1 Section 8.4 Exercises

Exercises with solutions from Section 8.4 of [UA].

Exercise 8.4.1. For $n \in \mathbf{N}$, let

$$n\# = n + (n-1) + (n-2) + \cdots + 2 + 1.$$

(a) Without looking ahead, decide if there is a natural way to define $0\#$. How about $(-2)\#$? Conjecture a reasonable value for $\frac{7}{2}\#$.

(b) Now prove $n\# = \frac{1}{2}n(n+1)$ for all $n \in \mathbf{N}$, and revisit part (a).

Solution. (a) We observe that $n\#$ satisfies the relation $n\# = n + (n-1)\#$ for $n \geq 2$; it seems reasonable to use this relation to extend the definition of $\#$. Thus

$$1\# = 1 + 0\# \quad \Longrightarrow \quad 0\# = 1 - 1\# = 0.$$

Similarly,

$$0\# = (-1)\# = -1 + (-2)\# \quad \Longrightarrow \quad (-2)\# = 1.$$

Some more calculations show that

$$1\# + (-1)\# = 1, \quad 2\# + (-2)\# = 4, \quad \text{and} \quad 3\# + (-3)\# = 9.$$

Given this, we might conjecture that $n\# + (-n)\# = n^2$ for $n \in \mathbf{N}$. Using this identity and the previous recurrence relation, we find that $\frac{1}{2}\# = \frac{1}{2} + \left(-\frac{1}{2}\right)\# = \frac{3}{8}$ and thus

$$\frac{7}{2}\# = \frac{15}{2} + \frac{1}{2}\# = \frac{63}{8}.$$

(b) This is a classic result, the proof of which is likely one of the first encountered by students learning mathematical induction, and so I won't repeat it here. Another method, perhaps more satisfying, is often attributed to Gauss (whether this story is true or not is unclear; he certainly wouldn't have been the first to find this formula).

Taking $n = 0, -2$, and $\frac{7}{2}$ in this formula confirms our conjectures from part (a).

Exercise 8.4.2. Verify that the series converges absolutely for all $x \in \mathbf{R}$, that $E(x)$ is differentiable on $\mathbf{R}$, and $E'(x) = E(x)$.

Solution. For a given non-zero $x \in \mathbf{R}$, note that

$$\frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1} \to 0;$$

it follows from the Ratio Test (Exercise 2.7.9) that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely. Theorem 6.5.7 now implies that E is differentiable on $\mathbf{R}$ and furthermore that

$$E'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = E(x).$$

Exercise 8.4.3. (a) Use the results of Exercise 2.8.7 and the binomial formula to show that $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbf{R}$.

(b) Show that $E(0) = 1, E(-x) = 1/E(x)$, and $E(x) > 0$ for all $x \in \mathbf{R}$.

Solution. (a) Let $x, y \in \mathbf{R}$ be given and for each $n \geq 0$ let $a_n = \frac{y^n}{n!}$ and $b_n = \frac{x^n}{n!}$. For each $k \geq 0$, define

$$d_k = a_0 b_k + \cdots + a_k b_0 = \sum_{n=0}^{k} a_n b_{k-n} = \sum_{n=0}^{k} \frac{x^{k-n} y^n}{(k-n)!n!} = \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} x^{k-n} y^n = \frac{(x+y)^k}{k!}.$$

It follows that for each $N \geq 0$ we have

$$\sum_{k=0}^{N} d_k = \sum_{k=0}^{N} \frac{(x+y)^k}{k!}.$$

On one hand, $\sum_{k=0}^{N} \frac{(x+y)^k}{k!} \to E(x+y)$ as $N \to \infty$; on the other hand,

$$\sum_{k=0}^{N} d_k \to \left(\sum_{n=0}^{\infty} b_n \right) \left(\sum_{n=0}^{\infty} a_n \right) = E(x)E(y) \text{ as } N \to \infty$$

by Exercise 2.8.7. We may conclude that $E(x+y) = E(x)E(y)$.

(b) $E(0) = 1$ is clear from the definition of E. Taking $y = -x$ in the identity $E(x+y) = E(x)E(y)$ shows that $E(0) = 1 = E(x)E(-x)$ for all $x \in \mathbf{R}$, which implies that $E(x) \neq 0$ for all $x \in \mathbf{R}$; since E is continuous and $E(0) = 1$, we must then have $E(x) > 0$ for all $x \in \mathbf{R}$.

Exercise 8.4.4. Define $e = E(1)$. Show $E(n) = e^n$ and $E(m/n) = (\sqrt[n]{e})^m$ for all $m, n \in \mathbf{Z}$.

Solution. By Exercise 8.4.3 (a) we have, for each $n \in \mathbf{N}$,

$$E(n) = E\left(\sum_{j=1}^{n} 1\right) = \prod_{j=1}^{n} E(1) = \prod_{j=1}^{n} e = e^n,$$

and by Exercise 8.4.3 (b) we have $E(0) = 1 = e^0$. Thus the identity $E(n) = e^n$ holds for all $n \geq 0$; extending this to all $n \in \mathbf{Z}$ now follows from the identity $E(-x) = \frac{1}{E(x)}$ from Exercise 8.4.3 (b).

For $n \in \mathbf{N}$, we have

$$e = E(1) = E\left(\sum_{j=1}^{n} \frac{1}{n}\right) = \prod_{j=1}^{n} E\left(\frac{1}{n}\right) = \left[E\left(\frac{1}{n}\right)\right]^n.$$

Because $E\left(\frac{1}{n}\right)$ is positive (Exercise 8.4.3 (b)), the above equation implies that $E\left(\frac{1}{n}\right)$ is the unique positive n^{th} root of e, i.e. $E\left(\frac{1}{n}\right) = \sqrt[n]{e}$. We can now argue as in the previous paragraph to see that $E\left(\frac{m}{n}\right) = (\sqrt[n]{e})^m$ for all $m \in \mathbf{Z}$ and $n \in \mathbf{N}$.

Exercise 8.4.5. Show $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n = 0, 1, 2, \dots$.

To get started notice that when $x \geq 0$, all the terms in (1) are positive.

