1 Section 5.C Exercises

Exercises with solutions from Section 5.C of [LADR].

Exercise 5.C.1. Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{null} T \oplus \operatorname{range} T$.

Solution. By 1.45 and 3.22, it will suffice to show that $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.

Since T is diagonalizable, there is a basis $v_1, \dots, v_n$ of V such that, for each j, $Tv_j = \lambda_j v_j$ for some eigenvalue λ_j. Suppose $v = a_1 v_1 + \dots + a_n v_n \in \operatorname{null} T \cap \operatorname{range} T$. Then

$$0 = Tv = a_1 \lambda_1 v_1 + \dots + a_n \lambda_n v_n,$$

whence $a_j \lambda_j = 0$ for each j by the linear independence of the basis $v_1, \dots, v_n$. Furthermore, since $0 = v - Tw$ for some $w = b_1 v_1 + \dots + b_n v_n$, we also have

$$0 = (a_1 - b_1 \lambda_1) v_1 + \dots + (a_n - b_n \lambda_n) v_n,$$

which implies that $a_j = b_j \lambda_j$ for each j, again by the linear independence of the basis $v_1, \dots, v_n$. Combining this with the relation $a_j \lambda_j = 0$, we see that $b_j \lambda_j^2 = 0$, which shows that either $b_j = 0$, $\lambda_j = 0$, or both are zero. In any case, since $a_j = b_j \lambda_j$, we have that each $a_j = 0$ and hence that $v = 0$.

Exercise 5.C.2. Prove the converse of the statement in the exercise above or give a counterexample to the converse.

Solution. The converse does not hold. For a counterexample, consider the operator $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x, y) = (x + y, y)$. With respect to the standard basis of $\mathbf{R}^2$, this operator has the upper-triangular matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus the only eigenvalue of T is 1 (5.32). It is straightforward to verify that $E(1, T) = \operatorname{span}((1, 0))$, which is one-dimensional and so cannot possibly equal $\mathbf{R}^2$. Hence T is not diagonalizable (5.41).

However, T is injective, i.e. $\operatorname{null} T = \{0\}$. It follows from 1.45 and 3.22 that $\mathbf{R}^2 = \operatorname{null} T \oplus \operatorname{range} T$.

Exercise 5.C.3. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:

(a) $V = \operatorname{null} T \oplus \operatorname{range} T$.

(b) $V = \operatorname{null} T + \operatorname{range} T$.

(c) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.

Solution. That (a) implies (b) is clear.

Suppose that (b) holds. We have

$$\dim(\operatorname{null} T \cap \operatorname{range} T) = \dim \operatorname{null} T + \dim \operatorname{range} T - \dim(\operatorname{null} T + \operatorname{range} T)$$

by 2.43. By assumption we have $\dim(\operatorname{null} T + \operatorname{range} T) = \dim V$ and by 3.22 we also have $\dim \operatorname{null} T + \dim \operatorname{range} T = \dim V$. Thus $\dim(\operatorname{null} T \cap \operatorname{range} T) = 0$, which is the case if and only if $\operatorname{null} T \cap \operatorname{range} T = \{0\}$. Hence (c) holds.

Suppose that (c) holds. Then the sum $\operatorname{null} T \oplus \operatorname{range} T$ is direct by 1.45 and furthermore we have

$$\dim(\operatorname{null} T + \dim \operatorname{range} T) = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim V$$

by 2.43 and 3.22. It follows that $V = \operatorname{null} T \oplus \operatorname{range} T$ by Exercise 2.C.1 and hence (a) holds.

Exercise 5.C.4. Give an example to show that the exercise above is false without the hypothesis that V is finite-dimensional.

Solution. Consider the forward-shift operator $T : \mathbf{C}^\infty \to \mathbf{C}^\infty$ given by

$$T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots).$$

Then

$$\operatorname{null} T = \{0\} \quad \text{and} \quad \operatorname{range} T = \{(0, z_2, z_3, \ldots) : z_j \in \mathbf{C}\}.$$

Thus $\operatorname{null} T \cap \operatorname{range} T = \{0\}$, however $\mathbf{C}^\infty \neq \operatorname{null} T + \operatorname{range} T = \operatorname{range} T$.

Exercise 5.C.5. Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if

$$V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$$

for every $\lambda \in \mathbf{C}$.

