1 Section 5.A Exercises

Exercises with solutions from Section 5.A of [LADR].

Exercise 5.A.1. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subset \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subset U$, then U is invariant under T.

Solution. (a) Suppose $u \in U \subseteq \operatorname{null} T$. Then $Tu = 0 \in U$ and thus U is invariant under T.

(b) Suppose $u \in U$. Then $Tu \in \operatorname{range} T \subseteq U$ and thus U is invariant under T.

Exercise 5.A.2. Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null} S$ is invariant under T.

Solution. Suppose $u \in \operatorname{null} S$. Then

$$S(Tu) = T(Su) = T(0) = 0,$$

so that $Tu \in \operatorname{null} S$ and thus $\operatorname{null} S$ is invariant under T.

Exercise 5.A.3. Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{range} S$ is invariant under T.

Solution. Suppose $Su \in \operatorname{range} S$ for some $u \in V$. Then

$$T(Su) = S(Tu) \in \operatorname{range} S$$

and thus $\operatorname{range} S$ is invariant under T.

Exercise 5.A.4. Suppose $T \in \mathcal{L}(V)$ and $U_1, \dots, U_m$ are subspaces of V invariant under T. Prove that $U_1 + \cdots + U_m$ is invariant under T.

Solution. Suppose $u_1 + \cdots + u_m \in U_1 + \cdots + U_m$, where each $u_j \in U_j$. Then

$$T(u_1 + \cdots + u_m) = Tu_1 + \cdots + Tu_m,$$

which belongs to $U_1 + \cdots + U_m$ since each $Tu_j \in U_j$. Thus $U_1 + \cdots + U_m$ is invariant under T.

Exercise 5.A.5. Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

Solution. Let $\mathscr{U}$ be a collection of subspaces of V invariant under T and let $W = \bigcap_{U \in \mathscr{U}} U$. Suppose $w \in W$. For each $U \in \mathscr{U}$, we have $w \in U$; since each U is invariant under T, it follows that Tw belongs to each U. Thus $Tw \in W$ and we see that W is invariant under T.

Exercise 5.A.6. Prove or give a counterexample: if V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Solution. This is true. It will suffice to show that if $U \neq \{0\}$, then $U = V$. Suppose therefore that there exists some $v_1 \in U$ with $v_1 \neq 0$. We can then extend this to a basis $v_1, \ldots, v_m$ of V. For each $1 \leq j \leq m$, define an operator $T_j : V \to V$ by $T_j v_1 = v_j$ and $T_j v_i = 0$ for $i \neq 1$. Then by assumption U is invariant under T_j, so we must have $T_j v_1 = v_j \in U$. Thus U contains the basis $v_1, \ldots, v_m$ of V and hence $U = V$.

Exercise 5.A.7. Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Solution. We can observe that T is a counterclockwise rotation by $90°$ about the origin followed by a dilation of the x-axis by a factor of 3. A similar argument to Example 5.8 (a) then shows that T has no eigenvalues.

Alternatively, for $\lambda \in \mathbf{R}$ we can try to solve the equation $T(x, y) = (-3y, x) = (\lambda x, \lambda y)$. Substituting $x = \lambda y$ into $-3y = \lambda x$ gives us $-3y = \lambda^2 y$. Since $y = 0$ implies that $x = 0$, and eigenvectors are non-zero, we may assume that $y \neq 0$ and thus obtain the equation $\lambda^2 + 3 = 0$. Since this has no real solutions, we see that T has no eigenvalues.

Exercise 5.A.8. Define $T \in \mathcal{L}(\mathbf{F}^2)$ by

$$T(w, z) = (z, w).$$

Find all eigenvalues and eigenvectors of T.

Solution. T is a reflection in the line $z = w$. An appeal to our geometric intuition suggests that 1 is an eigenvalue with corresponding eigenvector $(1, 1)$ and that -1 is an eigenvalue with corresponding eigenvector $(-1, 1)$. To see this algebraically, suppose $\lambda \in \mathbf{F}$ and $(w, z) \neq (0, 0)$ are such that $T(w, z) = (z, w) = (\lambda w, \lambda z)$. Substituting $z = \lambda w$ into $w = \lambda z$ gives us $w = \lambda^2 w$. Since $w = 0$ implies that $z = 0$, and eigenvectors are non-zero, we may assume that $w \neq 0$ and thus obtain the equation $\lambda^2 - 1 = 0$, which has solutions $\lambda = \pm 1$. These are both eigenvalues, since

$$T(1, 1) = (1, 1) \quad \text{and} \quad T(-1, 1) = (1, -1) = -(-1, 1).$$

Since $\dim \mathbf{F}^2 = 2$, 5.10 implies that there are no other eigenvectors of T linearly independent from these two. We may conclude that the eigenvalues and eigenvectors of T are:

eigenvalue	corresponding eigenvectors
1	(z, z) for $z \in \mathbf{F} \setminus \{0\}$
-1	$(-z, z)$ for $z \in \mathbf{F} \setminus \{0\}$

Exercise 5.A.9. Define $T \in \mathcal{L}(\mathbf{F}^3)$ by

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3).$$

Find all eigenvalues and eigenvectors of T.

Solution. T can be thought of as the composition of the following transformations:

1. a projection onto the z_2z_3-plane;
2. a clockwise rotation of 90° around the z_3-axis; after the projection onto the z_2z_3-plane, this is equivalent to a reflection in the plane $z_1 = z_2$;
3. a dilation of the z_1-axis by a factor of 2;
4. a dilation of the z_3-axis by a factor of 5.

