1 Section 7.3 Exercises

Exercises with solutions from Section 7.3 of [UA].

Exercise 7.3.1. Consider the function

$$h(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 2 & \text{for } x = 1 \end{cases}$$

over the interval $[0, 1]$.

(a) Show that $L(f, P) = 1$ for every partition P of $[0, 1]$.

(b) Construct a partition P for which $U(f, P) < 1 + 1/10$.

(c) Given $\epsilon > 0$, construct a partition P_ϵ for which $U(f, P_\epsilon) < 1 + \epsilon$.

Solution. (a) Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[0, 1]$. For any $1 \le k \le n$,

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = 1$$

and thus

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{n} \Delta x_k = 1 - 0 = 1.$$

(b) Set $x_0 = 0, x_1 = \frac{19}{20}, x_2 = 1$, and let P be the partition $\{x_0, x_1, x_2\}$ of $[0, 1]$. Since

$$M_1 = \sup\{f(x) : x \in [x_0, x_1]\} = 1 \quad \text{and} \quad M_2 = \sup\{f(x) : x \in [x_1, x_2]\} = 2,$$

we have

$$U(f, P) = M_1(x_1 - x_0) + M_2(x_2 - x_1) = 2 - x_1 = 2 - \frac{19}{20} = \frac{21}{20} = 1 + \frac{1}{20} < 1 + \frac{1}{10}.$$

(c) Set $x_0 = 0, x_1 = \max\{\frac{1}{2}, 1 - \frac{\epsilon}{2}\}, x_2 = 1$, and let P be the partition $\{x_0, x_1, x_2\}$ of $[0, 1]$. Since

$$M_1 = \sup\{f(x) : x \in [x_0, x_1]\} = 1 \quad \text{and} \quad M_2 = \sup\{f(x) : x \in [x_1, x_2]\} = 2,$$

we have

$$U(f, P) = M_1(x_1 - x_0) + M_2(x_2 - x_1) = 2 - x_1 \le 1 + \frac{\epsilon}{2} < 1 + \epsilon.$$

Exercise 7.3.2. Recall that Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

has a countable set of discontinuities occurring at precisely every rational number. Follow these steps to prove $t(x)$ is integrable on $[0,1]$ with $\int_0^1 t = 0$.

(a) First argue that $L(t, P) = 0$ for any partition P of $[0,1]$.

(b) Let $\epsilon > 0$, and consider the set of points $D_{\epsilon/2} = \{x \in [0,1] : t(x) \geq \epsilon/2\}$. How big is $D_{\epsilon/2}$?

(c) To complete the argument, explain how to construct a partition P_ϵ of $[0,1]$ so that $U(t, P_\epsilon) < \epsilon$.

Solution. (a) Let $P = \{x_0, x_1, \ldots x_n\}$ be an arbitrary partition of $[0,1]$. The irrationals are dense in $\mathbf{R}$, so any subinterval $[x_{k-1}, x_k]$ contains an irrational number y. Since $t(y) = 0$ and $t(x) \geq 0$ for all $x \in [0,1]$, it follows that $m_k = 0$, from which we see that $L(t, P) = 0$.

(b) Since $0 \leq t(x) \leq 1$ for all $x \in [0,1]$, if $\frac{\epsilon}{2} > 1$ then $D_{\epsilon/2}$ is empty. Suppose therefore that $0 < \frac{\epsilon}{2} \leq 1$ and let N be the smallest positive integer such that $\frac{1}{N} < \frac{\epsilon}{2}$. It follows that $D_{\epsilon/2}$ consists precisely of those rational numbers $\frac{m}{n} \in [0,1]$ (in lowest terms with $n > 0$) with $1 \leq n \leq N$, of which there are only finitely many. Thus $D_{\epsilon/2}$ is finite for any $\epsilon > 0$.

