1 Section 5.3 Exercises

Exercises with solutions from Section 5.3 of [UA].

Exercise 5.3.1. Recall from Exercise 4.4.9 that a function $f : A \to \mathbf{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x \neq y$ in A.

(a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

(b) Review the definition of a contractive function in Exercise 4.3.11. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?

Solution. (a) Note that $|f'|$ is continuous on $[a, b]$ since f' is continuous on $[a, b]$. The Extreme Value Theorem then implies that $|f'|$ attains a maximum on $[a, b]$, say $M = |f'(t)|$ for some $t \in [a, b]$. Let $x < y$ in $[a, b]$ be given. The Mean Value Theorem on the interval $[x, y]$ implies that there is a $c \in (x, y)$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)| \leq M.$$

Thus f is Lipschitz on $[a, b]$.

(b) If $|f'(x)| < 1$ on $[a, b]$, then the maximum value $M = |f'(t)|$ from part (a) must satisfy $M < 1$ and thus f is contractive on $[a, b]$.

Exercise 5.3.2. Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Solution. We will prove the contrapositive statement. Suppose that there exist $x < y$ in A such that $f(x) = f(y)$. Then Rolle's Theorem implies that there exists some $c \in (x, y)$ such that $f'(c) = 0$.

For a counterexample to the converse statement, consider the one-to-one function $f : (-1, 1) \to (-1, 1)$ given by $f(x) = x^3$, which satisfies $f'(0) = 0$.

Exercise 5.3.3. Let h be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1, h(1) = 2$, and $h(3) = 2$.

(a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$.

(b) Argue that at some point c we have $h'(c) = 1/3$.

(c) Argue that $h'(x) = 1/4$ at some point in the domain.

Solution. (a) Define $f : [0,3] \to \mathbf{R}$ by $f(x) = h(x) - x$ and note that f is continuous since h is continuous. Furthermore, since $f(1) = h(1) - 1 = 1$ and $f(3) = h(3) - 3 = -1$, the Intermediate Value Theorem implies that there exists some $d \in (1,3)$ such that $f(d) = 0$, i.e. $h(d) = d$.

(b) Since h is differentiable on $[0,3]$, the Mean Value Theorem implies that there exists some point $c \in (0,3)$ such that

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{1}{3}.$$

(c) Similarly to part (b), the Mean Value Theorem implies that there exists some point $b \in (1,3)$ such that

$$h'(b) = \frac{h(3) - h(1)}{3 - 1} = 0.$$

Combining this with part (b), we see that h' takes the values 0 and $\frac{1}{3}$. Since $0 < \frac{1}{4} < \frac{1}{3}$, Darboux's Theorem implies that h' takes the value $\frac{1}{4}$ at some point in the domain $[0,3]$.

Exercise 5.3.4. Let f be differentiable on an interval A containing zero, and assume (x_n) is a sequence in A with $(x_n) \to 0$ and $x_n \neq 0$.

(a) If $f(x_n) = 0$ for all $n \in \mathbf{N}$, show $f(0) = 0$ and $f'(0) = 0$.

(b) Add the assumption that f is twice-differentiable at zero and show that $f''(0) = 0$ as well.

Solution. (a) We have

$$0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} f(x_n) = f(0)$$

since f is continuous at zero.

Note that, for each $n \in \mathbf{N}$, since $x_n \neq 0$, the difference quotient $\frac{f(x_n)-f(0)}{x_n-0} = \frac{f(x_n)}{x_n}$ is well-defined and satisfies $\frac{f(x_n)}{x_n} = 0$. Since $f'(0)$ exists, it must then be the case that

$$f'(0) = \lim_{n\to\infty} \frac{f(x_n)}{x_n} = 0.$$

(b) We are given that the limit

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{f'(x)}{x}$$

exists. Since $f(0) = 0$, we may apply L'Hospital's Rule (Theorem 5.3.6) to see that

$$f''(0) = \lim_{x \to 0} \frac{2f(x)}{x^2}.$$

Similarly to part (a), note that for each $n \in \mathbf{N}$, since $x_n \neq 0$, the quotient $\frac{2f(x_n)}{x_n^2}$ is well-defined and satisfies $\frac{2f(x_n)}{x_n^2} = 0$. Since $\lim_{x \to 0} \frac{2f(x)}{x^2}$ exists, it must be the case that

$$f''(0) = \lim_{n \to \infty} \frac{2f(x_n)}{x_n^2} = 0.$$

Exercise 5.3.5. (a) Supply the details for the proof of Cauchy's Generalized Mean Value Theorem (Theorem 5.3.5).

(b) Give a graphical interpretation of the Generalized Mean Value Theorem analogous to the one given for the Mean Value Theorem at the beginning of Section 5.3. (Consider f and g as parametric equations for a curve.)

