1 Section 8.5 Exercises

Exercises with solutions from Section 8.5 of [UA].

Exercise 8.5.1. (a) Verify that

$$u(x,t) = b_n \sin(nx)\cos(nt)$$

satisfies equations (1), (2), and (3) for any choice of $n \in \mathbf{N}$ and $b_n \in \mathbf{R}$. What goes wrong if $n \notin \mathbf{N}$?

(b) Explain why any finite sum of functions of the form given in part (a) would also satisfy (1), (2), and (3). (Incidentally, it is possible to hear the different solutions in (a) for values of n up to 4 or 5 by isolating the harmonics on a well-made stringed instrument.)

Solution. (a) Let $n \in \mathbf{N}$ and $b_n \in \mathbf{R}$ be given. Calculations show that

$$\frac{\partial u}{\partial t} = -nb_n \sin(nx)\sin(nt), \quad \frac{\partial^2 u}{\partial t^2} = -n^2 b_n \sin(nx)\cos(nt),$$
$$\text{and} \quad \frac{\partial^2 u}{\partial x^2} = -n^2 b_n \sin(nx)\cos(nt).$$

It is then clear that u satisfies equations (1), (2), and (3). If $n \notin \mathbf{N}$, then it may no longer be the case that u satisfies equations (2) and (3).

(b) If u and v both satisfy equations (1), (2), and (3), then observe that

$$\frac{\partial^2}{\partial x^2}(u+v) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2}{\partial t^2}(u+v).$$

Thus $u+v$ also satisfies equation (1). Furthermore,

$$u(0,t) + v(0,t) = 0 \quad \text{and} \quad u(\pi,t) + v(\pi,t) = 0$$

for all $t \geq 0$, so that $u+v$ also satisfies equation (2). Finally,

$$\frac{\partial}{\partial t}[u+v](x,0) = \frac{\partial u}{\partial t}(x,0) + \frac{\partial v}{\partial t}(x,0) = 0$$

for all $x \in [0,\pi]$, so that $u+v$ also satisfies equation (3).

Exercise 8.5.2. Using trigonometric identities when necessary, verify the following integrals.

(a) For all $n \in \mathbf{N}$,

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(nx)\,dx = 0.$$

(b) For all $n \in \mathbf{N}$,

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi.$$

(c) For all $m, n \in \mathbf{N}$,

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0.$$

For $m \neq n$,

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = 0.$$

Solution. (a) Let $n \in \mathbf{N}$ be given. A calculation shows that

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = \frac{[\sin(nx)]_{x=-\pi}^{x=\pi}}{n} = 0.$$

Notice that $\sin(nx)$ is an odd function. An odd function integrated over an interval of the form $[-a, a]$ is necessarily zero and hence

$$\int_{-\pi}^{\pi} \sin(nx)\,dx = 0.$$

(b) Let $n \in \mathbf{N}$ be given. Using the identity $\cos^2(x) = \frac{1+\cos(2x)}{2}$, we calculate

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \frac{1+\cos(2nx)}{2}\,dx = \left[\frac{x}{2} + \frac{\sin(2nx)}{4n}\right]_{x=-\pi}^{x=\pi} = \pi.$$

Similarly, using the identity $\sin^2(x) = \frac{1-\cos(2x)}{2}$, we find that

$$\int_{-\pi}^{\pi} \sin^2(nx)\,dx = \int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2}\,dx = \left[\frac{x}{2} - \frac{\sin(2nx)}{4n}\right]_{x=-\pi}^{x=\pi} = \pi.$$

(c) For any $m, n \in \mathbf{N}$, notice that $\cos(mx)\sin(nx)$ is the product of an even function and an odd function and hence is itself an odd function; it follows that

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0.$$

Now suppose $m \neq n$. Using the identity $\cos(x)\cos(y) = \frac{\cos(x-y)+\cos(x+y)}{2}$, we calculate

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx &= \int_{-\pi}^{\pi} \frac{\cos((m-n)x)+\cos((m+n)x)}{2}\,dx \\ &= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} + \frac{\sin((m+n)x)}{m+n}\right]_{x=-\pi}^{x=\pi} = 0.\end{aligned}$$

Similarly, using the identity $\sin(x)\sin(y) = \frac{\cos(x-y)-\cos(x+y)}{2}$, we find that

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx &= \int_{-\pi}^{\pi} \frac{\cos((m-n)x)-\cos((m+n)x)}{2}\,dx \\ &= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right]_{x=-\pi}^{x=\pi} = 0.\end{aligned}$$

Exercise 8.5.3. Derive the formulas

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \quad \text{and} \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \tag{10}$$

for all $m \geq 1$.