(c) Let $\epsilon > 0$ be given. If $D_{\epsilon/2}$ is empty, i.e. if $0 \leq t(x) < \frac{\epsilon}{2}$ for all $x \in [0,1]$, then let P_ϵ be the partition $\{0, 1\}$ of $[0,1]$. For this partition we have

$$U(t, P_\epsilon) = \sup\{t(x) : x \in [0,1]\} \leq \frac{\epsilon}{2} < \epsilon.$$

Now suppose that $D_{\epsilon/2}$ is not empty; by part (b) it must be the case that $D_{\epsilon/2} = \{y_1, \ldots, y_m\}$ for some $m \in \mathbf{N}$ and some $y_1, \ldots, y_m \in [0,1]$. Let $P_\epsilon = \{x_0, \ldots, x_n\}$ be the evenly spaced partition of $[0,1]$ such that $\Delta x_k < \frac{\epsilon}{2(m+1)}$ for each $k \in \{1, \ldots, n\}$. Decompose the set $\{1, \ldots, n\}$ into the disjoint union $A \cup A^{\mathsf{c}}$, where

$$A = \{k \in \{1, \ldots, n\} : \text{there exists } j \in \{1, \ldots, m\} \text{ such that } y_j \in [x_{k-1}, x_k]\},$$

so that

$$U(t, P_\epsilon) = \sum_{k=1}^{n} M_k \Delta x_k = \sum_{k \in A} M_k \Delta x_k + \sum_{k \notin A} M_k \Delta x_k. \tag{1}$$

Note that A can contain at most $m+1$ elements and also that $M_k \leq 1$ for any $k \in \{1, \dots, n\}$. It follows that

$$\sum_{k \in A} M_k \Delta x_k < \sum_{k \in A} \frac{\epsilon}{2(m+1)} \leq (m+1) \frac{\epsilon}{2(m+1)} = \frac{\epsilon}{2}. \tag{2}$$

Now suppose that $k \in \{1, \dots, n\}$ is such that $k \notin A$, so that $f(x) < \frac{\epsilon}{2}$ for all $x \in [x_{k-1}, x_k]$. Then $M_k \leq \frac{\epsilon}{2}$ and it follows that

$$\sum_{k \notin A} M_k \Delta x_k \leq \frac{\epsilon}{2} \sum_{k \notin A} \Delta x_k \leq \frac{\epsilon}{2} \sum_{k=1}^{n} \Delta x_k = \frac{\epsilon}{2}. \tag{3}$$

Combining (1), (2), and (3), we see that $U(t, P_\epsilon) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

We have now shown that for any $\epsilon > 0$ there exists a partition P_ϵ of $[0, 1]$ such that $U(t, P_\epsilon) < \epsilon$. From part (a) we have $L(t, P_\epsilon) = 0$ and hence $U(t, P_\epsilon) - L(t, P_\epsilon) < \epsilon$; it follows that t is integrable on $[0, 1]$. Part (a) also shows that

$$\int_0^1 t = L(t) = 0.$$

Exercise 7.3.3. Let

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbf{N} \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable on $[0, 1]$ and compute $\int_0^1 f$.

Solution. Let $P = \{x_0, \dots, x_n\}$ be an arbitrary partition of $[0, 1]$. The irrationals are dense in $\mathbf{R}$, so any subinterval $[x_{k-1}, x_k]$ contains an irrational number y. Since $f(y) = 0$ and $f(x) \geq 0$ for all $x \in [0, 1]$, it follows that $m_k = 0$, from which we see that $L(f, P) = 0$. Because P was an arbitrary partition of $[0, 1]$, we have also shown that $L(f) = 0$; once we show that f is integrable on $[0, 1]$ it will follow that $\int_0^1 f = 0$.

