1 Section 6.7 Exercises

Exercises with solutions from Section 6.7 of [UA].

Exercise 6.7.1. Assuming WAT, show that if f is continuous on $[a, b]$, then there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on $[a, b]$.

Solution. The Weierstrass Approximation Theorem implies that for each $n \in \mathbf{N}$ there exists a polynomial p_n such that

$$|f(x) - p_n(x)| < \frac{1}{n}$$

for all $x \in [a, b]$. It follows that $p_n \to f$ uniformly on $[a, b]$.

Exercise 6.7.2. Prove Theorem 6.7.3.

Solution. Since f is a continuous function defined on the compact set $[a, b]$, Theorem 4.4.7 implies that f is uniformly continuous on $[a, b]$ and hence there exists a $\delta > 0$ such that

$$x, y \in [a, b] \text{ and } |x - y| < \delta \quad \implies \quad |f(x) - f(y)| < \frac{\epsilon}{2}.$$

Let $n \in \mathbf{N}$ be such that $\frac{1}{n} < \delta$ and for each $0 \leq j \leq n$ let $x_j = a + j\frac{b-a}{n}$. Let $\phi : [a, b] \to \mathbf{R}$ be the polygonal function which is linear on each subinterval $[x_j, x_{j+1}]$ and passes through the points $(x_j, f(x_j))$ and $(x_{j+1}, f(x_{j+1}))$. For $x \in [a, b]$, we have $x \in [x_j, x_{j+1}]$ for some $0 \leq j \leq n - 1$. It follows that

$$|f(x) - \phi(x)| \leq |f(x) - \phi(x_j)| + |\phi(x_j) - \phi(x)| \leq |f(x) - \phi(x_j)| + |\phi(x_j) - \phi(x_{j+1})|;$$

for the last inequality we are using that ϕ is a line segment on the interval $[x_j, x_{j+1}]$ and thus $|\phi(x) - \phi(y)| \leq |\phi(x_j) - \phi(x_{j+1})|$ for any $x, y \in [x_j, x_{j+1}]$. By definition we have $\phi(x_j) = f(x_j)$ for any j and so

$$|f(x) - \phi(x)| \leq |f(x) - f(x_j)| + |f(x_j) - f(x_{j+1})| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Exercise 6.7.3. (a) Find the second degree polynomial $p(x) = q_0 + q_1x + q_2x^2$ that interpolates the three points $(-1, 1), (0, 0)$, and $(1, 1)$ on the graph of $g(x) = |x|$. Sketch $g(x)$ and $p(x)$ over $[-1, 1]$ on the same set of axes.

(b) Find the fourth degree polynomial that interpolates $g(x) = |x|$ at the points $x = -1, -1/2, 0, 1/2$, and 1. Add a sketch of this polynomial to the graph from (a).

