1 Section 4 Exercises

Exercises with solutions from Section 4 of [LADR].

Exercise 4.1. Verify all the assertions in 4.5 except the last one.

Solution. Suppose $w = a + bi$ and $z = x + yi$ are complex numbers.

(a) The assertion is that $z + \overline{z} = 2\mathrm{Re}\, z$. Indeed,

$$z + \overline{z} = (x + yi) + (x - yi) = 2x = 2\mathrm{Re}\, z.$$

(b) The assertion is that $z - \overline{z} = 2(\mathrm{Im}\, z)i$. Indeed,

$$z - \overline{z} = (x + yi) - (x - yi) = 2yi = 2(\mathrm{Im}\, z)i.$$

(c) The assertion is that $z\overline{z} = |z|^2$. Indeed,

$$z\overline{z} = (x + yi)(x - yi) = x^2 + y^2 = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2 = |z|^2.$$

(d) The assertion is that $\overline{w + z} = \overline{w} + \overline{z}$ and that $\overline{wz} = \overline{w}\,\overline{z}$. Indeed,

$$\overline{w + z} = (a + x) - (b + y)i = (a - bi) + (x - yi) = \overline{w} + \overline{z},$$

$$\overline{wz} = (ax - by) - (ay + bx)i = (a - bi)(x - yi) = \overline{w}\,\overline{z}.$$

(e) The assertion is that $\overline{\overline{z}} = z$, which is clear.

(f) The assertion is that $|\mathrm{Re}\, z| \leq |z|$ and that $|\mathrm{Im}\, z| \leq |z|$. Since each quantity involved is positive, it will suffice to show that $|\mathrm{Re}\, z|^2 \leq |z|^2$ and that $|\mathrm{Im}\, z|^2 \leq |z|^2$. These two inequalities are clear since $|z|^2 = |\mathrm{Re}\, z|^2 + |\mathrm{Im}\, z|^2$.

(g) The assertion is that $|\overline{z}| = |z|$. This follows since $(-\mathrm{Im}\, z)^2 = (\mathrm{Im}\, z)^2$.

(h) The assertion is that $|wz| = |w||z|$. Since both sides are positive, it will suffice to show that $|wz|^2 = |w|^2|z|^2$. Then using parts (c) and (d), we have

$$|wz|^2 = wz\overline{wz} = wz\overline{w}\,\overline{z} = w\overline{w}z\overline{z} = |w|^2|z|^2.$$

Exercise 4.2. Suppose m is a positive integer. Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

Solution. Let U be the set in question. We have $x^m, 1 - x^m \in U$, but $x^m + 1 - x^m = 1 \notin U$. So U cannot be a subspace of $\mathcal{P}(\mathbf{F})$ since it is not closed under addition.

Exercise 4.3. Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p \text{ is even}\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

Solution. Let U be the set in question. We have $x^2, x - x^2 \in U$, but $x^2 + x - x^2 = x \notin U$. So U cannot be a subspace of $\mathcal{P}(\mathbf{F})$ since it is not closed under addition.

Exercise 4.4. Suppose m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that p has no other zeros.

Solution. Let $p(z) = (z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_m)^{n-m+1}$. Then $\deg p = m - 1 + (n - m + 1) = n$, each λ_j is a root of p, and p has no other zeros since $(z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_m)^{n-m+1}$ is zero if and only if $z \in \{\lambda_1, \ldots, \lambda_m\}$.

