1 Section 4.6 Exercises

Exercises with solutions from Section 4.6 of [UA].

Exercise 4.6.1. Using modifications of these functions, construct a function $f : \mathbf{R} \to \mathbf{R}$ so that

(a) $D_f = \mathbf{Z}^c$.

(b) $D_f = \{x : 0 < x \leq 1\}$.

Solution. (a) Since $\mathbf{Z}^c$ is an open set, the construction given in Exercise 4.3.14 (b) will result in an f such that $D_f = \mathbf{Z}^c$.

(b) By Exercise 4.3.14, there exist functions $g, h : \mathbf{R} \to \mathbf{R}$ such that $D_g = \left(0, \frac{1}{2}\right)$ and $D_h = \left[\frac{1}{2}, 1\right]$. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = g(x) + h(x)$; it follows from Theorem 4.3.4 that $D_f = (0, 1]$.

Exercise 4.6.2. Given a countable set $A = \{a_1, a_2, a_3, \ldots\}$, define $f(a_n) = 1/n$ and $f(x) = 0$ for all $x \notin A$. Find D_f.

Solution. Our claim is that $D_f = A$. First, fix $c \notin A$; we will show that f is continuous at c. Let $\epsilon > 0$ be given.

Case 1. If $\epsilon > 1$, then note that

$$|f(x) - f(c)| = f(x) \leq 1 < \epsilon$$

for any $x \in \mathbf{R}$. Hence any $\delta > 0$ will suffice; say, $\delta = 1$.

Case 2. If $0 < \epsilon \leq 1$, then let $N \in \mathbf{N}$ be such that $\frac{1}{N} < \epsilon$ and note that $N \geq 2$. Consider the set

$$E = \{|c - a_n| : 1 \leq n \leq N - 1\}.$$

This set is non-empty as $N \geq 2$ and clearly finite, so we are justified in letting $\delta = \min E$. Each element of E must be strictly positive as $c \notin A$ and hence δ is also strictly positive. Furthermore, the interval $(c - \delta, c + \delta)$ has the property that if $a_n \in (c - \delta, c + \delta)$, then $n \geq N$. It follows that

$$|f(a_n) - f(c)| = \tfrac{1}{n} \leq \tfrac{1}{N} < \epsilon.$$

Additionally, if $x \in (c - \delta, c + \delta)$ and $x \notin A$, then $|f(x) - f(c)| = 0 < \epsilon$.

We have now shown that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$x \in (c - \delta, c + \delta) \implies f(x) \in (f(c) - \epsilon, f(c) + \epsilon).$$

Hence f is continuous at each $c \notin A$.

Now fix $a_n \in A$; we will show that f is not continuous at a_n. Set $\epsilon = \frac{1}{n} > 0$ and let $\delta > 0$ be given. Since the interval $(a_n - \delta, a_n + \delta)$ is uncountable and A is countable, it must be the case that there exists $x \in (a_n - \delta, a_n + \delta)$ such that $x \notin A$. Then

$$|f(x) - f(a_n)| = \tfrac{1}{n} = \epsilon.$$

Thus f is not continuous at each element of A and our claim follows.

Exercise 4.6.3. State a similar definition for the left-hand limit

$$\lim_{x \to c^-} f(x) = L.$$

Solution. See Exercise 4.2.10 (a).

Exercise 4.6.4. Supply a proof for this proposition.

Solution. See Exercise 4.2.10 (b).

Exercise 4.6.5. Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

Solution. Suppose $f : \mathbf{R} \to \mathbf{R}$ is monotone. (For simplicity, we will assume that the domain of f is all of $\mathbf{R}$. A more general statement can certainly be made for monotone functions $A \to \mathbf{R}$ defined on any domain $A \subseteq \mathbf{R}$, but Abbott's definitions of left- and right-hand limits are slightly awkward here. For example, if $f : [0, 1] \to \mathbf{R}$ is a function, then Abbott's definition of the left-hand limit of f at 0 implies that $\lim_{x \to 0^-} f(x) = L$ for *any* $L \in \mathbf{R}$; we may choose any $\delta > 0$ we like and obtain a statement of the form $(\forall x \in \emptyset)$, which is always true. It would be better not to talk about $\lim_{x \to 0^-} f(x)$ at all in such a case.)

