Consequences of the least-upper-bound property of $\mathbb{R}$

Contents

The following is mostly paraphrased from Chapter 1 of [PMA].

Theorem 1. There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

For a proof of Theorem 1, see here. In this document, we shall present some consequences of Theorem 1.

1 The Archimedean property of $\mathbb{R}$

Theorem 2 (Archimedean property of $\mathbb{R}$)**.** Let $x > 0$ and y be real numbers. Then there exists a positive integer n such that $nx > y$.

Proof. Suppose to the contrary that for all positive integers n we have $nx \leq y$. Then the set $A = \{nx : n \in \mathbb{N}\}$ is non-empty and bounded above, so by the least-upper-bound property of $\mathbb{R}$ the supremum $\alpha = \sup A$ exists in $\mathbb{R}$. Since $x > 0$, we have $\alpha - x < \alpha$ so that $\alpha - x$ is not an upper bound of A. Hence there exists a positive integer m such that $\alpha - x < mx$, which gives $\alpha < (m+1)x$; but this contradicts the fact that α is the supremum of A. □

2 Density of $\mathbb{Q}$ and $\mathbb{Q}^{\mathsf{C}}$ in $\mathbb{R}$

Lemma 3. Any real number lies between two consecutive integers. That is, for any $x \in \mathbb{R}$ there exists an $m \in \mathbb{Z}$ such that $m - 1 \leq x < m$.

Proof. By the Archimedean property, there exist positive integers m_1, m_2 such that $m_1 > x$ and $m_2 > -x$, which gives $-m_2 < x < m_1$. This implies that the set $A = \{n \in \mathbb{Z} : x < n\}$ is non-empty ($m_1 \in A$) and bounded below (by $-m_2$). Then by the well-ordering principle, A has a least element; call it m. Since this is the least element of A, we must have $m - 1 \notin A$ and so $m - 1 \leq x < m$. □

Theorem 4. Between any two real numbers there exists a rational number. That is, for any $x, y \in \mathbb{R}$ with $x < y$ there exists a $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. By the Archimedean property, there exists a positive integer n such that $n(y - x) > 1$. By Lemma 3, there exists an integer m such that $m - 1 \leq nx < m$. Combining these inequalities gives $nx < m \leq 1 + nx < ny$, which implies that $x < \frac{m}{n} < y$. So the desired rational is $p = \frac{m}{n}$. □

Corollary 5. Between any two real numbers there exists an irrational number. That is, for any $x, y \in \mathbb{R}$ with $x < y$ there exists a $z \in \mathbb{Q}^{\mathsf{C}}$ such that $x < z < y$.

Proof. By Theorem 4, there exists a rational number p such that $x - \sqrt{2} < p < y - \sqrt{2}$, which gives $x < p + \sqrt{2} < y$. So the desired irrational number is $z = p + \sqrt{2}$. □

3 Existence of nth roots in $\mathbb{R}$

First, a useful inequality. Suppose n is a positive integer and a, b are real numbers such that $0 < a < b$. This implies that $0 < b^{n-2}a < b^{n-1}$. Furthermore, we have $0 < a^2 < b^2$, which gives $0 < b^{n-3}a^2 < b^{n-1}$, and so on. Combining this with the equality

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$$

gives us the inequality

$$b^n - a^n < (b-a)nb^{n-1}. \tag{1}$$

Theorem 6. For every real $x > 0$ and every positive integer n there is exactly one positive real y such that $y^n = x$.

Proof. Suppose y_1 and y_2 are positive real numbers such that $y_1 \neq y_2$. Without loss of generality, assume $0 < y_1 < y_2$. Then $0 < y_1^n < y_2^n$, so that $y_1^n \neq y_2^n$. Hence by the contrapositive, $y_1^n = y_2^n$ implies that $y_1 = y_2$. This gives us the uniqueness of any such y in Theorem 1.

For existence, let $E = \{t \in \mathbb{R} : t > 0, t^n < x\}$. Observe that $t = \frac{x}{1+x}$ satisfies $t < x$ and $0 < t < 1$, which gives $0 < t^n < t < x$. Hence $t \in E$ and so E is non-empty. Now suppose $t \geq 1 + x > 1$, so that $t^n > t \geq 1 + x > x$. Then by the contrapositive, $t^n < x$ implies that $t < 1+x$, and we see that E is bounded above by $1+x$. We may now invoke the least-upper-bound property of $\mathbb{R}$ and set $y = \sup E$. Note that y must be positive, since $\frac{x}{1+x}$ belongs to E. To show that $y^n = x$, we will show that both of the assumptions $y^n < x$ and $y^n > x$ lead to contradictions.

First, assume that $y^n < x$. Using the Archimedean property, choose h such that $0 < h < 1$ and $h < \frac{x-y^n}{n(y+1)^{n-1}}$. Now take $a = y$ and $b = y + h$ in inequality (1) to obtain

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n,$$

whence $(y+h)^n < x$ and so $y + h \in E$; but this contradicts the fact that y is the supremum of E, since $y + h > y$.

Next, assume that $y^n > x$ and set $k = \frac{y^n - x}{ny^{n-1}} < y$. Take $a = y - k$ and $b = y$ in inequality (1) to obtain

$$y^n - (y-k)^n < kny^{n-1} = y^n - x,$$

whence $(y-k)^n \geq x$. Then $t \geq y - k$ implies that $t^n \geq x$; the contrapositive of this shows that $y - k$ is an upper bound for E. This contradicts the fact that y is the least upper bound of E, since $y - k < y$. □

Corollary 7. Let a and b be positive real numbers and n a positive integer. Then

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}.$$

Proof. Let $\alpha = \sqrt[n]{a}$ and $\beta = \sqrt[n]{b}$. Then by the commutativity of multiplication, we have

$$(\alpha\beta)^n = \alpha^n \beta^n = ab.$$

The uniqueness part of Theorem 6 then implies that $\sqrt[n]{ab} = \alpha\beta = \sqrt[n]{a}\,\sqrt[n]{b}$. $\square$

[PMA] Rudin, W. (1976) *Principles of Mathematical Analysis.* 3rd edn.