1 Section 8.2 Exercises

Exercises with solutions from Section 8.2 of [UA].

Exercise 8.2.1. Decide which of the following are metrics on $X = \mathbf{R}^2$. For each, we let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points in the plane.

(a) $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(b) $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(c) $d(x,y) = |x_1 x_2 + y_1 y_2|$.

Solution. (a) This is a metric on $\mathbf{R}^2$. To see this, we shall verify each property in Definition 8.2.1. Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbf{R}^2$ be given.

(i) It is clear that $d(x,y) \geq 0$. Observe that

$$\begin{aligned} d(x,y) = 0 &\iff \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0 \\ &\iff (x_1 - y_1)^2 + (x_2 - y_2)^2 = 0 \\ &\iff (x_1 - y_1)^2 = 0 \text{ and } (x_2 - y_2)^2 = 0 \\ &\iff x_1 = y_1 \text{ and } x_2 = y_2 \\ &\iff x = y. \end{aligned}$$

(ii) We have

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} = d(y,x).$$

(iii) For $a = (a_1, a_2), b = (b_1, b_2) \in \mathbf{R}^2$, observe that

$$\begin{aligned} &\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2} \leq \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} \\ &\iff (a_1 + b_1)^2 + (a_2 + b_2)^2 \leq a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2} \\ &\iff a_1 b_1 + a_2 b_2 \leq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}. \end{aligned}$$

This last inequality follows from the Cauchy-Schwarz inequality. The desired triangle inequality for d can now be obtained by taking $a = x - z$ and $b = z - y$.

(b) This is a metric on $\mathbf{R}^2$. To see this, we shall verify each property in Definition 8.2.1. Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbf{R}^2$ be given.

(i) It is clear that $d(x, y) \geq 0$. Observe that

$$\begin{aligned} d(x,y) = 0 &\iff \max\{|x_1 - y_1|, |x_2 - y_2|\} = 0 \\ &\iff |x_1 - y_1| = 0 \text{ and } |x_2 - y_2| = 0 \\ &\iff x_1 = y_1 \text{ and } x_2 = y_2 \\ &\iff x = y. \end{aligned}$$

(ii) We have

$$d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} = \max\{|y_1 - x_1|, |y_2 - x_2|\} = d(y,x).$$

(iii) Let $z = (z_1, z_2) \in \mathbf{R}^2$ be given. Suppose that $d(x, y) = |x_1 - y_1|$ (the case where $d(x, y) = |x_2 - y_2|$ is handled similarly) and observe that

$$d(x,y) = |x_1 - y_1| \leq |x_1 - z_1| + |z_1 - y_1| \leq d(x,z) + d(z,y).$$

(c) This is not a metric on $\mathbf{R}^2$. To see this, observe that by taking $x = (1, 1)$ and $y = (-1, 1)$ we obtain $d(x, y) = 0$, but $x \neq y$. Thus property (i) of Definition 8.2.1 is not satisfied.

Exercise 8.2.2. Let $C[0, 1]$ be the collection of continuous functions on the closed interval $[0, 1]$. Decide which of the following are metrics on $C[0, 1]$.

(a) $d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}$.

(b) $d(f, g) = |f(1) - g(1)|$.

(c) $d(f, g) = \int_0^1 |f - g|$.

Solution. (a) This is a metric on $C[0, 1]$. Note that by the Extreme Value Theorem (Theorem 4.4.2), the supremum is actually a maximum.

(i) Because each element of $\{|f(x) - g(x)| : x \in [0, 1]\}$ is non-negative, we must have $d(f, g) \geq 0$. Observe that

$$\begin{aligned} d(f,g) = 0 &\iff \max\{|f(x) - g(x)| : x \in [0,1]\} = 0 \\ &\iff |f(x) - g(x)| = 0 \text{ for all } x \in [0,1] \\ &\iff f(x) = g(x) \text{ for all } x \in [0,1] \\ &\iff f = g. \end{aligned}$$

(ii) As $|f(x) - g(x)| = |g(x) - f(x)|$ for each $x \in [0,1]$, we see that $d(f,g) = d(g,f)$.

(iii) Let $h \in C[0,1]$ be given and suppose that $|f - g|$ attains its maximum at some $t \in [0,1]$, so that $d(f,g) = |f(t) - g(t)|$. Then:

$$d(f,g) = |f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)| \leq d(f,h) + d(h,g).$$

(b) This is not a metric on $C[0,1]$. To see this, let $f, g \in C[0,1]$ be given by $f(x) = 0$ and $g(x) = 1 - x$. Then

$$d(f,g) = |f(1) - g(1)| = 0$$

and yet $f \neq g$, so that d fails to satisfy property (i) in Definition 8.2.1.

(c) This is a metric on $C[0,1]$:

(i) As $|f - g| \geq 0$, Theorem 7.4.2 (iv) shows that $d(f,g) \geq 0$. Observe that

$$\begin{aligned} d(f,g) = 0 &\iff \int_0^1 |f - g| = 0 \\ &\iff |f(x) - g(x)| = 0 \text{ for all } x \in [0,1] \\ &\iff f(x) = g(x) \text{ for all } x \in [0,1] \\ &\iff f = g, \end{aligned}$$

where we have used the contrapositive of Exercise 7.4.3 (c) for the second equivalence.

(ii) We have $d(f,g) = d(g,f)$ since $|f - g| = |g - f|$.

(iii) Let $h \in C[0,1]$ be given. For any $x \in [0,1]$ we have the inequality

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|.$$

Theorem 7.4.2 (iv) then implies that

$$\int_0^1 |f - g| \leq \int_0^1 |f - h| + \int_0^1 |h - g|,$$

i.e. $d(f,g) \leq d(f,h) + d(h,g)$.

