1 Section 2.C Exercises

Exercises with solutions from Section 2.C of [LADR].

Exercise 2.C.1. Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Solution. U is also finite-dimensional by 2.26, so U has a basis B by 2.32. Since B has length $\dim U = \dim V$ and B is linearly independent, B is a basis of V by 2.39. It follows that $V = \operatorname{span} B = U$.

Exercise 2.C.2. Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}, \mathbf{R}^2$, and all lines in $\mathbf{R}^2$ through the origin.

Solution. It is easily verified that $\{0\}, \mathbf{R}^2$, and all lines in $\mathbf{R}^2$ through the origin are indeed subspaces of $\mathbf{R}^2$. Suppose that U is a subspace of $\mathbf{R}^2$. Then since $\dim \mathbf{R}^2 = 2$, by 2.38 it must be the case that $\dim U \in \{0, 1, 2\}$. If $\dim U = 0$ then $U = \{0\}$ and if $\dim U = 2$ then $U = \mathbf{R}^2$, by Exercise 2.C.1. Suppose therefore that $\dim U = 1$. Then there exists a basis u of U, so that $U = \operatorname{span}(u) = \{\lambda u : \lambda \in \mathbf{R}\}$, and we see that U is a line through the origin with direction vector u (this is indeed a direction vector, i.e. $u \neq 0$, since the list u is linearly independent). We have now shown that if U is a subspace of $\mathbf{R}^2$, then U is either $\{0\}, \mathbf{R}^2$, or a line in $\mathbf{R}^2$ through the origin. We may conclude that these are precisely the subspaces of $\mathbf{R}^2$.

Exercise 2.C.3. Show that the subspaces of $\mathbf{R}^3$ are precisely $\{0\}, \mathbf{R}^3$, all lines in $\mathbf{R}^3$ through the origin, and all planes in $\mathbf{R}^3$ through the origin.

Solution. It is easily verified that $\{0\}, \mathbf{R}^3$, all lines in $\mathbf{R}^3$ through the origin, and all planes in $\mathbf{R}^3$ through the origin are indeed subspaces of $\mathbf{R}^3$. Suppose that U is a subspace of $\mathbf{R}^3$. Then since $\dim \mathbf{R}^3 = 3$, by 2.38 it must be the case that $\dim U \in \{0, 1, 2, 3\}$. If $\dim U = 0$ then $U = \{0\}$ and if $\dim U = 3$ then $U = \mathbf{R}^3$, by Exercise 2.C.1. Suppose therefore that $\dim U = 1$. Then there exists a basis u of U, so that $U = \operatorname{span}(u) = \{\lambda u : \lambda \in \mathbf{R}\}$, and we see that U is a line through the origin with direction vector u (this is indeed a direction vector, i.e. $u \neq 0$, since the list u is linearly independent). Now suppose that $\dim U = 2$. Then there exists a basis u_1, u_2 of U, so that $U = \operatorname{span}(u_1, u_2) = \{\lambda_1 u_1 + \lambda_2 u_2 : \lambda_1, \lambda_2 \in \mathbf{R}\}$, and we see that U is a plane through the origin with direction vectors u_1 and u_2 (these are indeed distinct direction vectors, i.e. neither is a scalar multiple of the other, since the list u_1, u_2 is linearly independent). We have now shown that if U is a subspace of $\mathbf{R}^3$, then U is either $\{0\}, \mathbf{R}^3$, a line in $\mathbf{R}^3$ through the origin, or a plane in $\mathbf{R}^3$ through the origin. We may conclude that these are precisely the subspaces of $\mathbf{R}^3$.

Exercise 2.C.4. (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution. (a) Let $B = x-6, (x-6)^2, (x-6)^3, (x-6)^4$; clearly each vector in B belongs to U. Suppose we have scalars a_1, a_2, a_3, a_4 such that

$$a_1(x-6) + a_2(x-6)^2 + a_3(x-6)^3 + a_4(x-6)^4 = 0$$

for each $x \in \mathbf{F}$. Using the reasoning of Example 2.41, we see that the left-hand side has an a_4x^4 term but the right-hand side has no x^4 term. Hence we must have $a_4 = 0$, and a similar argument with the x^3 term, the x^2 term, and the x term shows that $a_3 = a_2 = a_1 = 0$. It follows that B is linearly independent and thus by 2.23 we have $\dim U \geq 4$. We also have $\dim U \leq \dim \mathcal{P}_4(\mathbf{F}) = 5$ by 2.38. However, it cannot be the case that $\dim U = 5$, since by Exercise 2.C.1 this would imply that $U = \mathcal{P}_4(\mathbf{F})$, but $U \neq \mathcal{P}_4(\mathbf{F})$ since not all polynomials $p \in \mathcal{P}_4(\mathbf{F})$ satisfy $p(6) = 0$. So $\dim U = 4$ and by 2.39 we may conclude that B is a basis of U.