Solution. (a) Define $h : [a, b] \to \mathbf{R}$ by

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$$

and note that h is continuous on $[a, b]$ and differentiable on (a, b) since f and g are. The Mean Value Theorem implies that there exists some $c \in (a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a},$$

or equivalently

$$\begin{aligned}
&[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) \\
&= \frac{[f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) - [f(b) - f(a)]g(a) + [g(b) - g(a)]f(a)}{b - a} \\
&= \frac{[f(b) - f(a)][g(b) - g(a)] - [g(b) - g(a)][f(b) - f(a)]}{b - a} \\
&= 0.
\end{aligned}$$

(b) If $f'(c) \neq 0$ and $f(b) \neq f(a)$, so that

$$\frac{g'(c)}{f'(c)} = \frac{g(b) - g(a)}{f(b) - f(a)},$$

then the Generalized Mean Value Theorem can be geometrically interpreted as asserting the existence of a tangent line to the graph of the curve $\gamma : [a, b] \to \mathbf{R}^2;\ \gamma(t) = (f(t), g(t))$ at the point $(f(c), g(c))$ which is parallel to the line through the points $(f(a), g(a))$ and $(f(b), g(b))$.

Exercise 5.3.6. (a) Let $g : [0, a] \to \mathbf{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in [0, a]$. Show $|g(x)| \leq Mx$ for all $x \in [0, a]$.

(b) Let $h : [0, a] \to \mathbf{R}$ be twice differentiable, $h'(0) = h(0) = 0$ and $|h''(x)| \leq M$ for all $x \in [0, a]$. Show $|h(x)| \leq Mx^2/2$ for all $x \in [0, a]$.

(c) Conjecture and prove an analogous result for a function that is differentiable three times on $[0, a]$.

Solution. (a) Since $g(0) = 0$, the inequality $|g(x)| \leq Mx$ is clear when $x = 0$. Suppose $x \in (0, a]$. By the Mean Value Theorem on the interval $[0, x]$, there exists some $c \in (0, x)$ such that

$$|g'(c)| = \left|\frac{g(x)}{x}\right| \implies |g(x)| = |g'(c)|x \leq Mx.$$

(b) Since $h(0) = 0$, the inequality $|h(x)| \leq Mx^2/2$ is clear when $x = 0$. Suppose $x \in (0, a]$. Using the Generalized Mean Value Theorem on the interval $[0, x]$ with the functions h and $\frac{1}{2}x^2$, we can find some $c \in (0, x)$ such that

$$\frac{h(x)}{\frac{1}{2}x^2} = \frac{h'(c)}{c}.$$

Now we can use the Mean Value Theorem on the interval $[0, c]$ with the function h' to find some $d \in (0, c)$ such that

$$h''(d) = \frac{h'(c)}{c}.$$

Combining this with the previous equality, we see that

$$h''(d) = \frac{h(x)}{\frac{1}{2}x^2} \implies |h(x)| = \tfrac{1}{2}|h''(d)|x^2 \leq \tfrac{1}{2}Mx^2.$$

(c) Suppose $f : [0, a] \to \mathbf{R}$ is three times differentiable, $f''(0) = f'(0) = f(0) = 0$, and $f'''(x) \leq M$ for all $x \in [0, a]$. We claim that $|f(x)| \leq \frac{1}{6}Mx^3$ for all $x \in [0, a]$. To see this, we proceed as in part (b). Since $f(0) = 0$, the inequality $|f(x)| \leq \frac{1}{6}Mx^3$ is clear when $x = 0$.

Suppose $x \in (0, a]$. Using the Generalized Mean Value Theorem on the interval $[0, x]$ with the functions f and $\frac{1}{6}x^3$, we can find some $b \in (0, x)$ such that

$$\frac{f(x)}{\frac{1}{6}x^3} = \frac{f'(b)}{\frac{1}{2}b^2}.$$

Using the Generalized Mean Value Theorem on the interval $[0, b]$ with the functions f' and $\frac{1}{2}x^2$, we can find some $c \in (0, b)$ such that

$$\frac{f'(b)}{\frac{1}{2}b^2} = \frac{f''(c)}{c}.$$

Now we can use the Mean Value Theorem on the interval $[0, c]$ with the function f'' to find some $d \in (0, c)$ such that

$$f'''(d) = \frac{f''(c)}{c}.$$

Combining all of these equalities, we see that

$$f'''(d) = \frac{f(x)}{\frac{1}{6}x^3} \implies |f(x)| = \tfrac{1}{6}|f'''(d)|x^3 \leq \tfrac{1}{6}Mx^3.$$

Exercise 5.3.7. A *fixed point* of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Solution. We will prove the contrapositive statement. Suppose that $x < y$ belong to the domain of f and are such that $f(x) = x$ and $f(y) = y$. By the Mean Value Theorem on the interval $[x, y]$, there exists some $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{x - y} = \frac{y - x}{y - x} = 1.$$

Exercise 5.3.8. Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x \to 0} f'(x) = L$, show $f'(0)$ exists and equals L.