Exercise 8.2.3. Verify that the discrete metric is actually a metric.

Solution. Properties (i) and (ii) in Definition 8.2.1 are clear. For the triangle inequality, let $x, y, z \in X$ be given, and suppose that all three are distinct. Then:

$$\rho(x, y) = 1 < 2 = \rho(x, z) + \rho(z, y).$$

Now suppose that $x \neq y$ and $y = z$. Then:

$$\rho(x, y) = 1 = \rho(x, z) + \rho(z, y).$$

The other cases are handled similarly.

Exercise 8.2.4. Show that a convergent sequence is Cauchy.

Solution. Suppose that (x_n) is a convergent sequence in a metric space (X, d), with $\lim x_n = x \in X$, and let $\epsilon > 0$ be given. There exists an $N \in \mathbf{N}$ such that $d(x_n, x) < \frac{\epsilon}{2}$ whenever $n \geq N$. Suppose that $m, n \geq N$ and observe that

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \epsilon.$$

Thus (x_n) is Cauchy.

Exercise 8.2.5. (a) Consider $\mathbf{R}^2$ with the discrete metric $\rho(x, y)$ examined in Exercise 8.2.3. What do Cauchy sequences look like in this space? Is $\mathbf{R}^2$ complete with respect to this metric?

(b) Show that $C[0, 1]$ is complete with respect to the metric in Exercise 8.2.2 (a).

(c) Define $C^1[0, 1]$ to be the collection of differentiable functions on $[0, 1]$ whose derivatives are also continuous. Is $C^1[0, 1]$ complete with respect to the metric defined in Exercise 8.2.2 (a)?

Solution. (a) Suppose (x_n) is a Cauchy sequence in $(\mathbf{R}^2, \rho)$. There exists an $N \in \mathbf{N}$ such that $\rho(x_m, x_n) < \frac{1}{2}$ for any $m, n \geq N$. Since ρ takes values in $\{0, 1\}$, we have $\rho(x, y) < \frac{1}{2}$ if and only if $\rho(x, y) = 0$, which is the case if and only if $x = y$. Thus $x_m = x_n$ for all $m, n \geq N$; in particular, $x_n = x_N$ for all $n \geq N$, i.e. the sequence (x_n) is eventually constant. It is straightforward to prove that eventually constant sequences converge to that constant (in any metric space) and thus $(\mathbf{R}^2, \rho)$ is complete.

(b) Let d be the metric from Exercise 8.2.2 (a). Here is a useful lemma, the proof of which is essentially immediate from the definitions.

Lemma 1. Suppose (f_n) is a sequence of functions in $C[a, b]$ and $f \in C[a, b]$. Then (f_n) converges to f in the metric space $(C[a, b], d)$ (in the sense of Definition 8.2.2) if and only if (f_n) converges to f uniformly (in the sense of Definition 6.2.3).

Suppose that (f_n) is a Cauchy sequence in $(C[0,1],d)$ and let $\epsilon > 0$ be given. There exists an $N \in \mathbf{N}$ such that $d(f_m, f_n) < \epsilon$ whenever $m, n \geq N$. Thus, for any $m, n \geq N$ and $x \in [0,1]$, we have

$$|f_m(x) - f_n(x)| \leq d(f_m, f_n) < \epsilon.$$

It follows from Theorem 6.2.5 that there is a function $f : [0,1] \to \mathbf{R}$ such that $f_n \to f$ uniformly; note that f must belong to $C[0,1]$ by Theorem 6.2.6. Lemma 1 now implies that (f_n) converges to f in the metric space $(C[0,1],d)$ and we may conclude that this metric space is complete.

(c) This metric space is not complete. To see this, consider the sequence of functions (f_n) in $C^1[0,1]$ given by $f_n(x) = \sqrt{x + \frac{1}{n}}$; we claim that this is a Cauchy sequence in $(C^1[0,1],d)$. For a given $\epsilon > 0$, let $N \in \mathbf{N}$ be such that $N > \frac{4}{\epsilon^2}$ and suppose that $n \geq m \geq N$. Then for any $x \in [0,1]$, we have

$$\begin{aligned}|f_m(x) - f_n(x)| = \sqrt{x + \frac{1}{m}} - \sqrt{x + \frac{1}{n}} &= \frac{\frac{1}{m} - \frac{1}{n}}{\sqrt{x + \frac{1}{m}} + \sqrt{x + \frac{1}{n}}} \\ &\leq \frac{\frac{1}{m}}{\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}}} = \frac{\frac{1}{\sqrt{m}}}{1 + \frac{\sqrt{m}}{\sqrt{n}}} \leq \frac{1}{\sqrt{m}} < \frac{\epsilon}{2}.\end{aligned}$$

As $x \in [0,1]$ was arbitrary, we see that

$$n \geq m \geq N \quad \Longrightarrow \quad d(f_m, f_n) \leq \frac{\epsilon}{2} < \epsilon$$

and our claim follows.

Now we claim that (f_n) is not a convergent sequence in $(C^1[0,1],d)$. To see this, we will argue by contradiction: suppose that there is some $f \in C^1[0,1]$ such that $d(f_n, f) \to 0$. Fix $x \in [0,1]$ and observe that $|f_n(x) - f(x)| \leq d(f_n, f)$; the Squeeze Theorem then implies that the sequence of real numbers $(f_n(x))$ converges to $f(x)$ (i.e. in the metric space $\mathbf{R}$ with the usual metric). However, it is evident that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x + \frac{1}{n}} = \sqrt{x}.$$

Since limits are unique (Theorem 2.2.7; this actually holds in any metric space), we must have $f(x) = \sqrt{x}$ for each $x \in [0,1]$—but this implies that f is not differentiable at $x = 0$, contradicting that $f \in C^1[0,1]$. We must conclude that (f_n) does not converge in $(C^1[0,1],d)$.

