1 Section 6.C Exercises

Exercises with solutions from Section 6.C of [LADR].

Exercise 6.C.1. Suppose $v_1, \ldots, v_m \in V$. Prove that

$$\{v_1, \ldots, v_m\}^\perp = (\operatorname{span}(v_1, \ldots, v_m))^\perp.$$

Solution. Suppose $v \in (\operatorname{span}(v_1, \ldots, v_m))^\perp$. Since each $v_j \in \operatorname{span}(v_1, \ldots, v_m)$, this implies that $\langle v, v_j \rangle = 0$ for each $1 \leq j \leq m$. It follows that $v \in \{v_1, \ldots, v_m\}^\perp$ and hence that $(\operatorname{span}(v_1, \ldots, v_m))^\perp \subseteq \{v_1, \ldots, v_m\}^\perp$.

Now suppose that $v \in \{v_1, \ldots, v_m\}^\perp$ and let $a_1 v_1 + \cdots + a_m v_m \in \operatorname{span}(v_1, \ldots, v_m)$ be given. We then have

$$\langle v, a_1 v_1 + \cdots + a_m v_m \rangle = \overline{a_1} \langle v, v_1 \rangle + \cdots + \overline{a_m} \langle v, v_m \rangle = 0.$$

It follows that $v \in (\operatorname{span}(v_1, \ldots, v_m))^\perp$ and hence that $\{v_1, \ldots, v_m\}^\perp \subseteq (\operatorname{span}(v_1, \ldots, v_m))^\perp$. We may conclude that

$$\{v_1, \ldots, v_m\}^\perp = (\operatorname{span}(v_1, \ldots, v_m))^\perp.$$

Exercise 6.C.2. Suppose U is a finite-dimensional subspace of V. Prove that $U^\perp = \{0\}$ if and only if $U = V$.
[*Exercise 14(a) shows that the result above is not true without the hypothesis that U is finite-dimensional.*]

Solution. The implication $U = V \implies U^\perp = \{0\}$ is the content of 6.46 (c).

For the converse implication, we will prove the contrapositive statement. Suppose therefore that $U \neq V$ and let $u_1, \ldots, u_m$ be a basis of U. Since $U \neq V$, there must exist some $v \in V \setminus U$ such that the list $u_1, \ldots, u_m, v$ is linearly independent. Perform the Gram-Schmidt procedure (6.31) on this list to obtain an orthonormal list $e_1, \ldots, e_m, e_{m+1}$ such that

$$\operatorname{span}(e_1, \ldots, e_m) = \operatorname{span}(u_1, \ldots, u_m) = U$$

and such that e_{m+1} is orthogonal to each vector in the list $e_1, \ldots, e_m$, i.e. $e_{m+1} \in \{e_1, \ldots, e_m\}^\perp$. By Exercise 6.C.1, this is equivalent to saying that $e_{m+1} \in (\operatorname{span}(e_1, \ldots, e_m))^\perp = U^\perp$. Since $e_{m+1} \neq 0$, we see that $U^\perp \neq \{0\}$.

Exercise 6.C.3. Suppose U is a subspace of V with basis $u_1, \ldots, u_m$ and

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V. Prove that if the Gram-Schmidt Procedure is applied to the basis of V above, producing a list $e_1, \ldots, e_m, f_1, \ldots, f_n$, then $e_1, \ldots, e_m$ is an orthonormal basis of U and $f_1, \ldots, f_n$ is an orthonormal basis of $U^\perp$.

Solution. The Gram-Schmidt procedure guarantees that

$$\operatorname{span}(e_1, \ldots, e_m) = \operatorname{span}(u_1, \ldots, u_m) = U.$$

Any orthonormal list is linearly independent, so we have a linearly independent list $e_1, \ldots, e_m$ of length m contained inside a subspace of dimension m; it follows that $e_1, \ldots, e_m$ is an orthonormal basis of U.

The Gram-Schmidt procedure also guarantees that for any $1 \leq j \leq n$ the vector f_j is orthogonal to each vector in the list $e_1, \ldots, e_n$. By Exercise 6.C.1, this implies that

$$f_j \in (\operatorname{span}(e_1, \ldots, e_m))^{\perp} = U^{\perp}.$$

As before, the list $f_1, \ldots f_n$ is orthonormal and hence linearly independent. Consequently, we have a linearly independent list $f_1, \ldots, f_n$ of length n contained inside a subspace of dimension n (6.50); it follows that $f_1, \ldots, f_n$ is an orthonormal basis of $U^{\perp}$.

