

1 Section 4.3 Exercises

Exercises with solutions from Section 4.3 of [UA].

Exercise 4.3.1. Let $g(x) = \sqrt[3]{x}$.

- (a) Prove that g is continuous at $c = 0$.
- (b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ will be helpful.)

Solution. (a) Let $\epsilon > 0$ be given and set $\delta = \epsilon^3$. Then provided we take $x \in \mathbf{R}$ such that $|x| < \delta$, we will have

$$|\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\delta} = \epsilon.$$

Thus g is continuous at $c = 0$.

- (b) Taking $a = x^{1/3}$ and $b = c^{1/3}$ in the identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ gives

$$\begin{aligned} x - c &= (x^{1/3} - c^{1/3})(x^{2/3} + (xc)^{1/3} + c^{2/3}) \\ \implies |x - c| &= |x^{1/3} - c^{1/3}| |x^{2/3} + (xc)^{1/3} + c^{2/3}|. \end{aligned}$$

If we take x close enough to c so that x and c have the same sign, i.e. take x such that $|x - c| < |c|$, then $xc > 0$ and so

$$|x^{2/3} + (xc)^{1/3} + c^{2/3}| = x^{2/3} + (xc)^{1/3} + c^{2/3} \geq c^{2/3}.$$

Set $\delta = \min\{|c|, c^{2/3}\epsilon\}$ and suppose $x \in \mathbf{R}$ is such that $|x - c| < \delta$. By the previous discussion, we then have

$$|x^{1/3} - c^{1/3}| \leq \frac{|x - c|}{c^{2/3}} < \frac{\delta}{c^{2/3}} < \epsilon.$$

Thus g is continuous at c .

Exercise 4.3.2. To gain a deeper understanding of the relationship between ϵ and δ in the definition of continuity, let's explore some modest variations of Definition 4.3.1. In all of these, let f be a function defined on all of \mathbf{R} .

- (a) Let's say f is *onetinuuous* at c if for all $\epsilon > 0$ we can choose $\delta = 1$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is onetinuuous on all of \mathbf{R} .

- (b) Let's say f is *equaltinuous* at c if for all $\epsilon > 0$ we can choose $\delta = \epsilon$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is equaltinuous on \mathbf{R} that is nowhere onetinuuous, or explain why there is no such function.
- (c) Let's say f is *lesstinuous* at c if for all $\epsilon > 0$ we can choose $0 < \delta < \epsilon$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is lesstinuous on \mathbf{R} that is nowhere equaltinuous, or explain why there is no such function.
- (d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

Solution. (a) Let f be given by $f(x) = 0$ for all $x \in \mathbf{R}$. Fix $c \in \mathbf{R}$ and let $\epsilon > 0$ be given. If $x \in \mathbf{R}$ is such that $|x - c| < 1$, then

$$|f(x) - f(c)| = |0 - 0| = 0 < \epsilon.$$

Thus f is onetinuuous on \mathbf{R} .

- (b) Let f be given by $f(x) = x$ for all $x \in \mathbf{R}$. Fix $c \in \mathbf{R}$ and let $\epsilon > 0$ be given. If $x \in \mathbf{R}$ is such that $|x - c| < \epsilon$, then

$$|f(x) - f(c)| = |x - c| < \epsilon.$$

Thus f is equaltinuous on \mathbf{R} . However, f is nowhere onetinuuous. Fix $c \in \mathbf{R}$ again and consider $\epsilon = 1/4$. Note that $x = c + 1/2$ satisfies $|x - c| = |c + 1/2 - c| = 1/2 < 1$, however

$$|f(x) - f(c)| = |x - c| = 1/2 > 1/4 = \epsilon.$$

Thus f is nowhere onetinuuous.

