1 Section 2.8 Exercises

Exercises with solutions from Section 2.8 of [UA].

Exercise 2.8.1. Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{nn}$. How does this value compare to the two iterated values for the sum already computed?

Solution. The array in question is

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & -1 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $a_{ij} = 1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$. If we let $f(j)$ be the sum of the first row up to the j^{th} column, then using the formula for the partial sums of a geometric series, we find that

$$\begin{aligned} f(j) &= \begin{cases} -1 & \text{if } j = 1, \\ -1 + \frac{1}{2} + \cdots + \frac{1}{2^{j-1}} = -\frac{1}{2^{j-1}} & \text{if } j \geq 2 \end{cases} \\ &= -\frac{1}{2^{j-1}}. \end{aligned}$$

Since subsequent rows are simply the first row shifted along, it is clear that $s_{11} = f(1), s_{22} = f(1) + f(2), s_{33} = f(1) + f(2) + f(3)$, and in general

$$s_{nn} = \sum_{j=1}^{n} f(j) = \sum_{j=1}^{n} \frac{-1}{2^{j-1}} = -\sum_{j=0}^{n-1} \frac{1}{2^j}.$$

It follows that

$$\lim_{n\to\infty} s_{nn} = -\sum_{j=0}^{\infty} \frac{1}{2^j} = -2.$$

At the beginning of Section 2.1, we found that summing along the rows first gave a value for the double sum of 0, whereas summing down the columns first gave a value of -2.

Exercise 2.8.2. Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

converges (meaning for each fixed $i \in \mathbf{N}$ the series $\sum_{j=1}^{\infty}|a_{ij}|$ converges to some real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

converges.

Solution. For each $i \in \mathbf{N}$, Theorem 2.7.6 implies that the series $\sum_{j=1}^{\infty} a_{ij}$ converges to some real number c_i. Observe that

$$0 \leq |c_i| = \left|\sum_{j=1}^{\infty} a_{ij}\right| \leq \sum_{j=1}^{\infty}|a_{ij}| = b_i.$$

Since $\sum_{i=1}^{\infty} b_i$ converges, the Comparison Test implies that the series $\sum_{i=1}^{\infty} c_i$ is absolutely convergent and hence convergent.

Exercise 2.8.3. (a) Prove that (t_{nn}) converges.

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges.

Solution. (a) Since $|a_{ij}| \geq 0$ for all positive integers i and j, the sequence (t_{nn}) is increasing and bounded above by the real number $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$. Hence by the Monotone Convergence Theorem, (t_{nn}) converges.

(b) Suppose $n > m$ are positive integers. By examining the array

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} & \cdots & a_{mn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} & \cdots & a_{nn} \end{bmatrix},$$

we see that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} = \sum_{i=1}^{m}\sum_{j=m+1}^{n} a_{ij} + \sum_{i=m+1}^{n}\sum_{j=1}^{n} a_{ij}.$$

(The sum of the top right "square" and the bottom "rectangle" of the array.) Let $\epsilon > 0$ be given. Since (t_{nn}) is a Cauchy sequence, there exists an $N \in \mathbf{N}$ such that $n > m \geq N$ implies that

$$|t_{nn} - t_{mm}| = t_{nn} - t_{mm} < \epsilon.$$

For such n and m, observe that

$$\begin{aligned}
|s_{nn} - s_{mm}| &= \left| \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \right| \\
&= \left| \sum_{i=1}^{m} \sum_{j=m+1}^{n} a_{ij} + \sum_{i=m+1}^{n} \sum_{j=1}^{n} a_{ij} \right| \\
&\leq \sum_{i=1}^{m} \sum_{j=m+1}^{n} |a_{ij}| + \sum_{i=m+1}^{n} \sum_{j=1}^{n} |a_{ij}| \\
&= t_{nn} - t_{mm} \\
&< \epsilon.
\end{aligned}$$

It follows that (s_{nn}) is a Cauchy sequence and hence convergent.