Exercise 8.2.6. Which of these functions from $C[0,1]$ to $\mathbf{R}$ (with the usual metric) are continuous?

(a) $g(f) = \int_0^1 fk$, where k is some fixed function in $C[0,1]$.

(b) $g(f) = f(1/2)$.

(c) $g(f) = f(1/2)$, but this time with respect to the metric on $C[0,1]$, from Exercise 8.2.2 (c).

Solution. (a) This function is continuous. Fix $f \in C[0,1]$, let $\epsilon > 0$ be given and set $\delta = \frac{\epsilon}{1+\int_0^1 |k|}$. Then for any $h \in C[0,1]$ satisfying $d(f,h) < \delta$, we have

$$|g(f) - g(h)| = \left| \int_0^1 fk - \int_0^1 hk \right| = \left| \int_0^1 (f-h)k \right| \leq d(f,h) \int_0^1 |k| < \delta \int_0^1 |k| < \epsilon.$$

Thus g is continuous at any $f \in C[0,1]$.

(b) This function is continuous. Fix $f \in C[0,1]$, let $\epsilon > 0$ be given and set $\delta = \epsilon$. Then for any $h \in C[0,1]$ satisfying $d(f,h) < \delta$, we have

$$|g(f) - g(h)| = |f(1/2) - h(1/2)| \leq d(f,h) < \epsilon.$$

Thus g is continuous at any $f \in C[0,1]$.

(c) This function is not continuous; we will show that g is not continuous at the constant function $f(x) = 0$. For any $\delta > 0$, pick $n \in \mathbf{N}$ such that $\frac{1}{n+1} < \delta$ and define $h : [0,1] \to \mathbf{R}$ by

$$h(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2} - \frac{1}{n+1}\right) \cup \left[\frac{1}{2} + \frac{1}{n+1}, 1\right], \\ (n+1)x - \frac{n}{2} + \frac{1}{2} & \text{if } x \in \left[\frac{1}{2} - \frac{1}{n+1}, \frac{1}{2}\right), \\ (n-1)x - \frac{n}{2} + \frac{3}{2} & \text{if } x \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{n+1}\right); \end{cases}$$

see Figure 1. Then

$$d(f,h) = \int_0^1 |f - h| = \int_0^1 h = \frac{1}{n+1} < \delta$$

and yet $|g(f) - g(h)| = \left| f\left(\frac{1}{2}\right) - h\left(\frac{1}{2}\right) \right| = 1$. Thus g is not continuous at f.

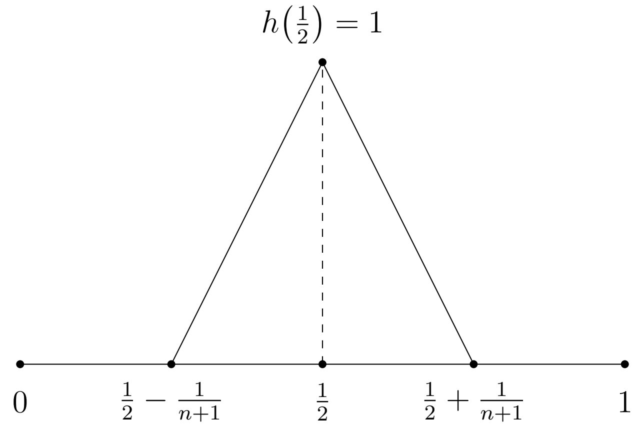


Figure 1: h on $[0, 1]$

Exercise 8.2.7. Describe the ϵ-neighborhoods in $\mathbf{R}^2$ for each of the different metrics described in Exercise 8.2.1. How about for the discrete metric?

Solution. Let d be the metric from Exercise 8.2.1 (a) and let d' be the metric from Exercise 8.2.2 (b). With respect to d, a typical ϵ-neighbourhood of some $x = (x_1, x_2) \in \mathbf{R}^2$ is the set

$$V_\epsilon(x) = \left\{ y = (y_1, y_2) \in \mathbf{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < \epsilon \right\}.$$

This consists of all the points contained strictly inside the circle of radius ϵ centred at x; see Figure 2a, which displays $V_1(0)$ with respect to d.

With respect to d', a typical ϵ-neighbourhood of some $x = (x_1, x_2) \in \mathbf{R}^2$ is the set

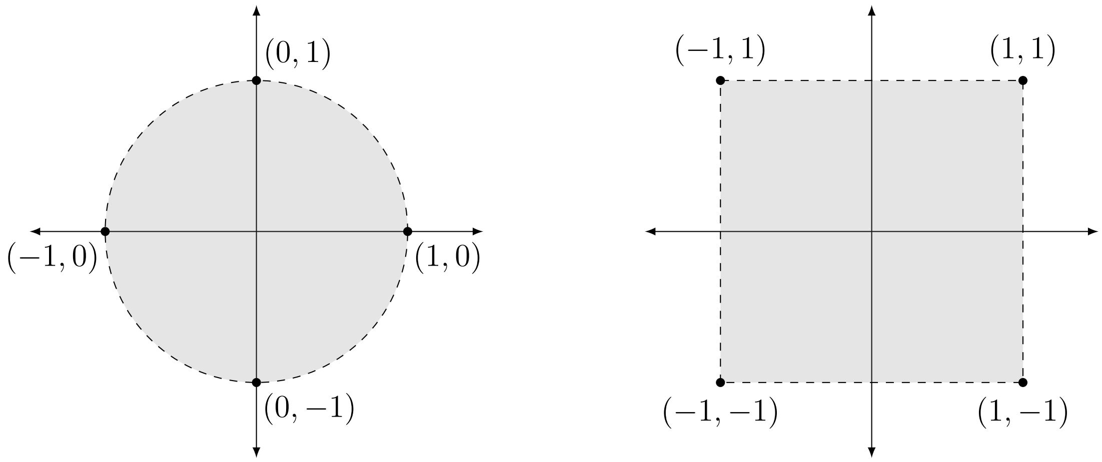
$$V_\epsilon(x) = \left\{ y = (y_1, y_2) \in \mathbf{R}^2 : \max\{|x_1 - y_1|, |x_2 - y_2|\} < \epsilon \right\}.$$

This consists of all the points contained strictly inside the square of side length 2ϵ centred at x; see Figure 2b, which displays $V_1(0)$ with respect to d'.

