1 Section 8.3 Exercises

Exercises with solutions from Section 8.3 of [UA].

Exercise 8.3.1. Supply the details to show that when $x = \pi/2$ the product formula in (2) is equivalent to

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right), \tag{3}$$

where the infinite product in (2) is interpreted to be a limit of partial products. (Although it is not necessary for what follows, it might be useful to review the treatment of infinite products in Exercises 2.4.10 and 2.7.10.)

Solution. Let's express the product in (2) as

$$\sin(x) = x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots = x\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{x}{k\pi}\right)\left(1+\frac{x}{k\pi}\right).$$

Taking $x = \frac{\pi}{2}$ gives us

$$1 = \frac{\pi}{2}\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{1}{2k}\right)\left(1+\frac{1}{2k}\right) = \frac{\pi}{2}\lim_{n\to\infty}\prod_{k=1}^{n}\frac{(2k-1)(2k+1)}{2k\cdot 2k}.$$

If we let $p_n = \prod_{k=1}^{n}\frac{(2k-1)(2k+1)}{2k\cdot 2k}$, then the equation above becomes $\frac{2}{\pi} = \lim_{n\to\infty} p_n$. Note that each p_n is positive; using the continuity of $x \mapsto \frac{1}{x}$, we then have

$$\frac{\pi}{2} = \frac{1}{\lim_{n\to\infty} p_n} = \lim_{n\to\infty}\frac{1}{p_n} = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{2k\cdot 2k}{(2k-1)(2k+1)}.$$

Exercise 8.3.2. Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a,b]$ and derive the integration-by-parts formula

$$\int_a^b h(t)k'(t)\,dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\,dt\,.$$

Solution. See Exercise 7.5.6 (a).

Exercise 8.3.3. (a) Using the simple identity $\sin^n(x) = \sin^{n-1}(x)\sin(x)$ and the previous exercise, derive the recurrence relation

$$b_n = \frac{n-1}{n}b_{n-2} \quad \text{for all } n \geq 2.$$

(b) Use this relation to generate the first three even terms and the first three odd terms of the sequence (b_n).

(c) Write a general expression for b_{2n} and b_{2n+1}.

Solution. (a) Using integration-by-parts, we have

$$\begin{aligned}
b_n &= \int_0^{\frac{\pi}{2}} \sin^n(x)\,dx \\
&= \int_0^{\frac{\pi}{2}} \sin^{n-1}(x)\sin(x)\,dx \\
&= \int_0^{\frac{\pi}{2}} \sin^{n-1}(x)(-\cos(x))'\,dx \\
&= -\sin^{n-1}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \sin^{n-1}(0)\cos(0) + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\cos^2(x)\,dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)[1-\sin^2(x)]\,dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\,dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n(x)\,dx \\
&= (n-1)b_{n-2} - (n-1)b_n.
\end{aligned}$$

The desired recurrence relation follows.

(b) Some calculations reveal that

$$b_0 = \frac{\pi}{2}, \quad b_2 = \frac{\pi}{2}\cdot\frac{1}{2}, \quad b_4 = \frac{\pi}{2}\cdot\frac{1\cdot 3}{2\cdot 4} \quad \text{and} \quad b_1 = 1, \quad b_3 = \frac{2}{1\cdot 3}, \quad b_5 = \frac{2\cdot 4}{1\cdot 3\cdot 5}.$$

(c) Simple induction arguments show that

$$b_{2n} = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \quad \text{and} \quad b_{2n+1} = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)}.$$

Exercise 8.3.4. Show

$$\lim_{n\to\infty}\frac{b_{2n}}{b_{2n+1}} = 1,$$

and use this fact to finish the proof of Wallis's product formula in (3).

Solution. As noted in the textbook, the sequence (b_n) is decreasing and thus

$$0 < b_{2n+1} \le b_{2n} \le b_{2n-1} \quad \Longrightarrow \quad 1 \le \frac{b_{2n}}{b_{2n+1}} \le \frac{b_{2n-1}}{b_{2n+1}} \tag{$*$}$$

for each $n \in \mathbf{N}$. Using the recurrence relation from Exercise 8.3.3 (a), we have

$$\frac{b_{2n-1}}{b_{2n+1}} = 1 + \frac{1}{2n} \to 1$$

and hence it follows from $(*)$ and the Squeeze Theorem that $\lim_{n\to\infty} \frac{b_{2n}}{b_{2n+1}} = 1$.

