1 Section 5.4 Exercises

Exercises with solutions from Section 5.4 of [UA].

Exercise 5.4.1. Sketch a graph of $(1/2)h(2x)$ on $[-2,3]$. Give a qualitative description of the functions

$$h_n(x) = \frac{1}{2^n}h(2^n x)$$

as n gets larger.

Solution. See Figure 1 for the sketch. Each h_n is a periodic "sawtooth" function; as n gets larger,

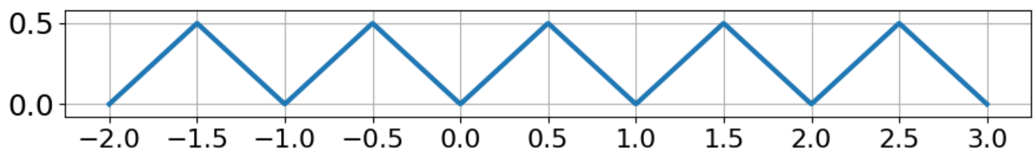


Figure 1: $(1/2)h(2x)$

the "teeth" get more densely packed and the peaks get lower.

Exercise 5.4.2. Fix $x \in \mathbf{R}$. Argue that the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n}h(2^n x)$$

converges and thus $g(x)$ is properly defined.

Solution. Note that for each $n \in \mathbf{N}$ we have $0 \le 2^{-n}h(2^n x) \le 2^{-n}$ since $0 \le h(x) \le 1$. As the series $\sum_{n=0}^{\infty} 2^{-n}$ is convergent (Example 2.7.5), the series $\sum_{n=0}^{\infty} 2^{-n}h(2^n x)$ is also convergent by the Comparison Test (Theorem 2.7.4).

Exercise 5.4.3. Taking the continuity of $h(x)$ as given, reference the proper theorems from Chapter 4 that imply that the *finite* sum

$$g_m(x) = \sum_{n=0}^{m} \frac{1}{2^n}h(2^n x)$$

is continuous on $\mathbf{R}$.

Solution. The continuity of g_m follows from Theorem 4.3.4 and Theorem 4.3.9.

Exercise 5.4.4. As the graph of Figure 5.7 suggests, the structure of $g(x)$ is quite intricate. Answer the following questions, assuming that $g(x)$ is indeed continuous.

(a) How do we know g attains a maximum value M on $[0, 2]$? What is this value?

(b) Let D be the set of points in $[0, 2]$ where g attains its maximum. That is $D = \{x \in [0, 2] : g(x) = M\}$. Find one point in D.

(c) Is D finite, countable, or uncountable?

Solution. (a) Since g is continuous on the compact set $[0, 2]$, we know it attains a maximum here by the Extreme Value Theorem (Theorem 4.4.2). To find this maximum value M, for each non-negative integer n let $f_n(x) = 2^{-2n}h(2^{2n}x) + 2^{-2n-1}h(2^{2n+1}x)$, so that

$$f_0(x) = h(x) + \frac{1}{2}h(2x), \quad f_1(x) = \frac{1}{4}h(4x) + \frac{1}{8}h(8x), \quad \text{etc.}$$

Thus

$$g(x) = h(x) + \frac{1}{2}h(2x) + \frac{1}{4}h(4x) + \frac{1}{8}h(8x) + \cdots = f_0(x) + f_1(x) + \cdots .$$

(For any given x, such a regrouping of terms is justified since we showed in Exercise 5.4.2 that the series defining $g(x)$ is convergent; see Exercise 2.5.3.)

See Figure 2a for a graph of f_0 on $[0, 2]$ and note that $f_0(x) = 1$ on the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$. Furthermore, observe that $f_1(x) = \frac{1}{4}f_0(4x)$, so that on the interval $[0, 2]$ the function f_1 is given by four copies of f_0 scaled by a factor of $\frac{1}{4}$. The interval $\left[\frac{1}{2}, \frac{3}{2}\right]$, where f_0 is constant, contains two of the intervals of length $\frac{1}{4}$ where f_1 is also constant; see Figure 2b. On these intervals, we then have $f_0(x) + f_1(x) = 1 + \frac{1}{4}$. Similarly, f_2 is given by $f_2(x) = \frac{1}{16}f_0(16x)$. Furthermore, there are further subintervals of the previous subintervals where f_2 is also constant and thus, on these subintervals, we have $f_0(x) + f_1(x) + f_2(x) = 1 + \frac{1}{4} + \frac{1}{16}$; see Figure 2c.

