Understanding Analysis Solutions

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Contents

Notation

I will sometimes use notation and terminology that differ from Abbott's choices. I will try to collect these differences here; please refer back to this section if you are unfamiliar with a term I have used.

Functions

Suppose $f : A \to B$ is a function. If $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B, then I will refer to f as injective or as an injection. Abbott uses the term one-to-one (1-1); both injective and one-to-one are common terms for this property of a function.

If given any $b \in B$ there exists an $a \in A$ such that $f(a) = b$, then I will refer to f as surjective or as a surjection. Abbott uses the term onto; both surjective and onto are common terms for this property of a function.

If f is both injective and surjective, then I will refer to f as bijective, or a bijection. Abbott simply calls such a function 1-1 and onto.

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1. (a) Prove that $\sqrt{3}$ is irrational. Does the same argument work to show that $\sqrt{6}$ is irrational?

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Solution. (a) Suppose there was a rational number $p = \frac{m}{n}$, which we may assume is in lowest terms, such that $p^2 = 3$. Then $m^2 = 3n^2$, so that m^2 is divisible by 3. This implies that m is divisible by 3. To see this, observe that for any $k \in \mathbf{Z}$ we have

$$(3k+1)^2 = 3(3k^2+2k)+1 \qquad \text{and} \qquad (3k+2)^2 = 3(3k^2+4k+1)+1.$$

Since m is of the form $3k+1$ or $3k+2$ for some integer k if m is not divisible by 3, it follows that

$$\text{if } m \text{ is not divisible by 3, then } m^2 \text{ is not divisible by 3;}$$

the contrapositive of this statement is what we wanted to see.

Thus we may write $m = 3k$ for some $k \in \mathbf{Z}$ and substitute this into the equation $m^2 = 3n^2$ to obtain the equation $n^2 = 3k^2$, which implies that n is also divisible by 3. So m and n share the factor 3; this is a contradiction since we assumed that m and n had no common factors. We may conclude that there is no rational number whose square is 3.

The same argument works to show that there is no rational number whose square is 6; the crux of this argument is the implication

$$\text{if } m^2 \text{ is divisible by 6, then } m \text{ is divisible by 6.}$$

This can be seen using what we have already proved. If m^2 is divisible by $6 = 2 \cdot 3$, then m^2 is divisible by 2 and 3. It follows that m is divisible by 2 and 3 and hence that m is divisible by 6.

(b) The argument breaks down when we try to assert that

$$\text{if } m^2 \text{ is divisible by 4, then } m \text{ is divisible by 4.}$$

This implication is false. For example, $2^2 = 4$ is divisible by 4 but 2 is not divisible by 4.

Exercise 1.2.2. Show that there is no rational number r satisfying $2^r = 3$.

Solution. Suppose there was a rational number $r = \frac{m}{n}$, which we may assume is in lowest terms with $n > 0$, such that $2^r = 3$. This implies that $2^m = 3^n$. Since $n > 0$ gives $3^n \geq 3$ and $2^m < 2$ for $m \leq 0$, it must be the case that $m > 0$. Then the left-hand side of the equation $2^m = 3^n$ is a positive even integer whereas the right-hand side is a positive odd integer, which is a contradiction. We may conclude that there is no rational number r such that $2^r = 3$.

Exercise 1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution. (a) This is false, as Example 1.2.2 shows.

(b) This is true and we can use the following lemma to prove it.

Lemma L.1. If $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of positive integers, i.e., $a_{n+1} \leq a_n$ and $a_n \geq 1$ for all $n \in \mathbf{N}$, then $(a_n)_{n=1}^{\infty}$ must be eventually constant. That is, there exists an $N \in \mathbf{N}$ such that $a_n = a_N$ for all $n > N$.

Proof. Let A be the set $\{a_n : n \in \mathbf{N}\}$, which is non-empty and bounded below by 1. It follows from the well-ordering principle that A has a least element, say $\min A = a_N$ for some $N \in \mathbf{N}$. Let $n > N$ be given. It cannot be the case that $a_n < a_N$, since this would contradict that a_N is the least element of A, so we must have $a_n \geq a_N$. By assumption $a_n \leq a_N$ and so we may conclude that $a_n = a_N$. □

Consider the sequence $(|A_n|)_{n=1}^{\infty}$, where $|A_n|$ is the number of elements contained in A_n. This is a sequence of positive integers, because each A_n is finite and non-empty, and furthermore this sequence is decreasing because the sets $(A_n)_{n=1}^{\infty}$ are nested:

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots .$$

We may now invoke Lemma L.1 to obtain an $N \in \mathbf{N}$ such that $|A_n| = |A_N|$ for all $n > N$. Combining this equality with the inclusion $A_n \subseteq A_N$ for each $n > N$, we see that $A_n = A_N$ for all $n > N$. It follows that $\bigcap_{n=1}^{\infty} A_n = A_N$, which by assumption is finite and non-empty.

(c) This is false. Consider $A = B = \emptyset$ and $C = \{0\}$. Then

$$A \cap (B \cup C) = \emptyset \neq \{0\} = (A \cap B) \cup C.$$

(d) This is true, since

$$x \in A \cap (B \cap C) \iff x \in A \text{ and } x \in (B \cap C) \iff x \in A \text{ and } (x \in B \text{ and } x \in C)$$

$$\iff (x \in A \text{ and } x \in B) \text{ and } x \in C \iff x \in (A \cap B) \text{ and } x \in C \iff x \in (A \cap B) \cap C,$$

where we have used that logical conjunction ("and") is associative for the third equivalence. It follows that x belongs to $A \cap (B \cap C)$ if and only if x belongs to $(A \cap B) \cap C$, which is to say that $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) This is true, since

$$x \in A \cap (B \cup C) \iff x \in A \text{ and } x \in (B \cup C) \iff x \in A \text{ and } (x \in B \text{ or } x \in C)$$
$$\iff (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \iff x \in (A \cap B) \text{ or } x \in (A \cap C)$$
$$\iff x \in (A \cap B) \cup (A \cap C),$$

where we have used that logical conjunction ("and") distributes over logical disjunction ("or") for the third equivalence. It follows that x belongs to $A \cap (B \cup C)$ if and only if x belongs to $(A \cap B) \cup (A \cap C)$, which is to say that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1.2.4. Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$.

Solution. Arrange $\mathbf{N}$ in a grid like so:

A_1	A_2	A_3	A_4	$\cdots$
1	3	6	10	$\cdots$
2	5	9	14	$\cdots$
4	8	13	19	$\cdots$
7	12	18	25	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now take A_i to be the set of numbers appearing in the i^{th} column.

Exercise 1.2.5 (De Morgan's Laws). Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^{\mathsf{c}}$, explain why $x \in A^{\mathsf{c}} \cup B^{\mathsf{c}}$. This shows that $(A \cap B)^{\mathsf{c}} \subseteq A^{\mathsf{c}} \cup B^{\mathsf{c}}$.

(b) Prove the reverse inclusion $(A \cap B)^{\mathsf{c}} \supseteq A^{\mathsf{c}} \cup B^{\mathsf{c}}$, and conclude that $(A \cap B)^{\mathsf{c}} = A^{\mathsf{c}} \cup B^{\mathsf{c}}$.

(c) Show $(A \cup B)^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}}$ by demonstrating inclusion both ways.

Solution. (a) Observe that

$$x \in (A \cap B)^{\mathsf{c}} \iff x \notin A \cap B \iff \text{not } (x \in A \text{ and } x \in B)$$
$$\iff x \notin A \text{ or } x \notin B \iff x \in A^{\mathsf{c}} \cup B^{\mathsf{c}}.$$

(b) See part (a).

(c) The proof is similar to the one given in parts (a) and (b):

$$x \in (A \cup B)^{\mathsf{c}} \iff x \notin A \cup B \iff \text{not } (x \in A \text{ or } x \in B)$$
$$\iff x \notin A \text{ and } x \notin B \iff x \in A^{\mathsf{c}} \cap B^{\mathsf{c}}.$$

Exercise 1.2.6. (a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution. (a) First suppose that a and b are both non-negative, so that $a+b$ is also non-negative; it follows that $|a+b| = a+b$ and $|a|+|b| = a+b$. Thus the triangle inequality in this case reduces to the evidently true statement $a+b \leq a+b$.

Now suppose that a and b are both negative, so that $a+b$ is also negative; it follows that $|a+b| = -a-b$ and $|a|+|b| = -a-b$. Thus the triangle inequality in this case reduces to the evidently true statement $-a-b \leq -a-b$.

(b) Starting from the true statement $ab \leq |ab|$ and using that $a^2 = |a|^2$ and $|ab| = |a||b|$ for any real numbers a and b, observe that

$$2ab \leq 2|ab| \iff a^2 + 2ab + b^2 \leq |a|^2 + 2|a||b| + |b|^2$$
$$\iff (a+b)^2 \leq (|a|+|b|)^2 \iff |a+b|^2 \leq (|a|+|b|)^2.$$

Because both $|a+b|$ and $|a|+|b|$ are non-negative, the inequality $|a+b|^2 \leq (|a|+|b|)^2$ is equivalent to $|a+b| \leq |a|+|b|$, as desired.

(c) We apply the triangle inequality twice:

$$|a-b| = |a-c+c-b| \leq |a-c| + |c-b| \leq |a-c| + |c-d| + |d-b|.$$

(d) Using the triangle inequality and the fact that $|-a| = |a|$ for any $a \in \mathbf{R}$, we find that

$$|a| = |a-b+b| \leq |a-b| + |b| \iff |a| - |b| \leq |a-b|,$$

$$|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a| \iff |b| - |a| \leq |a - b|.$$

Since $||a| - |b||$ equals either $|a| - |b|$ or $|b| - |a|$, it follows that $||a| - |b|| \leq |a - b|$.

Exercise 1.2.7. Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbf{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution. (a) Some straightforward calculations reveal that

$$f(A) = [0, 4], \qquad f(A \cap B) = f([1, 2]) = [1, 4], \qquad f(A \cup B) = f([0, 4]) = [0, 16],$$

$$f(B) = [1, 16], \qquad f(A) \cap f(B) = [1, 4], \qquad f(A) \cup f(B) = [0, 16].$$

From this we see that $f(A \cap B) = f(A) \cap f(B)$ and $f(A \cup B) = f(A) \cup f(B)$.

(b) Let $A = \{-1\}$ and $B = \{1\}$. Then $f(A \cap B) = f(\emptyset) = \emptyset$ but

$$f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\} \neq \emptyset.$$

(c) Observe that

$$y \in g(A \cap B) \iff y = g(x) \text{ for some } x \in A \cap B$$

$$\implies (y = g(x_1) \text{ for some } x_1 \in A) \text{ and } (y = g(x_2) \text{ for some } x_2 \in B)$$

$$\iff y \in g(A) \text{ and } y \in g(B) \iff y \in g(A) \cap g(B).$$

It follows that y belongs to $g(A) \cap g(B)$ whenever y belongs to $g(A \cap B)$, which is to say that $g(A \cap B) \subseteq g(A) \cap g(B)$.

(d) We always have $g(A \cup B) = g(A) \cup g(B)$; indeed,

$$\begin{aligned} y \in g(A \cup B) &\iff y = g(x) \text{ for some } x \in A \cup B \\ &\iff y = g(x) \text{ for some } x \text{ such that } (x \in A \text{ or } x \in B) \\ &\iff (y = g(x_1) \text{ for some } x_1 \in A) \text{ or } (y = g(x_2) \text{ for some } x_2 \in B) \\ &\iff y \in g(A) \text{ or } y \in g(B) \iff y \in g(A) \cup g(B). \end{aligned}$$

It follows that $g(A \cup B) = g(A) \cup g(B)$.

Exercise 1.2.8. Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is 1-1 but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not 1-1.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is 1-1 and onto.

Solution. (I much prefer the terms injective/surjective/bijective rather than one-to-one and onto; see notation. I will use these terms throughout this document.)

(a) Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(n) = 2n$. Then f is injective since $n = m$ if and only if $2n = 2m$, but f is not surjective since the range of f contains only even numbers.

$$\begin{tikzcd} \mathbf{N} \arrow[d, "f"'] & 1 \arrow[dr] & 2 \arrow[drr] & 3 \arrow[drrr] & \cdots & & & \\ \mathbf{N} & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{tikzcd}$$

(b) Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(1) = 1$ and $f(n) = n - 1$ for $n \geq 2$. Then $f(n + 1) = n$ for any $n \in \mathbf{N}$, so that f is surjective, but f is not injective since $f(1) = f(2) = 1$.

$$\begin{tikzcd} \mathbf{N} \arrow[d, "f"'] & 1 \arrow[d] & 2 \arrow[dl] & 3 \arrow[dl] & 4 \arrow[dl] & 5 \arrow[dl] & \cdots \\ \mathbf{N} & 1 & 2 & 3 & 4 & \cdots & \end{tikzcd}$$

(c) Let $f : \mathbf{N} \to \mathbf{Z}$ be given by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\begin{tikzcd}[row sep=large] \mathbf{N} \arrow[d, "f"'] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & 4 \arrow[d] & 5 \arrow[d] & \cdots \\ \mathbf{Z} & 0 & 1 & -1 & 2 & -2 & \cdots \end{tikzcd}$$

To see that f is injective, let $n \neq m$ be given and consider these cases.

Case 1. If n and m are both even, then $f(n) \neq f(m)$ since $n \neq m$ if and only if $\frac{n}{2} \neq \frac{m}{2}$.

Case 2. If n and m are both odd, then $f(n) \neq f(m)$ since $n \neq m$ if and only if $-\frac{n-1}{2} \neq -\frac{m-1}{2}$.

Case 3. If n and m have opposite signs, say n is even and m is odd, then $f(n) \neq f(m)$ since $f(n) > 0$ and $f(m) \leq 0$.

To see that f is surjective, let $n \in \mathbf{Z}$ be given. If $n > 0$, then $f(2n) = n$, and if $n \leq 0$ then $f(-2n+1) = n$.