Now let $q_n = \prod_{k=1}^{n} \frac{2k\cdot 2k}{(2k-1)(2k+1)}$; our goal is to show that $\frac{\pi}{2} = \lim_{n\to\infty} q_n$. Using the expressions for b_{2n} and b_{2n+1} derived in Exercise 8.3.3 (c), we find that

$$\frac{b_{2n}}{b_{2n+1}} = \frac{\pi}{2}\cdot\frac{1}{q_n} \quad \Longleftrightarrow \quad q_n = \frac{\pi}{2}\cdot\frac{b_{2n+1}}{b_{2n}}.$$

It follows from the previous paragraph that $\lim_{n\to\infty} q_n = \frac{\pi}{2}$.

Exercise 8.3.5. Derive the following alternative form of Wallis's product formula:

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}.$$

Solution. Letting $q_n = \prod_{k=1}^{n} \frac{2k\cdot 2k}{(2k-1)(2k+1)}$, some calculations reveal that

$$q_n = \frac{1}{2}\cdot\frac{2^{4n}(n!)^4}{[(2n)!]^2 n}\cdot\frac{2n}{2n+1},$$

from which we obtain

$$\sqrt{2q_n}\sqrt{1+\frac{1}{2n}} = \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}.$$

The alternative formula now follows as $\sqrt{2q_n} \to \sqrt{\pi}$ (Exercise 8.3.4) and $\sqrt{1+\frac{1}{2n}} \to 1$.

Exercise 8.3.6. Show that $1/\sqrt{1-x}$ has Taylor expansion $\sum_{n=0}^{\infty} c_n x^n$, where $c_0 = 1$ and

$$c_n = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$$

for $n \ge 1$.

Solution. See Exercise 6.6.10 (a).

Exercise 8.3.7. Show that $\lim c_n = 0$ but $\sum_{n=0}^{\infty} c_n$ diverges.

Solution. If we let $a_n = \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}} > 0$, then $c_n = \frac{a_n^{-1}}{\sqrt{n}}$. Since $a_n^{-1} \to \frac{1}{\sqrt{\pi}}$ (Exercise 8.3.5) and $\frac{1}{\sqrt{n}} \to 0$, we see that $\lim_{n\to\infty} c_n = 0$.

Because (a_n) is convergent (Exercise 8.3.5), there is some $M > 0$ such that $a_n \le M$ for each $n \in \mathbf{N}$, which implies that

$$c_n = \frac{a_n^{-1}}{\sqrt{n}} \ge \frac{M^{-1}}{\sqrt{n}}$$

for each $n \in \mathbf{N}$. Since $\sum_{n=1}^{\infty} \frac{M^{-1}}{\sqrt{n}}$ is divergent (Corollary 2.4.7), the Comparison Test (Theorem 2.7.4) shows that $\sum_{n=1}^{\infty} c_n$ is divergent. It follows that $\sum_{n=0}^{\infty} c_n$ is divergent.

Exercise 8.3.8. Using the expression for $E_N(x)$ from Lagrange's Remainder Theorem, show that equation (4) is valid for all $|x| < 1/2$. What goes wrong when we try to use this method to prove (4) for $x \in (1/2, 1)$?

Solution. See Exercise 6.6.10 (a); a small modification of that argument shows that equation (4) is valid for all $|x| \le \frac{1}{2}$.

Exercise 8.3.9. (a) Show

$$f(x) = f(0) + \int_0^x f'(t)\,dt\,.$$

(b) Now use a previous result from this section to show

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)\,dt\,.$$

(c) Continue in this fashion to complete the proof of the theorem.

Solution. (a) The Fundamental Theorem of Calculus (Theorem 7.5.1 (i)) shows that

$$\int_0^x f'(t)\,dt = f(x) - f(0).$$

(b) Using integration-by-parts, we have

$$\int_0^x f'(t)\,dt = \int_0^x f'(t)\cdot 1\,dt = xf'(x) - \int_0^x f''(t)t\,dt.$$

The Fundamental Theorem of Calculus shows that $\int_0^x f''(t)\,dt = f'(x) - f'(0)$; combining this with the equation above and part (a) gives

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)\,dt.$$

(c) Applying integration-by-parts again, we see that

$$\int_0^x f''(t)(x-t)\,dt = \frac{1}{2}f''(0)x^2 + \frac{1}{2}\int_0^x f^{(3)}(t)(x-t)^2\,dt,$$

so that

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{2}\int_0^x f^{(3)}(t)(x-t)^2\,dt.$$

If we continue applying integration-by-parts, we obtain

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \cdots + \frac{1}{N!}f^{(N)}(0)x^N + \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\,dt \\ &= S_N(x) + \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\,dt. \end{aligned}$$

The desired result follows.

