The following is mostly paraphrased from Section 2.2 of [UA].

1 Algebraic limit theorem

Theorem 1. Suppose (a_n) and (b_n) are sequences of complex numbers such that $\lim a_n = a$ and $\lim b_n = b$ for some a and b in $\mathbf{C}$.

(i) For any $c \in \mathbf{C}$, $\lim(ca_n) = ca$.

(ii) $\lim(a_n + b_n) = a + b$.

(iii) $\lim(a_n b_n) = ab$.

(iv) If $b \neq 0$, $\lim(a_n/b_n) = a/b$.

Proof.

(i) Let $\epsilon > 0$ be given. Then there exists an $N \in \mathbf{N}$ such that

$$n \geq N \implies |a_n - a| < \frac{\epsilon}{1 + |c|}.$$

Then provided $n \geq N$, we have

$$|ca_n - ca| = |c||a_n - a| < \frac{|c|\epsilon}{1 + |c|} < \epsilon.$$

It follows that $\lim(ca_n) = ca$.

(ii) Let $\epsilon > 0$ be given. Then there are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |a_n - a| < \frac{\epsilon}{2} \qquad \text{and} \qquad n \geq N_2 \implies |b_n - b| < \frac{\epsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows that $\lim(a_n + b_n) = a + b$.

(iii) Since (a_n) is convergent it is also bounded, say by $M > 0$. Let $\epsilon > 0$ be given. Then there are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |a_n - a| < \frac{\epsilon}{2(1+|b|)} \qquad \text{and} \qquad n \geq N_2 \implies |b_n - b| < \frac{\epsilon}{2M}.$$

Set $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \leq |a_n||b_n - b| + |b||a_n - a| \\ &\leq M|b_n - b| + |b||a_n - a| < \frac{M\epsilon}{2M} + \frac{|b|\epsilon}{2(1+|b|)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

It follows that $\lim(a_n b_n) = ab$.

(iv) It will suffice to prove that $\lim(1/b_n) = 1/b$. The general case will then follow from part (iii). Since $b \neq 0$, there exists an $N_1 \in \mathbf{N}$ such that $n \geq N_1 \implies |b_n - b| < \frac{1}{2}|b|$. Observe that for $n \geq N_1$ we have

$$0 < |b| \leq |b - b_n| + |b_n| < \tfrac{1}{2}|b| + |b_n| \implies 0 < \tfrac{1}{2}|b| < |b_n| \implies 0 < \frac{1}{|b_n|} < \frac{2}{|b|}.$$

Let $\epsilon > 0$ be given. Then there exists an $N_2 \in \mathbf{N}$ such that

$$n \geq N_2 \implies |b_n - b| < \frac{|b|^2 \epsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b\, b_n}\right| = \frac{|b_n - b|}{|b||b_n|} < \frac{2|b_n - b|}{|b|^2} < \epsilon.$$

It follows that $\lim(1/b_n) = 1/b$. □

2 Order limit theorem

Theorem 2. Suppose (a_n) and (b_n) are sequences of real numbers such that $\lim a_n = a$ and $\lim b_n = b$ for some a and b in $\mathbf{R}$.

(i) If $a_n \geq 0$ for all $n \in \mathbf{N}$, then $a \geq 0$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbf{N}$, then $a \leq b$.

(iii) If there exists $c \in \mathbf{R}$ for which $c \leq b_n$ for all $n \in \mathbf{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbf{N}$, then $a \leq c$.

Proof.

(i) Let us prove the contrapositive statement:

$$\text{If } a < 0, \text{ then } a_N < 0 \text{ for some } N \in \mathbf{N}.$$

There is an $N \in \mathbf{N}$ such that $n \geq N \implies |a_n - a| < -a$. Then observe that

$$a_N - a \leq |a_N - a| < -a \implies a_N < 0.$$

(ii) The sequence $(b_n - a_n)$ has limit $b - a$ by Theorem 1. Furthermore, this sequence satisfies $b_n - a_n \geq 0$ for all $n \in \mathbf{N}$; part (i) then implies that $b - a \geq 0$, i.e. $a \leq b$.

(iii) This follows by taking the constant sequence $(c, c, c, \ldots)$ in part (ii) for (a_n) and (b_n) respectively. □

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edn.