Exercise 1.2.9. Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0,4]$ and B is the closed interval $[-1,1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

Solution. (a) Some straightforward calculations reveal that

$$\begin{array}{lll} f^{-1}(A) = [-2,2], & f^{-1}(A \cap B) = [-1,1], & f^{-1}(A \cup B) = [-2,2], \\ f^{-1}(B) = [-1,1], & f^{-1}(A) \cap f^{-1}(B) = [-1,1], & f^{-1}(A) \cup f^{-1}(B) = [-2,2]. \end{array}$$

From this we see that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) Observe that

$$x \in g^{-1}(A \cap B) \iff g(x) \in A \cap B \iff (g(x) \in A) \text{ and } (g(x) \in B)$$
$$\iff (x \in g^{-1}(A)) \text{ and } (x \in g^{-1}(B)) \iff x \in g^{-1}(A) \cap g^{-1}(B).$$

Similarly,

$$x \in g^{-1}(A \cup B) \iff g(x) \in A \cup B \iff (g(x) \in A) \text{ or } (g(x) \in B)$$
$$\iff (x \in g^{-1}(A)) \text{ or } (x \in g^{-1}(B)) \iff x \in g^{-1}(A) \cup g^{-1}(B).$$

Exercise 1.2.10. Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution. (a) This is false; the implication

$$\text{if } a < b + \epsilon \text{ for every } \epsilon > 0, \text{ then } a < b$$

does not hold. The problem occurs when we consider the case where $a = b$. For example, we certainly have $1 < 1 + \epsilon$ for every $\epsilon > 0$ but of course $1 < 1$ is false.

(b) See part (a).

(c) This is true. The implication

$$\text{if } a \leq b, \text{ then } a < b + \epsilon \text{ for every } \epsilon > 0$$

follows since $a \leq b < b + \epsilon$ for every $\epsilon > 0$ and the implication

$$\text{if } a > b, \text{ then } a \geq b + \epsilon \text{ for some } \epsilon > 0$$

can be seen by taking $\epsilon = a - b > 0$, so that $b + \epsilon = a \leq a$.

Exercise 1.2.11. Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution. (a) The negated statement is:

$$\text{there exist real numbers } a < b \text{ such that } a + \tfrac{1}{n} \geq b \text{ for all } n \in \mathbf{N}.$$

The original statement is true and follows from the Archimedean Property (Theorem 1.4.2).

(b) The negated statement is:

$$\text{for all } x > 0, \text{ there exists an } n \in \mathbf{N} \text{ such that } \tfrac{1}{n} \leq x.$$

The negated statement is true and again follows from the Archimedean Property (Theorem 1.4.2).

(c) The negated statement is:

there are two distinct real numbers with no rational number between them.

The original statement is true; this is the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3).

Exercise 1.2.12. Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution. (a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $y_n > -6$. Since $y_1 = 6$, the truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and observe that

$$y_{n+1} = \frac{2}{3}y_n - 2 > \frac{2}{3}(-6) - 2 = -6,$$

i.e., $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

(b) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $y_{n+1} \leq y_n$. Since $y_1 = 6$ and $y_2 = 2$, the truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and observe that

$$y_{n+2} = \frac{2}{3}y_{n+1} - 2 \leq \frac{2}{3}y_n - 2 = y_{n+1},$$

i.e., $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

Exercise 1.2.13. For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^{\mathsf{c}} = A_1^{\mathsf{c}} \cap A_2^{\mathsf{c}} \cap \cdots \cap A_n^{\mathsf{c}}$$

for any finite $n \in \mathbf{N}$.

(b) It is tempting to appeal to induction to conclude that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathsf{c}} = \bigcap_{i=1}^{\infty} A_i^{\mathsf{c}},$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution. (a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $(A_1 \cup \cdots \cup A_n)^{\mathsf{c}} = A_1^{\mathsf{c}} \cap \cdots \cap A_n^{\mathsf{c}}$ for any sets $A_1, \ldots, A_n$. The truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$, let $A_1, \ldots, A_n, A_{n+1}$ be given, and observe that

$$\begin{aligned}
(A_1 \cup \cdots \cup A_n \cup A_{n+1})^{\mathsf{c}} &= ((A_1 \cup \cdots \cup A_n) \cup (A_{n+1}))^{\mathsf{c}} \\
&= (A_1 \cup \cdots \cup A_n)^{\mathsf{c}} \cap A_{n+1}^{\mathsf{c}} && \text{(Exercise 1.2.5)} \\
&= A_1^{\mathsf{c}} \cap \cdots \cap A_n^{\mathsf{c}} \cap A_{n+1}^{\mathsf{c}}, && \text{(induction hypothesis)}
\end{aligned}$$

i.e., $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

(b) Let $B_i = \{i, i+1, i+2, \ldots\}$, so that

$$B_1 = \{1, 2, 3, \ldots\}, \quad B_2 = \{2, 3, 4, \ldots\}, \quad B_3 = \{3, 4, 5, \ldots\}, \quad \text{etc.}$$

It is straightforward to verify that $\bigcap_{i=1}^{n} B_i = B_n \neq \emptyset$ for any $n \in \mathbf{N}$; however, as Example 1.2.2 shows, the intersection $\bigcap_{i=1}^{\infty} B_i$ is empty.

(c) Observe that

$$x \in \left(\bigcup_{i=1}^{\infty} A_i \right)^{\mathsf{c}} \iff x \notin \bigcup_{i=1}^{\infty} A_i \iff x \notin A_i \text{ for every } i \in \mathbf{N} \iff x \in \bigcap_{i=1}^{\infty} A_i^{\mathsf{c}}.$$

It follows that

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^{\mathsf{c}} = \bigcap_{i=1}^{\infty} A_i^{\mathsf{c}}.$$

1.3 The Axiom of Completeness

Exercise 1.3.1. (a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution. (a) A real number t is the *greatest lower bound* for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

(i) t is a lower bound for A;

(ii) if b is any lower bound for A, then $b \leq t$.

(b) Here is a version of Lemma 1.3.8 for greatest lower bounds.

Lemma L.2. Assume $t \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then $t = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $a < t + \epsilon$.

Proof. First, let us prove the implication

$$\text{if } t = \inf A, \text{ then for every } \epsilon > 0 \text{ there exists an } a \in A \text{ such that } a < t + \epsilon$$

by proving the contrapositive statement

$$\text{if there exists an } \epsilon > 0 \text{ such that } t + \epsilon \le a \text{ for every } a \in A \text{ then } t \neq \inf A.$$

If such an $\epsilon > 0$ exists, then $t + \epsilon$ is a lower bound for A strictly greater than t; it follows that t is not the greatest lower bound for A, i.e., $t \neq \inf A$.

Now let us prove the converse:

$$\text{if for every } \epsilon > 0 \text{ there exists an } a \in A \text{ such that } a < t + \epsilon, \text{ then } t = \inf A.$$

Suppose $b \in \mathbf{R}$ is such that $b > t$. Taking $\epsilon = b - t > 0$, by assumption we are guaranteed the existence of an $a \in A$ such that $a < t + \epsilon = b$. Thus b is not a lower bound for A; this proves the contrapositive of criterion (ii) in part (a) and we may conclude that $t = \inf A$. □

Exercise 1.3.2. Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \ge \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution. (a) Take $B = \{0\}$, so that $\inf B = \sup B = 0$.

(b) This is impossible. To see this, let us first use induction to show that any non-empty finite subset of $\mathbf{R}$ contains a minimum and a maximum element.

Lemma L.3. If $E \subseteq \mathbf{R}$ is non-empty and finite, then E contains a minimum and a maximum element.

Proof. For $n \in \mathbf{N}$, let $P(n)$ be the statement that any subset of $\mathbf{R}$ containing n elements has a minimum and a maximum element. For the base case $P(1)$, simply observe that $\min\{x\} = \max\{x\} = x$ for any $x \in \mathbf{R}$.

Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and let $E \subseteq \mathbf{R}$ be a set containing $n + 1$ elements. Fix some $x \in E$ and consider the set $F = E \setminus \{x\}$, which contains n elements. Our induction hypothesis guarantees the existence of a minimum element $a := \min F$ and a maximum element $b := \max F$, which must satisfy $a \le b$. There are then three cases; the conclusion in each case is straightforward to verify.

Case 1. If $x < a$, then $\min E = x$ and $\max E = b$.

Case 2. If $x > b$, then $\min E = a$ and $\max E = x$.

Case 3. If $a \leq x \leq b$, then $\min E = a$ and $\max E = b$.

In any case, the set E has a minimum and a maximum element, i.e., $P(n+1)$ holds. This completes the induction step and the proof. □

It is immediate from the definition of the supremum and the maximum of a set $E \subseteq \mathbf{R}$ that if $\max E$ exists then $\sup E = \max E$ (see Exercise 1.3.7); similarly, if $\min E$ exists then $\inf E = \min E$. It follows that the given request is impossible: if $E \subseteq \mathbf{R}$ is finite, then Lemma L.3 implies that $\min E = \inf E$ and $\max E = \sup E$ both exist and hence E contains both its infimum and its supremum.

(c) Consider the bounded set $E = \{p \in \mathbf{Q} : 0 < p \leq 1\}$, which satisfies $\sup E = 1 \in E$ and $\inf E = 0 \notin E$.

Exercise 1.3.3. (a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b$ is a lower bound for $A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution. (a) B is non-empty since A is bounded below, and B is bounded above by any $x \in A$; there exists at least one such x since A is non-empty. It follows from the Axiom of Completeness that $\sup B$ exists. To see that $\sup B = \inf A$, we need to show that $\sup B$ satisfies criteria (i) and (ii) from Exercise 1.3.1 (a).

(i) First we need to prove that $\sup B$ is a lower bound for A, i.e., if $x \in A$, then $\sup B \leq x$. We will prove the contrapositive statement: if $x < \sup B$, then $x \notin A$. If x is strictly less than $\sup B$, then x cannot be an upper bound for B. Thus there exists some $b \in B$ such that $x < b$. Since b is a lower bound for A, it follows that $x \notin A$.

(ii) Suppose $y \in \mathbf{R}$ is a lower bound of A, so that y belongs to B; it follows that $y \leq \sup B$.

We may conclude that $\sup B = \inf A$.

(b) Part (a) shows that the existence of the greatest lower bound for non-empty bounded below subsets of $\mathbf{R}$ is implied by the Axiom of Completeness; adding this existence as part of the Axiom of Completeness would be redundant.

Exercise 1.3.4. Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^{n} A_k)$.

(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

Solution. (a) Let $n \in \mathbf{N}$ be given. For each $k \in \{1, \ldots, n\}$, the Axiom of Completeness guarantees that $\sup A_k$ exists. By Lemma L.3, the finite set $\{\sup A_1, \ldots, \sup A_k\}$ has a maximum element, say M; we claim that $\sup(\bigcup_{k=1}^{n} A_k) = M$. To prove this, we must verify criteria (i) and (ii) from Definition 1.3.2.

(i) If $x \in \bigcup_{k=1}^{n} A_k$, then $x \in A_k$ for some $k \in \{1, \ldots, n\}$; it follows that $x \le \sup A_k \le M$. Since x was arbitrary, we see that M is an upper bound for $\bigcup_{k=1}^{n} A_k$.

(ii) If $b \in \mathbf{R}$ is an upper bound for $\bigcup_{k=1}^{n} A_k$, then b must be an upper bound for each A_k. It follows that $\sup A_k \le b$ for each $k \in \{1, \ldots, n\}$ and hence that $M \le b$.

We may conclude that $\sup(\bigcup_{k=1}^{n} A_k) = M$.

(b) The proof given above does not extend to the infinite case, since in general the set $\{\sup A_1, \sup A_2, \ldots\}$ need not have a maximum. Indeed, it may be the case that $\sup(\bigcup_{k=1}^{\infty} A_k)$ does not exist. For example, take $A_k = [0, k]$. Then each A_k is non-empty and bounded above with $\sup A_k = k$, but $\bigcup_{k=1}^{\infty} A_k = [0, \infty)$, which does not have a supremum in $\mathbf{R}$.

Exercise 1.3.5. As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \ge 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution. (a) If $c = 0$ then the result is clear, so suppose that $c > 0$. For any $x \in A$, notice that

$$x \le \sup A \iff cx \le c \sup A.$$

This demonstrates that $c \sup A$ is an upper bound for cA.

Suppose $b \in \mathbf{R}$ is an upper bound for cA, i.e., $cx \le b$ for all $x \in A$. Then $x \le c^{-1} b$ for all $x \in A$, i.e., $c^{-1} b$ is an upper bound for A. It follows that $\sup A \le c^{-1} b$ and hence that $c \sup A \le b$. We may conclude that $\sup(cA) = c \sup A$.

(b) If $c < 0$ and $\inf A$ exists then $\sup(cA) = c \inf A$. The proof is similar to part (a). For any $x \in A$, we have

$$\inf A \leq x \iff cx \leq c \inf A,$$

so that $c \inf A$ is an upper bound for cA.

Suppose $b \in \mathbf{R}$ is an upper bound for cA, i.e., $cx \leq b$ for all $x \in A$. Then $c^{-1}b \leq x$ for all $x \in A$, so that $c^{-1}b$ is a lower bound for A. It follows that $c^{-1}b \leq \inf A$ and hence that $c \inf A \leq b$. We may conclude that $\sup(cA) = c \inf A$.

If $\inf A$ doesn't exist then $\sup(cA)$ doesn't exist either, since for $c < 0$ the set A is bounded below if and only if cA is bounded above. For example, $A = (-\infty, 0)$ and $c = -1$ gives $cA = (0, \infty)$.

Exercise 1.3.6. Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A+B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution. (a) For any $a \in A$ and $b \in B$ we have $a \leq s$ and $b \leq t$. It follows that $a + b \leq s + t$ and thus $s + t$ is an upper bound for $A + B$.

(b) For any $b \in B$ we have $a + b \leq u$, which gives $b \leq u - a$. This demonstrates that $u - a$ is an upper bound for B and so it follows that $t \leq u - a$.

(c) Part (b) implies that for any $a \in A$ we have $t \leq u - a$, which gives $a \leq u - t$. This shows that $u - t$ is an upper bound for A and it follows that $s \leq u - t$, i.e., $s + t \leq u$. Since u was an arbitrary upper bound for $A + B$, we may conclude that

$$\sup(A + B) = s + t = \sup A + \sup B.$$

(d) Let $\epsilon > 0$ be given. By Lemma 1.3.8, there exist elements $a \in A$ and $b \in B$ such that $s - \frac{\epsilon}{2} < a$ and $t - \frac{\epsilon}{2} < b$, which implies that $s + t - \epsilon < a + b$. We showed in part (a) that $s + t$ is an upper bound for $A + B$, so we may invoke Lemma 1.3.8 to conclude that $\sup(A + B) = \sup A + \sup B$.