Let $\epsilon > 0$ be given. If $\frac{\epsilon}{2} > 1$, then $f(x) \leq \frac{\epsilon}{2}$ for all $x \in [0, 1]$. Take the partition $P_\epsilon = \{0, 1\}$ of $[0, 1]$, so that

$$U(f, P_\epsilon) = \sup\{f(x) : x \in [0, 1]\} \leq \frac{\epsilon}{2} < \epsilon.$$

As noted above, we have $L(f, P_\epsilon) = 0$ and thus $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

Now suppose that $\frac{\epsilon}{2} \leq 1$. Our argument here is similar to the one we gave in Exercise 7.3.2 (c). Choose $N \in \mathbf{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$; note that $N \geq 2$. Let $P_\epsilon = \{x_0, x_1, \dots, x_n\}$ be the partition of $[0, 1]$ where $x_0 = 0, x_1 = \frac{1}{N}, x_n = 1$, and $x_2, \dots, x_{n-1}$ are chosen to be evenly spaced between $\frac{1}{N}$ and 1, such that $\Delta x_k < \frac{\epsilon}{2N}$ for $k \geq 2$. Then

$$U(f, P_\epsilon) = \sum_{k=1}^{n} M_k \Delta x_k = M_1 \Delta x_1 + \sum_{k=2}^{n} M_k \Delta x_k = \frac{1}{N} + \sum_{k=2}^{n} M_k \Delta x_k < \frac{\epsilon}{2} + \sum_{k=2}^{n} M_k \Delta x_k. \tag{1}$$

Decompose the set $\{2, \ldots, n\}$ into the disjoint union $A \cup A^{\mathsf{c}}$, where

$$A = \left\{ k \in \{2, \ldots, n\} : \text{there exists } j \in \{1, \ldots, N-1\} \text{ such that } \tfrac{1}{j} \in [x_{k-1}, x_k] \right\},$$

so that

$$\sum_{k=2}^{n} M_k \Delta x_k = \sum_{k \in A} M_k \Delta x_k + \sum_{k \notin A} M_k \Delta x_k. \tag{2}$$

Note that A can contain at most N elements and also that $M_k \le 1$ for any $k \in \{2, \ldots, n\}$. It follows that

$$\sum_{k \in A} M_k \Delta x_k < \sum_{k \in A} \frac{\epsilon}{2N} \le N \frac{\epsilon}{2N} = \frac{\epsilon}{2}. \tag{3}$$

Now suppose that $k \in \{2, \ldots, n\}$ is such that $k \notin A$, so that $f(x) = 0$ for all $x \in [x_{k-1}, x_k]$. Thus $M_k = 0$ and it follows that

$$\sum_{k \notin A} M_k \Delta x_k = 0. \tag{4}$$

Combining (1), (2), (3), and (4), we see that $U(f, P_\epsilon) < \epsilon$. As noted above, we have $L(f, P_\epsilon) = 0$ and thus $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

We have now shown that for any $\epsilon > 0$ there exists a partition P_ϵ of $[0, 1]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. We may conclude that f is integrable on $[0, 1]$.

Exercise 7.3.4. Let f and g be functions defined on (possibly different) closed intervals, and assume the range of f is contained in the domain of g so that the composition $g \circ f$ is properly defined.

(a) Show, by example, that it is not the case that if f and g are integrable, then $g \circ f$ is integrable.

Now decide on the validity of each of the following conjectures, supplying a proof or counterexample as appropriate.

(b) If f is increasing and g is integrable, then $g \circ f$ is integrable.

(c) If f is integrable and g is increasing, then $g \circ f$ is integrable.

Solution. (a) Let $f : [0, 1] \to \mathbf{R}$ be Thomae's function as defined in Exercise 7.3.2; as we showed there, f is integrable. Let $g : [0, 1] \to \mathbf{R}$ be given by

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x \le 1. \end{cases}$$

Theorem 7.3.2 shows that g is also integrable. However, note that since $f(x) = 0$ for irrational x and $0 < f(x) \leq 1$ for rational x, the composition $g \circ f : [0,1] \to \mathbf{R}$ is nothing but Dirichlet's function, which was shown to be non-integrable in Example 7.3.3.

(b) This is actually false, however the only counterexample I know of is quite involved and uses material from Section 7.6.

(c) See part (a) for a counterexample.

Exercise 7.3.5. Provide an example or give a reason why the request is impossible.

(a) A sequence $(f_n) \to f$ pointwise, where each f_n has at most a finite number of discontinuities but f is not integrable.

