1 Section 8.A Exercises

Exercises with solutions from Section 8.A of [LADR].

Exercise 8.A.1. Define $T \in \mathcal{L}(\mathbf{C}^2)$ by

$$T(w, z) = (z, 0).$$

Find all generalized eigenvectors of T.

Solution. The matrix of T with respect to the standard basis of $\mathbf{C}^2$ is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

from which we see that the only eigenvalue of T is 0. Note that, by 8.11,

$$G(0, T) = \operatorname{null} T^2 = \mathbf{C}^2;$$

thus every non-zero $v \in \mathbf{C}^2$ is a generalized eigenvector of T corresponding to the eigenvalue 0.

Exercise 8.A.2. Define $T \in \mathcal{L}(\mathbf{C}^2)$ by

$$T(w, z) = (-z, w).$$

Find the generalized eigenspaces corresponding to the distinct eigenvalues of T.

Solution. Some routine calculations reveal that the eigenvalues of T are $\pm i$, and furthermore that

$$E(-i, T) = \operatorname{null}(T + iI) = \operatorname{span}((1, i)) \quad \text{and} \quad E(i, T) = \operatorname{null}(T - iI) = \operatorname{span}((1, -i)).$$

Thus $\mathbf{C}^2 = E(-i, T) \oplus E(i, T)$. It follows that $G(-i, T) = E(-i, T)$; if this were not the case, then we would obtain at least two linearly independent generalized eigenvectors corresponding to the eigenvalue $-i$. Together with the eigenvector $(1, -i)$ corresponding to the eigenvalue i, which by 8.13 must be linearly independent from the generalized eigenvectors corresponding to the eigenvalue $-i$, we would obtain at least three linearly independent vectors in $\mathbf{C}^2$, which is of course impossible since $\dim \mathbf{C}^2 = 2$. Thus $G(-i, T) = E(-i, T)$ and similarly we see that $G(i, T) = E(i, T)$.

Exercise 8.A.3. Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $G(\lambda, T) = G\big(\frac{1}{\lambda}, T^{-1}\big)$ for every $\lambda \in \mathbf{F}$ with $\lambda \neq 0$.

Solution. Let $n = \dim V$ and suppose $v \in G(\lambda, T)$, so that $(T - \lambda I)^n v = 0$ (8.11). Using the binomial theorem (which holds here because T and I commute), this is equivalent to

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \lambda^k T^{n-k} v = 0.$$

Now apply T^{-1} to both sides of this equation n times, multiply through by $(-1)^n \lambda^{-n}$, and use that $(-1)^k = (-1)^{-k}$ to obtain

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left(\lambda^{-1}\right)^{n-k} \left(T^{-1}\right)^{k} v = 0.$$

Using the binomial theorem again, the above expression is equivalent to

$$\left(-\lambda^{-1} I + T^{-1}\right)^n v = 0,$$

so that $v \in \operatorname{null}(T^{-1} - \lambda^{-1} I)^n = G(\lambda^{-1}, T^{-1})$. Thus $G(\lambda, T) \subseteq G(\lambda^{-1}, T^{-1})$. Replacing T with T^{-1} and λ with λ^{-1} in the above argument and using that $\left(T^{-1}\right)^{-1} = T$ and $\left(\lambda^{-1}\right)^{-1} = \lambda$ gives us the reverse inclusion and we may conclude that $G(\lambda, T) = G(\lambda^{-1}, T^{-1})$.

Exercise 8.A.4. Suppose $T \in \mathcal{L}(V)$ and $\alpha, \beta \in \mathbf{F}$ with $\alpha \neq \beta$. Prove that

$$G(\alpha, T) \cap G(\beta, T) = \{0\}.$$

Solution. If α is not an eigenvalue of T, then $G(\alpha, T) = \{0\}$ and thus $G(\alpha, T) \cap G(\beta, T) = \{0\}$; a similar argument holds if β is not an eigenvalue of T. Suppose therefore that α and β are eigenvalues of T and assume for the sake of contradiction that there exists a non-zero $v \in G(\alpha, T) \cap G(\beta, T)$. Then v is a generalized eigenvector of T corresponding to α and also a generalized eigenvector of T corresponding to β. It follows from 8.13 that the list v, v is linearly independent, which is clearly not true. Thus $G(\alpha, T) \cap G(\beta, T) = \{0\}$.