Exercise 1.3.7. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution. Let $b \in \mathbf{R}$ be an upper bound of A. Since $a \in A$, we must have $a \leq b$; it follows that $a = \sup A$.

Exercise 1.3.8. Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.

(c) $\{n/(3n+1) : n \in \mathbf{N}\}$.

(d) $\{m/(m+n) : m, n \in \mathbf{N}\}$.

Solution. (a) The supremum is 1 and the infimum is 0.

(b) The supremum is 1 and the infimum is -1.

(c) The supremum is $\frac{1}{3}$ and the infimum is $\frac{1}{4}$.

(d) The supremum is 1 and the infimum is 0.

Exercise 1.3.9. (a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Solution. (a) Let $\epsilon = \sup B - \sup A > 0$. By Lemma 1.3.8, there exists a $b \in B$ such that $\sup B - \epsilon = \sup A < b$. It follows that b is an upper bound for A.

(b) Take $A = B = (0, 1)$. Then $\sup A = \sup B = 1$, but no element of B is an upper bound for A (the interval $(0, 1)$ has no maximum element).

Exercise 1.3.10 (Cut Property). The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution. (a) Suppose that A and B are non-empty disjoint subsets of $\mathbf{R}$ such that $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$. Notice that A is non-empty (by assumption) and bounded above (because B is non-empty); the Axiom of Completeness then implies that $c := \sup A$ exists. It follows that $x \leq c$ for all $x \in A$ and, since each element of B is an upper bound for A, we also have $x \geq c$ for all $x \in B$.

(b) Suppose that $E \subseteq \mathbf{R}$ is non-empty and bounded above. Define

$$A = \{a \in \mathbf{R} : a \text{ is not an upper bound of } E\},$$

$$B = A^{\mathsf{c}} = \{b \in \mathbf{R} : b \text{ is an upper bound of } E\}.$$

Notice that B is non-empty as E is bounded above and A is non-empty because $x - 1 \in A$ for any $x \in E$; we are guaranteed the existence of at least one $x \in E$ as E is non-empty. Furthermore, A and B are evidently disjoint and satisfy $A \cup B = \mathbf{R}$.

Let $a \in A$ and $b \in B$ be given. Since a is not an upper bound for E there exists some $x \in E$ such that $a < x$ and since b is an upper bound for E, we must then have $x \leq b$; it follows that $a < b$. We may now invoke the Cut Property to obtain a $c \in \mathbf{R}$ such that $x \leq c$ for all $x \in A$ and $x \geq c$ for all $x \in B$.

We claim that $c = \sup E$. Since $A \cup B = \mathbf{R}$ and $A \cap B = \emptyset$, exactly one of $c \in A$ or $c \in B$ holds. Suppose that $c \in A$, i.e., c is not an upper bound of E, which is the case if and only if there is some $z \in E$ such that $c < z$. Observe that $y := \frac{c+z}{2}$ satisfies $c < y < z$, so that $y \in A$; but this contradicts the fact that $x \leq c$ for all $x \in A$.

So it must be the case that $c \in B$, i.e., c is an upper bound for E. The Cut Property says that $c \leq x$ for all $x \in B$; in other words, c is less than all other upper bounds of E. We may conclude that $c = \sup E$.

(c) A concrete example is given in the following lemma, which will also be useful for later exercises.

Lemma L.4. Let

$$A = \{p \in \mathbf{Q} : p < 0 \text{ or } p^2 < 2\} \quad \text{and} \quad B = \{p \in \mathbf{Q} : p > 0 \text{ and } p^2 > 2\}.$$

Then:

(i) A and B are non-empty, $A \cup B = \mathbf{Q}$, and $A \cap B = \emptyset$.

(ii) $p < q$ for all $p \in A$ and $q \in B$.

(iii) A has no maximum element and B has no minimum element.

Proof. (i) It is clear that A and B are non-empty. The negation of the statement "$p < 0$ or $p^2 < 2$" is "$p > 0$ and $p^2 \geq 2$"; by Theorem 1.1.1, this negated statement is equivalent to "$p > 0$ and $p^2 > 2$" for $p \in \mathbf{Q}$. Thus $B = \mathbf{Q} \setminus A$, from which it follows that $A \cup B = \mathbf{Q}$ and $A \cap B = \emptyset$.

(ii) Let $p \in A$ and $q \in B$ be given. If $p \leq 0$ then evidently $p < q$, so suppose that $p > 0$. It must then be the case that $p^2 < 2$, whence $p^2 < q^2$. Since p and q are positive, this implies that $p < q$.

(iii) Let $p \in A$ be given. We need to show that there exists some $q \in A$ such that $p < q$. If $p \leq 0$, we can take $q = 1$; if $p > 0$, so that $p^2 < 2$, then define

$$q = p + \frac{2 - p^2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{1}$$

Notice that $0 < \frac{2-p^2}{p+2}$, since $p^2 < 2$, from which it follows that $p < q$. A straightforward calculation yields

$$2 - q^2 = \frac{2(2 - p^2)}{(p + 2)^2};$$

using again that $p^2 < 2$, we see that $2 - q^2 > 0$ and thus $q \in A$.

Now let $p \in B$ be given. We need to show that there exists some $q \in B$ such that $q < p$. In fact, we can define q by equation (1) again; an argument similar to the one just given shows that $q < p$ and $q \in B$. $\square$

Parts (i) and (ii) of Lemma L.4 show that the sets A and B satisfy the hypotheses of the Cut Property. If the Cut Property held for $\mathbf{Q}$, then we would be able to obtain a $c \in \mathbf{Q}$

such that $p \le c$ for all $p \in A$ and $c \le q$ for all $q \in B$. Since $A \cup B = \mathbf{Q}$ and $A \cap B = \emptyset$, this implies that c is either the maximum of A or the minimum of B—but this contradicts part (iii) of Lemma L.4. We may conclude that the Cut Property does not hold for $\mathbf{Q}$.

Exercise 1.3.11. Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \le \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution. (a) This is true. The Axiom of Completeness guarantees that $\sup A$ and $\sup B$ both exist. Furthermore, since each element of A is an element of B, any upper bound of B must be an upper bound of A also. In particular, $\sup B$ must be an upper bound of A; it follows that $\sup A \le \sup B$.

(b) This is true. Let $c = \frac{\sup A + \inf B}{2}$, so that $\sup A < c < \inf B$, and notice that for any $a \in A$ and $b \in B$ we have

$$a \le \sup A < c < \inf B \le b.$$

(c) This is false. Consider $A = (-1, 0)$ and $B = (0, 1)$, and notice that $c = 0$ satisfies $a < c < b$ for all $a \in A$ and $b \in B$, but $\sup A = \inf B = 0$.

1.4 Consequences of Completeness

Exercise 1.4.1. Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a + b$ are elements of $\mathbf{Q}$ as well.

(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a + t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \ne 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Solution. (a) Suppose $a = \frac{m}{n}$ and $b = \frac{p}{q}$. Then

$$ab = \frac{mp}{nq} \qquad \text{and} \qquad a + b = \frac{mq + np}{nq},$$

which are rational numbers.

(b) Let $a \in \mathbf{Q}$ be fixed. We want to prove that

$$\text{if } t \in \mathbf{I}, \text{ then } a + t \in \mathbf{I}.$$

To do this, we will prove the contrapositive statement

$$\text{if } a + t \in \mathbf{Q}, \text{ then } t \in \mathbf{Q}.$$

Simply observe that $t = (a + t) - a$; it follows from part (a) that $t \in \mathbf{Q}$.

Similarly, let $a \in \mathbf{Q}$ be non-zero. We can show that

$$\text{if } at \in \mathbf{Q}, \text{ then } t \in \mathbf{Q}$$

by observing that $t = a^{-1}(at)$ and appealing to part (a) to conclude that $t \in \mathbf{Q}$.

(c) $\mathbf{I}$ is not closed under addition or multiplication. For example, $-\sqrt{2}$ and $\sqrt{2}$ are irrational numbers, but their sum is the rational number 0 and their product is the rational number -2. The sum or product of two irrational numbers may be irrational; for example, it can be shown that $\sqrt{2} + \sqrt{3}$ and $\sqrt{2}\sqrt{3} = \sqrt{6}$ are irrational:

- For the irrationality of $\sqrt{6}$, see Exercise 1.2.1 (a).
- For the irrationality of $\sqrt{2} + \sqrt{3}$, observe that $\sqrt{2} + \sqrt{3}$ is a root of the polynomial $x^4 - 10x^2 + 1$; the rational root theorem says that the only possible rational roots of this polynomial are ± 1—but neither of these solve the equation $x^4 - 10x^2 + 1 = 0$.

So in general, we cannot say anything about the sum or product of two irrational numbers without more information.

Exercise 1.4.2. Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution. If s is not an upper bound for A then there must exist some $x \in A$ such that $s < x$. By the Archimedean Property (Theorem 1.4.2), there exists a natural number n such that $s + \frac{1}{n} < x$, which implies that $s + \frac{1}{n}$ is not an upper bound for A. Given our hypothesis that $s + \frac{1}{n}$ is an upper bound for A for all $n \in \mathbf{N}$, we see that s must be an upper bound for A.

Now let $\epsilon > 0$ be given and using the Archimedean Property (Theorem 1.4.2), pick a natural number n such that $\frac{1}{n} < \epsilon$. By assumption $s - \frac{1}{n}$ is not an upper bound for A, so there must exist some $x \in A$ such that $s - \frac{1}{n} < x$; this implies that $s - \epsilon < x$ since $\frac{1}{n} < \epsilon$. Because $\epsilon > 0$ was arbitrary, we may invoke Lemma 1.3.8 to conclude that $s = \sup A$.

Exercise 1.4.3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution. It is clear that any $x \leq 0$ does not belong to $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$. Let $x > 0$ be given and use the Archimedean Property (Theorem 1.4.2) to choose an $N \in \mathbf{N}$ such that $\frac{1}{N} < x$. It follows that $x \notin \left(0, \frac{1}{N}\right)$ and hence that $x \notin \bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$. We may conclude that $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right) = \emptyset$.

Exercise 1.4.4. Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$.

Solution. It is clear that b is an upper bound for T. Let $\epsilon > 0$ be given. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), there exists a rational number p satisfying

$$\max\{a, b - \epsilon\} < p < b.$$

It follows that $p \in T$ and $b - \epsilon < p$ and hence, by Lemma 1.3.8, we may conclude that $\sup T = b$.

Exercise 1.4.5. Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), there exists a rational number p satisfying $a - \sqrt{2} < p < b - \sqrt{2}$, which gives $a < p + \sqrt{2} < b$. Since $p + \sqrt{2}$ is irrational (Exercise 1.4.1 (b)), the corollary is proved.

Exercise 1.4.6. Recall that a set B is *dense* in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2.

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution. (a) This set is not dense in $\mathbf{R}$. For $1 \leq q \leq 10$, observe that if $p \geq 1$ then $\frac{p}{q} \geq \frac{1}{10}$, if $p \leq -1$ then $\frac{p}{q} \leq -\frac{1}{10}$, and if $p = 0$ then $\frac{p}{q} = 0$. So there is no element of this set between the real numbers $\frac{1}{1000}$ and $\frac{1}{100}$, for example.

(b) This set is dense in $\mathbf{R}$. Let $a < b$ be given real numbers. Using the Archimedean Property (Theorem 1.4.2), let $n \in \mathbf{N}$ be such that $\frac{1}{n} < b - a$, which implies that $\frac{1}{2^n} < b - a$. Now let p be the smallest integer greater than $2^n a$, so that $p - 1 \leq 2^n a < p$, and observe that

$$2^n a < p \leq 1 + 2^n a < 2^n b;$$

it follows that $\frac{p}{2^n}$ lies between a and b.

(c) This set is not dense in $\mathbf{R}$. If $p > 0$ then

$$10|p| \geq q \iff 10p \geq q \iff \frac{p}{q} \geq \frac{1}{10},$$

and if $p < 0$ then

$$10|p| \geq q \iff -10p \geq q \iff \frac{p}{q} \leq -\frac{1}{10}.$$

We cannot have $p = 0$ since q is a positive integer. Thus there is no element of this set between the real numbers 0 and $\frac{1}{100}$, for example.

Exercise 1.4.7. Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Solution. Assuming that $\alpha^2 - 2 > 0$, the Archimedean Property (Theorem 1.4.2) implies that there is an $n \in \mathbf{N}$ such that

$$\frac{2\alpha}{n} < \alpha^2 - 2 \iff 2 < \alpha^2 - \frac{2\alpha}{n}.$$

Let $\beta = \alpha - \frac{1}{n}$ and note that since $1 \in T$ we have $\alpha \geq 1$ and hence $\beta \geq 0$; it follows that $t \leq \beta$ for all $t \in T$ such that $t < 0$. Now observe that

$$\beta^2 = \left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n} > 2,$$

so that for any $t \in T$ we have $t^2 < 2 < \beta^2$. If $t \in T$ is such that $t \geq 0$ then the inequality $t^2 < \beta^2$ implies that $t < \beta$, as β is also non-negative.

We have now shown that $t \leq \beta$ for all $t \in T$, i.e., β is an upper bound for T—but this contradicts the fact that α is the supremum of T since $\beta < \alpha$.

Exercise 1.4.8. Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbf{R} : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution. (a) Let

$$A = \left\{ -\frac{1}{2n} : n \in \mathbf{N} \right\} = \left\{ -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots \right\}$$

$$\text{and} \qquad B = \left\{ -\frac{1}{2n-1} : n \in \mathbf{N} \right\} = \left\{ -1, -\frac{1}{3}, -\frac{1}{5}, \ldots \right\}.$$

Then $A \cap B = \emptyset$ and $\sup A = \sup B = 0$, which belongs to neither A nor B.

(b) If we let $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbf{N}$, then $\bigcap_{n=1}^{\infty} J_n = \{0\}$.

(c) For $n \in \mathbf{N}$, let $L_n = [n, \infty)$.