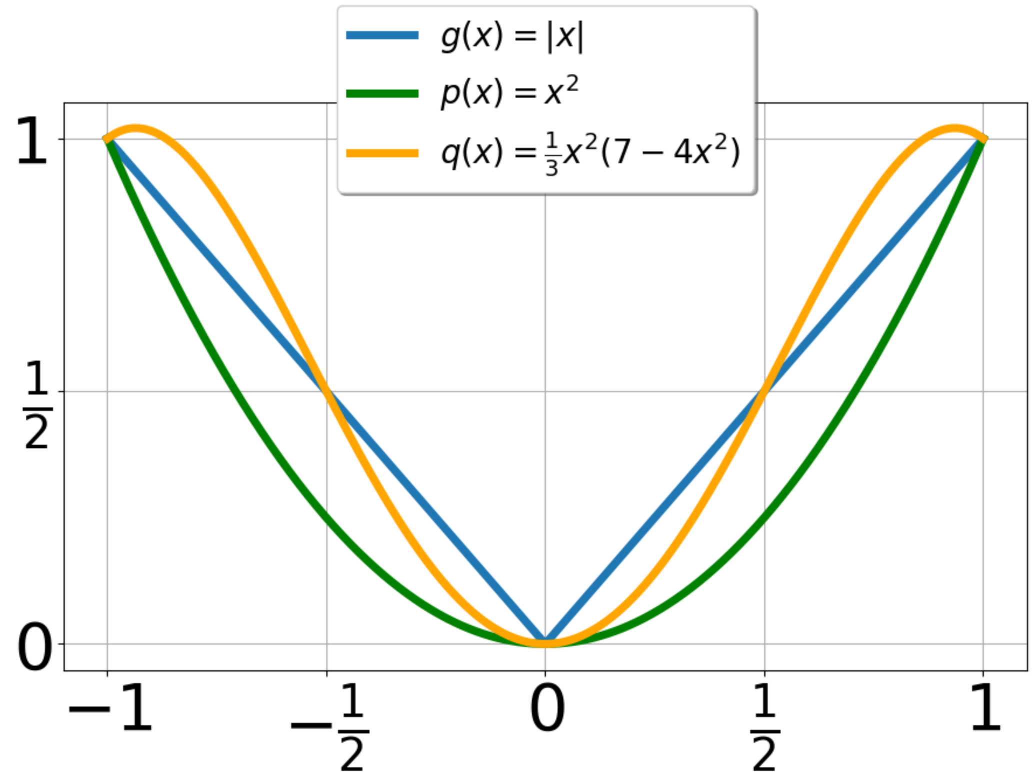

Figure 1: g, p, and q on $[-1, 1]$

Solution. (a) It is clear that the desired second degree polynomial is $p(x) = x^2$. See Figure 1 for the sketches.

(b) We are looking for a polynomial $q(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ such that $q(-1) = 1, q\left(-\frac{1}{2}\right) = \frac{1}{2}, q(0) = 0, q\left(\frac{1}{2}\right) = \frac{1}{2}$, and $q(1) = 1$. The condition $q(0) = 0$ immediately gives us $a_0 = 0$ and the remaining four conditions give us the linear system

$$\begin{cases} -a_1 + \phantom{\frac{1}{4}}a_2 - \phantom{\frac{1}{8}}a_3 + \phantom{\frac{1}{16}}a_4 = 1 \\ \frac{1}{2}a_1 + \frac{1}{4}a_2 - \frac{1}{8}a_3 + \frac{1}{16}a_4 = \frac{1}{2} \\ \frac{1}{2}a_1 + \frac{1}{4}a_2 + \frac{1}{8}a_3 + \frac{1}{16}a_4 = \frac{1}{2} \\ a_1 + \phantom{\frac{1}{4}}a_2 + \phantom{\frac{1}{8}}a_3 + \phantom{\frac{1}{16}}a_4 = 1 \end{cases}.$$

Using Gaussian elimination, or otherwise, this system can be solved to obtain $a_1 = 0, a_2 = \frac{7}{3}, a_3 = 0$, and $a_4 = -\frac{4}{3}$ and thus $q(x) = \frac{1}{3}x^2(7 - 4x^2)$. See Figure 1 for the sketch.

Exercise 6.7.4. Show that $f(x) = \sqrt{1-x}$ has Taylor series coefficients a_n where $a_0 = 1$ and

$$a_n = \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$.

Solution. We have $f(0) = a_0 = 1$ and it is a straightforward calculation to see that

$$f^{(n)}(x) = \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}(1-x)^{-n-1/2}$$

for $n \geq 1$. It follows from this that

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} = a_n = \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$.

Exercise 6.7.5. (a) Follow the advice in Exercise 6.6.9 to prove the Cauchy form of the remainder:

$$E_N(x) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x.

(b) Use this result to prove equation (1) is valid for all $x \in (-1, 1)$.

Solution. (a) See Exercise 6.6.9.

(b) Suppose $0 < |x| < 1$. For $n \in \mathbf{N}$, the Cauchy Remainder Theorem implies that there exists some c_n such that $0 < |c_n| < |x|$ and

$$\begin{aligned}
E_n(x) &= \frac{f^{(n+1)}(c_n)}{n!}(x-c_n)^n x \\
&= \frac{-1 \cdot 3 \cdots (2n-3)(2n-1)}{2^{n+1} n!}(1-c_n)^{-n-3/2}(x-c_n)^n x \\
&= -\frac{1}{2} \cdot \frac{1 \cdot 3 \cdots (2n-3)(2n-1)}{2 \cdot 4 \cdots (2n-2)(2n)} \left(\frac{x-c_n}{1-c_n}\right)^n \frac{x}{(1-c_n)^{3/2}} \\
&= -\frac{1}{2}\left(\prod_{j=1}^{n} \frac{2j-1}{2j}\right)\left(\frac{x-c_n}{1-c_n}\right)^n \frac{x}{(1-c_n)^{3/2}}.
\end{aligned}$$