In other words, T maps $(z_1, z_2, z_3) \in \mathbf{F}^3$ like so:

$$(z_1, z_2, z_3) \mapsto (0, z_2, z_3) \mapsto (z_2, 0, z_3) \mapsto (2z_2, 0, z_3) \mapsto (2z_2, 0, 5z_3).$$

An appeal to our geometric intuition suggests that 5 is an eigenvalue with corresponding eigenvector $(0, 0, 1)$ and that 0 is an eigenvector with corresponding eigenvector $(1, 0, 0)$. To prove this, suppose that $\lambda \in \mathbf{F}$ and $(z_1, z_2, z_3) \neq (0, 0, 0)$ are such that

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = (\lambda z_1, \lambda z_2, \lambda z_3).$$

If $\lambda \neq 0$, then the equation $\lambda z_2 = 0$ implies that $z_2 = 0$ and thus the equation $2z_2 = \lambda z_1$ implies that $z_1 = 0$. Since eigenvectors are non-zero, it must be the case that $z_3 \neq 0$ and so the equation $5z_3 = \lambda z_3$ gives us $\lambda = 5$. So the only possible eigenvalues are 0 and 5, which are indeed eigenvalues since

$$T(0, 0, 1) = (0, 0, 5) = 5(0, 0, 1) \quad \text{and} \quad T(1, 0, 0) = (0, 0, 0) = 0(1, 0, 0).$$

We claim that there are no other eigenvectors of T linearly independent from these two. First, we consider the eigenvalue 5. As we just showed, any eigenvector corresponding to this eigenvalue must satisfy $z_1 = z_2 = 0$ and thus each such eigenvector is a scalar multiple of $(0, 0, 1)$. Next, we consider the eigenvalue 0; this is equivalent to considering the nullspace of T. It is straightforward to verify that $(1, 0, 0)$ is a basis for $\operatorname{null} T$ and so we may conclude that the eigenvalues and eigenvectors of T are:

eigenvalue	corresponding eigenvectors
5	$(0,0,z)$ for $z \in \mathbf{F} \setminus \{0\}$
0	$(z,0,0)$ for $z \in \mathbf{F} \setminus \{0\}$

Exercise 5.A.10. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by

$$T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n).$$

(a) Find all eigenvalues and eigenvectors of T.

(b) Find all invariant subspaces of T.

Solution. (a) Letting e_j be the j^{th} standard basis vector of $\mathbf{F}^n$, we notice that for each $1 \le j \le n$

$$Te_j = je_j.$$

Thus j is an eigenvalue of T with corresponding eigenvector e_j. By 5.10 and 5.13, we can be sure that these are all of the eigenvalues and eigenvectors of T, i.e.

eigenvalue	corresponding eigenvectors
1	$(z,0,\ldots,0)$ for $z \in \mathbf{F} \setminus \{0\}$
2	$(0,z,\ldots,0)$ for $z \in \mathbf{F} \setminus \{0\}$
$\vdots$	$\vdots$
n	$(0,0,\ldots,z)$ for $z \in \mathbf{F} \setminus \{0\}$

(b) First, let us prove some useful results.

Lemma 1. Suppose $T : V \to V$ is a linear operator and U is a subspace of V invariant under T. If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of T with corresponding eigenvectors $v_1, \ldots, v_k$, then

$$v_1 + \cdots + v_k \in U \iff v_j \in U \text{ for each } 1 \le j \le k.$$

Proof. We will prove this by induction on k. The base case $k = 1$ is clear, so suppose the result is true for some k and suppose we have distinct eigenvalues $\lambda_1, \ldots, \lambda_{k+1}$ of T with corresponding eigenvectors $v_1, \ldots, v_{k+1}$. If $v_j \in U$ for each $1 \le j \le k+1$, then $v_1 + \cdots + v_{k+1} \in U$ since U is a subspace of V. Suppose that $v := v_1 + \cdots + v_{k+1} \in U$. Since U is invariant under T we have

$$Tv = \lambda_1 v_1 + \cdots + \lambda_{k+1} v_{k+1} \in U.$$

This gives us
$$Tv - \lambda_{k+1}v = (\lambda_1 - \lambda_{k+1})v_1 + \cdots + (\lambda_k - \lambda_{k+1})v_k \in U.$$
By assumption, the eigenvalues $\lambda_1, \ldots, \lambda_{k+1}$ are distinct and so for each $1 \leq j \leq k$ we have $\lambda_j - \lambda_{k+1} \neq 0$. It follows that each $(\lambda_j - \lambda_{k+1})v_j$ is an eigenvector of T corresponding to the eigenvalue λ_j. Our induction hypothesis then guarantees that each $(\lambda_j - \lambda_{k+1})v_j$ belongs to the subspace U and thus each v_j belongs to U, which gives us
$$v_{k+1} = v - (v_1 + \cdots + v_k) \in U.$$
This completes the induction step and the proof. □

Lemma 2. Suppose $T : V \to V$ is a linear operator with $\dim V = n$ and $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of T with corresponding eigenvectors $v_1, \ldots, v_n$, so that
$$V = E_1 \oplus \cdots \oplus E_n,$$
where $E_j = \operatorname{span}(v_j)$. If U is a subspace of V invariant under T, then
$$U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_n).$$

Proof. Since $V = E_1 \oplus \cdots \oplus E_n$, for any $u \in U$ we have $u = e_1 + \cdots + e_n$, where each $e_j \in E_j$. If any $e_j = 0$ then certainly $e_j \in U$; otherwise, e_j is an eigenvector of T corresponding to the eigenvalue λ_j and so Lemma 1 implies that the non-zero e_j's belong to U also. It follows that $u \in (U \cap E_1) + \cdots + (U \cap E_n)$ and hence that
$$U = (U \cap E_1) + \cdots + (U \cap E_n).$$
The directness of this sum follows immediately from the directness of the sum $V = E_1 \oplus \cdots \oplus E_n$. □

Theorem 1. Suppose $T : V \to V$ is a linear operator with $\dim V = n$ and $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of T with corresponding eigenvectors $v_1, \ldots, v_n$, so that
$$V = E_1 \oplus \cdots \oplus E_n,$$
where $E_j = \operatorname{span}(v_j)$. Then the subspaces of V which are invariant under T are precisely those of the form
$$E_{j_1,\ldots,j_k} := E_{j_1} \oplus \cdots \oplus E_{j_k} = \operatorname{span}(v_{j_1}, \ldots, v_{j_k}),$$
where $1 \leq j_1 < \cdots < j_k \leq n$ are positive integers and $0 \leq k \leq n$; when $k = 0$ define $E_0 := \{0\}$.

