1 Section 6.4 Exercises

Exercises with solutions from Section 6.4 of [UA].

Exercise 6.4.1. Supply the details for the proof of the Weierstrass M-Test (Corollary 6.4.5).

Solution. Let $\epsilon > 0$ be given. Since the series $\sum_{n=1}^{\infty} M_n$ is convergent, its sequence of partial sums is a Cauchy sequence. Consequently, there exists an $N \in \mathbf{N}$ such that

$$n > m \geq N \quad \Longrightarrow \quad M_{m+1} + \cdots + M_n < \epsilon.$$

Suppose $x \in A$ and $n > m \geq N$. Then

$$|f_{m+1}(x) + \cdots + f_n(x)| \leq |f_{m+1}(x)| + \cdots + |f_n(x)| \leq M_{m+1} + \cdots + M_n < \epsilon.$$

It follows from Theorem 6.4.4 that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Exercise 6.4.2. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

(b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Solution. (a) This is true. Suppose that each g_n is defined on some domain $A \subseteq \mathbf{R}$. Note that Theorem 6.4.4 implies in particular that for any $\epsilon > 0$ there is an $N \in N$ such that

$$x \in A \text{ and } n \geq N \quad \Longrightarrow \quad |g_n(x)| \leq \epsilon.$$

Thus g_n converges uniformly to the zero function.

(b) This is true. Suppose that each f_n and each g_n is defined on some domain $A \subseteq \mathbf{R}$. Theorem 6.4.4 implies that for any $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$x \in A \text{ and } n > m \geq N \quad \Longrightarrow \quad g_{m+1}(x) + \cdots + g_n(x) < \epsilon;$$

note that we have used the non-negativity of each g_n here. Suppose $x \in A$ and $n > m \geq N$. By hypothesis we have

$$f_{m+1}(x) + \cdots + f_n(x) \leq g_{m+1}(x) + \cdots + g_n(x) < \epsilon.$$

Combining the above inequality with the non-negativity of each f_n and Theorem 6.4.4, we see that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

(c) This is false. For each $n \in \mathbf{N}$ define a function $f_n : \mathbf{R} \to \mathbf{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{x} & \text{if } x = n, \\ 0 & \text{otherwise,} \end{cases}$$

and let $f : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\sum_{n=1}^{\infty} f_n$ converges to f uniformly on $\mathbf{R}$. Observe that the partial sum function is

$$s_n(x) = f_1(x) + \cdots + f_n(x) = \begin{cases} \frac{1}{x} & \text{if } x = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that for any $x \in \mathbf{R}$

$$|s_n(x) - f(x)| \leq \tfrac{1}{n+1};$$

since this bound converges to zero and does not depend on x our claim follows.

Clearly, $\sup_{x \in \mathbf{R}} |f_n(x)| = \frac{1}{n}$. Since the harmonic series diverges, we see that the converse of the Weierstrass M-Test does not hold.

Exercise 6.4.3. (a) Show that

$$g(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all of $\mathbf{R}$.

(b) The function g was cited in Section 5.4 as an example of a continuous nowhere differentiable function. What happens if we try to use Theorem 6.4.3 to explore whether g is differentiable?

Solution. (a) Observe that

$$\left| \frac{\cos(2^n x)}{2^n} \right| \leq \frac{1}{2^n}$$

for every $x \in \mathbf{R}$. Since the series $\sum_{n=0}^{\infty} 2^{-n}$ is convergent, the Weierstrass M-Test implies that $g(x) = \sum_{n=0}^{\infty} 2^{-n} \cos(2^n x)$ converges uniformly on $\mathbf{R}$. Each $2^{-n} \cos(2^n x)$ is continuous on $\mathbf{R}$, so Theorem 6.4.2 implies that g is continuous on $\mathbf{R}$.

(b) To use Theorem 6.4.3, we should show that the series

$$\sum_{n=0}^{\infty}\left(\frac{\cos(2^n x)}{2^n}\right)' = -\sum_{n=0}^{\infty}\sin(2^n x).$$

converges uniformly on $\mathbf{R}$. However, this series does not even converge pointwise on $\mathbf{R}$. For example, consider the series of real numbers

$$\sum_{n=0}^{\infty}\sin(2^n).$$

To show that this series is divergent, we will show that the sequence $(\sin(2^n))$ does not converge to zero. To see this, consider the following two cases.