Solution. Suppose that T is diagonalizable, so that there is some basis such that the matrix of T with respect to this basis is diagonal:

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

For any $\lambda \in \mathbf{C}$, the matrix of the operator $T - \lambda I$ with respect to this same basis is also diagonal:

$$\begin{pmatrix} \lambda_1 - \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda_n - \lambda \end{pmatrix}.$$

So $T - \lambda I$ is also diagonalizable and thus by Exercise 5.C.1 we have

$$V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I).$$

We will prove the converse statement by strong induction on the dimension of V. For a non-negative integer n, let $P(n)$ be the following statement: if V is an n-dimensional complex vector space and $T \in \mathcal{L}(V)$ is such that $V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$ for all $\lambda \in \mathbf{C}$, then T is diagonalizable. The truth of $P(0)$ is clear. Suppose that $P(0), \ldots, P(n)$ all hold for some non-negative integer n, let V be an $(n+1)$-dimensional complex vector space, and let $T \in \mathcal{L}(V)$ be such that $V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$ for all $\lambda \in \mathbf{C}$. Our aim is to show that T is diagonalizable.

Since V is a complex vector space, there is some eigenvalue λ_0 of T (5.21). Set

$$U := \operatorname{null}(T - \lambda_0 I) = E(\lambda_0, T) \quad \text{and} \quad W := \operatorname{range}(T - \lambda_0 I).$$

It is straightforward to verify that W is invariant under T (see the first proof of 5.21). We may then consider the restriction operator $S := T|_W : W \to W$. Let $\lambda \in \mathbf{C}$ be given. Since S is a restriction of T, we have

$$\operatorname{null}(S - \lambda I) \subseteq \operatorname{null}(T - \lambda I) \quad \text{and} \quad \operatorname{range}(S - \lambda I) \subseteq \operatorname{range}(T - \lambda I),$$

which implies that

$$\operatorname{null}(S - \lambda I) \cap \operatorname{range}(S - \lambda I) \subseteq \operatorname{null}(T - \lambda I) \cap \operatorname{range}(T - \lambda I). \tag{1}$$

By assumption we have $V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$; the equivalence of (a) and (c) in Exercise 5.C.3 thus shows that $\operatorname{null}(T - \lambda I) \cap \operatorname{range}(T - \lambda I) = \{0\}$. It follows from (1) that $\operatorname{null}(S - \lambda I) \cap \operatorname{range}(S - \lambda I) = \{0\}$ and another application of Exercise 5.C.3 allows us to conclude that $W = \operatorname{null}(S - \lambda I) \oplus \operatorname{range}(S - \lambda I)$.

By assumption we have $V = U \oplus W$; since λ_0 is an eigenvalue of T we must have $\dim U \geq 1$ and thus $k := \dim W \leq n$. We may now invoke the induction hypothesis $P(k)$ to see that S is diagonalizable and hence obtain a basis $w_1, \ldots, w_k$ for W consisting of eigenvectors of S (5.41), which must also be eigenvectors of T. Let $u_1, \ldots, u_m$ be a basis of U; evidently these are eigenvectors of T corresponding to the eigenvalue λ_0. Since $V = U \oplus W$, it follows from Exercise

2.B.8 that the list $u_1, \ldots, u_m, w_1, \ldots, w_k$ is a basis for V consisting of eigenvectors of T and thus T is diagonalizable (5.41).

This completes the induction step and we may conclude that $P(n)$ holds for all non-negative integers n.

Exercise 5.C.6. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, and $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

Solution. T is diagonalizable (5.44), so there is a basis $v_1, \ldots, v_n$ of V such that, for each j, $Tv_j = \lambda_j v_j$ for some eigenvalue λ_j. By assumption each eigenvector of T is also an eigenvector of S, so for each j we also have $Sv_j = \mu_j v_j$ for some eigenvalue μ_j. Let $v = a_1 v_1 + \cdots + a_n v_n$ be given. Then

$$\begin{aligned}(ST)v = a_1(ST)v_1 + \cdots + a_n(ST)v_n &= a_1\lambda_1\mu_1 v_1 + \cdots + a_n\lambda_n\mu_n v_n \\ &= a_1\mu_1\lambda_1 v_1 + \cdots + a_n\mu_n\lambda_n v_n = a_1(TS)v_1 + \cdots + a_n(TS)v_n = (TS)v.\end{aligned}$$

Thus $ST = TS$.

Exercise 5.C.7. Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V and that $\lambda \in \mathbf{F}$. Prove that λ appears on the diagonal of A precisely $\dim E(\lambda, T)$ times.