- (c) Let f be given by $f(x) = 2x$ for all $x \in \mathbf{R}$. Fix $c \in \mathbf{R}$ and let $\epsilon > 0$ be given. Set $\delta = \frac{\epsilon}{2} < \epsilon$. If $x \in \mathbf{R}$ is such that $|x - c| < \delta$, then

$$|f(x) - f(c)| = 2|x - c| < 2\delta = \epsilon.$$

Thus f is lesstinuous on \mathbf{R} . However, f is nowhere equaltinuous. Fix $c \in \mathbf{R}$ again and let $\epsilon = 1$. Note that $x = c + 3/4$ satisfies $|x - c| = 3/4 < \epsilon$, however

$$|f(x) - f(c)| = 2|x - c| = 3/2 > \epsilon.$$

Thus f is nowhere equaltinuous.

- (d) It is clear that every lesstinuous function is continuous. We claim that every continuous function is lesstinuous. To see this, let f be a continuous function. Fix $c \in \mathbf{R}$ and $\epsilon > 0$. Since f is continuous at c , there is a $\delta' > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta'$. Set $\delta = \min\{\delta', \frac{\epsilon}{2}\}$. Then $0 < \delta < \epsilon$ and if $x \in \mathbf{R}$ is such that $|x - c| < \delta$, then x also satisfies $|x - c| < \delta'$ and hence $|f(x) - f(c)| < \epsilon$.

Exercise 4.3.3. (a) Supply a proof for Theorem 4.3.9 using the ϵ - δ characterization of continuity.

(b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2 (iii)).

Solution. (a) Let $a \in A$ and $\epsilon > 0$ be given. By assumption we have $f(a) \in B$, so g is continuous at $f(a)$. There then exists a $\delta_1 > 0$ such that

$$|y - f(a)| < \delta_1 \text{ and } y \in B \implies |g(y) - g(f(a))| < \epsilon. \quad (1)$$

Since f is continuous at a , there exists a $\delta_2 > 0$ such that

$$|x - a| < \delta_2 \text{ and } x \in A \implies |f(x) - f(a)| < \delta_1. \quad (2)$$

Combining (1) and (2), we have

$$\begin{aligned} |x - a| < \delta_2 \text{ and } x \in A &\implies |f(x) - f(a)| < \delta_1 \text{ and } f(x) \in B \\ &\implies |g(f(x)) - g(f(a))| < \epsilon. \end{aligned}$$

Thus $g \circ f$ is continuous at each $a \in A$.

(b) Let $a \in A$ be given and suppose $(a_n) \subseteq A$ is such that $\lim a_n = a$. Since f is continuous at a , Theorem 4.3.2 (iii) gives us $\lim f(a_n) = f(a)$. By assumption g is continuous at $f(a) \in B$ and $(f(a_n)) \subseteq B$, so Theorem 4.3.2 (iii) again gives us $\lim g(f(a_n)) = g(f(a))$. One more application of Theorem 4.3.2 (iii) allows us to conclude that $g \circ f$ is continuous at each $a \in A$.

Exercise 4.3.4. Assume f and g are defined on all of \mathbf{R} and that $\lim_{x \rightarrow p} f(x) = q$ and $\lim_{x \rightarrow q} g(x) = r$.

(a) Give an example to show that it may not be true that

$$\lim_{x \rightarrow p} g(f(x)) = r.$$

(b) Show that the result in (a) does follow if we assume f and g are continuous.

(c) Does the result in (a) hold if we only assume f is continuous? How about if we only assume that g is continuous?

Solution. (a) Let f be given by $f(x) = 0$ for all $x \in \mathbf{R}$ and let g be given by

$$g(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

We have $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = 0$, however note that $g(f(x)) = g(0) = 1$ for all $x \in \mathbf{R}$. It follows that

$$\lim_{x \rightarrow 0} g(f(x)) = 1 \neq 0.$$

- (b) By Theorem 4.3.9, the composition $g \circ f$ is continuous. Since f and g are defined on all of \mathbf{R} , Theorem 4.3.2 (iv) lets us write

$$\lim_{x \rightarrow p} g(f(x)) = g(f(p)) = g\left(\lim_{x \rightarrow p} f(x)\right) = g(q) = \lim_{x \rightarrow q} g(x).$$