Exercise 2.8.4. (a) Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbf{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

(b) Now, show that there exists an N such that

$$|s_{mn} - S| < \epsilon$$

for all $m, n \geq N$.

Solution. (a) By the approximation property for suprema, there exist positive integers m', n' such that $B - \frac{\epsilon}{2} < t_{m'n'} \leq B$. Set $N_1 = \max\{m', n'\}$. Since each $|a_{ij}|$ is positive, (t_{mn}) is increasing in both m and n; it follows that for $m, n \geq N_1$ we have $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

(b) Since $\lim_{n\to\infty} s_{nn} = S$, there is an $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies that $|s_{nn} - S| < \frac{\epsilon}{2}$.

Set $N = \max\{N_1, N_2\}$ and suppose that $m, n > N$. Similarly to Exercise 2.8.3 (b), we have

$$\begin{aligned}
|s_{mn} - s_{NN}| &= \left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} - \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\right| \\
&= \left|\sum_{i=1}^{N}\sum_{j=N+1}^{n} a_{ij} + \sum_{i=N+1}^{m}\sum_{j=1}^{n} a_{ij}\right| \\
&\le \sum_{i=1}^{N}\sum_{j=N+1}^{n} |a_{ij}| + \sum_{i=N+1}^{m}\sum_{j=1}^{n} |a_{ij}| \\
&= t_{mn} - t_{NN} \\
&\le B - t_{NN} \\
&< \tfrac{\epsilon}{2}.
\end{aligned}$$

It follows that

$$|s_{mn} - S| \le |s_{mn} - s_{NN}| + |s_{NN} - S| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

Exercise 2.8.5. (a) Show that for all $m \ge N$

$$|(r_1 + r_2 + \cdots + r_m) - S| \le \epsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges to S.

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$, converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

Solution. (a) Suppose that $n \ge N$. Then

$$\begin{aligned}
|(r_1 + r_2 + \cdots + r_m) - S| &\le |(r_1 + r_2 + \cdots + r_m) - s_{mn}| + |s_{mn} - S| \\
&< \left|(r_1 + r_2 + \cdots + r_m) - \left(\sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj}\right)\right| + \epsilon \\
&\le \left|r_1 - \sum_{j=1}^{n} a_{1j}\right| + \left|r_2 - \sum_{j=1}^{n} a_{2j}\right| + \cdots + \left|r_m - \sum_{j=1}^{n} a_{mj}\right| + \epsilon.
\end{aligned}$$

Since this is true for any $n \ge N$ and for any given i we have $\sum_{j=1}^{\infty} a_{ij} = r_i$, taking the limit in n on both sides of the inequality

$$|(r_1 + r_2 + \cdots + r_m) - S| < \left|r_1 - \sum_{j=1}^{n} a_{1j}\right| + \left|r_2 - \sum_{j=1}^{n} a_{2j}\right| + \cdots + \left|r_m - \sum_{j=1}^{n} a_{mj}\right| + \epsilon$$

gives us

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \epsilon.$$

It follows that $\lim_{m\to\infty} \left(\sum_{i=1}^m r_i\right) = S$, i.e. $\sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} = S$.

(b) Fix $j \in \mathbf{N}$ and let (x_n) be the sequence of partial sums of the series $\sum_{i=1}^\infty |a_{ij}|$, i.e.

$$x_n = |a_{1j}| + |a_{2j}| + \cdots + |a_{nj}|.$$

Since each $|a_{ij}|$ is a term of the convergent series $\sum_{j=1}^\infty |a_{ij}| = r_i$, which has only non-negative terms, we see that $|a_{ij}| \leq r_i$, so that

$$x_n \leq r_1 + r_2 + \cdots + r_n \leq \sum_{i=1}^\infty r_i,$$

where the last inequality follows since each r_i is non-negative. So (x_n) is an increasing and bounded sequence and hence converges by the Monotone Convergence Theorem. It follows that $\sum_{i=1}^\infty a_{ij}$ converges to some (non-negative) real number c_j.