Solution. Let $m \geq 1$ be given. Multiply both sides of equation (6) by $\cos(mx)$ and integrate over $[-\pi, \pi]$ to obtain

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \int_{-\pi}^{\pi}\left(a_0\cos(mx) + \sum_{n=1}^{\infty} a_n\cos(mx)\cos(nx) + b_n\cos(mx)\sin(nx)\right).$$

Now, assuming we are justified in doing so, we swap the integral with the sum and use Exercise 8.5.2 to find that

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \pi a_m.$$

We can find b_m similarly, multiplying equation (6) by $\sin(mx)$ instead.

Exercise 8.5.4. (a) Referring to the previous example, explain why we can be sure that the convergence of the partial sums to $f(x)$ is *not* uniform on any interval containing 0.

(b) Repeat the computations of Example 8.5.1 for the function $g(x) = |x|$ and examine graphs for some partial sums. This time, make use of the fact that g is even ($g(x) = g(-x)$) to simplify the calculations. By just looking at the coefficients, how do we know this series converges uniformly to something?

(c) Use graphs to collect some empirical evidence regarding the question of term-by-term differentiation in our two examples to this point. Is it possible to conclude convergence or divergence of either differentiated series by looking at the resulting coefficients? Theorem 6.4.3 is about the legitimacy of term-by-term differentiation. Can it be applied to either of these examples?

Solution. (a) Each partial sum S_N is continuous at 0, whereas f is not continuous at 0. It follows from the contrapositive of Theorem 6.2.6 that the convergence is not uniform.

(b) The fact that g is even implies that each b_n is zero. We calculate

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \begin{cases} -\frac{4}{n^2\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus the Fourier series for g is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2};$$

see Figure 1 for a graph of g, S_1, and S_2 over $[-\pi, \pi]$.

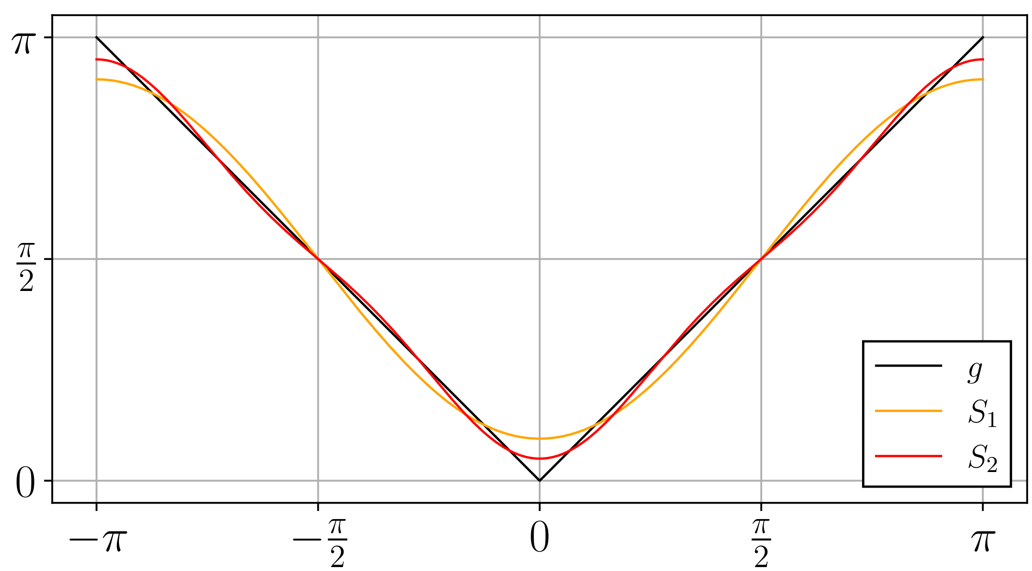

Figure 1: g, S_1, and S_2 on $[-\pi, \pi]$

Notice that
$$\left|\frac{\cos((2n-1)x)}{(2n-1)^2}\right| \leq \frac{1}{(2n-1)^2}$$
for each $n \in \mathbf{N}$ and $x \in [-\pi, \pi]$. It follows from the Weierstrass M-Test that the series converges uniformly.