(b) We claim that $1 \notin \operatorname{span} B$. To see this, suppose that we have scalars a_1, a_2, a_3, a_4 such that

$$a_1(x-6) + a_2(x-6)^2 + a_3(x-6)^3 + a_4(x-6)^4 = 1$$

for each $x \in \mathbf{F}$. The same argument used in part (a) shows that we must have $a_4 = a_3 = a_2 = a_1 = 0$ and so we arrive at the contradiction $0 = 1$. Thus $1 \notin \operatorname{span} B$, so the list $B' := 1, x-6, (x-6)^2, (x-6)^3, (x-6)^4$ is linearly independent by Exercise 2.A.11. Then since $\dim \mathcal{P}_4(\mathbf{F}) = 5$, 2.42 allows us to conclude that B' is a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Let $W = \operatorname{span}(1)$, i.e. the subspace of all constant polynomials. As the proof of 2.34 shows, we then have $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Exercise 2.C.5. (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{R}) : p''(6) = 0\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

Solution. (a) Let $B = 1, x, (x-6)^3, (x-6)^4$; clearly each vector in B belongs to U. Suppose we have scalars a_0, a_1, a_3, a_4 such that

$$a_0 + a_1x + a_3(x-6)^3 + a_4(x-6)^4 = 0$$

for each $x \in \mathbf{F}$. Using the reasoning of Example 2.41, we see that the left-hand side has an a_4x^4 term but the right-hand side has no x^4 term. Hence we must have $a_4 = 0$, and a similar argument with the x^3 term, the x term, and the constant term shows that $a_3 = a_1 = a_0 = 0$.

It follows that B is linearly independent and thus by 2.23 we have $\dim U \geq 4$. We also have $\dim U \leq 5$ by 2.38. However, it cannot be the case that $\dim U = 5$, since by Exercise 2.C.1 this would imply that $U = \mathcal{P}_4(\mathbf{R})$, but $U \neq \mathcal{P}_4(\mathbf{R})$ since not all polynomials $p \in \mathcal{P}_4(\mathbf{R})$ satisfy $p''(6) = 0$. So $\dim U = 4$ and by 2.39 we may conclude that B is a basis of U.

(b) We claim that $x^2 \notin \operatorname{span} B$. To see this, suppose that we have scalars a_0, a_1, a_3, a_4 such that

$$a_0 + a_1 x + a_3(x-6)^3 + a_4(x-6)^4 = x^2$$

for each $x \in \mathbf{F}$. The same argument used in part (a) shows that we must have $a_4 = a_3 = a_1 = a_0 = 0$ and so we arrive at the contradiction $0 = x^2$ for every $x \in \mathbf{F}$. Thus $x^2 \notin \operatorname{span} B$, so the list $B' := 1, x, x^2, (x-6)^3, (x-6)^4$ is linearly independent by Exercise 2.A.11. Then since $\dim \mathcal{P}_4(\mathbf{R}) = 5$, 2.42 allows us to conclude that B' is a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Let $W = \operatorname{span}(x^2)$. As the proof of 2.34 shows, we then have $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

Exercise 2.C.6. (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution. (a) Let $B = 1, (x-2)(x-5), (x-2)^2(x-5), (x-2)^2(x-5)^2$; clearly each vector in B belongs to U. Suppose we have scalars a_0, a_2, a_3, a_4 such that

$$a_0 + a_2(x-2)(x-5) + a_3(x-2)^2(x-5) + a_4(x-2)^2(x-5)^2 = 0$$

for each $x \in \mathbf{F}$. Using the reasoning of Example 2.41, we see that the left-hand side has an $a_4 x^4$ term but the right-hand side has no x^4 term. Hence we must have $a_4 = 0$, and a similar argument with the x^3 term, the x^2 term, and the constant term shows that $a_3 = a_2 = a_0 = 0$. It follows that B is linearly independent and thus by 2.23 we have $\dim U \geq 4$. We also have $\dim U \leq 5$ by 2.38. However, it cannot be the case that $\dim U = 5$, since by Exercise 2.C.1 this would imply that $U = \mathcal{P}_4(\mathbf{F})$, but $U \neq \mathcal{P}_4(\mathbf{F})$ since not all polynomials $p \in \mathcal{P}_4(\mathbf{F})$ satisfy $p(2) = p(5)$. So $\dim U = 4$ and by 2.39 we may conclude that B is a basis of U.

