Linear Algebra Done Right Solutions

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Contents

Chapter 1. Vector Spaces

1.A. $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 1.A.1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi$ and $\beta = u + vi$, then

$$\alpha + \beta = (x + u) + (y + v)i = (u + x) + (v + y)i = \beta + \alpha,$$

where we have used the commutativity of addition in $\mathbf{R}$.

Exercise 1.A.2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}(\alpha + \beta) + \lambda &= ((x + u) + (y + v))i + \lambda = ((x + u) + s) + ((y + v) + t)i \\ &= (x + (u + s)) + (y + (v + t))i = \alpha + ((u + s) + (v + t)i) = \alpha + (\beta + \lambda),\end{aligned}$$

where we have used the associativity of addition in $\mathbf{R}$.

Exercise 1.A.3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}(\alpha\beta)\lambda &= [(xu - yv) + (xv + yu)i]\lambda \\ &= [(xu - yv)s - (xv + yu)t] + [(xu - yv)t + (xv + yu)s]i \\ &= [(xu)s - (yv)s - (xv)t - (yu)t] + [(xu)t - (yv)t + (xv)s + (yu)s]i \\ &= [x(us) - x(vt) - y(ut) - y(vs)] + [x(ut) + x(vs) + y(us) - y(vt)]i \\ &= [x(us - vt) - y(ut + vs)] + [x(ut + vs) + y(us - vt)]i \\ &= \alpha[(us - vt) + (ut + vs)i] \\ &= \alpha(\beta\lambda),\end{aligned}$$

where we have used several algebraic properties of $\mathbf{R}$.

Exercise 1.A.4. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}\lambda(\alpha + \beta) &= [s(x + u) - t(y + v)] + [s(y + v) + t(x + u)i] \\ &= (sx + su - ty - tv) + (sy + sv + tx + tu)i \\ &= [(sx - ty) + (sy + tx)i] + [(su - tv) + (sv + tu)i] \\ &= \lambda\alpha + \lambda\beta,\end{aligned}$$

where we have used distributivity in $\mathbf{R}$.

Exercise 1.A.5. Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Solution. Suppose that $\alpha = x + yi$. Let $\beta = -x - yi$ and observe that

$$\alpha + \beta = (x - x) + (y - y)i = 0 + 0i = 0.$$

To see that β is unique, suppose that β' also satisfies $\alpha + \beta' = 0$ and notice that

$$\beta = \beta = 0 = \beta + (\alpha + \beta') = (\alpha + \beta) + \beta' = 0 + \beta' = \beta',$$

where we have used the associativity of addition in $\mathbf{C}$ (Exercise 1.A.2) and the commutativity of addition in $\mathbf{C}$ (Exercise 1.A.1).

Exercise 1.A.6. Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Solution. Suppose that $\alpha = x + yi$. Since $\alpha \neq 0$, it must be the case that x and y are not both zero, so that $x^2 + y^2 \neq 0$. Let $\beta = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$ and observe that

$$\alpha\beta = (x + yi)\left(\frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i\right) = \frac{x^2 + y^2}{x^2 + y^2} + \frac{xy - xy}{x^2 + y^2}i = 1 + 0i = 1.$$

To see that β is unique, suppose β' also satisfies $\alpha\beta' = 1$ and notice that

$$\beta = \beta 1 = \beta(\alpha\beta') = (\alpha\beta)\beta' = 1\beta' = \beta',$$

where we have used the associativity of multiplication in $\mathbf{C}$ (Exercise 1.A.3) and the commutativity of multiplication in $\mathbf{C}$ (1.4).

Exercise 1.A.7. Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution. Let $z = \frac{-1+\sqrt{3}i}{2}$, so that $2z = -1+\sqrt{3}i$. Observe that

$$\begin{aligned} (2z)^2 = 4z^2 = \left(-1+\sqrt{3}i\right)^2 &= 1 - 2\sqrt{3}i - 3 = -2 - 2\sqrt{3}i \\ \Rightarrow (2z)^3 = (4z^2)(2z) = \left(-2-2\sqrt{3}i\right)\left(-1+\sqrt{3}i\right) &= 2 - 2\sqrt{3}i + 2\sqrt{3}i + 6 = 8, \end{aligned}$$

i.e. $8z^3 = 8$. It follows that $z^3 = 1$.

Exercise 1.A.8. Find two distinct square roots of i.

Solution. Let $z_1 = \frac{1+i}{\sqrt{2}}$ and $z_2 = -z_1$ (z_1 and z_2 are distinct since $z_1 \neq 0$) and observe that

$$2z_1^2 = (1+i)^2 = 2i \quad \Rightarrow \quad z_1^2 = i,$$

i.e. z_1 is a square root of i. Furthermore, $z_2^2 = (-z_1)^2 = z_1^2 = i$, so that z_2 is a square root of i also.

Exercise 1.A.9. Find $x \in \mathbf{R}^4$ such that

$$(4,-3,1,7) + 2x = (5,9,-6,8).$$

Solution. The unique solution is $x = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right)$.

Exercise 1.A.10. Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i).$$

Solution. If there was such a λ, then

$$\lambda(2-3i) = 12-5i \quad \Rightarrow \quad \lambda = \frac{12-5i}{2-3i} = 3+2i.$$

However,

$$(3+2i)(-6+7i) = -32+9i \neq -32-9i.$$

Exercise 1.A.11. Show that $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$, and $z = (z_1, ..., z_n)$, then

$$\begin{aligned}(x+y)+z &= (x_1+y_1, ..., x_n+y_n)+z = ((x_1+y_1)+z_1, ..., (x_n+y_n)+z_n) \\ &= (x_1+(y_1+z_1), ..., x_n+(y_n+z_n)) = x+(y_1+z_1, ..., y_n+z_n) = x+(y+z),\end{aligned}$$

where we have used the associativity of addition in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.2).

Exercise 1.A.12. Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Solution. If $x = (x_1, ..., x_n)$, then

$$(ab)x = ((ab)x_1, ..., (ab)x_n) = (a(bx_1), ..., a(bx_n)) = a(bx_1, ..., bx_n) = a(bx),$$

where we have used the associativity of multiplication in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.3).

Exercise 1.A.13. Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$, then

$$1x = (1x_1, ..., 1x_n) = (x_1, ..., x_n) = x,$$

where we have used that $1x_j = x_j$ for any $x_j \in \mathbf{F}$.

Exercise 1.A.14. Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, then

$$
\begin{aligned}
\lambda(x+y) &= \lambda(x_1+y_1,\dots,x_n+y_n) \\
&= (\lambda(x_1+y_1),\dots,\lambda(x_n+y_n)) \\
&= (\lambda x_1+\lambda y_1,\dots,\lambda x_n+\lambda y_n) \\
&= (\lambda x_1,\dots,\lambda x_n)+(\lambda y_1,\dots,\lambda y_n) \\
&= \lambda(x_1,\dots,x_n)+\lambda(y_1,\dots,y_n) \\
&= \lambda x+\lambda y,
\end{aligned}
$$

where we have used distributivity in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.4).

Exercise 1.A.15. Show that $(a+b)x = ax+bx$ for all $a,b\in\mathbf{F}$ and all $x\in\mathbf{F}^n$.

