The following is paraphrased from pages 10-11 of [PMA].

1 Existence of nth roots in $\mathbb{R}$

First, a useful inequality. Suppose n is a positive integer and a, b are real numbers such that $0 < a < b$. This implies that $0 < b^{n-2}a < b^{n-1}$. Furthermore, we have $0 < a^2 < b^2$, which gives $0 < b^{n-3}a^2 < b^{n-1}$, and so on. Combining this with the equality

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$$

gives us the inequality

$$b^n - a^n < (b-a)nb^{n-1}. \tag{1}$$

Theorem 1. For every real $x > 0$ and every positive integer n there is exactly one positive real y such that $y^n = x$.

Proof. Suppose y_1 and y_2 are positive real numbers such that $y_1 \neq y_2$. Without loss of generality, assume $0 < y_1 < y_2$. Then $0 < y_1^n < y_2^n$, so that $y_1^n \neq y_2^n$. Hence by the contrapositive, $y_1^n = y_2^n$ implies that $y_1 = y_2$. This gives us the uniqueness of any such y in Theorem 1.

For existence, let $E = \{t \in \mathbb{R} : t > 0, t^n < x\}$. Observe that $t = \frac{x}{1+x}$ satisfies $t < x$ and $0 < t < 1$, which gives $0 < t^n < t < x$. Hence $t \in E$ and so E is non-empty. Now suppose $t \geq 1 + x > 1$, so that $t^n > t \geq 1 + x > x$. Then by the contrapositive, $t^n < x$ implies that $t < 1 + x$, and we see that E is bounded above by $1 + x$. We may now invoke the least-upper-bound property of $\mathbb{R}$ and set $y = \sup E$. Note that y must be positive, since $\frac{x}{1+x}$ belongs to E. To show that $y^n = x$, we will show that both of the assumptions $y^n < x$ and $y^n > x$ lead to contradictions.

First, assume that $y^n < x$. Using the Archimedean property, choose h such that $0 < h < 1$ and $h < \frac{x - y^n}{n(y+1)^{n-1}}$. Now take $a = y$ and $b = y + h$ in inequality (1) to obtain

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n,$$

whence $(y+h)^n < x$ and so $y + h \in E$; but this contradicts the fact that y is the supremum of E, since $y + h > y$.

Next, assume that $y^n > x$ and set $k = \frac{y^n - x}{ny^{n-1}} < y$. Take $a = y - k$ and $b = y$ in inequality (1) to obtain

$$y^n - (y-k)^n < kny^{n-1} = y^n - x,$$

whence $(y-k)^n \geq x$. Then $t \geq y - k$ implies that $t^n \geq x$; the contrapositive of this shows that $y - k$ is an upper bound for E. This contradicts the fact that y is the least upper bound of E, since $y - k < y$. $\square$

2 A corollary

Theorem 2. Let a and b be positive real numbers and n a positive integer. Then

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}.$$

Proof. Let $\alpha = \sqrt[n]{a}$ and $\beta = \sqrt[n]{b}$. Then by the commutativity of multiplication, we have

$$(\alpha\beta)^n = \alpha^n \beta^n = ab.$$

The uniqueness part of Theorem 1 then implies that $\sqrt[n]{ab} = \alpha\beta = \sqrt[n]{a}\,\sqrt[n]{b}$. $\square$

[PMA] Rudin, W. (1976) *Principles of Mathematical Analysis.* 3rd edn.