We claim the following. If f is increasing, then for each $c \in \mathbf{R}$

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : x < c\} \quad \text{and} \quad \lim_{x \to c^+} f(x) = \inf\{f(x) : c < x\}.$$

If f is decreasing, then the sup and inf should be swapped.

Proof. We will prove that for an increasing function $f : \mathbf{R} \to \mathbf{R}$ and some $c \in \mathbf{R}$, the left-hand limit is given by

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : x < c\}.$$

The other cases are handled similarly.

First, note that the set $\{f(x) : x < c\}$ is certainly non-empty. Furthermore, it is bounded above by $f(c)$ as f is increasing, so by completeness we are justified in setting $s := \sup\{f(x) : x < c\}$.

Let $\epsilon > 0$ be given. There is a $y < c$ such that $s - \epsilon < f(y) \leq s$ (Lemma 1.3.8). Since f is increasing, for each $x \in (y, c)$ we have $s - \epsilon < f(y) \leq f(x) \leq s$. In other words, letting $\delta = c - y$, we have

$$c - \delta < x < c \implies |f(x) - s| < \epsilon.$$

It follows that $\lim_{x \to c^-} f(x) = s$, as claimed. □

So for a monotone function $f : \mathbf{R} \to \mathbf{R}$, the left- and right-hand limits at some point $c \in \mathbf{R}$ always exist. It follows that if f is discontinuous at c, it must be the case that these left- and right-hand limits are not equal (Theorem 4.6.3/Exercise 4.6.4), i.e. f has a jump discontinuity at c.

Exercise 4.6.6. Construct a bijection between the set of jump discontinuities of a monotone function f and a subset of $\mathbf{Q}$. Conclude that D_f for a monotone function f must either be finite or countable, but not uncountable.

Solution. Suppose $f : \mathbf{R} \to \mathbf{R}$ is monotone increasing (the case where f is decreasing is handled similarly) and let D_f be the set of jump discontinuities of f (by Exercise 4.6.5, D_f is the set of all discontinuities of f). Fix $c \in D_f$. As we showed in Exercise 4.6.5, we have

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : x < c\} \quad \text{and} \quad \lim_{x \to c^+} f(x) = \inf\{f(x) : c < x\}.$$

Let $l_c = \lim_{x \to c^-} f(x)$ and $u_c = \lim_{x \to c^+} f(x)$. Since f is discontinuous at c and increasing, we must have $l_c < u_c$ and hence (l_c, u_c) is a proper open interval. Suppose $d \in D_f$ is such that $c < d$. Then $u_c \leq f\left(\frac{c+d}{2}\right) \leq l_d$, so that the open intervals (l_c, u_c) and (l_d, u_d) are disjoint. It follows that the set

$$\{(l_c, u_c) : c \in D_f\}$$

consists of pairwise disjoint open intervals. Given this, for each $c \in D_f$ we can choose a rational number $r_c \in (l_c, u_c)$ and be sure that the function $g : D_f \to \mathbf{Q}$ mapping $c \mapsto r_c$ is injective. This sets up a bijection between D_f and $g(D_f) \subseteq \mathbf{Q}$. It follows that D_f is finite or countable, but not uncountable.

Exercise 4.6.7. (a) Show that in each of the above cases we get an F_σ set as the set where the function is discontinuous.

(b) Show that the two sets of discontinuity in Exercise 4.6.1 are F_σ sets.

