1 Section 3.C Exercises

Exercises with solutions from Section 3.C of [LADR].

Exercise 3.C.1. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

Solution. Let $v_1, \ldots, v_n$ be a basis for V and $w_1, \ldots, w_m$ be a basis for W, so that the matrix of T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

Set $p = \dim \operatorname{null} T$ and $q = \dim \operatorname{range} T$, so that $p + q = n$. Since the list $v_1, \ldots, v_n$ is linearly independent, at most p of these vectors can belong to $\operatorname{null} T$. Equivalently, at least $n - p = q$ of these vectors do not belong to $\operatorname{null} T$. Let v_k be such a vector, i.e.

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m \neq 0.$$

Since this is non-zero, at least one of the scalars $A_{j,k}$ must be non-zero; this is true for each of the vectors from the list $v_1, \ldots, v_n$ which do not belong to $\operatorname{null} T$, of which there are at least q. Thus $\mathcal{M}(T)$ has at least $q = \dim \operatorname{range} T$ non-zero entries.

Exercise 3.C.2. Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbf{R})$ and a basis of $\mathcal{P}_2(\mathbf{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

[*Compare the exercise above to Example 3.34.*
The next exercise generalizes the exercise above.]

Solution. Take $\frac{1}{3}x^3, \frac{1}{2}x^2, x, 1$ as a basis of $\mathcal{P}_3(\mathbf{R})$ and $x^2, x, 1$ as a basis of $\mathcal{P}_2(\mathbf{R})$. Then

$$D\left(\tfrac{1}{3}x^3\right) = x^2, \quad D\left(\tfrac{1}{2}x^2\right) = x, \quad D(x) = 1, \quad \text{and} \quad D(1) = 0.$$

Thus the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Exercise 3.C.3. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row j, column j, equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

Solution. As shown in Exercise 3.B.12, there is a subspace U of V such that $V = U \oplus \operatorname{null} T$ and $\operatorname{range} T = \{Tu : u \in U\}$. Let $u_1, \ldots, u_k$ be a basis of U and let $x_1, \ldots, x_n$ be a basis of $\operatorname{null} T$; the list $u_1, \ldots, u_k, x_1, \ldots, x_n$ is a basis of V since the sum $V = U \oplus \operatorname{null} T$ is direct (see Exercise 2.B.8). We claim that the list $Tu_1, \ldots, Tu_k$ is a basis of $\operatorname{range} T$. Since $\operatorname{range} T = \{Tu : u \in U\}$, it is clear that this list spans $\operatorname{range} T$. Suppose we have scalars $a_1, \ldots, a_k$ such that

$$a_1 Tu_1 + \cdots = a_k Tu_k = T(a_1 u_1 + \cdots + a_k u_k) = 0.$$

Then $a_1 u_1 + \cdots + a_k u_k$ belongs to $\operatorname{null} T$ as well as U and so we must have $a_1 u_1 + \cdots + a_k u_k = 0$. The linear independence of the list $u_1, \ldots, u_k$ implies that $a_1 = \cdots = a_k = 0$ and hence the list $Tu_1, \ldots, Tu_k$ is linearly independent and our claim follows.

Extend the basis $Tu_1, \ldots, Tu_k$ to a basis $Tu_1, \ldots, Tu_k, y_1, \ldots, y_m$ for W. Then:

$$Tu_j = 0Tu_1 + \cdots + 1Tu_j + \cdots + 0Tu_k + 0y_1 + \cdots + 0y_m \quad \text{and} \quad Tx_j = 0.$$

Thus the matrix of T with respect to the bases $u_1, \ldots, u_k, x_1, \ldots, x_n$ and $Tu_1, \ldots, Tu_k, y_1, \ldots, y_m$ has the desired form.

Exercise 3.C.4. Suppose $v_1, \ldots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_n$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except for possibly a 1 in the first row, first column.
[*In this exercise, unlike Exercise 3, you are given the basis of V instead of being able to choose a basis of V.*]

Solution. There are two cases.

