

1 Section 7.C Exercises

Exercises with solutions from Section 7.C of [LADR].

Exercise 7.C.1. Prove or give a counterexample: If $T \in \mathcal{L}(V)$ is self-adjoint and there exists an orthonormal basis e_1, \dots, e_n of V such that $\langle Te_j, e_j \rangle \geq 0$ for each j , then T is a positive operator.

Solution. This is false. For a counterexample, consider the operator $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ given by $T(x, y) = (-x, y)$. The matrix of T with respect to the standard orthonormal basis of \mathbf{R}^2 is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

from which we see that T is self-adjoint and that -1 is an eigenvalue of T . It follows from the equivalence of (a) and (b) in 7.35 that T is not a positive operator. However, consider the orthonormal basis

$$e_1 = \left(\cos \frac{\pi}{4}, \sin \frac{\pi}{4} \right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad e_2 = \left(-\sin \frac{\pi}{4}, \cos \frac{\pi}{4} \right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

of \mathbf{R}^2 . We then have $Te_1 = e_2$ and $Te_2 = e_1$, so that

$$\langle Te_1, e_1 \rangle = \langle Te_2, e_2 \rangle = 0.$$

Exercise 7.C.2. Suppose T is a positive operator on V . Suppose $v, w \in V$ are such that

$$Tv = w \quad \text{and} \quad Tw = v.$$

Prove that $v = w$.

Solution. Note that

$$\langle v - w, v - w \rangle = \langle Tw - Tv, v - w \rangle = -\langle T(v - w), v - w \rangle \leq 0,$$

where we have used that T is a positive operator. It follows that $\langle v - w, v - w \rangle = 0$, which is the case if and only if $v = w$.

Exercise 7.C.3. Suppose T is a positive operator on V and U is a subspace of V invariant under T . Prove that $T|_U \in \mathcal{L}(U)$ is a positive operator on U .

Solution. For $u \in U$ we have

$$\langle T|_U(u), u \rangle = \langle Tu, u \rangle \geq 0,$$

where we have used that T is a positive operator.

Exercise 7.C.4. Suppose $T \in \mathcal{L}(V, W)$. Prove that T^*T is a positive operator on V and TT^* is a positive operator on W .

Solution. Suppose $v \in V$. Then

$$\langle T^*Tv, v \rangle = \langle Tv, Tv \rangle \geq 0.$$

Thus T^*T is a positive operator on V . Similarly, suppose $w \in W$. Then

$$\langle TT^*w, w \rangle = \langle T^*w, T^*w \rangle \geq 0;$$

we are using here that $T = (T^*)^*$ (7.6 (c)). Thus TT^* is a positive operator on W .

Exercise 7.C.5. Prove that the sum of two positive operators on V is positive.

Solution. Suppose that $S, T \in \mathcal{L}(V)$ are positive operators and let $v \in V$ be given. Then

$$\langle (S + T)v, v \rangle = \langle Sv + Tv, v \rangle = \langle Sv, v \rangle + \langle Tv, v \rangle \geq 0.$$

Exercise 7.C.6. Suppose T is a positive operator on V . Prove that T^k is positive for every positive integer k .

Solution. Suppose that k is a positive even integer, say $k = 2m$, and let $v \in V$ be given. Then

$$\langle T^k v, v \rangle = \langle T^{2m} v, v \rangle = \langle T^{2m-1} v, Tv \rangle = \cdots = \langle T^m v, T^m v \rangle \geq 0,$$

where we have repeatedly used that T is self-adjoint (note that this did not require T to be positive; this is analagous to the fact that any real number raised to an even integer power is non-negative). Now suppose that k is a positive odd integer, say $k = 2m + 1$, and let $v \in V$ be given. Then

$$\langle T^k v, v \rangle = \langle T^{2m+1} v, v \rangle = \langle T^{2m} v, Tv \rangle = \cdots = \langle T^{m+1} v, T^m v \rangle = \langle T(T^m v), T^m v \rangle \geq 0,$$

where we have again repeatedly used that T is self-adjoint and also that T is a positive operator.

Exercise 7.C.7. Suppose T is a positive operator on V . Prove that T is invertible if and only if

$$\langle Tv, v \rangle > 0$$

for every $v \in V$ with $v \neq 0$.

Solution. Suppose that T is invertible. By the equivalence of (a) and (d) in 7.35, there exists a self-adjoint operator $R \in \mathcal{L}(V)$ such that $T = R^2$. Note that

$$T \text{ invertible} \iff R \text{ invertible} \iff R \text{ injective};$$

the first equivalence follows from [Exercise 3.D.9](#) and the second equivalence follows from 3.69. Let $v \in V$ be non-zero. Then Rv is non-zero since R is injective and it follows that

$$\langle Tv, v \rangle = \langle R^2v, v \rangle = \langle Rv, Rv \rangle > 0,$$

where we have used that R is self-adjoint for the second equality.