(d) This is impossible. To see this, let $(I_n)_{n=1}^{\infty}$ be a sequence of closed bounded intervals satisfying $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for every $N \in \mathbf{N}$. Define $J_N = \bigcap_{n=1}^{N} I_n$ for $N \in \mathbf{N}$ and note that any finite intersection of closed bounded intervals is a (possibly empty) closed bounded interval. Thus:

- each J_N is a closed bounded interval;
- these intervals are non-empty and nested, i.e., $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$;

- $\bigcap_{n=1}^{\infty} I_n = \bigcap_{N=1}^{\infty} J_N$.

It then follows from the Nested Interval Property (Theorem 1.4.1) that $\bigcap_{n=1}^{\infty} I_n = \bigcap_{N=1}^{\infty} J_N$ is non-empty.

1.5 Cardinality

Exercise 1.5.1. Finish the following proof for Theorem 1.5.7.

Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$ which is 1-1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbf{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbf{N}$ onto A.

Solution. Given $n_1 = \min f^{-1}(A) = \min\{n \in \mathbf{N} : f(n) \in A\}$, we can construct a sequence $(n_k)_{k=1}^{\infty}$ of natural numbers recursively by defining

$$n_k = \min\bigl(f^{-1}(A) \setminus \{n_1, \ldots, n_{k-1}\}\bigr) = \min(\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, \ldots, n_{k-1}\})$$

for $k \geq 2$. Because A is infinite and f is surjective, the set $\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, \ldots, n_{k-1}\}$ is non-empty (indeed, it must be infinite) for each $k \geq 2$; it follows that each n_k is well-defined. See Figure F.1 for an example construction of the sequence $(n_k)_{k=1}^{\infty}$, for some bijection $f : \mathbf{N} \to B$.

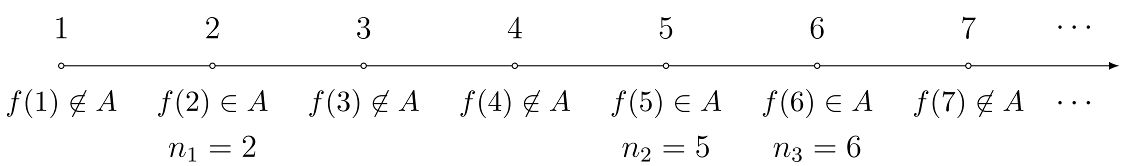

Figure F.1: Example construction of $(n_k)_{k=1}^{\infty}$

It is clear from this construction that $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence.

Define $g : \mathbf{N} \to A$ by $g(k) = f(n_k)$; we claim that g is a bijection. For injectivity, observe that

$$g(\ell) = g(k) \iff f(n_\ell) = f(n_k) \iff n_\ell = n_k \iff \ell = k,$$

where we have used the injectivity of f for the second equivalence and the strict monotonicity of the sequence $(n_k)_{k=1}^{\infty}$ for the third equivalence.

For the surjectivity of g, let $a \in A$ be given. Since f is surjective, there is a positive integer N such that $f(N) = a$; we need to find some $k \in \mathbf{N}$ such that $n_k = N$. It cannot be the case that $N < n_1$, otherwise n_1 would not be the minimum of $\{n \in \mathbf{N} : f(n) \in A\}$, so we must have $n_1 \leq N$. Given this, and the fact that $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers, there must exist a $k \in \mathbf{N}$ such that $n_k \leq N < n_{k+1}$. In fact, it must be the case that $n_k = N$, otherwise n_{k+1} would not be the minimum of $\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, \ldots, n_k\}$. Thus $g(k) = f(n_k) = f(N) = a$.

Exercise 1.5.2. Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbf{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbf{Q}$ is uncountable:

Assume, for contradiction, that $\mathbf{Q}$ is countable. Thus we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbf{Q}$ must therefore be uncountable.

Solution. The construction does not imply that $\bigcap_{n=1}^{\infty} I_n = \emptyset$; it only guarantees that this intersection does not contain any rational numbers.

Exercise 1.5.3. Use the following outline to supply proofs for the statements in Theorem 1.5.8.

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Now, explain how the more general statement in (i) follows.

(b) Explain why induction *cannot* be used to prove part (ii) of Theorem 1.5.8 from part (i).

(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$\begin{array}{cccccc}
1 & 3 & 6 & 10 & 15 & \cdots \\
2 & 5 & 9 & 14 & \cdots & \\
4 & 8 & 13 & \cdots & & \\
7 & 12 & \cdots & & & \\
11 & \cdots & & & & \\
\vdots & & & & &
\end{array}$$

leads to a proof of Theorem 1.5.8 (ii).

Solution. (a) As noted, it will suffice to show that $A_1 \cup B_2$ is countable, where $B_2 = A_2 \setminus A_1$. Since A_1 is countable, there exists a bijection $f : \mathbf{N} \to A_1$. Consider the following cases.

Case 1. If B_2 is empty, then $A_1 \cup B_2 = A_1$, which is countable by assumption.

Case 2. Suppose that B_2 is non-empty and finite, say $B_2 = \{x_1, \ldots, x_k\}$ for some $k \in \mathbf{N}$. Define $g : \mathbf{N} \to A_1 \cup B_2$ by

$$g(n) = \begin{cases} x_n & \text{if } 1 \leq n \leq k, \\ f(n-k) & \text{if } k < n. \end{cases}$$

The injectivity of g follows as A_1 and B_2 are disjoint and f is injective. For the surjectivity of g, it is clear that every element of B_2 belongs to the range of g; the surjectivity of f implies that the elements of A_1 belong to the range of g also.

Case 3. Suppose that B_2 is infinite. Since B_2 is a subset of the countable set A_2, Exercise 1.5.1 implies that B_2 is countable, i.e., there exists a bijection $h : \mathbf{N} \to B_2$. Define $g : \mathbf{N} \to A_1 \cup B_2$ by

$$g(n) = \begin{cases} f\left(\frac{n}{2}\right) & \text{if } n \text{ is even}, \\ h\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd}. \end{cases}$$

To see that g is injective, suppose that m and n are distinct positive integers.

Case 3.1 If both of m and n are even then $g(m) \neq g(n)$ since f is injective.

Case 3.2 If both of m and n are odd then $g(m) \neq g(n)$ since h is injective.

Case 3.3 If one of m and n is even and the other is odd then $g(m) \neq g(n)$ since f maps into A_1, h maps into B_2, and $A_1 \cap B_2 = \emptyset$.

To see that g is surjective, let $x \in A_1 \cup B_2$ be given. Since $A_1 \cap B_2 = \emptyset$, exactly one of the statements $x \in A_1$ or $x \in B_2$ holds. Suppose $x \in A_1$. Because f is surjective, there is a positive integer n such that $f(n) = x$; it follows that $g(2n) = f(n) = x$. If $x \in B_2$, then the surjectivity of h implies that there is a positive integer n such that $h(n) = x$; it follows that $g(2n-1) = h(n) = x$. We may conclude that g is a bijection and hence that $A_1 \cup B_2$ is countable.

A simple induction argument proves the more general statement in Theorem 1.5.8 (i). Let $P(n)$ be the statement that for countable sets $A_1, \dots, A_n$, the union $A_1 \cup \cdots \cup A_n$ is countable. The truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and suppose we have countable sets $A_1, \dots, A_n, A_{n+1}$. Let $A' = A_1 \cup \cdots \cup A_n$; the induction hypothesis guarantees that A' is countable. Observe that

$$A_1 \cup \cdots \cup A_n \cup A_{n+1} = A' \cup A_{n+1}.$$

Since A' and A_{n+1} are countable, the union $A' \cup A_{n+1}$ is also countable by our previous proof, i.e., $P(n+1)$ holds. This completes the induction step and the proof.

(b) Induction can only be used to show that a particular statement $P(n)$ holds for each value of $n \in \mathbf{N}$.

(c) For each $n \in \mathbf{N}$ there exists a bijection $f_n : \mathbf{N} \to A_n$. Let $a_{mn} = f_n(m)$ and arrange these into another two-dimensional array like so:

$$\begin{array}{cccccc}
A_1 & A_2 & A_3 & A_4 & A_5 & \cdots \\
\hline
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\
a_{21} & a_{22} & a_{23} & a_{24} & \ddots & \\
a_{31} & a_{32} & a_{33} & \ddots & & \\
a_{41} & a_{42} & \ddots & & & \\
a_{51} & \ddots & & & & \\
\vdots & & & & &
\end{array}
\qquad
\begin{array}{cccccc}
1 & 3 & 6 & 10 & 15 & \cdots \\
2 & 5 & 9 & 14 & \ddots & \\
4 & 8 & 13 & \ddots & & \\
7 & 12 & \ddots & & & \\
11 & \ddots & & & & \\
\vdots & & & & &
\end{array}$$

Since each f_n is surjective, each element of $\bigcup_{n=1}^{\infty} A_n$ appears somewhere in the left array. We define a function $g : \bigcup_{n=1}^{\infty} A_n \to \mathbf{N}$ by working through the grid along the diagonals (first a_{11}, then a_{21}, then a_{12}, then a_{31}, and so on), mapping an element a_{mn} to the natural number appearing in the corresponding position in the right array. The A_n's may have elements in common; if we encounter an element a_{mn} that we have already seen before, we simply skip this element and move on to the next one. In this way, we obtain an injective function g. If we denote the range of g by $B \subseteq \mathbf{N}$, then $g : \bigcup_{n=1}^{\infty} A_n \to B$ is a bijection. Since the infinite set A_1 is contained in the union $\bigcup_{n=1}^{\infty} A_n$ and g is injective, it must be the case that B is infinite; Exercise 1.5.1 then implies that B is countable, i.e., there is a bijection $h : \mathbf{N} \to B$. It follows that the function $g^{-1} \circ h : \mathbf{N} \to \bigcup_{n=1}^{\infty} A_n$ is a bijection and we may conclude that $\bigcup_{n=1}^{\infty} A_n$ is countable.

Exercise 1.5.4. (a) Show $(a, b) \sim \mathbf{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbf{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Solution. (a) Let $f : (-1, 1) \to \mathbf{R}$ be the bijection given by $f(x) = \frac{x}{x^2-1}$ (see Example 1.5.4) and let $g : (a, b) \to (-1, 1)$ be given by $g(x) = \frac{2(x-a)}{b-a} - 1$; it is straightforward to verify that g is a bijection. Thus $(a, b) \sim (-1, 1) \sim \mathbf{R}$ and it follows that $(a, b) \sim \mathbf{R}$ (Exercise 1.5.5).

(b) Let $f : (a, \infty) \to (0, 1)$ be the bijection given by $f(x) = \frac{1}{x+1-a}$. Thus $(a, \infty) \sim (0, 1)$ and, by part (a), $(0, 1) \sim \mathbf{R}$; it follows that $(a, \infty) \sim \mathbf{R}$.

(c) It is clear that $[0, 1) \sim (0, 1]$ via the map $x \mapsto 1 - x$. Define a function $f : (0, 1) \to (0, 1]$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n+1} \text{ for some } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

This function is a bijection since it has an inverse $f^{-1} : (0, 1] \to (0, 1)$ given by

$$f^{-1}(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

It follows that $(0, 1) \sim (0, 1]$ and hence that $(0, 1) \sim [0, 1)$.

Exercise 1.5.5. (a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

Solution. (a) The identity function $f : A \to A$ given by $f(x) = x$ is a bijection.

(b) Since $A \sim B$, there is a bijection $f : A \to B$. A function is bijective if and only if it has an inverse function $f^{-1} : B \to A$, which must also be bijective.

(c) There are bijections $f : A \to B$ and $g : B \to C$. It follows that the composition $g \circ f : A \to C$ is also a bijection.

Exercise 1.5.6. (a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Solution. (a) Take $A_n = (n, n+1)$ for $n \in \mathbf{N}$.

(b) No such collection exists. To see this, suppose there was such a collection $\{I_a : a \in A\}$ for some uncountable set A. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exists a rational number $r_a \in I_a$ for each $a \in A$. Since the intervals are disjoint, each r_a must be distinct and hence the collection $\{r_a : a \in A\}$ must be an uncountable subset of $\mathbf{Q}$—but this contradicts Exercise 1.5.1.

Exercise 1.5.7. Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to produce a 1-1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999 . . . .)

The Schröder-Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

Solution. (a) Take $f : (0, 1) \to S$ given by $f(x) = \left(x, \frac{1}{2}\right)$.

(b) For $(x, y) \in S$, suppose x has decimal representation $0.x_1x_2x_3\ldots$ and y has decimal representation $0.y_1y_2y_3\ldots$, where if necessary we choose the decimal representation terminating in 0's. To define $g : S \to (0, 1)$, let $g(x, y) = 0.x_1y_1x_2y_2x_3y_3\ldots$

For the injectivity of g, suppose we have $(x, y) \neq (a, b)$ in S, so that at least one of $x \neq a$ or $y \neq b$ holds. Assuming $x \neq a$ (the case where $y \neq b$ is handled similarly), let $0.x_1x_2x_3\ldots$

be the decimal representation of x and let $0.a_1a_2a_3\ldots$ be the decimal representation of a. Since $x \neq a$, there must be some index n such that $x_n \neq a_n$. If $g(x,y)$ has decimal representation $0.s_1s_2s_3\ldots$ and $g(a,b)$ has decimal representation $0.t_1t_2t_3\ldots$, then

$$s_{2n-1} = x_n \neq a_n = t_{2n-1}.$$

This implies that $g(x,y) \neq g(a,b)$, provided it is not the case that $g(x,y)$ terminates in 0's and $g(a,b)$ terminates in 9's, or vice versa. To rule this out, note that $g(a,b)$ terminates in 9's only if both a and b terminate in 9's—but our construction specifically chooses the decimal representations for a and b terminating in 0's if necessary. The case where $g(x,y)$ terminates in 9's is handled similarly.

This function g is not surjective since 0.1 does not belong to the range of g. Indeed,

$$g(x,y) = 0.x_1y_1x_2y_2\ldots = 0.1000\ldots$$

implies that $y = 0$, but $(x,0) \notin S$ for any $x \in (0,1)$.

Exercise 1.5.8. Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

Solution. Suppose $a \in (0,1]$; we claim that $B \cap (a,2]$ must be a (possibly empty) finite set. By the Archimedean Property (Theorem 1.4.2), there is an $n \in \mathbf{N}$ such that $na > 2$. If $B \cap (a,2]$ contains at least n elements, say $\{b_1,\ldots,b_n\}$, then since each $b_i > a$ we have

$$b_1 + \cdots + b_n > na > 2.$$

This contradicts our hypotheses, so it must be the case that $B \cap (a,2]$ contains less than n elements and our claim follows.