Exercise 8.A.5. Suppose $T \in \mathcal{L}(V)$, m is a positive integer, and $v \in V$ is such that $T^{m-1} v \neq 0$ but $T^m v = 0$. Prove that

$$v, Tv, T^2 v, \ldots, T^{m-1} v$$

is linearly independent.

Solution. Note that the given hypothesis implies that

$$T^k v = 0 \quad \iff \quad k \geq m. \tag{1}$$

Suppose that $a_0, \ldots, a_{m-1}$ are scalars such that

$$a_0 v + a_1 Tv + a_2 T^2 v + \cdots + a_{m-1} T^{m-1} v = 0.$$

Apply T to both sides of this equation $m-1$ times to obtain

$$a_0 T^{m-1} v + a_1 T^m v + a_2 T^{m+1} v + \cdots + a_{m-1} T^{2m-2} v = a_0 T^{m-1} v = 0,$$

where we have used (1) for the first equality. Again by (1) we have $T^{m-1} v \neq 0$ and thus $a_0 = 0$, which leaves us with the equality

$$a_1 Tv + a_2 T^2 v + \cdots + a_{m-1} T^{m-1} v = 0.$$

Apply T to both sides of this equation $m-2$ times and, as before, obtain $a_1 = 0$. Continuing in this manner, we see that $a_0 = \cdots = a_{m-1} = 0$ and the linear independence of the list

$$v, Tv, T^2 v, \ldots, T^{m-1} v$$

follows.

Exercise 8.A.6. Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is defined by $T(z_1, z_2, z_3) = (z_2, z_3, 0)$. Prove that T has no square root. More precisely, prove that there does not exist $S \in \mathcal{L}(\mathbf{C}^3)$ such that $S^2 = T$.

Solution. Suppose such an operator S exists and observe that $T^3 = S^6 = 0$. Thus S is nilpotent and it follows from 8.18 that $S^4 = 0$; but $S^4 = T^2$ is the operator

$$(z_1, z_2, z_3) \mapsto (z_3, 0, 0),$$

which is non-zero.

Exercise 8.A.7. Suppose $N \in \mathcal{L}(V)$ is nilpotent. Prove that 0 is the only eigenvalue of N.

Solution. This is immediate from 8.19 and 5.32.

Exercise 8.A.8. Prove or give a counterexample: The set of nilpotent operators on V is a subspace of $\mathcal{L}(V)$.

Solution. Let $\mathcal{N} = \{N \in \mathcal{L}(V) : N \text{ is nilpotent}\}$. If $\dim V = 1$, then 8.18 implies that $\mathcal{N} = \{0\}$, which is a subspace of V. Suppose that $\dim V = n \geq 2$ and let $v_1, v_2, \ldots, v_n$ be a basis of V. Let N and M be the operators on V defined by

$$N v_j = \begin{cases} v_1 & \text{if } j = 2, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad M v_j = \begin{cases} v_2 & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $N^2 = M^2 = 0$ and thus $N, M \in \mathcal{N}$. However, note that

$$(N + M)(v_1 + v_2) = v_1 + v_2.$$

Note further that $v_1+v_2 \neq 0$ since v_1, v_2 is linearly independent. It follows that 1 is an eigenvalue of $N+M$ with a corresponding eigenvector v_1+v_2 and hence by Exercise 8.A.7 we have $N+M \notin \mathcal{N}$. Thus $\mathcal{N}$ is not closed under addition and hence is not a subspace of V.

Exercise 8.A.9. Suppose $S, T \in \mathcal{L}(V)$ and ST is nilpotent. Prove that TS is nilpotent.

Solution. Let $n = \dim V$, so that $(ST)^n = 0$ (8.18). That is,

$$0 = STST \cdots ST.$$

This implies that

$$0 = T(STST \cdots ST)S = (TS)^{n+1}$$

and thus TS is nilpotent.