Exercise 6.C.4. Suppose U is the subspace of $\mathbf{R}^4$ defined by

$$U = \operatorname{span}((1, 2, 3, -4), (-5, 4, 3, 2)).$$

Find an orthonormal basis of U and an orthonormal basis of $U^{\perp}$.

Solution. It is straightforward to verify that

$$u_1 = (1, 2, 3, -4), \quad u_2 = (-5, 4, 3, 2), \quad v_1 = (1, 0, 0, 0), \quad v_2 = (0, 1, 0, 0)$$

is a basis of $\mathbf{R}^4$; clearly u_1, u_2 is a basis of U. Performing the Gram-Schmidt procedure on this list yields the orthonormal list

$$e_1 = \frac{1}{\sqrt{30}}(1, 2, 3, -4), \quad e_2 = \frac{1}{\sqrt{12030}}(-77, 56, 39, 38),$$
$$f_1 = \frac{1}{\sqrt{76190}}(190, 117, 60, 151), \quad f_2 = \frac{1}{9\sqrt{190}}(0, 81, -90, 27).$$

As we showed in Exercise 6.C.3, e_1, e_2 must be an orthonormal basis of U and f_1, f_2 must be an orthonormal basis of $U^{\perp}$.

Exercise 6.C.5. Suppose V is finite-dimensional and U is a subspace of V. Show that $P_{U^{\perp}} = I - P_U$, where I is the identity operator on V.

Solution. For $v \in V$, we can write $v = w + u$ for unique vectors $w \in U^\perp$ and $u \in (U^\perp)^\perp = U$ (6.47 and 6.51). Note that $P_{U^\perp} v = w$ and $P_U v = u$. It follows that

$$P_{U^\perp} v = w = v - u = Iv - P_U v = (I - P_U)v$$

and hence that $P_{U^\perp} = I - P_U$.

Exercise 6.C.6. Suppose U and W are finite-dimensional subspaces of V. Prove that $P_U P_W = 0$ if and only if $\langle u, w \rangle = 0$ for all $u \in U$ and $w \in W$.

Solution. Suppose that $\langle u, w \rangle = 0$ for all $u \in U$ and $w \in W$. For $v \in V$, write $v = w + y$, where $w \in W$ and $y \in W^\perp$, so that $P_W v = w$. Our hypothesis ensures that $w \in U^\perp$ and thus $P_U P_W v = P_U w = 0$ by 6.55 (c).

For the converse implication, suppose that $P_U P_W = 0$ and let $u \in U$ and $w \in W$ be given. On one hand, we have $P_U P_W w = 0$ by assumption; on the other hand we have $P_U P_W w = P_U w$ by 6.55 (b). Thus $P_U w = 0$, so that $w \in \operatorname{null} P_U$. By 6.55 (e), this is equivalent to $w \in U^\perp$, whence $\langle u, w \rangle = 0$.

Exercise 6.C.7. Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in $\operatorname{null} P$ is orthogonal to every vector in $\operatorname{range} P$. Prove that there exists a subspace U of V such that $P = P_U$.

Solution. By Exercise 5.B.4 and 6.47 we have the decompositions

$$V = \operatorname{range} P \oplus \operatorname{null} P = \operatorname{range} P \oplus (\operatorname{range} P)^\perp,$$

which implies that $\dim \operatorname{null} P = \dim(\operatorname{range} P)^\perp$. Combining this with the hypothesis $\operatorname{null} P \subseteq (\operatorname{range} P)^\perp$ we see that $\operatorname{null} P = (\operatorname{range} P)^\perp$. Let $U = \operatorname{range} P$; we claim that $P = P_U$. To see this, let $v = Px + w$ be given, where $Px \in \operatorname{range} P$ and $w \in (\operatorname{range} P)^\perp = \operatorname{null} P$ are unique. Then

$$P_U v = Px = P(Px + w) = Pv,$$

where we have used $P^2 = P$ and $w \in \operatorname{null} P$ for the third equality.

Exercise 6.C.8. Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and

$$\|Pv\| \leq \|v\|$$

for every $v \in V$. Prove that there exists a subspace U of V such that $P = P_U$.