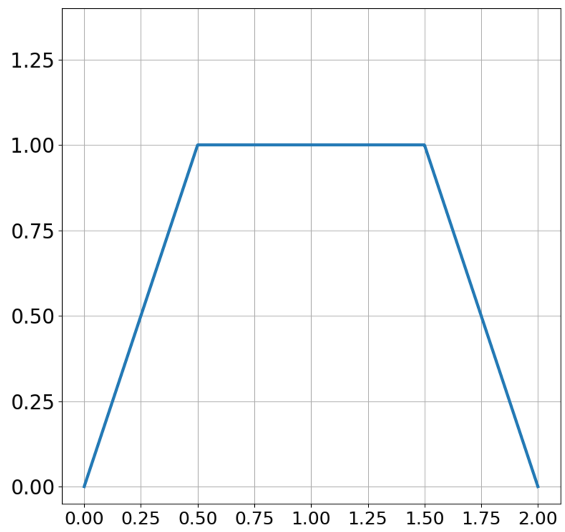
We can continue arguing in this manner to see that $M \geq 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{4}{3}$. On the other hand, since each f_n satisfies $f_n(x) \leq 4^{-n}$ on $[0, 2]$, we have

$$g(x) = f_0(x) + f_1(x) + f_2(x) + \cdots \leq 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{4}{3}.$$

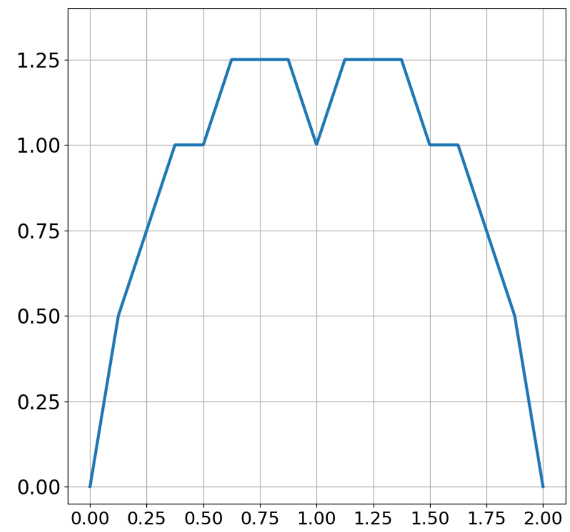
We may conclude that $M = \frac{4}{3}$.

(b) Let us show that for every non-negative integer n, we have $h\left(\frac{2^{n+1}}{3}\right) = \frac{2}{3}$. The base case $n = 0$ is clear. Suppose that the result is true for some n. Observe that

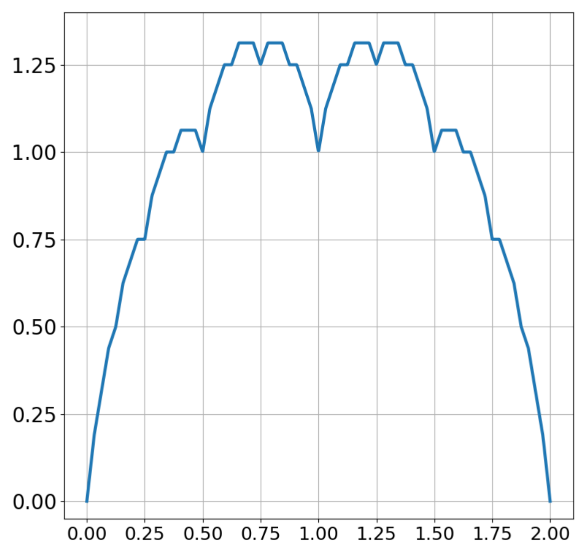
$$h\left(\tfrac{2^{n+2}}{3}\right) = h\left(\tfrac{2^{n+2}}{3} - 2^{n+1}\right) = h\left(\tfrac{2^{n+1}(2-3)}{3}\right) = h\left(-\tfrac{2^{n+1}}{3}\right) = h\left(\tfrac{2^{n+1}}{3}\right) = \frac{2}{3},$$



(a) $f_0(x)$ on $[0, 2]$



(b) $f_0(x) + f_1(x)$ on $[0, 2]$



(c) $f_0(x) + f_1(x) + f_2(x)$ on $[0, 2]$

Figure 2: Function graphs for Exercise 5.4.4

where we have used our induction hypothesis and the fact that h is an even 2-periodic function. It follows by induction that $h\left(\frac{2^{n+1}}{3}\right) = \frac{2}{3}$ for all non-negative integers n.