Exercise 8.3.10. (a) Make a rough sketch of $1/\sqrt{1-x}$ and $S_2(x)$ over the interval $(-1,1)$, and compute $E_2(x)$ for $x = 1/2, 3/4$, and $8/9$.

(b) For a general x satisfying $-1 < x < 1$, show

$$E_2(x) = \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\,dt\,.$$

(c) Explain why the inequality

$$\left|\frac{x-t}{1-t}\right| \leq |x|$$

is valid, and use this to find an overestimate for $|E_2(x)|$ that no longer involves an integral. Note that this estimate will necessarily depend on x. Confirm that things are going well by checking this overestimate is in fact larger than $|E_2(x)|$ at the three computed values from part (a).

(d) Finally, show $E_N(x) \to 0$ as $N \to \infty$ for an arbitrary $x \in (-1,1)$.

Solution. (a) See Figure 1 for the sketch. The errors are:

$$E_2\left(\frac{1}{2}\right) \approx 0.0705, \quad E_2\left(\frac{3}{4}\right) \approx 0.4141, \quad E_2\left(\frac{8}{9}\right) \approx 1.2593.$$

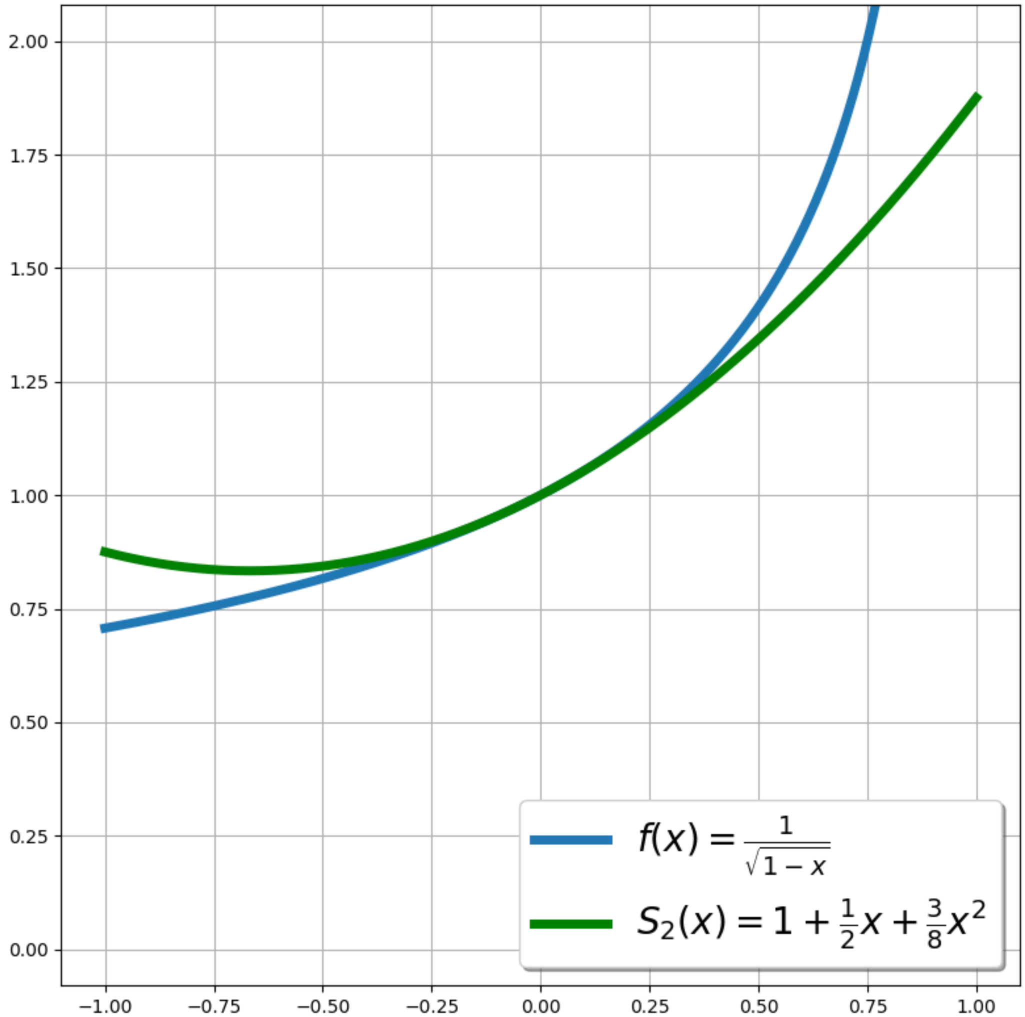

Figure 1: f and S_2 on $(-1, 1)$

(b) Using that

$$f^{(N)}(t) = \frac{1 \cdot 3 \cdots (2n-1)}{2^n}(1-t)^{-n-1/2}$$

and Theorem 8.3.1, we have

$$E_2(x) = \frac{1}{2}\int_0^x f^{(3)}(t)(x-t)^2\,dt = \frac{1}{2}\int_0^x \frac{15}{8}(1-t)^{-2-3/2}(x-t)^2\,dt$$
$$= \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\,dt.$$

