Linear Algebra Done Right Solutions

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Contents

Chapter 1. Vector Spaces

1.A. $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 1.A.1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi$ and $\beta = u + vi$, then

$$\alpha + \beta = (x+u) + (y+v)i = (u+x) + (v+y)i = \beta + \alpha,$$

where we have used the commutativity of addition in $\mathbf{R}$.

Exercise 1.A.2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}(\alpha + \beta) + \lambda &= ((x+u) + (y+v))i + \lambda = ((x+u)+s) + ((y+v)+t)i \\ &= (x + (u+s)) + (y + (v+t))i = \alpha + ((u+s) + (v+t)i) = \alpha + (\beta + \lambda),\end{aligned}$$

where we have used the associativity of addition in $\mathbf{R}$.

Exercise 1.A.3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}(\alpha\beta)\lambda &= [(xu - yv) + (xv + yu)i]\lambda \\ &= [(xu - yv)s - (xv + yu)t] + [(xu - yv)t + (xv + yu)s]i \\ &= [(xu)s - (yv)s - (xv)t - (yu)t] + [(xu)t - (yv)t + (xv)s + (yu)s]i \\ &= [x(us) - x(vt) - y(ut) - y(vs)] + [x(ut) + x(vs) + y(us) - y(vt)]i \\ &= [x(us - vt) - y(ut + vs)] + [x(ut + vs) + y(us - vt)]i \\ &= \alpha[(us - vt) + (ut + vs)i] \\ &= \alpha(\beta\lambda),\end{aligned}$$

where we have used several algebraic properties of $\mathbf{R}$.

Exercise 1.A.4. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}\lambda(\alpha + \beta) &= [s(x+u) - t(y+v)] + [s(y+v) + t(x+u)i] \\ &= (sx + su - ty - tv) + (sy + sv + tx + tu)i \\ &= [(sx - ty) + (sy + tx)i] + [(su - tv) + (sv + tu)i] \\ &= \lambda\alpha + \lambda\beta,\end{aligned}$$

where we have used distributivity in $\mathbf{R}$.

Exercise 1.A.5. Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Solution. Suppose that $\alpha = x + yi$. Let $\beta = -x - yi$ and observe that

$$\alpha + \beta = (x - x) + (y - y)i = 0 + 0i = 0.$$

To see that β is unique, suppose that β' also satisfies $\alpha + \beta' = 0$ and notice that

$$\beta = \beta = 0 = \beta + (\alpha + \beta') = (\alpha + \beta) + \beta' = 0 + \beta' = \beta',$$

where we have used the associativity of addition in $\mathbf{C}$ (Exercise 1.A.2) and the commutativity of addition in $\mathbf{C}$ (Exercise 1.A.1).

Exercise 1.A.6. Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Solution. Suppose that $\alpha = x + yi$. Since $\alpha \neq 0$, it must be the case that x and y are not both zero, so that $x^2 + y^2 \neq 0$. Let $\beta = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$ and observe that

$$\alpha\beta = (x + yi)\left(\frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i\right) = \frac{x^2 + y^2}{x^2 + y^2} + \frac{xy - xy}{x^2 + y^2}i = 1 + 0i = 1.$$

To see that β is unique, suppose β' also satisfies $\alpha\beta' = 1$ and notice that

$$\beta = \beta 1 = \beta(\alpha\beta') = (\alpha\beta)\beta' = 1\beta' = \beta',$$

where we have used the associativity of multiplication in $\mathbf{C}$ (Exercise 1.A.3) and the commutativity of multiplication in $\mathbf{C}$ (1.4).

Exercise 1.A.7. Show that

$$\frac{-1 + \sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution. Let $z = \frac{-1+\sqrt{3}i}{2}$, so that $2z = -1 + \sqrt{3}i$. Observe that

$$\begin{aligned}(2z)^2 = 4z^2 &= \left(-1 + \sqrt{3}i\right)^2 = 1 - 2\sqrt{3}i - 3 = -2 - 2\sqrt{3}i \\ \Rightarrow (2z)^3 = (4z^2)(2z) &= \left(-2 - 2\sqrt{3}i\right)\left(-1 + \sqrt{3}i\right) = 2 - 2\sqrt{3}i + 2\sqrt{3}i + 6 = 8,\end{aligned}$$

i.e. $8z^3 = 8$. It follows that $z^3 = 1$.

Exercise 1.A.8. Find two distinct square roots of i.

Solution. Let $z_1 = \frac{1+i}{\sqrt{2}}$ and $z_2 = -z_1$ (z_1 and z_2 are distinct since $z_1 \neq 0$) and observe that

$$2z_1^2 = (1 + i)^2 = 2i \quad \Rightarrow \quad z_1^2 = i,$$

i.e. z_1 is a square root of i. Furthermore, $z_2^2 = (-z_1)^2 = z_1^2 = i$, so that z_2 is a square root of i also.

Exercise 1.A.9. Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution. The unique solution is $x = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right)$.