Exercise 4.5. Suppose m is a nonnegative integer, $z_1, \ldots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \ldots, w_{m+1} \in \mathbf{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbf{F})$ such that

$$p(z_j) = w_j$$

for $j = 1, \ldots, m + 1$.
[*This result can be proved without using linear algebra. However, try to find the clearer, shorter proof that uses some linear algebra.*]

Solution. Define a map $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by

$$Tp = (p(z_1), p(z_2), p(z_3), \ldots, p(z_{m+1}));$$

it is straightforward to verify that T is linear. Consider the list $\mathscr{B} := p_0, p_1, p_2, \ldots, p_m$ in $\mathcal{P}_m(\mathbf{F})$ given by

$$\begin{aligned}
p_0(z) &= 1, \\
p_1(z) &= z - z_1, \\
p_2(z) &= (z - z_1)(z - z_2), \\
&\ \vdots \\
p_m(z) &= (z - z_1)(z - z_2)\cdots(z - z_m).
\end{aligned}$$

Since each p_j satisfies $\deg p_j = j$, Exercise 2.C.10 shows that $\mathscr{B}$ is a basis of $\mathcal{P}_m(\mathbf{F})$. Observe that since the elements $z_1, \ldots, z_{m+1}$ are distinct we have

$$\begin{aligned} Tp_0 &= (1, 1, 1, \ldots, 1, 1), \\ Tp_1 &= (0, 1, 1, \ldots, 1, 1), \\ Tp_2 &= (0, 0, 1, \ldots, 1, 1), \\ &\vdots \\ Tp_m &= (0, 0, 0, \ldots, 0, 1). \end{aligned}$$

It is easily verified that the list $Tp_0, \ldots, Tp_m$ is a basis of $\mathbf{F}^{m+1}$. Since T maps a basis to a basis, it must be an isomorphism; it follows that there exists a unique $p \in \mathcal{P}_m(\mathbf{F})$ such that

$$Tp = (p(z_1), p(z_2), \ldots, p(z_{m+1})) = (w_1, w_2, \ldots, w_{m+1}).$$

Exercise 4.6. Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m. Prove that p has m distinct zeros if and only if p and its derivative p' have no zeros in common.

Solution. Suppose that p and p' have a zero in common, say $\lambda \in \mathbf{F}$, so that

$$p(z) = (z - \lambda)q(z) \quad \text{and} \quad p'(z) = (z - \lambda)r(z)$$

for some $q, r \in \mathcal{P}(\mathbf{C})$ satisfying $\deg q = m - 1$ and $\deg r = m - 2$. Using the product rule, we have

$$p'(z) = q(z) + (z - \lambda)q'(z) = (z - \lambda)r(z).$$

Evaluating this at $z = \lambda$, we see that $q(\lambda) = 0$. Thus $z - \lambda$ is a factor of q; it follows that p is of the form $p(z) = (z - \lambda)^2 t(z)$ for some $t \in \mathcal{P}(\mathbf{C})$ satisfying $\deg t = m - 2$ and hence that p has strictly less than m zeros.

Now suppose that p has strictly less than m zeros. Then it must be the case that p has a zero $\lambda \in \mathbf{F}$ such that $p(z) = (z - \lambda)^k q(z)$ for some positive integer $k \geq 2$. It follows that

$$p'(z) = k(z - \lambda)^{k-1} q(z) + (z - \lambda)^k q'(z)$$

and hence that $p'(\lambda) = 0$, since $k \geq 2$. Thus p and p' have the zero λ in common.

Exercise 4.7. Prove that every polynomial of odd degree with real coefficients has a real zero.

Solution. Let $p \in \mathcal{P}(\mathbf{R})$ be a polynomial of odd degree. By 4.17, p is of the form

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M),$$

where $c, \lambda_1, \ldots, \lambda_m, b_1, \ldots, b_M, c_1, \ldots, c_M \in \mathbf{R}$, with $b_j^2 < 4c_j$ for each j (either of m or M could be zero). This implies that $\deg p = m + 2M$. Since $\deg p$ is given as odd, it must be the case that $m > 0$ and hence p has at least one real zero.

Exercise 4.8. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^{\mathbf{R}}$ by

$$Tp = \begin{cases} \dfrac{p - p(3)}{x - 3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}$$

Show that $Tp \in \mathcal{P}(\mathbf{R})$ for every polynomial $p \in \mathcal{P}(\mathbf{R})$ and that T is a linear map.