Proof. It is straightforward to verify that each $E_{j_1,\dots,j_k}$ is indeed a subspace of V invariant under T. To see that each such subspace is of this form, let U be a subspace of V invariant under T. By Lemma 2, we have

$$U = (U \cap E_1) \oplus \cdots \oplus (U \cap E_n).$$

For each j, since $\dim E_j = 1$, we can either have $U \cap E_j = \{0\}$ or $U \cap E_j = E_j$. If each $U \cap E_j = \{0\}$, then $U = \{0\} = E_0$. Otherwise, let $1 \leq j_1 < \cdots < j_k \leq n$ be those indices for which $U \cap E_j = E_j$. Then U is nothing but $E_{j_1,\dots,j_k}$. □

Now let us return to the exercise. As we showed in part (a), the eigenvalues of T are $1, 2, 3, \dots, n$ with corresponding eigenvectors $e_1, e_2, e_3, \dots, e_n$, where e_j is the j^{th} standard basis vector of $\mathbf{F}^n$. We can now appeal to Theorem 1 above to obtain all subspaces of $\mathbf{F}^n$ invariant under T.

Exercise 5.A.11. Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

Solution. We notice that for $p_0(x) = 1$,

$$Tp_0 = 0 = 0p_0.$$

Thus 0 is an eigenvalue of T with corresponding eigenvector $p_0(x) = 1$. Moreover, the only polynomials whose derivative is zero are the constant polynomials.

Suppose $p \in \mathcal{P}(\mathbf{R})$ satisfies $\deg p \geq 1$. If $\lambda \neq 0$, then $\deg(\lambda p) = \deg p$, whereas $\deg p' = \deg p - 1$. Thus we cannot have $Tp = p' = \lambda p$ and we may conclude that the eigenvalues and eigenvectors of T are:

eigenvalue	corresponding eigenvectors
0	$\alpha \in \mathbf{F} \setminus \{0\}$

Exercise 5.A.12. Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by

$$(Tp)(x) = xp'(x)$$

for all $x \in \mathbf{R}$. Find all eigenvalues and eigenvectors of T.

Solution. Letting $p_j \in \mathcal{P}_4(\mathbf{R})$ be given by $p_j(x) = x^j$ for $0 \leq j \leq 4$, we notice that

$$(Tp_j)(x) = jx^j = jp_j(x).$$

By 5.10 and 5.13, we may conclude that the eigenvalues and eigenvectors of T are:

eigenvalue	corresponding eigenvectors
0	αp_0 for $\alpha \in \mathbf{F} \setminus \{0\}$
1	αp_1 for $\alpha \in \mathbf{F} \setminus \{0\}$
2	αp_2 for $\alpha \in \mathbf{F} \setminus \{0\}$
3	αp_3 for $\alpha \in \mathbf{F} \setminus \{0\}$
4	αp_4 for $\alpha \in \mathbf{F} \setminus \{0\}$

Exercise 5.A.13. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Prove that there exists $\alpha \in \mathbf{F}$ such that $|\alpha - \lambda| < \frac{1}{1000}$ and $T - \alpha I$ is invertible.

Solution. Seeking a contradiction, suppose that for all $\alpha \in \mathbf{F}$ such that $|\alpha - \lambda| < \frac{1}{1000}$, the operator $T - \alpha I$ is not invertible. By 5.6 each such α, of which there are infinitely many, must be an eigenvalue of T; but this contradicts 5.13, which says that since V is finite-dimensional, T can have at most $\dim V$ eigenvalues.

Exercise 5.A.14. Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of P.

Solution. We notice that $Pu = u$ for any $u \in U$. Since $U \neq \{0\}$, it follows that 1 is an eigenvalue of P with corresponding eigenvectors $u \in U \setminus \{0\}$. Similarly, we notice that $Pw = 0$ for any $w \in W$. Since $W \neq \{0\}$, it follows that 0 is an eigenvalue of P with corresponding eigenvectors $w \in W \setminus \{0\}$.

We claim that 1 and 0 are the only eigenvalues of P. To see this, suppose $\lambda \in \mathbf{F}$ and $u + w \neq 0$ are such that

$$P(u + w) = u = \lambda u + \lambda w \iff (1 - \lambda)u = \lambda w \in U \cap W.$$

Since the sum $V = U \oplus W$ is direct, we have $U \cap W = \{0\}$. It follows that $(1 - \lambda)u = \lambda w = 0$. If $\lambda \neq 1$ and $\lambda \neq 0$, then this equation can only be satisfied by $u = w = 0$; but then $u + w = 0$ is not an eigenvector.

Now we claim that the only eigenvectors corresponding to the eigenvalue 1 are those of the form $u \in U \setminus \{0\}$. Indeed, if $v \neq 0$ satisfies $v \notin U$, then we must have $v = u + w$ with $w \neq 0$. It follows that $Pv = P(u + w) = u \neq u + w$ since w is non-zero.