Exercise 1.A.10. Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Solution. If there was such a λ, then

$$\lambda(2 - 3i) = 12 - 5i \quad \Rightarrow \quad \lambda = \frac{12 - 5i}{2 - 3i} = 3 + 2i.$$

However,

$$(3 + 2i)(-6 + 7i) = -32 + 9i \neq -32 - 9i.$$

Exercise 1.A.11. Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$, and $z = (z_1, ..., z_n)$, then

$$\begin{aligned}(x + y) + z &= (x_1 + y_1, ..., x_n + y_n) + z = ((x_1 + y_1) + z_1, ..., (x_n + y_n) + z_n) \\ &= (x_1 + (y_1 + z_1), ..., x_n + (y_n + z_n)) = x + (y_1 + z_1, ..., y_n + z_n) = x + (y + z),\end{aligned}$$

where we have used the associativity of addition in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.2).

Exercise 1.A.12. Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Solution. If $x = (x_1, ..., x_n)$, then

$$(ab)x = ((ab)x_1, ..., (ab)x_n) = (a(bx_1), ..., a(bx_n)) = a(bx_1, ..., bx_n) = a(bx),$$

where we have used the associativity of multiplication in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.3).

Exercise 1.A.13. Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$, then

$$1x = (1x_1, ..., 1x_n) = (x_1, ..., x_n) = x,$$

where we have used that $1x_j = x_j$ for any $x_j \in \mathbf{F}$.

Exercise 1.A.14. Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, then

$$\begin{aligned}\lambda(x + y) &= \lambda(x_1 + y_1, ..., x_n + y_n) \\ &= (\lambda(x_1 + y_1), ..., \lambda(x_n + y_n)) \\ &= (\lambda x_1 + \lambda y_1, ..., \lambda x_n + \lambda y_n) \\ &= (\lambda x_1, ..., \lambda x_n) + (\lambda y_1, ..., \lambda y_n) \\ &= \lambda(x_1, ..., x_n) + \lambda(y_1, ..., y_n) \\ &= \lambda x + \lambda y,\end{aligned}$$

where we have used distributivity in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.4).

Exercise 1.A.15. Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$, then

$$\begin{aligned}(a + b)x &= (a + b)(x_1, ..., x_n) \\ &= ((a + b)x_1, ..., (a + b)x_n) \\ &= (ax_1 + bx_1, ..., ax_n + bx_n) \\ &= (ax_1, ..., ax_n) + (bx_1, ..., bx_n) \\ &= a(x_1, ..., x_n) + b(x_1, ..., x_n) \\ &= ax + bx,\end{aligned}$$

where we have used distributivity in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.4).

1.B. Definition of Vector Space

Exercise 1.B.1. Show that $-(-v) = v$ for every $v \in V$.

Solution. Since $v + (-v) = 0$ and the additive inverse of a vector is unique (1.27), it must be the case that $-(-v) = v$.

Exercise 1.B.2. Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution. It will suffice to show that if $av = 0$ and $a \neq 0$, so that a^{-1} exists, then $v = 0$. Indeed,

$$av = 0 \quad \Rightarrow \quad a^{-1}(av) = 0 \quad \Rightarrow \quad (a^{-1}a)v = 0 \quad \Rightarrow \quad 1v = 0 \quad \Rightarrow \quad v = 0.$$

Exercise 1.B.3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution. For $v, w, x \in V$, notice that

$$v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \tfrac{1}{3}(w - v).$$

Exercise 1.B.4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Solution. The empty set does not contain an additive identity.

Exercise 1.B.5. Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

> *The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.*

Solution. If V satisfies all of the conditions in (1.20), then as shown in (1.30) we have $0v = 0$ for all $v \in V$. Suppose that V satisfies all of the conditions in (1.20), except we have replaced the additive inverse condition with the condition that $0v = 0$ for all $v \in V$. We want to show that for each $v \in V$, there exists an element $w \in V$ such that $v + w = 0$. Indeed, for $v \in V$, let $w = (-1)v$ and observe that

$$v + w = 1v + (-1)v = (1 - 1)v = 0v = 0.$$

Exercise 1.B.6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t &= \infty + \infty = \infty, \\ t + (-\infty) = (-\infty) + t &= (-\infty) + (-\infty) = -\infty, \\ \infty + (-\infty) &= (-\infty) + \infty = 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution. This is not a vector space over $\mathbf{R}$, since addition is not associative:

$$(1 + \infty) + (-\infty) = \infty + (-\infty) = 0 \neq 1 = 1 + 0 = 1 + (\infty + (-\infty)).$$

Exercise 1.B.7. Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Solution. We define addition and scalar multiplication on V^S as in (1.24), i.e. for $f, g \in V^S$ the sum $f + g \in V^S$ is the function

$$\begin{aligned} f + g \,:\, S &\to V \\ x &\mapsto f(x) + g(x); \end{aligned}$$

the addition $f(x) + g(x)$ is vector addition in V. Similarly, for $\lambda \in \mathbf{F}$ and $f \in V^S$, the product $\lambda f \in V^S$ is the function

$$\begin{aligned} \lambda f \,:\, S &\to V \\ x &\mapsto \lambda f(x); \end{aligned}$$

the product $\lambda f(x)$ is scalar multiplication in V. We now show that V^S with these definitions satisfies each condition in definition (1.20).

