1 Section 3.5 Exercises

Exercises with solutions from Section 3.5 of [UA].

Exercise 3.5.1. Argue that a set A is a G_δ set if and only if its complement is an F_σ set.

Solution. This is immediate from De Morgan's Laws (see Exercise 3.2.9).

Exercise 3.5.2. Replace each ____ with the word *finite* or *countable* depending on which is more appropriate.

(a) The ____ union of F_σ sets is an F_σ set.

(b) The ____ intersection of F_σ sets is an F_σ set.

(c) The ____ union of G_δ sets is a G_δ set.

(d) The ____ intersection of G_δ sets is a G_δ set.

Solution. (a) The countable union of F_σ sets is an F_σ set. Suppose we have a countable collection $\{A_m : m \in \mathbf{N}\}$ of F_σ sets, i.e. for each $m \in \mathbf{N}$ there is a countable collection $\{B_{m,n} : n \in \mathbf{N}\}$ of closed sets such that $A_m = \bigcup_{n=1}^\infty B_{m,n}$. Then

$$\bigcup_{m=1}^\infty A_m = \bigcup_{m=1}^\infty \bigcup_{n=1}^\infty B_{m,n} = \bigcup_{(m,n)\in\mathbf{N}^2} B_{m,n}.$$

$\mathbf{N}^2$ is countable by Theorem 1.5.8 (ii), so we have expressed $\bigcup_{m=1}^\infty A_m$ as a countable union of closed sets and hence $\bigcup_{m=1}^\infty A_m$ is an F_σ set.

(b) The finite intersection of F_σ sets is an F_σ set. To see this, it will suffice to show that if A and B are F_σ sets, then $A \cap B$ is an F_σ set; the general case will follow from an induction argument. Suppose therefore that $A = \bigcup_{m=1}^\infty A_m$ and $B = \bigcup_{n=1}^\infty B_n$, where $\{A_m : m \in \mathbf{N}\}$ and $\{B_n : n \in \mathbf{N}\}$ are countable collections of closed sets. We claim that

$$A \cap B = \left(\bigcup_{m=1}^\infty A_m\right) \cap \left(\bigcup_{n=1}^\infty B_n\right) = \bigcup_{(m,n)\in\mathbf{N}^2} (A_m \cap B_n).$$

Indeed,

$$\begin{aligned}
x \in \left(\textstyle\bigcup_{m=1}^\infty A_m\right) \cap \left(\textstyle\bigcup_{n=1}^\infty B_n\right) &\iff \left(x \in \textstyle\bigcup_{m=1}^\infty A_m\right) \text{ and } \left(x \in \textstyle\bigcup_{n=1}^\infty B_n\right)\\
&\iff (\exists m \in \mathbf{N} : x \in A_m) \text{ and } (\exists n \in \mathbf{N} : x \in B_n)\\
&\iff (\exists (m,n) \in \mathbf{N}^2 : x \in A_m \text{ and } x \in B_n)\\
&\iff (\exists (m,n) \in \mathbf{N}^2 : x \in A_m \cap B_n)\\
&\iff x \in \textstyle\bigcup_{(m,n)\in\mathbf{N}^2} (A_m \cap B_n).
\end{aligned}$$

For any $(m,n) \in \mathbf{N}^2$, the intersection $A_m \cap B_n$ is closed since both A_m and B_n are closed. Thus we have expressed $A \cap B$ as a countable union of closed sets and hence $A \cap B$ is an F_σ set.