Similarly, the only eigenvectors corresponding to the eigenvalue 0 are those of the form $w \in W \setminus \{0\}$.; it is straightforward to verify that the nullspace of P is exactly W. We may conclude that the eigenvalues and eigenvectors of T are:

eigenvalue	corresponding eigenvectors
1	$u \in U \setminus \{0\}$
0	$w \in W \setminus \{0\}$

Exercise 5.A.15. Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Solution. (a) Suppose $\lambda \in \mathbf{F}$ is an eigenvalue of T with a corresponding eigenvector $v \in V$. Since S is surjective, there is a $u \in V$ such that $v = Su$. Since $v \neq 0$, it must be the case that $u \neq 0$. Furthermore,

$$Tv = \lambda v \iff (TS)(u) = \lambda Su \iff (S^{-1}TS)(u) = \lambda u.$$

Thus λ is an eigenvalue of $S^{-1}TS$ with a corresponding eigenvector u.

Similarly, suppose $\lambda \in \mathbf{F}$ is an eigenvalue of $S^{-1}TS$ with a corresponding eigenvector $u \in V$. Since S^{-1} is surjective, there is a $v \in V$ such that $u = S^{-1}v$. Since $u \neq 0$, it must be the case that $v \neq 0$. Furthermore,

$$(S^{-1}TS)(u) = \lambda u \iff (S^{-1}T)(v) = \lambda S^{-1}v \iff Tv = \lambda v.$$

Thus λ is an eigenvalue of T with a corresponding eigenvector v.

(b) Let λ be an eigenvalue of T; as we showed in part (a), this is the case if and only if λ is an eigenvalue of $S^{-1}TS$. Define

$$E(T;\lambda) = \{v \in V : v \neq 0 \text{ and } Tv = \lambda v\},$$
$$E(S^{-1}TS;\lambda) = \{u \in V : u \neq 0 \text{ and } (S^{-1}TS)u = \lambda u\}.$$

Then by part (a), we have

$$E(S^{-1}TS;\lambda) = \{S^{-1}v : v \in E(T;\lambda)\} \quad \text{and} \quad E(T;\lambda) = \{Su : u \in E(S^{-1}TS;\lambda)\}.$$

Exercise 5.A.16. Suppose V is a complex vector space, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V contains only real entries. Show that if λ is an eigenvalue of T, then so is $\overline{\lambda}$.

Solution. Let $v_1, \ldots, v_n$ be a basis of V such that the matrix of T with respect to this basis contains only real entries, i.e. if this matrix has entries $A_{i,j}$, then each $A_{i,j} \in \mathbf{R}$. Suppose that $\lambda \in \mathbf{C}$ is an eigenvalue of T with a corresponding eigenvector $x = \sum_{i=1}^{n} x_i v_i \in V$, so that

$$Tx = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x_j A_{i,j} \right) v_i = \sum_{i=1}^{n} \lambda x_i v_i = \lambda x.$$

By unique representation, for each $1 \leq i \leq n$ we then have

$$\sum_{j=1}^{n} x_j A_{i,j} = \lambda x_i \iff \overline{\sum_{j=1}^{n} x_j A_{i,j}} = \overline{\lambda x_i} \iff \sum_{j=1}^{n} \overline{x_j} A_{i,j} = \overline{\lambda}\,\overline{x_i},$$

where we have used that $\overline{A_{i,j}} = A_{i,j}$ since each $A_{i,j} \in \mathbf{R}$. Define $\overline{x} = \sum_{i=1}^{n} \overline{x_i} v_i$ and note that $\overline{x} \neq 0$ since $x \neq 0$. Furthermore,

$$T\overline{x} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \overline{x_j} A_{i,j} \right) v_i = \sum_{i=1}^{n} \overline{\lambda}\,\overline{x_i} v_i = \overline{\lambda}\,\overline{x},$$

demonstrating that $\overline{\lambda}$ is an eigenvector of T with a corresponding eigenvector $\overline{x}$.

Exercise 5.A.17. Give an example of an operator $T \in \mathcal{L}(\mathbf{R}^4)$ such that T has no (real) eigenvalues.

Solution. Define $T : \mathbf{R}^4 \to \mathbf{R}^4$ by $T(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, -x_1)$ and suppose λ is such that

$$T(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, -x_1) = \lambda(x_1, x_2, x_3, x_4).$$

We then have

$$-x_1 = \lambda x_4 = \lambda^2 x_3 = \lambda^3 x_2 = \lambda^4 x_1.$$

Since

$$x_1 = 0 \implies x_2 = 0 \implies x_3 = 0 \implies x_4 = 0$$

and we are looking for eigenvectors, we may assume that $x_1 \neq 0$ and arrive at the equation $\lambda^4 + 1 = 0$, which has no real solutions. It follows that T has no real eigenvalues.

Exercise 5.A.18. Show that the operator $T \in \mathcal{L}(\mathbf{C}^\infty)$ defined by

$$T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

Solution. We are looking for solutions to the equation

$$(0, z_1, z_2, \ldots) = (\lambda z_1, \lambda z_2, \lambda z_3, \ldots).$$

where $(z_1, z_2, \ldots) \neq 0$ and $\lambda \in \mathbf{C}$. If $\lambda = 0$, then $z_1 = z_2 = \cdots = 0$ and so we may assume that $\lambda \neq 0$. From the equation $0 = \lambda z_1$ we can then deduce that $z_1 = 0$, which in turn gives us the equation $0 = \lambda z_2$, which similarly implies that $z_2 = 0$, and so on. Since both assumptions $\lambda = 0$ and $\lambda \neq 0$ imply that $(z_1, z_2, \ldots) = 0$, we may conclude that T has no eigenvalues.