Now suppose that $\langle Tv, v \rangle > 0$ for every $v \in V$ with $v \neq 0$. Since T is self-adjoint (we do not actually require that T is positive for the implication in this direction), the relevant Spectral Theorem (7.24 or 7.29) implies that there is an orthonormal basis e_1, \dots, e_n of V such that $Te_j = \lambda_j e_j$. By assumption, we have

$$\langle Te_j, e_j \rangle > 0 \iff \langle \lambda_j e_j, e_j \rangle > 0 \iff \lambda_j > 0.$$

Thus 0 is not an eigenvalue of T , i.e. $\text{null } T = \{0\}$, i.e. T is invertible (3.69).

Exercise 7.C.8. Suppose $T \in \mathcal{L}(V)$. For $u, v \in V$, define $\langle u, v \rangle_T$ by

$$\langle u, v \rangle_T = \langle Tu, v \rangle.$$

Prove that $\langle \cdot, \cdot \rangle_T$ is an inner product on V if and only if T is an invertible positive operator (with respect to the original inner product $\langle \cdot, \cdot \rangle$).

Solution. Suppose that T is an invertible positive operator.

Positivity. Let $v \in V$ be given. Then $\langle v, v \rangle_T = \langle Tv, v \rangle \geq 0$ since T is a positive operator.

Definiteness. We have $\langle 0, 0 \rangle_T = \langle T(0), 0 \rangle = 0$ and the fact that $\langle v, v \rangle_T = \langle Tv, v \rangle \neq 0$ for $v \neq 0$ follows from [Exercise 7.C.7](#).

Additivity in first slot. For $u, v, w \in V$ we have

$$\langle u + v, w \rangle_T = \langle T(u + v), w \rangle = \langle Tu + Tv, w \rangle = \langle Tu, w \rangle + \langle Tv, w \rangle = \langle u, w \rangle_T + \langle v, w \rangle_T.$$

Homogeneity in first slot. For $u, v \in V$ and $\lambda \in \mathbf{F}$ we have

$$\langle \lambda u, v \rangle_T = \langle T(\lambda u), v \rangle = \langle \lambda Tu, v \rangle = \lambda \langle Tu, v \rangle = \lambda \langle u, v \rangle_T.$$

Conjugate symmetry. For $u, v \in V$ we have

$$\langle u, v \rangle_T = \langle Tu, v \rangle = \overline{\langle v, Tu \rangle} = \overline{\langle Tv, u \rangle} = \overline{\langle v, u \rangle}_T,$$

where we have used that T is self-adjoint for the third equality.

Thus $\langle \cdot, \cdot \rangle_T$ is an inner product on V .

Now suppose that $\langle \cdot, \cdot \rangle_T$ is an inner product on V and let $u, v \in V$ be given. Then

$$\langle Tu, v \rangle = \langle u, v \rangle_T = \overline{\langle v, u \rangle_T} = \overline{\langle Tv, u \rangle} = \langle u, Tv \rangle$$

and thus T is a self-adjoint operator. Furthermore, for $v \in V$ we have

$$\langle Tv, v \rangle = \langle v, v \rangle_T \geq 0$$

and thus T is a positive operator. Finally, if $v \neq 0$ then

$$\langle Tv, v \rangle = \langle v, v \rangle_T > 0$$

and it follows from [Exercise 7.C.7](#) that T is invertible.

Exercise 7.C.9. Prove or disprove: the identity operator on \mathbf{F}^2 has infinitely many self-adjoint square roots.

Solution. This is true. For a given $t \in (0, 1)$, let R be the operator on \mathbf{F}^2 whose matrix with respect to the standard orthonormal basis of \mathbf{F}^2 is

$$A = \begin{pmatrix} \sqrt{1-t^2} & t \\ t & -\sqrt{1-t^2} \end{pmatrix}.$$

Then R is self-adjoint since $A = A^*$ and the matrix of R^2 is

$$A^2 = \begin{pmatrix} \sqrt{1-t^2} & t \\ t & -\sqrt{1-t^2} \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise 7.C.10. Suppose $S \in \mathcal{L}(V)$. Prove that the following are equivalent:

- (a) S is an isometry;
- (b) $\langle S^*u, S^*v \rangle = \langle u, v \rangle$ for all $u, v \in V$;
- (c) S^*e_1, \dots, S^*e_m is an orthonormal list for every orthonormal list of vectors e_1, \dots, e_m in V ;

(d) S^*e_1, \dots, S^*e_n is an orthonormal basis for some orthonormal basis e_1, \dots, e_n of V .