Exercise 8.A.10. Suppose that $T \in \mathcal{L}(V)$ is not nilpotent. Let $n = \dim V$. Show that $V = \operatorname{null} T^{n-1} \oplus \operatorname{range} T^{n-1}$.

Solution. Since T is not nilpotent, it must be the case that $\operatorname{null} T^n \neq V$ and thus $\dim \operatorname{null} T^n \leq n - 1$. Using the same logic as in 8.4, it follows that

$$\operatorname{null} T^{n-1} = \operatorname{null} T^n = \operatorname{null} T^{n+1} = \cdots .$$

Since $\dim \operatorname{null} T^{n-1} = \dim \operatorname{null} T^n$, 3.22 implies that

$$n = \dim \operatorname{null} T^n + \dim \operatorname{range} T^n = \dim \operatorname{null} T^{n-1} + \dim \operatorname{range} T^{n-1}$$
$$\implies \quad \dim \operatorname{range} T^n = \dim \operatorname{range} T^{n-1}.$$

Combining this with the fact that $\operatorname{range} T^n$ is a subspace of $\operatorname{range} T^{n-1}$, we see that $\operatorname{range} T^{n-1} = \operatorname{range} T^n$ and hence by 8.5

$$V = \operatorname{null} T^n \oplus \operatorname{range} T^n = \operatorname{null} T^{n-1} \oplus \operatorname{range} T^{n-1}.$$

Exercise 8.A.11. Prove or give a counterexample: If V is a complex vector space and $\dim V = n$ and $T \in \mathcal{L}(V)$, then T^n is diagonalizable.

Solution. This is false. For a counterexample, consider the operator $T : \mathbf{C}^2 \to \mathbf{C}^2$ given by $T(w, z) = (w + z, z)$, which satisfies $T^2(w, z) = (w + 2z, z)$. With respect to the standard basis, T^2 has the upper-triangular matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

from which we see that the only eigenvalue of T is 1. It is straightforward to verify that $E(1, T) = \operatorname{span}((1, 0))$, which is one-dimensional and so cannot possibly equal $\mathbf{C}^2$. Hence T is not diagonalizable (5.41).

Exercise 8.A.12. Suppose $N \in \mathcal{L}(V)$ and there exists a basis of V with respect to which N has an upper-triangular matrix with only 0's on the diagonal. Prove that N is nilpotent.

Solution. Suppose that the basis in question is $v_1, \ldots, v_n$ and the matrix in question has entries $A_{i,j}$ satisfying $A_{i,j} = 0$ for $i \geq j$. Then $Nv_1 = 0$, which gives $N^n v_1 = 0$, and

$$Nv_2 = A_{1,2}v_1 \quad \Longrightarrow \quad N^2 v_2 = 0 \quad \Longrightarrow \quad N^n v_2 = 0.$$

Similarly,

$$Nv_3 = A_{1,3}v_1 + A_{2,3}v_2 \quad \Longrightarrow \quad N^2 v_3 = A_{1,2}A_{2,3}v_1 \quad \Longrightarrow \quad N^3 v_3 = 0 \quad \Longrightarrow \quad N^n v_3 = 0.$$

We can continue in this manner to see that $N^n v_i = 0$ for each $1 \leq i \leq n$, demonstrating that $N^n = 0$.

Exercise 8.A.13. Suppose V is an inner product space and $N \in \mathcal{L}(V)$ is normal and nilpotent. Prove that $N = 0$.

Solution. By 8.19, there is a basis $v_1, \ldots, v_n$ of V such that the matrix of N with respect to this basis is upper-triangular with only 0's on the diagonal. By applying the Gram-Schmidt procedure to the basis $v_1, \ldots, v_n$, we obtain an orthonormal basis $e_1, \ldots, e_n$ of V such that the matrix of N with respect to this basis is upper-triangular with only 0's on the diagonal (see 6.37). As the proof of the Complex Spectral Theorem (7.24) shows, the fact that N is normal implies that this upper-triangular matrix is actually diagonal. Since all the diagonal entries are 0, we see that the matrix of N with respect to $e_1, \ldots, e_n$ is the zero matrix and it follows that N is the zero operator.