Any element of B must be less than or equal to 2, so $B \subseteq (0,2]$ and it follows that

$$B = \bigcup_{n=1}^{\infty} \left(B \cap \left(\tfrac{1}{n}, 2\right]\right).$$

This expresses B as a countable union of finite sets and thus B is either finite or countable (Theorem 1.5.8).

Exercise 1.5.9. A real number $x \in \mathbf{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbf{Z}$, not all zero, such that

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3}+\sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbf{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution. (a) $\sqrt{2}$ is a root of the polynomial x^2-2, $\sqrt[3]{2}$ is a root of the polynomial x^3-2, and $\sqrt{3}+\sqrt{2}$ is a root of the polynomial x^4-10x^2+1.

(b) We will use the following useful corollary of Theorem 1.5.8 (ii).

Lemma L.5. If $A_1, \ldots, A_n$ are countable sets, then $A_1 \times \cdots \times A_n$ is also countable.

Proof. Suppose that A and B are countable sets, so that $B = \{b_1, b_2, b_3, \ldots\}$. For each $n \in \mathbf{N}$, it is clear that the set $A \times \{b_n\}$ is countable. Now observe that

$$A \times B = \bigcup_{n=1}^{\infty} (A \times \{b_n\}).$$

It follows from Theorem 1.5.8 (ii) that $A \times B$ is countable. A straightforward induction argument proves the general case. □

Let P_n be the collection of polynomials with integer coefficients that have degree n, i.e. $P_n = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 : a_n, \ldots, a_0 \in \mathbf{Z}, a_n \neq 0\}$. Notice that

$$P_n \sim (\mathbf{Z} \setminus \{0\}) \times \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{n \text{ times}}$$

via the map

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \mapsto (a_n, a_{n-1}, \ldots, a_1, a_0).$$

It then follows from Lemma L.5 that P_n is countable. For a polynomial $p \in P_n$, let R_p be the set of its roots, i.e., $R_p = \{x \in \mathbf{R} : p(x) = 0\}$, and note that R_p is always a finite set. Now observe that

$$A_n = \bigcup_{p \in P_n} R_p,$$

demonstrating that A_n is a countable union of finite sets; it follows from Theorem 1.5.8 that A_n is either finite or countable. Since $\sqrt[n]{k} \in A_n$ for each $k \in \mathbf{N}$ (it is a root of the polynomial $x^n - k$), we see that A_n must be infinite and hence countable.

(c) If we let A be the set of all algebraic numbers then $A = \bigcup_{n=1}^{\infty} A_n$, i.e., A is a countable union of countable sets. It follows from Theorem 1.5.8 (ii) that A is countable.

A consequence of this is that the set of transcendental numbers A^{c} must be uncountable. To see this, note that $\mathbf{R} = A \cup A^{\mathsf{c}}$, the union of two countable sets is countable, and $\mathbf{R}$ is not countable.

Exercise 1.5.10. (a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(b) Now let A be the set of all $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable, and let $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

Solution. (a) If we suppose that for each $a \in (0, 1)$ the set $C \cap [a, 1]$ is countable, then we can express C as a countable union of countable sets:

$$C = \bigcup_{n=2}^{\infty} \left(C \cap \left[\frac{1}{n}, 1 \right] \right).$$

This implies that C is countable (Theorem 1.5.8 (ii)). Thus, given that C is uncountable, there must exist some $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(b) Not necessarily. Suppose $C = [0, 1]$. Then for all $a \in (0, 1)$, we have $C \cap [a, 1] = [a, 1]$, which is uncountable. Thus $A = (0, 1)$ and it follows that $\alpha = \sup A = 1$, but $C \cap [\alpha, 1] = \{1\}$ is not uncountable.

(c) The statement is no longer true in general. If we let $C = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ then no matter which $a \in (0,1)$ we choose, the intersection $C \cap [a,1]$ is a finite set (since there are only finitely many positive integers less than or equal to a^{-1}, there are only finitely many reciprocals of positive integers greater than or equal to a).

Exercise 1.5.11 (Schröder-Bernstein Theorem). Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$.

The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbf{N}\}$ is a similar collection in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

Solution. (a) Abusing notation slightly, we have bijections $f : A \to B$ and $g : B' \to A'$, and their inverses $f^{-1} : B \to A$ and $g^{-1} : A' \to B'$. Since $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, the functions $h : X \to Y$ and $h' : Y \to X$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g^{-1}(x) & \text{if } x \in A', \end{cases} \qquad h'(y) = \begin{cases} f^{-1}(y) & \text{if } y \in B, \\ g(y) & \text{if } y \in B' \end{cases}$$

are well-defined. It is straightforward to verify that h and h' are mutual inverses and thus $X \sim Y$.

(b) If A_1 is empty then $X = g(Y)$, i.e., g is surjective. Since g is injective by assumption, it immediately follows that $X \sim Y$ via g.

Let $P(n)$ be the statement that $\{A_1, \ldots, A_n\}$ is a pairwise disjoint collection of sets; to prove that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection, we will first use induction to prove that $P(n)$ holds for all $n \in \mathbf{N}$. The truth of $P(1)$ is clear, so suppose that $P(n)$ holds for some $n \in \mathbf{N}$. To demonstrate the truth of $P(n+1)$, we need to show that $A_k \cap A_{n+1} = \emptyset$ for all $1 \leq k \leq n$. Because $A_{n+1} = g(f(A_n)) \subseteq g(Y)$ and $A_1 = X \setminus g(Y)$, we see that $A_1 \cap A_{n+1} = \emptyset$. If $n \geq 2$, suppose that $2 \leq k \leq n$ and observe that

$$\begin{aligned} A_k \cap A_{n+1} &= g(f(A_{k-1})) \cap g(f(A_n)) \\ &= g(f(A_{k-1} \cap A_n)) && (f \text{ and } g \text{ are injective}) \\ &= g(f(\emptyset)) && (\text{induction hypothesis}) \\ &= \emptyset. \end{aligned}$$

Hence $P(n+1)$ holds; this completes the induction step and it follows that $P(n)$ holds for all $n \in \mathbf{N}$.

We can now show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of sets. Let A_m and A_n be given and suppose without loss of generality that $m < n$. By the previous paragraph the collection $\{A_1, \ldots, A_m, \ldots A_n\}$ is pairwise disjoint and thus $A_m \cap A_n = \emptyset$.

That $\{f(A_n) : n \in \mathbf{N}\}$ is a pairwise disjoint collection now follows immediately from the injectivity of f.

(c) Observe that

$$f(A) = f\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f(A_n) = B,$$

where we have used that the image of a union is the union of the images; the proof of this is similar to the proof of the special case given in Exercise 1.2.7 (d).

(d) Notice that

$$\begin{aligned} b \in B' &\iff b \notin f(A_n) \text{ for all } n \in \mathbf{N} \\ &\iff g(b) \notin g(f(A_n)) \text{ for all } n \in \mathbf{N} && (g \text{ is injective}) \\ &\iff g(b) \notin A_{n+1} \text{ for all } n \in \mathbf{N} \\ &\iff g(b) \notin A_n \text{ for all } n \geq 2. \end{aligned}$$

Notice further that $g(y) \notin X \setminus g(Y) = A_1$ for any $y \in Y$. It follows that

$$b \in B' \iff g(b) \notin A_n \text{ for all } n \in \mathbf{N} \iff g(b) \in A'. \tag{$*$}$$

Thus g maps B' into A'. To see that $g : B' \to A'$ is surjective, observe that for any $a \in A'$ we have, in particular, $a \notin A_1 = X \setminus g(Y)$, so that $a \in g(Y)$, i.e., $a = g(y)$ for some $y \in Y$. It then follows from $(*)$ that $y \in B'$.

1.6 Cantor's Theorem

Exercise 1.6.1. Show that $(0, 1)$ is uncountable if and only if $\mathbf{R}$ is uncountable. This shows that Theorem 1.6.1 is equivalent to Theorem 1.5.6.

Solution. We have $(0, 1) \sim \mathbf{R}$ by Exercise 1.5.4 (a).

Exercise 1.6.2. (a) Explain why the real number $x = .b_1b_2b_3b_4\ldots$ cannot be $f(1)$.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbf{N}$.

(c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

Solution. (a) We have decimal expansions

$$f(1) = .a_{11}a_{12}a_{13}a_{14}\ldots \qquad \text{and} \qquad x = .b_1b_2b_3b_4\ldots.$$

By construction, $b_1 \neq a_{11}$. This implies that $f(1) \neq x$, provided these decimal expansions are not two different representations of the same real number (for example, .3 and .2999...). However, since the only way this can occur is when one decimal expansion terminates in repeating 0's and the other terminates in repeating 9's, and the digits b_n are always either 2 or 3, we see that $.b_1b_2b_3b_4\ldots$ must be the unique decimal representation of a real number.

(b) Since $.b_1b_2b_3b_4\ldots$ is the unique decimal expansion of the real number x (see part (a)) and $b_n \neq a_{nn}$, we have $x \neq f(n)$ for every $n \in \mathbf{N}$.

(c) The real number x belongs to $(0, 1)$ but not to the image of f, which contradicts our assumption that f was surjective. It follows that there cannot exist a bijection between $\mathbf{N}$ and $(0, 1)$. Since $(0, 1)$ is clearly infinite, we may conclude that $(0, 1)$ is uncountable.

Exercise 1.6.3. Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbf{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.

(b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, 1/2 can also be written as .5 or as .4999.... Doesn't this cause some problems?

Solution. (a) The problem with this reasoning is that the real number

$$x = .b_1 b_2 b_3 b_4 \ldots$$

that we construct may not be rational. For example, consider the function $f : \mathbf{N} \to (0,1) \cap \mathbf{Q}$ given by

$$\begin{array}{lll} f(1) = .3, & f(6) = .000003, & \\ f(2) = .02, & f(7) = .0000003, & \\ f(3) = .003, & f(8) = .00000003, & \cdots \\ f(4) = .0003, & f(9) = .000000002, & \\ f(5) = .00002, & f(10) = .0000000003, & \end{array}$$

This results in $x = .2322322232\ldots$, which is not rational since its decimal expansion does not repeat. So while x does not belong to the image of f, this is not a problem because x does not belong to $(0,1) \cap \mathbf{Q}$ either.

(b) We addressed this issue in Exercise 1.6.2 (a).

Exercise 1.6.4. Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1,0,1,0,1,0,1,0,\ldots)$ is an element of S, as is the sequence $(1,1,1,1,1,1,\ldots)$.

Give a rigorous argument showing that S is uncountable.

Solution. Suppose we have a function $f : \mathbf{N} \to S$. For each $m \in \mathbf{N}$, let a_{mn} be the element in the n^{th} position of $f(m)$, so that

$$f(m) = (a_{m1}, a_{m2}, a_{m3}, a_{m4}, \ldots) \in S.$$

Let $b = (b_1, b_2, b_3, b_4, \ldots)$ be the sequence given by

$$b_n = \begin{cases} 0 & \text{if } a_{nn} = 1, \\ 1 & \text{if } a_{nn} = 0. \end{cases}$$

Then $b \in S$ but $b \neq f(n)$ for any $n \in \mathbf{N}$, since b differs from $f(n)$ in the n^{th} position. It follows that f is not a surjection and hence that there can be no bijection between $\mathbf{N}$ and S. It is clear that S is infinite, so we may conclude that S is uncountable.

Exercise 1.6.5. (a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution. (a) We have

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

(b) To form a subset B of A, for each element $a \in A$, we must decide whether to include a in B or not. This is a binary choice to be made for each of the n elements of A; it follows that there are 2^n subsets of A.

Exercise 1.6.6. (a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1-1 mappings from A into $P(A)$.

(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1-1 map $g : C \to P(C)$.

(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are *onto*.

Solution. (a) Here are two injections $f : A \to P(A)$ and $g : A \to P(A)$:

$$\begin{aligned} f(a) &= \{a\}, & g(a) &= \{a, b\}, \\ f(b) &= \{b\}, & g(b) &= \{b, c\}, \\ f(c) &= \{c\}, & g(c) &= \{a, c\}. \end{aligned}$$

(b) Let g be given by

$$\begin{aligned} g(1) &= \{1\}, & g(3) &= \{3\}, \\ g(2) &= \{2\}, & g(4) &= \{4\}. \end{aligned}$$

(c) The power set of a finite set A always contains strictly more elements than A (Exercise 1.6.5 (b)). For finite sets, it is impossible to construct a surjective function from a set A to a set B if B contains strictly more elements than A.

Exercise 1.6.7. Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule. In each case, note that B is not in the range of the function used.

Solution. For all three functions from Exercise 1.6.6 we have $B = \emptyset$, which does not belong to the range of any of the functions.

Exercise 1.6.8. (a) First, show that the case $a' \in B$ leads to a contradiction.

(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

Solution. (a) and (b). We have $a' \in B$ if and only if $a' \notin f(a') = B$, which is clearly a contradiction since a' either does or does not belong to B.

Exercise 1.6.9. Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $P(\mathbf{N}) \sim \mathbf{R}$.

Solution. First, let us show that $P(\mathbf{N}) \sim S$, where S is the set of all binary sequences defined in Exercise 1.6.4. Consider the function $f : P(\mathbf{N}) \to S$ given by $f(E) = (a_1, a_2, a_3, \dots)$ where

$$a_n = \begin{cases} 1 & \text{if } n \in E, \\ 0 & \text{if } n \notin E. \end{cases}$$

This function is a bijection since it has an inverse $f^{-1} : S \to P(\mathbf{N})$ given by

$$f^{-1}(a_1, a_2, a_3, \dots) = \{n \in \mathbf{N} : a_n = 1\}.$$

Now let us show that $S \sim (0, 1)$. Consider the function $g : S \to (0, 1)$ given by

$$g(a_1, a_2, a_3, \dots) = 0.5a_1a_2a_3\dots,$$

where $0.5a_1a_2a_3\dots$ is a decimal expansion (for example, $g(1, 0, 1, 0, 0, 0, \dots) = 0.5101$). This function is injective since if $a = (a_1, a_2, a_3, \dots) \neq b = (b_1, b_2, b_3, \dots)$, there must exist some

$n \in \mathbf{N}$ such that $a_n \neq b_n$. It follows that $g(a) \neq g(b)$, provided $g(a) = 0.5a_1a_2a_3\ldots$ and $g(b) = 0.5b_1b_2b_3\ldots$ are not two different decimal expansions of the same real number. This cannot be the case since each a_i and b_i is either 0 or 1, and never 9.