Exercise 5.A.19. Suppose n is a positive integer and $T \in \mathcal{L}(\mathbf{F}^n)$ is defined by

$$T(x_1, \dots, x_n) = (x_1 + \dots + x_n, \dots, x_1 + \dots + x_n);$$

in other words, T is the operator whose matrix (with respect to the standard basis) consists of all 1's. Find all eigenvalues and eigenvectors of T.

Solution. If $n = 1$ then T is the identity operator on $\mathbf{F}$, whose only eigenvalue is 1 with corresponding eigenvectors $x \in \mathbf{F} \setminus \{0\}$.

Suppose $n \geq 2$ and let $e_1, \dots, e_n$ be the standard basis of $\mathbf{F}^n$. Then

$$\begin{aligned}\operatorname{null} T = \{(-(x_2 + \dots + x_n), x_2, \dots, x_n) \in \mathbf{F}^n : x_2, \dots, x_n \in \mathbf{F}\} \\ = \operatorname{span}(e_2 - e_1, e_3 - e_1, \dots, e_n - e_1),\end{aligned}$$

$$\operatorname{range} T = \operatorname{span}(e_1 + \dots + e_n) = \operatorname{span}((1, 1, \dots, 1)).$$

Thus 0 is an eigenvalue of T with corresponding eigenvectors $x \in \operatorname{null} T \setminus \{0\}$ and n is an eigenvalue of T with corresponding eigenvectors $x \in \operatorname{span}(e_1 + \dots + e_n) \setminus \{0\}$, since

$$T(1, 1, \dots, 1) = (n, n, \dots, n) = n(1, 1, \dots, 1).$$

We claim that these are the only eigenvalues of T. Indeed, if $x \neq 0$ and $\lambda \neq 0$ are such that $Tx = \lambda x$, then since $\operatorname{range} T = \operatorname{span}((1, \dots, 1))$, there must exist some $\alpha \in \mathbf{F}$ such that

$$Tx = \lambda x = \alpha(1, \dots, 1) \implies x = \lambda^{-1}\alpha(1, \dots, 1).$$

Thus the eigenvector x, which corresponds to the eigenvalue λ, and the eigenvector $(1, \dots, 1)$, which corresponds to the eigenvalue n, are linearly dependent. By the contrapositive of 5.10, it must be the case that $\lambda = n$.

Since $\dim \operatorname{null} T + \dim \operatorname{range} T = \dim V$, 5.10 allows us to conclude that

eigenvalue	corresponding eigenvectors
0	$x \in \operatorname{null} T \setminus \{0\}$
n	$\alpha(1, \dots, 1)$ for $\alpha \in \mathbf{F} \setminus \{0\}$

Exercise 5.A.20. Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by

$$T(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots).$$

Solution. Observe that for any $\lambda \in \mathbf{F}$, we have $(1, \lambda, \lambda^2, \lambda^3, \ldots) \neq 0$ and

$$T(1, \lambda, \lambda^2, \lambda^3, \ldots) = (\lambda, \lambda^2, \lambda^3, \lambda^4, \ldots) = \lambda(1, \lambda, \lambda^2, \lambda^3, \ldots).$$

It follows that each $\lambda \in \mathbf{F}$ is an eigenvalue of T with a corresponding eigenvector $(1, \lambda, \lambda^2, \lambda^3, \ldots)$.

Fix $\lambda \in \mathbf{F}$. We claim that if $z = (z_1, z_2, z_3, \ldots)$ is an eigenvector of T corresponding to λ, then $z = \alpha v$, where $\alpha \in \mathbf{F} \setminus \{0\}$ and $v = (1, \lambda, \lambda^2, \lambda^3, \ldots)$. Indeed,

$$Tz = (z_2, z_3, z_4, \ldots) = (\lambda z_1, \lambda z_2, \lambda z_3, \ldots) = \lambda z$$

implies that $z_2 = \lambda z_1$, which gives $z_3 = \lambda z_2 = \lambda^2 z_1$, and so on. In general, $z_n = \lambda^{n-1} z_1$ for each positive integer n, i.e.

$$z = (z_1, \lambda z_1, \lambda^2 z_1, \ldots) = z_1(1, \lambda, \lambda^2, \ldots) = z_1 v.$$

Note that we must have $z_1 \neq 0$ since z is an eigenvector of T. We may conclude that

eigenvalue	corresponding eigenvectors
$\lambda \in \mathbf{F}$	$\alpha(1, \lambda, \lambda^2, \ldots)$ for $\alpha \in \mathbf{F} \setminus \{0\}$

Exercise 5.A.21. Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

Solution. (a) For $\lambda \neq 0$ and $v \neq 0$, we have

$$Tv = \lambda v \iff v = \lambda T^{-1} v \iff \lambda^{-1} v = T^{-1} v.$$

(b) See part (a).

Exercise 5.A.22. Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors v and w in V such that

$$Tv = 3w \quad \text{and} \quad Tw = 3v.$$

Prove that 3 or -3 is an eigenvalue of T.

Solution. Applying T to both sides of the equation $Tv = 3w$ shows that $T^2 v = 9v$ or equivalently that $(T^2 - 9I)(v) = 0$. Since $v \neq 0$, this demonstrates that the operator $T^2 - 9I = (T - 3I)(T + 3I)$ is not injective. It must then be the case that at least one of the operators $T - 3I$ and $T + 3I$ is not injective and thus 3 or -3 is an eigenvalue of T.

Exercise 5.A.23. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Solution. Suppose that 0 is an eigenvalue of ST. It must then be the case that ST is not invertible (5.6), hence TS is not invertible (Exercise 3.D.9; here we use that V is finite-dimensional), and hence 0 is an eigenvalue of TS (5.6 again). By symmetry, we see that 0 is an eigenvalue of ST if and only if 0 is an eigenvalue of TS.