(c) For the function f from Example 8.5.1, notice that f is not differentiable at $x = 0$ or at $x = \pm\pi$, but satisfies $f'(x) = 0$ for all $x \in (-\pi, 0) \cup (0, \pi)$. The term-by-term differentiated series is
$$\frac{4}{\pi}\sum_{n=0}^{\infty} \cos((2n+1)x).$$
See Figure 2 for a graph of S_{40} and S_{80} over $[-\pi, \pi]$.

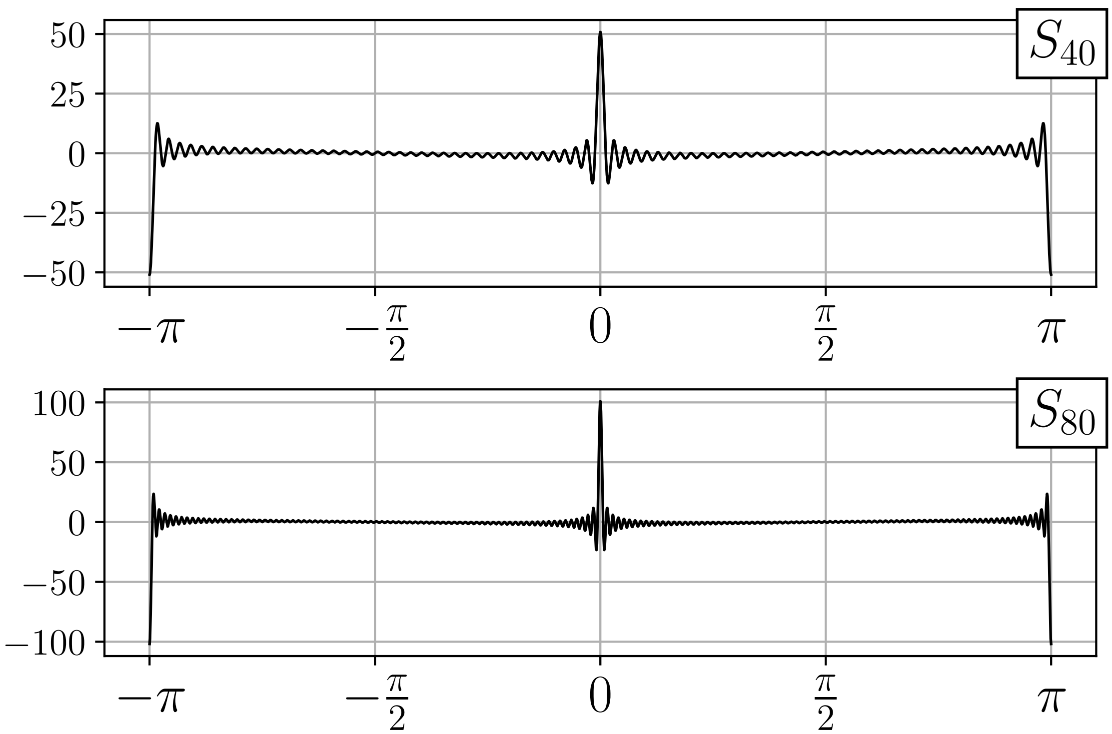

Figure 2: S_{40} and S_{80} on $[-\pi, \pi]$

The series clearly diverges for $x = 0$ and $x = \pm\pi$; this behaviour is reflected in the graph. However, based on the graph we might naively believe that the series is converging to

$f'(x) = 0$ for all $x \in (-\pi, 0) \cup (0, \pi)$. In fact, this series converges if and only if $x = m\pi + \frac{\pi}{2}$ for some $m \in \mathbf{Z}$. If we put the term-by-term differentiated series in the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

then the coefficients are given by

$$b_n = 0, \quad a_0 = 0, \quad \text{and} \quad a_n = \begin{cases} \frac{4}{\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

which certainly do not allow us to conclude convergence of the term-by-term differentiated series. Furthermore, we cannot use Theorem 6.4.3 since the term-by-term differentiated series does not even converge pointwise, let alone uniformly.

For the function g from part (b), notice that g is not differentiable at $x = 0$ or at $x = \pm\pi$, but satisfies

$$g'(x) = \begin{cases} -1 & \text{if } -\pi < x < 0, \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

Notice the similarity to f; indeed, the term-by-term differentiated series is identical to the Fourier series for f:

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}.$$

See Figure 3 for a graph of S_4 and S_{20} on $[-\pi, \pi]$.

