1 Section 7.5 Exercises

Exercises with solutions from Section 7.5 of [UA].

Exercise 7.5.1. (a) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^{x} f$. Find a piecewise algebraic formula for $F(x)$ for all x. Where is F continuous? Where is F differentiable? Where does $F'(x) = f(x)$?

(b) Repeat part (a) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Solution. (a) Some calculations reveal that $F : [-1, \infty) \to \mathbf{R}$ is given by

$$F(x) = \begin{cases} \frac{1}{2}(1 - x^2) & \text{if } -1 \leq x \leq 0, \\ \frac{1}{2}(1 + x^2) & \text{if } x > 0. \end{cases}$$

It is straightforward to manually check that F is differentiable (and hence continuous) on its domain, with derivative given by $F'(x) = f(x)$. However, note that the Fundamental Theorem of Calculus part (ii) (FToC, Theorem 7.5.1 (ii)) immediately implies that F is continuous on any interval of the form $[-1, b]$ for $b \in \mathbf{R}$ (in fact, Lipschitz on such intervals) and hence is continuous on its domain. Furthermore, as f is continuous everywhere, the FToC also implies that F is differentiable on its domain with derivative given by $F'(x) = f(x)$.

(b) In this case, the function $F : [-1, \infty) \to \mathbf{R}$ is given by

$$F(x) = \begin{cases} 1 + x & \text{if } -1 \leq x \leq 0, \\ 1 + 2x & \text{if } x > 0. \end{cases}$$

As in part (a), the FToC part (ii) implies that F is continuous on its domain. Furthermore, since f is continuous on $A = [-1, 0) \cup (0, \infty)$, the FToC implies that F is differentiable on A with derivative given by $F'(x) = f(x)$. However, because f is not continuous at 0 the FToC does not allow us to conclude that F is differentiable at 0. Indeed, F fails to be differentiable here:

$$\lim_{x \to 0^-} \frac{F(x) - F(0)}{x} = 1 \neq 2 = \lim_{x \to 0^+} \frac{F(x) - F(0)}{x}.$$

Exercise 7.5.2. Decide whether each statement is true or false, providing a short justification for each conclusion.

(a) If $g = h'$ for some h on $[a, b]$, then g is continuous on $[a, b]$.

(b) If g is continuous on $[a, b]$, then $g = h'$ for some h on $[a, b]$.

(c) If $H(x) = \int_a^x h$ is differentiable at $c \in [a, b]$, then h is continuous at c.

Solution. (a) This is false. For a counterexample, consider the function $h : [-1, 1] \to \mathbf{R}$ given by

$$h(x) = \begin{cases} x^{5/3} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then, as we showed in Exercise 5.2.7 (a), h is differentiable but h' is not continuous at 0.

(b) This is true. Theorem 7.2.9 implies that g is integrable on $[a, b]$ and so we are justified in defining $h : [a, b] \to \mathbf{R}$ by $h(x) = \int_a^x g$; the continuity of g on $[a, b]$ then allows us to use the FToC part (ii) to conclude that $g = h'$.

(c) This is false. For a counterexample, consider $h : [-1, 1] \to \mathbf{R}$ given by

$$h(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then $H : [-1, 1] \to \mathbf{R}$ defined by $H(x) = \int_{-1}^x h(t)\, dt$ is identically zero and hence differentiable at 0, but h is not continuous at 0.

Exercise 7.5.3. The hypothesis in Theorem 7.5.1 (i) that $F'(x) = f(x)$ for all $x \in [a, b]$ is slightly stronger than it needs to be. Carefully read the proof and state exactly what needs to be assumed with regard to the relationship between f and F for the proof to be valid.

Solution. In light of Theorem 7.4.1, it would suffice for $F'(x) = f(x)$ to hold for all but finitely many $x \in [a, b]$.

Exercise 7.5.4. Show that if $f : [a, b] \to \mathbf{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a, b]$, then $f(x) = 0$ everywhere on $[a, b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

Solution. Define $F : [a, b] \to \mathbf{R}$ by $F(x) = \int_a^x f$. On one hand, since by assumption F is identically zero on $[a, b]$, we have that F is differentiable on $[a, b]$ and satisfies $F'(x) = 0$ for all $x \in [a, b]$. On the other hand, because f is continuous on $[a,b]$, the FToC part (ii) implies that $F'(x) = f(x)$ for all $x \in [a,b]$. Thus f is identically zero on $[a,b]$.