Case 1. If there exists an $N \in \mathbf{N}$ such that $|\sin(2^{n+1})| > |\sin(2^n)|$ for all $n \geq N$, then it must be the case that $\sin(2^n) \not\to 0$.

Case 2. If there does not exist such an N, then there must be infinitely many natural numbers n such that $|\sin(2^{n+1})| \leq |\sin(2^n)|$. Consider such an n. Using the identity

$$\sin(2^{n+1}) = 2\sin(2^n)\cos(2^n)$$

and the fact that $\sin(2^n) \neq 0$ for any $n \in \mathbf{N}$, we see that $|\cos(2^n)| \leq \frac{1}{2}$. The Pythagorean identity then implies that $|\sin(2^n)| \geq \frac{\sqrt{3}}{2}$. So in this case, the sequence $(\sin(2^n))$ satisfies $|\sin(2^n)| \geq \frac{\sqrt{3}}{2}$ infinitely often and hence $\sin(2^n) \not\to 0$.

So Theorem 6.4.3 does not allow us to conclude anything about the differentiability of g.

Exercise 6.4.4. Define

$$g(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{(1+x^{2n})}.$$

Find the values of x where the series converges and show that we get a continuous function on this set.

Solution. For $|x| = 1$ we have $g(x) = \sum_{n=0}^{\infty}\frac{1}{2}$, which diverges. For $|x| > 1$ we have

$$\frac{x^{2n}}{1+x^{2n}} = \frac{1}{x^{-2n}+1} \to 1 \text{ as } n \to \infty$$

and thus $g(x)$ diverges.

Now suppose that $r > 0$ is such that $0 \leq r^2 < 1$. Observe that for all $x \in [-r, r]$ we have

$$0 \leq \frac{x^{2n}}{1 + x^{2n}} \leq x^{2n} \leq r^{2n}.$$

Since $\sum_{n=0}^{\infty} r^{2n}$ is a convergent geometric series, the Weierstrass M-Test implies that g converges uniformly on $[-r, r]$. Since any $x \in (-1, 1)$ is contained inside an interval of this form, we may conclude that g converges and is continuous at each $x \in (-1, 1)$ (Theorem 6.4.2). Combining this with our previous discussion, we see that g converges pointwise precisely on the open interval $(-1, 1)$.

Exercise 6.4.5. (a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots$$

is continuous on $[-1, 1]$.

(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges for every x in the half-open interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Solution. (a) For any $x \in [-1, 1]$ we have

$$\left| \frac{x^n}{n^2} \right| \leq \frac{1}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, the Weierstrass M-Test allows us to conclude that h converges uniformly on $[-1, 1]$ and Theorem 6.4.2 then implies that h is continuous on $[-1, 1]$, since each $\frac{x^n}{n^2}$ is continuous here.

(b) Observe that

$$\left| \frac{x^n}{n} \right| \leq |x_0|^n$$

for every $x \in [-x_0, x_0]$. Since $\sum_{n=1}^{\infty} |x_0|^n$ is a convergent geometric series, the Weierstrass M-Test implies that f converges uniformly on $[-x_0, x_0]$. Theorem 6.4.2 allows us to conclude that f is continuous on $[-x_0, x_0]$, since each $\frac{x^n}{n}$ is continuous here.

Exercise 6.4.6. Let

$$f(x) = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \frac{1}{x+4} - \cdots.$$

Show f is defined for all $x > 0$. Is f continuous on $(0, \infty)$? How about differentiable?

Solution. Observe that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n}.$$

The term-by-term differentiated series is

$$-\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(x+n)^2}.$$

Note that

$$\left| \frac{(-1)^{n+1}}{(x+n)^2} \right| \leq \frac{1}{n^2}$$

for any $x \in (0, \infty)$ and $n \in \mathbf{N}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, the Weierstrass M-Test implies that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(x+n)^2}$ converges uniformly on $(0, \infty)$. It follows that the term-by-term differentiated series converges uniformly on $(0, \infty)$ and hence we may invoke Theorem 6.4.3 to see that the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{x+n}$ converges uniformly on $(0, \infty)$ to a differentiable function f; this also implies that f is defined and continuous on $(0, \infty)$.