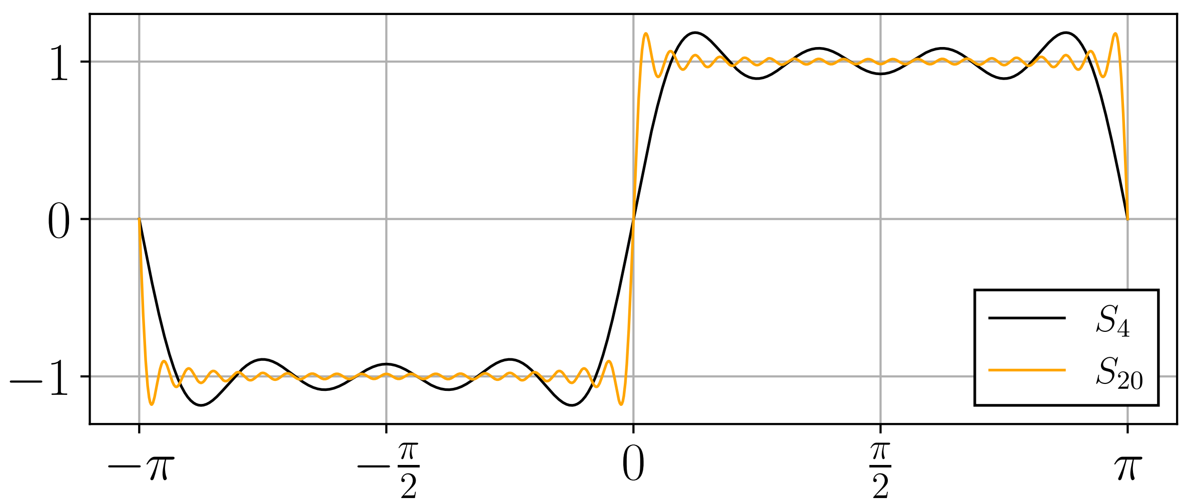

Figure 3: S_4 and S_{20} on $[-\pi, \pi]$

If we put the term-by-term differentiated series in the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

then the coefficients are given by

$$a_n = 0 \quad \text{and} \quad b_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

which certainly do not allow us to conclude convergence of the term-by-term differentiated series. To use Theorem 6.4.3, we would have to show that the term-by-term differentiated series converges uniformly. At this stage, it is not clear how to do so.

Exercise 8.5.5. Explain why h is uniformly continuous on $\mathbf{R}$.

Solution. By assumption h is continuous on the compact set $[-\pi, \pi]$ and thus, by Theorem 4.4.7, h is uniformly continuous on $[-\pi, \pi]$. This is sufficient to show that h is uniformly continuous on $\mathbf{R}$, since, by the 2π-periodicity of h, for any $x, y \in \mathbf{R}$ there exist integers m, n such that $x + 2m\pi \in [-\pi, \pi]$ and $y + 2n\pi \in [-\pi, \pi]$.

Exercise 8.5.6. Show that $\left|\int_a^b h(x)\sin(nx)\,dx\right| < \epsilon/n$, and use this fact to complete the proof.

Solution. Let's slightly modify the start of the proof by instead choosing a $\delta > 0$ such that $|h(x) - h(y)| < \frac{\epsilon}{2\pi}$ whenever $|x - y| < \delta$. For $x \in [a, b]$, define $g(x) = h(x) - h\big(\frac{a+b}{2}\big)$ and note that $|g(x)| < \frac{\epsilon}{2\pi}$, since

$$\left|x - \frac{a+b}{2}\right| \leq \frac{b-a}{2} = \frac{\pi}{n} < \delta.$$

By $\frac{2\pi}{n}$-periodicity, we have

$$\int_a^b \sin(nx)\,dx = \int_{-\pi/n}^{\pi/n} \sin(nx)\,dx = 0.$$

Since $|\sin(nx)| \leq 1$ for all $x \in \mathbf{R}$, it follows that

$$\begin{aligned} \left|\int_a^b h(x)\sin(nx)\,dx\right| &\leq \left|h\left(\frac{a+b}{2}\right)\int_a^b \sin(nx)\,dx\right| + \left|\int_a^b g(x)\sin(nx)\,dx\right| \\ &\leq \int_a^b |g(x)||\sin(nx)|\,dx \leq \int_a^b \frac{\epsilon}{2\pi}\,dx = \frac{\epsilon}{2\pi}\cdot\frac{2\pi}{n} = \frac{\epsilon}{n}. \end{aligned}$$