Now consider the function $h : (0,1) \to S$ given by

$$h(x) = h(0.a_1a_2a_3\ldots) = (a_1, a_2, a_3, \ldots),$$

where $0.a_1a_2a_3\ldots$ is the **binary** expansion of $x \in (0,1)$, choosing that expansion which terminates in 0's if x has two different binary expansions. This function is injective since if $x = 0.a_1a_2a_3\ldots \neq y = 0.b_1b_2b_3\ldots$, then there must be some $n \in \mathbf{N}$ such that $a_n \neq b_n$. It follows that $h(x) \neq h(y)$.

The Schröder-Bernstein Theorem (Exercise 1.5.11) now implies that $S \sim (0,1)$. We showed in Exercise 1.5.4 that $(0,1) \sim \mathbf{R}$ and thus

$$P(\mathbf{N}) \sim S \sim (0,1) \sim \mathbf{R}.$$

In Exercise 1.5.5 we showed that $\sim$ is an equivalence relation, so the chain of equivalences above allows us to conclude that $P(\mathbf{N}) \sim \mathbf{R}$.

Exercise 1.6.10. As a final exercise, answer each of the following by establishing a 1-1 correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0,1\}$ to $\mathbf{N}$ countable or uncountable?

(b) Is the set of all functions from $\mathbf{N}$ to $\{0,1\}$ countable of uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an *antichain* if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbf{N})$ contain an uncountable antichain?

Solution. (a) Let $\mathbf{N}^{\{0,1\}}$ be the set of all functions from $\{0,1\}$ to $\mathbf{N}$. Consider the function $F : \mathbf{N}^{\{0,1\}} \to \mathbf{N} \times \mathbf{N}$ given by $F(f) = (f(0), f(1))$. This function is a bijection since it has an inverse $F^{-1} : \mathbf{N} \times \mathbf{N} \to \mathbf{N}^{\{0,1\}}$ given by $F^{-1}(a,b) = f$, where $f : \{0,1\} \to \mathbf{N}$ is the function satisfying $f(0) = a, f(1) = b$. Thus

$$\mathbf{N}^{\{0,1\}} \sim \mathbf{N} \times \mathbf{N} \sim \mathbf{N},$$

where we have used Lemma L.5 for the second equivalence. We may conclude that $\mathbf{N}^{\{0,1\}}$ is countable.

(b) The set of all functions from $\mathbf{N}$ to $\{0,1\}$ is nothing but the set of all binary sequences S defined in Exercise 1.6.4, since a function $f : \mathbf{N} \to \{0,1\}$ can be identified with the sequence $(f(0), f(1), f(2), \ldots)$. Thus the set of all functions from $\mathbf{N}$ to $\{0,1\}$ is uncountable, since we showed that S is uncountable in Exercise 1.6.4.

(c) Consider the following collection of subsets of $P(\mathbf{Q})$:

$$\mathcal{A} := \{(a, a+1) \cap \mathbf{Q} : a \in \mathbf{R}\}.$$

For real numbers $a < b$, it follows from the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3) that there exist rational numbers p and q such that $a < p < b$ and $a < q < a+1$. Let $r = \min\{p, q\}$ and notice that $a < r < b$ and $a < r < a+1$. It follows that $r \in (a, a+1)$ and $r \notin (b, b+1)$, whence

$$(a, a+1) \cap \mathbf{Q} \not\subseteq (b, b+1) \cap \mathbf{Q}.$$

A similar argument shows that this non-inclusion still holds if $b < a$ and so it follows that for any real numbers $a \neq b$ we have

$$(a, a+1) \cap \mathbf{Q} \not\subseteq (b, b+1) \cap \mathbf{Q},$$

i.e., $\mathcal{A}$ is an antichain.

Another consequence of the previous paragraph is that if a and b are distinct real numbers, then

$$(a, a+1) \cap \mathbf{Q} \neq (b, b+1) \cap \mathbf{Q}.$$

It follows that the map $g : \mathbf{R} \to \mathcal{A}$ defined by $g(a) = (a, a+1) \cap \mathbf{Q}$ is injective. Since g is evidently surjective, we have that $\mathbf{R} \sim \mathcal{A}$.

To finish the exercise, we will need the following two lemmas.

Lemma L.6. Suppose A and B are sets and $f : A \to B$ is a bijection. Define $F : P(A) \to P(B)$

$$F(X) = f(X) = \{f(x) : x \in X\}.$$

Then F is a bijection.

Proof. Suppose $X, Y \in P(A)$ are such that $X \neq Y$. Without loss of generality suppose that $X \not\subseteq Y$, so that there is some $x \in X$ such that $x \notin Y$. The injectivity of f then implies that $f(x) \notin f(Y)$, whence $F(X) \neq F(Y)$. Thus F is injective.

Now let $Y \in P(B)$ be given. For each $y \in Y$, the surjectivity of f implies that there is some $x \in A$ such that $f(x) = y$; let X be the collection of these x. It follows that $F(X) = Y$ and hence that F is surjective. $\square$

Lemma L.7. Suppose A and B are sets and $f : A \to B$ is injective. Then if $\mathcal{A} \subseteq P(A)$ is an antichain, so is $\mathcal{A}' := \{f(X) : X \in \mathcal{A}\} \subseteq P(B)$.

Proof. Suppose we have two elements $f(X)$ and $f(Y)$ in $\mathcal{A}'$, where X and Y belong to $\mathcal{A}$. Since $\mathcal{A}$ is an antichain, we have $X \nsubseteq Y$, which can be the case if and only if there is some $x \in X$ such that $x \notin Y$. The injectivity of f then implies that $f(x) \in f(X)$ but $f(x) \notin f(Y)$. It follows that $f(X)$ is not a subset of $f(Y)$ and we may conclude that $\mathcal{A}'$ is an antichain. $\square$

Returning to the exercise, let $f : \mathbf{Q} \to \mathbf{N}$ be a bijection (such a function exists by Theorem 1.5.6 (i)). By Lemma L.6, the function $F : P(\mathbf{Q}) \to P(\mathbf{N})$ defined by $F(X) = f(X)$ is also a bijection, which restricts to a bijection $F : \mathcal{A} \to F(\mathcal{A})$. Thus $F(\mathcal{A}) \sim \mathcal{A} \sim \mathbf{R}$, so that $F(\mathcal{A})$ is uncountable. We may now use Lemma L.7 to conclude that $F(\mathcal{A}) \subseteq P(\mathbf{N})$ is an uncountable antichain.

Chapter 2

Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.1. What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution. First observe that the statement

$$\text{for all } N \in \mathbf{N},\, n \geq N \implies |x_n - x| < \epsilon$$

is equivalent to

$$\text{for all } n \in \mathbf{N},\, |x_n - x| < \epsilon.$$

So a sequence verconges to x if there exists an $\epsilon > 0$ such that $|x_n - x| < \epsilon$, or equivalently such that $x_n \in (x - \epsilon, x + \epsilon)$, for all $n \in \mathbf{N}$.

For an example of a vercongent sequence that diverges, consider $(x_n) = (1, 0, 1, 0, \ldots)$. This sequence verconges to $\frac{1}{2}$ since $\left|x_n - \frac{1}{2}\right| = \frac{1}{2} < 1$ for all $n \in \mathbf{N}$. To see that this sequence diverges, suppose there was some $x \in \mathbf{R}$ such that $\lim x_n = x$. Then there must exist some $N \in \mathbf{N}$ such that $n \geq N$ implies that $|x_n - x| < \frac{1}{2}$. Observe that

$$1 = |x_N - x_{N+1}| \leq |x_N - x| + |x_{N+1} - x| < \frac{1}{2} + \frac{1}{2} = 1,$$

i.e., $1 < 1$, which is a contradiction.

A sequence can verconge to two different values. The sequence $(x_n) = (1, 1, 1, 1, \ldots)$ verconges to 1:

$$|x_n - 1| = 0 < 1 \text{ for all } n \in \mathbf{N},$$

and also to 0:

$$|x_n| = 1 < 2 \text{ for all } n \in \mathbf{N}.$$

This definition describes the bounded sequences (see Definition 2.3.1); a sequence which verconges to some $x \in \mathbf{R}$ must be bounded, and conversely any bounded sequence verconges to some $x \in \mathbf{R}$.

Exercise 2.2.2. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution. (a) Let $\epsilon > 0$ be given. Choose $N \in \mathbf{N}$ such that $N > \frac{3}{25\epsilon}$ and observe that for $n \geq N$ we have

$$\left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| = \frac{3}{25n+20} < \frac{3}{25n} \leq \frac{3}{25N} < \epsilon.$$

It follows that $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) Let $\epsilon > 0$ be given. Choose $N \in \mathbf{N}$ such that $N > \frac{2}{\epsilon}$ and observe that for $n \geq N$ we have

$$\left| \frac{2n^2}{n^3+3} \right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

It follows that $\lim \frac{2n^2}{n^3+3} = 0$.

(c) Let $\epsilon > 0$ be given. Choose $N \in \mathbf{N}$ such that $N > \frac{1}{\epsilon^3}$ and observe that for $n \geq N$ we have

$$\left| \frac{\sin(n^2)}{\sqrt[3]{n}} \right| = \frac{|\sin(n^2)|}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \epsilon.$$

It follows that $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Exercise 2.2.3. Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution. (a) We would have to find a college in the United States where every student is less than seven feet tall.

(b) We would have to find a college in the United States where each professor gives at least one student a grade of C or worse.

(c) We would have to show that every college in the United States has a student who is less than six feet tall.

Exercise 2.2.4. Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution. (a) Consider $(x_n) = (1, 0, 1, 0, \ldots)$. This sequence has an infinite number of ones but, as shown in Exercise 2.2.1, diverges.

(b) This is impossible. Suppose (x_n) is such a sequence with $\lim x_n = x \neq 1$. Then there must exist some $N \in \mathbf{N}$ such that for all $n \geq N$ we have $|x_n - x| < |1 - x|$. Since this sequence contains infinitely many ones, it must be the case that there is some $m \geq N$ such that $x_m = 1$. This implies that $|x_m - x| = |1 - x| < |1 - x|$, which is a contradiction.

(c) Consider the sequence

$$(x_n) = (1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \ldots).$$

For each $n \in \mathbf{N}$ we can find n consecutive ones starting at the m^{th} position and, for $n \geq 2$, we can find a zero at the $(m-1)^{\text{th}}$ position, where $m = \frac{n(n+1)}{2}$. Furthermore, the sequence is divergent. To see this, suppose there was some $x \in \mathbf{R}$ such that $\lim x_n = x$. It follows that there is an $N \in \mathbf{N}$ such that $n \geq N$ implies that $|x_n - x| < \frac{1}{2}$. Since the sequence contains infinitely many ones and zeros, we can find indices $k, \ell \geq N$ such that $x_k = 1$ and $x_\ell = 0$. Then

$$1 = |x_k - x_\ell| \leq |x_k - x| + |x_\ell - x| < \frac{1}{2} + \frac{1}{2} = 1,$$

i.e., $1 < 1$, which is a contradiction.

Exercise 2.2.5. Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$,

(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution. (a) We claim that $\lim a_n = 0$. Let $\epsilon > 0$ be given and observe that if $n \geq 6$, then

$$0 < \frac{5}{n} < 1 \implies \left[\left[\frac{5}{n}\right]\right] = 0.$$

So if we take $N = 6$, then $n \geq N$ implies that $\left|\left[\left[\frac{5}{n}\right]\right]\right| = 0 < \epsilon$.

(b) We claim that $\lim a_n = 1$. Let $\epsilon > 0$ be given and observe that if $n \geq 7$, then

$$\frac{1}{n} < \frac{1}{6} \iff \frac{4}{n} < \frac{2}{3} \iff \frac{4}{n} + \frac{1}{3} < 1.$$

Hence for $n \geq 7$ we have

$$0 < \frac{4}{n} + \frac{1}{3} < 1 \implies \left[\left[\frac{4}{n} + \frac{1}{3}\right]\right] = 0.$$

So if we take $N = 7$, then $n \geq N$ implies that

$$\left[\left[\frac{12 + 4n}{3n} - 1\right]\right] = \left[\left[\frac{4}{n} + \frac{1}{3}\right]\right] = 0 < \epsilon.$$

These examples demonstrate that taking smaller ϵ-neighbourhoods may not require us to take larger values of N; the same value of N in each example works for every ϵ-neighbourhood that we choose.

Exercise 2.2.6. Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and $(a_n) \to b$. Now argue $a = b$.

Solution. Let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |a_n - a| < \frac{\epsilon}{2} \qquad \text{and} \qquad n \geq N_2 \implies |a_n - b| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$|a - b| = |a - a_n + a_n - b| \leq |a_n - a| + |a_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So we have shown that $|a - b| < \epsilon$ for any $\epsilon > 0$; it follows from Theorem 1.2.6 that $a = b$.

Exercise 2.2.7. Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution. (a) The sequence $(-1)^n$ is frequently but not eventually in the set $\{1\}$. To see this, let $N \in \mathbf{N}$ be given. If N is even, then $(-1)^N \in \{1\}$ and $(-1)^{N+1} \notin \{1\}$, and if N is odd then $(-1)^N \notin \{1\}$ and $(-1)^{N+1} \in \{1\}$. In any case, we can always find indices $m, n \geq N$ such that $(-1)^m \notin \{1\}$ (this shows that the sequence is not eventually in $\{1\}$) and such that $(-1)^n \in \{1\}$ (this shows that the sequence is frequently in $\{1\}$).

(b) Eventually is the stronger definition. Frequently does not imply eventually, as part (a) shows, but eventually does imply frequently. To see this, suppose that (a_n) is eventually in a set A, i.e., there is an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$. Let $M \in \mathbf{N}$ be given. Set $n = \max\{M, N\}$ and observe that $n \geq M$ and $a_n \in A$. Hence (a_n) is frequently in A.

(c) The term we want is eventually. Here is a rephrasing of Definition 2.2.3B: a sequence (a_n) converges to a if, given any $\epsilon > 0$, the sequence (a_n) is eventually in the ϵ-neighbourhood $V_\epsilon(a)$ of a.