- (c) As the counterexample in part (a) shows, the result does not hold if we only assume f is continuous. However, it does hold if we only assume g is continuous. To see this, let (x_n) be some sequence satisfying $\lim x_n = p$ and $x_n \neq p$. Theorem 4.2.3 shows that $\lim f(x_n) = q$, and since g is continuous the sequential characterization of continuity implies that

$$\lim g(f(x_n)) = g(q) = r,$$

where the last equality also follows from the continuity of g . Theorem 4.2.3 allows us to conclude that

$$\lim_{x \rightarrow p} g(f(x)) = r.$$

Exercise 4.3.5. Show using Definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbf{R}$, then $f : A \rightarrow \mathbf{R}$ is continuous at c .

Solution. Since c is an isolated point of A , there exists a $\delta > 0$ such that $(c - \delta, c + \delta) \cap A = \{c\}$. Let $\epsilon > 0$ be given. If $x \in A$ is such that $|x - c| < \delta$, then it must be the case that $x = c$, which gives us

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon.$$

Thus f is continuous at c .

Exercise 4.3.6. Provide an example of each or explain why the request is impossible.

- (a) Two functions f and g , neither of which is continuous at 0 such that $f(x)g(x)$ and $f(x)+g(x)$ are continuous at 0.

- (b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0.
- (c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.
- (d) A function $f(x)$ not continuous at 0 such that $f(x) + \frac{1}{f(x)}$ is continuous at 0.
- (e) A function $f(x)$ not continuous at 0 such that $[f(x)]^3$ is continuous at 0.

Solution. (a) Let $f, g : \mathbf{R} \rightarrow \mathbf{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases} \quad g(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Neither f nor g is continuous at 0, however note that for all $x \in \mathbf{R}$ we have

$$f(x)g(x) = 0 \quad \text{and} \quad f(x) + g(x) = 1.$$

Thus fg and $f + g$ are continuous at 0.

- (b) This is impossible. If f and $f + g$ are continuous at 0 then Theorem 4.3.4 implies that $g = f + g - f$ is continuous at 0.
- (c) Take g as in part (a) and let $f(x) = 0$ for all $x \in \mathbf{R}$. Then g is not continuous at 0 but $f = fg$ is continuous at 0.
- (d) Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be given by

$$f(x) = \begin{cases} \sqrt{2} - 1 & \text{if } x \neq 0, \\ \sqrt{2} + 1 & \text{if } x = 0. \end{cases}$$

Then f is discontinuous at 0, but one can verify that $f(x) + \frac{1}{f(x)} = 2\sqrt{2}$ for all $x \in \mathbf{R}$ and hence $f + \frac{1}{f}$ is continuous at 0.

- (e) This is impossible. As we showed in [Exercise 4.3.1](#), the function $g(x) = \sqrt[3]{x}$ is continuous everywhere. Thus if $[f(x)]^3$ is continuous at 0, then by Theorem 4.3.9 the composition

$$f(x) = \sqrt[3]{[f(x)]^3}$$

must also be continuous at 0.

- Exercise 4.3.7.** (a) Referring to the proper theorems, give a formal argument that Dirichlet's function from Section 4.1 is nowhere-continuous on \mathbf{R} .
- (b) Review the definition of Thomae's function in Section 4.1 and demonstrate that it fails to be continuous at every rational point.
- (c) Use the characterization of continuity in Theorem 4.3.2 (iii) to show that Thomae's function is continuous at every irrational point in \mathbf{R} . (Given $\epsilon > 0$, consider the set of points $\{x \in \mathbf{R} : t(x) \geq \epsilon\}$.)

Solution. (a) Let $g : \mathbf{R} \rightarrow \mathbf{R}$ be Dirichlet's function, i.e.