Now observe that

$$g\left(\tfrac{2}{3}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n} h\left(\tfrac{2^{n+1}}{3}\right) = \frac{2}{3}\sum_{n=0}^{\infty}\frac{1}{2^n} = \frac{4}{3} = M.$$

Thus $\frac{2}{3} \in D$.

(c) We will show that D is uncountable. In fact, we will prove a stronger statement: D is in bijection with $\mathbf{R}$. To do this, we will inject a space of binary sequences into D; after appealing to results we proved in Section 1.5 and Section 1.6, this will allow us to conclude the desired result.

First, suppose that $b : \{0, 1, 2, \ldots\} \to \{0, 1\}$ satisfies $b(0) = 0$. We claim that

$$x_b := \sum_{k=0}^{\infty} \frac{(-1)^{b(k)}}{4^k} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{b(k)}}{4^k}$$

belongs to D, i.e. satisfies $x_b \in [0, 2]$ and $g(x_b) = M = \frac{4}{3}$. (For the intuition here, see Figure 3. The choice of $b(0) = 0$ guarantees that $x_b \in [0, 2]$; a choice of $b(0) = 1$ would give us $x_b \in [-2, 0]$.) To see this, observe that

$$x_b \leq \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{4}{3} \quad \text{and} \quad x_b \geq 1 - \sum_{k=1}^{\infty}\frac{1}{4^k} = \frac{2}{3}$$

and thus $x_b \in [0, 2]$. Now let us express g as

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{4^n} f_0(4^n x).$$

(See part (a) for the definition of f_0 and the justification for this expression; also see Figure 2a for a graph of f_0 on the interval $[0, 2]$.) Suppose that K is a non-negative integer and $n \geq K + 1$. Observe that

$$4^n\left(1 + \frac{(-1)^{b(1)}}{4} + \cdots + \frac{(-1)^{b(K)}}{4^K}\right)$$

is an even integer and thus by the 2-periodicity of f_0 we have

$$f_0\left(4^n\left(1 + \frac{(-1)^{b(1)}}{4} + \cdots + \frac{(-1)^{b(K)}}{4^K}\right)\right) = f_0(0) = 0.$$