Exercise 8.A.14. Suppose V is an inner product space and $N \in \mathcal{L}(V)$ is nilpotent. Prove that there exists an orthonormal basis of V with respect to which N has an upper-triangular matrix. [*If $\mathbf{F} = \mathbf{C}$ then the result above follows from Schur's Theorem (6.38) without the hypothesis that N is nilpotent. Thus the exercise above needs to be proved only when $\mathbf{F} = \mathbf{R}$.*]

Solution. See Exercise 8.A.13.

Exercise 8.A.15. Suppose $N \in \mathcal{L}(V)$ is such that $\operatorname{null} N^{\dim V-1} \neq \operatorname{null} N^{\dim V}$. Prove that N is nilpotent and that

$$\dim \operatorname{null} N^j = j$$

for every integer j with $0 \leq j \leq \dim V$.

Solution. Let $n = \dim V$. By 8.2 and 8.3, it must be the case that

$$\{0\} = \operatorname{null} N^0 \subsetneq \operatorname{null} N^1 \subsetneq \operatorname{null} N^2 \subsetneq \cdots \subsetneq \operatorname{null} N^{n-1} \subsetneq \operatorname{null} N^n.$$

Combining this chain of strict inclusions with the fact that $\dim \operatorname{null} N^0 = \dim\{0\} = 0$ and the fact that $\dim \operatorname{null} N^n \leq n$, we see that both of the inequalities

$$\dim \operatorname{null} N^j \geq j \quad \text{and} \quad \dim \operatorname{null} N^j \leq j$$

hold for each $0 \leq j \leq n$, which is the case if and only if $\dim \operatorname{null} N^j = j$. In particular we have

$$\dim \operatorname{null} N^n = n \quad \iff \quad \operatorname{null} N^n = V \quad \iff \quad N^n = 0.$$

Exercise 8.A.16. Suppose $T \in \mathcal{L}(V)$. Show that

$$V = \operatorname{range} T^0 \supset \operatorname{range} T^1 \supset \cdots \supset \operatorname{range} T^k \supset \operatorname{range} T^{k+1} \supset \cdots .$$

Solution 1. Suppose k is a non-negative integer and $v \in \operatorname{range} T^{k+1}$, so that $v = T^{k+1}w$ for some $w \in V$. Then $v = T^k(Tw)$, so that $v \in \operatorname{range} T^k$ also. Thus $\operatorname{range} T^{k+1} \subseteq \operatorname{range} T^k$.

Solution 2. Here is another solution for the special case where V is an inner product space. By 8.2 we have

$$\{0\} = \operatorname{null}(T^*)^0 \subseteq \operatorname{null}(T^*)^1 \subseteq \operatorname{null}(T^*)^2 \subseteq \cdots \subseteq \operatorname{null}(T^*)^k \subseteq \operatorname{null}(T^*)^{k+1} \subseteq \cdots .$$

By 7.6 (e), this is equivalent to

$$\{0\} = \operatorname{null}\left(T^0\right)^* \subseteq \operatorname{null}\left(T^1\right)^* \subseteq \operatorname{null}\left(T^2\right)^* \subseteq \cdots \subseteq \operatorname{null}\left(T^k\right)^* \subseteq \operatorname{null}\left(T^{k+1}\right)^* \subseteq \cdots ,$$

which, by 7.7 (a), is equivalent to

$$\{0\}^\perp = \left(\operatorname{range} T^0\right)^\perp \subseteq \left(\operatorname{range} T^1\right)^\perp \subseteq \left(\operatorname{range} T^2\right)^\perp \subseteq \cdots \subseteq \left(\operatorname{range} T^k\right)^\perp \subseteq \left(\operatorname{range} T^{k+1}\right)^\perp \subseteq \cdots .$$