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Suppose $c \in \mathbf{Q}$. By the density of \mathbf{I} in \mathbf{R} , for any $\delta > 0$ there is an irrational number $x \in \mathbf{I}$ such that $x \in V_\delta(c)$; it follows that $g(x) = 0 \notin V_1(1) = V_1(g(c))$. Thus by (the negation of) Theorem 4.3.2 (ii), g is not continuous at c .

Similarly, suppose $c \in \mathbf{I}$. By the density of \mathbf{Q} in \mathbf{R} , for any $\delta > 0$ there is a rational number $x \in \mathbf{Q}$ such that $x \in V_\delta(c)$; it follows that $g(x) = 1 \notin V_1(0) = V_1(g(c))$. Thus by (the negation of) Theorem 4.3.2 (ii), g is not continuous at c .

We have now shown that g fails to be continuous at each $c \in \mathbf{R}$, i.e. that g is nowhere-continuous on \mathbf{R} .

- (b) Let $t : \mathbf{R} \rightarrow \mathbf{R}$ be Thomae's function, i.e.

$$t(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1/n & \text{if } x = m/n \in \mathbf{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Suppose $c \in \mathbf{Q}$. The density of \mathbf{I} in \mathbf{R} allows us to pick a sequence of irrational numbers (x_n) such that $\lim x_n = c$. We then have $t(x_n) = 0$ for each $n \in \mathbf{N}$ and so $\lim t(x_n) = 0$. However, $t(c)$ is strictly positive; it follows that $\lim t(x_n) \neq t(c)$. Corollary 4.3.3 allows us to conclude that t is not continuous at $c \in \mathbf{Q}$. Thus t fails to be continuous on \mathbf{Q} .

- (c) We will use the following lemma, which was proved in the solution to [Exercise 4.2.3 \(c\)](#).

Lemma 1. Suppose $c \in \mathbf{R}$ and $K \in \mathbf{N}$. There exists a $\delta > 0$ such that if $\frac{a}{b} \neq c$ is a rational number contained in $V_\delta(c)$ with $b > 0$, then $b > K$.

Suppose $c \in \mathbf{I}$ and suppose we have a sequence (x_n) such that $\lim x_n = c$. Our aim is to show that $\lim t(x_n) = t(c) = 0$. Let $\epsilon > 0$ be given and choose $K \in \mathbf{N}$ such that $\frac{1}{K} < \epsilon$. By Lemma 1, there exists a $\delta > 0$ such that if $y = \frac{a}{b}$ is a rational number contained in $V_\delta(c)$ with $b > 0$, then $b > K$. For such a y , we then have $t(y) = \frac{1}{b} < \frac{1}{K} < \epsilon$. Since $\lim x_n = c$, there is an $N \in \mathbf{N}$ such that $x_n \in V_\delta(c)$ for all $n \geq N$. Suppose $n \in \mathbf{N}$ satisfies $n \geq N$. There are two cases.

Case 1. $x_n \in \mathbf{I}$. Then $|t(x_n)| = 0 < \epsilon$.

Case 2. $x_n \in \mathbf{Q}$. Then since $x_n \in V_\delta(c)$, as noted before we have $|t(x_n)| < \frac{1}{K} < \epsilon$.

In either case we have $|t(x_n)| < \epsilon$ and thus $\lim t(x_n) = t(c) = 0$ as desired. Theorem 4.3.2 (iii) allows us to conclude that t is continuous at $c \in \mathbf{I}$ and hence that t is continuous on \mathbf{I} .

Exercise 4.3.8. Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of \mathbf{R} .

- (a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.
- (b) If $g(r) = 0$ for all $r \in \mathbf{Q}$, then $g(x) = 0$ for all $x \in \mathbf{R}$.
- (c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbf{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

Solution. (a) This is true. Let (x_n) be the sequence given by $x_n = 1 - \frac{1}{n}$. Since g is continuous at 1 and $\lim x_n = 1$, Theorem 4.3.2 (iii) implies that $\lim g(x_n) = g(1)$. Note that $x_n < 1$ for each $n \in \mathbf{N}$, so that $g(x_n) \geq 0$ for each $n \in \mathbf{N}$. The Order Limit Theorem (Theorem 2.3.4) allows us to conclude that $\lim g(x_n) = g(1) \geq 0$ also.