Since $\frac{2j-1}{2j} < 1$ for each $1 \le j \le n$, we have $\prod_{j=1}^{n} \frac{2j-1}{2j} < 1$ and thus

$$|E_n(x)| < \left|\frac{x - c_n}{1 - c_n}\right|^n \frac{|x|}{(1 - c_n)^{3/2}};$$

we have used that $|c_n| < 1 \implies 0 < 1 - c_n < 2$ to obtain $|1 - c_n| = 1 - c_n$. Note that

$$c_n \le |c_n| < |x| \quad \implies \quad -|x| < -c_n \quad \implies \quad \frac{1}{(1 - c_n)^{3/2}} < \frac{1}{(1 - |x|)^{3/2}}.$$

Note further that if $0 < c_n < x < 1$ then

$$xc_n < c_n \quad \implies \quad \frac{x - c_n}{1 - c_n} < x \quad \implies \quad \left|\frac{x - c_n}{1 - c_n}\right| < |x|,$$

and if $-1 < x < c_n < 0$ then

$$c_n < xc_n \quad \implies \quad \frac{c_n - x}{1 - c_n} < -x \quad \implies \quad \left|\frac{x - c_n}{1 - c_n}\right| < |x|.$$

Combining these inequalities, we see that

$$|E_n(x)| < \frac{|x|^{n+1}}{(1 - |x|)^{3/2}}$$

and it follows that $\lim_{n\to\infty} E_n(x) = 0$ since $|x| < 1$.

Exercise 6.7.6. (a) Let

$$c_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \ge 1$. Show $c_n < \frac{2}{\sqrt{2n+1}}$.

(b) Use (a) to show that $\sum_{n=0}^{\infty} a_n$ converges (absolutely, in fact) where a_n is the sequence of Taylor coefficients generated in Exercise 6.7.4.

(c) Carefully explain how this verifies that equation (1) holds for all $x \in [-1, 1]$.

Solution. (a) We will prove this by induction. For the base case $n = 1$, we have

$$c_1 = \frac{1}{2} < \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{2(1) + 1}}.$$

Suppose the inequality holds for some $n \in \mathbf{N}$, so that

$$c_{n+1} = c_n \cdot \frac{2n+1}{2n+2} < \frac{2}{\sqrt{2n+1}} \cdot \frac{2n+1}{2n+2} = \frac{2\sqrt{2n+1}}{2n+2}.$$

Now observe that

$$\begin{aligned} \frac{2\sqrt{2n+1}}{2n+2} < \frac{2}{\sqrt{2n+3}} &\iff \frac{\sqrt{2n+1}}{2n+2} < \frac{1}{\sqrt{2n+3}} \\ &\iff \frac{2n+1}{4n^2+8n+4} < \frac{1}{2n+3} \\ &\iff 4n^2+8n+3 < 4n^2+8n+4 \\ &\iff 0 < 1. \end{aligned}$$

Thus $c_{n+1} < \frac{2}{\sqrt{2n+3}}$. This completes the induction step and the proof.

(b) Since

$$\sum_{n=0}^{\infty} |a_n| = 1 + \sum_{n=1}^{\infty} |a_n|,$$

it will suffice to show that $\sum_{n=1}^{\infty} |a_n|$ is convergent. Note that for $n \geq 1$ we have by part (a)

$$|a_n| = \frac{c_n}{2n-1} < \frac{2}{(2n-1)\sqrt{2n+1}} < \frac{2}{(2n-1)^{3/2}} \leq \frac{2}{n^{3/2}}.$$

Since the series $\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$ is convergent (Corollary 2.4.7), we see by comparison that the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(c) Part (b) shows that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at the points $x = -1$ and $x = 1$. It follows from Abel's Theorem (Theorem 6.5.4) that the power series converges uniformly and hence is continuous on $[-1, 1]$. Thus the function $h : [-1, 1] \to \mathbf{R}$ given by

$$h(x) = \sqrt{1-x} - \sum_{n=0}^{\infty} a_n x^n$$

is continuous on its domain and, by Exercise 6.7.5, satisfies $h(x) = 0$ for all $x \in (-1, 1)$. It must then be the case that $h(-1) = h(1) = 0$ also.