Let us now consider non-zero eigenvalues. Suppose that $\lambda \neq 0$ and $v \neq 0$ are such that $(ST)(v) = \lambda v$. Note that we must have $Tv \neq 0$, otherwise this equation becomes $0 = \lambda v$, which cannot be the case if $\lambda \neq 0$ and $v \neq 0$. Applying T to both sides of the equation $(ST)(v) = \lambda v$ gives us $(TS)(Tv) = \lambda(Tv)$ and thus λ is also an eigenvalue of TS with a corresponding eigenvector Tv. By symmetry, we see that $\lambda \neq 0$ is an eigenvalue of ST if and only if λ is an eigenvalue of TS.

Exercise 5.A.24. Suppose A is an n-by-n matrix with entries in $\mathbf{F}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = Ax$, where elements of $\mathbf{F}^n$ are thought of as n-by-1 column vectors.

(a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of the entries in each column of A equals 1. Prove that 1 is an eigenvalue of T.

Solution. (a) Let $A_{i,j}$ be the entries of A; our assumption is that $\sum_{j=1}^{n} A_{i,j} = 1$ for each $1 \leq i \leq n$. Observe that

$$T(1, \dots, 1) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} A_{1,j} \\ \vdots \\ \sum_{j=1}^{n} A_{n,j} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Thus 1 is an eigenvalue of T with a corresponding eigenvector $(1, \dots, 1)$.

(b) Let $e_1, \dots, e_n$ be the standard basis of $\mathbf{F}^n$ and let $\psi : \mathbf{F}^n \to \mathbf{F}$ be the linear functional given by $\psi(e_j) = 1$. Clearly, the matrix of T with respect to the standard basis of $\mathbf{F}^n$ is A. It follows that

$$(\psi(T - I))(e_j) = \psi\left(\left(\sum_{i=1}^{n} A_{i,j} e_i\right) - e_j\right) = \left(\sum_{i=1}^{n} A_{i,j}\right) - 1 = 0,$$

where we have used our assumption that the sum of the entries in each column of A equals 1. So $\psi \circ (T - I) : \mathbf{F}^n \to \mathbf{F}$ is the zero map; if the operator $T - I$ were invertible then it would have to be the case that ψ was zero. Since ψ is non-zero, we see that $T - I$ is not invertible and hence 1 is an eigenvalue of T (5.6).

Exercise 5.A.25. Suppose $T \in \mathcal{L}(V)$ and u, v are eigenvectors of T such that $u+v$ is also an eigenvector of T. Prove that u and v are eigenvectors of T corresponding to the same eigenvalue.

Solution. Suppose u, v, and $u+v$ are eigenvectors corresponding to the eigenvalues λ, μ, and γ respectively. Since the list $u, v, u+v$ is linearly dependent, the contrapositive of 5.10 shows that the eigenvalues λ, μ, and γ must not be distinct, i.e. at least two of them are equal. In fact, all three must be equal: if $\lambda = \mu$ then

$$\lambda(u+v) = \lambda u + \lambda v = \lambda u + \mu v = Tu + Tv = T(u+v) = \gamma(u+v)$$

and thus $\lambda = \mu = \gamma$ since $u + v \neq 0$; if $\lambda = \gamma$ then

$$\lambda u + \lambda v = \lambda(u+v) = \gamma(u+v) = T(u+v) = Tu + Tv = \lambda u + \mu v \implies \lambda v = \mu v,$$

and thus $\lambda = \mu = \gamma$ since $v \neq 0$; similarly, $\mu = \gamma$ implies $\lambda = \mu = \gamma$.

Exercise 5.A.26. Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Solution. The case where $V = \{0\}$ is easily handled, so assume that $V \neq \{0\}$. Fix some non-zero $u \in V$; by assumption we have $Tu = \lambda u$ for some $\lambda \in \mathbf{F}$. Suppose $v \in V$ is non-zero. If $u + v \neq 0$, then by assumption v and $u+v$ are both eigenvectors of T, so Exercise 5.A.25 implies that u and v are eigenvectors corresponding the same eigenvalue λ, so that $Tv = \lambda v$. If $v = -u$, then $Tv = -Tu = -\lambda u = \lambda v$. Thus we have $Tv = \lambda v$ for all $v \in V$, i.e. $T = \lambda I$.

Exercise 5.A.27. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ is such that every subspace of V with dimension $\dim V - 1$ is invariant under T. Prove that T is a scalar multiple of the identity operator.

Solution. If $V = \{0\}$, then $T = 0I$. If $\dim V = 1$, then $V = \operatorname{span}(v)$ for some $v \neq 0$. It follows that $Tv = \lambda v$ for some $\lambda \in \mathbf{F}$ and thus $T = \lambda I$.

Suppose that $\dim V = n \geq 2$. Let $v_1 \in V$ be non-zero and extend this to a basis $v_1, v_2, \ldots, v_n$ of V. For each $2 \leq j \leq n$, let U_j be the span of the vectors $v_1, v_2, \ldots, v_n$ except for v_j, so that

$$U_2 = \operatorname{span}(v_1, v_3, \ldots, v_n), \quad U_3 = \operatorname{span}(v_1, v_2, v_4, \ldots, v_n), \quad \text{etc.}$$