Let $\epsilon > 0$ be given. As in Exercise 2.8.4, there is an $N \in \mathbf{N}$ such that $|s_{mn} - S| < \epsilon$ for all $m, n \geq N$. We can write s_{mn} as

$$s_{mn} = \sum_{i=1}^m a_{i1} + \sum_{i=1}^m a_{i2} + \cdots + \sum_{i=1}^m a_{in}.$$

Suppose that $m, n \geq N$. Then

$$\begin{aligned}
|(c_1 + c_2 + \cdots + c_n) - S| &\leq |(c_1 + c_2 + \cdots + c_n) - s_{mn}| + |s_{mn} - S| \\
&< \left|(c_1 + c_2 + \cdots + c_n) - \left(\sum_{i=1}^m a_{i1} + \sum_{i=1}^m a_{i2} + \cdots + \sum_{i=1}^m a_{in}\right)\right| + \epsilon \\
&\leq \left|c_1 - \sum_{i=1}^m a_{i1}\right| + \left|c_2 - \sum_{i=1}^m a_{i2}\right| + \cdots + \left|c_n - \sum_{i=1}^m a_{in}\right| + \epsilon.
\end{aligned}$$

Since this is true for any $m \geq N$ and for any given j we have $\sum_{i=1}^\infty a_{ij} = c_j$, taking the limit in m on both sides of the inequality

$$|(c_1 + c_2 + \cdots + c_n) - S| < \left|c_1 - \sum_{i=1}^m a_{i1}\right| + \left|c_2 - \sum_{i=1}^m a_{i2}\right| + \cdots + \left|c_n - \sum_{i=1}^m a_{in}\right| + \epsilon$$

gives us

$$|(c_1 + c_2 + \cdots + c_n) - S| \leq \epsilon.$$

It follows that $\lim_{n\to\infty} \left(\sum_{j=1}^n c_j\right) = S$, i.e. $\sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij} = S$.

Exercise 2.8.6. (a) Assuming the hypothesis—and hence the conclusion—of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

(b) Imitate the strategy in the proof of Theorem 2.8.1 to show that $\sum_{k=2}^{\infty} d_k$ converges to $S = \lim_{n\to\infty} s_{nn}$.

Solution. (a) Observe that

$$\begin{aligned}
|d_2| &= |a_{11}| \\
|d_2| + |d_3| &= |a_{11}| + |a_{12} + a_{21}| \le (|a_{11}| + |a_{12}|) + |a_{21}| \\
|d_2| + |d_3| + |d_4| &= |a_{11}| + |a_{12} + a_{21}| + |a_{13} + a_{22} + a_{31}| \\
&\le (|a_{11}| + |a_{12}| + |a_{13}|) + (|a_{21}| + |a_{22}|) + |a_{31}|,
\end{aligned}$$

and in general for $n \ge 2$,

$$\sum_{k=2}^{n} |d_k| \le \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} |a_{ij}| \le \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|,$$

where the last inequality follows since each $|a_{ij}|$ is non-negative. By assumption $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$ is some real number, so the sequence $\sum_{k=2}^{n} |d_k|$ is increasing and bounded above and hence converges by the Monotone Convergence Theorem.