(b) We claim that $x \notin \operatorname{span} B$. To see this, suppose that we have scalars a_0, a_2, a_3, a_4 such that

$$a_0 + a_2(x-2)(x-5) + a_3(x-2)^2(x-5) + a_4(x-2)^2(x-5)^2 = x$$

for each $x \in \mathbf{F}$. The same argument used in part (a) shows that we must have $a_4 = a_3 = a_2 = a_0 = 0$ and so we arrive at the contradiction $0 = x$ for every $x \in \mathbf{F}$. Thus $x \notin \operatorname{span} B$, so the list $B' := 1, x, (x-2)(x-5), (x-2)^2(x-5), (x-2)^2(x-5)^2$ is linearly independent by Exercise 2.A.11. Then since $\dim \mathcal{P}_4(\mathbf{F}) = 5$, 2.42 allows us to conclude that B' is a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Let $W = \operatorname{span}(x)$. As the proof of 2.34 shows, we then have $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Exercise 2.C.7. (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution. (a) Let $B = 1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$; clearly each vector in B belongs to U. Suppose we have scalars a_0, a_3, a_4 such that

$$a_0 + a_3(x-2)(x-5)(x-6) + a_4(x-2)^2(x-5)(x-6) = 0$$

for each $x \in \mathbf{F}$. Using the reasoning of Example 2.41, we see that the left-hand side has an a_4x^4 term but the right-hand side has no x^4 term. Hence we must have $a_4 = 0$, and a similar argument with the x^3 term and the constant term shows that $a_3 = a_0 = 0$. It follows that B is linearly independent and thus by 2.23 we have $\dim U \geq 3$. Note that U is a subspace of the subspace from Exercise 2.C.6, which has dimension 4, so we also have $\dim U \leq 4$ by 2.38. However, it cannot be the case that $\dim U = 4$, since by Exercise 2.C.1 this would imply that U was equal to the subspace of Exercise 2.C.6, but this cannot be true since, for example, $p(x) = (x-2)(x-5)$ satisfies $p(2) = p(5)$ but does not satisfy $p(2) = p(5) = p(6)$. So $\dim U = 4$ and by 2.39 we may conclude that B is a basis of U.

(b) We claim that $x \notin \operatorname{span} B$. To see this, suppose that we have scalars a_0, a_3, a_4 such that

$$a_0 + a_3(x-2)(x-5)(x-6) + a_4(x-2)^2(x-5)(x-6) = x$$

for each $x \in \mathbf{F}$. The same argument used in part (a) shows that we must have $a_4 = a_3 = a_0 = 0$ and so we arrive at the contradiction $0 = x$ for every $x \in \mathbf{F}$. Thus $x \notin \operatorname{span} B$, so the list $B' := 1, x, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$ is linearly independent by Exercise 2.A.11.

Next we claim that $x^2 \notin \operatorname{span} B'$. To see this, suppose that we have scalars a_0, a_1, a_3, a_4 such that

$$a_0 + a_1x + a_3(x-2)(x-5)(x-6) + a_4(x-2)^2(x-5)(x-6) = x^2$$

for each $x \in \mathbf{F}$. The same argument used in part (a) shows that we must have $a_4 = a_3 = a_1 = a_0 = 0$ and so we arrive at the contradiction $0 = x^2$ for every $x \in \mathbf{F}$. Thus $x^2 \notin \operatorname{span} B'$, so the list $B'' := 1, x, x^2, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$ is linearly independent by Exercise 2.A.11.

Then since $\dim \mathcal{P}_4(\mathbf{F}) = 5$, 2.42 allows us to conclude that B'' is a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Let $W = \operatorname{span}(x, x^2)$. As the proof of 2.34 shows, we then have $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Exercise 2.C.8. (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{R}) : \int_{-1}^{1} p = 0\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution. (a) Let $B = x, x^2 - \frac{1}{3}, x^3, x^4 - \frac{1}{5}$; it is easily verified that each vector in B belongs to U. Suppose we have scalars a_1, a_2, a_3, a_4 such that