Solution. We would like to see that the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x}$$

exists and equals L. Letting I denote the interval domain of f, note that the numerator and denominator of this fraction are both continuous on I, differentiable on $I \setminus \{0\}$, and vanish at zero. That is, we have satisfied the hypotheses of the 0/0 case of L'Hospital's Rule (Theorem 5.3.6) and hence

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} f'(x) = L.$$

Exercise 5.3.9. Assume f and g are as described in Theorem 5.3.6, but now add the assumption that f and g are differentiable at a, and f' and g' are continuous at a with $g'(a) \neq 0$. Find a short proof for the 0/0 case of L'Hospital's Rule under this stronger hypothesis.

Solution. Note that for all $x \neq a$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}.$$

By assumption, the limits

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

both exist and $g'(a) \neq 0$. It follows from Corollary 4.2.4 (iv) that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

Now we can use our assumption that f' and g' are continuous at a with $g'(a) \neq 0$ to see that

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}.$$

Exercise 5.3.10. Let $f(x) = x \sin(1/x^4) e^{-1/x^2}$ and $g(x) = e^{-1/x^2}$. Using the familiar properties of these functions, compute the limit as x approaches zero of $f(x)$, $g(x)$, $f(x)/g(x)$, and $f'(x)/g'(x)$. Explain why the results are surprising but not in conflict with the content of Theorem 5.3.6.

Solution. Some algebra reveals that

$$\frac{f(x)}{g(x)} = x \sin\left(\tfrac{1}{x^4}\right) \quad \text{and} \quad \frac{f'(x)}{g'(x)} = \sin\left(\tfrac{1}{x^4}\right)\left(\frac{x^3}{2} + x\right) - \frac{2\cos\left(\frac{1}{x^4}\right)}{x}.$$

Given an $\epsilon > 0$, we have

$$|x| < \sqrt{\frac{1}{\log\left(\frac{1}{\epsilon}\right)}} \implies e^{-\frac{1}{x^2}} < \epsilon$$

and thus $\lim_{x \to 0} g(x) = 0$. Combining this with various applications of the Squeeze Theorem, we see that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{f(x)}{g(x)} = 0.$$

However, we claim that $\frac{f'(x)}{g'(x)}$ does not converge to zero as $x \to 0$. To see this, consider the sequence (x_n) given by

$$x_n = \frac{1}{\sqrt[4]{2n\pi}},$$

which satisfies $\lim_{n\to\infty} x_n = 0$. Then

$$\frac{f'(x_n)}{g'(x_n)} = -2\sqrt[4]{2n\pi} \to -\infty \text{ as } n \to \infty.$$

This does not conflict with the content of Theorem 5.3.6 since f and g are not continuous at zero; they are not even defined at zero.

Exercise 5.3.11. (a) Use the Generalized Mean Value Theorem to furnish a proof of the 0/0 case of L'Hospital's Rule (Theorem 5.3.6).

(b) If we keep the first part of the hypothesis of Theorem 5.3.6 the same but we assume that

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \infty,$$

does it necessarily follow that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \infty?$$

Solution. (a) Let $\epsilon > 0$ be given. Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon.$$

Suppose $x \in (a - \delta, a)$. By the Generalized Mean Value Theorem on the interval $[x, a]$, there exists some $c \in (x, a)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(a) - f(x)}{g(a) - g(x)} = \frac{f(x)}{g(x)};$$

note we are using that g' does not vanish on (x, a). Since $c \in (a - \delta, a)$, we then have

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \epsilon.$$

We can similarly handle the case where $x \in (a, a+\delta)$ by using the Generalized Mean Value Theorem on the interval $[a, x]$. In any case, we have shown that if x satisfies $0 < |x-a| < \delta$ then

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$$

and thus

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

(b) It does necessarily follow; the proof from part (a) needs only slight modifications. Let $M > 0$ be given. Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \infty$, there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \frac{f'(x)}{g'(x)} \geq M.$$

Suppose $x \in (a-\delta, a)$. By the Generalized Mean Value Theorem on the interval $[x, a]$, there exists some $c \in (x, a)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(a) - f(x)}{g(a) - g(x)} = \frac{f(x)}{g(x)};$$

note we are using that g' does not vanish on (x, a). Since $c \in (a-\delta, a)$, we then have

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \geq M.$$

We can similarly handle the case where $x \in (a, a+\delta)$ by using the Generalized Mean Value Theorem on the interval $[a, x]$. In any case, we have shown that if x satisfies $0 < |x-a| < \delta$ then

$$\frac{f(x)}{g(x)} \geq M$$

and thus

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \infty.$$

Exercise 5.3.12. If f is twice differentiable on an open interval containing a and f'' is continuous at a, show

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a).$$

(Compare this to Exercise 5.2.6(b).)

Solution. We have by the 0/0 case of L'Hospital's Rule (Theorem 5.3.6) that

$$\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2}=\lim_{h\to 0}\frac{f'(a+h)-f'(a-h)}{2h}.$$

Since f' is differentiable at a, we may apply Exercise 5.2.6 (b) to see that

$$\lim_{h\to 0}\frac{f'(a+h)-f'(a-h)}{2h}=f''(a).$$

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.