Case 1. $Tv_1 = 0$, i.e. $v_1 \in \operatorname{null} T$. In this case, let $w_1, \ldots, w_n$ be any basis of W. By linear independence of this basis, we have

$$Tv_1 = 0 = 0w_1 + \cdots + 0w_n.$$

Thus the entries in the first column of $\mathcal{M}(T)$ are all 0.

Case 2. $Tv_1 \neq 0$, i.e. $v_1 \notin \operatorname{null} T$. In this case, let $w_1 := Tv_1$. The list w_1 is linearly independent since $w_1 \neq 0$ and can hence be extended to a basis $w_1, \ldots, w_n$ for W. We then have

$$Tv_1 = w_1 = 1w_1 + 0w_2 + \cdots + 0w_n.$$

Thus the entries in the first column of $\mathcal{M}(T)$ are all 0 except for a 1 in the first row.

Exercise 3.C.5. Suppose $w_1, \ldots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of V such that all the entries in the first row of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except for possibly a 1 in the first row, first column.
[*In this exercise, unlike Exercise 3, you are given the basis of W instead of being able to choose a basis of W.*]

Solution. Let $u_1, \ldots, u_m$ be any basis of V and let M_1 be the matrix of T with respect to the bases $u_1, \ldots, u_m$ and $w_1, \ldots, w_n$, with entries $A_{j,k}$, i.e.

$$M_1 = \mathcal{M}(T, (u_1, \ldots, u_m), (w_1, \ldots, w_n)) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} \end{pmatrix}.$$

If the entries in the first row of M_1 are all 0, then we are done. Otherwise, there exists some $1 \leq i \leq m$ such that $A_{1,i}$ is non-zero, so that $\lambda := A_{1,i}^{-1}$ exists. Define

$$v_1 := \lambda u_i, \quad v_i := u_1 - \lambda A_{1,1} u_i, \quad \text{and} \quad v_k := u_k - \lambda A_{1,k} u_i \text{ for } 2 \leq k \leq m, k \neq i.$$

We claim that $v_1, \ldots, v_m$ is a basis for V. Observe that

$$u_1 = v_i + A_{1,1} v_1, \quad u_i = A_{1,i} v_1, \quad \text{and} \quad u_k = v_k + A_{1,k} v_1 \text{ for } 2 \leq k \leq m, k \neq i.$$

So each vector from the basis $u_1, \ldots, u_m$ can be expressed as a linear combination of vectors from the list $v_1, \ldots, v_m$. Since $V = \operatorname{span}(u_1, \ldots, u_m)$, it follows that $V = \operatorname{span}(v_1, \ldots, v_m)$. By 2.42, we may now conclude that this list is a basis for V. Observe that

$$\begin{aligned} Tv_1 &= \lambda T u_i = \lambda(A_{1,i} w_1 + \cdots + A_{n,i} w_n) \\ &= 1 w_1 + \cdots + \lambda A_{n,i} w_n, \\ Tv_i &= Tu_1 - A_{1,1}(\lambda T u_i) \\ &= A_{1,1} w_1 + \cdots + A_{n,1} w_n - A_{1,1}(w_1 + \cdots + \lambda A_{n,i} w_n) \\ &= 0 w_1 + \cdots + (A_{n,1} - \lambda A_{1,1} A_{n,i}) w_n, \end{aligned}$$

and for $2 \leq k \leq m, k \neq i$,

$$\begin{aligned} Tv_k &= Tu_k - A_{1,k}(\lambda T u_i) \\ &= A_{1,k} w_1 + \cdots + A_{n,k} w_n - A_{1,k}(w_1 + \cdots + \lambda A_{n,i} w_n) \\ &= 0 w_1 + \cdots + (A_{n,k} - \lambda A_{1,k} A_{n,i}) w_n. \end{aligned}$$

Thus the entries in the first row of the matrix of T with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are 0, except for a 1 in the first column.

