Linear Algebra Done Right Solutions

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Contents

Chapter 1 Vector Spaces

1.A. $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 1.A.1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi$ and $\beta = u + vi$, then

$$\alpha + \beta = (x+u) + (y+v)i = (u+x) + (v+y)i = \beta + \alpha,$$

where we have used the commutativity of addition in $\mathbf{R}$.

Exercise 1.A.2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}(\alpha + \beta) + \lambda &= ((x+u) + (y+v))i + \lambda = ((x+u)+s) + ((y+v)+t)i \\ &= (x + (u+s)) + (y + (v+t))i = \alpha + ((u+s) + (v+t)i) = \alpha + (\beta + \lambda),\end{aligned}$$

where we have used the associativity of addition in $\mathbf{R}$.

Exercise 1.A.3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}(\alpha\beta)\lambda &= [(xu - yv) + (xv + yu)i]\lambda \\ &= [(xu - yv)s - (xv + yu)t] + [(xu - yv)t + (xv + yu)s]i \\ &= [(xu)s - (yv)s - (xv)t - (yu)t] + [(xu)t - (yv)t + (xv)s + (yu)s]i \\ &= [x(us) - x(vt) - y(ut) - y(vs)] + [x(ut) + x(vs) + y(us) - y(vt)]i \\ &= [x(us - vt) - y(ut + vs)] + [x(ut + vs) + y(us - vt)]i \\ &= \alpha[(us - vt) + (ut + vs)i] \\ &= \alpha(\beta\lambda),\end{aligned}$$

where we have used several algebraic properties of $\mathbf{R}$.

Exercise 1.A.4. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}\lambda(\alpha + \beta) &= [s(x+u) - t(y+v)] + [s(y+v) + t(x+u)i] \\ &= (sx + su - ty - tv) + (sy + sv + tx + tu)i \\ &= [(sx - ty) + (sy + tx)i] + [(su - tv) + (sv + tu)i] \\ &= \lambda\alpha + \lambda\beta,\end{aligned}$$

where we have used distributivity in $\mathbf{R}$.

Exercise 1.A.5. Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Solution. Suppose that $\alpha = x + yi$. Let $\beta = -x - yi$ and observe that

$$\alpha + \beta = (x - x) + (y - y)i = 0 + 0i = 0.$$

To see that β is unique, suppose that β' also satisfies $\alpha + \beta' = 0$ and notice that

$$\beta = \beta = 0 = \beta + (\alpha + \beta') = (\alpha + \beta) + \beta' = 0 + \beta' = \beta',$$

where we have used the associativity of addition in $\mathbf{C}$ (Exercise 1.A.2) and the commutativity of addition in $\mathbf{C}$ (Exercise 1.A.1).

Exercise 1.A.6. Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Solution. Suppose that $\alpha = x + yi$. Since $\alpha \neq 0$, it must be the case that x and y are not both zero, so that $x^2 + y^2 \neq 0$. Let $\beta = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$ and observe that

$$\alpha\beta = (x + yi)\left(\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i\right) = \frac{x^2+y^2}{x^2+y^2} + \frac{xy - xy}{x^2+y^2}i = 1 + 0i = 1.$$

To see that β is unique, suppose β' also satisfies $\alpha\beta' = 1$ and notice that

$$\beta = \beta 1 = \beta(\alpha\beta') = (\alpha\beta)\beta' = 1\beta' = \beta',$$

where we have used the associativity of multiplication in $\mathbf{C}$ (Exercise 1.A.3) and the commutativity of multiplication in $\mathbf{C}$ (1.4).

Exercise 1.A.7. Show that

$$\frac{-1 + \sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution. Let $z = \frac{-1+\sqrt{3}i}{2}$, so that $2z = -1 + \sqrt{3}i$. Observe that

$$\begin{aligned}&(2z)^2 = 4z^2 = \left(-1 + \sqrt{3}i\right)^2 = 1 - 2\sqrt{3}i - 3 = -2 - 2\sqrt{3}i \\ \Rightarrow\ &(2z)^3 = (4z^2)(2z) = \left(-2 - 2\sqrt{3}i\right)\left(-1 + \sqrt{3}i\right) = 2 - 2\sqrt{3}i + 2\sqrt{3}i + 6 = 8,\end{aligned}$$

i.e., $8z^3 = 8$. It follows that $z^3 = 1$.

Exercise 1.A.8. Find two distinct square roots of i.

Solution. Let $z_1 = \frac{1+i}{\sqrt{2}}$ and $z_2 = -z_1$ (z_1 and z_2 are distinct since $z_1 \neq 0$) and observe that

$$2z_1^2 = (1+i)^2 = 2i \quad \Rightarrow \quad z_1^2 = i,$$

i.e. z_1 is a square root of i. Furthermore, $z_2^2 = (-z_1)^2 = z_1^2 = i$, so that z_2 is a square root of i also.

Exercise 1.A.9. Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution. The unique solution is $x = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right)$.