$$a_1 x + a_2(x^2 - \tfrac{1}{3}) + a_3 x^3 + a_4(x^4 - \tfrac{1}{5}) = 0$$

for each $x \in \mathbf{F}$. Using the reasoning of Example 2.41, we see that the left-hand side has an $a_4 x^4$ term but the right-hand side has no x^4 term. Hence we must have $a_4 = 0$, and a similar argument with the x^3 term, the x^2 term, and the x term shows that $a_3 = a_2 = a_1 = 0$. It follows that B is linearly independent and thus by 2.23 we have $\dim U \geq 4$. We also have $\dim U \leq 5$ by 2.38. However, it cannot be the case that $\dim U = 5$, since by Exercise 2.C.1 this would imply that $U = \mathcal{P}_4(\mathbf{R})$, but $U \neq \mathcal{P}_4(\mathbf{R})$ since not all polynomials $p \in \mathcal{P}_4(\mathbf{R})$ satisfy $\int_{-1}^{1} p = 0$. So $\dim U = 4$ and by 2.39 we may conclude that B is a basis of U.

(b) We claim that $1 \notin \operatorname{span} B$. To see this, suppose that we have scalars a_1, a_2, a_3, a_4 such that

$$a_1 x + a_2(x^2 - \tfrac{1}{3}) + a_3 x^3 + a_4(x^4 - \tfrac{1}{5}) = 1$$

for each $x \in \mathbf{F}$. The same argument used in part (a) shows that we must have $a_4 = a_3 = a_2 = a_1 = 0$ and so we arrive at the contradiction $0 = 1$. Thus $1 \notin \operatorname{span} B$, so the list $B' := 1, x, x^2 - \frac{1}{3}, x^3, x^4 - \frac{1}{5}$ is linearly independent by Exercise 2.A.11. Then since $\dim \mathcal{P}_4(\mathbf{R}) = 5$, 2.42 allows us to conclude that B' is a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Let $W = \operatorname{span}(1)$. As the proof of 2.34 shows, we then have $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

Exercise 2.C.9. Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1.$$

Solution. Define $B := v_2 - v_1, \dots, v_m - v_1$. We claim that B is linearly independent. To see this, suppose we have scalars $a_2, \dots, a_m$ such that

$$a_2(v_2 - v_1) + \cdots + a_m(v_m - v_1) = -(a_2 + \cdots + a_m)v_1 + a_2 v_2 + \cdots + a_m v_m = 0.$$

Since the list $v_1, \ldots, v_m$ is linearly independent, this implies that $a_2 = \cdots = a_m = 0$, whence B is linearly independent. Now observe that for any $2 \leq i \leq m$ we have

$$v_i - v_1 = (v_i + w) - (v_1 + w) \in \operatorname{span}(v_1 + w, \ldots, v_m + w).$$

So we have shown that B is a list of $m-1$ linearly independent vectors in $\operatorname{span}(v_1+w, \ldots, v_m+w)$. Thus by 2.23 we have

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

Exercise 2.C.10. Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_j has degree j. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Solution. Let us show that the list $p_0, p_1, \ldots, p_m$ is linearly independent. Suppose we have scalars $a_0, a_1, \ldots, a_m$ such that

$$a_0 p_0(x) + a_1 p_1(x) + \cdots + a_m p_m(x) = 0 \tag{1}$$

for all $x \in \mathbf{F}$. Suppose that c_m is the coefficient of x^m in the polynomial p_m; we have $c_m \neq 0$ since p_m has degree m. Since each p_j has degree j, we see that the coefficient of x^m in the polynomial p_j for $j < m$ must be zero. Hence the left-hand side of (1) has an $a_m c_m x^m$ term whereas the right-hand side has no x^m term. It follows that $a_m c_m = 0$, and since $c_m \neq 0$, it must be the case that $a_m = 0$. Repeating this argument for the x^{m-1} term, then the x^{m-2} term, and so on, we find that $a_0 = a_1 = \cdots = a_m = 0$. Thus $p_0, p_1, \ldots, p_m$ is linearly independent. Since this is a list of $m+1$ linearly independent vectors in $\mathcal{P}_m(\mathbf{F})$, which has dimension $m+1$, 2.39 allows us to conclude that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Exercise 2.C.11. Suppose that U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3, \dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.

Solution. By 2.43, we have

$$8 = \dim \mathbf{R}^8 = \dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 8 - \dim(U \cap W).$$

It follows that $\dim(U \cap W) = 0$ and hence that $U \cap W = \{0\}$. Then by 1.45, the sum $U + W$ is direct.