(d) Such a sequence is not necessarily eventually in $(1.9, 2.1)$; consider the sequence $(x_n) = (2, 0, 2, 0, 2, \ldots)$ for example. For any $N \in \mathbf{N}$, we can always find an index $n \geq N$ (either $n = N$ or $n = N + 1$) such that $x_n = 0 \notin (1.9, 2.1)$. However, such a sequence must be frequently in $(1.9, 2.1)$. To see this, let $N \in \mathbf{N}$ be given. Then there must exist an index $n \geq N$ such that $x_n = 2 \in (1.9, 2.1)$ (otherwise there would be only finitely many twos in the sequence).

Exercise 2.2.8. For some additional practice with nested quantifiers, consider the following invented defintion:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero-heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counterexample.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if ....

Solution. (a) This sequence is zero-heavy; $M = 1$ works. Indeed, let $N \in \mathbf{N}$ be given. If N is odd then let $n = N$ and if N is even then let $n = N + 1$. In either case, we have $N \leq n \leq N + 1$ and $x_n = 0$.

(b) A zero-heavy sequence must contain an infinite number of zeros. To see this, suppose (x_n) is a sequence with a finite number of zeros, i.e. there is an $N \in \mathbf{N}$ such that $x_n \neq 0$ for all $n \geq N$. Then no matter which M we choose, we will never be able to find $n \in \mathbf{N}$ with $N \leq n \leq N + M$ and $x_n = 0$. Thus the sequence (x_n) is not zero-heavy.

(c) A sequence with an infinite number of zeros is not necessarily zero-heavy. For a counterexample, consider the sequence

$$(x_n) = (1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \ldots).$$

This sequence contains infinitely many zeros, but is not zero-heavy. To see this, let $M \in \mathbf{N}$ be given. It is always possible to find M consecutive ones in the sequence (x_n) (see Exercise 2.2.4 (c)); suppose this string of ones starts at $x_N = 1$. Then for each $n \in \mathbf{N}$ satisfying $N \leq n \leq N + M$, we have $x_n = 1 \neq 0$.

(d) A sequence is not zero-heavy if for every $M \in \mathbf{N}$ there exists an $N \in \mathbf{N}$ such that $x_n \neq 0$ for each $n \in \mathbf{N}$ satisfying $N \leq n \leq N + M$.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Solution. (a) Let $\epsilon > 0$ be given. Since $x_n \to 0$, there exists an $N \in \mathbf{N}$ such that

$$n \geq N \implies |x_n| = x_n < \epsilon^2 \iff \sqrt{x_n} < \epsilon.$$

It follows that $\lim(\sqrt{x_n}) = 0$.

(b) By Theorem 2.3.4, we must have $x \geq 0$. The case $x = 0$ was handled in part (a), so suppose that $x > 0$, which gives $\sqrt{x} > 0$. For each $n \in \mathbf{N}$, observe that

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \frac{\left|\sqrt{x_n} - \sqrt{x}\right|\left(\sqrt{x_n} + \sqrt{x}\right)}{\sqrt{x_n} + \sqrt{x}} = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}.$$

Let $\epsilon > 0$ be given. Since $x_n \to x$, there exists an $N \in \mathbf{N}$ such that $|x_n - x| < \epsilon\sqrt{x}$ whenever $n \geq N$. For $n \geq N$, it follows that

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leq \frac{|x_n - x|}{\sqrt{x}} < \epsilon.$$

Thus $\lim(\sqrt{x_n}) = \sqrt{x}$.

Exercise 2.3.2. Using only Definition 2.2.3, prove that if $(x_n) \to 2$ then

(a) $\left(\frac{2x_n-1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

Solution. (a) Let $\epsilon > 0$ be given. Since $x_n \to 2$, there exists an $N \in \mathbf{N}$ such that $n \geq N$ implies that $|x_n - 2| < \frac{3\epsilon}{2}$. For $n \geq N$ we then have

$$\left|\frac{2x_n-1}{3} - 1\right| = \left|\frac{2x_n-4}{3}\right| = \frac{2}{3}|x_n - 2| < \epsilon.$$

It follows that $\left(\frac{2x_n-1}{3}\right) \to 1$.

(b) Since $x_n \to 2$, there is an $N_1 \in \mathbf{N}$ such that $n \geq N_1 \implies |x_n - 2| < 1$. For $n \geq N_1$ we then have

$$2 \leq |x_n - 2| + |x_n| < 1 + |x_n| \implies 1 < |x_n| \implies \frac{1}{|x_n|} < 1.$$

Let $\epsilon > 0$ be given. Since $x_n \to 2$, there is an $N_2 \in \mathbf{N}$ such that $|x_n - 2| < 2\epsilon$ whenever $n \geq N_2$. Set $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{2 - x_n}{2x_n}\right| = \frac{|x_n - 2|}{2|x_n|} < \frac{|x_n - 2|}{2} < \epsilon.$$

It follows that $\frac{1}{x_n} \to \frac{1}{2}$.

Exercise 2.3.3 (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution. Let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |x_n - l| < \epsilon \iff -\epsilon < x_n - l < \epsilon,$$

$$n \geq N_2 \implies |z_n - l| < \epsilon \iff -\epsilon < z_n - l < \epsilon.$$

Let $N = \max\{N_1, N_2\}$. Then since $x_n - l \leq y_n - l \leq z_n - l$ for all $n \in \mathbf{N}$, for $n \geq N$ we have

$$-\epsilon < y_n - l < \epsilon \iff |y_n - l| < \epsilon.$$

It follows that $\lim y_n = l$.

Exercise 2.3.4. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$.

Solution. The manipulations of limits in these solutions are justified by the Algebraic Limit Theorem (Theorem 2.3.3).

(a) We have

$$\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) = \frac{1+2\lim a_n}{1+3\lim a_n - 4(\lim a_n)^2} = \frac{1}{1} = 1.$$

(b) We have

$$\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) = \lim\left(\frac{a_n^2+4a_n}{a_n}\right) = \lim(a_n+4) = \lim a_n + 4 = 4.$$

(c) We have

$$\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) = \lim\left(\frac{2+3a_n}{1+5a_n}\right) = \frac{2+3\lim a_n}{1+5\lim a_n} = \frac{2}{1} = 2.$$

Exercise 2.3.5. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution. (z_n) is the sequence given by

$$z_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd,} \\ y_{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$$

Suppose that (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n = L$ for some $L \in \mathbf{R}$ and let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |x_n - L| < \epsilon \qquad \text{and} \qquad n \geq N_2 \implies |y_n - L| < \epsilon.$$

Let $N = \max\{N_1, N_2\}$ and suppose $n \in \mathbf{N}$ is such that $n \geq 2N$. If n is odd then $\frac{n+1}{2} \in \mathbf{N}$ and

$$n \geq 2N > 2N - 1 \implies \frac{n+1}{2} > N \geq N_1 \implies \left|x_{\frac{n+1}{2}} - L\right| < \epsilon.$$

Hence

$$|z_n - L| = \left|x_{\frac{n+1}{2}} - L\right| < \epsilon.$$

If n is even then $\frac{n}{2} \in \mathbf{N}$ and

$$n \geq 2N \implies \frac{n}{2} \geq N \geq N_2 \implies \left|y_{\frac{n}{2}} - L\right| < \epsilon.$$

Hence

$$|z_n - L| = \left|y_{\frac{n}{2}} - L\right| < \epsilon.$$

In either case we have $|z_n - L| < \epsilon$, i.e.,

$$n \geq 2N \implies |z_n - L| < \epsilon.$$

It follows that $\lim z_n = L$.

Now suppose that (z_n) is convergent with $\lim z_n = L$ for some $L \in \mathbf{R}$. Let $\epsilon > 0$ be given. Since $z_n \to L$, there exists an $N \in \mathbf{N}$ such that $|z_n - L| < \epsilon$ whenever $n \geq N$. For such n, we have $2n > 2n - 1 \geq n \geq N$ and so

$$|x_n - L| = |z_{2n-1} - L| < \epsilon \qquad \text{and} \qquad |y_n - L| = |z_{2n} - L| < \epsilon.$$

It follows that $\lim x_n = \lim y_n = L$.

Exercise 2.3.6. Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Solution. Observe that

$$b_n = n - \sqrt{n^2 + 2n} = \frac{(n - \sqrt{n^2 + 2n})(n + \sqrt{n^2 + 2n})}{n + \sqrt{n^2 + 2n}} = \frac{-2n}{n + \sqrt{n^2 + 2n}} = \frac{-2}{1 + \sqrt{1 + \frac{2}{n}}}.$$

Hence, using Exercise 2.3.1,

$$\lim b_n = \lim \left(\frac{-2}{1 + \sqrt{1 + \frac{2}{n}}} \right) = \frac{-2}{1 + \sqrt{1 + 2 \lim \frac{1}{n}}} = \frac{-2}{1 + \sqrt{1}} = -1.$$

Exercise 2.3.7. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Solution. (a) Take $x_n = n$ and $y_n = -n$.

(b) This is impossible. If (x_n) and $(x_n + y_n)$ both converge, then by the Algebraic Limit Theorem (Theorem 2.3.3) (y_n) must be convergent with limit $\lim y_n = \lim(x_n + y_n) - \lim x_n$.

(c) Take $b_n = \frac{1}{n}$.

(d) This is impossible; $(a_n - b_n)$ must be unbounded. Since (b_n) is convergent, it must be bounded (Theorem 2.3.2), i.e., there exists some $B \geq 0$ such that $|b_n| \leq B$ for all $n \in \mathbf{N}$. Let $M \geq 0$ be given. Since (a_n) is unbounded, there exists some $N \in \mathbf{N}$ such that $|a_N| \geq M + B$. Then observe that

$$|a_N - b_N| \geq ||a_N| - |b_N|| \geq |a_N| - |b_N| \geq M + B - B = M,$$

where we have used Exercise 1.2.6 (d) for the first inequality. Since M was arbitrary, we see that the sequence $(a_n - b_n)$ is unbounded.

(e) Take $a_n = \frac{1}{n^2}$ and $b_n = n$.

Exercise 2.3.8. Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Solution. (a) Suppose $p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$. The Algebraic Limit Theorem (Theorem 2.3.3) and some simple induction arguments allow us to make the following manipulations:

$$\lim(p(x_n)) = \lim(a_m x_n^m + a_{m-1} x_n^{m-1} + \cdots + a_1 x_n + a_0)$$

$$= a_m(\lim x_n)^m + a_{m-1}(\lim x_n)^{m-1} + \cdots + a_1 \lim x_n + a_0$$
$$= a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$
$$= p(x).$$

(b) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and the convergent sequence $x_n = \frac{1}{n} \to 0$. We then have $(f(x_n)) = (1, 1, 1, \ldots)$, which converges to $1 \neq 0 = f(0)$.

Exercise 2.3.9. (a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution. (a) There is an $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$. Let $\epsilon > 0$ be given. Since $b_n \to 0$, there is an $N \in \mathbf{N}$ such that

$$n \geq N \implies |b_n| < \frac{\epsilon}{M}.$$

Observe that for $n \geq N$ we have

$$|a_n b_n| = |a_n||b_n| \leq M|b_n| < \frac{M\epsilon}{M} = \epsilon.$$

It follows that $\lim(a_n b_n) = 0$. We may not use the Algebraic Limit Theorem here since the sequence (a_n) is not necessarily convergent; the hypotheses of that theorem require both sequences (a_n) and (b_n) to be convergent.

(b) If the sequence (a_n) converges to some a then we may use the Algebraic Limit Theorem to conclude that $\lim(a_n b_n) = ab$. If the sequence (a_n) is divergent, then $(a_n b_n)$ must also be divergent. To see this, we will prove the contrapositive, i.e., if (a_nb_n) converges to some $x \in \mathbf{R}$ then (a_n) is convergent. Indeed, since $b \neq 0$, the Algebraic Limit Theorem implies that

$$\lim a_n = \lim\left(\frac{a_nb_n}{b_n}\right) = \frac{x}{b}.$$

(c) Since (b_n) is convergent, it is bounded (Theorem 2.3.2). So we may apply part (a) (we have swapped the roles of (a_n) and (b_n)) to conclude that

$$\lim(a_nb_n) = 0 = 0b = ab.$$

Exercise 2.3.10. Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Solution. (a) This is false; consider $a_n = b_n = (-1)^n$.

(b) This is true. Let $\epsilon > 0$ be given. Since $b_n \to b$, there is an $N \in \mathbf{N}$ such that $|b_n - b| < \epsilon$ whenever $n \geq N$. For such n, the reverse triangle inequality (Exercise 1.2.6 (d)) gives

$$||b_n| - |b|| \leq |b_n - b| < \epsilon.$$

It follows that $\lim|b_n| = |b|$.

(c) This is true. Using the Algebraic Limit Theorem (Theorem 2.3.3), we have

$$\lim b_n = \lim(b_n - a_n + a_n) = \lim(b_n - a_n) + \lim a_n = 0 + a = a.$$

(d) This is true. Since $0 \leq |b_n - b| \leq a_n$ for every $n \in \mathbf{N}$, the Squeeze Theorem (Exercise 2.3.3) implies that $\lim|b_n - b| = 0$, i.e., for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n \geq N \implies ||b_n - b| - 0| = |b_n - b| < \epsilon,$$

which is exactly the statement $\lim b_n = b$.

Exercise 2.3.11 (Cesaro Means). (a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution. (a) Suppose $\lim x_n = x$ and let $\epsilon > 0$ be given. Since $x_n \to x$, there is a positive integer $N_1 \in \mathbf{N}$ such that

$$n \geq N_1 \implies |x_n - x| < \frac{\epsilon}{2}.$$

Given this N_1, notice that the sequence

$$\left(\frac{|x_1 - x| + \cdots + |x_{N_1} - x|}{n} \right)_{n=1}^{\infty}$$

has non-negative terms and converges to zero (the numerator is a constant); it follows that there is an $N_2 \in \mathbf{N}$ such that

$$n \geq N_2 \implies \frac{|x_1 - x| + \cdots + |x_{N_1} - x|}{n} < \frac{\epsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$ and observe that for $n \geq N + 1$ we have

$$\begin{aligned}
|y_n - x| &= \left| \frac{x_1 + \cdots + x_n}{n} - \frac{nx}{n} \right| \\
&= \left| \frac{(x_1 - x) + \cdots + (x_n - x)}{n} \right| \\
&\leq \frac{|x_1 - x| + \cdots + |x_{N_1} - x|}{n} + \frac{|x_{N_1+1} - x| + \cdots + |x_n - x|}{n} \\
&< \frac{\epsilon}{2} + \frac{n - N_1}{n} \cdot \frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

It follows that $\lim y_n = x$.

