Understanding Analysis Solutions

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Contents

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1. (a) Prove that $\sqrt{3}$ is irrational. Does the same argument work to show that $\sqrt{6}$ is irrational?

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Solution. (a) Suppose there was a rational number $p = \frac{m}{n}$, which we may assume is in lowest terms, such that $p^2 = 3$. Then $m^2 = 3n^2$, so that m^2 is divisible by 3. This implies that m is divisible by 3. To see this, observe that for any $k \in \mathbf{Z}$ we have

$$(3k+1)^2 = 3(3k^2+2k)+1 \qquad \text{and} \qquad (3k+2)^2 = 3(3k^2+4k+1)+1.$$

Since m is of the form $3k+1$ or $3k+2$ for some integer k if m is not divisible by 3, it follows that

if m is not divisible by 3, then m^2 is not divisible by 3;

the contrapositive of this statement is what we wanted to see.

Thus we may write $m = 3k$ for some $k \in \mathbf{Z}$ and substitute this into the equation $m^2 = 3n^2$ to obtain the equation $n^2 = 3k^2$, which implies that n is also divisible by 3. So m and n share the factor 3; this is a contradiction since we assumed that m and n had no common factors. We may conclude that there is no rational number whose square is 3.

The same argument works to show that there is no rational number whose square is 6; the crux of this argument is the implication

if m^2 is divisible by 6, then m is divisible by 6.

This can be seen using what we have already proved. If m^2 is divisible by $6 = 2 \cdot 3$, then m^2 is divisible by 2 and 3. It follows that m is divisible by 2 and 3 and hence that m is divisible by 6.

(b) The argument breaks down when we try to assert that

if m^2 is divisible by 4, then m is divisible by 4.

This implication is false. For example, $2^2 = 4$ is divisible by 4 but 2 is not divisible by 4.

Exercise 1.2.2. Show that there is no rational number r satisfying $2^r = 3$.

Solution. Suppose there was a rational number $r = \frac{m}{n}$, which we may assume is in lowest terms with $n > 0$, such that $2^r = 3$. This implies that $2^m = 3^n$. Since $n > 0$ gives $3^n \geq 3$ and $2^m < 2$ for $m \leq 0$, it must be the case that $m > 0$. Then the left-hand side of the equation $2^m = 3^n$ is a positive even integer whereas the right-hand side is a positive odd integer, which is a contradiction. We may conclude that there is no rational number r such that $2^r = 3$.

Exercise 1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution. (a) This is false, as Example 1.2.2 shows.

(b) This is true and we can use the following lemma to prove it.

Lemma L.1. If $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of positive integers, i.e., $a_{n+1} \leq a_n$ and $a_n \geq 1$ for all $n \in \mathbf{N}$, then $(a_n)_{n=1}^{\infty}$ must be eventually constant. That is, there exists an $N \in \mathbf{N}$ such that $a_n = a_N$ for all $n > N$.

Proof. Let A be the set $\{a_n : n \in \mathbf{N}\}$, which is non-empty and bounded below by 1. It follows from the well-ordering principle that A has a least element, say $\min A = a_N$ for some $N \in \mathbf{N}$. Let $n > N$ be given. It cannot be the case that $a_n < a_N$, since this would contradict that a_N is the least element of A, so we must have $a_n \geq a_N$. By assumption $a_n \leq a_N$ and so we may conclude that $a_n = a_N$. $\square$

Consider the sequence $(|A_n|)_{n=1}^{\infty}$, where $|A_n|$ is the number of elements contained in A_n. This is a sequence of positive integers, because each A_n is finite and non-empty, and furthermore this sequence is decreasing because the sets $(A_n)_{n=1}^{\infty}$ are nested:

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots .$$

We may now invoke Lemma L.1 to obtain an $N \in \mathbf{N}$ such that $|A_n| = |A_N|$ for all $n > N$. Combining this equality with the inclusion $A_n \subseteq A_N$ for each $n > N$, we see that $A_n = A_N$ for all $n > N$. It follows that $\bigcap_{n=1}^{\infty} A_n = A_N$, which by assumption is finite and non-empty.

(c) This is false. Consider $A = B = \emptyset$ and $C = \{0\}$. Then

$$A \cap (B \cup C) = \emptyset \neq \{0\} = (A \cap B) \cup C.$$

(d) This is true, since

$$x \in A \cap (B \cap C) \iff x \in A \text{ and } x \in (B \cap C) \iff x \in A \text{ and } (x \in B \text{ and } x \in C)$$

$$\iff (x \in A \text{ and } x \in B) \text{ and } x \in C \iff x \in (A \cap B) \text{ and } x \in C \iff x \in (A \cap B) \cap C,$$

where we have used that logical conjunction ("and") is associative for the third equivalence. It follows that x belongs to $A \cap (B \cap C)$ if and only if x belongs to $(A \cap B) \cap C$, which is to say that $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) This is true, since

$$x \in A \cap (B \cup C) \iff x \in A \text{ and } x \in (B \cup C) \iff x \in A \text{ and } (x \in B \text{ or } x \in C)$$
$$\iff (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \iff x \in (A \cap B) \text{ or } x \in (A \cap C)$$
$$\iff x \in (A \cap B) \cup (A \cap C),$$

where we have used that logical conjunction ("and") distributes over logical disjunction ("or") for the third equivalence. It follows that x belongs to $A \cap (B \cup C)$ if and only if x belongs to $(A \cap B) \cup (A \cap C)$, which is to say that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.