Now let $x_0 < x_1 < \cdots < x_n$ be the evenly spaced partition of $[-\pi, \pi]$ such that each subinterval has length $\frac{2\pi}{n}$. Then

$$\left|\int_{-\pi}^{\pi} h(x)\sin(nx)\,dx\right| = \left|\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} h(x)\sin(nx)\,dx\right| \leq \sum_{j=1}^{n}\left|\int_{x_{j-1}}^{x_j} h(x)\sin(nx)\,dx\right| < \sum_{j=1}^{n}\frac{\epsilon}{n} = \epsilon.$$

Thus $\int_{-\pi}^{\pi} h(x)\sin(nx)\,dx \to 0$ and by repeating this argument with sin replaced by cos, we can show that $\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx \to 0$.

Exercise 8.5.7. (a) First, argue why the integral involving $q_x(u)$ tends to zero as $N \to \infty$.

(b) The first integral is a little more subtle because the function $p_x(u)$ has the $\sin(u/2)$ term in the denominator. Use the fact that f is differentiable at x (and a familiar limit from calculus) to prove that the first integral goes to zero as well.

Solution. (a) The continuity of f implies the continuity of q_x and thus by the Riemann-Lebesgue Lemma (Theorem 8.5.2) we have

$$\int_{-\pi}^{\pi} q_x(u)\cos(Nu)\,du \to 0 \text{ as } N \to \infty.$$

(b) The continuity of p_x on $(-\pi, 0) \cup (0, \pi]$ follows as f, sin, and cos are continuous everywhere and sin is non-zero on $\left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. Strictly speaking, p_x is not defined at $u = 0$. We claim that defining $p_x(0) = 2f'(x)$ results in p_x also being continuous at zero. Observe that for $u \neq 0$:

$$\frac{f(u+x) - f(x)}{\sin(u/2)} = 2 \cdot \frac{f(u+x) - f(x)}{u} \cdot \frac{u/2}{\sin(u/2)} \to 2f'(x) \text{ as } u \to 0,$$

where we have used that f is differentiable at x and also that $\lim_{u\to\infty} \frac{u}{\sin(u)} = 1$. Thus p_x is continuous on $(-\pi, \pi]$ and so we may again use the Riemann-Lebesgue Lemma to conclude that

$$\int_{-\pi}^{\pi} p_x(u)\sin(Nu)\,du \to 0 \text{ as } N \to \infty.$$

Exercise 8.5.8. Prove that if a sequence of real numbers (x_n) converges, then the arithmetic means

$$y_n = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

also converge to the same limit. Give an example to show that it is possible for the sequence of means (y_n) to converge even if the original sequence (x_n) does not.

Solution. Suppose that $\lim x_n = x$ and let $\epsilon > 0$ be given. There exists an $N_1 \in \mathbf{N}$ such that $|x_n - x| < \frac{\epsilon}{2}$ whenever $n \geq N_1$. Choose $N_2 \in \mathbf{N}$ such that

$$\frac{|x_1 - x| + \cdots + |x_{N_1} - x|}{N_2} < \frac{\epsilon}{2}$$

and suppose that $n > \max\{N_1, N_2\}$. It follows that

$$\begin{aligned}
|y_n - x| &= \left| \frac{x_1 + \cdots + x_{N_1} + x_{N_1+1} + \cdots + x_n}{n} - \frac{nx}{n} \right| \\
&= \left| \frac{(x_1 - x) + \cdots + (x_{N_1} - x)}{n} + \frac{(x_{N_1+1} - x) + \cdots + (x_n - x)}{n} \right| \\
&\leq \frac{|x_1 - x| + \cdots + |x_{N_1} - x|}{n} + \frac{|x_{N_1+1} - x| + \cdots + |x_n - x|}{n} \\
&< \frac{\epsilon}{2} + \frac{n - N_1}{n} \cdot \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

Thus $\lim y_n = x$. For an example where (x_n) does not converge but (y_n) does, let $x_n = (-1)^n$. Then

$$y_n = \begin{cases} -\frac{1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

which converges to zero.

