1 Section 1.4 Exercises

Exercises with solutions from Section 1.4 of [UA].

Exercise 1.4.1. Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a+b$ are elements of $\mathbf{Q}$ as well.

(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a+t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

Solution. (a) Suppose $a = \frac{k}{l}$ and $b = \frac{m}{n}$. Then

$$ab = \tfrac{km}{ln} \quad \text{and} \quad a+b = \tfrac{kn+lm}{ln},$$

which are rational numbers.

(b) Let $a \in \mathbf{Q}$ be fixed. We want to prove that

$$t \in \mathbf{I} \implies a+t \in \mathbf{I}.$$

To do this, we will prove the contrapositive statement

$$a+t \in \mathbf{Q} \implies t \in \mathbf{Q}.$$

Simply observe that $t = (a+t) - a$. Then by part (a), $t \in \mathbf{Q}$.

Similarly, let $a \in \mathbf{Q}$ be non-zero. We can show that

$$at \in \mathbf{Q} \implies t \in \mathbf{Q}$$

by observing that $t = a^{-1}(at)$ and appealing to part (a) to conclude that $t \in \mathbf{Q}$.

(c) $\mathbf{I}$ is not closed under addition or multiplication. For example, $-\sqrt{2}$ and $\sqrt{2}$ are irrational numbers, but their sum is the rational number 0 and their product is the rational number -2. The sum or product of two irrational numbers may be irrational; for example, it can be shown that $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}\sqrt{3}$ are irrational. So in general, we cannot say anything about the sum or product of two irrational numbers without more information.

Exercise 1.4.2. Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution. First, let us show that s is an upper bound for A. Seeking a contradiction, suppose this is not the case. Then there must exist some $x \in A$ such that $s < x$. By the Archimedean property of $\mathbf{R}$, there exists a natural number n such that $\frac{1}{n} < x - s \iff s + \frac{1}{n} < x$. This implies that $s + \frac{1}{n}$ is not an upper bound for A, which contradicts our hypotheses. Hence it must be that s is an upper bound for A.

Now let $\epsilon > 0$ be given and using the Archimedean property, pick a natural number n such that $\frac{1}{n} < \epsilon$. By assumption, $s - \frac{1}{n}$ is not an upper bound for A, so there must exist some $x \in A$ such that $s - \frac{1}{n} < x$; this implies that $s - \epsilon < x$ since $\frac{1}{n} < \epsilon$. Hence by Lemma 1.3.8, $s = \sup A$.

Exercise 1.4.3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution. Suppose there was some $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$, i.e. some $x \in \mathbf{R}$ such that for all $n \in \mathbf{N}$, $0 < x < 1/n$. This directly contradicts the Archimedean property of $\mathbf{R}$, which says that there must exist an $N \in \mathbf{N}$ such that $1/N < x$. We may conclude that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

Exercise 1.4.4. Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$.

Solution. It is clear that b is an upper bound for T. Let $\epsilon > 0$ be given. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exists a rational number p satisfying $b - \epsilon < p < b$ and a rational number q satisfying $a < q < b$. Let $r = \max\{p, q\}$. Then $a < r < b$, so that $r \in T$, and $b - \epsilon < r$. Hence by Lemma 1.3.8, $\sup T = b$.

Exercise 1.4.5. Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exists a rational number p satisfying $a - \sqrt{2} < p < b - \sqrt{2}$, which gives $a < p + \sqrt{2} < b$. Since $p + \sqrt{2}$ is irrational by Exercise 1.4.1, the corollary is proved.

Exercise 1.4.6. Recall that a set B is *dense* in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2.

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution. (a) This set is not dense in $\mathbf{R}$. Observe that

$$q \leq 10 \iff \tfrac{1}{q} \geq \tfrac{1}{10}.$$

Then if $p > 0$ we have $p/q \geq 1/10$, if $p < 0$ we have $p/q \leq -1/10$, and of course if $p = 0$ we have $p/q = 0$. So there is no element of this set between the real numbers $1/1000$ and $1/100$, for example.