Exercise 2.C.12. Suppose U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

Solution. By 2.43, we have

$$9 = \dim \mathbf{R}^9 \geq \dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 10 - \dim(U \cap W).$$

It follows that $\dim(U \cap W) \geq 1$ and hence that $U \cap W \neq \{0\}$.

Exercise 2.C.13. Suppose U and W are both 4-dimensional subspaces of $\mathbf{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

Solution. By 2.43, we have

$$6 = \dim \mathbf{C}^6 \geq \dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 8 - \dim(U \cap W).$$

It follows that $\dim(U \cap W) \geq 2$ and hence we can find a linearly independent list v_1, v_2 in $U \cap W$. Then by Exercise 2.A.2 (b), neither one of these vectors is a scalar multiple of the other.

Exercise 2.C.14. Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V. Prove that $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

Solution. Since each U_i is finite-dimensional, it has a basis B_i, so that $U_i = \operatorname{span} B_i$. If we let B be the list of vectors $B_1, B_2, \ldots, B_m$, then it is clear that $U_1 + \cdots + U_m$ is spanned by B. It follows that $U_1 + \cdots + U_m$ is finite-dimensional. Since B has length $\dim U_1 + \cdots + \dim U_m$, 2.23 implies that

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

Exercise 2.C.15. Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

Solution. Since $n \geq 1$, V has a non-empty basis $u_1, \ldots, u_n$. Let $U_i = \operatorname{span}(u_i)$; since each $u_i \neq 0$, we have $\dim U_i = 1$. Then by the definition of a direct sum (1.40) and 2.29, which says that each vector in V is a unique linear combination of the basis vectors $u_1, \ldots, u_n$, we have

$$V = U_1 \oplus \cdots \oplus U_n.$$

Exercise 2.C.16. Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

[*The exercise above deepens the analogy between direct sums of subspaces and disjoint unions of subsets. Specifically, compare this exercise to the following obvious statement: if a set is written as a disjoint union of finite subsets, then the number of elements in the set equals the sum of the numbers of elements in the disjoint subsets.*]

Solution. Since each U_i is finite-dimensional, it has a basis $u_1^{(i)}, \dots, u_{n_i}^{(i)}$. It is clear that the list

$$u_1^{(1)}, \dots, u_{n_1}^{(1)}, \dots, u_1^{(m)}, \dots, u_{n_m}^{(m)}$$

spans $U_1 \oplus \cdots \oplus U_m$, whence $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional. Suppose we have scalars $a_1^{(1)}, \dots, a_{n_m}^{(m)}$ such that

$$a_1^{(1)} u_1^{(1)} + \cdots + a_{n_1}^{(1)} u_{n_1}^{(1)} + \cdots + a_1^{(m)} u_1^{(m)} + \ldots + a_{n_m}^{(m)} u_{n_m}^{(m)} = 0.$$

Since $a_1^{(i)} u_1^{(i)} + \cdots + a_{n_i}^{(i)} u_{n_i}^{(i)} \in U_i$ for each i, and the sum $U_1 \oplus \cdots \oplus U_m$ is direct, by 1.44 it must be the case that $a_1^{(i)} u_1^{(i)} + \cdots + a_{n_i}^{(i)} u_{n_i}^{(i)} = 0$ for each i. Then since the list $u_1^{(i)}, \dots, u_{n_i}^{(i)}$ is linearly independent for each i, we must have $a_1^{(i)} = \cdots = a_{n_i}^{(i)} = 0$ for each i. It follows that the list

$$u_1^{(1)}, \dots, u_{n_1}^{(1)}, \dots, u_1^{(m)}, \dots, u_{n_m}^{(m)}$$

is linearly independent and hence is a basis of $U_1 \oplus \cdots \oplus U_m$. Then since $n_1 + \cdots + n_m = \dim U_1 + \cdots + \dim U_m$, we have

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

Exercise 2.C.17. You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned} \dim(U_1 + U_2 + U_3) \\ &= \dim U_1 + \dim U_2 + \dim U_3 \\ &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &\quad + \dim(U_1 \cap U_2 \cap U_3). \end{aligned}$$

Prove this or give a counterexmaple.

Solution. This is false. Consider the vector space $\mathbf{R}^2$ and suppose U_1, U_2, U_3 are three distinct lines through the origin. It is easily verified that $U_1 + U_2 + U_3 = \mathbf{R}^2$ and that $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = U_1 \cap U_2 \cap U_3 = \{0\}$. Then the left-hand side of the equation in question is 2, whereas the right-hand side is

$$1 + 1 + 1 - 0 - 0 - 0 - 0 = 3 \neq 2.$$

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edn.