6.46 (c) and (e) now imply that

$$\begin{aligned} V = \left(\left(\operatorname{range} T^0\right)^\perp\right)^\perp \supseteq \left(\left(\operatorname{range} T^1\right)^\perp\right)^\perp \supseteq \left(\left(\operatorname{range} T^2\right)^\perp\right)^\perp \\ \supseteq \cdots \supseteq \left(\left(\operatorname{range} T^k\right)^\perp\right)^\perp \supseteq \left(\left(\operatorname{range} T^{k+1}\right)^\perp\right)^\perp \supseteq \cdots . \end{aligned}$$

Finally, 6.51 shows that this is equivalent to

$$V = \operatorname{range} T^0 \supseteq \operatorname{range} T^1 \supseteq \operatorname{range} T^2 \supseteq \cdots \supseteq \operatorname{range} T^k \supseteq \operatorname{range} T^{k+1} \supseteq \cdots .$$

Exercise 8.A.17. Suppose $T \in \mathcal{L}(V)$ and m is a non-negative integer such that

$$\operatorname{range} T^m = \operatorname{range} T^{m+1}.$$

Prove that $\operatorname{range} T^k = \operatorname{range} T^m$ for all $k > m$.

Solution 1. It will suffice to show that $\operatorname{range} T^{m+n} = \operatorname{range} T^{m+1}$ for all positive integers n. The inclusion $\operatorname{range} T^{m+n} \subseteq \operatorname{range} T^{m+1}$ follows from Exercise 8.A.16. We will prove the reverse inclusion by induction on n. The base case $n = 1$ is clear, so suppose that the inclusion holds for some positive integer n and let $v \in \operatorname{range} T^{m+1}$ be given. The induction hypothesis then implies that $v \in \operatorname{range} T^{m+n}$, so that $v = T^{m+n}w$ for some $w \in V$. By assumption we have $\operatorname{range} T^m = \operatorname{range} T^{m+1}$, so we must have $T^m w = T^{m+1}u$ for some $u \in V$. This implies that $v = T^{m+n+1}u$ and thus $v \in \operatorname{range} T^{m+n+1}$. Hence $\operatorname{range} T^{m+1} \subseteq \operatorname{range} T^{m+n+1}$; this completes the induction step and the proof.

Solution 2. Here is another solution for the special case where V is an inner product space. Observe that

$$\begin{aligned}
\operatorname{range} T^m = \operatorname{range} T^{m+1} &\implies (\operatorname{range} T^m)^\perp = \left(\operatorname{range} T^{m+1}\right)^\perp \\
&\implies \operatorname{null}(T^m)^* = \operatorname{null}\left(T^{m+1}\right)^* && (7.7\text{ (a)}) \\
&\implies \operatorname{null}(T^*)^m = \operatorname{null}(T^*)^{m+1} && (7.6\text{ (e)}) \\
&\implies \operatorname{null}(T^*)^m = \operatorname{null}(T^*)^k \quad \text{for all } k > m && (8.3) \\
&\implies \operatorname{null}(T^m)^* = \operatorname{null}\left(T^k\right)^* \quad \text{for all } k > m && (7.6\text{ (e)}) \\
&\implies (\operatorname{range} T^m)^\perp = \left(\operatorname{range} T^k\right)^\perp \quad \text{for all } k > m && (7.7\text{ (a)}) \\
&\implies \operatorname{range} T^m = \operatorname{range} T^k \quad \text{for all } k > m. && (6.51)
\end{aligned}$$

Exercise 8.A.18. Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Prove that

$$\operatorname{range} T^n = \operatorname{range} T^{n+1} = \operatorname{range} T^{n+2} = \cdots.$$

Solution 1. By Exercise 8.A.17, it will suffice to show that $\operatorname{range} T^n = \operatorname{range} T^{n+1}$. Seeking a contradiction, suppose that this is not the case. It then follows from Exercise 8.A.16 and Exercise 8.A.17 that

$$\operatorname{range} T^{n+1} \subsetneq \operatorname{range} T^n \subsetneq \cdots \subsetneq \operatorname{range} T^2 \subsetneq \operatorname{range} T^1 \subsetneq \operatorname{range} T^0 = V.$$

This implies that $\dim \operatorname{range} T^1 \leq n - 1, \dim \operatorname{range} T^2 \leq n - 2, \ldots, \dim \operatorname{range} T^n \leq 0$, and hence that $\dim \operatorname{range} T^{n+1} \leq -1$, which is a contradiction.

