(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_{\mathrm{o}}(x) = \frac{f(x) - f(-x)}{2}.$$

It is easily verified that f_{e} is an even function, that f_{o} is an odd function, and that $f = f_{\mathrm{e}} + f_{\mathrm{o}}$. It follows that $\mathbf{R}^{\mathbf{R}} = U_{\mathrm{e}} \oplus U_{\mathrm{o}}$.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edn.