For an example demonstrating that this conclusion does not follow if f is not continuous, consider $f : [0,1] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$$

Then $\int_0^x f = 0$ for all $x \in [0,1]$, but f is not identically zero.

Exercise 7.5.5. The Fundamental Theorem of Calculus can be used to supply a shorter argument for Theorem 6.3.1 under the additional assumption that the sequence of derivatives is continuous.

Assume $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a,b]$. Assuming each f_n' is continuous, we can apply Theorem 7.5.1 (i) to get

$$\int_a^x f_n' = f_n(x) - f_n(a)$$

for all $x \in [a,b]$. Show that $g(x) = f'(x)$.

Solution. Let $x \in [a,b]$ be given. Because $f_n' \to g$ uniformly on $[a,x]$, Theorem 7.4.4 shows that

$$\lim_{n\to\infty} \int_a^x f_n' = \int_a^x g.$$

We can then take the limit as $n \to \infty$ on both sides of the equation $\int_a^x f_n' = f_n(x) - f_n(a)$ and use the pointwise convergence $f_n \to f$ to see that

$$f(x) = f(a) + \int_a^x g$$

for all $x \in [a,b]$. Since g is the uniform limit of a sequence of continuous functions it is itself continuous (Theorem 6.2.6) and so we may invoke the FToC part (ii) to conclude that $f'(x) = g(x)$ for all $x \in [a,b]$.

Exercise 7.5.6 (Integration-by-parts). (a) Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a,b]$ and derive the familiar integration-by-parts formula

$$\int_a^b h(t)k'(t)\,dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\,dt.$$

(b) Explain how the result in Exercise 7.4.6 can be used to slightly weaken the hypothesis in part (a).

Solution. (a) By assumption the functions h, h', k, and k' are continuous on $[a, b]$; it follows that $(hk)' = hk' + h'k$ is continuous on $[a, b]$. Theorem 7.2.9 then implies that $(hk)'$ is integrable on $[a, b]$ and so we may use the FToC part (i) to see that

$$\int_a^b h(t)k'(t) + h'(t)k(t)\, dt = \int_a^b (h(t)k(t))'\, dt = h(b)k(b) - h(a)k(a).$$

(b) In light of Exercise 7.4.6, we need only assume that h' and k' are integrable on $[a, b]$.

Exercise 7.5.7. Use part (ii) of Theorem 7.5.1 to construct another proof of part (i) of Theorem 7.5.1 under the stronger hypothesis that f is continuous. (To get started, set $G(x) = \int_a^x f$.)

Solution. It will suffice to show that $G(b) = F(b) - F(a)$. Because f is continuous on $[a, b]$, the FToC part (ii) implies that $G'(x) = f(x) = F'(x)$ for all $x \in [a, b]$; it follows from Corollary 5.3.4 that $G(x) = F(x) + k$ for some constant k. Substituting $x = a$, we see that $k = -F(a)$ and thus $G(b) = F(b) - F(a)$, as desired.

Exercise 7.5.8 (Natural Logarithm and Euler's Constant). Let

$$L(x) = \int_1^x \frac{1}{t}\, dt,$$

where we consider only $x > 0$.

(a) What is $L(1)$? Explain why L is differentiable and find $L'(x)$.

(b) Show that $L(xy) = L(x) + L(y)$. (Think of y as a constant and differentiate $g(x) = L(xy)$.)

(c) Show $L(x/y) = L(x) - L(y)$.

(d) Let

$$\gamma_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - L(n).$$

Prove that (γ_n) converges. The constant $\gamma = \lim \gamma_n$ is called Euler's constant.

(e) Show how consideration of the sequence $\gamma_{2n} - \gamma_n$ leads to the interesting identity

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

Solution. (a) We have $L(1) = 0$. Because t^{-1} is continuous on $(0, \infty)$, the FToC part (ii) shows that L is differentiable on $(0, \infty)$ and satisfies $L'(x) = x^{-1}$.