Solution. (a) For Dirichlet's function, $\mathbf{R}$ is a closed set. For the modified Dirichlet function, we have

$$\mathbf{R} \setminus \{0\} = \bigcup_{n=1}^{\infty} \left(-\infty, -\tfrac{1}{n}\right] \cup \left[\tfrac{1}{n}, \infty\right).$$

For Thomae's function, we have

$$\mathbf{Q} = \bigcup_{q \in \mathbf{Q}} \{q\}.$$

(b) Observe that

$$\mathbf{Z}^{\mathsf{c}} = \bigcup_{(m,n) \in \mathbf{Z} \times \mathbf{N}} \left[m + \tfrac{1}{n+1}, m + 1 - \tfrac{1}{n+1}\right] \quad \text{and} \quad (0,1] = \bigcup_{n=1}^{\infty} \left[\tfrac{1}{n}, 1\right].$$

Exercise 4.6.8. Prove that, for a fixed $\alpha > 0$, the set D_f^α is closed.

Solution. First, let us write down the negation of α-continuity. A function f is not α-continuous at a point $x \in \mathbf{R}$ if for all $\delta > 0$ there exist $y, z \in (x - \delta, x + \delta)$ such that $|f(y) - f(z)| \geq \alpha$.

Now, to show that D_f^α is closed, let (x_n) be a sequence contained in D_f^α such that $\lim x_n = x$ for some $x \in \mathbf{R}$. Our aim is to show that f is not α-continuous at x. Let $\delta > 0$ be given. Since $\lim x_n = x$, there is an $N \in \mathbf{N}$ such that $x_N \in \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)$, and since f is not α-continuous at x_N, there exist $y, z \in \left(x_N - \frac{\delta}{2}, x_N + \frac{\delta}{2}\right)$ such that $|f(y) - f(z)| \geq \alpha$. In fact, the triangle inequality implies that $y, z \in (x - \delta, x + \delta)$ and thus f is not α-continuous at x.

It follows that D_f^α contains its limit points and hence that D_f^α is a closed set.

Exercise 4.6.9. If $\alpha < \alpha'$, show that $D_f^{\alpha'} \subseteq D_f^\alpha$.

Solution. A function f is not α'-continuous at a point $x \in \mathbf{R}$ if for all $\delta > 0$ there exist $y, z \in (x - \delta, x + \delta)$ such that $|f(y) - f(z)| \geq \alpha' > \alpha$; it follows that f is also not α-continuous at x.

Exercise 4.6.10. Let $\alpha > 0$ be given. Show that if f is continuous at x, then it is α-continuous at x as well. Explain how it follows that $D_f^\alpha \subseteq D_f$.

Solution. Since f is continuous at x, there is a $\delta > 0$ such that

$$y \in (x - \delta, x + \delta) \implies |f(y) - f(x)| < \tfrac{\alpha}{2}.$$

If $y, z \in (x - \delta, x + \delta)$, then

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(z) - f(x)| < \tfrac{\alpha}{2} + \tfrac{\alpha}{2} = \alpha.$$

Thus f is α-continuous at x. The contrapositive of this result states that if f is not α-continuous at x, then f is not continuous at x. It follows that $D_f^\alpha \subseteq D_f$.

Exercise 4.6.11. Show that if f is not continuous at x, then f is not α-continuous for some $\alpha > 0$. Now explain why this guarantees that

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n},$$

where $\alpha_n = 1/n$.

Solution. If f is not continuous at x, then there exists an $\epsilon > 0$ such that for all $\delta > 0$, there is a $y \in (x - \delta, x + \delta)$ such that $|f(y) - f(x)| \geq \epsilon$. It follows that f is not α-continuous at x, where we take $\alpha = \epsilon$.

Suppose $x \in D_f$. As we just showed, there exists an $\alpha > 0$ such that $x \in D_f^{\alpha}$. Let $n \in \mathbf{N}$ be such that $\frac{1}{n} < \alpha$. We then have $D_f^{\alpha} \subseteq D_f^{\alpha_n}$ (Exercise 4.6.9) and so $x \in D_f^{\alpha_n}$. It follows that

$$D_f \subseteq \bigcup_{n=1}^{\infty} D_f^{\alpha_n}.$$

For the reverse inclusion, note that for each $n \in \mathbf{N}$, we have $D_f^{\alpha_n} \subseteq D_f$ by Exercise 4.6.10. We may conclude that

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n}.$$

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.