For each $2 \leq j \leq n$, the subspace U_j has dimension $n-1$ and so by assumption is invariant under T. Since v_1 belongs to U_j, we then have $Tv_1 \in U_j$. Thus

$$Tv_1 = A_{j,1}v_1 + A_{j,2}v_2 + \cdots + A_{j,j-1}v_{j-1} + 0v_j + A_{j,j+1}v_{j+1} + \cdots + A_{j,n}v_n$$

for some scalars $A_{j,1}, \dots, A_{j,j-1}, A_{j,j+1}, \dots, A_{j,n}$. We can put these scalars in a matrix:

$$\begin{pmatrix} A_{2,1} & 0 & A_{2,3} & \cdots & A_{2,n-2} & A_{2,n-1} & A_{2,n} \\ A_{3,1} & A_{3,2} & 0 & \cdots & A_{3,n-2} & A_{3,n-1} & A_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ A_{n-1,1} & A_{n-1,2} & A_{n-1,3} & \cdots & A_{n-1,n-2} & 0 & A_{n-1,n} \\ A_{n,1} & A_{n,2} & A_{n,3} & \cdots & A_{n,n-2} & A_{n,n-1} & 0 \end{pmatrix}.$$

Each row of this matrix represents the coefficients with respect to the basis $v_1, \dots, v_n$ of the same vector Tv_1 and thus by unique representation, the entries in a given column must be equal, i.e. this matrix is nothing but

$$\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

where $\lambda = A_{2,1}$. Thus $Tv_1 = \lambda v_1$, demonstrating that v_1 is an eigenvector of T corresponding to the eigenvalue λ. Since v_1 was arbitrary, we have shown that each non-zero vector in V is an eigenvector of T and so Exercise 5.A.26 allows us to conclude that T is a scalar multiple of the identity operator.

Exercise 5.A.28. Suppose V is finite-dimensional with $\dim V \geq 3$ and $T \in \mathcal{L}(V)$ is such that every 2-dimensional subspace of V is invariant under T. Prove that T is a scalar multiple of the identity operator.

Solution. The proof is similar to Exercise 5.A.27. Suppose $\dim V = n \geq 3$. Let $v_1 \in V$ be non-zero and extend this to a basis $v_1, v_2, \dots, v_n$ of V. For each $2 \leq j \leq n$, let $U_j = \operatorname{span}(v_1, v_j)$; each U_j is 2-dimensional and hence by assumption is invariant under T. Since $v_1 \in U_j$, we then have $Tv_1 \in U_j$ and thus

$$Tv_1 = A_{j,1}v_1 + 0v_2 + \cdots + 0v_{j-1} + A_{j,j}v_j + 0v_{j+1} + \cdots + 0v_n$$

for some scalars $A_{j,1}$ and $A_{j,j}$. We can put these scalars in a matrix:

$$\begin{pmatrix} A_{2,1} & A_{2,2} & 0 & \cdots & 0 & 0 & 0 \\ A_{3,1} & 0 & A_{3,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ A_{n-1,1} & 0 & 0 & \cdots & 0 & A_{n-1,n-1} & 0 \\ A_{n,1} & 0 & 0 & \cdots & 0 & 0 & A_{n,n} \end{pmatrix}.$$

(Note that this matrix has at least 2 rows since $n \geq 3$.) Each row of this matrix represents the coefficients with respect to the basis $v_1, \dots, v_n$ of the same vector Tv_1 and thus by unique representation, the entries in a given column must be equal, i.e. this matrix is nothing but

$$\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

where $\lambda = A_{2,1}$. Thus $Tv_1 = \lambda v_1$, demonstrating that v_1 is an eigenvector of T corresponding to the eigenvalue λ. Since v_1 was arbitrary, we have shown that each non-zero vector in V is an eigenvector of T and so Exercise 5.A.26 allows us to conclude that T is a scalar multiple of the identity operator.

Exercise 5.A.29. Suppose $T \in \mathcal{L}(V)$ and $\dim \operatorname{range} T = k$. Prove that T has at most $k+1$ distinct eigenvalues.

Solution. Suppose T has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding eigenvectors $v_1, \dots, v_n$, so that

$$Tv_1 = \lambda_1 v_1, \dots, Tv_n = \lambda_n v_n \in \operatorname{range} T.$$

The list $v_1, \dots, v_n$ is linearly independent (5.10) and thus the list $\lambda_1 v_1, \dots, \lambda_n v_n$ is also linearly independent, provided each eigenvalue λ_j is non-zero; if some eigenvalue $\lambda_j = 0$ (since the eigenvalues are distinct there can be at most one such λ_j), we can discard $\lambda_j v_j$ from the list and be left with a linearly independent list of $n-1$ vectors. In either case, there are at least $n-1$ linearly independent vectors in $\operatorname{range} T$ and thus $\dim \operatorname{range} T \geq n-1$.

If T has $n \geq k+2$ distinct eigenvalues, then by the previous discussion we must have $\dim \operatorname{range} T \geq k+1$. Thus $\dim \operatorname{range} T = k$ implies that T has at most $k+1$ distinct eigenvalues.

Exercise 5.A.30. Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5$, and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

Solution. T has $3 = \dim \mathbf{R}^3$ eigenvalues $-4, 5$, and $\sqrt{7}$; it follows that 9 cannot be an eigenvalue of T (5.13) and hence the operator $T - 9I$ is invertible (5.6). The desired $x \in \mathbf{R}^3$ is then $(T - 9I)^{-1}(-4, 5, \sqrt{7})$.

Exercise 5.A.31. Suppose V is finite-dimensional and $v_1, \dots, v_m$ is a list of vectors in V. Prove that $v_1, \dots, v_m$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, \dots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

Solution. Suppose $v_1, \ldots, v_m$ is linearly independent. Extend this to a basis $v_1, \ldots, v_m, w_1, \ldots, w_n$ for V and define a linear operator $T : V \to V$ by

$$Tv_1 = v_1, Tv_2 = 2v_2, \ldots, Tv_m = mv_m, \text{ and } Tw_1 = \cdots = Tw_n = 0.$$

Then T is such that $v_1, \ldots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

The converse implication is the content of 5.10.