Exercise 8.5.9. Use the previous identity to show that

$$\frac{1/2 + D_1(\theta) + D_2(\theta) + \cdots + D_N(\theta)}{N+1} = \frac{1}{2(N+1)} \left[\frac{\sin\left((N+1)\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \right]^2.$$

Solution. It will suffice to show that

$$1 + 2D_1(\theta) + \cdots + 2D_N(\theta) = \frac{\sin^2\left((N+1)\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}.$$

Indeed, using the identities $\sin(\alpha)\sin(\theta) = \frac{1}{2}(\cos(\alpha - \theta) - \cos(\alpha + \theta))$ and $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$, we find that

$$\begin{aligned}
2\sum_{k=0}^{N} D_k(\theta) &= \sum_{k=0}^{N} \frac{\sin\left(\left(k+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} \sum_{k=0}^{N} \left[\sin\left(\left(k+\tfrac{1}{2}\right)\theta\right) \sin\left(\tfrac{\theta}{2}\right)\right] \\
&= \frac{1}{2\sin^2\left(\frac{\theta}{2}\right)} \sum_{k=0}^{N} [\cos(k\theta) - \cos((k+1)\theta)] \\
&= \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} \cdot \frac{1-\cos((N+1)\theta)}{2} \\
&= \frac{\sin^2\left((N+1)\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}.
\end{aligned}$$

Exercise 8.5.10. (a) Show that

$$\sigma_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x) F_N(u)\, du.$$

(b) Graph the function $F_N(u)$ for several values of N. Where is F_N large, and where is it close to zero? Compare this function to the Dirichlet kernel $D_N(u)$. Now, prove that $F_N \to 0$ uniformly on any set of the form $\{u : |u| \geq \delta\}$, where $\delta > 0$ is fixed (and u is restricted to the interval $(-\pi, \pi]$).

(c) Prove that $\int_{-\pi}^{\pi} F_N(u)\, du = \pi$.

(d) To finish the proof of Fejér's Theorem, first choose a $\delta > 0$ so that

$$|u| < \delta \quad \text{implies} \quad |f(x+u) - f(x)| < \epsilon.$$

Set up a single integral that represents the difference $\sigma_N(x) - f(x)$ and divide this integral into sets where $|u| \leq \delta$ and $|u| \geq \delta$. Explain why it is possible to make each of these integrals sufficiently small, independently of the choice of x.

Solution. (a) Using the expression for $S_n(x)$ derived previously in the textbook, we have

$$\sigma_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} S_n(x) = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)D_n(u)\,du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\frac{1}{N+1}\sum_{n=0}^{N} D_n(u)\,du = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)F_N(u)\,du.$$

(b) See Figure 4 for a graph of F_4, F_8, and F_{12} over the interval $[-\pi, \pi]$.