(b) For $k \ge 2$, let

$$e_k = |a_{1,k-1}| + |a_{2,k-2}| + \cdots + |a_{k-1,1}|.$$

By considering Figure 1, which shows the special case $n = 6$, we see that for each $n \ge 2$,

$$s_{nn} - \sum_{k=2}^{n} d_k = \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} a_{ij} \quad \text{and} \quad t_{nn} - \sum_{k=2}^{n} e_k = \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} |a_{ij}|.$$

This implies that

$$\left| s_{nn} - \sum_{k=2}^{n} d_k \right| = \left| \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} a_{ij} \right| \le \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} |a_{ij}| = t_{nn} - \sum_{k=2}^{n} e_k. \tag{1}$$

Let $\epsilon > 0$ be given. Since $\lim_n s_{nn} = S$ and (t_{nn}) is an increasing Cauchy sequence, there are positive integers N_1, N_2 such that

$$n \ge N_1 \implies |s_{nn} - S| < \tfrac{\epsilon}{2} \quad \text{and} \quad n > m \ge N_2 \implies t_{nn} - t_{mm} < \tfrac{\epsilon}{2}. \tag{2}$$

Set $N = \max\{N_1, 2N_2\}$ and suppose $n \geq N$. Since $n \geq 2N_2$, each term of $t_{N_2N_2}$ appears in $\sum_{k=2}^{n} e_k$ (see Figure 1, which has the special case $N_2 = 2$); it follows that $t_{N_2N_2} \leq \sum_{k=2}^{n} e_k$. Then by (1) and (2) we have

$$\left| s_{nn} - \sum_{k=2}^{n} d_k \right| \leq t_{nn} - t_{N_2N_2} < \tfrac{\epsilon}{2},$$

which implies

$$\left| \sum_{k=2}^{n} d_k - S \right| \leq |s_{nn} - S| + \left| s_{nn} - \sum_{k=2}^{n} d_k \right| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

It follows that $\lim_n \sum_{k=2}^{n} d_k = S$.

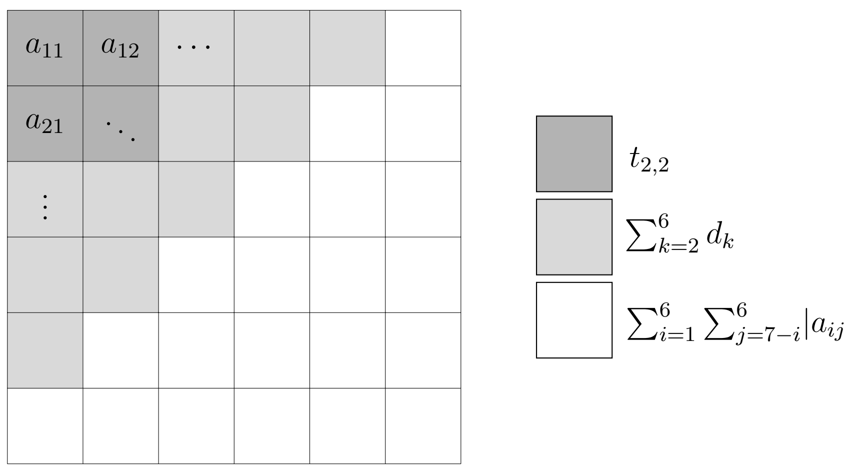

Figure 1: $s_{6,6}$

Exercise 2.8.7. Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges so that we may apply Theorem 2.8.1.

(b) Let $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1$.

Solution. (a) Let $A' = \sum_{i=1}^{\infty} |a_i|$ and $B' = \sum_{j=1}^{\infty} |b_j|$. Suppose $i \in \mathbf{N}$ is fixed. Then

$$\sum_{j=1}^{n} |a_i b_j| = |a_i| \sum_{j=1}^{n} |b_j| \to |a_i| B' \text{ as } n \to \infty.$$

It follows that

$$\sum_{i=1}^{n} |a_i| B' = B' \sum_{i=1}^{n} |a_i| \to A'B' \text{ as } n \to \infty,$$

i.e.

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j| = A'B'.$$

(b) For each $n \in \mathbf{N}$ we have

$$s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j = \left(\sum_{i=1}^{n} a_i \right) \left(\sum_{j=1}^{n} b_j \right).$$

The Algebraic Limit Theorem now implies that $\lim_{n \to \infty} s_{nn} = AB$, and Theorem 2.8.1 then gives the desired result.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.