Solution 2. Here is another solution for the special case where V is an inner product space. By 8.4 we have

$$\operatorname{null}(T^*)^n = \operatorname{null}(T^*)^{n+1} = \operatorname{null}(T^*)^{n+2} = \cdots .$$

By 7.6 (e), this is equivalent to

$$\operatorname{null}(T^n)^* = \operatorname{null}\left(T^{n+1}\right)^* = \operatorname{null}\left(T^{n+2}\right)^* = \cdots ,$$

which, by 7.7 (a), is equivalent to

$$(\operatorname{range} T^n)^\perp = \left(\operatorname{range} T^{n+1}\right)^\perp = \left(\operatorname{range} T^{n+2}\right)^\perp = \cdots .$$

Finally, 6.51 shows that this is equivalent to

$$\operatorname{range} T^n = \operatorname{range} T^{n+1} = \operatorname{range} T^{n+2} = \cdots .$$

Exercise 8.A.19. Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer. Prove that

$$\operatorname{null} T^m = \operatorname{null} T^{m+1} \quad \text{if and only if} \quad \operatorname{range} T^m = \operatorname{range} T^{m+1}.$$

Solution. Suppose that $\operatorname{null} T^m = \operatorname{null} T^{m+1}$, so that $\dim \operatorname{null} T^m = \dim \operatorname{null} T^{m+1}$. By 3.22, this implies that $\dim \operatorname{range} T^m = \dim \operatorname{range} T^{m+1}$. Combining this with the fact that $\operatorname{range} T^{m+1} \subseteq \operatorname{range} T^m$, shown in Exercise 8.A.16, we see that $\operatorname{range} T^m = \operatorname{range} T^{m+1}$. A similar argument gives the reverse implication.

Exercise 8.A.20. Suppose $T \in \mathcal{L}(\mathbf{C}^5)$ is such that $\operatorname{range} T^4 \neq \operatorname{range} T^5$. Prove that T is nilpotent.

Solution. By Exercise 8.A.19, it must be the case that $\operatorname{null} T^4 \neq \operatorname{null} T^5$. Since $\dim \mathbf{C}^5 = 5$, it follows from Exercise 8.A.15 that T is nilpotent.

Exercise 8.A.21. Find a vector space W and $T \in \mathcal{L}(W)$ such that $\operatorname{null} T^k \subsetneq \operatorname{null} T^{k+1}$ and $\operatorname{range} T^k \supsetneq \operatorname{range} T^{k+1}$ for every positive integer k.

Solution. Let $L : \mathbf{C}^\infty \to \mathbf{C}^\infty$ and $R : \mathbf{C}^\infty \to \mathbf{C}^\infty$ be the left- and right-shift operators respectively, i.e.

$$L(z_1, z_2, z_3, \ldots) = (z_2, z_3, z_4, \ldots) \quad \text{and} \quad R(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, \ldots).$$

It is then straightforward to verify that

$$\operatorname{null} L^k \subsetneq \operatorname{null} L^{k+1}, \quad \operatorname{range} L^k = \mathbf{C}^\infty, \quad \operatorname{null} R^k = \{0\}, \quad \text{and} \quad \operatorname{range} R^k \supsetneq R^{k+1}$$

for every positive integer k. Define $T : \mathbf{C}^\infty \times \mathbf{C}^\infty \to \mathbf{C}^\infty \times \mathbf{C}^\infty$ by

$$T(z, w) = (Lz, Rw).$$

It follows from the previous discussion that

$$\operatorname{null} T^k \subsetneq \operatorname{null} T^{k+1} \quad \text{and} \quad \operatorname{range} T^k \supsetneq \operatorname{range} T^{k+1}$$

for every positive integer k.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edition.