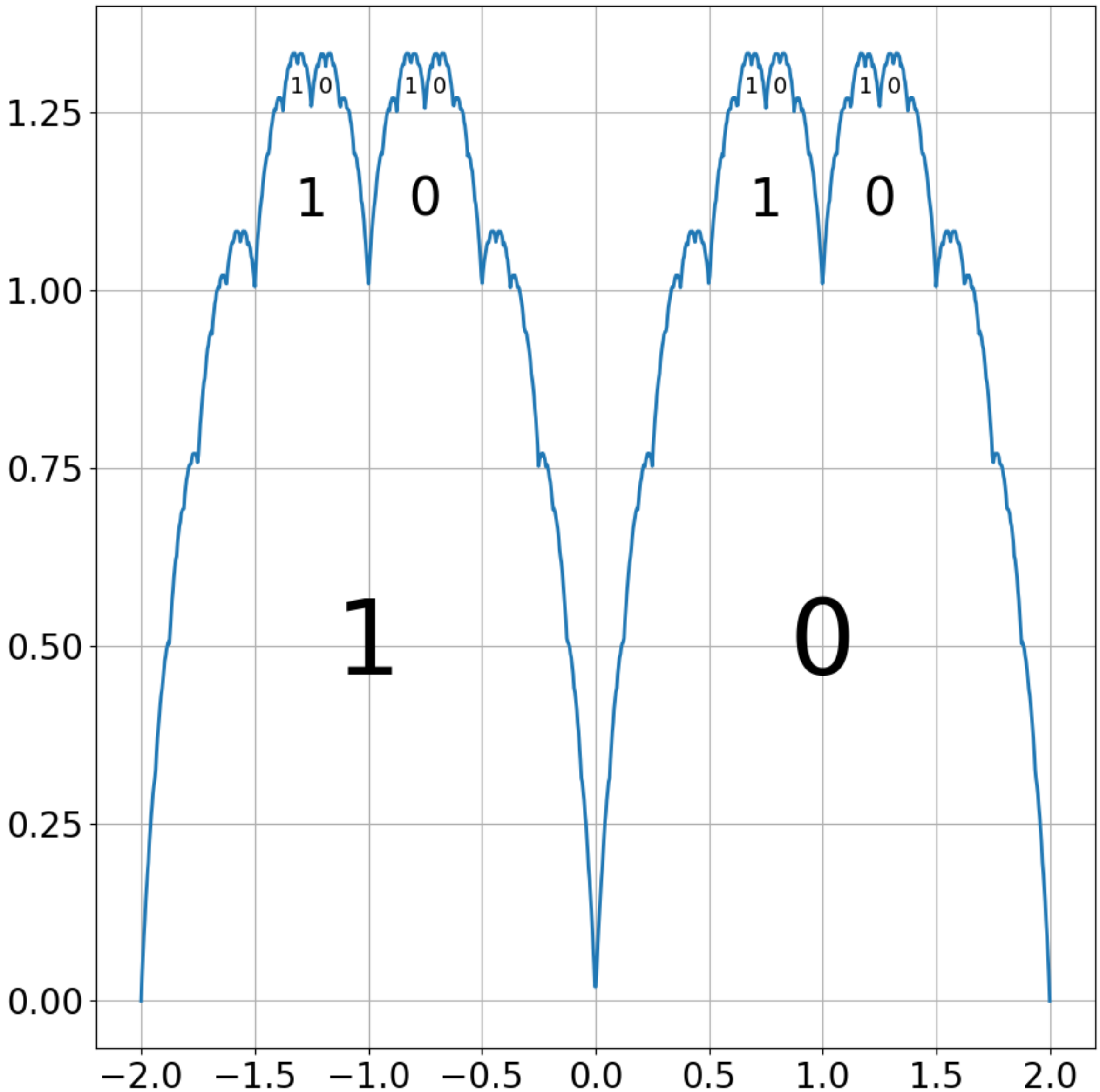


Figure 3: g on $[-2, 2]$

Furthermore, observe that

$$4^K\left(1+\frac{(-1)^{b(1)}}{4}+\cdots+\frac{(-1)^{b(K)}}{4^K}\right)$$

is an odd integer and thus by the 2-periodicity of f_0 we have

$$f_0\left(4^K\left(1+\frac{(-1)^{b(1)}}{4}+\cdots+\frac{(-1)^{b(K)}}{4^K}\right)\right)=f_0(1)=1.$$

Now, if $K\geq 1$, suppose that $0\leq n\leq K-1$. Then

$$\begin{aligned}4^n&\left(1+\frac{(-1)^{b(1)}}{4}+\cdots+\frac{(-1)^{b(K)}}{4^K}\right)\\&=\underbrace{4^n+(-1)^{b(1)}4^{n-1}+\cdots+(-1)^{b(n)}}_{\text{odd integer}}+\frac{(-1)^{b(n+1)}}{4}+\cdots+\frac{(-1)^{b(k)}}{4^{K-n}}.\end{aligned}$$

It follows from the 2-periodicity of f_0 that

$$f_0\left(4^n\left(1+\frac{(-1)^{b(1)}}{4}+\cdots+\frac{(-1)^{b(K)}}{4^K}\right)\right)=f_0\left(1+\frac{(-1)^{b(n+1)}}{4}+\cdots+\frac{(-1)^{b(k)}}{4^{K-n}}\right).$$

Note that

$$\frac{2}{3}=1-\sum_{k=0}^{\infty}\frac{1}{4^k}\leq 1+\frac{(-1)^{b(n+1)}}{4}+\cdots+\frac{(-1)^{b(k)}}{4^{K-n}}\leq\sum_{k=0}^{\infty}\frac{1}{4^k}=\frac{4}{3}.$$

Since $f_0(x)=1$ on the interval $\left[\frac{2}{3},\frac{4}{3}\right]$, we see that

$$f_0\left(1+\frac{(-1)^{b(n+1)}}{4}+\cdots+\frac{(-1)^{b(k)}}{4^{K-n}}\right)=1.$$