Exercise 1.A.10. Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Solution. If there was such a λ, then

$$\lambda(2 - 3i) = 12 - 5i \quad \Rightarrow \quad \lambda = \frac{12 - 5i}{2 - 3i} = 3 + 2i.$$

However,

$$(3 + 2i)(-6 + 7i) = -32 + 9i \neq -32 - 9i.$$

Exercise 1.A.11. Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$, and $z = (z_1, ..., z_n)$, then

$$\begin{aligned}(x + y) + z &= (x_1 + y_1, ..., x_n + y_n) + z = ((x_1 + y_1) + z_1, ..., (x_n + y_n) + z_n) \\ &= (x_1 + (y_1 + z_1), ..., x_n + (y_n + z_n)) = x + (y_1 + z_1, ..., y_n + z_n) = x + (y + z),\end{aligned}$$

where we have used the associativity of addition in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.2).

Exercise 1.A.12. Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Solution. If $x = (x_1, ..., x_n)$, then

$$(ab)x = ((ab)x_1, ..., (ab)x_n) = (a(bx_1), ..., a(bx_n)) = a(bx_1, ..., bx_n) = a(bx),$$

where we have used the associativity of multiplication in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.3).

Exercise 1.A.13. Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$, then

$$1x = (1x_1, ..., 1x_n) = (x_1, ..., x_n) = x,$$

where we have used that $1x_j = x_j$ for any $x_j \in \mathbf{F}$.

Exercise 1.A.14. Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, then

$$\begin{aligned}\lambda(x + y) &= \lambda(x_1 + y_1, ..., x_n + y_n) \\ &= (\lambda(x_1 + y_1), ..., \lambda(x_n + y_n)) \\ &= (\lambda x_1 + \lambda y_1, ..., \lambda x_n + \lambda y_n) \\ &= (\lambda x_1, ..., \lambda x_n) + (\lambda y_1, ..., \lambda y_n) \\ &= \lambda(x_1, ..., x_n) + \lambda(y_1, ..., y_n) \\ &= \lambda x + \lambda y,\end{aligned}$$

where we have used distributivity in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.4).

Exercise 1.A.15. Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$, then

$$\begin{aligned}(a + b)x &= (a + b)(x_1, ..., x_n) \\ &= ((a + b)x_1, ..., (a + b)x_n) \\ &= (ax_1 + bx_1, ..., ax_n + bx_n) \\ &= (ax_1, ..., ax_n) + (bx_1, ..., bx_n) \\ &= a(x_1, ..., x_n) + b(x_1, ..., x_n) \\ &= ax + bx,\end{aligned}$$

where we have used distributivity in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.4).

1.B. Definition of Vector Space

Exercise 1.B.1. Show that $-(-v) = v$ for every $v \in V$.

Solution. Since $v + (-v) = 0$ and the additive inverse of a vector is unique (1.27), it must be the case that $-(-v) = v$.

Exercise 1.B.2. Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution. It will suffice to show that if $av = 0$ and $a \neq 0$, so that a^{-1} exists, then $v = 0$. Indeed,

$$av = 0 \quad \Rightarrow \quad a^{-1}(av) = 0 \quad \Rightarrow \quad (a^{-1}a)v = 0 \quad \Rightarrow \quad 1v = 0 \quad \Rightarrow \quad v = 0.$$

Exercise 1.B.3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution. For $v, w, x \in V$, notice that

$$v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \tfrac{1}{3}(w - v).$$

Exercise 1.B.4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Solution. The empty set does not contain an additive identity.

Exercise 1.B.5. Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

> *The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.*

Solution. If V satisfies all of the conditions in (1.20), then as shown in (1.30) we have $0v = 0$ for all $v \in V$. Suppose that V satisfies all of the conditions in (1.20), except we have replaced the additive inverse condition with the condition that $0v = 0$ for all $v \in V$. We want to show that for each $v \in V$, there exists an element $w \in V$ such that $v + w = 0$. Indeed, for $v \in V$, let $w = (-1)v$ and observe that

$$v + w = 1v + (-1)v = (1 - 1)v = 0v = 0.$$

Exercise 1.B.6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) = (-\infty) + \infty &= 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution. This is not a vector space over $\mathbf{R}$, since addition is not associative:

$$(1 + \infty) + (-\infty) = \infty + (-\infty) = 0 \neq 1 = 1 + 0 = 1 + (\infty + (-\infty)).$$

Exercise 1.B.7. Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Solution. We define addition and scalar multiplication on V^S as in (1.24), i.e. for $f, g \in V^S$ the sum $f + g \in V^S$ is the function

$$\begin{aligned} f + g : S &\to V \\ x &\mapsto f(x) + g(x); \end{aligned}$$

the addition $f(x) + g(x)$ is vector addition in V. Similarly, for $\lambda \in \mathbf{F}$ and $f \in V^S$, the product $\lambda f \in V^S$ is the function

$$\begin{aligned} \lambda f : S &\to V \\ x &\mapsto \lambda f(x); \end{aligned}$$

the product $\lambda f(x)$ is scalar multiplication in V. We now show that V^S with these definitions satisfies each condition in definition (1.20).