Solution. If $x=(x_1,\dots,x_n)$, then

$$
\begin{aligned}
(a+b)x &= (a+b)(x_1,\dots,x_n) \\
&= ((a+b)x_1,\dots,(a+b)x_n) \\
&= (ax_1+bx_1,\dots,ax_n+bx_n) \\
&= (ax_1,\dots,ax_n)+(bx_1,\dots,bx_n) \\
&= a(x_1,\dots,x_n)+b(x_1,\dots,x_n) \\
&= ax+bx,
\end{aligned}
$$

where we have used distributivity in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.4).

1.B. Definition of Vector Space

Exercise 1.B.1. Show that $-(-v)=v$ for every $v\in V$.

Solution. Since $v+(-v)=0$ and the additive inverse of a vector is unique (1.27), it must be the case that $-(-v)=v$.

Exercise 1.B.2. Suppose $a\in\mathbf{F}, v\in V$, and $av=0$. Prove that $a=0$ or $v=0$.

Solution. It will suffice to show that if $av = 0$ and $a \neq 0$, so that a^{-1} exists, then $v = 0$. Indeed,

$$av = 0 \quad \Rightarrow \quad a^{-1}(av) = 0 \quad \Rightarrow \quad (a^{-1}a)v = 0 \quad \Rightarrow \quad 1v = 0 \quad \Rightarrow \quad v = 0.$$

Exercise 1.B.3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution. For $v, w, x \in V$, notice that

$$v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \tfrac{1}{3}(w - v).$$

Exercise 1.B.4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Solution. The empty set does not contain an additive identity.

Exercise 1.B.5. Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

> *The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.*

Solution. If V satisfies all of the conditions in (1.20), then as shown in (1.30) we have $0v = 0$ for all $v \in V$. Suppose that V satisfies all of the conditions in (1.20), except we have replaced the additive inverse condition with the condition that $0v = 0$ for all $v \in V$. We want to show that for each $v \in V$, there exists an element $w \in V$ such that $v + w = 0$. Indeed, for $v \in V$, let $w = (-1)v$ and observe that

$$v + w = 1v + (-1)v = (1 - 1)v = 0v = 0.$$

Exercise 1.B.6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) = (-\infty) + \infty &= 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution. This is not a vector space over $\mathbf{R}$, since addition is not associative:

$$(1 + \infty) + (-\infty) = \infty + (-\infty) = 0 \neq 1 = 1 + 0 = 1 + (\infty + (-\infty)).$$

Exercise 1.B.7. Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Solution. We define addition and scalar multiplication on V^S as in (1.24), i.e. for $f, g \in V^S$ the sum $f + g \in V^S$ is the function

$$\begin{aligned} f + g \,:\, S &\to V \\ x &\mapsto f(x) + g(x); \end{aligned}$$

the addition $f(x) + g(x)$ is vector addition in V. Similarly, for $\lambda \in \mathbf{F}$ and $f \in V^S$, the product $\lambda f \in V^S$ is the function

$$\begin{aligned} \lambda f \,:\, S &\to V \\ x &\mapsto \lambda f(x); \end{aligned}$$

the product $\lambda f(x)$ is scalar multiplication in V. We now show that V^S with these definitions satisfies each condition in definition (1.20).

Commutativity. Let $f, g \in V^S$ and $x \in S$ be given. Observe that

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x),$$

where we have used the commutativity of addition in V for the second equality. It follows that $f + g = g + f$.

Associativity. Let $f, g, h \in V^S$ and $x \in S$ be given. Observe that

$$\begin{aligned}((f+g)+h)(x) &= (f+g)(x) + h(x) = (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) = f(x) + (g+h)(x) = (f + (g+h))(x),\end{aligned}$$

where we have used the associativity of addition in V for the third equality. It follows that $(f+g)+h = f+(g+h)$. Similarly, let $f \in V^S$ and $a, b \in \mathbf{F}$ be given. Observe that, for any $x \in S$,

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = a((bf)(x)) = (a(bf))(x),$$

where we have used the associativity of scalar multiplication in V for the second equality. It follows that $(ab)f = a(bf)$.

Additive identity. We claim that the additive identity in V^S is the function $0 : S \to V$ given by $0(x) = 0$ for any $x \in S$; the 0 on the right-hand side is the additive identity in V. Indeed, for any $f \in V^S$ and $x \in S$ we have

$$(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x).$$

It follows that $f + 0 = f$.

Additive inverse. For $f \in V^S$, define $g : S \to V$ by $g(x) = -f(x)$ for $x \in S$, where $-f(x)$ is the additive inverse in V of $f(x)$. We claim that g is the additive inverse of f. To see this, let $x \in S$ be given and observe that

$$(f+g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0 = 0(x);$$

it follows that $f + g = 0$.

Multiplicative identity. Let $f \in V^S$ and $x \in S$ be given. Observe that

$$(1f)(x) = 1f(x) = f(x),$$

where we have used that $1v = v$ for any $v \in V$. It follows that $1f = f$.

Distributive properties. Let $a \in \mathbf{F}$ and $f, g \in V^S$ be given. Observe that, for any $x \in S$,

$$\begin{aligned}(a(f+g))(x) &= a(f+g)(x) = a((f(x) + g(x)) \\ &= af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x),\end{aligned}$$

where we have used the first distributive property in V for the third equality. It follows that $a(f+g) = af + ag$. Similarly, let $a, b \in \mathbf{F}$ and $f \in V^S$ be given. For any $x \in S$, observe that

$$((a+b)f)(x) = (a+b)f(x) = af(x) + bf(x) = (af)(x) + (bf)(x) = (af+bf)(x),$$

where we have used the second distributive property in V for the second equality. It follows that $(a+b)f = af + bf$.

We may conclude that V^S is a vector space over $\mathbf{F}$.

Exercise 1.B.8. Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

  for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

  for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

> *Think of V as a subset of $V_{\mathbf{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbf{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.*

Solution. We need to verify each condition in definition (1.20). The algebraic manipulations required to show that commutativity, associativity, and the first distributive property hold for $V_{\mathbf{C}}$ are essentially the same algebraic manipulations we performed in Exercise 1.A.1, Exercise 1.A.2, Exercise 1.A.3, and Exercise 1.A.4, except instead of using the algebraic properties of $\mathbf{R}$, we use the algebraic properties of V (i.e. the properties listed in (1.20)); we will avoid repeating ourselves and instead verify the remaining conditions.

Additive identity. We claim that the additive identity in $V_{\mathbf{C}}$ is $0 + i0$, where 0 is the additive identity in V. Indeed, for any $u + iv \in V_{\mathbf{C}}$ we have

$$(u + iv) + (0 + i0) = (u + 0) + i(v + 0) = u + iv.$$

Additive inverse. We claim that the additive inverse of an element $u + iv \in V_{\mathbf{C}}$ is the element $(-u) + i(-v)$, where $-u$ is the additive inverse of u in V. Indeed,

$$(u + iv) + ((-u) + i(-v)) = (u + (-u)) + i(v + (-v)) = 0 + i0.$$

Multiplicative identity. For any $u + iv \in V_{\mathbf{C}}$, we have

$$(1 + 0i)(u + iv) = (1u - 0v) + i(1v + 0u) = u + iv.$$

Distributive properties. For the second distributive property, let $a + bi, c + di \in \mathbf{C}$ and $u + iv \in V_{\mathbf{C}}$ be given. Observe that

$$\begin{aligned}
((a + bi) + (c + di))(u + iv) &= ((a + c) + (b + d)i)(u + iv) \\
&= ((a + c)u - (b + d)v) + i((a + c)v + (b + d)u) \\
&= (au + cu - bv - dv) + i(av + cv + bu + du) \\
&= ((au - bv) + i(av + bu)) + ((cu - dv) + i(cv + du)) \\
&= (a + bi)(u + iv) + (c + di)(u + iv),
\end{aligned}$$

where we have used the second distributive property for V for the third equality.