(c) First suppose that $x \in [0,1)$. The inequality $0 \le t \le x < 1$ implies that

$$t \ge xt \quad \Longrightarrow \quad -t \le -xt \quad \Longrightarrow \quad x-t \le x-xt = x(1-t) \quad \Longrightarrow \quad \frac{x-t}{1-t} \le x.$$

Since $\frac{x-t}{1-t}$ and x are both non-negative in this case, we obtain the inequality $\left|\frac{x-t}{1-t}\right| \le |x|$. Now suppose that $x \in (-1,0)$. The inequality $-1 < x \le t < 0$ implies that

$$1 \ge x \quad \Longrightarrow \quad t \le xt \quad \Longrightarrow \quad t-x \le xt-x = (-x)(1-t) \quad \Longrightarrow \quad \frac{t-x}{1-t} \le -x.$$

Since $\frac{x-t}{1-t}$ and x are both negative in this case, we again obtain the inequality $\left|\frac{x-t}{1-t}\right| \le |x|$. Using this inequality and the expression for E_2 found in part (b), we have for $x \in [0,1)$:

$$\begin{aligned}
|E_2(x)| &= \frac{15}{16}\left|\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\,dt\right| \\
&\le \frac{15}{16}\int_0^x \left|\left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\right| dt \\
&\le \frac{15}{16}x^2 \int_0^x \frac{1}{(1-t)^{3/2}}\,dt \\
&= \frac{15}{8}x^2 \left[\frac{1}{\sqrt{1-t}}\right]_{t=0}^{t=x} \\
&= \frac{15}{8}x^2 \left(\frac{1}{\sqrt{1-x}} - 1\right).
\end{aligned}$$

Similarly, for $x \in (-1, 0)$:

$$\begin{aligned}
|E_2(x)| &= \frac{15}{16}\left|\int_x^0 \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}\,dt\right| \\
&\le \frac{15}{16}x^2 \int_x^0 \frac{1}{(1-t)^{3/2}}\,dt \\
&= \frac{15}{8}x^2 \left[\frac{1}{\sqrt{1-t}}\right]_{t=x}^{t=0} \\
&= \frac{15}{8}x^2\left(1 - \frac{1}{\sqrt{1-x}}\right).
\end{aligned}$$

Hence for all $x \in (-1, 1)$ we have the overestimate

$$|E_2(x)| \le \frac{15}{8}x^2\left|\frac{1}{\sqrt{1-x}} - 1\right|.$$

Denoting this overestimate by $G(x)$ and comparing with the values from part (a), we find that

$$\begin{aligned}
E_2\left(\frac{1}{2}\right) &\approx 0.0705 \le 0.1942 \approx G\left(\frac{1}{2}\right), \\
E_2\left(\frac{3}{4}\right) &\approx 0.4141 \le 1.0547 \approx G\left(\frac{3}{4}\right), \\
E_2\left(\frac{8}{9}\right) &\approx 1.2593 \le 2.9630 \approx G\left(\frac{8}{9}\right).
\end{aligned}$$

(d) Using that

$$f^{(N)}(t) = \frac{1 \cdot 3 \cdots (2n-1)}{2^n}(1-t)^{-n-1/2} = \frac{(2N)!}{2^{2N}N!}(1-t)^{-N-1/2}$$

and Theorem 8.3.1, we have for a fixed $x \in (-1, 1)$:

$$\begin{aligned}
E_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N\,dt &= \frac{1}{N!} \cdot \frac{(2N+2)!}{2^{2N+2}(N+1)!}\int_0^x \left(\frac{x-t}{1-t}\right)^N \frac{1}{(1-t)^{3/2}}\,dt \\
&= c_{N+1}(N+1)\int_0^x \left(\frac{x-t}{1-t}\right)^N \frac{1}{(1-t)^{3/2}}\,dt,
\end{aligned}$$

where (c_n) was defined in Exercise 8.3.6. From this expression we can derive, as in part (c), the estimate

$$|E_N(x)| \leq c_{N+1}(N+1)|x|^N \left| \frac{1}{\sqrt{1-x}} - 1 \right|.$$

Since $|x| < 1$ we have $\lim_{N\to\infty}(N+1)|x|^N = 0$ and we showed in Exercise 8.3.7 that $\lim_{N\to\infty} c_{N+1} = 0$; it now follows from the Squeeze Theorem that $\lim_{n\to\infty}|E_N(x)| = 0$.