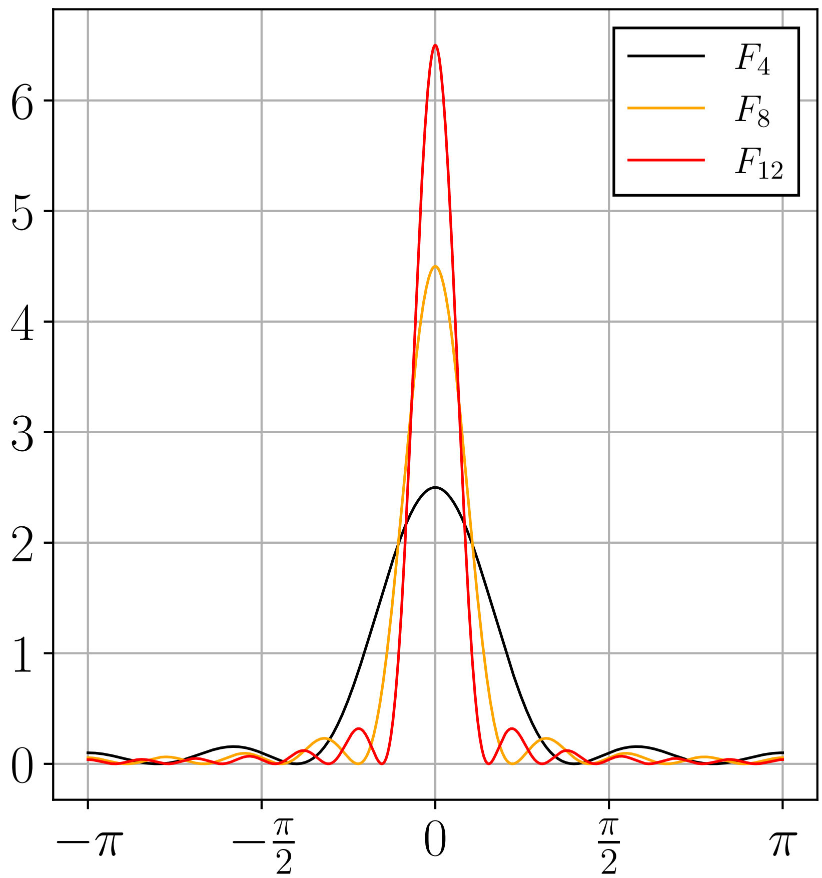

Figure 4: F_4, F_8, and F_{12} on $[-\pi, \pi]$

Like the Dirichlet kernel, the Fejér kernel has a large peak at 0 and decays away from 0; unlike the Dirichlet kernel, the Fejér kernel is non-negative.

Let $0 < \delta < \pi$ be given and set $A = \{u \in [-\pi, \pi] : \delta \leq |u|\}$. For any $u \in A$, observe that $\sin^2\left(\frac{\delta}{2}\right) \leq \sin^2\left(\frac{u}{2}\right)$. Since $\delta \in (0, \pi)$, we have $\sin^2\left(\frac{\delta}{2}\right) > 0$ and thus

$$\frac{1}{\sin^2\left(\frac{u}{2}\right)} \leq \frac{1}{\sin^2\left(\frac{\delta}{2}\right)}$$

for each $u \in A$. It follows that

$$|F_N(u)| = \frac{1}{2(N+1)} \cdot \frac{\sin^2\left((N+1)\frac{u}{2}\right)}{\sin^2\left(\frac{u}{2}\right)} \leq \frac{1}{2(N+1)} \cdot \frac{1}{\sin^2\left(\frac{\delta}{2}\right)}$$

for all $u \in A$. It is clear from this bound that $F_N \to 0$ uniformly on A.

(c) Recalling that $\int_{-\pi}^{\pi} D_n(u)\,du = \pi$ for any $n \geq 0$, we have

$$\int_{-\pi}^{\pi} F_N(u)\,du = \int_{-\pi}^{\pi} \frac{1}{N+1} \sum_{n=0}^{N} D_n(u)\,du = \frac{1}{N+1} \sum_{n=0}^{N} \int_{-\pi}^{\pi} D_n(u)\,du = \frac{(N+1)\pi}{N+1} = \pi.$$

(d) By assumption f is continuous on $[-\pi, \pi]$ and hence is uniformly continuous here. Thus, for any $\epsilon > 0$, we can choose a $0 < \delta < \pi$ such that

$$|u| < \delta \qquad \Longrightarrow \qquad |f(x+u) - f(x)| < \epsilon.$$

For any $x \in (-\pi, \pi]$ and $N \in \mathbf{N}$, parts (a) and (c) imply that

$$\sigma_N(x) - f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+u) - f(x)] F_N(u)\,du.$$

Observe that

$$\begin{aligned} \left| \frac{1}{\pi} \int_{|u|<\delta} [f(x+u) - f(x)] F_N(u)\,du \right| &\leq \frac{1}{\pi} \int_{|u|<\delta} |f(x+u) - f(x)| F_N(u)\,du \\ &< \frac{\epsilon}{\pi} \int_{|u|<\delta} F_N(u)\,du < \frac{\epsilon}{\pi} \int_{-\pi}^{\pi} F_N(u)\,du = \epsilon. \end{aligned}$$

Let $M > 0$ be a bound on f over $[-\pi, \pi]$. By part (b), there exists a $K \in \mathbf{N}$ such that $F_N(u) \leq \frac{\epsilon}{4M}$ for all $\delta \leq u \leq \pi$ and $N \geq K$. For such N, observe that

$$\begin{aligned} \left| \frac{1}{\pi} \int_{\delta \leq |u| \leq \pi} [f(x+u) - f(x)] F_N(u)\,du \right| &\leq \frac{1}{\pi} \int_{\delta \leq |u| \leq \pi} |f(x+u) - f(x)| F_N(u)\,du \\ &\leq \frac{2M\epsilon}{4M\pi} \int_{\delta \leq |u| \leq \pi} du < \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} du = \epsilon. \end{aligned}$$

It now follows that for any $x \in (-\pi, \pi]$ and $N \geq K$, we have

$$|\sigma_N(x) - f(x)| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+u) - f(x)] F_N(u)\, du \right|$$
$$\leq \left| \frac{1}{\pi} \int_{|u|<\delta} [f(x+u) - f(x)] F_N(u)\, du \right| + \left| \frac{1}{\pi} \int_{\delta \leq |u| \leq \pi} [f(x+u) - f(x)] F_N(u)\, du \right| < 2\epsilon.$$

We may conclude that $\sigma_N \to f$ uniformly on $(-\pi, \pi]$.

