1 Section 2.A Exercises

Exercises with solutions from Section 2.A of [LADR].

Exercise 2.A.1. Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Solution. Let $v \in V$ be given. Since $V = \operatorname{span}(v_1, v_2, v_3, v_4)$, there are scalars a_1, a_2, a_3, a_4 such that $v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$. Observe that

$$\begin{aligned}
&a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4\\
&= a_1v_1 + (a_1 + a_2 - a_1)v_2 + (a_1 + a_2 + a_3 - a_1 - a_2)v_3 + (a_1 + a_2 + a_3 + a_4 - a_1 - a_2 - a_3)v_4\\
&= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4\\
&= v.
\end{aligned}$$

Hence $v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$. It follows that $V = \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$.

Exercise 2.A.2. Verify the assertions in Example 2.18.

Solution. (a) The assertion is that a list v of one vector $v \in V$ is linearly independent if and only if $v \neq 0$. To see this, first suppose that $v \neq 0$ and $a \in \mathbf{F}$ is such that $av = 0$. Then by Exercise 1.B.2, it must be the case that $a = 0$, demonstrating that the list v is linearly independent. Now suppose that $v = 0$, so that any choice of non-zero $a \in \mathbf{F}$ will satisfy $av = 0$; $1v = 0$ for example. It follows that the list v is linearly dependent.

(b) The assertion is that a list of two vectors in V is linearly independent if and only if neither vector is a scalar multiple of the other. Suppose that u, v is the list of two vectors in V, and suppose that one is a scalar multiple of the other, say $v = \lambda u$ for some scalar λ. Then $v - \lambda u = 0$; the coefficient of v in this linear combination is non-zero, so the list u, v is linearly dependent. Now suppose that the list u, v is linearly dependent, so that $\mu v + \lambda u = 0$ where at least one of μ and λ is non-zero, say $\mu \neq 0$: then $v = -\frac{\lambda}{\mu}u$.

(c) The assertion is that the list $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ is linearly independent in $\mathbf{F}^4$. This is easily seen, since

$$a(1, 0, 0, 0) + b(0, 1, 0, 0) + c(0, 0, 1, 0) = (a, b, c, 0) = (0, 0, 0, 0)$$

forces $a = b = c = 0$.

(d) The assertion is that the list $1, z, \ldots, z^m$ is linearly independent in $\mathcal{P}(\mathbf{F})$ for each nonnegative integer m. Suppose that we have scalars $a_0, a_1, \ldots, a_m$ such that

$$a_0 + a_1 z + \cdots + a_m z^m = 0$$

for all $z \in \mathbf{F}$. Since the degree of the right-hand side is $-\infty$ and any non-zero coefficient on the left-hand side would result in a polynomial with degree zero or greater, it must be the case that $a_0 = a_1 = \cdots = a_m = 0$.

Exercise 2.A.3. Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbf{R}^3$.

Solution. Let $t = 2$. Then observe that

$$3(3, 1, 4) - 2(2, -3, 5) - (5, 9, 2) = (0, 0, 0).$$

Exercise 2.A.4. Verify the assertion in the second bullet point in Example 2.20.

Solution. The assertion is that the list $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$. If $c = 8$ then, as shown in the first bullet point in Example 2.20, we have

$$2(2, 3, 1) + 3(1, -1, 2) + (-1)(7, 3, 8) = (0, 0, 0).$$

Now suppose that the list is linearly dependent. Since $(1, -1, 2)$ is clearly not a scalar multiple of $(2, 3, 1)$, the Linear Dependence Lemma implies that $(7, 3, c)$ lies in the span of $(2, 3, 1)$ and $(1, -1, 2)$, i.e. there are scalars x and y such that

$$x(2, 3, 1) + y(1, -1, 2) = (7, 3, c).$$

Solving the equations $2x + y = 7$ and $3x - y = 3$ gives $x = 2$ and $y = 3$, whence $c = x + 2y = 8$.