Exercise 6.4.7. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}.$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

(b) Can we determine if f is twice-differentiable?

Solution. (a) Let $f_k : \mathbf{R} \to \mathbf{R}$ be given by $f_k(x) = \frac{\sin(kx)}{k^3}$. Observe that

$$|f_k'(x)| = \left| \frac{\cos(kx)}{k^2} \right| \leq \frac{1}{k^2}$$

for all $x \in \mathbf{R}$. The Weierstrass M-Test then implies that the series

$$\sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

converges uniformly on $\mathbf{R}$; since each f_k' is continuous on $\mathbf{R}$, Theorem 6.4.2 implies that $\sum_{k=1}^{\infty} f_k'(x)$ is also continuous on $\mathbf{R}$. Combining our previous discussion with Theorem 6.4.3 and the fact that $f(0) = 0$, we see that $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$ converges uniformly on $\mathbf{R}$ to a differentiable function f, that

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2},$$

and that f' is continuous on $\mathbf{R}$.

(b) We will show that Theorem 6.4.3 cannot be used to determine if f is twice-differentiable on $\mathbf{R}$, by showing that the series of second derivatives

$$\sum_{k=1}^{\infty} f_k''(x) = -\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$

does not converge uniformly on $\mathbf{R}$. To see this, we will use the negation of Theorem 6.4.4. Let $N \in \mathbf{N}$ be given and set $x = \frac{\pi}{4N}$. For any $N + 1 \le k \le 2N$, we then have $\frac{\pi}{4} \le kx \le \frac{\pi}{2}$ and hence $\sin(kx) \ge \frac{1}{\sqrt{2}}$. Now observe that

$$\left| \sum_{k=N+1}^{2N} \frac{\sin(kx)}{k} \right| \ge \frac{1}{\sqrt{2}} \sum_{k=N+1}^{2N} \frac{1}{k} \ge \frac{1}{\sqrt{2}} \sum_{k=N+1}^{2N} \frac{1}{2N} = \frac{1}{2\sqrt{2}}.$$

It follows from Theorem 6.4.4 that the convergence of the series $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$ is not uniform on $\mathbf{R}$. Consequently, we may not use Theorem 6.4.3 to conclude anything about the twice-differentiability of f.

Exercise 6.4.8. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}.$$

Where is f defined? Continuous? Differentiable? Twice-differentiable?

Solution. Let $f_k : \mathbf{R} \to \mathbf{R}$ be given by $f_k(x) = \frac{\sin(x/k)}{k}$, so that $f(x) = \sum_{k=1}^{\infty} f_k(x)$. Observe that

$$f_k'(x) = \frac{\cos\left(\frac{x}{k}\right)}{k^2} \quad \text{and} \quad f_k''(x) = -\frac{\sin\left(\frac{x}{k}\right)}{k^3}.$$

The bound $|f_k''(x)| \le \frac{1}{k^3}$ for all $x \in \mathbf{R}$ combined with the Weierstrass M-Test shows that the series $\sum_{k=1}^{\infty} f_k''(x)$ converges uniformly on $\mathbf{R}$. Since

$$f(0) = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} f_k'(0) = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

are both convergent, two applications of Theorem 6.4.3 show that $\sum_{k=1}^{\infty} f_k'(x)$ and $\sum_{k=1}^{\infty} f_k(x)$ converge uniformly on $\mathbf{R}$. Furthermore,

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x) \quad \text{and} \quad f''(x) = \sum_{k=1}^{\infty} f_k''(x).$$

In particular, f is defined and continuous on $\mathbf{R}$.

Exercise 6.4.9. Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

(a) Show that h is a continuous function defined on all of $\mathbf{R}$.

(b) Is h differentiable? If so, is the derivative function h' continuous?

Solution. (a) We have the bound

$$\frac{1}{x^2 + n^2} \le \frac{1}{n^2}$$

for all $x \in \mathbf{R}$; the Weierstrass M-Test now implies that the series $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ converges uniformly on $\mathbf{R}$. Since each $\frac{1}{x^2+n^2}$ is continuous on $\mathbf{R}$, Theorem 6.4.2 allows us to conclude that h is also continuous on $\mathbf{R}$.