Solution. We will prove the more general result that $\lim_{x\to\infty} x^n e^{-yx} = 0$ for $n \geq 0$ and $y > 0$, which will be useful later. For $x > 0$, observe that $x^n e^{-yx}$ is positive. Furthermore,

$$\begin{aligned} x^{-n}e^{yx} &= x^{-n}\left(1 + yx + \cdots + \frac{y^n x^n}{n!} + \frac{y^{n+1}x^{n+1}}{(n+1)!} + \cdots\right) \\ &= \left(\frac{1}{x^n} + \frac{y}{x^{n-1}} + \cdots + \frac{y^n}{n!} + \frac{y^{n+1}x}{(n+1)!} + \cdots\right) > \frac{y^{n+1}x}{(n+1)!}. \end{aligned}$$

Let $\epsilon > 0$ be given and set $M = \frac{(n+1)!}{y^{n+1}\epsilon} > 0$. Then for $x \geq M$, we have

$$x^{-n}e^{yx} > \frac{y^{n+1}x}{(n+1)!} \geq \frac{y^{n+1}M}{(n+1)!} = \frac{1}{\epsilon} \quad \iff \quad x^n e^{-yx} < \epsilon.$$

We may conclude that $\lim_{x\to\infty} x^n e^{-yx} = 0$.

Exercise 8.4.6. (a) Explain why we know e^x has an inverse function—let's call it $\log x$—defined on the strictly positive real numbers and satisfying

(i) $\log(e^y) = y$ for all $y \in \mathbf{R}$ and

(ii) $e^{\log x} = x$, for all $x > 0$.

(b) Prove $(\log x)' = 1/x$. (See Exercise 5.2.12.)

(c) Fix $y > 0$ and differentiate $\log(xy)$ with respect to x. Conclude that

$$\log(xy) = \log x + \log y \quad \text{for all } x, y > 0.$$

(d) For $t > 0$ and $n \in \mathbf{N}, t^n$ has the usual interpretation as $t \cdot t \cdots t$ (n times). Show that

$$t^n = e^{n \log t} \quad \text{for all } n \in \mathbf{N}. \tag{2}$$

Solution. For notation, we will use either $E(x)$ or e^x depending on which is more convenient.

(a) Because $(e^x)' = e^x > 0$ (Exercise 8.4.2 and Exercise 8.4.3 (b)), we see that E is injective (Exercise 5.3.2). For any $y > 0$, we have

$$e^y = \left(1 + y + \frac{y^2}{2!} + \cdots\right) > y$$

and Exercise 8.4.5 shows that there is some $z < 0$ such that $e^z < y$; it follows from the Intermediate Value Theorem (Theorem 4.5.1) that there exists some $x \in (z, y)$ such that $e^x = y$. We have now shown that $E : \mathbf{R} \to (0, \infty)$ is a bijection and thus there exists an inverse function.

(b) By Exercise 5.2.12 and Exercise 8.4.2, we have

$$(\log x)' = \frac{1}{E'(\log x)} = \frac{1}{E(\log x)} = \frac{1}{x}.$$

(c) Using the chain rule and part (b), we have

$$(\log(xy))' = \frac{y}{xy} = \frac{1}{x} = (\log x)'.$$

It follows from Corollary 5.3.4 that $\log(xy) = \log x + k$ for some $k \in \mathbf{R}$; taking $x = 1$ shows that $k = \log y$.

(d) For a given $n \in \mathbf{N}$, the identity $\log(xy) = \log x + \log y$ from part (c) shows that $n \log t = \log(t^n)$ and thus

$$e^{n \log t} = e^{\log(t^n)} = t^n.$$

Exercise 8.4.7. (a) Show $t^{m/n} = \left(\sqrt[n]{t}\right)^m$ for all $m, n \in \mathbf{N}$.

(b) Show $\log(t^x) = x \log t$, for all $t > 0$ and $x \in \mathbf{R}$.

(c) Show t^x is differentiable on $\mathbf{R}$ and find the derivative.

Solution. For notation, we will use either $E(x)$ or e^x depending on which is more convenient.

(a) Let $n \in \mathbf{N}$ be given. By Exercise 8.4.3 (a), we have

$$\left(E\left(\frac{1}{n}\log t\right)\right)^n = \prod_{j=1}^{n} E\left(\frac{1}{n}\log t\right) = E\left(\sum_{j=1}^{n}\frac{1}{n}\log t\right) = E(\log t) = t.$$

As $E\left(\frac{1}{n}\log t\right)$ is positive, it follows from the equation above that $E\left(\frac{1}{n}\log t\right)$ is the unique positive n^{th} root of t, i.e.

$$t^{1/n} = E\left(\frac{1}{n}\log t\right) = \sqrt[n]{t}.$$

Now let $m, n \in \mathbf{N}$ be given. By Exercise 8.4.3 (a) and the previous paragraph, we have

$$t^{m/n} = E\left(\frac{m}{n}\log t\right) = \left(E\left(\frac{1}{n}\log t\right)\right)^m = \left(\sqrt[n]{t}\right)^m.$$

(b) This is immediate from the definition of t^x:

$$\log(t^x) = \log(E(x\log t)) = x\log t.$$

(c) Using the chain rule, we find that

$$(t^x)' = (E(x\log t))' = (\log t)E'(x\log t) = (\log t)E(x\log t) = (\log t)t^x.$$

Exercise 8.4.8. Inspired by the fact that $0! = 1$ and $1! = 1$, let $h(x)$ satisfy

(i) $h(x) = 1 \quad$ for all $0 \le x \le 1$, and

(ii) $h(x) = xh(x-1) \quad$ for all $x \in \mathbf{R}$.

- (a) Find a formula for $h(x)$ on $[1, 2]$, $[2, 3]$, and $[n, n+1]$ for arbitrary $n \in \mathbf{N}$.
- (b) Now do the same for $[-1, 0]$, $[-2, -1]$, and $[-n, -n+1]$.
- (c) Sketch h over the domain $[-4, 4]$.

Solution. (a) On $[1,2]$ we find that $h(x) = x$, on $[2,3]$ we find that $h(x) = x(x-1)$, and in general we obtain $h(x) = x(x-1)\cdots(x-n+1)$ on $[n, n+1]$ for $n \in \mathbf{N}$.

(b) Replacing x with $x+1$ in (ii), we see that $h(x) = \frac{h(x+1)}{x}$ for all $x \neq 0$. Using this and (i), we see that $h(x) = \frac{1}{x}$ for $x \in [-1, 0)$. Similarly, $h(x) = \frac{1}{x(x+1)}$ for $x \in [-2, -1)$ and in general $h(x) = \frac{1}{x(x+1)\cdots(x+n-1)}$ on $[-n, -n+1)$ for $n \in \mathbf{N}$.

(c) See Figure 1 for the sketch.