- (b) This is true. Let $x \in \mathbf{R}$ be given. By the density of \mathbf{Q} in \mathbf{R} , there is a sequence (r_n) of rational numbers such that $\lim r_n = x$. On the one hand, by the continuity of g at x , we must have $\lim g(r_n) = g(x)$. On the other hand, $g(r_n) = 0$ for all $n \in \mathbf{N}$ and thus $\lim g(r_n) = 0$. Since the limit of a sequence is unique, we see that $g(x) = 0$.
- (c) This is true. Since g is continuous at x_0 , for $\epsilon = g(x_0) > 0$ there is a $\delta > 0$ such that $g(x) \in V_\epsilon(g(x_0)) = (0, 2g(x_0))$ whenever $x \in V_\delta(x_0)$. In other words, for each of the uncountably many $x \in (x_0 - \delta, x_0 + \delta)$, we have $g(x) > 0$.

Exercise 4.3.9. Assume $h : \mathbf{R} \rightarrow \mathbf{R}$ is continuous on \mathbf{R} and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Solution. Suppose that $(x_n) \subseteq K$ is a convergent sequence with $\lim x_n = x$ for some $x \in \mathbf{R}$. The continuity of h implies that $\lim h(x_n) = h(x)$. Since each $x_n \in K$, we have $h(x_n) = 0$ for each $n \in \mathbf{N}$ and thus $\lim h(x_n) = 0$. The uniqueness of the limit of a sequence implies that $h(x) = 0$, i.e. that $x \in K$. Theorem 3.2.8 allows us to conclude that K is closed.

Exercise 4.3.10. Observe that if a and b are real numbers, then

$$\max\{a, b\} = \frac{1}{2}[(a + b) + |a - b|].$$

(a) Show that if f_1, f_2, \dots, f_n are continuous functions, then

$$g(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\}$$

is a continuous function.

(b) Let's explore whether the result in (a) extends to the infinite case. For each $n \in \mathbf{N}$, define f_n on \mathbf{R} by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \geq 1/n \\ n|x| & \text{if } |x| < 1/n. \end{cases}$$

Now explicitly compute $h(x) = \sup\{f_1(x), f_2(x), f_3(x), \dots\}$.

Solution. (a) First, let us show that the function $x \mapsto |x|$ is continuous. If $y \in \mathbf{R}$ and $\epsilon > 0$, set $\delta = \epsilon$ and suppose that $|x - y| < \delta$. By [Exercise 1.2.6 \(d\)](#) (the reverse triangle inequality), we have

$$||x| - |y|| \leq |x - y| < \delta = \epsilon.$$

Hence $x \mapsto |x|$ is continuous on \mathbf{R} .

Now suppose that $f_1, f_2 : A \rightarrow \mathbf{R}$ are two continuous functions defined on some domain $A \subseteq \mathbf{R}$. For any $x \in A$, note that

$$g(x) = \max\{f_1(x), f_2(x)\} = \frac{1}{2}[(f_1(x) + f_2(x)) + |f_1(x) - f_2(x)|].$$

Since f_1 and f_2 are continuous on A , and we showed that $x \mapsto |x|$ is continuous everywhere, Theorem 4.3.9 and several applications of Theorem 4.3.4 show that g is also continuous on A .

Using the observation that

$$\max\{f_1(x), f_2(x), \dots, f_n(x)\} = \max\{\max\{f_1(x), f_2(x), \dots, f_{n-1}(x)\}, f_n(x)\},$$

a simple induction argument on n shows that the maximum of n continuous functions is a continuous function.