Exercise 8.3.11. Assuming that the derivative of $\arcsin(x)$ is indeed $1/\sqrt{1-x^2}$, supply the justification that allows us to conclude

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1} \quad \text{for all } |x| < 1\,. \tag{5}$$

Solution. Because the power series $\sum_{n=0}^{\infty} c_n x^{2n}$ converges to $\frac{1}{\sqrt{1-x^2}}$ on $(-1,1)$, Exercise 6.5.4 (a) shows that the power series

$$\sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1}$$

converges on $(-1,1)$ and has derivative $\frac{1}{\sqrt{1-x^2}}$. As this is also the derivative of $\arcsin(x)$, Corollary 5.3.4 implies that

$$\arcsin(x) = k + \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1}$$

for all $x \in (-1,1)$ and some $k \in \mathbf{R}$; taking $x = 0$ shows that $k = 0$.

Exercise 8.3.12. Our work thus far shows that the Taylor series in (5) is valid for all $|x| < 1$, but note that $\arcsin(x)$ is continuous for all $|x| \leq 1$. Carefully explain why the series in (5) converges uniformly to $\arcsin(x)$ on the closed interval $[-1,1]$.

Solution. Letting $a_n = \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}$, we have

$$\frac{c_n}{2n+1} = \frac{a_n^{-1}}{\sqrt{n}(2n+1)}$$

for each $n \in \mathbf{N}$. As (a_n) converges to $\sqrt{\pi}$ (Exercise 8.3.5) and consists of strictly positive terms, there is some $L > 0$ such that $a_n \geq L$ for all $n \in \mathbf{N}$. It follows that

$$\frac{c_n}{2n+1} = \frac{a_n^{-1}}{\sqrt{n}(2n+1)} \leq \frac{L^{-1}}{\sqrt{n}(2n+1)}$$

and hence the series $\sum_{n=0}^{\infty} \frac{c_n}{2n+1}$ converges by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. Since each term of the series $\sum_{n=0}^{\infty} \frac{c_n}{2n+1}$ is positive, we have shown that the power series $\sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1}$ converges absolutely at $x = 1$; it follows from Theorem 6.5.2 that the convergence is uniform on $[-1, 1]$. This implies that the power series is continuous on $[-1, 1]$. Since arcsin is also continuous on this interval, the function

$$D(x) = \arcsin(x) - \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1}$$

is continuous on $[-1, 1]$ and satisfies, by Exercise 8.3.11, $D(x) = 0$ for all $x \in (-1, 1)$; by continuity, it must then be the case that $D(-1) = D(1) = 0$ also.

Exercise 8.3.13. (a) Show

$$\int_0^{\pi/2} \theta \, d\theta = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1},$$

being careful to justify each step in the argument. The term b_{2n+1} refers back to our earlier work on Wallis's product.

(b) Deduce

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},$$

and use this to finish the proof that $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$.

Solution. (a) We have

$$\int_0^{\pi/2} \theta \, d\theta = \int_0^{\pi/2} \left(\sum_{n=0}^{\infty} \frac{c_n}{2n+1} \sin^{2n+1}(\theta) \right) d\theta.$$

The uniform convergence of $\sum_{n=0}^{\infty} \frac{c_n}{2n+1} \sin^{2n+1}(\theta)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, Theorem 7.4.4, and the linearity of the integral (Theorem 7.4.2 (i)) allow us to interchange the integral with the series, obtaining

$$\int_0^{\pi/2} \theta \, d\theta = \sum_{n=0}^{\infty} \left(\int_0^{\pi/2} \frac{c_n}{2n+1} \sin^{2n+1}(\theta) \, d\theta \right) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1}.$$

(b) Using our formula for b_{2n+1} obtained in Exercise 8.3.3 (c), we see that $c_n b_{2n+1} = \frac{1}{2n+1}$ and hence by part (a):

$$\frac{\pi^2}{8} = \int_0^{\pi/2} \theta \, d\theta = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} b_{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Now we split the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ over the odd and even positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2};$$

these manipulations are valid because these are convergent series. It follows from the above expression that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.