For the discrete metric ρ, we have

$$V_\epsilon(x) = \begin{cases} \{x\} & \text{if } 0 < \epsilon \leq 1, \\ \mathbf{R}^2 & \text{if } \epsilon > 1. \end{cases}$$

This situation is typical for a discrete metric space.



(a) $V_1(0)$ with respect to d

(b) $V_1(0)$ with respect to d'

Figure 2: $V_1(0)$ with respect to d and d'

Exercise 8.2.8. Let (X, d) be a metric space.

(a) Verify that a typical ϵ-neighborhood $V_\epsilon(x)$ is an open set. Is the set

$$C_\epsilon(x) = \{y \in X : d(x, y) \leq \epsilon\}$$

a closed set?

(b) Show that a set $E \subseteq X$ is open if and only if its complement is closed.

Solution. (a) Let $\epsilon > 0$ and $x \in X$ be fixed. Given a $y \in V_\epsilon(x)$, let $\delta = \epsilon - d(x, y) > 0$; we claim that $V_\delta(y) \subseteq V_\epsilon(x)$. To see this, suppose that $z \in V_\delta(y)$, so that

$$d(z, y) < \delta = \epsilon - d(x, y) \qquad \Longleftrightarrow \qquad d(z, y) + d(x, y) < \epsilon.$$

The triangle inequality now implies that

$$d(z, x) \leq d(z, y) + d(x, y) < \epsilon.$$

Thus $z \in V_\epsilon(x)$ and it follows that $V_\delta(y) \subseteq V_\epsilon(x)$; see Figure 3, which shows the special case of $\mathbf{R}^2$ with the usual metric. As $y \in V_\epsilon(x)$ was arbitrary, we may conclude that $V_\epsilon(x)$ is an open set.

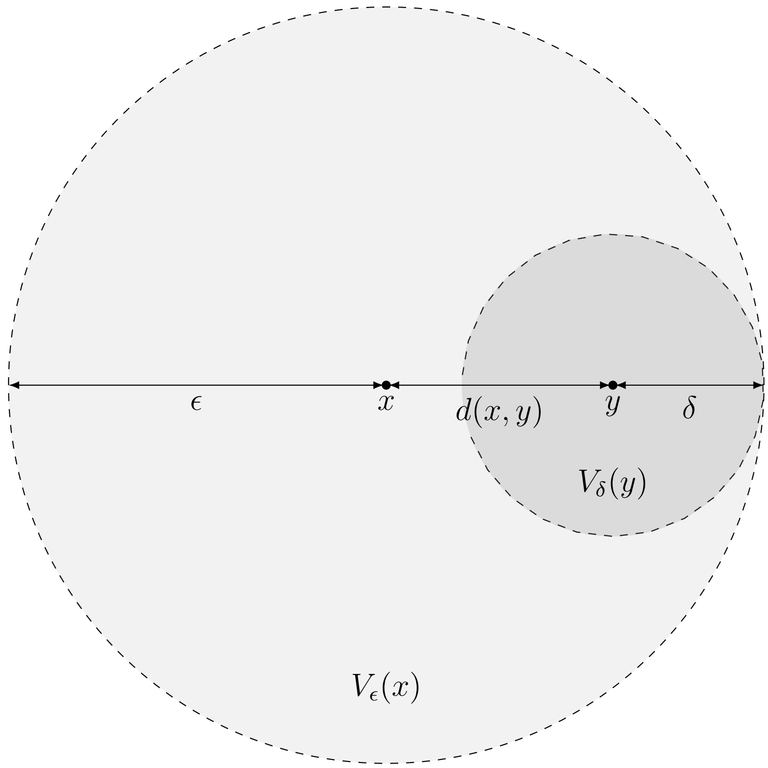


Figure 3: $V_\epsilon(x)$ is open

Now we will show that, for $\epsilon > 0$ and $x \in X$, the set $C_\epsilon(x)$ is closed. To see this, let's prove the following:

$$\text{if } y \in X \text{ is such that } d(x,y) > \epsilon \text{ then } y \text{ is not a limit point of } C_\epsilon(x).$$

Let $\delta = d(x,y) - \epsilon > 0$ and suppose $z \in V_\delta(y)$, so that

$$d(z,y) < \delta = d(x,y) - \epsilon \quad \iff \quad d(x,y) - d(z,y) > \epsilon.$$

By the triangle inequality, we have

$$d(x,y) \le d(z,x) + d(z,y) \quad \Longrightarrow \quad d(z,x) \ge d(x,y) - d(z,y) > \epsilon.$$

Thus $d(z,x) > \epsilon$, so that $z \notin C_\epsilon(x)$. We have now shown that there is a $\delta > 0$ such that $V_\delta(y) \cap C_\epsilon(x) = \emptyset$; see Figure 4, which shows the special case of $\mathbf{R}^2$ with the usual metric. It follows that y is not a limit point of $C_\epsilon(x)$.