Solution. Let $v_1, \ldots, v_n$ be the basis of V in question, so that, for each j, $Tv_j = \lambda_j v_j$ for some eigenvalue λ_j. Let $k := |\{1 \le j \le n : \lambda_j = \lambda\}|$; our aim is to show that $k = \dim E(\lambda, T)$. Note that, since each $v_j \neq 0$, we have $\lambda_j = \lambda$ if and only if $v_j \in E(\lambda, T)$. It follows that

$$k = |\{1 \le j \le n : v_j \in E(\lambda, T)\}|.$$

Since the list $v_1, \ldots, v_n$ is linearly independent, it is immediate that $k \le \dim E(\lambda, T)$. For simplicity (and without loss of generality), let's assume that those vectors from the basis $v_1, \ldots, v_n$ which belong to $E(\lambda, T)$ are the first k vectors, i.e. $v_1, \ldots, v_k$ all belong to $E(\lambda, T)$ and $v_{k+1}, \ldots, v_n$ do not (note that either of the lists $v_1, \ldots, v_k$ and $v_{k+1}, \ldots, v_n$ could be empty; if one is empty, the other is not, provided $n \ge 1$).

Suppose that $v = a_1 v_1 + \cdots + a_k v_k + a_{k+1} v_{k+1} + \cdots + a_n v_n$ belongs to $E(\lambda, T)$. Then

$$\begin{aligned}Tv = a_1\lambda v_1 + \cdots + a_k\lambda v_k &+ a_{k+1}\lambda_{k+1} v_{k+1} + \cdots + a_n\lambda_n v_n \\ &= a_1\lambda v_1 + \cdots + a_k\lambda v_k + a_{k+1}\lambda v_{k+1} + \cdots + a_n\lambda v_n = \lambda v,\end{aligned}$$

which implies that

$$a_{k+1}(\lambda_{k+1} - \lambda)v_{k+1} + \cdots + a_n(\lambda_n - \lambda)v_n = 0.$$

Since each vector v_j in the list $v_{k+1}, \ldots, v_n$ does not belong to $E(\lambda, T)$, which we noted earlier is the case if and only if $\lambda_j \neq \lambda$, the linear independence of this list implies that $a_{k+1} = \cdots = a_n = 0$ and thus

$$v = a_1 v_1 + \cdots + a_k v_k,$$

demonstrating that the list $v_1, \ldots, v_k$ spans $E(\lambda, T)$. It follows that $k \geq \dim E(\lambda, T)$ and we may conclude that $k = \dim E(\lambda, T)$.

Exercise 5.C.8. Suppose $T \in \mathcal{L}(\mathbf{F}^5)$ and $\dim E(8, T) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.

Solution. We will prove the contrapositive statement; suppose that neither $T - 2I$ nor $T - 6I$ is invertible. Then 2 and 6 are eigenvalues of T (5.6), so

$$\dim E(2, T) \geq 1 \quad \text{and} \quad \dim E(6, T) \geq 1.$$

By 5.38, we have

$$\begin{aligned} &\dim E(8, T) + \dim E(2, T) + \dim E(6, T) \leq \dim \mathbf{F}^5 = 5 \\ &\qquad \implies \dim E(8, T) \leq 5 - \dim E(2, T) - \dim E(6, T) \leq 3 < 4. \end{aligned}$$

Thus $\dim E(8, T) \neq 4$.

Exercise 5.C.9. Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $E(\lambda, T) = E\big(\frac{1}{\lambda}, T^{-1}\big)$ for every $\lambda \in \mathbf{F}$ with $\lambda \neq 0$.

Solution. This is immediate from the fact that, for $\lambda \neq 0$ and $v \neq 0$, one has

$$Tv = \lambda v \iff T^{-1} v = \lambda^{-1} v.$$

(See Exercise 5.A.21.)

Exercise 5.C.10. Suppose that V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct nonzero eigenvalues of T. Prove that

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \operatorname{range}(T).$$

Solution. Note that

$$\dim E(0, T) + \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V$$

follows from 5.38 (if 0 is an eigenvalue of T, the inequality is the second part of 5.38; if 0 is not an eigenvalue of T then $\dim E(0, T) = 0$ and thus we can add $\dim E(0, T)$ to the left-hand side of the inequality in 5.38). By 3.22 we have

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim E(0, T) + \dim \operatorname{range} T$$

and thus

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \operatorname{range}(T).$$

Exercise 5.C.11. Verify the assertion in Example 5.40.