Solution. Suppose $w \in \operatorname{null} P$ and $Px \in \operatorname{range} P$. Our hypothesis implies the inequality

$$\|Px\| = \|P(Px + \lambda w)\| \leq \|Px + \lambda w\|$$

for any $\lambda \in \mathbf{F}$. It follows from Exercise 6.A.6 that $\langle w, Px \rangle = 0$ and hence that $\operatorname{null} P \subseteq (\operatorname{range} P)^\perp$. We can now set $U = \operatorname{range} P$ and proceed as in Exercise 6.C.7 to see that $P = P_U$.

Exercise 6.C.9. Suppose $T \in \mathcal{L}(V)$ and U is a finite-dimensional subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

Solution. Suppose that U is invariant under T and let $v \in V$ be given. Then

$$P_U v \in U \quad \Longrightarrow \quad T P_U v \in U \quad \Longrightarrow \quad P_U T P_U v = T P_U v$$

where the last implication follows from 6.55 (b). Now suppose that U is not invariant under T, i.e. there is some $u \in U$ such that $Tu \notin U$. Then

$$T P_U u = Tu \notin U \quad \text{and} \quad P_U T P_U u \in U,$$

where we have used 6.55 (b) and (d). It follows that $P_U T P_U u \neq T P_U u$ and hence that $P_U T P_U \neq P_U T$.

Exercise 6.C.10. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V. Prove that U and $U^\perp$ are both invariant under T if and only if $P_U T = T P_U$.

Solution. Suppose that U and $U^\perp$ are both invariant under T and let $v = u + w \in V$ be given, where $u \in U$ and $w \in U^\perp$ are unique. By assumption we have $Tu \in U$ and $Tw \in U^\perp$; it follows that

$$P_U T v = P_U(Tu + Tw) = Tu = T P_U v.$$

Now suppose that U is not invariant under T, i.e. there is some $u \in U$ such that $Tu \notin U$. As in Exercise 6.C.9, we have

$$T P_U u = Tu \notin U \quad \text{and} \quad P_U T u \in U,$$

so that $T P_U \neq P_U T$. Similarly, suppose that $U^\perp$ is not invariant under T, i.e. there is some $w \in U^\perp$ such that $Tw \notin U^\perp$. Then

$$T P_U w = T(0) = 0 \quad \text{and} \quad P_U T w \neq 0,$$

where we have used 6.55 (e). It follows that $T P_U \neq P_U T$.

Exercise 6.C.11. In $\mathbf{R}^4$, let

$$U = \operatorname{span}((1,1,0,0),(1,1,1,2)).$$

Find $u \in U$ such that $\|u - (1,2,3,4)\|$ is as small as possible.

Solution. Let $u_1 = (1,1,0,0)$ and $u_2 = (1,1,1,2)$, so that $U = \operatorname{span}(u_1, u_2)$. Performing the Gram-Schmidt procedure on the list u_1, u_2 yields the orthonormal basis

$$e_1 = \frac{1}{\sqrt{2}}(1,1,0,0), \quad e_2 = \frac{1}{\sqrt{5}}(0,0,1,2)$$

for U. Let $v = (1, 2, 3, 4)$. According to 6.56, to minimize $\|u - v\|$ we should take $u = P_U v$. This can be calculated using 6.55 (i):

$$P_U v = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 = \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5} \right).$$

Exercise 6.C.12. Find $p \in \mathcal{P}_3(\mathbf{R})$ such that $p(0) = 0, p'(0) = 0$, and

$$\int_0^1 |2 + 3x - p(x)|^2 \, \mathrm{d}x$$

is as small as possible.

Solution. Equip $\mathcal{P}_3(\mathbf{R})$ with the inner product

$$\langle p, q \rangle = \int_0^1 p(x) q(x) \, \mathrm{d}x$$

and let

$$U = \{p \in \mathcal{P}_3(\mathbf{R}) : p(0) = p'(0) = 0\}.$$

It is straightforward to verify that U is a subspace of $\mathcal{P}_3(\mathbf{R})$ and that x^2, x^3 is a basis of U. Performing the Gram-Schmidt procedure on this basis yields the orthonormal basis

$$e_1(x) = \sqrt{5} x^2, \quad e_2(x) = 6\sqrt{7}\big(x^3 - \tfrac{5}{6} x^2\big)$$

for U. Let $q(x) = 2 + 3x$. According to 6.56, to minimize $\|q - p\|^2 = \int_0^1 |2 + 3x - p(x)|^2 \, \mathrm{d}x$ we should take $p = P_U q$. This can be calculated using 6.55 (i):

$$P_U q = \langle q, e_1 \rangle e_1 + \langle q, e_2 \rangle e_2 = 24x^2 - \frac{203}{10} x^3.$$