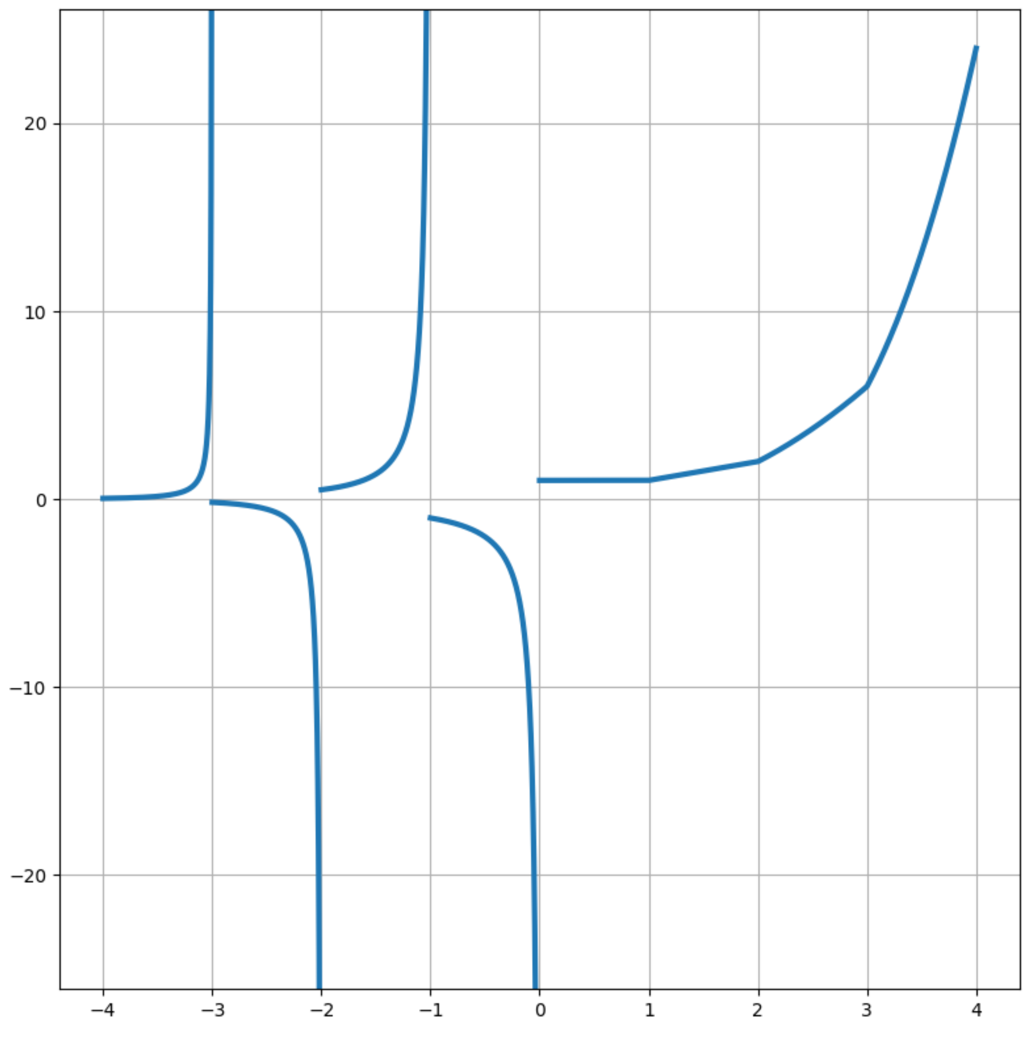

Figure 1: h on $[-4, 4]$

Exercise 8.4.9. (a) Show that the improper integral $\int_a^\infty f$ converges if and only if, for all $\epsilon > 0$ there exists $M > a$ such that whenever $d > c \geq M$ it follows that

$$\left| \int_c^d f \right| < \epsilon.$$

(In one direction it will be useful to consider the sequence $a_n = \int_a^{a+n} f$.)

(b) Show that if $0 \leq f \leq g$ and $\int_a^\infty g$ converges then $\int_a^\infty f$ converges.

(c) Part (a) is a Cauchy criterion, and part (b) is a comparison test. State and prove an absolute convergence test for improper integrals.

Solution. (a) Suppose that $\int_a^\infty f$ converges to some $L \in \mathbf{R}$ and let $\epsilon > 0$ be given. There exists an $M > a$ such that

$$b \geq M \quad \Longrightarrow \quad \left| \int_a^b f - L \right| < \frac{\epsilon}{2}.$$

It follows that for $d > c \geq M$ we have

$$\left| \int_c^d f \right| = \left| \int_a^d f - \int_a^c f - L + L \right| \leq \left| \int_a^c f - L \right| + \left| \int_a^d f - L \right| < \epsilon.$$

Now suppose that

$$\text{for all } \epsilon > 0 \text{ there exists an } M > a \text{ such that} \quad d \geq c \geq M \quad \Longrightarrow \quad \left| \int_c^d f \right| < \epsilon. \tag{$*$}$$

For each $n \in \mathbf{N}$ define $a_n = \int_a^{a+n} f$. Given an $\epsilon > 0$, obtain an M from $(*)$ and let $N \in \mathbf{N}$ be such that $a + N \geq M$. If $n \geq m \geq N$, then by $(*)$ we have

$$|a_n - a_m| = \left| \int_{a+m}^{a+n} f \right| < \epsilon.$$

Thus (a_n) is Cauchy and hence convergent, say $\lim_{n\to\infty} a_n = L$.

We claim that $\int_a^\infty f = L$. To see this, let $\epsilon > 0$ be given. By $(*)$, there is an $M > a$ such that

$$d \geq c \geq M \quad \Longrightarrow \quad \left| \int_c^d f \right| < \frac{\epsilon}{2}. \tag{$\dagger$}$$

Let $N_1 \in \mathbf{N}$ be such that $a + N_1 \geq M$. Since $\lim_{n\to\infty} a_n = L$, there is an $N_2 \in \mathbf{N}$ such that

$$n \geq N_2 \quad \Longrightarrow \quad \left| \int_a^{a+n} f - L \right| < \frac{\epsilon}{2}. \tag{$\ddagger$}$$

Let $N = \max\{N_1, N_2\}$ and suppose that $b \geq a + N$. Then by (†) and (‡) we have

$$\left|\int_a^b f - L\right| \leq \left|\int_a^{a+N} f - L\right| + \left|\int_{a+N}^b f\right| < \epsilon.$$

Our claim follows.

(b) The inequality $0 \leq f \leq g$ implies that $0 \leq \int_c^d f \leq \int_c^d g$ for any $d \geq c \geq a$. Let $\epsilon > 0$ be given. By part (a), there is an $M > a$ such that $\left|\int_c^d g\right| = \int_c^d g < \epsilon$ whenever $d \geq c \geq M$. For such d and c we then have $\left|\int_c^d f\right| = \int_c^d f \leq \int_c^d g < \epsilon$. It follows from part (a) that $\int_a^\infty f$ converges.