1.C. Subspaces

Exercise 1.C.1. For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$
(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$
(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$
(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

Solution. Let U denote the set in each part of this question.

(a) This is a subspace of $\mathbf{F}^3$. Certainly the zero vector belongs to U. Suppose that $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in U$ and $\alpha \in \mathbf{F}$ and observe that

$$(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) = 0 + 0 = 0,$$
$$\alpha x_1 + 2(\alpha x_2) + 3(\alpha x_3) = \alpha(x_1 + 2x_2 + 3x_3) = \alpha 0 = 0.$$

Thus $x + y$ and αx also belong to U. It follows from (1.34) that U is a subspace of V.

(b) This is not a subspace of $\mathbf{F}^3$ because it does not contain the zero vector.

(c) This is not a subspace of $\mathbf{F}^3$ because it is not closed under addition: $(1, 1, 0)$ and $(0, 0, 1)$ belong to U, but $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$ does not belong to U.

(d) This is a subspace of $\mathbf{F}^3$. Note that $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 - 5x_3 = 0\}$. Certainly the zero vector belongs to U. Suppose that $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in U$ and $\alpha \in \mathbf{F}$ and observe that

$$(x_1 + y_1) - 5(x_3 + y_3) = (x_1 - 5x_3) + (y_1 - 5y_3) = 0 + 0 = 0,$$
$$\alpha x_1 - 5(\alpha x_3) = \alpha(x_1 - 5x_3) = \alpha 0 = 0.$$

Thus $x + y$ and αx also belong to U. It follows from (1.34) that U is a subspace of V.

Exercise 1.C.2. Verify all assertions about subspaces in Example 1.35.

Solution.

(a) The assertion is that if $b \in \mathbf{F}$, then

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\} = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 - 5x_4 = b\}$$

is a subspace of $\mathbf{F}^4$ if and only if $b = 0$. Indeed, if $b \neq 0$ then U is not a subspace of $\mathbf{F}^4$ because the zero vector does not belong to U, and if $b = 0$ then we may argue as in Exercise 1.C.1 (d) to see that U is a subspace of $\mathbf{F}^4$.

(b) The assertion is that the set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$, i.e.

$$U = \{f : [0, 1] \to \mathbf{R}, f \text{ continuous}\}$$

is a subspace of $\mathbf{R}^{[0,1]}$. Certainly the zero function $x \mapsto 0$ on $[0, 1]$ is continuous and hence belongs to U, and it is well-known from elementary real analysis that sums and constant multiples of continuous functions are again continuous. It follows from (1.34) that U is a subspace of $\mathbf{R}^{[0,1]}$.

(c) The assertion is that the set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$, i.e.

$$U = \{f : \mathbf{R} \to \mathbf{R}, f \text{ differentiable}\}$$

is a subspace of $\mathbf{R}^{\mathbf{R}}$. Certainly the zero function $x \mapsto 0$ on $\mathbf{R}$ is differentiable and hence belongs to U, and it is well-known from elementary real analysis that sums and constant multiples of differentiable functions are again differentiable. It follows from (1.34) that U is a subspace of $\mathbf{R}^{\mathbf{R}}$.

(d) The assertion is that the set U of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$. If $b \neq 0$, then the zero function $x \mapsto 0$ on $(0, 3)$, which has derivative $0 \neq b$ at $x = 2$, does not belong to U and thus U is not a subspace of $\mathbf{R}^{(0,3)}$.

Suppose that $b = 0$ and note that the zero function now belongs to U. If $f, g \in U$ and $\alpha \in \mathbf{R}$, then

$$(f+g)'(2) = f'(2) + g'(2) = 0 + 0 = 0 \quad \text{and} \quad (\alpha f)'(2) = \alpha f'(2) = \alpha 0 = 0.$$

Thus $f + g$ and αf belong to U. It follows from (1.34) that U is a subspace of $\mathbf{R}^{(0,3)}$.

(e) The assertion is that the set U of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^{\infty}$. Certainly the zero sequence $(0, 0, 0, ...)$ has limit 0 and hence belongs to U. Suppose that $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$ belong to U and $\alpha \in \mathbf{C}$. Using basic results about limits, observe that

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0 + 0 = 0$$

$$\text{and} \quad \lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n = \alpha 0 = 0.$$

Thus $x + y$ and αx belong to U. It follows from (1.34) that U is a subspace of $\mathbf{C}^{(0,3)}$.

Exercise 1.C.3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

Solution. Let U be the set in question; it is straightforward to verify that the zero function belongs to U. Suppose that $f, g \in U$ and $\alpha \in \mathbf{R}$. Observe that

$$(f+g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f(2) + g(2)) = 3(f+g)(2)$$

$$\text{and} \quad (\alpha f)'(-1) = \alpha f'(-1) = \alpha(3f(2)) = 3(\alpha f(2)) = 3(\alpha f)(2).$$

Thus $f + g$ and αf belong to U. It follows from (1.34) that U is a subspace of $\mathbf{R}^{(-4,4)}$.

Exercise 1.C.4. Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

Solution. Let U be the set in question. If $b \neq 0$ then the zero function $x \mapsto 0$ on $[0, 1]$, which has integral $0 \neq b$ over $[0, 1]$, does not belong to U and thus U is not a subspace of $\mathbf{R}^{[0,1]}$.

Suppose that $b = 0$ and note that the zero function now belongs to U. If $f, g \in U$ and $\alpha \in \mathbf{R}$, then using basic properties of integration we have

$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0 + 0 = 0 \quad \text{and} \quad \int_0^1 (\alpha f) = \alpha \int_0^1 f = \alpha 0 = 0.$$

Thus $f + g$ and αf belong to U. It follows from (1.34) that U is a subspace of $\mathbf{R}^{[0,1]}$.

Exercise 1.C.5. Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

Solution. The question is whether the subset

$$\mathbf{R}^2 = \{(x, y) : x, y \in \mathbf{R}\} \subseteq \{(z, w) : z, w \in \mathbf{C}\} = \mathbf{C}^2$$

is a subspace, where we are taking complex scalars in $\mathbf{C}^2$. This is not a subspace because it is not closed under scalar multiplication: $(1, 0) \in \mathbf{R}^2$ but $i(1, 0) = (i, 0) \notin \mathbf{R}^2$.

Exercise 1.C.6.

(a) Is $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?

(b) Is $\{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of $\mathbf{C}^3$?

Solution.