(b) A sequence $(g_n) \to g$ uniformly where each g_n has at most a finite number of discontinuities and g is not integrable.

(c) A sequence $(h_n) \to h$ uniformly where each h_n is not integrable but h is integrable.

Solution. (a) For each $n \in \mathbf{N}$ define $f_n : [0,1] \to \mathbf{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \left[\frac{1}{n}, 1\right], \\ 0 & \text{if } x \in \left[0, \frac{1}{n}\right), \end{cases}$$

and define $f : [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$$

Then $(f_n) \to f$ pointwise, each f_n has exactly one discontinuity at $x = \frac{1}{n}$, but f is not bounded and hence is not integrable.

(b) This is impossible. As discussed after Theorem 7.3.2, each g_n must be integrable. Exercise 7.2.5 then implies that g is integrable.

(c) For each $n \in \mathbf{N}$ define $h_n : [0,1] \to \mathbf{R}$ by

$$h_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

and let $h : [0,1] \to \mathbf{R}$ be identically zero. Then h is certainly integrable and a small modification of the argument given in Example 7.3.3 shows that each h_n is not integrable. Furthermore, since

$$\sup\{|h_n(x) - h(x)| : x \in [0,1]\} = \frac{1}{n} \to 0,$$

we have uniform convergence $(h_n) \to h$.

Exercise 7.3.6. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of all the rationals in $[0,1]$, and define

$$g_n(x) = \begin{cases} 1 & \text{if } x = r_n \\ 0 & \text{otherwise.} \end{cases}$$

(a) Is $G(x) = \sum_{n=1}^{\infty} g_n(x)$ integrable on $[0,1]$?

(b) Is $F(x) = \sum_{n=1}^{\infty} g_n(x)/n$ integrable on $[0,1]$?

Solution. (a) For irrational $x \in [0,1]$, we have $g_n(x) = 0$ for all $n \in \mathbf{N}$ and thus $G(x) = 0$. If $x \in [0,1]$ is rational, then $x = r_N$ for some $N \in \mathbf{N}$. Since $g_N(r_N) = 1$ and $g_n(r_N) = 0$ for $n \neq N$, we then have $G(r_N) = 1$. Hence G is in fact Dirichlet's function, which is not integrable (Example 7.3.3).

(b) We claim that F is integrable on $[0,1]$; notice that

$$F(x) = \begin{cases} \frac{1}{n} & \text{if } x = r_n \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

The density of the irrationals in $\mathbf{R}$ implies that $L(F,P) = 0$ for any partition P of $[0,1]$. Let $\epsilon > 0$ be given and set

$$D_{\epsilon/2} = \left\{x \in [0,1] : F(x) \geq \tfrac{\epsilon}{2}\right\}.$$

If $\frac{\epsilon}{2} > 1$ then $D_{\epsilon/2}$ is empty, since $0 \leq F(x) \leq 1$ for all $x \in [0,1]$. If $\frac{\epsilon}{2} \leq 1$ then let N be the smallest positive integer such that $\frac{1}{N} < \frac{\epsilon}{2}$; note that $N \geq 2$. It follows that

$$D_{\epsilon/2} = \{r_1, \ldots, r_{N-1}\},$$

so that $D_{\epsilon/2}$ is finite. We may now argue as in Exercise 7.3.2 (c) to obtain a partition P_ϵ of $[0,1]$ such that $U(F, P_\epsilon) < \epsilon$. Since $L(F, P_\epsilon) = 0$ we then have

$$U(F, P_\epsilon) - L(F, P_\epsilon) < \epsilon$$

and thus F is integrable on $[0,1]$. Furthermore, $\int_0^1 F = L(F) = 0$.

Exercise 7.3.7. Assume $f : [a,b] \to \mathbf{R}$ is integrable.

(a) Show that if g satisfies $g(x) = f(x)$ for all but a finite number of points in $[a,b]$, then g is integrable as well.

(b) Find an example to show that g may fail to be integrable if it differs from f at a countable number of points.