(b) Consider the divergent sequence $x_n = (-1)^{n+1}$. The sequence of averages (y_n) is then

$$y_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

which satisfies $\lim y_n = 0$.

Exercise 2.3.12. A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0,1)$, then a is also in the complement of $(0,1)$.

(c) If every a_n is rational, then a is rational.

Solution. (a) This is true. For any $b \in B$ we have $b \leq a_n$ for all $n \in \mathbf{N}$; the Order Limit Theorem (Theorem 2.3.4) then implies that $b \leq a$ and it follows that a is an upper bound for B.

(b) This is true. Observe that for a real number x we have

$$x \notin (0,1) \iff x \leq 0 \text{ or } x \geq 1 \iff \left|x - \frac{1}{2}\right| \geq \frac{1}{2}.$$

So for each $n \in \mathbf{N}$ we have $\left|a_n - \frac{1}{2}\right| \geq \frac{1}{2}$. The Algebraic Limit Theorem (Theorem 2.3.3) and Exercise 2.3.10 (b) imply that $\lim\left|a_n - \frac{1}{2}\right| = \left|a - \frac{1}{2}\right|$, and hence the Order Limit Theorem (Theorem 2.3.4) implies that $\left|a - \frac{1}{2}\right| \geq \frac{1}{2}$. It follows that a belongs to the complement of $(0,1)$.

(c) This is false. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), for each $n \in \mathbf{N}$ we may pick a rational number a_n satisfying $\sqrt{2} < a_n < \sqrt{2} + \frac{1}{n}$. The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim a_n = \sqrt{2}$, which is an irrational number.

Exercise 2.3.13 (Iterated Limits). Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the *iterated* limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \qquad \text{and} \qquad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right).$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2+n^2)$.

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) = b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

Solution. (a) We apply the Algebraic Limit Theorem (Theorem 2.3.3):

$$\lim_{m\to\infty} a_{mn} = \lim_{m\to\infty}\left(\frac{m}{m+n}\right) = \lim_{m\to\infty}\left(\frac{1}{1+\frac{n}{m}}\right) = \frac{1}{1+n\lim_{m\to\infty}\left(\frac{1}{m}\right)} = \frac{1}{1} = 1.$$

Hence $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn}) = \lim_{n\to\infty}(1) = 1$. Similarly,

$$\lim_{n\to\infty} a_{mn} = \lim_{n\to\infty}\left(\frac{m}{m+n}\right) = \lim_{n\to\infty}\left(\frac{\frac{m}{n}}{1+\frac{m}{n}}\right) = \frac{m\lim_{n\to\infty}\left(\frac{1}{n}\right)}{1+m\lim_{n\to\infty}\left(\frac{1}{n}\right)} = \frac{0}{1} = 0.$$

Thus $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn}) = \lim_{m\to\infty}(0) = 0$.

(b) For $a_{mn} = \frac{1}{m+n}$, we have $\lim_{m,n\to\infty} a_{mn} = 0$. To see this, let $\epsilon > 0$ be given. There is an $N \in \mathbf{N}$ such that $\frac{1}{n} < \epsilon$ whenever $n \geq N$, so that for $m, n \geq N$ we have

$$|a_{mn}| = \frac{1}{m+n} < \frac{1}{n} < \epsilon.$$

Thus $\lim_{m,n\to\infty} a_{mn} = 0$. The two iterated limits also exist and are equal to 0. Indeed, observe that for all $m, n \in \mathbf{N}$ we have $0 < \frac{1}{m+n} < \frac{1}{m}$. The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim_{m\to\infty} a_{mn} = \lim_{m\to\infty} \frac{1}{m+n} = 0$ and it follows that $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn}) = \lim_{n\to\infty}(0) = 0$. A similar argument shows that $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn}) = 0$.

Now let $a_{mn} = \frac{mn}{m^2+n^2}$; we claim that $\lim_{m,n\to\infty} a_{mn}$ does not exist. To see this, let us seek a contradiction and suppose that $\lim_{m,n\to\infty} a_{mn} = x$ for some $x \in \mathbf{R}$. There then exists an $N \in \mathbf{N}$ such that $|a_{mn} - x| < \frac{1}{20}$ whenever $m, n \geq N$. In particular, taking $n = m$,

$$m \geq N \implies \left|\frac{m^2}{m^2+m^2} - x\right| = \left|\frac{1}{2} - x\right| < \frac{1}{20} \iff x \in \left(\frac{9}{20}, \frac{11}{20}\right).$$

Similarly, taking $n = 2m$,

$$m \geq N \implies \left|\frac{2m^2}{m^2+4m^2} - x\right| = \left|\frac{2}{5} - x\right| < \frac{1}{20} \iff x \in \left(\frac{7}{20}, \frac{9}{20}\right).$$

So assuming that $\lim_{m,n\to\infty} a_{mn} = x$ for some $x \in \mathbf{R}$ leads us to the contradiction that $x < \frac{9}{20}$ and $x > \frac{9}{20}$; it follows that $\lim_{m,n\to\infty} a_{mn}$ does not exist. However, the two iterated limits do exist and are equal to 0. Using the Algebraic Limit Theorem (Theorem 2.3.3), for any $n \in \mathbf{N}$ we have

$$\lim_{m\to\infty}\left(\frac{mn}{m^2+n^2}\right) = \lim_{m\to\infty}\left(\frac{\frac{n}{m}}{1+\frac{n^2}{m^2}}\right) = \frac{n\lim_{m\to\infty}\left(\frac{1}{m}\right)}{1+n^2\lim_{m\to\infty}\left(\frac{1}{m^2}\right)} = \frac{0}{1} = 0.$$

It follows that $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn}) = 0$ and we can use a similar argument to show that $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn}) = 0$.

(c) Let $a_{mn} = (-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right)$; we claim that $\lim_{m,n\to\infty} a_{mn} = 0$. To see this, let $\epsilon > 0$ be given. There is an $N \in \mathbf{N}$ such that $\frac{1}{n} < \frac{\epsilon}{2}$ whenever $n \geq N$. For $m, n \geq N$ we then have

$$|a_{mn}| = \left|(-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right)\right| = \frac{1}{m} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\lim_{m,n\to\infty} a_{mn} = 0$. However, neither iterated limit exists. Fix $n \in \mathbf{N}$ and observe that

$$\begin{aligned}
|a_{mn} - a_{m+1,n}| &= \left|(-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right) - (-1)^{m+n+1}\left(\frac{1}{m+1} + \frac{1}{n}\right)\right| \\
&= \left|(-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n} + \frac{1}{m+1} + \frac{1}{n}\right)\right| \\
&= \left|\frac{1}{m} + \frac{1}{n} + \frac{1}{m+1} + \frac{1}{n}\right|
\end{aligned}$$

$$= \frac{1}{m} + \frac{1}{m+1} + \frac{2}{n}$$
$$\geq \frac{2}{n}.$$

Since $n \in \mathbf{N}$ is fixed, this implies that the sequence $(a_{mn} - a_{m+1,n})_{m=1}^{\infty}$ cannot converge to 0. Now observe that for any sequence (b_m), the Algebraic Limit Theorem (Theorem 2.3.3) implies that

$$\lim_{m\to\infty} b_m \text{ exists} \implies \lim_{m\to\infty}(b_m - b_{m+1}) = 0.$$

The contrapositive of this statement then implies that the limit $\lim_{m\to\infty} a_{mn}$ does not exist for any $n \in \mathbf{N}$ and it follows that the iterated limit $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn})$ does not exist. Using the symmetry of a_{mn} and swapping the roles of m and n in our argument shows that the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn})$ does not exist either.

(d) Seeking a contradiction, suppose that (b_m) does not converge to a, i.e., there is some $\epsilon > 0$ such that for all $N \in \mathbf{N}$ there is an $M \geq N$ such that $|b_M - a| \geq \epsilon$. Since $\lim_{m,n\to\infty} a_{mn} = a$, there exists some $N_1 \in \mathbf{N}$ such that

$$m, n \geq N_1 \implies |a_{mn} - a| < \frac{\epsilon}{2}. \tag{1}$$

By the previous discussion, there is an $M \geq N_1$ such that $|b_M - a| \geq \epsilon$. By assumption we have $\lim_{n\to\infty} a_{Mn} = b_M$, so there is an $N_2 \in \mathbf{N}$ such that $|a_{Mn} - b_M| < \frac{\epsilon}{2}$ whenever $n \geq N_2$. Let $N = \max\{N_1, N_2\}$ and observe that $|a_{MN} - a| < \frac{\epsilon}{2}$ by (1). However, the reverse triangle inequality (Exercise 1.2.6 (d)) gives us

$$\begin{aligned} |a_{MN} - a| &= |a_{MN} - b_M + b_M - a| \\ &\geq ||b_M - a| - |a_{MN} - b_M|| \\ &\geq |b_M - a| - |a_{MN} - b_M| \\ &> \epsilon - \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

So assuming that (b_m) does not converge to a leads us the contradiction that there exist positive integers M and N such that $|a_{MN} - a|$ is both less than and greater than $\frac{\epsilon}{2}$. We may conclude that $\lim_{m\to\infty} b_m = a$.

(e) If the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn})$ exists, then it must be the case that for each fixed $m \in \mathbf{N}$, the limit $\lim_{n\to\infty} a_{mn}$ exists. Part (d) then implies that

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = \lim_{m,n\to\infty} a_{mn}.$$

Swapping the roles of m and n and repeating the above argument shows that

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{m,n\to\infty} a_{mn}.$$

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1. (a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Solution. (a) Let $P(n)$ be the statement that $x_{n+1} \leq x_n$ and $x_n \geq -1$; we will use strong induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. Since $x_1 = 3$ and $x_2 = 1$, we see that $P(1)$ holds. Now suppose that $P(1), \ldots, P(n)$ all hold for some $n \in \mathbf{N}$ and observe that

$$x_{n+1} \leq x_n \leq 3 \implies 1 \leq 4 - x_n \leq 4 - x_{n+1} \implies \frac{1}{4 - x_{n+1}} \leq \frac{1}{4 - x_n},$$

i.e., $x_{n+2} \leq x_{n+1}$. Furthermore,

$$-1 \leq x_n \leq 3 \implies 1 \leq 4 - x_n \leq 5 \implies x_{n+1} = \frac{1}{4 - x_n} \geq \frac{1}{5} > -1.$$

Thus $P(n+1)$ holds. This completes the induction step and it follows that $P(n)$ holds for all $n \in \mathbf{N}$.

We have now shown that the sequence (x_n) is bounded below and decreasing and hence by the Monotone Convergence Theorem (Theorem 2.4.2) we may conclude that the sequence converges.

(b) If (x_n) is any convergent sequence with $\lim x_n = x$, then the sequence (y_n) given by $y_n = x_{n+k}$ for any $k \in \mathbf{N}$ is also convergent with $\lim y_n = x$. To see this, let $\epsilon > 0$ be given. Since $x_n \to x$, there exists an $N \in \mathbf{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq N$. Suppose $n \geq \max\{N-k, 1\}$, so that $n + k \geq N$. It follows that

$$|y_n - x| = |x_{n+k} - x| < \epsilon.$$

Thus $\lim y_n = x$.

(c) By parts (a) and (b) we have $\lim x_n = \lim x_{n+1} = x$ for some $x \in \mathbf{R}$. It then follows from the Algebraic Limit Theorem (Theorem 2.3.3) that

$$x_{n+1} = \frac{1}{4 - x_n} \implies \lim x_{n+1} = \frac{1}{4 - \lim x_n} \iff x = \frac{1}{4-x} \iff x^2 - 4x + 1 = 0.$$

This quadratic equation has solutions $x = 2 \pm \sqrt{3}$. Since (x_n) is decreasing and $x_2 = 1$, the Order Limit Theorem (Theorem 2.3.4) implies that $\lim x_n = x \leq 1 < 2 + \sqrt{3}$ and so we may discard the solution $x = 2 + \sqrt{3}$ to conclude that $\lim x_n = 2 - \sqrt{3}$.

Exercise 2.4.2. (a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$.

What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Solution. (a) The problem is we have assumed that $\lim y_n$ exists. Looking at the first few terms of the sequence $y_1 = 1, y_2 = 2, y_3 = 1, y_4 = 2, \ldots$, we see that in fact the sequence oscillates and does not converge.

(b) The strategy works this time. Let $P(n)$ be the statement that $y_{n+1} \geq y_n$ and $y_n \leq 3$; we will use strong induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. Since $y_1 = 1$ and $y_2 = 2$, we see that $P(1)$ holds. Suppose that $P(1), \ldots, P(n)$ all hold for some $n \in \mathbf{N}$ and observe that

$$y_{n+1} \geq y_n \geq 1 \implies \frac{1}{y_{n+1}} \leq \frac{1}{y_n} \implies 3 - \frac{1}{y_{n+1}} \geq 3 - \frac{1}{y_n},$$

i.e., $y_{n+2} \geq y_{n+1}$. Furthermore,

$$1 \leq y_n \leq 3 \implies \frac{1}{3} \leq \frac{1}{y_n} \implies y_{n+1} = 3 - \frac{1}{y_n} \leq \frac{8}{3} < 3.$$

Thus $P(n+1)$ holds. This completes the induction step and it follows that $P(n)$ holds for all $n \in \mathbf{N}$.

We have now shown that (y_n) is bounded above and increasing, so by the Monotone Convergence Theorem (Theorem 2.4.2) we have $\lim y_n = y$ for some $y \in \mathbf{R}$. Given this, the following manipulations are valid:

$$y_{n+1} = 3 - \frac{1}{y_n} \implies y = 3 - \frac{1}{y} \iff y^2 - 3y + 1 = 0.$$

This quadratic equation has solutions $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$. Since (y_n) is increasing and $y_2 = 2$, we must have $y \geq 2 > \frac{3}{2} - \frac{1}{2}\sqrt{5}$ and so we may discard the solution $y = \frac{3}{2} - \frac{1}{2}\sqrt{5}$ to conclude that $\lim y_n = \frac{3}{2} + \frac{1}{2}\sqrt{5}$.