(a) Let U be the set in question. For $a, b \in \mathbf{R}$ we have $a^3 = b^3$ if and only if $a = b$ and thus the set U can be expressed as

$$U = \{(a, a, c) \in \mathbf{R}^3 : a, c \in \mathbf{R}\}.$$

Certainly $(0, 0, 0) \in U$. If $(a, a, c), (x, x, y) \in U$ and $\lambda \in \mathbf{R}$, then observe that

$$(a, a, c) + (x, x, y) = (a + x, a + x, c + y) \in U \quad \text{and} \quad \lambda(a, a, c) = (\lambda a, \lambda a, \lambda c) \in U.$$

It follows from (1.34) that U is a subspace of $\mathbf{R}^3$.

(b) Let U be the set in question. Observe that

$$\left(\frac{-1 + \sqrt{3}i}{2}\right)^3 = \left(\frac{-1 - \sqrt{3}i}{2}\right)^3 = 1.$$

It follows that $u := \left(\frac{-1+\sqrt{3}i}{2}, 1, 0\right)$ and $v := \left(\frac{-1-\sqrt{3}i}{2}, 1, 0\right)$ belong to U. However,

$$u + v = (-1, 2, 0) \notin U.$$

Thus U is not a subspace of $\mathbf{C}^3$ because it is not closed under addition.

Exercise 1.C.7. Prove or give a counterexample: If U is a nonempty subset of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbf{R}^2$.

Solution. For a counterexample, consider $U = \{(a, b) : a, b \in \mathbf{Q}\} \subseteq \mathbf{R}^2$, which satisfies the required conditions since the sum of two rational numbers is a rational number and the additive inverse of a rational number is a rational number. However, U is not a subspace of $\mathbf{R}^2$ because it is not closed under scalar multiplication: $(1, 0) \in U$ but $\sqrt{2}(1, 0) = \left(\sqrt{2}, 0\right) \notin U$.

Exercise 1.C.8. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

Solution. Let U be the union of the x- and y-axes, i.e.

$$U = \{(x, 0) : x \in \mathbf{R}\} \cup \{(0, y) : y \in \mathbf{R}\}.$$

It is straightforward to verify that U is closed under scalar multiplication. However, U is not a subspace of $\mathbf{R}^2$ because it is not closed under addition: $(1, 0)$ and $(0, 1)$ belong to U, but $(1, 0) + (0, 1) = (1, 1)$ does not.

Exercise 1.C.9. A function $f : \mathbf{R} \to \mathbf{R}$ is called *periodic* if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution. Consider the periodic functions $\sin(x)$ and $\sin\left(\sqrt{2}x\right)$ and let $f(x) = \sin(x) + \sin\left(\sqrt{2}x\right)$. We will show that f is not periodic.

Suppose there was a positive real number p such that $f(x) = f(x + p)$ for all $x \in \mathbf{R}$, i.e.

$$\sin(x) + \sin\left(\sqrt{2}x\right) = \sin(x + p) + \sin\left(\sqrt{2}x + \sqrt{2}p\right) \text{ for all } x \in \mathbf{R}. \tag{1}$$

By differentiating this equation twice, we see that

$$\sin(x) + 2\sin\left(\sqrt{2}x\right) = \sin(x + p) + 2\sin\left(\sqrt{2}x + \sqrt{2}p\right) \text{ for all } x \in \mathbf{R}. \tag{2}$$

Subtracting equation (1) from equation (2) gives us

$$\sin\left(\sqrt{2}x\right) = \sin\left(\sqrt{2}x + \sqrt{2}p\right) \text{ for all } x \in \mathbf{R}, \tag{3}$$

which together with equation (1) implies that

$$\sin(x) = \sin(x + p) \text{ for all } x \in \mathbf{R}. \tag{4}$$

By taking $x = 0$ in equation (4) we see that $0 = \sin(p)$, which is the case if and only if $p = n\pi$ for some positive integer n (p was assumed to be positive). Substituting this value of p and $x = 0$ into equation (3) gives $0 = \sin\left(n\sqrt{2}\pi\right)$, which is the case if and only if $n\sqrt{2}\pi = m\pi$

for some integer m, which must be positive since n is positive. It follows that $\sqrt{2} = \frac{m}{n}$, contradicting the irrationality of $\sqrt{2}$.

Thus f is not periodic and we may conclude that the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ is not a subspace of $\mathbf{R}^{\mathbf{R}}$ because it is not closed under addition.

Exercise 1.C.10. Suppose V_1 and V_2 are subspaces of V. Prove that the intersection $V_1 \cap V_2$ is a subspace of V.

Solution. Because V_1 and V_2 are subspaces of V, we have $0 \in V_1$ and $0 \in V_2$ and thus $0 \in V_1 \cap V_2$. Suppose $u, v \in V_1 \cap V_2$ and $\lambda \in \mathbf{F}$. Since $u, v \in V_1$ and V_1 is a subspace of V, we have $u + v \in V_1$ and $\lambda u \in V_1$. Similarly, $u + v \in V_2$ and $\lambda u \in V_2$. Thus $u + v \in V_1 \cap V_2$ and $\lambda u \in V_1 \cap V_2$. We may use (1.34) to conclude that $V_1 \cap V_2$ is a subspace of V.

Exercise 1.C.11. Prove that the intersection of every collection of subspaces of V is a subspace of V.

Solution. Let $\mathcal{U}$ be an arbitrary collection of subspaces of V. We will show that $\bigcap \mathcal{U}$ is a subspace of V. The zero vector belongs to $\bigcap \mathcal{U}$ because each $U \in \mathcal{U}$ is a subspace of V and hence contains the zero vector. If $u, v \in \bigcap \mathcal{U}, \lambda \in \mathbf{F}$, and $U \in \mathcal{U}$, then $u, v \in U$ and thus $u + v \in U$ and $\lambda u \in U$ since U is a subspace of V. Because $U \in \mathcal{U}$ was arbitrary, it follows that $u + v \in \bigcap \mathcal{U}$ and $\lambda u \in \bigcap \mathcal{U}$. We may use (1.34) to conclude that $\bigcap \mathcal{U}$ is a subspace of V.

Exercise 1.C.12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution. Suppose that U and W are subspaces of V. We want to show that $U \cup W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$. If one of U or W is contained in the other then either $U \cup W = U$ or $U \cup W = W$; in either case, $U \cup W$ is then a subspace of V by assumption.

For the converse, it will suffice to show that if $U \cup W$ is a subspace of V and $U \nsubseteq W$, then $W \subseteq U$. Since $U \nsubseteq W$, there is some $u \in U$ such that $u \notin W$. Let $w \in W$ be given and note that, because $U \cup W$ is a subspace of V and $u, w \in U \cup W$, we must have $u + w \in U \cup W$. It cannot be the case that $u + w \in W$, since then $u + w - w = u \in W$, so it must be the case that $u + w \in U$. It follows that $u + w - u = w \in U$ and hence that $W \subseteq U$, as desired.

Exercise 1.C.13. Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

This exercise is surprisingly harder than Exercise 1.C.12, possibly because this exercise is not true if we replace $\mathbf{F}$ *with a field containing only two elements.*

Solution. Let U_1, U_2, and U_3 be subspaces of V. We want to show that $U = U_1 \cup U_2 \cup U_3$ is a subspace of V if and only if some U_j contains the other two. If some U_j contains the other two, then $U = U_j$ is a subspace of V by assumption.