Commutativity. Let $f, g \in V^S$ and $x \in S$ be given. Observe that

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x),$$

where we have used the commutativity of addition in V for the second equality. It follows that $f + g = g + f$.

Associativity. Let $f, g, h \in V^S$ and $x \in S$ be given. Observe that

$$\begin{aligned}((f + g) + h)(x) &= (f + g)(x) + h(x) = (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) = f(x) + (g + h)(x) = (f + (g + h))(x),\end{aligned}$$

where we have used the associativity of addition in V for the third equality. It follows that $(f + g) + h = f + (g + h)$. Similarly, let $f \in V^S$ and $a, b \in \mathbf{F}$ be given. Observe that, for any $x \in S$,

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = a((bf)(x)) = (a(bf))(x),$$

where we have used the associativity of scalar multiplication in V for the second equality. It follows that $(ab)f = a(bf)$.

Additive identity. We claim that the additive identity in V^S is the function $0 : S \to V$ given by $0(x) = 0$ for any $x \in S$; the 0 on the right-hand side is the additive identity in V. Indeed, for any $f \in V^S$ and $x \in S$ we have

$$(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x).$$

It follows that $f + 0 = f$.

Additive inverse. For $f \in V^S$, define $g : S \to V$ by $g(x) = -f(x)$ for $x \in S$, where $-f(x)$ is the additive inverse in V of $f(x)$. We claim that g is the additive inverse of f. To see this, let $x \in S$ be given and observe that

$$(f + g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0 = 0(x);$$

it follows that $f + g = 0$.

Multiplicative identity. Let $f \in V^S$ and $x \in S$ be given. Observe that

$$(1f)(x) = 1f(x) = f(x),$$

where we have used that $1v = v$ for any $v \in V$. It follows that $1f = f$.

Distributive properties. Let $a \in \mathbf{F}$ and $f, g \in V^S$ be given. Observe that, for any $x \in S$,

$$\begin{aligned}(a(f + g))(x) &= a(f + g)(x) = a((f(x) + g(x)) \\ &= af(x) + ag(x) = (af)(x) + (ag)(x) = (af + ag)(x),\end{aligned}$$

where we have used the first distributive property in V for the third equality. It follows that $a(f + g) = af + ag$. Similarly, let $a, b \in \mathbf{F}$ and $f \in V^S$ be given. For any $x \in S$, observe that

$$((a + b)f)(x) = (a + b)f(x) = af(x) + bf(x) = (af)(x) + (bf)(x) = (af + bf)(x),$$

where we have used the second distributive property in V for the second equality. It follows that $(a + b)f = af + bf$.

We may conclude that V^S is a vector space over $\mathbf{F}$.

Exercise 1.B.8. Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by
  $$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
  for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by
  $$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$
  for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

> *Think of V as a subset of $V_{\mathbf{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbf{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.*

Solution. We need to verify each condition in definition (1.20). The algebraic manipulations required to show that commutativity, associativity, and the first distributive property hold for $V_{\mathbf{C}}$ are essentially the same algebraic manipulations we performed in Exercise 1.A.1, Exercise 1.A.2, Exercise 1.A.3, and Exercise 1.A.4, except instead of using the algebraic properties of $\mathbf{R}$, we use the algebraic properties of V (i.e. the properties listed in (1.20)); we will avoid repeating ourselves and instead verify the remaining conditions.

Additive identity. We claim that the additive identity in $V_{\mathbf{C}}$ is $0 + i0$, where 0 is the additive identity in V. Indeed, for any $u + iv \in V_{\mathbf{C}}$ we have

$$(u + iv) + (0 + i0) = (u + 0) + i(v + 0) = u + iv.$$

Additive inverse. We claim that the additive inverse of an element $u + iv \in V_{\mathbf{C}}$ is the element $(-u) + i(-v)$, where $-u$ is the additive inverse of u in V. Indeed,

$$(u + iv) + ((-u) + i(-v)) = (u + (-u)) + i(v + (-v)) = 0 + i0.$$

Multiplicative identity. For any $u + iv \in V_{\mathbf{C}}$, we have

$$(1 + 0i)(u + iv) = (1u - 0v) + i(1v + 0u) = u + iv.$$

Distributive properties. For the second distributive property, let $a + bi, c + di \in \mathbf{C}$ and $u + iv \in V_{\mathbf{C}}$ be given. Observe that

$$\begin{aligned}((a + bi) + (c + di))(u + iv) &= ((a + c) + (b + d)i)(u + iv) \\ &= ((a + c)u - (b + d)v) + i((a + c)v + (b + d)u) \\ &= (au + cu - bv - dv) + i(av + cv + bu + du) \\ &= ((au - bv) + i(av + bu)) + ((cu - dv) + i(cv + du)) \\ &= (a + bi)(u + iv) + (c + di)(u + iv),\end{aligned}$$

where we have used the second distributive property for V for the third equality.