(c) We will show that if $\int_a^\infty |f|$ converges then so does $\int_a^\infty f$. For any $\epsilon > 0$, part (a) implies that there is an $M > a$ such that $\left|\int_c^d |f|\right| = \int_c^d |f| < \epsilon$ for any $d \geq c \geq M$. For such d and c it follows that $\left|\int_c^d f\right| \leq \int_c^d |f| < \epsilon$ and part (a) allows us to conclude that $\int_a^\infty f$ converges.

Exercise 8.4.10. (a) Use the properties of e^t previously discussed to show

$$\int_0^\infty e^{-t}\, dt = 1.$$

(b) Show

$$\frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}\, dt, \quad \text{for all } \alpha > 0. \tag{3}$$

Solution. (a) As $(-e^{-t})' = e^{-t}$ (chain rule and Exercise 8.4.2), the Fundamental Theorem of Calculus gives us

$$\lim_{b\to\infty} \int_0^b e^{-t}\, dt = \lim_{b\to\infty} \left(e^0 - e^{-b}\right) = 1 - \lim_{b\to\infty} e^{-b} = 1,$$

where we have used that $e^0 = 1$ (Exercise 8.4.3 (b)) and that $\lim_{b\to\infty} e^{-b} = 0$ (Exercise 8.4.5).

(b) Similarly to part (a), this time using change of variables:

$$\lim_{b\to\infty} \int_0^b e^{-\alpha t}\, dt = \lim_{b\to\infty} \alpha^{-1}\left(e^0 - e^{-b}\right) = \alpha^{-1}\left(1 - \lim_{b\to\infty} e^{-b}\right) = \alpha^{-1}.$$

Exercise 8.4.11. (a) Evaluate $\int_0^b te^{-\alpha t}\,dt$ using the integration-by-parts formula from Exercise 7.5.6. The result will be an expression in α and b.

(b) Now compute $\int_0^\infty te^{-\alpha t}\,dt$ and verify equation (4).

Solution. (a) After applying integration-by-parts and simplifying, we find that

$$\int_0^b te^{-\alpha t}\,dt = \alpha^{-2} - \alpha^{-1}be^{-\alpha b} - \alpha^{-2}e^{-\alpha b}.$$

(b) Using the expression from part (a) and Exercise 8.4.5, we see that

$$\lim_{b\to\infty}\int_0^b te^{-\alpha t}\,dt = \lim_{b\to\infty}\left(\alpha^{-2} - \alpha^{-1}be^{-\alpha b} - \alpha^{-2}e^{-\alpha b}\right) = \alpha^{-2}.$$

Exercise 8.4.12. Assume the function $f(x,t)$ is continuous on the rectangle $D = \{(x,t) : a \leq x \leq b, c \leq t \leq d\}$. Explain why the function

$$F(x) = \int_c^d f(x,t)\,dt$$

is properly defined for all $x \in [a,b]$.

Solution. Here is a useful lemma.

Lemma 1. Suppose $f : D \to \mathbf{R}$ is continuous, where

$$D = \{(x,t) \in \mathbf{R}^2 : a \leq x \leq b, c \leq t \leq d\}.$$

Then for a fixed $x_0 \in [a,b]$, the function $g : [c,d] \to \mathbf{R}$ given by $g(t) = f(x_0,t)$ is continuous.

Proof. Fix $t_0 \in [c,d]$; we aim to show that g is continuous at t_0, so let $\epsilon > 0$ be given. By assumption f is continuous at $(x_0,t_0) \in D$ and thus there is a $\delta > 0$ such that $|f(x,t) - f(x_0,t_0)| < \epsilon$ whenever $(x,t) \in D$ and

$$\|(x,t) - (x_0,t_0)\| = \sqrt{(x-x_0)^2 - (t-t_0)^2} < \delta.$$

Now suppose that $t \in [c,d]$ is such that $|t - t_0| < \delta$. Notice that

$$\|(x_0,t) - (x_0,t_0)\| = \sqrt{(t-t_0)^2} = |t - t_0| < \delta.$$

It follows that

$$|f(x_0,t) - f(x_0,t_0)| = |g(t) - g(t_0)| < \epsilon$$

and hence that g is continuous at t_0, as desired. □

If we fix $x \in [a,b]$, Lemma 1 implies that $f(x,t)$ is a continuous function of t on the interval $[c,d]$ and hence by Theorem 7.2.9 is integrable on $[c,d]$. Thus F is properly defined for each $x \in [a,b]$.

Exercise 8.4.13. Prove Theorem 8.4.5.

Solution. Fix $x_0 \in [a,b]$; we claim that F is continuous at x_0. Let $\epsilon > 0$ be given. Theorem 4.4.7 is easily adapted to show that f must be uniformly continuous on D and thus there exists a $\delta > 0$ such that

$$(x,t),(y,z) \in D \text{ and } \|(x,t) - (y,z)\| < \delta \quad\implies\quad |f(x,t) - f(y,z)| < \frac{\epsilon}{d-c}.$$

Suppose that $x \in [a,b]$ is such that $|x - x_0| < \delta$. Then for any $t \in [c,d]$ we have

$$\|(x,t) - (x_0,t)\| = |x - x_0| < \delta$$

and hence $|f(x,t) - f(x_0,t)| < \frac{\epsilon}{d-c}$. It follows that

$$|F(x) - F(x_0)| = \left|\int_c^d f(x,t) - f(x_0,t)\,dt\right| \leq \int_c^d |f(x,t) - f(x_0,t)|\,dt \leq \int_c^d \frac{\epsilon}{d-c}\,dt = \epsilon.$$

Thus F is continuous on the compact set $[a,b]$; Theorem 4.4.7 then implies that F is uniformly continuous on $[a,b]$.

Exercise 8.4.14. Finish the proof of Theorem 8.4.6.