Exercise 6.C.13. Find $p \in \mathcal{P}_5(\mathbf{R})$ that makes

$$\int_{-\pi}^{\pi} |\sin x - p(x)|^2 \, \mathrm{d}x$$

as small as possible.
[*The polynomial 6.60 is an excellent approximation to the answer to this exercise, but here you are asked to find the exact solution, which involves powers of π. A computer that can perform symbolic integration will be useful.*]

Solution. Equip $C_{\mathbf{R}}([-\pi,\pi])$ with the inner product

$$\langle p,q\rangle = \int_{-\pi}^{\pi} p(x)q(x)\,\mathrm{d}x$$

and let $U = \mathcal{P}_5(\mathbf{R})$. Performing the Gram-Schmidt procedure on the basis $1, x, x^2, x^3, x^4, x^5$ of U yields the orthonormal basis

$$e_1(x) = \frac{1}{\sqrt{2\pi}}, \quad e_2(x) = \sqrt{\frac{3}{2\pi^3}}x, \quad e_3(x) = -\frac{1}{2}\sqrt{\frac{5}{2\pi^5}}(\pi^2 - 3x^2),$$

$$e_4(x) = -\frac{1}{2}\sqrt{\frac{7}{2\pi^7}}(3\pi^2 x - 5x^3), \quad e_5(x) = \frac{3}{8\sqrt{2\pi^9}}(3\pi^4 - 30\pi^2 x^2 + 35x^4),$$

$$e_6(x) = -\frac{1}{8}\sqrt{\frac{11}{2\pi^{11}}}(15\pi^4 x - 70\pi^2 x^3 + 63x^5).$$

According to 6.56, to minimize $\|\sin x - p\|^2 = \int_{-\pi}^{\pi} |\sin x - p(x)|^2\,\mathrm{d}x$ we should take $p = P_U(\sin x)$. This can be calculated using 6.55 (i):

$$P_U(\sin x) = \frac{105(1465 - 153\pi^2 + \pi^4)}{8\pi^6}x - \frac{315(1155 - 125\pi^2 + \pi^4)}{4\pi^8}x^3 + \frac{693(945 - 105\pi^2 + \pi^4)}{8\pi^{10}}x^5.$$

Exercise 6.C.14. Suppose $C_{\mathbf{R}}([-1,1])$ is the vector space of continuous real-valued functions on the interval $[-1,1]$ with inner product given by

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$$

for $f, g \in C_{\mathbf{R}}([-1,1])$. Let U be the subspace of $C_{\mathbf{R}}([-1,1])$ defined by

$$U = \{f \in C_{\mathbf{R}}([-1,1]) : f(0) = 0\}.$$

(a) Show that $U^\perp = \{0\}$.

(b) Show that 6.47 and 6.51 do not hold without the finite-dimensional hypothesis.

Solution. (a) It is clear that $0 \in U^\perp$. For the reverse inclusion, suppose that $g \in U^\perp$, let $f : [-1,1] \to \mathbf{R}$ be given by $f(x) = x^2 g(x)$, and note that $f \in U$. It follows that

$$0 = \langle f,g\rangle = \int_{-1}^{1} [xg(x)]^2\,\mathrm{d}x.$$

Since the integrand $[xg(x)]^2$ is continuous and non-negative, we must have $xg(x) = 0$ for all $x \in [-1, 1]$, which implies that $g(x) = 0$ for all non-zero $x \in [-1, 1]$. The continuity of g implies that g must in fact be identically zero on $[-1, 1]$, i.e. $g = 0$. We may conclude that $U^\perp = \{0\}$.

(b) From part (a), we have $U \oplus U^\perp = U \neq C_{\mathbf{R}}([-1, 1])$, so that 6.47 does not hold. Part (a) and 6.46 (b) gives

$$(U^\perp)^\perp = \{0\}^\perp = C_{\mathbf{R}}([-1, 1]) \neq U,$$

so that 6.51 does not hold.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edition.