To summarize our findings, for each non-negative integer K we have

$$f_0\left(4^n\sum_{k=0}^{K}\frac{(-1)^{b(k)}}{4^k}\right)=\begin{cases}1 & \text{if } 0\leq n\leq K,\\ 0 & \text{if } n>K.\end{cases}\tag{1}$$

We can now show that $g(x_b) = \frac{4}{3}$:

$$\begin{aligned}
g(x_b) &= g\left(\lim_{K\to\infty}\sum_{k=0}^{K}\frac{(-1)^{b(k)}}{4^k}\right) \\
&= \lim_{K\to\infty} g\left(\sum_{k=0}^{K}\frac{(-1)^{b(k)}}{4^k}\right) && \text{(since g is continuous)} \\
&= \lim_{K\to\infty}\sum_{n=0}^{\infty}\frac{1}{4^n} f_0\left(4^n\sum_{k=0}^{K}\frac{(-1)^{b(k)}}{4^k}\right) \\
&= \lim_{K\to\infty}\sum_{n=0}^{K}\frac{1}{4^n} && \text{(by (1))} \\
&= \sum_{n=0}^{\infty}\frac{1}{4^n} \\
&= \frac{4}{3}.
\end{aligned}$$

If we let B be the following space of binary sequences

$$B := \{b : \{0,1,2,\ldots\} \to \{0,1\} \text{ such that } b(0) = 0\},$$

and define a function $\Psi : B \to D$ by $\Psi(b) = x_b$, we have now shown that Ψ is well-defined and maps into D. Our next claim is that Ψ is injective. To see this, suppose that $a, b \in B$ and $a \neq b$. Let

$$K := \min\{k \geq 0 : a(k) \neq b(k)\};$$

without loss of generality, we may assume that $a(K) = 1$ and $b(K) = 0$. Thus

$$\begin{aligned}
x_a &= 1 + \frac{(-1)^{a(1)}}{4} + \frac{(-1)^{a(2)}}{16} + \cdots - \frac{1}{4^K} + \frac{(-1)^{a(K+1)}}{4^{K+1}} + \cdots, \\
x_b &= 1 + \frac{(-1)^{a(1)}}{4} + \frac{(-1)^{a(2)}}{16} + \cdots + \frac{1}{4^K} + \frac{(-1)^{b(K+1)}}{4^{K+1}} + \cdots.
\end{aligned}$$

It follows that

$$x_b - x_a = \frac{2}{4^K} + \frac{(-1)^{b(K+1)} - (-1)^{a(K+1)}}{4^{K+1}} + \frac{(-1)^{b(K+2)} - (-1)^{a(K+2)}}{4^{K+2}} + \cdots$$

and hence that

$$\begin{aligned}4^{-K}(x_b - x_a) - 2 &= \frac{(-1)^{b(K+1)} - (-1)^{a(K+1)}}{4} + \frac{(-1)^{b(K+2)} - (-1)^{a(K+2)}}{16} + \cdots \\ &\geq -2\left(\frac{1}{4} + \frac{1}{16} + \cdots\right) \\ &= -\frac{2}{3}.\end{aligned}$$

Thus $4^{-K}(x_b - x_a) \geq \frac{4}{3} > 0$, which implies that $x_b > x_a$ and hence Ψ is injective.

It is straightforward to show that the map $B \to P(\mathbf{N})$, where $P(\mathbf{N})$ is the power set of $\mathbf{N}$, given by

$$b \mapsto \{n \in \mathbf{N} : b(n) = 1\}$$

has an inverse given by

$$A \subseteq \mathbf{N} \mapsto \left(n \mapsto \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A, \end{cases}\right)$$

so that B is in bijection with $P(\mathbf{N})$. As we showed in Exercise 1.6.9, $P(\mathbf{N})$ is in bijection with $\mathbf{R}$. The inclusion $D \hookrightarrow \mathbf{R}$ thus provides us with an injection $D \to B$. The Schröder-Bernstein Theorem (see Exercise 1.5.11) allows us to conclude that D is in bijection with $\mathbf{R}$ and hence is uncountable.

Exercise 5.4.5. Show that

$$\frac{g(x_m) - g(0)}{x_m - 0} = m + 1,$$

and use this to prove that $g'(0)$ does not exist.