Solution. As f_x is continuous on the compact set D, it must be uniformly continuous here. Thus there exists a $\delta > 0$ such that

$$(z,s),(x,t) \in D \text{ and } \|(z,s) - (x,t)\| < \delta \quad\implies\quad |f_x(z,s) - f_x(x,t)| < \frac{\epsilon}{d-c}. \tag{$*$}$$

Suppose that $z \in [a,b]$ is such that $0 < |z - x| < \delta$. For a given $t \in [c,d]$, the Mean Value Theorem (Theorem 5.3.2) implies that there exists some y_t strictly between z and x, so that $|y_t - x| < |z - x| < \delta$, satisfying

$$\frac{f(z,t) - f(x,t)}{z - x} = f_x(y_t,t).$$

Notice that $\|(y_t,t)-(x,t)\| = |y_t - x| < \delta$; it follows from $(*)$ that $|f_x(y_t,t) - f_x(x,t)| < \frac{\epsilon}{d-c}$ and hence that

$$\left|\frac{F(z)-F(x)}{z-x} - \int_c^d f_x(x,t)\,dt\right| = \left|\int_c^d \frac{f(z,t)-f(x,t)}{z-x} - f_x(x,t)\,dt\right|$$
$$= \left|\int_c^d f_x(y_t,t) - f_x(x,t)\,dt\right| \le \int_c^d |f_x(y_t,t)-f_x(x,t)|\,dt \le \int_c^d \frac{\epsilon}{d-c}\,dt = \epsilon.$$

Exercise 8.4.15. (a) Show that the improper integral $\int_0^\infty e^{-xt}\,dt$ converges uniformly to $1/x$ on the set $[1/2,\infty)$.

(b) Is the convergence uniform on $(0,\infty)$?

Solution. (a) Let $\epsilon > 0$ be given and set $M = \max\left\{-2\log\left(\frac{\epsilon}{2}\right), 0\right\}$. Then if $d \ge M$ and $x \ge \frac{1}{2}$, we have

$$\left|\frac{1}{x} - \int_0^d e^{-xt}\,dt\right| = \frac{e^{-xd}}{x} \le 2e^{-d/2} < \epsilon;$$

we are using here that E is strictly increasing, which implies that its inverse function log is also strictly increasing.

(b) The convergence is not uniform on $(0,\infty)$. For any $M > 0$, we have

$$\left|\frac{1}{x} - \int_0^M e^{-xt}\,dt\right| = \frac{e^{-Mx}}{x}.$$

Notice that $\lim_{x\to 0^+} \frac{e^{-Mx}}{x} = +\infty$, since $\lim_{x\to 0^+} e^{-Mx} = 1$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$. Thus there is an $x > 0$ such that

$$\left|\frac{1}{x} - \int_0^M e^{-xt}\,dt\right| = \frac{e^{-Mx}}{x} \ge 1.$$

Exercise 8.4.16. Prove the following analogue of the Weierstrass M-Test for improper integrals: If $f(x,t)$ satisfies $|f(x,t)| \le g(t)$ for all $x \in A$ and $\int_a^\infty g(t)\,dt$ converges, then $\int_a^\infty f(x,t)\,dt$ converges uniformly on A.

Solution. Here is a Cauchy criterion for the uniform convergence of an improper integral, an analogue of Theorem 6.4.4.

Lemma 2. Suppose $D = \{(x,t) \in \mathbf{R}^2 : x \in A, t \geq a\}$ for some $A \subseteq \mathbf{R}$ and $a \in \mathbf{R}$ and we have a function $f : D \to \mathbf{R}$. Then the improper integral $\int_a^\infty f(x,t)\,dt$ converges uniformly to some function $F : A \to \mathbf{R}$ if and only if for every $\epsilon > 0$ there exists an $M \geq a$ such that

$$x \in A \text{ and } c \geq b \geq M \quad \implies \quad \left|\int_b^c f(x,t)\,dt\right| < \epsilon. \tag{$*$}$$

First suppose that the improper integral $\int_a^\infty f(x,t)\,dt$ converges uniformly to some function $F : A \to \mathbf{R}$ and let $\epsilon > 0$ be given. There exists an $M \geq a$ such that

$$x \in A \text{ and } b \geq M \quad \implies \quad \left|F(x) - \int_a^b f(x,t)\,dt\right| < \frac{\epsilon}{2}.$$

Then provided $x \in A$ and $c \geq b \geq M$, we have

$$\begin{aligned}\left|\int_b^c f(x,t)\,dt\right| &= \left|-F(x) + \int_a^c f(x,t)\,dt + F(x) - \int_a^b f(x,t)\,dt + F(x)\right| \\ &\leq \left|F(x) - \int_a^c f(x,t)\,dt\right| + \left|F(x) - \int_a^b f(x,t)\,dt\right| < \epsilon.\end{aligned}$$

Now suppose that for each $\epsilon > 0$ there exists an $M \geq a$ such that $(*)$ holds. For each $x \in A$ we may invoke Exercise 8.4.9 (a) to see that the improper integral $\int_a^\infty f(x,t)\,dt$ converges; define $F(x)$ to be this value. We claim that the improper integral $\int_a^\infty f(x,t)\,dt$ converges uniformly to F on A. To see this, let $\epsilon > 0$ be given and obtain $M \geq a$ from $(*)$. If $x \in A$ and $c \geq b \geq M$, then

$$\begin{aligned}\left|F(x) - \int_a^b f(x,t)\,dt\right| &= \left|F(x) - \int_a^c f(x,t)\,dt + \int_b^c f(x,t)\,dt\right| \\ &\leq \left|F(x) - \int_a^c f(x,t)\,dt\right| + \left|\int_b^c f(x,t)\,dt\right| < \left|F(x) - \int_a^c f(x,t)\,dt\right| + \epsilon.\end{aligned}$$

Notice that this inequality holds for all $c \in [b, \infty)$. Since $\lim_{c\to\infty} g(c) = L$ implies $\lim_{c\to\infty}|g(c)| = |L|$, we can take the limit as $c \to \infty$ on both sides of the above inequality to obtain

$$\begin{aligned}\lim_{c\to\infty}\left|F(x) - \int_a^b f(x,t)\,dt\right| &= \left|F(x) - \int_a^b f(x,t)\,dt\right| \\ &\leq \lim_{c\to\infty}\left(\left|F(x) - \int_a^c f(x,t)\,dt\right| + \epsilon\right) = \left|F(x) - \lim_{c\to\infty}\int_a^c f(x,t)\,dt\right| + \epsilon = \epsilon,\end{aligned}$$

i.e. $\left|F(x) - \int_a^b f(x,t)\,dt\right| \leq \epsilon$. We may conclude that the improper integral $\int_a^\infty f(x,t)\,dt$ converges uniformly to F on A. $\square$

Returning to the exercise, let $\epsilon > 0$ be given. By Exercise 8.4.9 (a) there exists an $M \geq a$ such that

$$x \in A \text{ and } c \geq b \geq M \quad \Longrightarrow \quad \int_b^c g(t)\,dt < \epsilon.$$

It follows that for $x \in A$ and $c \geq b \geq M$ we have

$$\left| \int_b^c f(x,t)\,dt \right| \leq \int_b^c |f(x,t)|\,dt \leq \int_b^c g(t)\,dt < \epsilon.$$

Lemma 2 allows us to conclude that the improper integral $\int_a^\infty f(x,t)\,dt$ converges uniformly on A.

