Remark. $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Let F be an ordered field (for example, $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields). Suppose we have a sequence $(I_n)_{n\in\mathbb{N}}$ of closed bounded intervals

$$I_n = [a_n, b_n] = \{x \in F : a_n \leq x \leq b_n\}.$$

Consider the following two properties.

(i) For all $n \in \mathbb{N}, I_{n+1} \subseteq I_n$.

(ii) For all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|I_n| := b_n - a_n < \varepsilon$.

If $(I_n)_{n\in\mathbb{N}}$ satisfies property (i), then $(I_n)_{n\in\mathbb{N}}$ is known as a sequence of **nested intervals**. If $(I_n)_{n\in\mathbb{N}}$ satisfies both properties (i) and (ii), then $(I_n)_{n\in\mathbb{N}}$ is known as a sequence of **shrinking nested intervals**.

If F is such that any sequence $(I_n)_{n\in\mathbb{N}}$ of shrinking nested intervals has a singleton intersection, i.e. $\bigcap_{n=1}^{\infty} I_n = \{x\}$ for some $x \in F$, then F is said to have the **nested interval property**.

Let us first show that if F has the least-upper-bound property, then F has the nested interval property. Since $\mathbb{R}$ is the unique ordered field with the least-upper-bound property, the proposition we want to prove is the following.

Proposition 1. $\mathbb{R}$ has the nested interval property.

Proof. Let $(I_n)_{n\in\mathbb{N}}$ be a sequence of shrinking nested intervals, where $I_n = [a_n, b_n]$, and let A be the set of left endpoints, i.e. $A = \{a_n : n \in \mathbb{N}\}$. Note that for any $m \in \mathbb{N}$, b_m is an upper bound of A:

- if $n \leq m$, then $a_n \leq a_m \leq b_m$;
- if $m < n$, then $a_n \leq b_n \leq b_m$.

(The sequence of left endpoints is non-decreasing and the sequence of right endpoints is non-increasing since the intervals are nested.) Hence $x := \sup A$ exists in $\mathbb{R}$ and satisfies $a_n \leq x \leq b_n$ for each $n \in \mathbb{N}$, i.e. $x \in \bigcap_{n=1}^{\infty} I_n$. To see that this x is unique, suppose that $x_1, x_2 \in \bigcap_{n=1}^{\infty} I_n$ and without loss of generality suppose that $x_1 \leq x_2$. Let $\varepsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$ such that $b_N - a_N < \varepsilon$. Since $x_1, x_2 \in I_n$ for each $n \in \mathbb{N}$, we have

$$a_N \leq x_1, x_2 \leq b_N \implies x_2 - x_1 \leq b_N - a_N < \varepsilon.$$

Hence $0 \leq x_2 - x_1 < \varepsilon$ for any $\varepsilon > 0$; it follows that $x_1 = x_2$. $\square$

Note that the proof just given shows that if $(I_n)_{n\in\mathbb{N}}$ is only a sequence of nested intervals in $\mathbb{R}$, not necessarily shrinking, then $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

Next, let us show that if F has the nested interval property, then F has the least-upper-bound property, i.e. $F = \mathbb{R}$.

Proposition 2. If F has the nested interval property, then F has the least-upper-bound property.

Proof. Let $E \subseteq F$ be non-empty and bounded above by some $b_1 \in F$. If E has a maximum x, then $\sup E = x$ and we are done. Otherwise, we shall use an induction argument to construct a sequence $(I_n)_{n\in\mathbb{N}}$ of shrinking nested intervals. Pick some $a_1 \in E$; it must be the case that a_1 is not an upper bound of E since E has no maximum. Let $I_1 = [a_1, b_1]$. Then:

- a_1 is not an upper bound of E;
- b_1 is an upper bound of E;
- $|I_1| = 2^0(b_1 - a_1)$.

Suppose that after N steps we have chosen intervals $I_n = [a_n, b_n]$, $1 \le n \le N$, such that

- $a_1 \le \cdots \le a_N$ are not upper bounds of E;
- $b_N \le \cdots \le b_1$ are upper bounds of E;
- $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for $1 \le n \le N$.

Let $m = \frac{a_N + b_N}{2}$, the midpoint of the interval I_N. If m is not an upper bound of E, set

$$a_{N+1} = m, b_{N+1} = b_N, \text{and } I_{N+1} = [a_{N+1}, b_{N+1}].$$

If m is an upper bound of E, set

$$a_{N+1} = a_N, b_{N+1} = m, \text{and } I_{N+1} = [a_{N+1}, b_{N+1}].$$

In either case, we have chosen intervals $I_n = [a_n, b_n]$, $1 \le n \le N+1$, such that

- $a_1 \le \cdots \le a_{N+1}$ are not upper bounds of E;
- $b_{N+1} \le \cdots \le b_1$ are upper bounds of E;
- $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for $1 \le n \le N+1$.

In this way we obtain a sequence $(I_n)_{n\in\mathbb{N}}$ of intervals $I_n = [a_n, b_n]$ such that

- $a_1 \le \cdots \le a_n \le \cdots$ are not upper bounds of E;

- $\cdots \leq b_n \leq \cdots \leq b_1$ are upper bounds of E;
- $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for all $n \in \mathbb{N}$.

Hence $(I_n)_{n\in\mathbb{N}}$ is a sequence of shrinking nested intervals. By assumption, F has the nested interval property, so there exists an $x \in F$ such that $\bigcap_{n=1}^{\infty} I_n = \{x\}$. We claim that $x = \sup E$. For $y \in E$, suppose $x < y$. Then there is an $N \in \mathbb{N}$ such that

$$b_N - a_N < y - x \implies x + (b_N - a_N) < y.$$

Since $x \in \bigcap_{n=1}^{\infty} I_n$, we have

$$a_N \leq x \implies 0 \leq x - a_N \implies b_N \leq x + (b_N - a_N) < y.$$

This is a contradiction since b_N is an upper bound of E. It follows that $y \leq x$, so that x is an upper bound of E. Suppose that $z \in F$ is such that $z < x$. There is an $N \in \mathbb{N}$ such that

$$b_N - a_N < x - z \implies z < x - (b_N - a_N).$$

Since $x \in \bigcap_{n=1}^{\infty} I_n$, we have

$$x \leq b_N \implies x - b_N \leq 0 \implies x - (b_N - a_N) \leq a_N \implies z < a_N.$$

It follows that z is not an upper bound of E since a_N is not an upper bound of E. We may conclude that x is the least upper bound of E, i.e. $x = \sup E$. $\square$

So for ordered fields, the nested interval property and the least-upper-bound property are equivalent. In light of the fact that $\mathbb{R}$ is the unique ordered field with the least-upper-bound property, we see that $\mathbb{R}$ is the unique ordered field with the nested interval property.