The countable intersection of F_σ sets need not be an F_σ set. For a counterexample, let $\{r_1, r_2, \ldots\}$ be an enumeration of $\mathbf{Q}$ and for positive integers m and n, set

$$B_{m,n} := \left(-\infty, r_m - \tfrac{1}{n}\right] \cup \left[r_m + \tfrac{1}{n}, \infty\right).$$

Each $B_{m,n}$ is a closed set, so if we let $A_m := \bigcup_{n=1}^\infty B_{m,n}$ for each $m \in \mathbf{N}$, then each A_m is an F_σ set. We claim that $\bigcap_{m=1}^\infty A_m = \mathbf{I}$, the set of irrational numbers. To see this, we will show that $(\bigcap_{m=1}^\infty A_m)^{\mathsf{c}} = \mathbf{Q}$. By De Morgan's Laws, we have

$$\begin{aligned}
(\textstyle\bigcap_{m=1}^\infty A_m)^{\mathsf{c}} &= \textstyle\bigcup_{m=1}^\infty A_m^{\mathsf{c}} \\
&= \textstyle\bigcup_{m=1}^\infty (\bigcup_{n=1}^\infty B_{m,n})^{\mathsf{c}} \\
&= \textstyle\bigcup_{m=1}^\infty \bigcap_{n=1}^\infty B_{m,n}^{\mathsf{c}} \\
&= \textstyle\bigcup_{m=1}^\infty \bigcap_{n=1}^\infty \left(r_m - \frac{1}{n}, r_m + \frac{1}{n}\right) \\
&= \textstyle\bigcup_{m=1}^\infty \{r_m\} \\
&= \mathbf{Q}.
\end{aligned}$$

Thus $\bigcap_{m=1}^\infty A_m = \mathbf{I}$. As we will show in Exercise 3.5.6, $\mathbf{I}$ is not an F_σ set.

(c) The finite union of G_δ sets is a G_δ set, but the countable union of G_δ sets need not be a G_δ set; these statements follow from part (b) of this exercise, Exercise 3.5.1, and De Morgan's Laws.

(d) The countable intersection of G_δ sets is a G_δ set. Again, this follows from part (a) of this exercise, Exercise 3.5.1, and De Morgan's Laws.

Exercise 3.5.3. (This exercise has already appeared as Exercise 3.2.15.)

(a) Show that a closed interval $[a,b]$ is a G_δ set.

(b) Show that the half-open interval $(a,b]$ is both a G_δ and an F_σ set.

(c) Show that $\mathbf{Q}$ is an F_σ set, and the set of irrationals $\mathbf{I}$ forms a G_δ set.

Solution. See Exercise 3.2.15.

Exercise 3.5.4. Starting with $n = 1$, inductively construct a nested sequence of *closed* intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ satisfying $I_n \subseteq G_n$. Give special attention to the issue of the endpoints of each I_n. Show how this leads to a proof of the theorem.

Solution. Since G_1 is dense, it must be non-empty, i.e. there exists some $x_1 \in G_1$. Since G_1 is open, there exists an $\epsilon_1 > 0$ such that $(x_1 - \epsilon_1, x_1 + \epsilon_1) \subseteq G_1$. Set

$$a_1 := x_1 - \tfrac{\epsilon_1}{2}, \quad b_1 := x_1 + \tfrac{\epsilon_1}{2}, \quad \text{and} \quad I_1 = [a_1, b_1].$$

Then $I_1 \subseteq (x_1 - \epsilon_1, x_1 + \epsilon_1) \subseteq G_1$. This handles the base case. Now suppose that after n steps we have chosen nested, closed intervals $I_1 = [a_1, b_1] \supseteq \cdots \supseteq I_n = [a_n, b_n]$ such that $I_1 \subseteq G_1, \ldots, I_n \subseteq G_n$ and $a_1 < b_1, \ldots, a_n < b_n$. Since G_{n+1} is dense, there exists some $x_{n+1} \in G_{n+1}$ such that $a_n < x_{n+1} < b_n$, and since G_{n+1} is open, there exists some $\epsilon_{n+1} > 0$ such that $(x_{n+1} - \epsilon_{n+1}, x_{n+1} + \epsilon_{n+1}) \subseteq G_{n+1}$. Let $\delta = \min\left\{\frac{\epsilon_{n+1}}{2}, x_{n+1} - a_n, b_n - x_{n+1}\right\}$, and set

$$a_{n+1} := x_{n+1} - \delta, \quad b_{n+1} := x_{n+1} + \delta, \quad \text{and} \quad I_{n+1} = [a_{n+1}, b_{n+1}].$$

Then $a_{n+1} < b_{n+1}$, and since $\delta \leq x_{n+1} - a_n$ and $\delta \leq b_n - x_{n+1}$, we have $I_{n+1} \subseteq I_n$. Moreover, because $\delta \leq \frac{\epsilon_{n+1}}{2}$, we also have $I_{n+1} \subseteq (x_{n+1} - \epsilon_{n+1}, x_{n+1} + \epsilon_{n+1}) \subseteq G_{n+1}$. This completes the induction step.