- (b) If $x = 0$, then for each $n \in \mathbf{N}$ we have $f_n(0) = 0$ and thus $h(0) = 0$. If $x \neq 0$, then choose $n \in \mathbf{N}$ such that $\frac{1}{n} < |x|$. It follows that $f_n(x) = 1$ and thus $h(x) = 1$. So h is the function

$$h(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is not continuous at 0.

Exercise 4.3.11 (Contraction Mapping Theorem). Let f be a function defined on all of \mathbf{R} , and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbf{R}$.

- (a) Show that f is continuous on \mathbf{R} .
 (b) Pick some point $y_1 \in \mathbf{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \dots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

- (c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.
 (d) Finally, prove that if x is *any* arbitrary point in \mathbf{R} , then the sequence $(x, f(x), f(f(x)), \dots)$ converges to y defined by (b).

Solution. (a) Let $y \in \mathbf{R}$ and $\epsilon > 0$ be given. Set $\delta = \frac{\epsilon}{c}$ and suppose that $x \in \mathbf{R}$ is such that $|x - y| < \delta$. Then

$$|f(x) - f(y)| \leq c|x - y| < c\delta = \epsilon.$$

Hence f is continuous at each $y \in \mathbf{R}$.

- (b) Suppose $n > m$ are positive integers. Repeatedly applying the triangle inequality yields

$$|y_n - y_m| \leq |y_n - y_{n-1}| + \dots + |y_{m+1} - y_m| = \sum_{k=m}^{n-1} |y_{k+1} - y_k|.$$

Now we use the hypothesis that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in \mathbf{R}$ and the definition of the sequence $y_n = f(y_{n-1})$ to see that

$$\sum_{k=m}^{n-1} |y_{k+1} - y_k| \leq \sum_{k=m}^{n-1} c|y_k - y_{k-1}| \leq \dots \leq \sum_{k=m}^{n-1} c^{k-1}|y_2 - y_1| = c^{-2}|y_2 - y_1| \sum_{k=m+1}^n c^k.$$

If we let $s_n = \sum_{k=0}^n c^k$, then we have shown that for all positive integers $n > m$ we have the inequality

$$|y_n - y_m| \leq c^{-2}|y_2 - y_1|(s_n - s_m). \quad (1)$$

Let $\epsilon > 0$ be given. The series $\sum_{k=0}^{\infty} c^k$ is convergent since $0 < c < 1$, so the sequence (s_n) is Cauchy. There then exists an $N \in \mathbf{N}$ such that

$$n > m \geq N \implies |s_n - s_m| = s_n - s_m < \frac{c^2}{|y_2 - y_1| + 1} \epsilon. \quad (2)$$

Suppose n, m are positive integers such that $n > m \geq N$. By (1) and (2) we then have

$$|y_n - y_m| \leq c^{-2}|y_2 - y_1| \frac{c^2}{|y_2 - y_1| + 1} \epsilon < \epsilon.$$

Thus (y_n) is a Cauchy sequence.

(c) Since f is continuous at y , we have $\lim f(y_n) = f(y)$. It follows that

$$y = \lim y_{n+1} = \lim f(y_n) = f(y).$$

For uniqueness, observe that for any $x \in \mathbf{R}$ such that $x = f(x)$ we have

$$|x - y| = |f(x) - f(y)| \leq c|x - y|.$$

If $|x - y|$ were not zero, this would imply that $c \geq 1$. Since $0 < c < 1$, it must be the case that $|x - y| = 0$, i.e. that $x = y$.

(d) Let $x_1 = x$ and $x_{n+1} = f(x_n)$. As we just proved, (x_n) converges to some $y' \in \mathbf{R}$ such that $f(y') = y'$. The uniqueness part of (c) then implies that $y' = y$.

Exercise 4.3.12. Let $F \subseteq \mathbf{R}$ be a nonempty closed set and define $g(x) = \inf\{|x - a| : a \in F\}$. Show that g is continuous on all of \mathbf{R} and $g(x) \neq 0$ for all $x \notin F$.