Exercise 2.A.5. (a) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $(1 + i), (1 - i)$ is linearly independent.

(b) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $(1 + i), (1 - i)$ is linearly dependent.

Solution. (a) Suppose that x and y are real numbers such that

$$x(1+i)+y(1-i)=(x+y)+(x-y)i=0.$$

The real part of the left-hand side is $x+y$ and the imaginary part is $x-y$. A complex number is zero if and only if both its real and imaginary parts are zero, so we must have

$$x+y=0 \text{ and } x-y=0 \iff x=y=0.$$

Hence the list $(1+i),(1-i)$ is linearly independent.

(b) Observe that $i(1-i)=1+i$, so that $1+i$ is a scalar multiple of $1-i$. Then by Exercise 2.A.2 (b), the list $(1+i),(1-i)$ is linearly dependent.

Exercise 2.A.6. Suppose v_1,v_2,v_3,v_4 is linearly independent in V. Prove that the list

$$v_1-v_2,v_2-v_3,v_3-v_4,v_4$$

is also linearly independent.

Solution. Suppose that a_1,a_2,a_3,a_4 are scalars such that

$$a_1(v_1-v_2)+a_2(v_2-v_3)+a_3(v_3-v_4)+a_4v_4=0.$$

This is equivalent to

$$a_1v_1+(a_2-a_1)v_2+(a_3-a_2)v_3+(a_4-a_3)v_4=0.$$

Since the list v_1,v_2,v_3,v_4 is linearly independent, we must have

$$a_1=a_2-a_1=a_3-a_2=a_4-a_3=0,$$

which implies that $a_1=a_2=a_3=a_4=0$. It follows that the list $v_1-v_2,v_2-v_3,v_3-v_4,v_4$ is linearly independent.

Exercise 2.A.7. Prove or give a counterexample: If $v_1,v_2,\dots,v_m$ is a linearly independent list of vectors in V, then

$$5v_1-4v_2,v_2,v_3,\dots,v_m$$

is linearly independent.

Solution. (Assuming $m \geq 2$). Suppose that $a_1, a_2, \dots, a_m$ are scalars such that

$$a_1(5v_1 - 4v_2) + a_2v_2 + a_3v_3 + \cdots + a_mv_m = 0,$$

which is equivalent to

$$5a_1v_1 + (a_2 - 4a_1)v_2 + a_3v_3 + \cdots + a_mv_m = 0.$$

Since the list $v_1, v_2, \dots, v_m$ is linearly independent, this implies that

$$5a_1 = a_2 - 4a_1 = a_3 = \cdots = a_m = 0,$$

which implies that $a_1 = a_2 = a_3 = \cdots = a_m = 0$. It follows that the list $5v_1 - 4v_2, v_2, v_3, \dots, v_m$ is linearly independent.

Exercise 2.A.8. Prove or give a counterexample: If $v_1, v_2, \dots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \dots, \lambda v_m$ is linearly independent.

Solution. Suppose that $a_1, a_2, \dots, a_m$ are scalars such that

$$a_1\lambda v_1 + a_2\lambda v_2 + \cdots + a_m\lambda v_m = 0.$$

Since $\lambda \neq 0$, we may multiply both sides of this equation by λ^{-1} to obtain

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0.$$

Since the list $v_1, v_2, \dots, v_m$ is linearly independent, this implies that $a_1 = a_2 = \cdots = a_m = 0$. It follows that the list $\lambda v_1, \lambda v_2, \dots, \lambda v_m$ is linearly independent.

Exercise 2.A.9. Prove or give a counterexample: If $v_1, \dots, v_m$ and $w_1, \dots, w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, \dots, v_m + w_m$ is linearly independent.

Solution. This is false. Consider $\mathbf{R}$ as a vector space over itself and the two linearly independent lists 1 and -1. Then the list $1 + (-1) = 0$ is linearly dependent.