Exercise 8.5.11. (a) Use the fact that the Taylor series for $\sin(x)$ and $\cos(x)$ converge uniformly on any compact set to prove WAT under the added assumption that $[a, b]$ is $[0, \pi]$.

(b) Show how the case for an arbitrary interval $[a, b]$ follows from this one.

Solution. (a) First, let's prove the following result.

Lemma 1. Suppose that $T : \mathbf{R} \to \mathbf{R}$ is a trigonometric polynomial, i.e. T is either constant or of the form

$$T(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

for some $N \in \mathbf{N}$ and some coefficients $a_n, b_n \in \mathbf{R}$. Let $[a, b]$ be given. For any $\epsilon > 0$, there exists a polynomial p such that $|T(x) - p(x)| < \epsilon$ for all $x \in [a, b]$.

Proof. If T is constant the result is clear, so suppose that T is of the form

$$T(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

for some $N \in \mathbf{N}$ and some coefficients $a_n, b_n \in \mathbf{R}$. Let $1 \leq n \leq N$ be given. Because the Taylor series for $\cos(nx)$ converges uniformly on $[a, b]$, there exists a polynomial p_n (some partial sum of the Taylor series) such that

$$|\cos(nx) - p_n(x)| < \frac{\epsilon}{2N(1 + |a_n|)}$$

for each $x \in [a, b]$. Similarly, there exists a polynomial q_n such that

$$|\sin(nx) - q_n(x)| < \frac{\epsilon}{2N(1 + |b_n|)}$$

for each $x \in [a,b]$. Let p be the polynomial given by $p(x) = a_0 + \sum_{n=1}^{N} a_n p_n(x) + b_n q_n(x)$. Then for any $x \in [a,b]$, we have

$$\begin{aligned}
|T(x) - p(x)| &= \left|\sum_{n=1}^{N} a_n(\cos(nx) - p_n(x)) + b_n(\sin(nx) - q_n(x))\right| \\
&\leq \sum_{n=1}^{N} |a_n||\cos(nx) - p_n(x)| + |b_n||\sin(nx) - q_n(x)| \\
&< \sum_{n=1}^{N} \frac{\epsilon|a_n|}{2N(1+|a_n|)} + \frac{\epsilon|b_n|}{2N(1+|b_n|)} \\
&< \sum_{n=1}^{N} \frac{\epsilon}{N} \\
&= \epsilon.
\end{aligned}$$

□

Now let $f : [0,\pi] \to \mathbf{R}$ be continuous and let $\epsilon > 0$ be given. By Fejér's Theorem (Theorem 8.5.4), $\sigma_N \to f$ uniformly on $[0,\pi]$ and thus there exists an $M \in \mathbf{N}$ such that $|\sigma_M(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in [0,\pi]$. Notice that σ_M is a trigonometric polynomial; it follows from Lemma 1 that there exists a polynomial p such that $|\sigma_M(x) - p(x)| < \frac{\epsilon}{2}$ for all $x \in [0,\pi]$. Thus

$$|f(x) - p(x)| \leq |\sigma_M(x) - f(x)| + |\sigma_M(x) - p(x)| < \epsilon$$

for any $x \in [0,\pi]$.

(b) Let $f : [a,b] \to \mathbf{R}$ be continuous and define $g : [0,\pi] \to \mathbf{R}$ by $g(x) = f\big(\frac{b-a}{\pi}x + a\big)$; notice that g is continuous. Let $\epsilon > 0$ be given. By part (a), there exists a polynomial q such that $|g(x) - q(x)| < \epsilon$ for each $x \in [0,\pi]$. Define p by $p(x) = q\Big(\frac{\pi(x-a)}{b-a}\Big)$ and notice that p is a polynomial. For any $x \in [a,b]$ we have $\frac{\pi(x-a)}{b-a} \in [0,\pi]$ and thus

$$\left|g\left(\frac{\pi(x-a)}{b-a}\right) - q\left(\frac{\pi(x-a)}{b-a}\right)\right| = |f(x) - p(x)| < \epsilon.$$

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.