Solution. Fix $p \in \mathcal{P}(\mathbf{R})$ and notice that $p(x) - p(3)$ has a zero at $x = 3$, so that

$$p(x) - p(3) = (x - 3)q(x)$$

for some unique $q \in \mathcal{P}(\mathbf{R})$. It follows that for any $x \neq 3$ we have

$$q(x) = \frac{p(x) - p(3)}{x - 3}.$$

Differentiating the equality $p(x) - p(3) = (x - 3)q(x)$ shows that $p'(x) = q(x) + (x - 3)q'(x)$, whence $p'(3) = q(3)$. Thus $Tp = q \in \mathcal{P}(\mathbf{R})$.

To see that T is linear, let $p_1, p_2 \in \mathcal{P}(\mathbf{R})$ and $\lambda \in \mathbf{F}$ be given. There are unique polynomials $q_1, q_2 \in \mathcal{P}(\mathbf{R})$ such that

$$p_1(x) - p_1(3) = (x - 3)q_1(x) \quad \text{and} \quad p_2(x) - p_2(3) = (x - 3)q_2(x).$$

As we showed above, we must have $Tp_1 = q_1$ and $Tp_2 = q_2$. Note that

$$(p_1 + \lambda p_2)(x) - (p_1 + \lambda p_2)(3) = (x - 3)(q_1 + \lambda q_2)(x).$$

By uniqueness, we must have $T(p_1 + \lambda p_2) = q_1 + \lambda q_2 = Tp_1 + \lambda Tp_2$. Thus T is linear.

Exercise 4.9. Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by

$$q(z) = p(z)\overline{p(\overline{z})}.$$

Prove that q is a polynomial with real coefficients.

Solution. If $p = 0$ this is clear, so suppose that $\deg p = m \geq 0$. We will prove this by strong induction on m. For the base case $m = 0$, suppose that $p(z) = a_0 \in \mathbf{C}$ with $a_0 \neq 0$. Then

$$q(z) = a_0\overline{a_0} = |a_0|^2 \in \mathbf{R}.$$

Thus $q \in \mathcal{P}(\mathbf{R})$. Now suppose that the result is true for all $k \leq m$ and let $p(z) = a_0 + \cdots + a_{m+1}z^{m+1}$ be an arbitrary polynomial in $\mathcal{P}_{m+1}(\mathbf{C})$. Let $r(z) = a_0 + \cdots + a_m z^m$ and note that

$$\begin{aligned}
p(z)\overline{p(\overline{z})} &= r(z)\overline{p(\overline{z})} + a_{m+1}z^{m+1}\overline{p(\overline{z})} \\
&= r(z)\overline{r(\overline{z})} + r(z)\overline{a_{m+1}}z^{m+1} + a_{m+1}z^{m+1}\overline{r(\overline{z})} + (a_{m+1}z^{m+1})(\overline{a_{m+1}}z^{m+1}) \\
&= r(z)\overline{r(\overline{z})} + \left[r(z)\overline{a_{m+1}} + \overline{r(\overline{z})}a_{m+1}\right] z^{m+1} + |a_{m+1}|^2 z^{2(m+1)}. \qquad (1)
\end{aligned}$$

Observe that

$$r(z)\overline{a_{m+1}} + \overline{r(\overline{z})}a_{m+1} = \sum_{j=0}^{m} a_j\overline{a_{m+1}}z^j + \sum_{j=0}^{m} \overline{a_j}a_{m+1}z^j = \sum_{j=0}^{m} 2\mathrm{Re}\left(a_j\overline{a_{m+1}}\right) z^j \in \mathcal{P}(\mathbf{R}).$$

Our induction hypothesis guarantees that $r(z)\overline{r(\overline{z})}$ is a polynomial with real coefficients and so the expression for $q(z) = p(z)\overline{p(\overline{z})}$ in (1) shows that q is the sum of three polynomials with real coefficients and hence q is itself a polynomial with real coefficients. This completes the induction step and the proof.