Exercise 8.4.17. Prove Theorem 8.4.8.

Solution. For each $n \in \mathbf{N}$, define $F_n : [a,b] \to \mathbf{R}$ by

$$F_n(x) = \int_c^{c+n} f(x,t)\,dt.$$

By assumption f is continuous on $[a,b] \times [c, c+n]$ and so by Theorem 8.4.5 each F_n is uniformly continuous on $[a,b]$. As noted in the textbook, F_n converges to F uniformly on $[a,b]$. We may use Exercise 6.2.6 (a) to conclude that F is uniformly continuous on $[a,b]$.

Exercise 8.4.18. Prove Theorem 8.4.9.

Solution. For each $n \in \mathbf{N}$, define $F_n : [a,b] \to \mathbf{R}$ and $G : [a,b] \to \mathbf{R}$ by

$$F_n(x) = \int_c^{c+n} f(x,t)\,dt \quad \text{and} \quad G(x) = \int_c^\infty f_x(x,t)\,dt.$$

By Theorem 8.4.6 we have $F_n'(x) = \int_c^{c+n} f_x(x,t)\,dt$ and hence by assumption $F_n' \to G$ uniformly on $[a,b]$. Notice that our hypotheses imply

$$\lim_{n\to\infty} F_n(a) = \lim_{d\to\infty} \int_c^d f(a,t)\,dt = F(a).$$

We may now use Theorem 6.3.3 to see that $F_n \to F$ uniformly on $[a,b]$ and furthermore that

$$F'(x) = G(x) = \int_c^\infty f_x(x,t)\,dt.$$

Exercise 8.4.19. (a) Although we verified it directly, show how to use the theorems in this section to give a second justification for the formula

$$\frac{1}{\alpha^2} = \int_0^\infty te^{-\alpha t}\,dt, \quad \text{for all } \alpha > 0.$$

(b) Now derive the formula

$$\frac{n!}{\alpha^{n+1}} = \int_0^\infty t^n e^{-\alpha t}\,dt, \quad \text{for all } \alpha > 0. \tag{8}$$

Solution. We will need the following results about continuous functions, the proofs of which are straightforward and hence, for the sake of brevity, omitted.

Lemma 3. Suppose that $f, g : D \to \mathbf{R}$, where $D \subseteq \mathbf{R}^2$, are continuous functions.

(i) The function $(x, y) \mapsto f(x, y)g(x, y)$ is continuous on D.

(ii) The function $(x, y) \mapsto kf(x, y)$, for some $k \in \mathbf{R}$, is continuous on D.

(iii) If $h : A \to \mathbf{R}$ is continuous, where $A \subseteq f(D) \subseteq \mathbf{R}$, then the function $(x, y) \mapsto h(f(x, y))$ is continuous on D.

(a) Define $f : \mathbf{R}^2 \to \mathbf{R}$ by $f(\alpha, t) = e^{-\alpha t}$. It is easy to verify that the projections $(\alpha, t) \mapsto \alpha$ and $(\alpha, t) \mapsto t$ are continuous on all of $\mathbf{R}^2$; it follows from this fact and Lemma 3 that f is continuous on all of $\mathbf{R}^2$. Notice that $f_\alpha(\alpha, t) = -te^{-\alpha t}$ exists for all $(\alpha, t) \in \mathbf{R}^2$; we can argue as before to see that f_α is continuous on all of $\mathbf{R}^2$.

Let $0 < a < b$ be arbitrary and define $D = [a, b] \times [0, \infty)$; the previous paragraph shows that f and f_α are continuous on D. Furthermore, by Exercise 8.4.10 (b), the function $F : [a, b] \to \mathbf{R}$ given by

$$F(\alpha) = \int_0^\infty f(\alpha, t)\,dt$$

is well-defined and satisfies $F(\alpha) = \frac{1}{\alpha}$, so that $F'(\alpha) = -\frac{1}{\alpha^2}$.

Now we claim that the improper integral

$$\int_0^\infty f_\alpha(\alpha, t)\,dt = \int_0^\infty -te^{-\alpha t}\,dt$$

converges uniformly on $[a, b]$. Notice that

$$|f_\alpha(\alpha, t)| = te^{-\alpha t} \leq te^{-at}$$

for each $\alpha \in [a, b]$ and $t \geq 0$. Hence, by Exercise 8.4.16, it will suffice to show that the improper integral $\int_0^\infty te^{-at}\,dt$ converges. (Of course, we can show directly using integration-by-parts that it converges to $\frac{1}{a^2}$, as we did in Exercise 8.4.11, making this exercise redundant. However, since presumably the purpose of this exercise is to practice using the theorems and results of this section, we will proceed differently.) By Exercise 8.4.5 we have $\lim_{t\to\infty} te^{-at/2} = 0$ and so there exists an $M > 0$ such that

$$te^{-at/2} \leq 1 \qquad \Longleftrightarrow \qquad te^{-at} \leq e^{-at/2}$$

for all $t > M$. Since $t \mapsto te^{-at}$ is continuous on $[0, M]$ it must be bounded here, say by $L \geq 0$. Thus if we define $g : [0, \infty) \to \mathbf{R}$ by

$$g(t) = \begin{cases} L & \text{if } 0 \leq t \leq M, \\ e^{-at/2} & \text{if } t > M, \end{cases}$$

then $0 \leq te^{-at} \leq g(t)$ for all $t \geq 0$. A direct calculation shows that

$$\int_0^\infty g(t)\,dt = LM + \frac{2e^{-aM/2}}{a}$$

and hence by Exercise 8.4.9 (b) the improper integral $\int_0^\infty te^{-at}\,dt$ also converges. We may now apply Exercise 8.4.16 to see that the improper integral $\int_0^\infty f_\alpha(\alpha, t)\,dt$ converges uniformly on $[a, b]$.

We have now satisfied all the hypotheses of Theorem 8.4.9. Applying this theorem shows that

$$\frac{1}{\alpha^2} = -F'(\alpha) = -\int_0^\infty f_\alpha(\alpha, t)\,dt = \int_0^\infty te^{-\alpha t}\,dt$$

for all $\alpha \in [a, b]$. Since $0 < a < b$ were arbitrary, we may conclude that this formula holds for all $\alpha > 0$.