Solution. Define $T \in \mathcal{L}(\mathbf{R}^2)$ by

$$T(x, y) = (41x + 7y, -20x + 74y).$$

The claim is that with respect to the basis $(1, 4), (7, 5)$ the matrix of T is

$$\begin{pmatrix} 69 & 0 \\ 0 & 46 \end{pmatrix}.$$

Indeed,

$$T(1, 4) = (69, 276) = 69(1, 4) + 0(7, 5) \quad \text{and} \quad T(7, 5) = (322, 230) = 0(1, 4) + 46(7, 5).$$

Exercise 5.C.12. Suppose $R, T \in \mathcal{L}(\mathbf{F}^3)$ each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbf{F}^3)$ such that $R = S^{-1}TS$.

Solution. Since R and T both have $3 = \dim \mathbf{F}^3$ distinct eigenvalues, they are both diagonalizable, i.e. there exists a basis u_1, u_2, u_3 and a basis v_1, v_2, v_3 of V such that

$$Ru_1 = 2u_1,\ Ru_2 = 6u_2,\ Ru_3 = 7u_3,\ Tv_1 = 2v_1,\ Tv_2 = 6v_2,\ \text{and } Tv_3 = 7v_3.$$

Define $S \in \mathcal{L}(\mathbf{F}^3)$ by $Su_j = v_j$; S is invertible since it maps a basis to a basis. Furthermore,

$$S^{-1}TSu_1 = S^{-1}Tv_1 = 2S^{-1}v_1 = 2u_1 = Ru_1.$$

Similarly, we have $S^{-1}TSu_2 = Ru_2$ and $S^{-1}TSu_3 = Ru_3$. Since $S^{-1}TS$ and R agree on each basis vector u_1, u_2, u_3, we must have $S^{-1}TS = R$.

Exercise 5.C.13. Find $R, T \in \mathcal{L}(\mathbf{F}^4)$ such that R and T each have 2, 6, 7 as eigenvalues, R and T have no other eigenvalues, and there does not exist an invertible operator $S \in \mathcal{L}(\mathbf{F}^4)$ such that $R = S^{-1}TS$.

Solution. Let R and T be the operators which have the matrices

$$\mathcal{M}(R) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} \quad \text{and} \quad \mathcal{M}(T) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

with respect to the standard basis e_1, e_2, e_3, e_4 of $\mathbf{F}^4$. Since these matrices are upper-triangular, the eigenvalues of R and T are precisely 2, 6, 7 (5.32). To disprove the existence of an invertible operator $S \in \mathcal{L}(\mathbf{F}^4)$ such that $R = S^{-1}TS$, let $S \in \mathcal{L}(\mathbf{F}^4)$ be any invertible operator. By Exercise 5.A.15, the operator $S^{-1}TS$ also has 2 as an eigenvalue. Furthermore, the eigenspace $E(2, T)$ is the image under S of the eigenspace $E(2, S^{-1}TS)$. A restriction of the operator S thus provides us with an isomorphism between $E(2, T)$ and $E(2, S^{-1}TS)$; in particular, these eigenspaces must have the same dimension. However, note that

$$\dim E(2, T) = \dim \operatorname{span}(e_1) = 1 \neq 2 = \dim \operatorname{span}(e_1, e_2) = \dim E(2, R).$$

Thus there cannot exist an invertible operator $S \in \mathcal{L}(\mathbf{F}^4)$ such that $R = S^{-1}TS$.

Exercise 5.C.14. Find $T \in \mathcal{L}(\mathbf{C}^3)$ such that 6 and 7 are eigenvalues of T and such that T does not have a diagonal matrix with respect to any basis of $\mathbf{C}^3$.

Solution. Let T be the operator which has the matrix

$$\mathcal{M}(T) = \begin{pmatrix} 6 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

with respect to the standard basis e_1, e_2, e_3 of $\mathbf{C}^3$. Since this matrix is upper-triangular, the eigenvalues of T are precisely 6 and 7 (5.32). It is straightforward to verify that

$$E(6, T) = \operatorname{span}(e_1) \quad \text{and} \quad E(7, T) = \operatorname{span}(e_3),$$

so that

$$\dim E(6, T) + \dim E(7, T) = 2 \neq 3 = \dim \mathbf{C}^3.$$

It follows from the equivalence of (a) and (e) in 5.41 that T is not diagonalizable.

Exercise 5.C.15. Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is such that 6 and 7 are eigenvalues of T. Furthermore, suppose T does not have a diagonal matrix with respect to any basis of $\mathbf{C}^3$. Prove that there exists $(x, y, z) \in \mathbf{C}^3$ (see errata) such that $T(x, y, z) = (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z)$.

