# 1 Q does not have the least-upper-bound property

Let $A$ be the set of rational numbers which are either negative or have square less than 2 and let $B$ be the set of positive rational numbers with square greater than 2, i.e.

$$A = \{p \in \mathbf{Q} : p < 0 \text{ or } p^2 < 2\}, \quad B = \{p \in \mathbf{Q} : p > 0 \text{ and } p^2 > 2\}.$$

Note that $A$ and $B$ partition $\mathbf{Q}$ since there is no rational number whose square is 2.

**Lemma 1.** $A$ contains no greatest element and $B$ contains no least element. That is, for any $p \in A$ there exists a $q \in A$ with $q > p$ and for any $p \in B$ there exists a $q \in B$ with $q < p$.

*Proof.* If $p \in A$ and $p \le 0$, then take $q = 1$. Otherwise, for a positive rational number $p$, define

$$q = p + \frac{2 - p^2}{p + 2} = \frac{2p + 2}{p + 2}.$$

Then

$$2 - q^2 = 2 - \frac{(2p + 2)^2}{(p + 2)^2} = \frac{2(2 - p^2)}{(p + 2)^2}.$$

For $p \in A$ with $p > 0$, we have $2 - p^2 > 0$, so that $q > p$ and $q \in A$; for $p \in B$ we have $2 - p^2 < 0$, so that $q < p$ and $q \in B$. □

**Lemma 2.** The upper bounds of $A$ are exactly the elements of $B$.

*Proof.* Suppose $r \in \mathbf{Q}$ is an upper bound for $A$. Then certainly $r$ is positive, since $1 \in A$. Furthermore, exactly one of the following is true: $r^2 < 2$, $r^2 = 2$, or $r^2 > 2$. If $r^2 < 2$, then $r \in A$; but this implies that $r$ is the greatest element of $A$, contradicting Lemma 1. So $r^2 \ge 2$ and since $r^2 = 2$ is impossible for rational $r$, we must have $r^2 > 2$, i.e. $r \in B$.

Now suppose $r \in B$ and let $p$ be any element of $A$. If $r < p$, then $r^2 < p^2 < 2$ since $r$ is positive. This contradicts $r \in B$, so in fact we must have $r \ge p$, so that $r$ is an upper bound for $A$. □

Since $A$ and $B$ partition $\mathbf{Q}$, Lemma 1 and Lemma 2 give us the following corollary.

**Corollary 3.** The lower bounds of $B$ are exactly the elements of $A$.

**Theorem 4.** The set of rational numbers $\mathbf{Q}$ does not have the least-upper-bound property.

*Proof.* $A$ is non-empty and bounded above; the upper bounds of $A$ are exactly the elements of $B$. Since $B$ has no least element, it follows that $A$ has no least upper bound. □