Exercise 6.7.7. (a) Use the fact that $|a| = \sqrt{a^2}$ to prove that, given $\epsilon > 0$, there exists a polynomial $q(x)$ satisfying

$$||x| - q(x)| < \epsilon$$

for all $x \in [-1, 1]$.

(b) Generalize this conclusion to an arbitrary interval $[a, b]$.

Solution. (a) Note that $x \in [-1, 1]$ implies that $1 - x^2 \in [0, 1]$ and thus by Exercise 6.7.6 we have

$$\sqrt{1-(1-x^2)} = \sum_{n=0}^{\infty} a_n (1-x^2)^n.$$

As we showed in Exercise 6.7.6 this convergence is uniform, so there exists an $N \in \mathbf{N}$ such that

$$\left|\sqrt{1-(1-x^2)} - \sum_{n=0}^{N} a_n (1-x^2)^n\right| = \left||x| - \sum_{n=0}^{N} a_n (1-x^2)^n\right| < \epsilon$$

for all $x \in [-1, 1]$. Thus the desired polynomial is $q(x) = \sum_{n=0}^{N} a_n (1-x^2)^n$.

(b) For $a < b$ and $\epsilon > 0$, we would like to find a polynomial p such that $||x| - p(x)| < \epsilon$ for all $x \in [a, b]$. Let $c = \max\{|a|, |b|\}$ and note that $c > 0$. Note further that $x \in [a, b]$ implies that $\frac{x}{c} \in [-1, 1]$ and thus by part (a) there exists a polynomial q such that

$$\left|\left|\frac{x}{c}\right| - q\left(\frac{x}{c}\right)\right| < \frac{\epsilon}{c} \tag{1}$$

for all $\frac{x}{c} \in [-1, 1]$, i.e for all $x \in [-c, c]$. Let p be the polynomial given by $p(x) = cq\left(\frac{x}{c}\right)$. It follows from (1) that

$$||x| - p(x)| < \epsilon$$

for all $x \in [-c, c]$ and hence in particular for all $x \in [a, b]$.

Exercise 6.7.8. (a) Fix $a \in [-1, 1]$ and sketch

$$h_a(x) = \frac{1}{2}(|x-a| + (x-a))$$

over $[-1, 1]$. Note that h_a is polygonal and satisfies $h_a(x) = 0$ for all $x \in [-1, a]$.

(b) Explain why we know $h_a(x)$ can be uniformly approximated with a polynomial on $[-1, 1]$.

(c) Let ϕ be a polygonal function that is linear on each subinterval of the partition

$$-1 = a_0 < a_1 < a_2 < \cdots < a_n = 1.$$

Show there exist constants $b_0, b_1, \ldots, b_{n-1}$ so that

$$\phi(x) = \phi(-1) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x)$$

for all $x \in [-1, 1]$.

(d) Complete the proof of WAT for the interval $[-1, 1]$, and then generalize to an arbitrary interval $[a, b]$.