Commutativity. Let $f, g \in V^S$ and $x \in S$ be given. Observe that

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x),$$

where we have used the commutativity of addition in V for the second equality. It follows that $f + g = g + f$.

Associativity. Let $f, g, h \in V^S$ and $x \in S$ be given. Observe that

$$\begin{aligned}((f + g) + h)(x) &= (f + g)(x) + h(x) = (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) = f(x) + (g + h)(x) = (f + (g + h))(x),\end{aligned}$$

where we have used the associativity of addition in V for the third equality. It follows that $(f + g) + h = f + (g + h)$. Similarly, let $f \in V^S$ and $a, b \in \mathbf{F}$ be given. Observe that, for any $x \in S$,

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = a((bf)(x)) = (a(bf))(x),$$

where we have used the associativity of scalar multiplication in V for the second equality. It follows that $(ab)f = a(bf)$.

Additive identity. We claim that the additive identity in V^S is the function $0 : S \to V$ given by $0(x) = 0$ for any $x \in S$; the 0 on the right-hand side is the additive identity in V. Indeed, for any $f \in V^S$ and $x \in S$ we have

$$(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x).$$

It follows that $f + 0 = f$.

Additive inverse. For $f \in V^S$, define $g : S \to V$ by $g(x) = -f(x)$ for $x \in S$, where $-f(x)$ is the additive inverse in V of $f(x)$. We claim that g is the additive inverse of f. To see this, let $x \in S$ be given and observe that

$$(f + g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0 = 0(x);$$

it follows that $f + g = 0$.

Multiplicative identity. Let $f \in V^S$ and $x \in S$ be given. Observe that

$$(1f)(x) = 1f(x) = f(x),$$

where we have used that $1v = v$ for any $v \in V$. It follows that $1f = f$.

Distributive properties. Let $a \in \mathbf{F}$ and $f, g \in V^S$ be given. Observe that, for any $x \in S$,

$$\begin{aligned}(a(f + g))(x) &= a(f + g)(x) = a((f(x) + g(x)) \\ &= af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x),\end{aligned}$$

where we have used the first distributive property in V for the third equality. It follows that $a(f + g) = af + ag$. Similarly, let $a, b \in \mathbf{F}$ and $f \in V^S$ be given. For any $x \in S$, observe that

$$((a + b)f)(x) = (a + b)f(x) = af(x) + bf(x) = (af)(x) + (bf)(x) = (af + bf)(x),$$

where we have used the second distributive property in V for the second equality. It follows that $(a + b)f = af + bf$.

We may conclude that V^S is a vector space over $\mathbf{F}$.

Exercise 1.B.8. Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

  $$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

  for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

  $$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

  for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

> *Think of V as a subset of $V_{\mathbf{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbf{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.*

Solution. We need to verify each condition in definition (1.20).

Commutativity. The proof for commutativity is essentially the same as in Exercise 1.A.1, except instead of using the commutativity of addition in $\mathbf{R}$, we are using the commutativity of addition in V.

Associativity. The proofs for associativity are essentially the same as in Exercise 1.A.2 and Exercise 1.A.3, except instead of using the algebraic properties of $\mathbf{R}$, we are using the algebraic properties of V.

Additive identity. We claim that the additive identity in $V_{\mathbf{C}}$ is $0 + i0$, where 0 is the additive identity in V. Indeed, for any $u + iv \in V_{\mathbf{C}}$ we have

$$(u + iv) + (0 + i0) = (u + 0) + i(v + 0) = u + iv.$$

Additive inverse. We claim that the additive inverse of an element $u + iv \in V_{\mathbf{C}}$ is the element $(-u) + i(-v)$, where $-u$ is the additive inverse of u in V. Indeed,

$$(u + iv) + ((-u) + i(-v)) = (u + (-u)) + i(v + (-v)) = 0 + i0.$$

Multiplicative identity. For any $u + iv \in V_{\mathbf{C}}$, we have

$$(1 + 0i)(u + iv) = (1u - 0v) + i(1v + 0u) = u + iv.$$

Distributive properties. The proof for the first distributive property is essentially the same as in Exercise 1.A.4, except instead of using distributivity in $\mathbf{R}$, we are using the first distributive property of V. For the second distributive property, let $a + bi, c + di \in \mathbf{C}$ and $u + iv \in V_{\mathbf{C}}$ be given. Observe that

$$\begin{aligned}((a + bi) + (c + di))(u + iv) &= ((a + c) + (b + d)i)(u + iv) \\ &= ((a + c)u - (b + d)v) + i((a + c)v + (b + d)u) \\ &= (au + cu - bv - dv) + i(av + cv + bu + du) \\ &= ((au - bv) + i(av + bu)) + ((cu - dv) + i(cv + du)) \\ &= (a + bi)(u + iv) + (c + di)(u + iv),\end{aligned}$$

where we have used the second distributive property for V for the third equality.