Exercise 5.A.32. Suppose $\lambda_1, \ldots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbf{R}$.

Hint: Let $V = \operatorname{span}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$, and define an operator $T \in \mathcal{L}(V)$ by $Tf = f'$. Find eigenvalues and eigenvectors of T.

Solution. Following the hint, let $V = \operatorname{span}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$, and define an operator $T \in \mathcal{L}(V)$ by $Tf = f'$. Then for each $1 \leq j \leq n$,

$$T\left(e^{\lambda_j x}\right) = \left(e^{\lambda_j x}\right)' = \lambda_j e^{\lambda_j x}.$$

This demonstrates two things: that T really is an operator, i.e. T maps V into V, and also that λ_j is an eigenvalue of T with corresponding eigenvector $e^{\lambda_j x}$. Since the eigenvalues $\lambda_1, \ldots, \lambda_n$ are given as distinct, the corresponding eigenvectors $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ are linearly independent (5.10).

Exercise 5.A.33. Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{range} T) = 0$.

Solution. The operator $T/\operatorname{range} T : V/\operatorname{range} T \to V/\operatorname{range} T$ is defined by

$$(T/\operatorname{range} T)(v + \operatorname{range} T) = Tv + \operatorname{range} T.$$

Certainly $Tv \in \operatorname{range} T$ for any $v \in V$, so $Tv + \operatorname{range} T = 0$ by 3.85 and we see that $T/\operatorname{range} T$ is the zero map.

Exercise 5.A.34. Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{null} T)$ is injective if and only if $(\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$.

Solution. Suppose that $\operatorname{null} T \cap \operatorname{range} T = \{0\}$ and suppose that $v \in V$ is such that

$$(T/\operatorname{null} T)(v + \operatorname{null} T) = Tv + \operatorname{null} T = 0,$$

which is the case if and only if $Tv \in \operatorname{null} T$. Hence $Tv \in \operatorname{range} T \cap \operatorname{null} T$ and so by assumption we have $Tv = 0$. It follows that $v \in \operatorname{null} T$, which gives us $v + \operatorname{null} T = 0$, and we see that $T/\operatorname{null} T$ has trivial nullspace and so must be injective.

Suppose that $\operatorname{null} T \cap \operatorname{range} T \neq \{0\}$, i.e. there exists some $v \in \operatorname{null} T \cap \operatorname{range} T$ such that $v \neq 0$. Then $v = Tu$ for some $u \in V$; it must be the case that $u \notin \operatorname{null} T$ since $v \neq 0$. Thus

$$(T/\operatorname{null} T)(u + \operatorname{null} T) = Tu + \operatorname{null} T = v + \operatorname{null} T = 0,$$

where we have used that $v \in \operatorname{null} T$. Since $u \notin \operatorname{null} T$, we have $u + \operatorname{null} T \neq 0$, hence $T/\operatorname{null} T$ has non-trivial nullspace, and hence $T/\operatorname{null} T$ is not injective.

Exercise 5.A.35. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is invariant under T. Prove that each eigenvalue of T/U is an eigenvalue of T.
[*The exercise below asks you to verify that the hypothesis that V is finite-dimensional is needed for the exercise above.*]

Solution. Suppose $\lambda \in \mathbf{F}$ is an eigenvalue of T/U, i.e. there exists a non-zero $v + U \in V/U$ such that

$$(T/U)(v + U) = Tv + U = \lambda(v + U) = \lambda v + U.$$

Note that if $u \in U$, then since U is invariant under T we have $(T - \lambda I)(u) = Tu + \lambda u \in U$, i.e. U is also invariant under the operator $T - \lambda I : V \to V$. We can then consider the restriction operator $(T - \lambda I)|_U : U \to U$. There are two cases.

Case 1. Suppose that $(T - \lambda I)|_U$ fails to be surjective. By 3.69, it must be the case that $(T - \lambda I)|_U$ is not injective (here we use that V, and hence U, is finite-dimensional) and thus there exists some $u \neq 0$ such that $(T - \lambda I)|_U(u) = 0$, or equivalently $Tu = \lambda u$. Hence λ is an eigenvalue of T.

Case 2. Suppose that $(T - \lambda I)|_U$ is surjective. Since $Tv + U = \lambda v + U$, we have $Tv = \lambda v + u$ for some $u \in U$. The surjectivity of $(T - \lambda I)|_U$ implies that there exists some $u' \in U$ satisfying

$$(T - \lambda I)|_U(u') = Tu' - \lambda u' = -u.$$

Observe that

$$T(v + u') = Tv + Tu' = \lambda v + u + Tu' = \lambda v + \lambda u' = \lambda(v + u').$$

Furthermore, since $v + U$ is non-zero we must have $v \notin U$ and hence $v + u' \neq 0$. Thus λ is an eigenvalue of T.

Exercise 5.A.36. Give an example of a vector space V, an operator $T \in \mathcal{L}(V)$, and a subspace U of V that is invariant under T such that T/U has an eigenvalue that is not an eigenvalue of T.

Solution. Consider the forward-shift operator $T \in \mathcal{L}(\mathbf{C}^\infty)$ defined by

$$T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots).$$

As we showed in Exercise 5.A.18, T has no eigenvalues. Let

$$U = \operatorname{range} T = \{(0, z_2, z_3, z_4, \ldots) \in \mathbf{C}^\infty : z_j \in \mathbf{C}\}.$$

Then U is invariant under T and $T/U = 0$ by Exercise 5.A.33. Since $U \neq \mathbf{C}^\infty$, the quotient space $\mathbf{C}^\infty/U$ is not the trivial vector space and hence contains some non-zero vector $z+U$. Since $T/U = 0$, it follows that 0 is an eigenvalue of T/U with corresponding eigenvector $z+U$.

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