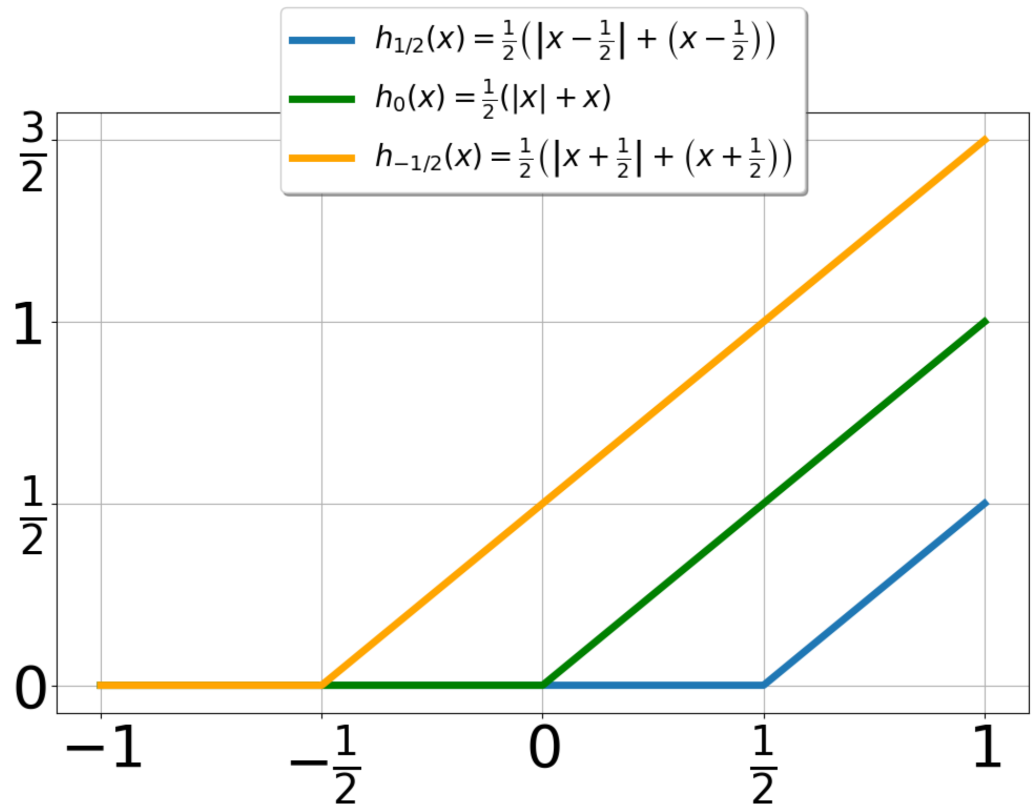

Figure 2: $h_{1/2}, h_0$, and $h_{1/2}$ on $[-1, 1]$

Solution. (a) See Figure 2 for a sketch of $h_{1/2}, h_0$, and $h_{1/2}$ on $[-1, 1]$.

(b) From Exercise 6.7.7 (b), for a given $\epsilon > 0$ we know that there exists a polynomial q such that

$$||x - a| - q(x - a)| < 2\epsilon$$

for all $x \in [-1, 1]$. Let $p(x) = \frac{1}{2}q(x - a) + \frac{1}{2}(x - a)$ and observe that

$$|h_a(x) - p(x)| = \frac{1}{2}||x - a| - q(x - a)| < \epsilon$$

for all $x \in [-1, 1]$.

(c) For $0 \le j \le n-1$, the polygonal function ϕ is given by a line segment on the subinterval $[a_j, a_{j+1}]$; let m_j be the slope of this line segment, i.e.

$$m_j = \frac{\phi(a_{j+1}) - \phi(a_j)}{a_{j+1} - a_j}.$$

Now set $b_0 = m_0$ and $b_j = m_j - m_{j-1}$ for $1 \le j \le n-1$ and let $\psi : [-1,1] \to \mathbf{R}$ be given by

$$\psi(x) = \phi(a_0) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x).$$

Our aim is to show that $\phi(x) = \psi(x)$ for all $x \in [-1,1]$. For such an x, we have $x \in [a_j, a_{j+1}]$ for some $0 \le j \le n-1$. Note that

$$\phi(x) = \phi(a_j) + m_j(x - a_j).$$

Note further that $h_{a_0}(x) = x - a_0, \ldots, h_{a_j}(x) = x - a_j$ and that $h_{a_{j+1}}(x) = \cdots = h_{a_{n-1}}(x) = 0$. Thus

$$\begin{aligned}
\psi(x) &= \phi(a_0) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_j h_{a_j}(x) \\
&= \phi(a_0) + m_0(x - a_0) + (m_1 - m_0)(x - a_1) + \cdots + (m_j - m_{j-1})(x - a_j) \\
&= \phi(a_0) + m_0(a_1 - a_0) + m_1(a_2 - a_1) + \cdots + m_{j-1}(a_j - a_{j-1}) + m_j(x - a_j) \\
&= \phi(a_1) + m_1(a_2 - a_1) + \cdots + m_{j-1}(a_j - a_{j-1}) + m_j(x - a_j) \\
&= \cdots \\
&= \phi(a_j) + m_j(x - a_j) \\
&= \phi(x).
\end{aligned}$$