Suppose that U is a subspace of V. If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $U = U_2 \cup U_3$ and we may apply Exercise 1.C.12 to see that either $U_2 \subseteq U_3$ or $U_3 \subseteq U_2$; in either case, one U_j contains the other two. Suppose therefore that no U_j is contained in the union of the other two. Seeking a contradiction, suppose further that no U_j contains the other two, so that

$$U_1 \nsubseteq (U_2 \cup U_3) \quad \text{and} \quad (U_2 \cup U_3) \nsubseteq U_1.$$

It follows that there exists some $u \in U_1$ such that $u \notin U_2 \cup U_3$ and some $v \in U_2 \cup U_3$ such that $v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq U$ and observe that no element of W belongs to U_1, for if $v + \lambda u \in U_1$ then $v + \lambda u - \lambda u = v \in U_1$—but $v \notin U_1$. Thus

$$W \cap U_1 = \emptyset \quad \text{and} \quad W \subseteq (U_1 \cup U_2 \cup U_3) \quad \Rightarrow \quad W \subseteq (U_2 \cup U_3).$$

Because W contains infinitely many elements, there must be some $i \in \{2, 3\}$ such that U_i contains infinitely many elements of W. There then exist $\lambda, \mu \in \mathbf{F}$ such that $\lambda \neq \mu$ and such that $v + \lambda u$ and $v + \mu u$ both belong to U_i, which implies that $(\lambda - \mu)u \in U_i$ since U_i is a subspace of V. This gives $u \in U_i$ since $\lambda - \mu \neq 0$, contradicting that $u \notin U_2 \cup U_3$. We may conclude that some U_j contains the other two.

Exercise 1.C.14. Suppose

$$U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}.$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

Solution. We claim that $U + W$ is the subspace

$$E = \{(x, y, 2x) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}.$$

To see this, let $(x, -x, 2x) \in U$ and $(y, y, 2y) \in W$ be given and notice that

$$(x, -x, 2x) + (y, y, 2y) = (x + y, -x + y, 2(x + y)) \in E.$$

Thus $U + W \subseteq E$. For the reverse inclusion, let $(x, y, 2x) \in E$ be given and observe that

$$(x, y, 2x) = \left(\frac{x-y}{2}, \frac{y-x}{2}, x - y\right) + \left(\frac{x+y}{2}, \frac{x+y}{2}, x + y\right) \in U + W.$$

Thus $U + W = E$, as claimed. In words, $U + W$ is the subspace of $\mathbf{F}^3$ consisting of those vectors whose third coordinate is twice their first coordinate.

Exercise 1.C.15. Suppose U is a subspace of V. What is $U + U$?

Solution. For $u + v \in U + U$ we have $u + v \in U$ since U is a subspace of V, and for $u \in U$ we have $u = u + 0 \in U + U$. Thus $U + U = U$.

Exercise 1.C.16. Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Solution. The operation is commutative, since addition of vectors in V is commutative. If $u + w \in U + W$, then $u + w = w + u \in W + U$, so that $U + W \subseteq W + U$. Similarly, $W + U \subseteq U + W$.

Exercise 1.C.17. Is the operation of addition on the subspaces of V associative? In other words, if V_1, V_2, V_3 are subspaces of V, is

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)?$$

Solution. The operation is associative, since addition of vectors in V is associative. If $(u_1 + u_2) + u_3 \in (U_1 + U_2) + U_3$, then

$$(u_1 + u_2) + u_3 = u_1 + (u_2 + u_3) \in U_1 + (U_2 + U_3),$$

so that $(U_1 + U_2) + U_3 \subseteq U_1 + (U_2 + U_3)$. Similarly, $U_1 + (U_2 + U_3) \subseteq (U_1 + U_2) + U_3$.

Exercise 1.C.18. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution. The subspace $\{0\}$ is the additive identity for the operation. If U is a subspace of V then $u + 0 = u$ for any $u \in U$; it follows that $U + \{0\} = U$.

Since $\{0\} + \{0\} = \{0\}$, the subspace $\{0\}$ is its own additive inverse. We claim that no other subspace of V has an additive inverse, i.e. if U is a subspace of V with $U \neq \{0\}$, then there does not exist a subspace W satisfying $U + W = \{0\}$. Indeed, simply observe that $U \subseteq U + W$ for any subspace W, so that $U + W \neq \{0\}$.

Exercise 1.C.19. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

Solution. This is false. For a counterexample, consider the real vector space $\mathbf{R}$ and observe that

$$\{0\} + \mathbf{R} = \mathbf{R} + \mathbf{R} = \mathbf{R},$$

but $\{0\} \neq \mathbf{R}$.

Exercise 1.C.20. Suppose

$$U = \big\{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\big\}.$$

Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

Solution. Let

$$W = \big\{(0, a, 0, b) \in \mathbf{F}^4 : a, b \in \mathbf{F}\big\};$$

it is straightforward to verify that W is a subspace of $\mathbf{F}^4$. If $v \in U \cap W$, then

$$v \in W \quad \Rightarrow \quad v = (0, a, 0, b) \text{ for some } a, b \in \mathbf{F},$$
$$v \in U \quad \Rightarrow \quad a = b = 0 \quad \Rightarrow \quad v = 0.$$

Thus $U \cap W = \{0\}$ and it follows from (1.46) that the sum $U + W$ is direct.

Let $(v_1, v_2, v_3, v_4) \in \mathbf{F}^4$ be given and observe that

$$(v_1, v_2, v_3, v_4) = (v_1, v_1, v_3, v_3) + (0, v_2 - v_1, 0, v_4 - v_3) \in U \oplus W.$$

Thus $\mathbf{F}^4 = U \oplus W$.

Exercise 1.C.21. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Solution. Let

$$W = \{(0, 0, a, b, c) \in \mathbf{F}^5 : a, b, c \in \mathbf{F}\};$$

it is straightforward to verify that W is a subspace of $\mathbf{F}^5$. If $v \in U \cap W$, then

$$v \in U \quad \Rightarrow \quad v = (x, y, x + y, x - y, 2x) \text{ for some } x, y \in \mathbf{F},$$
$$v \in W \quad \Rightarrow \quad x = y = 0 \quad \Rightarrow \quad v = 0.$$

Thus $U \cap W = \{0\}$ and it follows from (1.46) that the sum $U + W$ is direct.

Let $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbf{F}^5$ be given and observe that

$$\begin{aligned}(v_1, v_2, v_3, v_4, v_5) &= (v_1, v_2, v_1 + v_2, v_1 - v_2, 2v_1) \\ &\qquad + (0, 0, v_3 - (v_1 + v_2), v_4 - (v_1 - v_2), v_5 - 2v_1) \in U \oplus W.\end{aligned}$$

Thus $\mathbf{F}^5 = U \oplus W$.

Exercise 1.C.22. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution. Let

$$W_1 = \{(0, 0, a, 0, 0) \in \mathbf{F}^5 : a \in \mathbf{F}\}, \quad W_2 = \{(0, 0, 0, b, 0) \in \mathbf{F}^5 : b \in \mathbf{F}\},$$
$$W_3 = \{(0, 0, 0, 0, c) \in \mathbf{F}^5 : c \in \mathbf{F}\};$$

it is straightforward to verify that W_1, W_2, and W_3 are subspaces of $\mathbf{F}^5$. Suppose that

$$u = (x, y, x + y, x - y, 2x) \in U, \qquad w_1 = (0, 0, a, 0, 0) \in W_1,$$
$$w_2 = (0, 0, 0, b, 0) \in W_2, \quad \text{and} \quad w_3 = (0, 0, 0, 0, c) \in W_3$$

are such that $u + w_1 + w_2 + w_3 = 0$. That is,

$$(x, y, x + y + a, x - y + b, 2x + c) = (0, 0, 0, 0, 0),$$

from which it follows that $x = y = a = b = c = 0$. Thus the only way to express the zero vector as a sum $u + w_1 + w_2 + w_3 \in U + W_1 + W_2 + W_3$ is to take $u = w_1 = w_2 = w_3 = 0$ and so it follows from (1.45) that the sum $U + W_1 + W_2 + W_3$ is direct.