Exercise 2.A.10. Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \dots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \dots, v_m)$.

Solution. By the Linear Dependence Lemma, there is a $j \in \{1, 2, \dots, m\}$ such that $v_j + w \in \operatorname{span}(v_1 + w, \dots, v_{j-1} + w)$. If $j = 1$, then $v_1 + w \in \operatorname{span}() = \{0\}$, so that $w = -v_1$ and hence $w \in \operatorname{span}(v_1, \dots, v_m)$. If $j \geq 2$, then there are scalars $a_1, \dots, a_{j-1}$ such that

$$v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}(v_{j-1} + w),$$

which is equivalent to

$$v_j + \lambda w = a_1 v_1 + \cdots + a_{j-1} v_{j-1},$$

where $\lambda = 1 - (a_1 + \cdots + a_{j-1})$. Note that λ must be non-zero; if this were not the case, then v_j would lie in the span of $v_1, \ldots, v_{j-1}$, which cannot happen since the list $v_1, \ldots, v_j$ is linearly independent. It follows that

$$w = \lambda^{-1}(a_1 v_1 + \cdots + a_{j-1} v_{j-1} - v_j),$$

whence $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Exercise 2.A.11. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \ldots, v_m, w$ is linearly independent if and only if

$$w \notin \operatorname{span}(v_1, \ldots, v_m).$$

Solution. If $w \in \operatorname{span}(v_1, \ldots, v_m)$, then the list $v_1, \ldots, v_m, w$ is linearly dependent (third bullet point of Example 2.20). Conversely, suppose that the list $v_1, \ldots, v_m, w$ is linearly dependent. By the Linear Dependence Lemma, one of the vectors in the list must be in the span of the previous ones. It cannot be the case that some v_j belongs to $\operatorname{span}(v_1, \ldots, v_{j-1})$ since that would contradict the linear independence of the list $v_1, \ldots, v_m$, so it must be the case that $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Exercise 2.A.12. Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

Solution. It is easily verified that $\mathcal{P}_4(\mathbf{F})$ is spanned by the list $1, z, z^2, z^3, z^4$, which has length 5. Then by (2.23), any linearly independent list in $\mathcal{P}_4(\mathbf{F})$ can have length at most 5.

Exercise 2.A.13. Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

Solution. It is easily verified that the list $1, z, z^2, z^3, z^4$ is linearly independent in $\mathcal{P}_4(\mathbf{F})$. Then by (2.23), any list of vectors which spans $\mathcal{P}_4(\mathbf{F})$ must have length at least 5.

Exercise 2.A.14. Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Solution. Suppose that V is finite-dimensional, so that it is spanned by some list $w_1, \ldots, w_m$ of vectors, and let $v_1, v_2, \ldots$ be any sequence of vectors in V. Then by (2.23), the list $v_1, v_2, \ldots, v_{m+1}$ cannot be linearly independent.

Now suppose that V is infinite-dimensional, so that no list of vectors in V spans V. Certainly $V \neq \{0\}$, so pick any $v_1 \neq 0$ in V; the list v_1 is linearly independent. Suppose that after m steps, we have chosen vectors $v_1, \ldots, v_m$ such that the list $v_1, \ldots, v_m$ is linearly independent. Since $V \neq \operatorname{span}(v_1, \ldots, v_m)$, pick any $v_{m+1} \notin \operatorname{span}(v_1, \ldots, v_m)$. By Exercise 2.A.11, the list $v_1, \ldots, v_m, v_{m+1}$ is linearly independent. In this way, we inductively obtain a sequence $v_1, v_2, \ldots$ such that $v_1, \ldots, v_m$ is linearly independent for each positive integer m.

Exercise 2.A.15. Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

Solution. Consider the sequence of vectors $v_1, v_2, \ldots$, where v_i is the vector with a 1 in the i^{th} position and 0's elsewhere. For each positive integer m, it is not hard to see that the list $v_1, \ldots, v_m$ is linearly independent. It follows from Exercise 2.A.14 that $\mathbf{F}^\infty$ is infinite-dimensional.