Solution. For any $m \in \{0, 1, 2, \ldots\}$, we have

$$h(2^{n-m}) = \begin{cases} 2^{n-m} & \text{if } 0 \leq n \leq m, \\ 0 & \text{if } n > m. \end{cases}$$

(In the $0 \leq n \leq m$ case we have $0 < 2^{n-m} \leq 1$ and in the $n > m$ case we have that 2^{n-m} is an even integer; the 2-periodicity of h then implies that $h(2^{n-m}) = h(0) = 0$.) Thus

$$g(x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^{n-m}) = \sum_{n=0}^{m} \frac{1}{2^m} = \frac{m+1}{2^m},$$

which gives us

$$\frac{g(x_m) - g(0)}{x_m} = 2^m g(x_m) = m + 1.$$

Since $\lim_{m\to\infty} x_m = 0$ and

$$\lim_{m\to\infty} \frac{g(x_m)}{x_m} = \lim_{m\to\infty} m + 1 = +\infty,$$

it follows that the limit $\lim_{x\to 0} \frac{g(x)}{x}$ does not exist, i.e. $g'(0)$ does not exist.

Exercise 5.4.6. (a) Modify the previous argument to show that $g'(1)$ does not exist. Show that $g'(1/2)$ does not exist.

(b) Show that $g'(x)$ does not exist for any rational number of the form $x = p/2^k$ where $p \in \mathbf{Z}$ and $k \in \mathbf{N} \cup \{0\}$.

Solution. (a) These are both special cases of the result in part (b); for the sake of brevity, we omit these proofs.

(b) Let (x_m) be the sequence defined by $x_m = x + 2^{-m} = p2^{-k} + 2^{-m}$. Since we are interested in the limiting behaviour as $m \to \infty$, we may assume that $m > k$. Suppose $n \in \{0, 1, 2, \ldots\}$ is such that $n > m > k$. Then $p2^{n-k} + 2^{n-m}$ is an even integer and thus by the 2-periodicity of h we have

$$h(2^n x_m) = h(p2^{n-k} + 2^{n-m}) = h(0) = 0.$$

Now suppose that $k < n \le m$. Then $p2^{n-k}$ is an even integer and $0 < 2^{n-m} \le 1$, so

$$h(2^n x_m) = h(p2^{n-k} + 2^{n-m}) = h(2^{n-m}) = 2^{n-m}.$$

Finally, suppose that $0 \le n \le k < m$. Using Euclidean division, we can find integers q and r such that

$$p2^{n-k} = q + r2^{n-k} \quad \text{and} \quad 0 \le r2^{n-k} \le \tfrac{1}{2}.$$

Suppose q is even. Note that $0 < 2^{n-m} \le \frac{1}{2}$, so that $0 < r2^{n-k} + 2^{n-m} \le 1$. It follows that

$$\begin{aligned} h(2^n x_m) &= h(p2^{n-k} + 2^{n-m}) = h(q + r2^{n-k} + 2^{n-m}) \\ &= h(r2^{n-k} + 2^{n-m}) = r2^{n-k} + 2^{n-m} = h(p2^{n-k}) + 2^{n-m} = h(2^n x) + 2^{n-m}. \end{aligned}$$

Now suppose q is odd. Then $-1 < -1 + r2^{n-k} + 2^{n-m} \le 0$, so

$$\begin{aligned} h(2^n x_m) &= h(p2^{n-k} + 2^{n-m}) = h(q + r2^{n-k} + 2^{n-m}) \\ &= h(-1 + r2^{n-k} + 2^{n-m}) = 1 - r2^{n-k} - 2^{n-m} = h(p2^{n-k}) - 2^{n-m} = h(2^n x) - 2^{n-m}. \end{aligned}$$