Let $(v_1, v_2, v_3, v_4, v_5) \in \mathbf{F}^5$ be given and observe that

$$\begin{aligned}(v_1, v_2, v_3, v_4, v_5) &= (v_1, v_2, v_1 + v_2, v_1 - v_2, 2v_1) + (0, 0, v_3 - (v_1 + v_2), 0, 0) \\ &\quad + (0, 0, 0, v_4 - (v_1 - v_2), 0) + (0, 0, 0, 0, v_5 - 2v_1) \in U \oplus W_1 \oplus W_2 \oplus W_3.\end{aligned}$$

Thus $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Exercise 1.C.23. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$

then $V_1 = V_2$.

Hint: When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in $\mathbf{F}^2$.

Solution. This is false. For a counterexample, consider $V = \mathbf{R}^2$,

$$U = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, \quad V_1 = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, \quad V_2 = \{(y, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}.$$

It is straightforward to verify that $U \cap V_1 = U \cap V_2 = \{0\}$, so that $U + V_1$ and $U + V_2$ are both direct sums (1.46), and that $U \oplus V_1 = U \oplus V_2 = \mathbf{R}^2$. However, $V_1 \neq V_2$ since $(1, 1) \in V_2$ but $(1, 1) \notin V_1$.

Exercise 1.C.24. A function $f : \mathbf{R} \to \mathbf{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f : \mathbf{R} \to \mathbf{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$. Let V_e denote the set of real-valued even functions on $\mathbf{R}$ and let V_o denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^\mathbf{R} = V_\mathrm{e} \oplus V_\mathrm{o}$.

Solution. Suppose that $f \in V_\mathrm{e} \cap V_\mathrm{o}$, so that $f(x) = -f(x)$ for all $x \in \mathbf{R}$. This implies that $f(x) = 0$ for all $x \in \mathbf{R}$, i.e. $f = 0$. Thus $V_\mathrm{e} \cap V_\mathrm{o} = \{0\}$ and it follows from (1.46) that the sum $V_\mathrm{e} + V_\mathrm{o}$ is direct. For $f : \mathbf{R} \to \mathbf{R}$, define $f_\mathrm{e} : \mathbf{R} \to \mathbf{R}$ and $f_\mathrm{o} : \mathbf{R} \to \mathbf{R}$ by

$$f_\mathrm{e}(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_\mathrm{o}(x) = \frac{f(x) - f(-x)}{2}.$$

It is straightforward to verify that f_e is an even function, f_o is an odd function, and $f = f_\mathrm{e} + f_\mathrm{o}$. We may conclude that $\mathbf{R}^\mathbf{R} = V_\mathrm{e} \oplus V_\mathrm{o}$.

Chapter 2. Finite-Dimensional Vector Spaces

2.A. Span and Linear Independence

Exercise 2.A.1. Find a list of four distinct vectors in $\mathbf{F}^3$ whose span equals

$$\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}.$$

Solution. Let W be the subspace in question and consider the list

$$v_1 = (1, 0, -1), \quad v_2 = (0, 1, -1), \quad v_3 = (1, 1, -2), \quad v_4 = (1, -1, 0).$$

We claim that $\operatorname{span}(v_1, v_2, v_3, v_4) = W$. If $a_1, a_2, a_3, a_4 \in \mathbf{F}$, then

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = (a_1 + a_3 + a_4, a_2 + a_3 - a_4, -a_1 - a_2 - 2a_3) \in W$$
$$\text{since} \quad (a_1 + a_3 + a_4) + (a_2 + a_3 - a_4) + (-a_1 - a_2 - 2a_3) = 0.$$

Thus $\text{span}(v_1, v_2, v_3, v_4) \subseteq W$. Now suppose that $(x, y, z) \in W$ and observe that $z = -x - y$. It follows that

$$(x, y, z) = (x, y, -x - y) = xv_1 + yv_2 \in \text{span}(v_1, v_2, v_3, v_4).$$

Thus $W \subseteq \text{span}(v_1, v_2, v_3, v_4)$ and we may conclude that $\text{span}(v_1, v_2, v_3, v_4) = W$, as claimed.

Exercise 2.A.2. Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Solution. This is true. Let $v \in V$ be given. Since $V = \text{span}(v_1, v_2, v_3, v_4)$, there are scalars a_1, a_2, a_3, a_4 such that $v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$. Observe that

$$\begin{aligned} &a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4 \\ &\qquad = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = v. \end{aligned}$$

Thus $v \in \text{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$. It follows that

$$V = \text{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4).$$

Exercise 2.A.3. Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $\text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$.

Solution. For any scalars $a_1, \dots, a_m \in \mathbf{F}$, observe that

$$\begin{aligned} &a_1v_1 + a_2(v_1 + v_2) + \cdots + a_m(v_1 + \cdots + v_m) \\ &\qquad = (a_1 + \cdots + a_m)v_1 + (a_2 + \cdots + a_m)v_2 + \cdots + a_mv_m. \end{aligned}$$

It follows that $\text{span}(w_1, \dots, w_m) \subseteq \text{span}(v_1, \dots, v_m)$. Similarly, for any scalars $a_1, \dots, a_m \in \mathbf{F}$, notice that

$$\begin{aligned} a_1v_1 + a_2v_2 + \cdots + a_mv_m &= (a_1 - a_2)v_1 + (a_2 - a_3)(v_1 + v_2) \\ &\quad + \cdots + (a_{m-1} - a_m)(v_1 + \cdots + v_{m-1}) + a_m(v_1 + \cdots + v_m). \end{aligned}$$

Thus $\text{span}(v_1, \dots, v_m) \subseteq \text{span}(w_1, \dots, w_m)$.

Exercise 2.A.4.

(a) Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.

(b) Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

Solution.

(a) Suppose the list consists of the single vector $v \in V$. If $v \neq 0$ and $a \in \mathbf{F}$ is such that $av = 0$, then Exercise 1.B.2 shows that we must have $a = 0$; it follows that the list v is linearly independent. If $v = 0$ then simply observe that $1v = 0$, demonstrating that the list v is linearly dependent.

(b) Suppose that the list consists of the vectors $u, v \in V$. If one of these vectors is a scalar multiple of the other, say $v = \lambda u$ for some $\lambda \in \mathbf{F}$, then observe that $v - \lambda u = 0$. Because the coefficient of v in this linear combination is non-zero, we see that the list u, v is linearly dependent.

Conversely, suppose that the list u, v is linearly dependent, so that $\mu v + \lambda u = 0$ with at least one of the coefficients μ, λ non-zero, say $\mu \neq 0$; it follows that $v = -\frac{\lambda}{\mu} u$.