Solution. We will prove the following chain of implications:

$$(a) \implies (b) \implies (c) \implies (d) \implies (a).$$

Suppose that (a) holds. It follows from the equivalence of 7.42 (a), 7.42 (b), and 7.42 (g) that (b) holds. Suppose that (b) holds. It follows from the equivalence of 7.42 (b) and 7.42 (c) that (c) holds. Clearly (c) implies (d). Suppose that (d) holds. It follows from the equivalence of 7.42 (d) and 7.42 (a) that (a) holds.

Exercise 7.C.11. Suppose T_1, T_2 are normal operators on $\mathcal{L}(\mathbf{F}^3)$ and both operators have 2, 5, 7 as eigenvalues. Prove that there exists an isometry $S \in \mathcal{L}(\mathbf{F}^3)$ such that $T_1 = S^*T_2S$.

Solution. The relevant Spectral Theorem (7.24 or 7.29) implies that there are orthonormal bases e_1, e_2, e_3 and f_1, f_2, f_3 of \mathbf{F}^3 such that

$$T_1e_1 = 2e_1, \quad T_1e_2 = 5e_2, \quad T_1e_3 = 7e_3, \quad T_2f_1 = 2f_1, \quad T_2f_2 = 5f_2, \quad T_2f_3 = 7f_3.$$

Define an operator $S \in \mathcal{L}(\mathbf{F}^3)$ by $Se_j = f_j$. Since S maps an orthonormal basis to an orthonormal basis, it must be an isometry (7.42). Note that

$$S^*T_2Se_1 = S^*T_2f_1 = 2S^*f_1 = 2e_1 = T_1e_1;$$

here we have used that $S^* = S^{-1}$ for isometries (7.42). Similarly, we can show that $S^*T_2Se_j = T_1e_j$ for $j = 1, 2$ and thus $S^*T_2S = T_1$.

Exercise 7.C.12. Give an example of two self-adjoint operators $T_1, T_2 \in \mathcal{L}(\mathbf{F}^4)$ such that the eigenvalues of both operators are 2, 5, 7 but there does not exist an isometry $S \in \mathcal{L}(\mathbf{F}^4)$ such that $T_1 = S^*T_2S$. Be sure to explain why there is no isometry with the required property.

Solution. Let T_1 and T_2 be the operators on \mathbf{F}^4 whose matrices with respect to the standard orthonormal basis of \mathbf{F}^4 are

$$A_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix},$$

respectively. Since these matrices are diagonal, we see that T_1 and T_2 are self-adjoint and that their eigenvalues are precisely 2, 5, 7. Note that if there was an isometry $S \in \mathcal{L}(\mathbf{F}^4)$ such that $T_1 = S^*T_2S$, then since $S^*S = I$ (7.42), we would have

$$T_1 - 2I = S^*T_2S - 2I = S^*T_2S - S^*(2I)S = S^*(T_2 - 2I)S$$

and it would follow from [Exercise 3.D.6](#) that $\dim \text{null}(T_1 - 2I) = \dim \text{null}(T_2 - 2I)$. However, for the operators T_1 and T_2 defined above, we have

$$\dim \text{null}(T_1 - 2I) = 2 \neq 1 = \dim \text{null}(T_2 - 2I).$$

Exercise 7.C.13. Prove or give a counterexample: if $S \in \mathcal{L}(V)$ and there exists an orthonormal basis e_1, \dots, e_n of V such that $\|Se_j\| = 1$ for each e_j , then S is an isometry.

Solution. This is false. Let e_1, e_2 be the standard orthonormal basis of \mathbf{R}^2 and let $S \in \mathcal{L}(\mathbf{R}^2)$ be the operator on \mathbf{R}^2 defined by $Se_1 = Se_2 = e_1$. Then $\|Se_1\| = \|Se_2\| = \|e_1\| = 1$, however S is evidently not injective, hence not invertible, hence not an isometry (7.42).

Exercise 7.C.14. Let T be the second derivative operator in [Exercise 21 in Section 7.A](#). Show that $-T$ is a positive operator.

Solution. As in [Exercise 21](#), let

$$v = \frac{1}{\sqrt{2\pi}}, \quad e_j = \frac{\cos jx}{\sqrt{\pi}}, \quad \text{and} \quad f_j = \frac{\sin jx}{\sqrt{\pi}}$$

for each $1 \leq j \leq n$, and let $B := v, e_1, \dots, e_n, f_1, \dots, f_n$, so that B is an orthonormal basis of V ([Exercise 6.B.4](#)). As we showed in [Exercise 21](#), the matrix of T with respect to B is then

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -4 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -n^2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -n^2 \end{pmatrix},$$

from which we see that the operator $-T$ is diagonal, hence self-adjoint, and has only non-negative eigenvalues. It follows from the equivalence of (a) and (b) in 7.35 that $-T$ is a positive operator.