The following is paraphrased from pages 8-9 of [PMA].

1 Archimedean property of $\mathbb{R}$

Theorem 1 (Theorem 1.19, p. 8, [PMA])**.** There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

A consequence of Theorem 1 is:

Theorem 2 (Archimedean property of $\mathbb{R}$)**.** Let $x > 0$ and y be real numbers. Then there exists a positive integer n such that $nx > y$.

Proof. Suppose to the contrary that for all positive integers n we have $nx \leq y$. Then the set $A = \{nx : n \in \mathbb{N}\}$ is non-empty and bounded above, so by the least-upper-bound property of $\mathbb{R}$ the supremum $\alpha = \sup A$ exists in $\mathbb{R}$. Since $x > 0$, we have $\alpha - x < \alpha$ so that $\alpha - x$ is not an upper bound for A. Hence there exists a positive integer m such that $\alpha - x < mx$, which gives $\alpha < (m + 1)x$; but this contradicts the fact that α is the supremum of A. □

2 Density of $\mathbb{Q}$ in $\mathbb{R}$

Lemma 1. Any real number lies between two consecutive integers. That is, for any $x \in \mathbb{R}$ there exists an $m \in \mathbb{Z}$ such that $m - 1 \leq x < m$.

Proof. By the Archimedean property, there exist positive integers m_1, m_2 such that $m_1 > x$ and $m_2 > -x$, which gives $-m_2 < x < m_1$. This implies that the set $A = \{n \in \mathbb{Z} : x < n\}$ is non-empty ($m_1 \in A$) and bounded below (by $-m_2$). Then by the well-ordering principle, A has a least element; call it m. Since this is the least element of A, we must have $m - 1 \notin A$ and so $m - 1 \leq x < m$. □

Theorem 3. Between any two real numbers there exists a rational number. That is, for any $x, y \in \mathbb{R}$ with $x < y$ there exists a $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. By the Archimedean property, there exists a positive integer n such that $n(y - x) > 1$. By Lemma 1, there exists an integer m such that $m - 1 \leq nx < m$. Combining these inequalities gives $nx < m \leq 1 + nx < ny$, which implies $x < \frac{m}{n} < y$. So the desired rational is $p = \frac{m}{n}$. □

[PMA] Rudin, W. (1976) *Principles of Mathematical Analysis.* 3rd edn.