Thus we obtain a nested sequence of closed intervals (I_n) such that $I_n \subseteq G_n$ for each $n \in \mathbf{N}$. We may now appeal to the Nested Interval Property to obtain some $x \in \bigcap_{n=1}^{\infty} I_n$, which must also belong to $\bigcap_{n=1}^{\infty} G_n$.

Exercise 3.5.5. Show that it is impossible to write

$$\mathbf{R} = \bigcup_{n=1}^{\infty} F_n,$$

where for each $n \in \mathbf{N}, F_n$ is a closed set containing no nonempty open intervals.

Solution. Suppose that $\{F_n : n \in \mathbf{N}\}$ is a collection of closed sets, each of which contains no non-empty open intervals. Then for each $n \in \mathbf{N}, F_n^{\mathsf{c}}$ is an open set. Furthermore, we claim that F_n^{c} is dense. To see this, let $x < z$ be arbitrary real numbers. By assumption, $(x, z) \not\subseteq F_n$, so there exists some $y \in (x, z) \cap F_n^{\mathsf{c}}$; the claim follows.

Thus $\{F_n^{\mathsf{c}} : n \in \mathbf{N}\}$ is a collection of open, dense sets. Theorem 3.5.2 (Exercise 3.5.4) and De Morgan's Laws now imply that

$$\bigcap_{n=1}^{\infty} F_n^{\mathsf{c}} \neq \emptyset \iff \bigcup_{n=1}^{\infty} F_n \neq \mathbf{R}.$$

Exercise 3.5.6. Show how the previous exercise implies that the set $\mathbf{I}$ of irrationals cannot be an F_σ set, and $\mathbf{Q}$ cannot be a G_δ set.

Solution. We will argue by contradiction. Suppose that $\mathbf{I}$ is an F_σ set, so that $\mathbf{I} = \bigcup_{m=1}^\infty F_m$, where each F_m is closed. Note that for any $m \in \mathbf{N}$, it must be the case that F_m contains no non-empty open interval; otherwise, F_m would contain infinitely many rational numbers. Let $\{r_1, r_2, \dots\}$ be an enumeration of $\mathbf{Q}$, so that $\mathbf{Q} = \bigcup_{n=1}^\infty \{r_n\}$. For any $n \in \mathbf{N}$, the singleton $\{r_n\}$ is closed and contains no non-empty interval. Observe that

$$\mathbf{R} = \mathbf{I} \cup \mathbf{Q} = \left(\bigcup_{m=1}^\infty F_m \right) \cup \left(\bigcup_{n=1}^\infty \{r_n\} \right) = \bigcup_{(m,n) \in \mathbf{N}^2} (F_m \cup \{r_n\}).$$

For any $(m, n) \in \mathbf{N}^2$, the union $F_m \cup \{r_n\}$ is closed and contains no non-empty intervals. However, since $\mathbf{N}^2$ is countable, this expression for $\mathbf{R}$ contradicts Exercise 3.5.5. Hence it must be the case that $\mathbf{I}$ is not an F_σ set, which by Exercise 3.5.1 implies that $\mathbf{Q}$ cannot be a G_δ set.

Exercise 3.5.7. Using Exercise 3.5.6 and versions of the statements in Exercise 3.5.2, construct a set that is neither in F_σ nor in G_δ.