Exercise 3.C.6. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

Solution. Suppose there exists a basis $v_1, \ldots, v_m$ of V and a basis $w_1, \ldots, w_n$ of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1, i.e.

$$Tv_1 = \cdots = Tv_m = w_1 + \cdots + w_n.$$

We claim that Tv_1 is a basis for $\operatorname{range} T$. The linear independence of the basis $w_1, \ldots, w_n$ implies that $Tv_1 \neq 0$, so that the list Tv_1 is linearly independent. If $w \in \operatorname{range} T$, then $w = Tv$ for some $v \in V$. There are scalars $a_1, \ldots, a_m$ such that $v = a_1 v_1 + \cdots + a_m v_m$, which gives

$$w = Tv = a_1 Tv_1 + \cdots + a_m Tv_m = (a_1 + \cdots + a_m) Tv_1.$$

Thus the list Tv_1 spans $\operatorname{range} T$ and we may conclude that $\operatorname{range} T$ has a basis of length 1, i.e. $\dim \operatorname{range} T = 1$.

To prove the converse statement, let us first prove the following lemmas.

Lemma 1. Suppose U is a finite-dimensional vector space with $\dim U = m$, and suppose $u \in U$ is non-zero. Then there exists a basis $u_1, \ldots, u_m$ of U such that $u = u_1 + \cdots + u_m$.

Proof. If $m = 1$, then take $u_1 = u$ and we are done. Otherwise, since $u \neq 0$, we can extend the list u to a basis $u, u_1, \ldots, u_{m-1}$ of U. Define $u_m := u - u_1 - \cdots - u_{m-1}$ and consider the list $u_1, \ldots, u_{m-1}, u_m$. Since each vector in the basis $u, u_1, \ldots, u_{m-1}$ can be expressed as a linear combination of vectors from the list $u_1, \ldots, u_{m-1}, u_m$, it follows that $U = \operatorname{span}(u_1, \ldots, u_{m-1}, u_m)$. 2.42 allows us to conclude that $u_1, \ldots, u_m$ is a basis of U. From the definition of u_m, it is clear that $u = u_1 + \cdots + u_m$. $\square$

Lemma 2. Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. If $\operatorname{null} T \neq V$ then there exists a basis $v_1, \ldots, v_m$ of V such that $v_k \notin \operatorname{null} T$ for each $1 \leq k \leq m$.

Proof. By 2.34, we may write $V = U \oplus \operatorname{null} T$ for some subspace U of V. Since $\operatorname{null} T \neq V$, it must be the case that U is not the trivial subspace. Thus if we let $u_1, \ldots, u_{m-l}$ be a basis of U, then this basis contains at least one vector u_1. Letting $x_1, \ldots, x_l$ be a basis of $\operatorname{null} T$, Exercise 2.B.8 implies that

$$B := u_1, \ldots, u_{m-l}, x_1, \ldots, x_l$$

is a basis of V. Consider the list

$$B' := u_1, \ldots, u_{m-l}, x_1 + u_1, \ldots, x_l + u_1.$$

Since each vector in the basis B can be expressed as a linear combination of vectors from the list B', it follows that $V = \operatorname{span} B'$. 2.42 allows us to conclude that B' is a basis of V. Since the sum

$V = U \oplus \operatorname{null} T$ is direct, we have $U \cap \operatorname{null} T = \{0\}$; it follows that each u_k in the list $u_1, \ldots, u_{m-l}$ satisfies $u_k \notin \operatorname{null} T$. Furthermore, if $1 \leq k \leq l$, then

$$T(x_k + u_1) = Tx_k + Tu_1 = Tu_1 \neq 0,$$

so that $x_k + u_1 \notin \operatorname{null} T$ also. Thus B' is the desired basis of V. □