Solution. If A and B are non-empty and bounded below subsets of \mathbf{R} such that $a \leq b$ for all $a \in A$ and $b \in B$, then it is straightforward to verify that $\inf A \leq \inf B$.

Fix $c \in \mathbf{R}$ and note that for any $x \in \mathbf{R}$ and $a \in F$, we have $|x - a| \leq |x - c| + |c - a|$. By the previous paragraph, this implies that

$$\inf\{|x - a| : a \in F\} \leq \inf\{|x - c| + |c - a| : a \in F\}.$$

A statement analogous to Example 1.3.7 for infima then gives us

$$\inf\{|x - a| : a \in F\} \leq |x - c| + \inf\{|c - a| : a \in F\},$$

or equivalently $g(x) - g(c) \leq |x - c|$. We can similarly derive $g(c) - g(x) \leq |x - c|$ and hence

$$|g(x) - g(c)| \leq |x - c|.$$

Thus for any $\epsilon > 0$ we can take $\delta = \epsilon$ and obtain

$$|x - c| < \delta \implies |g(x) - g(c)| < \epsilon.$$

It follows that g is continuous at each $c \in \mathbf{R}$.

Suppose that $g(x) = 0$. Using Lemma 1.3.8, we can choose a sequence $(a_n) \subseteq F$ such that $\lim |x - a_n| = g(x) = 0$, which is equivalent to $\lim a_n = x$. Since F is closed, Theorem 3.2.8 implies that $x \in F$. Thus if $x \notin F$, it must be the case that $g(x) \neq 0$.

Exercise 4.3.13. Let f be a function defined on all of \mathbf{R} that satisfies the additive condition $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$.

- (a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.
- (b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbf{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbf{Z}$. Now, prove that $f(r) = kr$ for any rational number r .
- (c) Show that if f is continuous at $x = 0$, then f is continuous at every point in \mathbf{R} and conclude that $f(x) = kx$ for all $x \in \mathbf{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Solution. (a) We have $f(0) = f(0 + 0) = f(0) + f(0)$ and so $f(0) = 0$. Furthermore, for any $x \in \mathbf{R}$,

$$0 = f(0) = f(x - x) = f(x) + f(-x) \implies f(-x) = -f(x).$$

- (b) We will show that $f(n) = kn$ for all $n \in \mathbf{N}$ by induction on n . The base case is clear, so suppose that $f(n) = kn$ for some $n \in \mathbf{N}$. Then

$$f(n + 1) = f(n) + f(1) = kn + k = k(n + 1).$$

This completes the induction step and the proof.

Combining the identity $f(n) = kn$ with $f(-x) = -f(x)$ from part (a), we see that $f(z) = kz$ for all $z \in \mathbf{Z}$.

Now suppose that $r = \frac{m}{n}$ is a rational number. On one hand, using what we just proved,

$$f\left(n \frac{m}{n}\right) = f(m) = km.$$

On the other hand, using the additivity of f ,

$$f\left(n\frac{m}{n}\right) = f\left(\sum_{j=1}^n \frac{m}{n}\right) = \sum_{j=1}^n f\left(\frac{m}{n}\right) = nf\left(\frac{m}{n}\right).$$

Thus $nf\left(\frac{m}{n}\right) = km$, i.e. $f(r) = kr$.

- (c) Let $c \in \mathbf{R}$ be given and suppose (x_n) is a convergent sequence satisfying $\lim x_n = c$. Then since $\lim(x_n - c) = 0$ and f is continuous at 0, we must have $\lim f(x_n - c) = f(0) = 0$. By the additivity of f , for each $n \in \mathbf{N}$ we have $f(x_n - c) = f(x_n) - f(c)$. It follows that

$$0 = \lim f(x_n - c) = \lim(f(x_n) - f(c)) = (\lim f(x_n)) - f(c),$$

which implies $\lim f(x_n) = f(c)$. Thus f is continuous at each $c \in \mathbf{R}$.