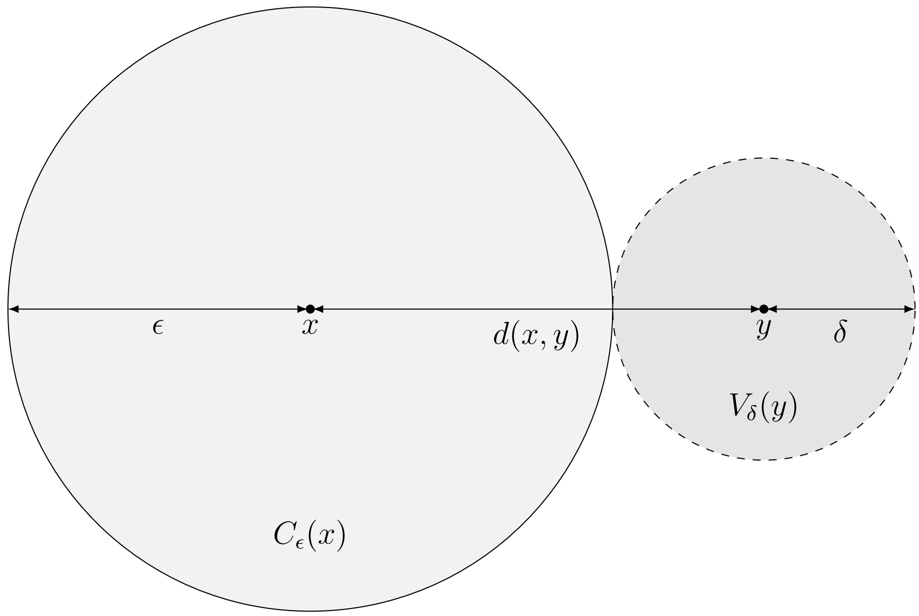


Figure 4: y is not a limit point of $C_\epsilon(x)$

The contrapositive of the statement just proven is:

$$\text{if } y \in X \text{ is a limit point of } C_\epsilon(x) \text{ then } d(x,y) \le \epsilon.$$

In other words, if y is a limit point of $C_\epsilon(x)$ then y belongs to $C_\epsilon(x)$. We may conclude that $C_\epsilon(x)$ is a closed set.

(b) Observe that

$$\begin{aligned} E \text{ is not open} &\iff (\exists x \in E)(\forall \epsilon > 0)(V_\epsilon(x) \not\subseteq E) \\ &\iff (\exists x \in E)(\forall \epsilon > 0)(V_\epsilon(x) \cap E^{\mathsf{c}} \neq \emptyset) \\ &\iff (\exists x \in E)(\forall \epsilon > 0)(V_\epsilon(x) \cap (E^{\mathsf{c}} \setminus \{x\}) \neq \emptyset) \\ &\iff (\exists x \in E)(x \text{ is a limit point of } E^{\mathsf{c}}) \\ &\iff E^{\mathsf{c}} \text{ does not contain all of its limit points} \\ &\iff E^{\mathsf{c}} \text{ is not closed.} \end{aligned}$$

Exercise 8.2.9. (a) Show that the set $Y = \{f \in C[0,1] : \|f\|_\infty \le 1\}$ is closed in $C[0,1]$.

(b) Is the set $T = \{f \in C[0,1] : f(0) = 0\}$ open, closed, or neither in $C[0,1]$?

Solution. (a) Using the notation of Exercise 8.2.2 (a), observe that $Y = C_1(0)$ (by 0 we mean the function which is identically zero on $[0,1]$). Thus, by Exercise 8.2.2 (a), Y is closed.

(b) T is not open. To see this, first observe that $0 \in T$. Now let $\epsilon > 0$ be given and define $f_\epsilon \in C[0,1]$ by $f_\epsilon(x) = \frac{\epsilon}{2}$. Then

$$d(f_\epsilon, 0) = \frac{\epsilon}{2} < \epsilon,$$

so that $f_\epsilon \in V_\epsilon(0)$. However, $f_\epsilon \notin T$ and so $V_\epsilon(0) \nsubseteq T$. As $\epsilon > 0$ was arbitrary, we may conclude that T is not open.

T is closed. To see this, suppose that $g \in C[0,1]$ is a limit point of T and let $\epsilon > 0$ be given. There exists some $f \in V_\epsilon(g) \cap T$ such that $f \neq g$. It follows that

$$|g(0)| = |g(0) - f(0)| \leq d(g, f) < \epsilon.$$

As $\epsilon > 0$ was arbitrary, we see that $g(0) = 0$, so that $g \in T$. Thus T contains its limit points, i.e. T is closed.

Exercise 8.2.10. (a) Supply a definition for *bounded* subsets of a metric space (X, d).

(b) Show that if K is a compact subset of the metric space (X, d), then K is closed and bounded.

(c) Show that $Y \subseteq C[0,1]$ from Exercise 8.2.9 (a) is closed and bounded but not compact.

Solution. (a) A subset $E \subseteq X$ is bounded if there exists some $y \in X$ and $M > 0$ such that $d(x, y) \leq M$ for all $x \in E$, i.e. $E \subseteq C_M(y)$.

(b) We will prove the contrapositive statement. First, suppose that K is not closed. Then there exists some $y \notin K$ such that y is a limit point of K. Thus, for each $n \in \mathbf{N}$, there exists some $x_n \in V_{n^{-1}}(y) \cap K$, i.e. there is some $x_n \in K$ such that $d(x_n, y) < \frac{1}{n}$. Given this, it is clear that (x_n) converges to y. It is straightforward to prove the analogous statement to Theorem 2.5.2 for metric spaces (the proof is almost identical) and hence any subsequence of (x_n) must also converge to y, which does not belong to K. Thus K is not compact.