(b) This set is dense in $\mathbf{R}$. To see this, let us first prove that for all $x \in \mathbf{R}$, there exists a $k \in \mathbf{N}$ such that $2^k > x$. Seeking a contradiction, suppose this is not the case. Then the set $K = \{2^k : k \in \mathbf{N}\}$ is non-empty and bounded above, so by the Axiom of Completeness $\alpha := \sup K$ exists in $\mathbf{R}$. Since $2^1 = 2$ belongs to K, it must be the case that α is positive. It follows that $\frac{\alpha}{2} < \alpha$, so that $\frac{\alpha}{2}$ cannot be an upper bound for K. Then there exists some $k \in \mathbf{N}$ such that $\frac{\alpha}{2} < 2^k$, which implies $\alpha < 2^{k+1}$; but this contradicts the fact that α is the supremum of K. Hence it must be the case that for all $x \in \mathbf{R}$, there exists a $k \in \mathbf{N}$ such that $2^k > x$.

Let $a < b$ be real numbers. By the previous paragraph, there exists a $k \in \mathbf{N}$ such that $2^k > \frac{1}{b-a}$, which implies that $\frac{1}{2^k} < b - a$. Now let p be the smallest integer greater than $2^k a$, so that $p - 1 \leq 2^k a < p$. Then observe that

$$2^k a < p \leq 1 + 2^k a < 2^k b \implies a < \tfrac{p}{2^k} < b.$$

(c) This set is not dense in $\mathbf{R}$. If $p > 0$ then

$$10|p| \geq q \iff 10p \geq q \iff p/q \geq 1/10,$$

and if $p < 0$ then

$$10|p| \geq q \iff -10p \geq q \iff p/q \leq -1/10.$$

We cannot have $p = 0$ since q is a positive integer. So as in part (a), there is no element of this set between the real numbers $1/1000$ and $1/100$, for example.

Exercise 1.4.7. Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Solution. By the Archimedean property of $\mathbf{R}$, there exists an $n \in \mathbf{N}$ such that $\frac{2\alpha}{n} < \alpha^2 - 2 \iff 2 < \alpha^2 - \frac{2\alpha}{n}$. Let $b = \alpha - \frac{1}{n}$. Note that since $1 \in T$, we have $\alpha \geq 1$ and hence $b \geq 0$. So if $t \in T$ is such that $t < 0$ then $t \leq b$. Now observe that

$$b^2 = \left(\alpha - \tfrac{1}{n}\right)^2 = \alpha^2 - \tfrac{2\alpha}{n} + \tfrac{1}{n^2} > \alpha^2 - \tfrac{2\alpha}{n} > 2,$$

so that for any $t \in T$ we have $t^2 < 2 < b^2$. It follows that if $t \geq 0$, we have $t \leq b$. We have now shown that $t \leq b$ for all $t \in T$, i.e. b is an upper bound for T; but this contradicts the fact that α is the supremum of T since $b < \alpha$.

Exercise 1.4.8. Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbf{R} : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution. (a) Take $A = \left\{-\frac{1}{2n} : n \in \mathbf{N}\right\}$ and $B = \left\{-\frac{1}{2n-1} : n \in \mathbf{N}\right\}$. Then $A \cap B = \emptyset, \sup A = \sup B = 0$ and 0 belongs to neither A nor B.

(b) Take $J_n = (-1/n, 1/n)$. Then $\bigcap_{n=1}^{\infty} J_n = \{0\}$.

(c) Take $L_n = [n, \infty)$.

(d) This is impossible. To see this, let $J_N = \bigcap_{n=1}^{N} I_n$ for $N \in \mathbf{N}$ and note that any finite intersection of closed bounded intervals is a (possibly empty) closed bounded interval. So: each J_N is a closed bounded interval; these intervals are nonempty and nested $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$; and $\bigcap_{n=1}^{\infty} I_n = \bigcap_{N=1}^{\infty} J_N$. Then by the Nested Interval Property of $\mathbf{R}$, we must have that $\bigcap_{n=1}^{\infty} I_n = \bigcap_{N=1}^{\infty} J_N$ is non-empty.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edn.