By Theorem 4.3.4, the function $f(x) - kx$ is continuous on all of \mathbf{R} and, by part (b), satisfies $f(r) - kr = 0$ for each $r \in \mathbf{Q}$. [Exercise 4.3.8](#) (b) allows us to conclude that $f(x) - kx = 0$, i.e. that $f(x) = kx$, for all $x \in \mathbf{R}$.

Exercise 4.3.14. (a) Let F be a closed set. Construct a function $f : \mathbf{R} \rightarrow \mathbf{R}$ such that the set of points where f fails to be continuous is precisely F . (The concept of the interior of a set, discussed in [Exercise 3.2.14](#), may be useful.)

- (b) Now consider an open set O . Construct a function $g : \mathbf{R} \rightarrow \mathbf{R}$ whose set of discontinuous points is precisely O . (For this problem, the function in [Exercise 4.3.12](#) may be useful.)

Solution. (a) Define $f : \mathbf{R} \rightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \cap F, \\ -1 & \text{if } x \in \mathbf{I} \cap F, \\ 0 & \text{if } x \notin F. \end{cases}$$

If $x \notin F$, then x belongs to the open set F^c and so there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq F^c$. Since f vanishes on this proper interval, we see that f is continuous at x .

Suppose $x \in \mathbf{Q} \cap F$ and let $\delta > 0$ be given. We consider two cases.

Case 1. If $(x - \delta, x + \delta) \subseteq F$, then we can find an irrational $y \in (x - \delta, x + \delta)$. It follows that

$$f(y) = -1 \notin (0, 2) = (f(x) - 1, f(x) + 1).$$

Case 2. If $(x - \delta, x + \delta) \not\subseteq F$, then we can find some $y \in (x - \delta, x + \delta)$ such that $y \notin F$. It follows that

$$f(y) = 0 \notin (0, 2) = (f(x) - 1, f(x) + 1).$$

Thus f is not continuous at x . A similar argument shows that f is not continuous at any $x \in \mathbf{I} \cap F$. We may conclude that the set of points where f fails to be continuous is precisely F .

(b) Let $d : \mathbf{R} \rightarrow \mathbf{R}$ be Dirichlet's function and let $h : \mathbf{R} \rightarrow \mathbf{R}$ be the function given by

$$h(x) = \inf\{|x - a| : a \in O^c\}.$$

In [Exercise 4.3.12](#) we showed that h is continuous everywhere. Furthermore, since O^c is closed, [Exercise 4.3.12](#) also shows that $h(x) > 0$ for all $x \in O$ and $h(x) = 0$ for all $x \notin O$. Define $g : \mathbf{R} \rightarrow \mathbf{R}$ by $g(x) = d(x)h(x)$ and suppose that $x \in O$. Since $h(x) > 0$ and h is continuous at x , by (the solution to) [Exercise 4.3.8](#) (c) there is some $\delta > 0$ such that h is strictly positive on the interval $I = (x - \delta, x + \delta)$. It follows that for all $t \in I$ we have $d(t) = \frac{g(t)}{h(t)}$. If g were continuous at x then Theorem 4.3.4 would imply that d is continuous at x . Since Dirichlet's function is nowhere-continuous, it must be the case that g fails to be continuous at x . Thus g is discontinuous on O .

Now suppose that $x \notin O$, so that $h(x) = 0$ and thus $g(x) = 0$. For any $y \in \mathbf{R}$, we then have

$$|g(y) - g(x)| = |g(y)| = |d(y)h(y)| = |d(y)||h(y)| \leq |h(y)|.$$

Since h is continuous at x and $h(x) = 0$, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|y - x| < \delta \implies |h(y)| < \epsilon.$$

It follows that $|g(y)| < \epsilon$ for such y and we see that g is continuous at x . We may conclude that the set of points where g fails to be continuous is precisely O .