(b) The term-by-term differentiated series is

$$-\sum_{n=1}^{\infty} \frac{2x}{(x^2 + n^2)^2}.$$

Note that

$$|x| \le 1 \text{ and } n \ge 2 \quad \Longrightarrow \quad \left| \frac{2x}{(x^2 + n^2)^2} \right| \le \frac{2}{n^4} \le \frac{1}{n^2}$$

and that

$$|x| > 1 \quad \Longrightarrow \quad \left| \frac{2x}{(x^2 + n^2)^2} \right| = \frac{2|x|}{x^4 + 2x^2n^2 + n^4} \le \frac{1}{|x|n^2} \le \frac{1}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and each summand $\frac{2x}{(x^2+n^2)^2}$ is continuous on $\mathbf{R}$, the Weierstrass M-Test and Theorem 6.4.2 imply that the series $\sum_{n=1}^{\infty} \frac{2x}{(x^2+n^2)^2}$ converges uniformly on $\mathbf{R}$ to a continuous function. We showed in part (a) that h converges uniformly on $\mathbf{R}$ and thus by Theorem 6.4.3 we have

$$h'(x) = -\sum_{n=1}^{\infty} \frac{2x}{(x^2+n^2)^2}.$$

Exercise 6.4.10. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the set of rational numbers. For each $r_n \in \mathbf{Q}$, define

$$u_n(x) = \begin{cases} 1/2^n & \text{for } x > r_n \\ 0 & \text{for } x \leq r_n. \end{cases}$$

Now, let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove that h is a monotone function defined on all of $\mathbf{R}$ that is continuous at every irrational point.

Solution. Observe that $|u_n(x)| \leq 2^{-n}$ for all $x \in \mathbf{R}$. Since $\sum_{n=1}^{\infty} 2^{-n}$ is a convergent geometric series, the Weierstrass M-Test implies that h converges uniformly on $\mathbf{R}$. To see that h is strictly increasing, let $x < y$ be given real numbers. There are a countable infinity of rational numbers contained in $[x, y)$, which we can enumerate as a subsequence $\{r_{n_1}, r_{n_2}, r_{n_3}, \ldots\}$ of the sequence $\{r_1, r_2, r_3, \ldots\}$. Now,

$$h(y) - h(x) = \sum_{n=1}^{\infty} (u_n(y) - u_n(x)).$$

Let $n \in \mathbf{N}$ be given and consider the following three cases.

Case 1. If $r_n < x < y$, then $u_n(y) = u_n(x) = 2^{-n}$ and thus $u_n(y) - u_n(x) = 0$.

Case 2. If $x < y \leq r_n$, then $u_n(y) = u_n(x) = 0$ and thus $u_n(y) - u_n(x) = 0$.

Case 3. If $x \leq r_n < y$, then $r_n \in \{r_{n_1}, r_{n_2}, r_{n_3}, \ldots\}$. Thus $n = n_k$ for some unique $k \in \mathbf{N}$, so that $u_n(y) = 2^{-n_k}$ and $u_n(x) = 0$, which gives us $u_n(y) - u_n(x) = 2^{-n_k}$.

It follows that

$$h(y) - h(x) = \sum_{k=1}^{\infty} 2^{-n_k} > 0.$$

To see that h is continuous at every irrational point, let us first show that each u_n is continuous at every irrational point. Fix $n \in \mathbf{N}, y \in \mathbf{I}$ (where $\mathbf{I}$ is the set of irrational numbers), and set $\delta := |y - r_n|$; δ must be positive since y is not rational. There are two cases:

Case 1. If $y < r_n$, then $u_n(x) = 0$ for all $x \in (y - \delta, y + \delta)$ and hence u_n is continuous at y.

Case 2. If $y > r_n$, then $u_n(x) = 2^{-n}$ for all $x \in (y - \delta, y + \delta)$ and hence u_n is continuous at y.

So each summand u_n is continuous on $\mathbf{I}$, and we showed earlier that h converges uniformly on $\mathbf{R}$ and so in particular uniformly on $\mathbf{I}$; Theorem 6.4.2 allows us to conclude that h is also continuous on $\mathbf{I}$.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.