Next, suppose that K is not bounded and fix some $y \in X$. For each $n \in \mathbf{N}$, there exists some $x_n \in K$ such that $d(x_n, y) > n$, so that any subsequence (x_{n_k}) must be unbounded (that is, the set $\{x_{n_k} : k \in \mathbf{N}\}$ must be unbounded). It is straightforward to prove the analogous statement to Theorem 2.3.2 for metric spaces and hence any subsequence of (x_n) must be divergent. Thus K is not compact.

(c) We showed in Exercise 8.2.9 (a) that Y is closed, and it is clearly bounded. To see that Y is not compact, consider the sequence of functions (f_n) given by $f_n(x) = x^n$, each of which is continuous on $[0,1]$, satisfies $\|f_n\|_\infty = 1$, and hence belongs to Y. We will argue by contradiction to show that (f_n) has no convergent subsequence. If (f_{n_k}) is a subsequence converging to some $f \in C[0,1]$, then in particular f is the pointwise limit of (f_{n_k}) on $[0,1]$. However, we can see directly that the pointwise limit of (f_{n_k}) is the function

$$x \mapsto \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Since limits are unique (Theorem 2.2.7), it must be the case that f is given by the function above, which is not continuous at $x = 1$, contradicting that $f \in C[0,1]$.

Exercise 8.2.11. (a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^{\circ} = E$.

(b) Show that $\overline{E}^{\mathsf{c}} = (E^{\mathsf{c}})^{\circ}$, and similarly that $(E^{\circ})^{\mathsf{c}} = \overline{E^{\mathsf{c}}}$.

Solution. (a) See Exercise 3.2.14 (a).

(b) See Exercise 3.2.14 (b).

Exercise 8.2.12. (a) Show

$$\overline{V_\epsilon(x)} \subseteq \{y \in X : d(x,y) \leq \epsilon\},$$

is an arbitrary metric space (X,d).

(b) To keep things from sounding too familiar, find an example of a specific metric space where

$$\overline{V_\epsilon(x)} \neq \{y \in X : d(x,y) \leq \epsilon\}.$$

Solution. (a) Using the notation from Exercise 8.2.8, note that $\{y \in X : d(x,y) \leq \epsilon\} = C_\epsilon(x)$. Clearly $V_\epsilon(x) \subseteq C_\epsilon(x)$ and thus if y is a limit point of $V_\epsilon(x)$ then y is also a limit point of $C_\epsilon(x)$. As we showed in Exercise 8.2.8, $C_\epsilon(x)$ is closed and hence $y \in C_\epsilon(x)$. We may conclude that $\overline{V_\epsilon(x)} \subseteq C_\epsilon(x)$.

(b) Consider the metric space $(\mathbf{R}, \rho)$, where ρ is the discrete metric. Then

$$\overline{V_1(0)} = \overline{\{0\}} = \overline{C_{1/2}(0)} = C_{1/2}(0) = \{0\} \neq \mathbf{R} = C_1(0).$$

Exercise 8.2.13. If E is a subset of a metric space (X,d), show that E is nowhere-dense in X if and only if $\overline{E}^{\mathsf{c}}$ is dense in X.

Solution. For the purposes of this exercise, let us denote by κE the closure of E, by ιE the interior of E, and by cE the complement of E. Observe that:

$$\begin{aligned} c\kappa E \text{ is dense in } X &\iff \kappa c \kappa E = X \\ &\iff c\kappa c\kappa E = \emptyset \\ &\iff \iota c c \kappa E = \emptyset && \text{(Exercise 8.2.11 (b))} \\ &\iff \iota\kappa E = \emptyset \\ &\iff E \text{ is nowhere-dense in } X. \end{aligned}$$

Exercise 8.2.14. (a) Give the details for why we know there exists a point $x_2 \in V_{\epsilon_1}(x_1) \cap O_2$ and an $\epsilon_2 > 0$ satisfying $\epsilon_2 < \epsilon_1/2$ with $V_{\epsilon_2}(x_2)$ contained in O_2 and

$$\overline{V_{\epsilon_2}(x_2)} \subseteq V_{\epsilon_1}(x_1).$$

(b) Proceed along this line and use the completeness of (X, d) to produce a single point $x \in O_n$ for every $n \in \mathbf{N}$.

Solution. (a) Note that x_1 must be a limit point of O_2 as O_2 is dense in X and thus there exists some $x_2 \in V_{\epsilon_1}(x_1) \cap O_2$. Since O_2 is open, there exists some $\delta > 0$ such that $V_\delta(x_2) \subseteq O_2$. If we let

$$\epsilon_2 = \min\left\{\delta, \frac{\epsilon_1}{4}, \frac{\epsilon_1 - d(x_1, x_2)}{2}\right\},$$

then $V_{\epsilon_2}(x_2) \subseteq O_2$, $\epsilon_2 < \frac{\epsilon_1}{2}$, and $\overline{V_{\epsilon_2}(x_2)} \subseteq V_{\epsilon_1}(x_1)$.