Solution. Since $\dim \mathbf{C}^3 = 3$, it must be the case that 6 and 7 are the only eigenvalues of T; if T had another distinct eigenvalue then T would be diagonalizable (5.44). It follows that the operator $T - 8I$ is surjective (5.6) and thus there exists $(x, y, z) \in \mathbf{C}^3$ such that $(T - 8I)(x, y, z) = (17, \sqrt{5}, 2\pi)$, or equivalently such that $T(x, y, z) = (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z)$.

Exercise 5.C.16. The ***Fibonacci sequence*** $F_1, F_2, \ldots$ is defined by

$$F_1 = 1,\ F_2 = 1, \quad \text{and} \quad F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 3.$$

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(x, y) = (y, x + y)$.

(a) Show that $T^n(0,1) = (F_n, F_{n+1})$ for each positive integer n.

(b) Find the eigenvalues of T.

(c) Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of T.

(d) Use the solution to part (c) to compute $T^n(0,1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each positive integer n.

(e) Use part (d) to conclude that for each positive integer n, the Fibonacci number F_n is the integer that is closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

Solution. (a) We will proceed by induction. The base case $n = 1$ is clear, so suppose that $T^n(0,1) = (F_n, F_{n+1})$ for some positive integer n. Then

$$T^{n+1}(0,1) = T(T^n(0,1)) = T(F_n, F_{n+1}) = (F_{n+1}, F_n + F_{n+1}) = (F_{n+1}, F_{n+2}).$$

This completes the induction step and the proof.

(b) We are looking for solutions $(x, y) \neq (0,0)$ and $\lambda \in \mathbf{R}$ of the equation

$$T(x,y) = (y, x+y) = (\lambda x, \lambda y).$$

From the equation $y = \lambda x$ we see that $x = 0$ if and only if $y = 0$, so we may assume that both of x and y are non-zero. Substituting $y = \lambda x$ into the equation $x + y = \lambda y$ and cancelling x gives us the equation $\lambda^2 - \lambda - 1 = 0$, which has two distinct real solutions:

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

These are indeed eigenvalues, since

$$T(1,\lambda_1) = (\lambda_1, \lambda_1 + 1) = (\lambda_1, \lambda_1^2) = \lambda_1(1, \lambda_1),$$

where we have used that λ_1 satisfies the equation $\lambda_1^2 - \lambda_1 - 1 = 0$. Similarly,

$$T(1,\lambda_2) = \lambda_2(1,\lambda_2).$$

Since $\dim \mathbf{R}^2 = 2$, we may conclude that the eigenvalues of T are precisely λ_1 and λ_2.

(c) Since $\lambda_1 \neq \lambda_2$, the eigenvectors $v_1 := (1, \lambda_1)$ and $v_2 := (1, \lambda_2)$ found in part (b) are linearly independent and thus form a basis of the 2-dimensional vector space $\mathbf{R}^2$.

(d) Observe that

$$v_1 - v_2 = (0, \lambda_1 - \lambda_2) = \left(0, \sqrt{5}\right).$$

Thus $(0,1) = \frac{1}{\sqrt{5}}(v_1 - v_2)$. Let n be a positive integer. We then have

$$T^n(0,1) = \frac{1}{\sqrt{5}}(T^n v_1 - T^n v_2) = \frac{1}{\sqrt{5}}(\lambda_1^n v_1 - \lambda_2^n v_2) = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n, \lambda_1^{n+1} - \lambda_2^{n+1}).$$

Given the result of part (a), we may conclude that

$$F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

(e) For any positive integer n, observe that

$$\begin{aligned} 2 < \sqrt{5} < 3 &\implies -1 < \frac{1-\sqrt{5}}{2} < -\frac{1}{2} \\ &\implies -1 < \left(\frac{1-\sqrt{5}}{2}\right)^n < 1 \\ &\implies -\frac{1}{\sqrt{5}} < -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n < \frac{1}{\sqrt{5}} \\ &\implies -\frac{1}{2} < -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n < \frac{1}{2}. \end{aligned}$$

It then follows from part (d) that

$$\frac{1}{\sqrt{5}}\lambda_1^n - \frac{1}{2} < F_n < \frac{1}{\sqrt{5}}\lambda_1^n + \frac{1}{2},$$

i.e. F_n is the sole integer belonging to the open interval $\left(\frac{1}{\sqrt{5}}\lambda_1^n - \frac{1}{2}, \frac{1}{\sqrt{5}}\lambda_1^n + \frac{1}{2}\right)$, which has length 1. We may conclude that F_n is the integer closest to $\frac{1}{\sqrt{5}}\lambda_1^n$.

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