Exercise 2.A.5. Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbf{R}^3$.

Solution. Let $t = 2$ and observe that

$$3(3, 1, 4) - 2(2, -3, 5) - (5, 9, 2) = (0, 0, 0).$$

Exercise 2.A.6. Show that the list $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$.

Solution. That the list is linearly dependent if $c = 8$ was shown in the first bullet point of (2.20). Conversely, suppose that the list is linearly dependent. Since $(1, -1, 2)$ is evidently not a scalar multiple of $(2, 3, 1)$, the linear dependence lemma (2.19) implies that $(7, 3, c)$ lies in the span of $(2, 3, 1)$ and $(1, -1, 2)$, i.e. there are scalars x and y such that

$$x(2, 3, 1) + y(1, -1, 2) = (7, 3, c).$$

Solving the equations $2x + y = 7$ and $3x - y = 3$ gives $x = 2$ and $y = 3$, whence $c = x + 2y = 8$.

Exercise 2.A.7.

(a) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $1 + i, 1 - i$ is linearly independent.

(b) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $1 + i, 1 - i$ is linearly dependent.

Solution.

(a) Suppose that x and y are real numbers such that

$$x(1 + i) + y(1 - i) = (x + y) + (x - y)i = 0.$$

Since a complex number is zero if and only if its real and imaginary parts are zero, we must have

$$x + y = 0 \text{ and } x - y = 0 \quad \Leftrightarrow \quad x = y = 0.$$

Thus the list $1 + i, 1 - i$ is linearly independent.

(b) Observe that $i(1 - i) = 1 + i$, so that $1 + i$ is a scalar multiple of $1 - i$. It follows from Exercise 2.A.4 (b) that the list $1 + i, 1 - i$ is linearly dependent.

Exercise 2.A.8. Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

Solution. Suppose that a_1, a_2, a_3, a_4 are scalars such that

$$\begin{aligned} &a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0 \\ &\qquad \Leftrightarrow \quad a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0. \end{aligned}$$

Since the list v_1, v_2, v_3, v_4 is linearly independent, we must have

$$a_1 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = 0,$$

which implies that $a_1 = a_2 = a_3 = a_4 = 0$. It follows that the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent.

Exercise 2.A.9. Prove or give a counterexample: If $v_1, v_2, ..., v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, ..., v_m$$

is linearly independent.

Solution. Suppose that $a_1, a_2, ..., a_m$ are scalars such that

$$\begin{aligned} & a_1(5v_1 - 4v_2) + a_2v_2 + a_3v_3 + \cdots + a_mv_m = 0 \\ & \qquad \Leftrightarrow \; 5a_1v_1 + (a_2 - 4a_1)v_2 + a_3v_3 + \cdots + a_mv_m = 0. \end{aligned}$$

Since the list $v_1, v_2, ..., v_m$ is linearly independent, we must have

$$5a_1 = a_2 - 4a_1 = a_3 = \cdots = a_m = 0,$$

which implies that $a_1 = a_2 = a_3 = \cdots = a_m = 0$. It follows that the list $5v_1 - 4v_2, v_2, v_3, ..., v_m$ is linearly independent.

Exercise 2.A.10. Prove or give a counterexample: If $v_1, v_2, ..., v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, ..., \lambda v_m$ is linearly independent.

Solution. Suppose that $a_1, a_2, ..., a_m$ are scalars such that

$$a_1\lambda v_1 + a_2\lambda v_2 + \cdots + a_m\lambda v_m = 0.$$

Since $\lambda \neq 0$, we may multiply both sides of this equation by λ^{-1} to obtain the equation

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0.$$

Since the list $v_1, v_2, ..., v_m$ is linearly independent, this implies that $a_1 = a_2 = \cdots = a_m = 0$. It follows that the list $\lambda v_1, \lambda v_2, ..., \lambda v_m$ is linearly independent.

Exercise 2.A.11. Prove or give a counterexample: If $v_1, ..., v_m$ and $w_1, ..., w_m$ are linearly independent lists of vectors in V, then the list $v_1 + w_1, ..., v_m + w_m$ is linearly independent.

Solution. This is false. Consider $\mathbf{R}$ as a vector space over itself. We have two linearly independent lists 1 and -1, but the list $1 + (-1) = 0$ is linearly dependent.

Exercise 2.A.12. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Solution. By the linear dependence lemma (2.19), there is a $j \in \{1, 2, \ldots, m\}$ such that $v_j + w \in \operatorname{span}\big(v_1 + w, \ldots, v_{j-1} + w\big)$. If $j = 1$ then $v_1 + w = 0$, i.e. $w = -v_1$. It follows that $w \in \operatorname{span}(v_1, \ldots, v_m)$.

If $j \geq 2$, then there are scalars $a_1, \ldots, a_{j-1}$ such that

$$v_j + w = a_1(v_1 + w) + \cdots + a_{j-1}\big(v_{j-1} + w\big) \quad \Leftrightarrow \quad v_j + \lambda w = a_1 v_1 + \cdots + a_{j-1} v_{j-1},$$

where $\lambda = 1 - \big(a_1 + \cdots + a_{j-1}\big)$. Note that λ must be non-zero: if this were not the case, then v_j would lie in the span of $v_1, \ldots, v_{j-1}$, which cannot happen since the list $v_1, \ldots, v_j$ is linearly independent. It follows that

$$w = \lambda^{-1}\big(a_1 v_1 + \cdots + a_{j-1} v_{j-1} - v_j\big),$$

so that $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Exercise 2.A.13. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, \ldots, v_m, w \text{ is linearly independent} \quad \Leftrightarrow \quad w \notin \operatorname{span}(v_1, \ldots, v_m).$$

Solution. If $w \in \operatorname{span}(v_1, \ldots, v_m)$ then the list $v_1, \ldots, v_m, w$ is linearly dependent by the third bullet point of (2.18). Conversely, suppose that the list $v_1, \ldots, v_m, w$ is linearly dependent. By the linear dependence lemma (2.19), one of the vectors in the list must be in the span of the previous vectors. It cannot be the case that some v_j belongs to $\operatorname{span}\big(v_1, \ldots, v_{j-1}\big)$ since this would contradict the linear independence of the list $v_1, \ldots, v_m$, so it must be the case that $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Exercise 2.A.14. Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that the list $v_1, \ldots, v_m$ is linearly independent if and only if the list $w_1, \ldots, w_m$ is linearly independent.

Solution. Let $W = \operatorname{span}(w_1, \ldots, w_m)$; by Exercise 2.A.3 we also have $W = \operatorname{span}(v_1, \ldots, v_m)$. If the list $w_1, \ldots, w_m$ is linearly dependent, then using the linear dependence lemma (2.19) we may remove some w_j from the list $w_1, \ldots, w_m$ to obtain a spanning list for W of length $m-1$. It follows from (2.22) that the list $v_1, \ldots, v_m$, which spans W, must be linearly dependent. A similar argument shows that the list $w_1, \ldots, w_m$ must be linearly dependent if the list $v_1, \ldots, v_m$ is linearly dependent.

Exercise 2.A.15. Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

Solution. As noted in the textbook, $\mathcal{P}_4(\mathbf{F})$ is spanned by the list $1, z, z^2, z^3, z^4$, which has length 5. It follows from (2.22) that any linearly independent list in $\mathcal{P}_4(\mathbf{F})$ can have length at most 5.