Exercise 2.A.16. Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

Solution. Consider the sequence of continuous functions $f_1, f_2, \ldots$, where $f_1(x) = 1$ for all $x \in [0, 1]$ and for $i \geq 2$, $f_i : [0, 1] \to \mathbf{R}$ is given by $f_i(x) = 1$ for $x \in [0, 1/i]$, $f_i(x) = 0$ for $x \in [1/(i-1), 1]$. On the interval $(1/i, 1/(i-1))$, f_i is a straight line segment joining the points $(1/i, 1)$ and $(1/(i-1), 0)$ to give a continuous function (see Figure 1).

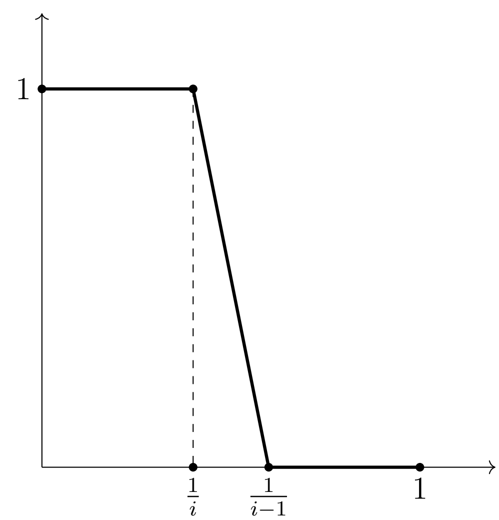

Figure 1: f_i

Let m be a positive integer and suppose we have real numbers $a_1, \ldots, a_m$ such that

$$a_1 f_1(x) + a_2 f_2(x) + \cdots + a_m f_m(x) = 0$$

for all $x \in [0, 1]$. In particular, taking $x = 1$ gives

$$a_1 f_1(1) + a_2 f_2(1) + \cdots + a_m f_m(1) = a_1 = 0.$$

Then taking $x = 1/2$ gives

$$a_2 f_2(1/2) + a_3 f_3(1/2) + \cdots + a_m f_m(1/2) = a_2 = 0.$$

Continuing in this way, taking $x = 1/i$ for each $i \in \{1, 2, \ldots, m\}$, we see that $a_1 = a_2 = \cdots = a_m = 0$. It follows that $f_1, f_2, \ldots, f_m$ is a linearly independent list for each positive integer m and hence by Exercise 2.A.14 the vector space in question is infinite-dimensional.

Exercise 2.A.17. Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_j(2) = 0$ for each j. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

Solution. Consider the list $p_0, p_1, \ldots, p_m, f$ of length $m + 2$, where $f(x) = x$. Since $\mathcal{P}_m(\mathbf{F})$ is spanned by a list of length $m + 1$ (Example 2.14), (2.23) implies that the list $p_0, p_1, \ldots, p_m, f$ is linearly dependent. Suppose that f belongs to $\operatorname{span}(p_0, p_1, \ldots, p_m)$, i.e. there are scalars $a_0, a_1, \ldots, a_m$ such that

$$x = a_0 p_0(x) + a_1 p_1(x) + \cdots + a_m p_m(x)$$

for all $x \in \mathbf{F}$. In particular,

$$2 = a_0 p_0(2) + a_1 p_1(2) + \cdots + a_m p_m(2) = 0,$$

which is a contradiction; it follows that $f \notin \operatorname{span}(p_0, p_1, \ldots, p_m)$. Hence by the Linear Dependence Lemma, there exists some $j \in \{0, 1, \ldots, m\}$ such that $p_j \in \operatorname{span}(p_0, p_1, \ldots, p_{j-1})$. The third bullet point of Example 2.20 now implies that the list $p_0, p_1, \ldots, p_m$ is linearly dependent.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edn.