(b) By continuing this process, we obtain a sequence (x_n) of points in X and a sequence (ϵ_n) of real numbers such that:

(i) $\epsilon_n < \frac{\epsilon_1}{2^{n-1}}$ for each $n \geq 2$;

(ii) $V_{\epsilon_n}(x_n) \subseteq O_n$ for each $n \in \mathbf{N}$;

(iii) the following chain of inclusions holds:

$$\cdots \subseteq V_{\epsilon_n}(x_n) \subseteq \overline{V_{\epsilon_n}(x_n)} \subseteq V_{\epsilon_{n-1}}(x_{n-1}) \subseteq \overline{V_{\epsilon_{n-1}}(x_{n-1})} \\ \subseteq \cdots \subseteq V_{\epsilon_2}(x_2) \subseteq \overline{V_{\epsilon_2}(x_2)} \subseteq V_{\epsilon_1}(x_1) \subseteq \overline{V_{\epsilon_1}(x_1)}.$$

By (i), for any $\epsilon > 0$ we can choose an $N \geq 2$ such that $2\epsilon_N < \epsilon$. Suppose $n \geq m \geq N$. By (iii) we have $x_m, x_n \in V_{\epsilon_N}(x_N)$ and thus

$$d(x_m, x_n) \leq d(x_m, x_N) + d(x_n, x_N) < 2\epsilon_N < \epsilon.$$

It follows that (x_n) is a Cauchy sequence. By assumption the metric space (X, d) is complete and so there exists some x_0 such that $\lim x_n = x_0$.

For any $m \in \mathbf{N}$, (iii) implies that the sequence (x_n) is eventually contained inside the set $\overline{V_{\epsilon_{m+1}}(x_{m+1})}$; it follows that x_0 is a limit point of $\overline{V_{\epsilon_{m+1}}(x_{m+1})}$. Since this set is closed, we have by (ii) and (iii):

$$x_0 \in \overline{V_{\epsilon_{m+1}}(x_{m+1})} \subseteq V_{\epsilon_m}(x_m) \subseteq O_m.$$

Thus $x_0 \in \bigcap_{m=1}^{\infty} O_m$.

Exercise 8.2.15. Complete the proof of the theorem.

Solution. Let (X, d) be a complete metric space and suppose $\{E_n : n \in \mathbf{N}\}$ is a countable collection of nowhere-dense sets. Notice that each $\overline{E_n}^{\mathsf{c}}$ is open (Exercise 8.2.8 (b)) and dense (Exercise 8.2.13); it follows from Theorem 8.2.10 that $\bigcap_{n=1}^{\infty} \overline{E_n}^{\mathsf{c}} \neq \emptyset$. Now observe that

$$E_n \subseteq \overline{E_n} \text{ for each } n \in \mathbf{N} \quad \implies \quad \overline{E_n}^{\mathsf{c}} \subseteq E_n^{\mathsf{c}} \text{ for each } n \in \mathbf{N} \quad \implies \quad \bigcap_{n=1}^{\infty} \overline{E_n}^{\mathsf{c}} \subseteq \bigcap_{n=1}^{\infty} E_n^{\mathsf{c}}.$$

Thus $\bigcap_{n=1}^{\infty} E_n^{\mathsf{c}} \neq \emptyset$, which implies that

$$X \neq \left(\bigcap_{n=1}^{\infty} E_n^{\mathsf{c}} \right)^{\mathsf{c}} = \bigcup_{n=1}^{\infty} E_n.$$

Exercise 8.2.16. Show that if $f \in C[0,1]$ is differentiable at a point $x \in [0,1]$, then $f \in A_{m,n}$ for some pair $m, n \in \mathbf{N}$.

Solution. By assumption we have

$$f'(x) = \lim_{t \to x} \frac{f(x) - f(t)}{x - t}$$

and thus there exists a $\delta > 0$ such that

$$0 < |x - t| < \delta \quad \implies \quad \left| \frac{f(x) - f(t)}{x - t} - f'(x) \right| < 1.$$

Let $m \in \mathbf{N}$ be such that $\frac{1}{m} < \delta$ and let $n \in \mathbf{N}$ be such that $1 + |f'(x)| \leq n$. Then:

$$0 < |x - t| < \frac{1}{m} < \delta \quad \implies \quad \left| \frac{f(x) - f(t)}{x - t} \right| \leq \left| \frac{f(x) - f(t)}{x - t} - f'(x) \right| + |f'(x)| < 1 + |f'(x)| \leq n.$$

Thus $f \in A_{m,n}$.

Exercise 8.2.17. (a) The sequence (x_k) does not necessarily converge, but explain why there exists a subsequence (x_{k_l}) that is convergent. Let $x = \lim(x_{k_l})$.

(b) Prove that $f_{k_l}(x_{k_l}) \to f(x)$.

(c) Now finish the proof that $A_{m,n}$ is closed.

Solution. (a) The sequence (x_n) is contained in the interval $[0, 1]$ and thus by the Bolzano-Weierstrass Theorem (Theorem 2.5.5) there exists a convergent subsequence (x_{k_l}).

(b) Let $\epsilon > 0$ be given. As $f_k \to f$ in $C[0, 1]$, there is an $L_1 \in \mathbf{N}$ such that

$$l \geq L_1 \quad \Longrightarrow \quad d(f_{k_l}, f) < \frac{\epsilon}{2}.$$

The continuity of f at x implies that $\lim_{l\to\infty} f(x_{k_l}) = f(x)$ and thus there is an $L_2 \in \mathbf{N}$ such that

$$l \geq L_2 \quad \Longrightarrow \quad |f(x_{k_l}) - f(x)| < \frac{\epsilon}{2}.$$

Now observe that for $l \geq \max\{L_1, L_2\}$ we have

$$|f_{k_l}(x_{k_l}) - f(x)| \leq |f_{k_l}(x_{k_l}) - f(x_{k_l})| + |f(x_{k_l}) - f(x)| \leq d(f_{k_l}, f) + \frac{\epsilon}{2} < \epsilon.$$

It follows that $f_{k_l}(x_{k_l}) \to f(x)$.