In either case, we have
$$h(2^n x_m) = h(2^n x) \pm 2^{n-m},$$
with the sign depending on the integer p (the sign will not be important in what follows). To summarize:
$$h(2^n x_m) = \begin{cases} h(2^n x) \pm 2^{n-m} & \text{if } 0 \le n \le k < m, \\ 2^{n-m} & \text{if } k < n \le m, \\ 0 & \text{if } n > m > k. \end{cases}$$
Notice that
$$\begin{aligned} g(x) &= g(p2^{-k}) \\ &= h(p2^{-k}) + 2^{-1}h(p2^{1-k}) + \cdots + 2^{-k}h(p) + 2^{-k-1}h(2p) + 2^{-k-2}h(2^2p) + \cdots \\ &= h(p2^{-k}) + 2^{-1}h(p2^{1-k}) + \cdots + 2^{-k}h(p) \\ &= \sum_{n=0}^{k} 2^{-n}h(p2^{n-k}) \\ &= \sum_{n=0}^{k} 2^{-n}h(2^n x). \end{aligned}$$
It follows that
$$\begin{aligned} g(x_m) = \sum_{n=0}^{\infty} 2^{-n}h(2^n x_m) = \sum_{n=0}^{k} 2^{-n}h(2^n x) \pm 2^{-m} + \sum_{n=k+1}^{m} 2^{-m} \\ = g(x) + (k+1)(\pm 2^{-m}) + (m-k)(2^{-m}). \end{aligned}$$
Thus
$$\frac{g(x_m) - g(x)}{x_m - x} = (k+1)(\pm 1) + m - k = m + K,$$
where $K = (k+1)(\pm 1) - k$ is some integer which depends only on x. Since $\lim_{m\to\infty} x_m = x$ and
$$\lim_{m\to\infty} \frac{g(x_m) - g(x)}{x_m - x} = \lim_{m\to\infty} m + K = +\infty,$$
it follows that the limit $\lim_{t\to x} \frac{g(t)-g(x)}{t-x}$ does not exist, i.e. $g'(x)$ does not exist.

Exercise 5.4.7. (a) First prove the following general lemma: Let f be defined on an open interval J and assume f is differentiable at $a \in J$. If (a_n) and (b_n) are sequences satisfying $a_n < a < b_n$ and $\lim a_n = \lim b_n = a$, show

$$f'(a) = \lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

(b) Now use this lemma to show that $g'(x)$ does not exist.

Solution. (a) Let us first prove an auxiliary result. Suppose $(x_n), (y_n)$, and (λ_n) are sequences such that $\lim x_n = \lim y_n = x$ and $|\lambda_n| \le B$ for all $n \in \mathbf{N}$ and some $B \ge 0$. We claim that $\lim(\lambda_n x_n + (1-\lambda_n)y_n) = x$. To see this, observe that

$$\begin{aligned} |\lambda_n x_n + (1-\lambda_n)y_n - x| &= |\lambda_n(x_n - x) + (1-\lambda_n)(y_n - x)| \\ &\le |\lambda_n||x_n - x| + |1-\lambda_n||y_n - x| \\ &\le (1+B)(|x_n - x| + |y_n - x|). \end{aligned}$$

Since $(1+B)(|x_n - x| + |y_n - x|) \to 0$, the Squeeze Theorem proves our claim.

Returning to the exercise, Theorem 4.2.3 implies that

$$\lim_{n\to\infty} \frac{f(a_n) - f(a)}{a_n - a} = \lim_{n\to\infty} \frac{f(b_n) - f(a)}{b_n - a} = f'(a).$$

Note that for each $n \in \mathbf{N}$ we have

$$1 - \frac{a_n - a}{a_n - b_n} = \frac{b_n - a}{b_n - a_n} \quad \text{and} \quad \left|\frac{a_n - a}{a_n - b_n}\right| < 1.$$

Furthermore,

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = \frac{a_n - a}{a_n - b_n}\frac{f(a_n) - f(a)}{a_n - a} + \frac{b_n - a}{b_n - a_n}\frac{f(b_n) - f(a)}{b_n - a}$$

for each $n \in \mathbf{N}$. It follows from our auxiliary result, taking

$$x_n = \frac{f(a_n) - f(a)}{a_n - a}, \quad y_n = \frac{f(b_n) - f(a)}{b_n - a}, \quad \text{and} \quad \lambda_n = \frac{a_n - a}{a_n - b_n},$$

that

$$f'(a) = \lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

(b) Recall that for each $n \in \{0, 1, 2, \ldots\}$, the function $h_n : \mathbf{R} \to \mathbf{R}$ is given by $h_n(x) = h(2^n x)$. Each h_n is a piecewise linear function which has corners, i.e. fails to be differentiable, at each dyadic rational $a2^{-n}$. Note that h_n is linear on each interval of the form $[a2^{-n}, (a+1)2^{-n}]$; in particular, h_n is differentiable on $(a2^{-n}, (a+1)2^{-n})$, with slope given by ± 1. Recall also that for each $m \in \{0, 1, 2, \ldots\}$, the function $g_m : \mathbf{R} \to \mathbf{R}$ is defined as