Now suppose that $\dim \operatorname{range} T = 1$, so that $\operatorname{range} T$ has a basis w. By Lemma 1, there is a basis $w_1, \ldots, w_n$ of W such that $w = w_1 + \cdots + w_n$, and by Lemma 2 there is a basis $u_1, \ldots, u_m$ of V such that $Tu_k \neq 0$ for $1 \leq k \leq m$. Then for each $1 \leq k \leq m$, we have

$$Tu_k = \lambda_k w$$

for some non-zero scalar λ_k. Set $v_k = \lambda_k^{-1} u_k$; it is easily verified that $v_1, \ldots v_m$ is a basis of V since each λ_k^{-1} is non-zero. Then

$$Tv_k = w = w_1 + \cdots + w_n.$$

Thus with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$, all entries of $\mathcal{M}(T)$ equal 1.

Exercise 3.C.7. Verify 3.36.

Solution. Suppose $S, T \in \mathcal{L}(V, W)$. Let $v_1, \ldots, v_m$ be a basis of V and let $w_1, \ldots, w_n$ be a basis of W. We wish to verify that, with respect to these bases, we have $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$. Suppose $\mathcal{M}(S)$ has entries $A_{j,k}$ and $\mathcal{M}(T)$ has entries $B_{j,k}$, i.e.

$$Sv_k = A_{1,k}w_1 + \cdots + A_{n,k}w_n \quad \text{and} \quad Tv_k = B_{1,k}w_1 + \cdots + B_{n,k}w_n.$$

Then

$$(S+T)(v_k) = Sv_k + Tv_k = (A_{1,k} + B_{1,k})w_1 + \cdots + (A_{n,k} + B_{n,k})w_n.$$

Thus $\mathcal{M}(S+T)$ has entries $A_{j,k} + B_{j,k}$.

Exercise 3.C.8. Verify 3.38.

Solution. Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Let $v_1, \ldots, v_m$ be a basis of V and let $w_1, \ldots, w_n$ be a basis of W. We wish to verify that, with respect to these bases, we have $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$. Suppose $\mathcal{M}(T)$ has entries $A_{j,k}$, i.e.

$$Tv_k = A_{1,k}w_1 + \cdots + A_{n,k}w_n.$$

Then

$$(\lambda T)(v_k) = \lambda T v_k = (\lambda A_{1,k})w_1 + \cdots + (\lambda A_{n,k})w_n.$$

Thus $\mathcal{M}(\lambda T)$ has entries $\lambda A_{j,k}$.

Exercise 3.C.9. Prove 3.52.

Solution. Suppose A is an m-by-n matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an n-by-1 matrix. Then Ac is an m-by-1 matrix whose entry in the j^{th} row is

$$(Ac)_{j,1} = \sum_{r=1}^{n} A_{j,r}c_r = c_1 A_{j,1} + \cdots + c_n A_{j,n}.$$

Thus

$$Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}.$$

Exercise 3.C.10. Suppose A is an m-by-n matrix and C is an n-by-p matrix. Prove that

$$(AC)_{j,\cdot} = A_{j,\cdot}C$$

for $1 \leq j \leq m$. In other words, show that row j of AC equals (row j of A) times C.

Solution. $(AC)_{j,\cdot}$ is a 1-by-p matrix whose entry in the k^{th} column is

$$((AC)_{j,\cdot})_{1,k} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k}.$$

$A_{j,\cdot}$ is a 1-by-n matrix and so $A_{j,\cdot}C$ is a 1-by-p matrix whose entry in the k^{th} column is

$$(A_{j,\cdot}C)_{1,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}C_{r,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k}.$$

Thus $(AC)_{j,\cdot} = A_{j,\cdot}C$.

Exercise 3.C.11. Suppose $a = (a_1 \cdots a_n)$ is a 1-by-n matrix and C is an n-by-p matrix. Prove that

$$aC = a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot}.$$

In other words, show that aC is a linear combination of the rows of C, with the scalars that multiply the rows coming from a.