(c) Suppose t is such that $0 < |x - t| < \frac{1}{m}$. Because $x_{k_l} \to x$, there is an $L \in \mathbf{N}$ such that

$$l \geq L \quad \Longrightarrow \quad |x - x_{k_l}| < \frac{1}{m} - |x - t| \quad \Longrightarrow \quad |x_{k_l} - t| \leq |x - x_{k_l}| + |x - t| < \frac{1}{m}.$$

This implies that

$$\left| \frac{f_{k_l}(x_{k_l}) - f_{k_l}(t)}{x_{k_l} - t} \right| \leq n \quad \text{for all } l \geq L.$$

Taking the limit as $l \to \infty$ on both sides of this inequality and using part (b), we see that

$$\left| \frac{f(x) - f(t)}{x - t} \right| \leq n$$

and hence $f \in A_{m,n}$. We may conclude that $A_{m,n}$ contains its limit points and hence is closed.

Exercise 8.2.18. A continuous function is called *polygonal* if its graph consists of a finite number of line segments.

(a) Show that there exists a polygonal function $p \in C[0,1]$ satisfying $\|f - p\|_\infty < \epsilon/2$.

(b) Show that if h is any function in $C[0,1]$ that is bounded by 1, then the function

$$g(x) = p(x) + \frac{\epsilon}{2} h(x)$$

satisfies $g \in V_\epsilon(f)$.

(c) Construct a polygonal function $h(x)$ in $C[0,1]$ that is bounded by 1 and leads to the conclusion $g \notin A_{m,n}$, where g is defined as in (b). Explain how this completes the argument for Theorem 8.2.12.

Solution. (a) This follows from Theorem 6.7.3, which we proved in Exercise 6.7.2.

(b) Observe that

$$\|f - g\|_\infty = \left\|f - p - \tfrac{\epsilon}{2} h\right\|_\infty \leq \|f - p\|_\infty + \left\|\tfrac{\epsilon}{2} h\right\|_\infty < \epsilon.$$

(c) Because p is polygonal, there are points $0 = x_0 < \cdots < x_N = 1$ such that p is a line segment on $[x_{k-1}, x_k]$; for each $1 \leq k \leq N$, let M_k be the slope of this line segment. Define $M = \max\{|M_1|, \ldots, |M_N|\}$ and let $h \in C[0,1]$ be the sawtooth function whose slope has absolute value $\frac{2}{\epsilon}(M + n + 1)$ as in Figure 5.

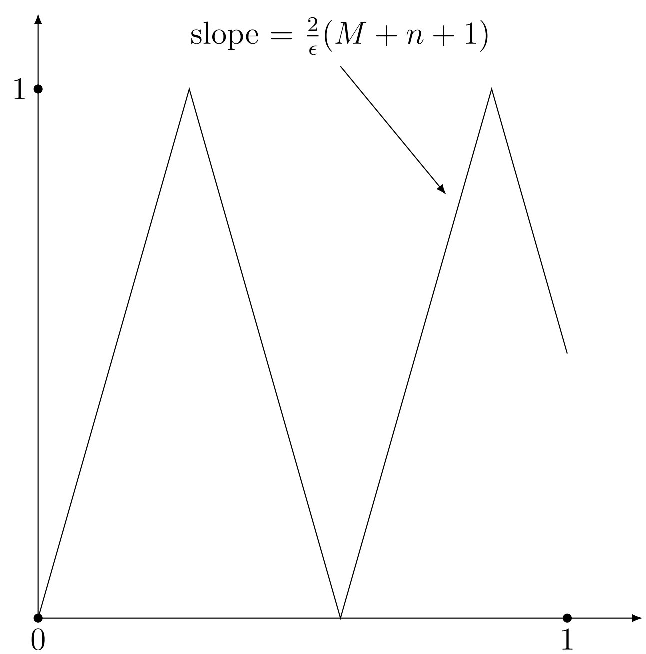


Figure 5: h on $[0,1]$

For any given $x \in [0,1]$, we have $x \in [x_{k-1}, x_k]$ for some $1 \leq k \leq N$. Note that we can always choose some $t \in [0,1]$ such that:

- $0 < |x - t| < \frac{1}{m}$;
- $t \in [x_{k-1}, x_k]$, so that x and t belong to the same line segment of p;
- x and t belong to the same line segment of h.

There are two cases. If x and t belong to a line segment of h which has slope $\frac{2}{\epsilon}(M+n+1)$, then

$$\left|\frac{g(x)-g(t)}{x-t}\right| = \left|\frac{p(x)-p(t)}{x-t} + \frac{\epsilon}{2}\frac{h(x)-h(t)}{x-t}\right|$$
$$= |M_k + M + n + 1| = M_k + M + n + 1 \geq n + 1 > n.$$

Similarly, if x and t belong to a line segment of h which has slope $-\frac{2}{\epsilon}(M+n+1)$, then

$$\left|\frac{g(x)-g(t)}{x-t}\right| = \left|\frac{p(x)-p(t)}{x-t} + \frac{\epsilon}{2}\frac{h(x)-h(t)}{x-t}\right|$$
$$= |M_k - M - n - 1| = n + 1 + M - M_k \geq n + 1 > n.$$

To summarize: for any $x \in [0,1]$ there exists a $t \in [0,1]$ such that $0 < |x-t| < \frac{1}{m}$ and

$$\left|\frac{g(x)-g(t)}{x-t}\right| > n;$$

it follows that $g \notin A_{m,n}$.

We have now shown that any ϵ-neighbourhood of f contains some function g which does not belong to $A_{m,n}$. As f was arbitrary, this implies that each $A_{m,n}$ has empty interior. We showed in Exercise 8.2.17 that each $A_{m,n}$ was a closed set and thus each $A_{m,n}$ is nowhere-dense in $C[0,1]$. It follows that the countable union

$$\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} A_{m,n}$$

is a set of first category; as we proved in Exercise 8.2.16, this union contains D and thus D is also a set of first category.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.