Solution. Define $E := (\mathbf{I} \cap (-\infty, 0]) \cup (\mathbf{Q} \cap [0, \infty))$; we claim that E is neither an F_σ nor a G_δ set. Seeking a contradiction, suppose that E is an F_σ set. It is not hard to see that any interval is an F_σ set (see Exercise 3.5.3), so by Exercise 3.5.2 (b) we have that

$$E \cap (-\infty, 0) = \mathbf{I} \cap (-\infty, 0)$$

is an F_σ set, i.e. there is a countable collection $\{F_m : m \in \mathbf{N}\}$ of closed sets such that

$$\mathbf{I} \cap (-\infty, 0) = \bigcup_{m=1}^\infty F_m.$$

For $m \in \mathbf{N}$, let $-F_m = \{-x : x \in F_m\}$. Then since $(x_n) \to x$ implies $(-x_n) \to -x$, each $-F_m$ is also closed. Furthermore, we have

$$\mathbf{I} \cap (0, \infty) = \bigcup_{m=1}^\infty -F_m.$$

It follows that $\mathbf{I} \cap (0, \infty)$ is an F_σ set. However, Exercise 3.5.2 (a) now implies that

$$\mathbf{I} = (\mathbf{I} \cap (-\infty, 0)) \cup (\mathbf{I} \cap (0, \infty))$$

is an F_σ set, contradicting Exercise 3.5.6. A similar argument with $\mathbf{Q}$ shows that E cannot be a G_δ set either.

Exercise 3.5.8. Show that a set E is nowhere-dense in $\mathbf{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbf{R}$.

Solution. We will show that $A \subseteq \mathbf{R}$ contains no non-empty open intervals if and only if A^{c} is dense in $\mathbf{R}$. By A containing no non-empty open intervals, we mean that for all $x, y \in \mathbf{R}$ such that $x < y$, we have $(x, y) \nsubseteq A$. This is equivalent to saying that for all $x, y \in \mathbf{R}$ such that $x < y$, there exists some $t \in \mathbf{R}$ such that $x < t < y$ and $t \notin A$. In other words, A^{c} is dense in $\mathbf{R}$.

Exercise 3.5.9. Decide whether the following sets are dense in $\mathbf{R}$, nowhere-dense in $\mathbf{R}$, or somewhere in between.

(a) $A = \mathbf{Q} \cap [0, 5]$.

(b) $B = \{1/n : n \in \mathbf{N}\}$.

(c) the set of irrationals.

(d) the Cantor set.

Solution. (a) We have $\overline{A} = [0, 5]$, which is not the entire real line and also contains non-empty open intervals. Thus A is neither dense nor nowhere-dense.

(b) We have $\overline{B} = \{0\} \cup B \neq \mathbf{R}$, so that B is not dense. Note that if $\overline{B}$ contained a non-empty open interval then $\overline{B}$ would contain at least one irrational number, but $\overline{B} \subseteq \mathbf{Q}$. Thus $\overline{B}$ contains no non-empty open intervals and hence B is nowhere-dense.

(c) $\mathbf{I}$ is dense in $\mathbf{R}$ (see Exercise 1.4.5) and hence cannot be nowhere-dense (a dense subset $E \subseteq \mathbf{R}$ certainly cannot be nowhere-dense; $\overline{E} = \mathbf{R}$ contains every non-empty open interval).

(d) The Cantor set is closed, so $\overline{C} = C \neq \mathbf{R}$. Thus C is not dense in $\mathbf{R}$. C also does not contain any non-empty open intervals; given any $x < y$ in C, it is always possible to find some $t \notin C$ such that $x < t < y$ (see Exercise 3.4.8). Thus C is nowhere-dense in $\mathbf{R}$.

Exercise 3.5.10. Finish the proof by finding a contradiction to the results in this section.

Solution. Since $E_n \subseteq \overline{E_n}$ for each $n \in \mathbf{N}$, we have $\mathbf{R} = \bigcup_{n=1}^{\infty} \overline{E_n}$. However, each $\overline{E_n}$ is closed and by assumption contains no non-empty open intervals, so this contradicts Exercise 3.5.5.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.