Exercise 2.A.16. Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

Solution. As shown in (2.16) (b), the list $1, z, z^2, z^3, z^4$, which has length 5, is linearly independent in $\mathcal{P}_4(\mathbf{F})$. It follows from (2.22) that any spanning list for $\mathcal{P}_4(\mathbf{F})$ must have length at least 5.

Exercise 2.A.17. Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Solution. First suppose that V is finite-dimensional, so that it is spanned by some list $w_1, \ldots, w_m$, and let $v_1, v_2, \ldots$ be any sequence of vectors in V; by (2.22), the list $v_1, v_2, \ldots, v_{m+1}$ must be linearly dependent.

Now suppose that V is infinite-dimensional, so that no list of vectors in V is a spanning list. Certainly $V \neq \{0\}$, so pick any $v_1 \neq 0$ in V and note that the list v_1 is linearly independent. Suppose that after m steps we have chosen a linearly independent list $v_1, \ldots, v_m$. By assumption $V \neq \operatorname{span}(v_1, \ldots, v_m)$, so pick any $v_{m+1} \notin \operatorname{span}(v_1, \ldots, v_m)$ and note that, by Exercise 2.A.13, the list $v_1, \ldots, v_m, v_{m+1}$ is linearly independent. This process recursively defines a sequence of vectors $v_1, v_2, \ldots$ such that $v_1, \ldots, v_m$ is linearly independent for each positive integer m.

Exercise 2.A.18. Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

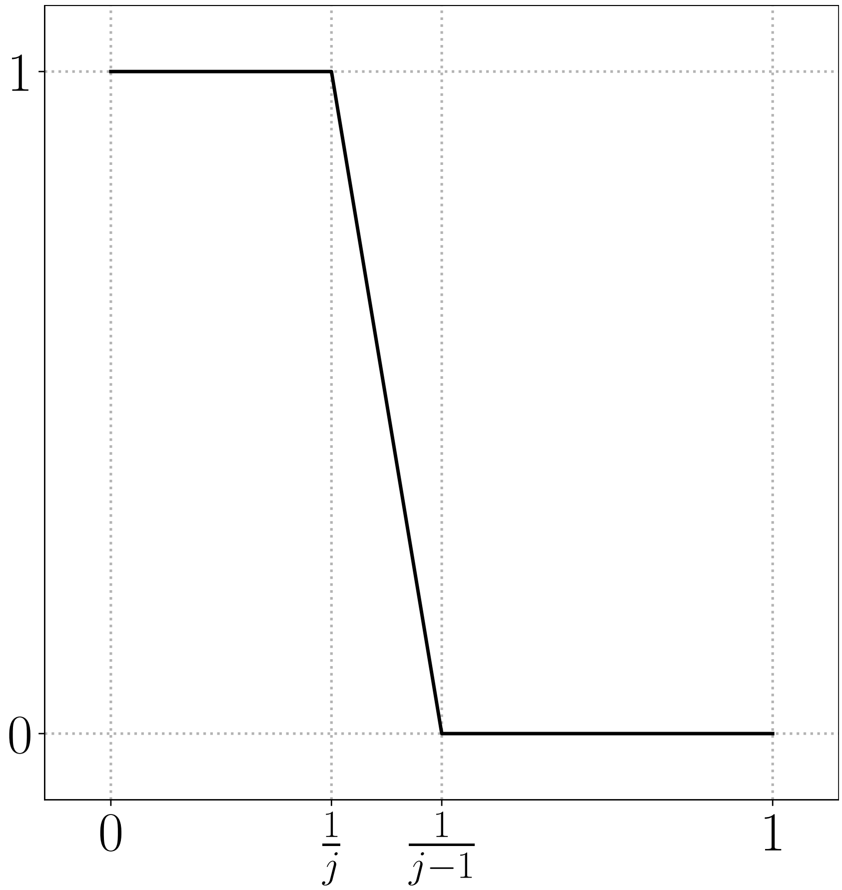
Solution. Consider the sequence of vectors $v_1, v_2, \ldots$, where $v_j \in \mathbf{F}^\infty$ is the sequence with a 1 in the j^{th} position and 0's elsewhere. For each positive integer m, it is straightforward to verify that the list $v_1, \ldots, v_m$ is linearly independent; it follows from Exercise 2.A.17 that $\mathbf{F}^\infty$ is infinite-dimensional.

Exercise 2.A.19. Prove that the real vector space of all continuous real-valued functions on the interval $[0,1]$ is infinite-dimensional.

Solution. Consider the sequence of continuous functions $f_1, f_2, \ldots$ on the interval $[0,1]$, where $f_1(x) = 1$ for all $x \in [0,1]$ and, for $j \geq 2$,

$$f_j(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq \frac{1}{j}, \\ 0 \text{ if } \frac{1}{j-1} \leq x \leq 1. \end{cases}$$

On the interval $\left(\frac{1}{j}, \frac{1}{j-1}\right)$, take f_j to be the line segment joining the points $\left(\frac{1}{j}, 1\right)$ and $\left(\frac{1}{j-1}, 0\right)$, so that f_j is a continuous function on $[0,1]$.



Let m be a positive integer and suppose we have real numbers $a_1, \ldots, a_m$ such that

$$a_1 f_1(x) + a_2 f_2(x) + \cdots + a_m f_m(x) = 0$$

for all $x \in [0, 1]$. Taking $x = 1$ gives us

$$a_1 f_1(1) + a_2 f_2(1) + \cdots + a_m f_m(1) = a_1 = 0.$$

Similarly, taking $x = \frac{1}{2}$ gives us

$$a_2 f_2\left(\tfrac{1}{2}\right) + \cdots + a_m f_m\left(\tfrac{1}{2}\right) = a_2 = 0.$$

By continuing in this fashion, taking $x = \frac{1}{j}$ for each $j \in \{1, 2, \dots, m\}$, we see that $a_1 = a_2 = \cdots = a_m = 0$. It follows that the list $f_1, f_2, \dots, f_m$ is linearly independent for each positive integer m and thus by Exercise 2.A.17 the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

Exercise 2.A.20. Suppose $p_0, p_1, \dots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \dots, m\}$. Prove that $p_0, p_1, \dots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

Solution. Since $\mathcal{P}_m(\mathbf{F})$ is spanned by the list $1, x, \dots, x^m$ of length $m + 1$, (2.22) implies that the list $p_0, p_1, \dots, p_m, x$ of length $m + 2$ is linearly dependent. The linear dependence lemma (2.19) implies that one of the vectors from this list belongs to the span of the previous vectors. Notice that for any scalars $a_0, \dots, a_m$,

$$x = a_0 p_0(x) + \cdots + a_m p_m(x) \text{ for all } x \in \mathbf{F} \quad \Rightarrow \quad 2 = a_0 p_0(2) + \cdots + a_m p_m(2) = 0,$$

which is a contradiction; it follows that $x \notin \operatorname{span}(p_0, p_1, \dots, p_m)$ and thus there must be some $j \in \{0, \dots, m\}$ such that $p_j \in \operatorname{span}(p_0, p_1, \dots, p_{j-1})$. The third bullet point of (2.18) then implies that the list $p_0, p_1, \dots, p_m$ is linearly dependent.