Solution. aC is a 1-by-p matrix whose entry in the k^{th} column is

$$(aC)_{1,k} = \sum_{r=1}^{n} a_r C_{r,k} = a_1 C_{1,k} + \cdots + a_n C_{n,k}.$$

Thus

$$aC = a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot}.$$

Exercise 3.C.12. Give an example with 2-by-2 matrices to show that matrix multiplication is not commutative. In other words, find 2-by-2 matrices A and C such that $AC \neq CA$.

Solution. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$AC = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = CA.$$

Exercise 3.C.13. Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E, and F are matrices whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense. Prove that $AB + AC$ and $DF + EF$ both make sense and that $A(B+C) = AB + AC$ and $(D+E)F = DF + EF$.

Solution. For $B + C$ to make sense, B and C must have the same sizes; suppose they are both n-by-p matrices. Then for $A(B+C)$ to make sense, A must be an m-by-n matrix. Given this, both AB and AC are m-by-p matrices and thus $AB + AC$ makes sense.

Similarly, suppose D and E are both m-by-n matrices. Then for $(D+E)F$ to make sense, F must be an n-by-p matrix. Given this, both DF and EF are m-by-p matrices and thus $DF + EF$ makes sense.

In what follows, the matrix of any linear map is understood to be with respect to the relevant standard bases of $\mathbf{F}^m, \mathbf{F}^n$, and $\mathbf{F}^p$. Given an m-by-n matrix A whose entries $A_{j,k}$ belong to $\mathbf{F}$, define a linear map $T_A : \mathbf{F}^n \to \mathbf{F}^m$ by

$$T_A e_k = A_{1,k} e_1 + \cdots + A_{m,k} e_m,$$

where e_k is the k^{th} standard basis vector. Evidently, the matrix of this linear map is A. Thus $A(B+C) = AB + AC$ if and only if $T_{A(B+C)} = T_{AB+AC}$. By 3.9, 3.36, and 3.43, we have

$$T_{A(B+C)} = T_A T_{B+C} = T_A(T_B + T_C) = T_A T_B + T_A T_C = T_{AB} + T_{AC} = T_{AB+AC}.$$

Similarly, $(D+E)F = DF + EF$ if and only if $T_{(D+E)F} = T_{DF+EF}$. By 3.9, 3.36, and 3.43, we have

$$T_{(D+E)F} = T_{D+E} T_F = (T_D + T_E) T_F = T_D T_F + T_E T_F = T_{DF} + T_{EF} = T_{DF+EF}.$$

Exercise 3.C.14. Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

Solution. If A is an m-by-n matrix, then for AB to make sense, B must be an n-by-p matrix, so that AB is an m-by-p matrix. Then for $(AB)C$ to make sense, C must be a p-by-q matrix. Thus BC is an n-by-q matrix and $A(BC)$ is an m-by-q matrix. For a given matrix A, define the linear map T_A as in Exercise 3.C.13. Then $(AB)C = A(BC)$ if and only if $T_{(AB)C} = T_{A(BC)}$. By 3.9 and 3.43, we have

$$T_{(AB)C} = T_{AB}T_C = (T_AT_B)T_C = T_A(T_BT_C) = T_AT_{BC} = T_{A(BC)}.$$

Exercise 3.C.15. Suppose A is an n-by-n matrix and $1 \leq j, k \leq n$. Show that the entry in row j, column k, of A^3 (which to defined to mean AAA) is

$$\sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}.$$

Solution. By the definition of matrix multiplication, we have

$$(A^3)_{j,k} = (A^2A)_{j,k} = \sum_{r=1}^{n}(A^2)_{j,r}A_{r,k} = \sum_{r=1}^{n}\sum_{p=1}^{n} A_{j,p}A_{p,r}A_{r,k} = \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}.$$

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