Solution. (a) Let $D = \{x \in [a,b] : f(x) \neq g(x)\}$. If D is empty then it is clear that g is integrable, so suppose that $D = \{c_1, \ldots, c_d\}$ for some $d \in \mathbf{N}$ and $c_1, \ldots, c_d \in [a,b]$. Let $\epsilon > 0$ be given. Because f is integrable, there exists a partition Q_ϵ of $[a,b]$ such that $U(f, Q_\epsilon) - L(f, Q_\epsilon) < \frac{\epsilon}{2}$. The integrability of f also implies that f is bounded; since g differs from f at only finitely many points, g must also be bounded, say by $R > 0$. Let $Q'_\epsilon = \{y_0, \ldots, y_l\}$ be the evenly spaced partition of $[a,b]$ such that

$$\Delta y_k < \frac{\epsilon}{4R(d+1)}$$

for each $k \in \{1, \ldots, l\}$, and let $P_\epsilon = Q_\epsilon \cup Q'_\epsilon = \{x_0, \ldots, x_n\}$ be the common refinement of Q_ϵ and Q'_ϵ, so that

$$\Delta x_k < \frac{\epsilon}{4R(d+1)}$$

for each $k \in \{1, \ldots, n\}$. Let

$$M_k^g = \sup\{g(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad m_k^g = \inf\{g(x) : x \in [x_{k-1}, x_k]\}$$

for each $k \in \{1, \ldots, n\}$, and define M_k^f and m_k^f similarly. Decompose the set $\{1, \ldots, n\}$ into the disjoint union $A \cup A^{\mathsf{c}}$, where

$$A = \{k \in \{1, \ldots, n\} : \text{there exists } j \in \{1, \ldots, d\} \text{ such that } c_j \in [x_{k-1}, x_k]\},$$

so that

$$U(g, P_\epsilon) - L(g, P_\epsilon) = \sum_{k=1}^{n} (M_k^g - m_k^g)\Delta x_k = \sum_{k \in A} (M_k^g - m_k^g)\Delta x_k + \sum_{k \notin A} (M_k^g - m_k^g)\Delta x_k. \quad (1)$$

Note that A can contain at most $d+1$ elements and also that $M_k^g - m_k^g \leq 2R$ for any $k \in \{1, \ldots, n\}$. It follows that

$$\sum_{k \in A} (M_k^g - m_k^g)\Delta x_k < \sum_{k \in A} 2R \frac{\epsilon}{4R(d+1)} \leq (d+1)\frac{\epsilon}{2(d+1)} = \frac{\epsilon}{2}. \quad (2)$$

Now suppose that $k \in \{1, \ldots, n\}$ is such that $k \notin A$, so that $f(x) = g(x)$ for all $x \in [x_{k-1}, x_k]$. It follows that $M_k^g - m_k^g = M_k^f - m_k^f$ and thus

$$\begin{aligned}\sum_{k \notin A} (M_k^g - m_k^g)\Delta x_k &= \sum_{k \notin A} (M_k^f - m_k^f)\Delta x_k \leq \sum_{k=1}^{n} (M_k^f - m_k^f)\Delta x_k \\ &= U(f, P_\epsilon) - L(f, P_\epsilon) \leq U(f, Q_\epsilon) - L(f, Q_\epsilon) < \frac{\epsilon}{2}. \quad (3)\end{aligned}$$

Combining (1), (2), and (3), we see that $U(g, P_\epsilon) - L(g, P_\epsilon) < \epsilon$. Because $\epsilon > 0$ was arbitrary, it follows that g is integrable on $[a, b]$.

(b) Let $f : [0,1] \to \mathbf{R}$ be identically zero, so that f is certainly integrable, and let $g : [0,1] \to \mathbf{R}$ be Dirichlet's function. Then g differs from f precisely on the countable set $\mathbf{Q} \cap [0,1]$ and yet g is not integrable.