(b) Let's prove this by induction; the case $n = 0$ was handled in Exercise 8.4.10 (b) and the case $n = 1$ was handled in Exercise 8.4.11 (and also part (a) of this exercise). Suppose that the result is true for some $n \geq 0$. Let $\alpha > 0$ be given and note that, for any $b > 0$, integration-by-parts gives us

$$\int_0^b t^{n+1}e^{-\alpha t}\,dt = -b^{n+1}e^{-\alpha b} + \frac{n+1}{\alpha}\int_0^b t^n e^{-\alpha t}\,dt.$$

Exercise 8.4.5 shows that $\lim_{b\to\infty} b^{n+1}e^{-\alpha b} = 0$ and our induction hypothesis ensures that $\int_0^\infty t^n e^{-\alpha t}\,dt = \frac{n!}{\alpha^{n+1}}$; it follows that

$$\int_0^\infty t^{n+1}e^{-\alpha t}\,dt = \frac{n+1}{\alpha} \cdot \frac{n!}{\alpha^{n+1}} = \frac{(n+1)!}{\alpha^{n+2}}.$$

This completes the induction step and the proof.

Exercise 8.4.20. (a) Show that $x!$ is an infinitely differentiable function on $(0, \infty)$ and produce a formula for the n^{th} derivative. In particular show that $(x!)'' > 0$.

(b) Use the integration-by-parts formula employed earlier to show that $x!$ satisfies the functional equation

$$(x + 1)! = (x + 1)x!.$$

Solution. The definition $x! = \int_0^\infty t^x e^{-t}\,dt$ involves an improper integral as defined in Definition 8.4.3. This definition requires the integrand $t^x e^{-t}$ to be defined on $[0, \infty)$, but in fact it is undefined for $t = 0$. I am going to ignore this issue.

(a) For $n \in \mathbf{N}$, let us denote the n^{th} derivative of $x!$ by $(x!)^{(n)}$. We will prove by induction that

$$(x!)^{(n)} = \int_0^\infty (\log t)^n t^x e^{-t}\,dt$$

for $x > 0$. For the base case $n = 1$, first observe that

$$\frac{d}{dx}\bigl(t^x e^{-t}\bigr) = (\log t) t^x e^{-t}.$$

Let $0 < a < b$ be arbitrary; we claim that the improper integral $\int_0^\infty (\log t) t^x e^{-t}\,dt$ converges uniformly on $[a, b]$. To see this, note that

$$\bigl|(\log t) t^x e^{-t}\bigr| = (\log t) t^x e^{-t} \le t^{x+1} e^{-t} \le t^{b+1} e^{-t}$$

for $x \in [a, b]$ and $t \ge 1$. Note further that

$$\bigl|(\log t) t^x e^{-t}\bigr| = |\log t| t^x e^{-t} \le |\log t| t^b$$

for $x \in [a, b]$ and $0 < t < 1$. Since

$$\lim_{t \to 0^+} |\log t| t^b = 0,$$

which can be seen using L'Hôpital's rule, there exists an $M > 0$ such that $|\log t| t^b \le M$ for all $x \in [a, b]$ and $0 < t < 1$. Thus, if we define

$$g(t) = \begin{cases} M & \text{if } 0 < t < 1, \\ t^{b+1} e^{-t} & \text{if } t \ge 1, \end{cases}$$

then $|(\log t)t^x e^{-t}| \leq g(t)$. It is straightforward to show that $\int_0^\infty g(t)\,dt$ converges and so it follows from Exercise 8.4.16 that $\int_0^\infty (\log t)t^x e^{-t}\,dt$ converges uniformly on $[a,b]$. We can now use Theorem 8.4.9 to see that

$$(x!)' = \int_0^\infty (\log t)t^x e^{-t}\,dt$$

for $x \in [a,b]$. Since $0 < a < b$ were arbitrary, we see that this formula holds for all $x > 0$.

The induction step is essentially identical to the base case; note that

$$\frac{d}{dx}\big((\log t)^n t^x e^{-t}\big) = (\log t)^{n+1} t^x e^{-t}.$$

For arbitrary $0 < a < b$, we can again bound $|(\log t)^{n+1} t^x e^{-t}|$ by

$$g(t) = \begin{cases} M & \text{if } 0 < t < 1, \\ t^{b+n+1} e^{-t} & \text{if } t \geq 1, \end{cases}$$

where $M > 0$ is some bound on $|(\log t)^{n+1} t^x e^{-t}|$ for $x \in [a,b]$ and $0 < t < 1$; the existence of this M follows since

$$\lim_{t \to 0^+} |\log t|^{n+1} t^b = 0,$$

which can be seen by repeated applications of L'Hôpital's rule. Then since $\int_0^\infty g(t)\,dt$ converges, Exercise 8.4.16 implies that the improper integral $\int_0^\infty (\log t)^{n+1} t^x e^{-t}\,dt$ converges uniformly on $[a,b]$ and hence by Theorem 8.4.9 we have

$$(x!)^{(n+1)} = \frac{d}{dx}(x!)^{(n)} = \int_0^\infty (\log t)^{n+1} t^x e^{-t}\,dt$$

for all $x \in [a,b]$. Since $0 < a < b$ were arbitrary, the formula holds for all $x > 0$. This completes the induction step and the proof.

In particular, we have

$$(x!)'' = \int_0^\infty (\log t)^2 t^x e^{-t}\,dt.$$

The integrand $(\log t)^2 t^x e^{-t}$ is strictly positive for all $x > 0$ and $t > 1$, whence $(x!)'' > 0$.

(b) For any $b > 0$, integration-by-parts gives

$$\int_0^b t^{x+1} e^{-t}\,dt = -b^{x+1} e^{-b} + (x+1)\int_0^b t^x e^{-t}\,dt,$$

which converges to $(x+1)x!$ as $b \to \infty$.

Exercise 8.4.21. (a) Use the convexity of $\log(f(x))$ and the three intervals $[n-1,n]$, $[n,n+x]$, and $[n,n+1]$ to show

$$x\log(n) \leq \log(f(n+x)) - \log(n!) \leq x\log(n+1).$$

(b) Show $\log(f(n+x)) = \log(f(x)) + \log((x+1)(x+2)\cdots(x+n))$.

(c) Now establish that

$$0 \leq \log(f(x)) - \log\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right) \leq x\log\left(1+\frac{1}{n}\right).$$

(d) Conclude that

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}, \quad \text{for all } x \in (0,1].$$

(e) Finally, show that the conclusion in (d) holds for all $x \geq 0$.