(d) Let $f : [-1,1] \to \mathbf{R}$ be continuous and let $\epsilon > 0$ be given. By Theorem 6.7.3 (see Exercise 6.7.2), there exists a polygonal function $\phi : [-1,1] \to \mathbf{R}$ which is linear on each subinterval of some partition

$$-1 = a_0 < a_1 < \cdots < a_n = 1$$

and which satisfies $|f(x) - \phi(x)| < \frac{\epsilon}{2}$ for all $x \in [-1,1]$. By part (c), there exist constants $b_0, \ldots, b_{n-1}$ such that

$$\phi(x) = \phi(a_0) + b_0 h_{a_0}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x)$$

for all $x \in [-1,1]$. Furthermore, by part (b), for each $0 \le j \le n-1$ there exists a polynomial p_j such that

$$\left| h_{a_j}(x) - p_j(x) \right| < \frac{\epsilon}{2n(1 + |b_j|)}.$$

Let p be the polynomial given by

$$p(x) = \phi(a_0) + b_0 p_0(x) + \cdots + b_{n-1} p_{n-1}(x)$$

and observe that for any $x \in [-1, 1]$ we have

$$\begin{aligned}
|\phi(x) - p(x)| &= \left|b_0 h_{a_0}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x) - b_0 p_0(x) - \cdots - b_{n-1} p_{n-1}(x)\right| \\
&\leq |b_0||h_{a_0}(x) - p_0(x)| + \cdots + |b_{n-1}|\left|h_{a_{n-1}}(x) - p_{n-1}(x)\right| \\
&< \frac{\epsilon |b_0|}{2n(1 + |b_0|)} + \cdots + \frac{\epsilon |b_{n-1}|}{2n(1 + |b_{n-1}|)} \\
&< \frac{\epsilon}{2n} + \cdots + \frac{\epsilon}{2n} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

It now follows that for any $x \in [-1, 1]$ we have

$$|f(x) - p(x)| \leq |f(x) - \phi(x)| + |\phi(x) - p(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We can now prove the general case. For $a < b$, let $f : [a, b] \to \mathbf{R}$ be continuous and let $\epsilon > 0$ be given. We would like to find a polynomial p such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$. Note that the function

$$\begin{aligned}
[-1, 1] &\to [a, b] \\
x &\mapsto \frac{b - a}{2}(x + 1) + a
\end{aligned}$$

is a continuous bijection with inverse

$$\begin{aligned}
[a, b] &\to [-1, 1] \\
x &\mapsto \frac{2(x - a)}{b - a} - 1.
\end{aligned}$$

Thus $g : [-1, 1] \to \mathbf{R}$ given by

$$g(x) = f\left(\frac{b - a}{2}(x + 1) + a\right)$$

is well-defined and, as a composition of continuous functions, is continuous on $[-1, 1]$. It follows from our previous discussion that there exists a polynomial q such that $|g(x) - q(x)| < \epsilon$ for all $x \in [-1, 1]$. Let p be the polynomial defined by

$$p(x) = q\left(\frac{2(x-a)}{b-a} - 1\right).$$

Since $x \in [a, b]$ implies that $\frac{2(x-a)}{b-a} - 1 \in [-1, 1]$, we have

$$\left|g\left(\frac{2(x-a)}{b-a} - 1\right) - q\left(\frac{2(x-a)}{b-a} - 1\right)\right| = |f(x) - p(x)| < \epsilon$$

for all $x \in [a, b]$.

Exercise 6.7.9. (a) Find a counterexample which shows that WAT is not true if we replace the closed interval $[a, b]$ with the open interval (a, b).

(b) What happens if we replace $[a, b]$ with the closed set $[a, \infty)$. Does the theorem still hold?

Solution. (a) Consider $f : (0, 1) \to \mathbf{R}$ given by $f(x) = x^{-1}$. Since any polynomial is bounded on $(0, 1)$, if we could uniformly approximate f with a polynomial on $(0, 1)$ then we would have that f is bounded on $(0, 1)$, which is not true.

(b) The theorem does not hold. Consider $g : [0, \infty) \to \mathbf{R}$ given by $g(x) = \sin(x)$. Evidently g cannot be uniformly approximated by a constant polynomial on $[0, \infty)$, and for a non-constant polynomial p we have $\lim_{x\to\infty}|p(x)| = +\infty$, whereas $|g(x)| \leq 1$ for all $x \in [0, \infty)$; it follows that we cannot uniformly approximate g with a non-constant polynomial on $[0, \infty)$ either.

Exercise 6.7.10. Is there a countable subset of polynomials $\mathcal{C}$ with the property that every continuous function on $[a, b]$ can be uniformly approximated by polynomials from $\mathcal{C}$?