$$g_m(x) = \sum_{n=0}^{m} 2^{-n} h_n(x) = \sum_{n=0}^{m} 2^{-n} h(2^n x).$$

Each g_m is a linear combination of piecewise linear functions and hence is itself a piecewise linear function. Consider two adjacent dyadic rationals $p2^{-m}$ and $(p+1)2^{-m}$. By our previous discussion, for each $0 \leq n \leq m$, the function h_n is linear on $[p2^{-m}, (p+1)2^{-m}]$ and hence differentiable on $(p2^{-m}, (p+1)2^{-m})$. It follows that g_m is linear on $[p2^{-m}, (p+1)2^{-m}]$ and hence differentiable on $(p2^{-m}, (p+1)2^{-m})$, with slope given by

$$g_m'(x) = \frac{g_m((p+1)2^{-m}) - g_m(p2^{-m})}{2^{-m}}$$

for $x \in (p2^{-m}, (p+1)2^{-m})$.

Let $x, (x_m)$, and (y_m) be defined as in the textbook. Given the previous discussion, for each $m \in \{0, 1, 2, \ldots\}$ we have

$$g_m'(x) = \frac{g_m(y_m) - g_m(x_m)}{y_m - x_m}.$$

In fact, since $h_n(x_m) = h_n(y_m) = 0$ for all $n > m$, we actually have $g(y_m) = g_m(y_m)$ and $g(x_m) = g_m(x_m)$, so that

$$\frac{g(y_m) - g(x_m)}{y_m - x_m} = \frac{g_m(y_m) - g_m(x_m)}{y_m - x_m} = g_m'(x).$$

Now observe that

$$g_{m+1}(t) - g_m(t) = 2^{-m-1} h_{m+1}(t).$$

As we noted earlier, each of the functions g_{m+1}, g_m, and h_{m+1} is differentiable at x since x is not a dyadic rational. It follows from the usual rules of differentiation that

$$\left| g_{m+1}'(x) - g_m'(x) \right| = \left| h_{m+1}'(x) \right| = |\pm 1| = 1.$$

This implies that the sequence $(g_m'(x))_{m=0}^{\infty}$ is not convergent, i.e. the sequence

$$\frac{g(y_m) - g(x_m)}{y_m - x_m}$$

does not converge. By the contrapositive of the result proved in part (a), we see that g is not differentiable at x.

Exercise 5.4.8. Review the argument for the nondifferentiability of $g(x)$ at nondyadic points. Does the argument still work if we replace $g(x)$ with the summation $\sum_{n=0}^{\infty}(1/2^n)h(3^n x)$? Does the argument work for the function $\sum_{n=0}^{\infty}(1/3^n)h(2^n x)$?

Solution. Let $g(x) = \sum_{n=0}^{\infty} 2^{-n}h(3^n x)$ and $g_m(x) = \sum_{n=0}^{m} 2^{-n}h(3^n x)$. The argument from Exercise 5.4.7 (b) should be repeated considering 3-adic rational numbers, i.e. rationals of the form $p3^{-k}$ for some $p \in \mathbf{Z}$ and $k \in \{0, 1, 2, \ldots\}$. The argument still works, with one small difference. If x is not a 3-adic rational number then similar reasoning shows that g_m is differentiable at x. The difference this time is that

$$\left|g'_{m+1}(x) - g'_m(x)\right| = \left(\tfrac{3}{2}\right)^{m+1}.$$

Since this does not converge to zero, we see that the sequence $(g'_m(x))_{m=0}^{\infty}$ is not convergent and we may conclude that $g'(x)$ does not exist.

Now let $g(x) = \sum_{n=0}^{\infty} 3^{-n}h(2^n x)$ and $g_m(x) = \sum_{n=0}^{m} 3^{-n}h(2^n x)$. We again consider dyadic rationals and arrive at

$$\left|g'_{m+1}(x) - g'_m(x)\right| = \left(\tfrac{2}{3}\right)^{m+1}$$

for an x which is not a dyadic rational number. Since this does converge to zero, our argument breaks down here. In fact, Theorem 6.4.3 implies that g is differentiable at every such x.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.