Exercise 7.3.8. As in Exercise 7.3.6, let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rationals in $[0,1]$, but this time define

$$h_n(x) = \begin{cases} 1 & \text{if } r_n < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq r_n. \end{cases}$$

Show $H(x) = \sum_{n=1}^{\infty} h_n(x)/2^n$ is integrable on $[0,1]$ even though it has discontinuities at every rational point.

Solution. For a given $N \in \mathbf{N}$ let $H_N(x) = \sum_{n=1}^{N} h_n(x)/2^n$ and order the rationals $\{r_1, \ldots, r_N\}$ as $0 \leq r_{i_1} < \cdots < r_{i_N} \leq 1$. Then

$$H_N(x) = \begin{cases} 0 & \text{if } x \in [0, r_{i_1}], \\ \frac{1}{2} & \text{if } x \in (r_{i_1}, r_{i_2}], \\ \frac{3}{4} & \text{if } x \in (r_{i_2}, r_{i_3}], \\ \vdots & \qquad \vdots \\ 1 - \frac{1}{2^N} & \text{if } x \in (r_{i_N}, 1]. \end{cases}$$

Thus H_N is piecewise-constant on $[0,1]$. It is straightforward to argue that such functions are integrable (this is implied by Theorem 7.4.1). Now observe that

$$\left| \frac{h_n(x)}{2^n} \right| \leq \frac{1}{2^n}$$

for each $n \in \mathbf{N}$. Since the series $\sum_{n=1}^{\infty} 2^{-n}$ is a convergent geometric series, the Weierstrass M-Test (Corollary 6.4.5) implies that H_N converges uniformly to H; it follows from Exercise 7.2.5 that H is integrable on $[0,1]$.

Exercise 7.3.9 (Content Zero). A set $A \subseteq [a,b]$ has *content zero* if for every $\epsilon > 0$ there exists a finite collection of open intervals $\{O_1, O_2, \ldots, O_N\}$ that contain A in their union and whose lengths sum to ϵ or less. Using $|O_n|$ to refer to the length of each interval, we have

$$A \subseteq \bigcup_{n=1}^{N} O_n \quad \text{and} \quad \sum_{n=1}^{N} |O_n| \leq \epsilon.$$

(a) Let f be bounded on $[a, b]$. Show that if the set of discontinuous points of f has content zero, then f is integrable.

(b) Show that any finite set has content zero.

(c) Content zero sets do not have to be finite. They do not have to be countable. Show that the Cantor set C defined in Section 3.1 has content zero.

(d) Prove that

$$h(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C, \end{cases}$$

is integrable, and find the value of the integral.

Solution. (a) Suppose f is bounded by $R > 0$ on $[a, b]$ and let $\epsilon > 0$ be given. Because the set of discontinuous points of f has content zero, we can choose a partition Q of $[a, b]$ such that the discontinuities of f are contained in the interiors of subintervals whose total length is strictly less than $\frac{\epsilon}{4R}$. Letting K be the union of the remaining subintervals, we see that f is continuous on K and also that K is compact, being a finite union of closed and bounded intervals. Thus f is uniformly continuous on K and, as in the proof of Theorem 7.2.9, we may refine the partition Q, subdividing K as necessary, to obtain a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that the indices $\{1, \ldots, n\}$ can be expressed as the disjoint union $A \cup B$, where:

(i) f is continuous on $\bigcup_{k\in A}[x_{k-1}, x_k]$ and $M_k - m_k < \frac{\epsilon}{2(b-a)}$ for $k \in A$;

(ii) the discontinuities of f are contained inside $\bigcup_{k\in B}(x_{k-1}, x_k)$ and $\sum_{k\in B} \Delta x_k < \frac{\epsilon}{4R}$.

It follows that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k \\ &= \sum_{k\in A} (M_k - m_k)\Delta x_k + \sum_{k\in B} (M_k - m_k)\Delta x_k \\ &< \frac{\epsilon}{2(b-a)} \sum_{k\in A} \Delta x_k + 2R \sum_{k\in B} \Delta x_k \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus f is integrable on $[a, b]$.