Solution. There is such a countable subset. Let $\mathcal{P}(\mathbf{R})$ be the collection of polynomials with real coefficients, let $\mathcal{P}(\mathbf{Q}) \subseteq \mathcal{P}(\mathbf{R})$ be the collection of polynomials with rational coefficients, and for each $n \geq 0$ let $\mathcal{P}_n(\mathbf{Q}) \subseteq \mathcal{P}(\mathbf{Q})$ be the collection of polynomials of degree n with rational coefficients. Then $\mathcal{P}_0(\mathbf{Q})$ is in bijection with $\mathbf{Q} \setminus \{0\}$ and $\mathcal{P}_n(\mathbf{Q})$ is in bijection with $\mathbf{Q}^{n-1} \times (\mathbf{Q} \setminus \{0\})$ for each $n \geq 1$. Thus each $\mathcal{P}_n(\mathbf{Q})$ is countable and it follows from the expression

$$\mathcal{P}(\mathbf{Q}) = \{0\} \cup \bigcup_{n=0}^{\infty} \mathcal{P}_n(\mathbf{Q})$$

(by 0 we mean the zero polynomial) and Theorem 1.5.8 (ii) that $\mathcal{P}(\mathbf{Q})$ is countable.

Now let $a < b$ be given and set $M = \max\{|a|, |b|, 1\}$. Suppose $f : [a, b] \to \mathbf{R}$ is continuous and let $\epsilon > 0$ be given. By the Weierstrass Approximation Theorem, there exists a polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{P}(\mathbf{R})$$

such that $|f(x) - p(x)| < \frac{\epsilon}{2}$ for all $x \in [a, b]$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, we can choose rational numbers $b_n, b_{n-1}, \ldots, b_1, b_0$ such that $|a_j - b_j| < \frac{\epsilon}{2M^n(n+1)}$ for each $0 \leq j \leq n$. Set

$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 \in \mathcal{P}(\mathbf{Q})$$

and observe that for any $x \in [a, b]$ we have

$$\begin{aligned} |p(x) - q(x)| &= \big|(a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \cdots + (a_1 - b_1)x + (a_0 - b_0)\big| \\ &\leq |a_n - b_n||x|^n + |a_{n-1} - b_{n-1}||x|^{n-1} + \cdots + |a_1 - b_1||x| + |a_0 - b_0| \\ &\leq |a_n - b_n|M^n + |a_{n-1} - b_{n-1}|M^{n-1} + \cdots + |a_1 - b_1|M + |a_0 - b_0| \\ &\leq |a_n - b_n|M^n + |a_{n-1} - b_{n-1}|M^n + \cdots + |a_1 - b_1|M^n + |a_0 - b_0|M^n \\ &< \frac{\epsilon}{2(n+1)} + \frac{\epsilon}{2(n+1)} + \cdots + \frac{\epsilon}{2(n+1)} + \frac{\epsilon}{2(n+1)} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

It follows that

$$|f(x) - q(x)| \leq |f(x) - p(x)| + |p(x) - q(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for any $x \in [a, b]$. Thus the desired countable subset $\mathcal{C}$ is $\mathcal{P}(\mathbf{Q})$.

Exercise 6.7.11. Assume that f has a continuous derivative on $[a, b]$. Show that there exists a polynomial $p(x)$ such that

$$|f(x) - p(x)| < \epsilon \quad \text{and} \quad |f'(x) - p'(x)| < \epsilon$$

for all $x \in [a, b]$.

Solution. By assumption f' is continuous on $[a, b]$, so the Weierstrass Approximation Theorem yields a polynomial q such that $|f'(x) - q(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a, b]$. Let p be the polynomial which satisfies $p' = q$ and $p(a) = f(a)$ and let $g : [a, b] \to \mathbf{R}$ be given by $g(x) = f(x) - p(x)$. Then $g(a) = 0$ and $g'(x) = f'(x) - q(x)$, so that $|g'(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a, b]$. Let $x \in (a, b]$ be given. By the Mean Value Theorem (Theorem 5.3.2), there exists some $c \in (a, x)$ such that

$$|f(x) - p(x)| = |g(x) - g(a)| = |g'(c)(x - a)| \leq (b - a)\frac{\epsilon}{b - a} = \epsilon.$$

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.