Exercise 4.10. Suppose m is a nonnegative integer and $p \in \mathcal{P}_m(\mathbf{C})$ is such that there exist distinct real numbers $x_0, x_1, \ldots, x_m$ such that $p(x_j) \in \mathbf{R}$ for $j = 0, 1, \ldots, m$. Prove that all the coefficients of p are real.

Solution. By Exercise 4.5, there is a unique polynomial $q \in \mathcal{P}_m(\mathbf{R})$ such that $q(x_j) = p(x_j)$ for each $j = 0, 1, \ldots, m$. Consider the polynomial $p - q \in \mathcal{P}_m(\mathbf{C})$. As we just showed, this polynomial has $m + 1$ distinct zeros. By 4.12, it must be the case that $p - q = 0$, i.e. $p = q \in \mathcal{P}_m(\mathbf{R})$.

Exercise 4.11. Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.

(a) Show that $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) Find a basis of $\dim \mathcal{P}(\mathbf{F})/U$.

Solution. (a) If $\deg p = 0$, so that p is a non-zero constant, then it is not hard to see that $U = \mathcal{P}(\mathbf{F})$. In this case, $\mathcal{P}(\mathbf{F})/U = \{0\}$ and so $\dim \mathcal{P}(\mathbf{F})/U = 0 = \deg p$.

Suppose that $\deg p \geq 1$ and let $m + 1 = \deg p$. Consider the quotient map $\pi : \mathcal{P}(\mathbf{F}) \to \mathcal{P}(\mathbf{F})/U$. If $s + U \in \mathcal{P}(\mathbf{F})/U$, then the Division Algorithm for Polynomials implies that there are unique polynomials $q, r \in \mathcal{P}(\mathbf{F})$ such that $s = pq + r$ and $\deg r < \deg p$. By 3.85 we have

$$s + U = (pq + r) + U = r + U.$$

So every element of $\mathcal{P}(\mathbf{F})/U$ is of the form $r + U$ with $\deg r < \deg p$, i.e. $\deg r \leq m$. It follows that the restriction $\pi : \mathcal{P}_m(\mathbf{F}) \to \mathcal{P}(\mathbf{F})/U$ is surjective (it is a slight abuse of notation to write π for this restriction). We claim that π is also injective. If $r \in \mathcal{P}_m(\mathbf{F})$ is such that $\pi(r) = r + U = 0 + U$, then it must be the case that $r \in U$, i.e. $r = pq$ for some $q \in \mathcal{P}(\mathbf{F})$. Since

$$q \neq 0 \implies \deg pq = \deg r \geq \deg p = m + 1$$

and $\deg r \leq m$, it must be the case that $q = 0$ and hence that $r = 0$. Thus π is injective.

We have now shown that π is an isomorphism. It follows that

$$\dim \mathcal{P}(\mathbf{F})/U = \dim \mathcal{P}_m(\mathbf{F}) = m + 1 = \deg p.$$

(b) If $\deg p = 0$ then, as shown in part (a), we have $\mathcal{P}(\mathbf{F})/U = \{0\}$ and so the empty list is the only basis of $\mathcal{P}(\mathbf{F})/U$.

If $\deg p \geq 1$, then let $m + 1 = \deg p$ and take the isomorphism $\pi : \mathcal{P}_m(\mathbf{F}) \to \mathcal{P}(\mathbf{F})/U$ from part (a). Since $1, z, z^2, \ldots, z^m$ is a basis of $\mathcal{P}_m(\mathbf{F})$, it follows that

$$\pi(1), \pi(z), \pi(z^2), \ldots, \pi(z^m) = 1 + U, z + U, z^2 + U, \ldots, z^m + U$$

is a basis of $\mathcal{P}(\mathbf{F})/U$.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edition.