(b) Let $A \subseteq \mathbf{R}$ be finite and let $\epsilon > 0$ be given. If A is empty then the open interval $\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right)$ suffices to show that A has content zero. Suppose therefore that A is not empty, say $A = \{x_1, \ldots, x_N\}$. For each $1 \le n \le N$, let

$$O_n = \left(x_n - \frac{\epsilon}{2N}, x_n + \frac{\epsilon}{2N}\right).$$

Then $A \subseteq \bigcup_{n=1}^{N} O_n$ and

$$\sum_{n=1}^{N} |O_n| = \sum_{n=1}^{N} \frac{\epsilon}{N} = \epsilon.$$

Thus A has content zero.

(c) Recall from Section 3.1 that the Cantor set C is defined as the intersection $C = \bigcap_{n=0}^{\infty} C_n$, where each C_n consists of 2^n closed intervals each of length 3^{-n} and such that

$$\cdots \subseteq C_2 \subseteq C_1 \subseteq C_0 = [0, 1].$$

Let $\epsilon > 0$ be given and choose $N \in \mathbf{N}$ such that

$$\left(\frac{2}{3}\right)^N + \left(\frac{1}{10}\right)^N < \epsilon.$$

The set C_N consists of 2^N closed intervals each of length 3^{-N}; suppose these intervals are $[x_k, y_k]$ for $1 \le k \le 2^N$, so that $y_k - x_k = 3^{-N}$. For each $1 \le k \le 2^N$, let

$$O_k = \left(x_k - \frac{1}{2^{N+1}10^N}, y_k + \frac{1}{2^{N+1}10^N}\right),$$

so that $[x_k, y_k] \subseteq O_k$ and

$$|O_k| = \frac{1}{3^N} + \frac{1}{2^N 10^N}.$$

Then

$$C = \bigcap_{n=0}^{\infty} C_n \subseteq C_N = \bigcup_{k=1}^{2^N} [x_k, y_k] \subseteq \bigcup_{k=1}^{2^N} O_k$$

$$\text{and} \quad \sum_{k=1}^{2^N} |O_k| = \sum_{k=1}^{2^N} \left(\frac{1}{3^N} + \frac{1}{2^N 10^N}\right) = \left(\frac{2}{3}\right)^N + \left(\frac{1}{10}\right)^N < \epsilon.$$

Thus C has content zero.

(d) Let

$$D_h = \{x \in \mathbf{R} : h \text{ is not continuous at } x\}.$$

We claim that $D_h = C$. First, suppose that $x \notin C$. Since C is closed, the complement of C is open and so there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq C^{\mathsf{c}}$. Thus h is constant on the proper interval $(x - \delta, x + \delta)$; it follows that h is continuous at x. Now suppose that $x \in C$. To show that h is not continuous at x, it will suffice to show that for any $\delta > 0$ there exists some $y \in (x - \delta, x + \delta)$ such that $y \notin C$. The existence of some δ such that this does not hold implies that C contains a proper interval. However, C cannot contain any proper intervals since it is totally disconnected (Exercise 3.4.8). Thus h is not continuous at x and our claim follows.

Abbott does not specify an interval to integrate h over, but in fact h is integrable over any interval $[a, b]$ for $a < b$. Let $g : [a, b] \to \mathbf{R}$ be the restriction of h to $[a, b]$. Then

$$D_g = \{x \in [a, b] : g \text{ is not continuous at } x\} = D_h \cap [a, b] = C \cap [a, b].$$

It is straightforward to verify that if a set has content zero, then the intersection of that set with any other set also has content zero. Thus, by part (c), D_g has content zero and it follows from part (a) that g is integrable. To calculate the integral of g, let P be any partition of $[a, b]$. As we noted before, C does not contain any proper intervals. It follows that any subinterval $[x_{k-1}, x_k]$ of the partition P contains some $x \notin C$, whence $g(x) = 0$. Thus $L(g, P) = 0$ and, because P was an arbitrary partition of $[a, b]$, it follows that

$$\int_a^b g = L(g) = 0.$$

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.