Solution. (a) First consider the intervals $[n-1,n]$ and $[n,n+x]$. Using the fact about convex functions mentioned previously in the textbook, we find the inequality

$$\log(f(n)) - \log(f(n-1)) \leq \frac{\log(f(n+x)) - \log(f(n))}{x}.$$

Since $f(n) = n!$ and $\log(a) - \log(b) = \log\left(\frac{a}{b}\right)$, we have

$$\log(f(n)) - \log(f(n-1)) = \log(n!) - \log((n-1)!) = \log\left(\frac{n!}{(n-1)!}\right) = \log(n).$$

Thus we obtain $x\log(n) \leq \log(f(n+x)) - \log(n!)$. A similar argument with the intervals $[n,n+x]$ and $[n,n+1]$ (remembering that $x \leq 1$) gives us the other desired inequality.

(b) Property (ii) implies that

$$f(x+n) = f(x)(x+1)(x+2)\cdots(x+n).$$

Now we can use that $\log(ab) = \log(a) + \log(b)$ to obtain the desired equality.

(c) Part (a) gives us

$$0 \le \log(f(n+x)) - \log(n!) - x\log(n) \le x\log(n+1) - x\log(n).$$

Part (b) and the usual properties of logarithms imply that

$$\begin{aligned}\log(f(n+x)) - \log(n!) - x\log(n) &= \log(f(x)) + \log((x+1)\cdots(x+n)) - \log(n^x n!)\\ &= \log(f(x)) - \log\!\left(\frac{n^x n!}{(x+1)\cdots(x+n)}\right).\end{aligned}$$

Similarly,

$$x\log(n+1) - x\log(n) = x(\log(n+1) - \log(n)) = x\log\!\left(\frac{n+1}{n}\right) = x\log\!\left(1+\frac{1}{n}\right).$$

Combining these gives the desired result.

(d) Since $\log\big(1+\frac{1}{n}\big) \to 0$, the Squeeze Theorem and part (c) imply that

$$\log(f(x)) = \lim_{n\to\infty} a_n \quad \text{where } a_n = \log\!\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right)$$

for each $x \in (0,1]$. Since the exponential function is continuous everywhere, the above equation implies that

$$f(x) = e^{\lim a_n} = \lim_{n\to\infty} e^{a_n} = \lim_{n\to\infty} \frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}$$

for each $x \in (0,1]$.

(e) For $x = 0$ we have

$$\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)} = \frac{n^0 n!}{n!} = 1 = f(0).$$

For $x > 0$, let $m \in \mathbf{N}$ be such that $x \in (0,m]$. By repeating our previous argument with the intervals $[n-1,n], [n,n+x]$, and $[n,n+m]$, we arrive at the inequality

$$0 \le \log(f(x)) - \log\!\left(\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}\right) \le \frac{x}{m}\log\!\left(\frac{(n+m)!}{n!n^m}\right).$$

Notice that

$$\frac{(n+m)!}{n!n^m} = \frac{(n+m)(n+m-1)\cdots(n+1)}{n^m} = \left(1+\frac{m}{n}\right)\left(1+\frac{m-1}{n}\right)\cdots\left(1+\frac{1}{n}\right).$$

Since each of the m terms in parentheses on the right-hand side converges to 1, we see that $\lim_{n\to\infty}\frac{(n+m)!}{n!n^m} = 1$ and thus

$$\lim_{n\to\infty}\frac{x}{m}\log\left(\frac{(n+m)!}{n!n^m}\right) = 0.$$

We can now argue as in part (d) using the Squeeze Theorem to see that

$$f(x) = \lim_{n\to\infty}\frac{n^x n!}{(x+1)(x+2)\cdots(x+n)}.$$

Exercise 8.4.22. (a) Where does $g(x) = \frac{x}{x!(-x)!}$ equal zero? What other familiar function has the same set of roots?

(b) The function e^{-x^2} provides the raw material for the all-important Gaussian bell curve from probability, where it is known that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. Use this fact (and some standard integration techniques) to evaluate $(1/2)!$.

(c) Now use (a) and (b) to conjecture a striking relationship between the factorial function and a well-known function from trigonometry.

Solution. (a) We are taking $\frac{1}{x!}$ to be zero when $x = -1, -2, -3, \ldots$ and thus g is zero at each integer. The function $\sin(\pi x)$ has the same set of roots.

(b) For any $b > 0$, standard integration techniques give us

$$\int_0^b \sqrt{t}e^{-t}\,dt = \int_0^{\sqrt{b}} 2u^2 e^{-u^2}\,du = -\sqrt{b}e^{-b} + \int_0^{\sqrt{b}} e^{-u^2}\,du,$$

which, given that $\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$, converges to $\frac{\sqrt{\pi}}{2}$ as $b\to\infty$. Thus

$$(1/2)! = \int_0^{\infty}\sqrt{t}e^{-t}\,dt = \frac{\sqrt{\pi}}{2}.$$

(c) We conjecture that $\frac{x}{x!(-x)!} = k\sin(\pi x)$ for some $k \in \mathbf{R}$. Taking $x = \frac{1}{2}$ gives us $k = \frac{1/2}{(1/2)!(-1/2)!}$. Using part (b) and the identity $(1/2)! = (1/2)(-1/2)!$, we find that $k = \frac{1}{\pi}$.

Exercise 8.4.23. As a parting shot, use the value for $(1/2)!$ and the Gauss product formula in equation (9) to derive the famous product formula for π discovered by John Wallis in the 1650s:

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right).$$

Solution. Taking $x = 1/2$ in equation (9) gives

$$(1/2)! = \frac{\sqrt{\pi}}{2} = \lim_{n\to\infty} \frac{\sqrt{n}(n!)}{\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{2n+1}{2}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}2^n(n!)}{3\cdot 5\cdots(2n+1)} = \lim_{n\to\infty} \frac{\sqrt{n}\cdot 2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}.$$

Squaring both sides of this equality, using the continuity of $x \mapsto x^2$, and multiplying through by 2 gives us

$$\begin{aligned}\frac{\pi}{2} &= \lim_{n\to\infty} \left(\frac{2\cdot 2}{3\cdot 3}\right)\left(\frac{4\cdot 4}{5\cdot 5}\right)\cdots\left(\frac{2n\cdot 2n}{(2n+1)(2n+1)}\right)(2n)\\ &= \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right)\left(\frac{2n}{2n+1}\right).\end{aligned}$$

Since $\lim_{n\to\infty} \frac{2n}{2n+1} = 1$, it must be the case that

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right).$$

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.