(b) Note that, by part (a),

$$\frac{d}{dx}L(xy) = yL'(xy) = \frac{y}{xy} = \frac{1}{x} = L'(x).$$

Corollary 5.3.4 then implies that $L(xy) = L(x) + k$ for some constant k. Substituting $x = 1$, we see that $k = L(y)$ and thus $L(xy) = L(x) + L(y)$, as desired.

(c) Observe that, by parts (a) and (b),

$$0 = L(1) = L\left(\frac{y}{y}\right) = L(y) + L\left(\frac{1}{y}\right),$$

so that $L\left(\frac{1}{y}\right) = -L(y)$ for any $y > 0$. Combining this with part (b) shows that $L\left(\frac{x}{y}\right) = L(x) - L(y)$.

(d) Let $n \geq 2$ be given and consider the partition $P = \{1, \dots, n\}$ of $[1, n]$. Then

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} > 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} = U\left(\frac{1}{t}, P\right) \geq U\left(\frac{1}{t}\right) = L(n).$$

Thus $\gamma_n \geq 0$ for each $n \in \mathbf{N}$, so that (γ_n) is bounded below.

Again, let $n \in \mathbf{N}$ be given and observe that

$$\gamma_n - \gamma_{n+1} = L\left(1 + \frac{1}{n}\right) - \frac{1}{n+1}.$$

Since $\frac{1}{t} \geq \frac{n}{n+1}$ on $\left[1, 1 + \frac{1}{n}\right]$, Theorem 7.4.2 (iii) shows that

$$L\left(1 + \frac{1}{n}\right) \geq \frac{1}{n+1}$$

and hence $\gamma_n \geq \gamma_{n+1}$ for each $n \in \mathbf{N}$, so that (γ_n) is decreasing; we can now appeal to the Monotone Convergence Theorem (Theorem 2.4.2) to conclude that (γ_n) converges.

(e) For $n \in \mathbf{N}$, observe that

$$\begin{aligned}\gamma_{2n} - \gamma_n &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - L(2n) + L(n) \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - \left(\frac{2}{2} + \frac{2}{4} + \cdots + \frac{2}{2n}\right) - L(2) - L(n) + L(n) \\ &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}\right) - L(2).\end{aligned}$$

Taking the limit as $n \to \infty$ on both sides gives the desired equality.

Exercise 7.5.9. Given a function f on $[a, b]$, define the *total variation* of f to be

$$Vf = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\right\},$$

where the supremum is taken over all partitions P of $[a, b]$.

(a) If f is continuously differentiable (f' exists as a continuous function), use the Fundamental Theorem of Calculus to show $Vf \leq \int_a^b |f'|$.

(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $Vf = \int_a^b |f'|$.

Solution. (a) Let $P = \{x_0, \dots, x_n\}$ be an arbitrary partition of $[a, b]$. Because f' is continuous on $[a, b]$, it is integrable on $[a, b]$ and so we may use the FToC part (i) and Theorem 7.4.2 (v) to see that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f' \right| \leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'| = \int_a^b |f'|.$$

As P was arbitrary, it follows that $Vf \leq \int_a^b |f'|$.

(b) For any $\epsilon > 0$, there exists a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ such that

$$\left(\int_a^b |f'|\right) - \epsilon = L(|f'|) - \epsilon < L(|f'|, P).$$

For $k \in \{1, \dots, n\}$, apply the Mean Value Theorem on the interval $[x_{k-1}, x_k]$ to obtain some $t_k \in (x_{k-1}, x_k)$ such that

$$|f'(t_k)|(x_k - x_{k-1}) = |f(x_k) - f(x_{k-1})|.$$

It follows that

$$\begin{aligned}
L(|f'|, P) &= \sum_{k=1}^{n} \inf\{|f'(t)| : t \in [x_{k-1}, x_k]\}(x_k - x_{k-1}) \\
&\le \sum_{k=1}^{n} |f'(t_k)|(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \\
&\le Vf.
\end{aligned}$$

We have now shown that for every $\epsilon > 0$ it holds that

$$\int_a^b |f'| \le Vf + \epsilon$$

and thus we obtain the inequality $\int_a^b |f'| \le Vf$. Given part (a), we may conclude that $Vf = \int_a^b |f'|$.

Exercise 7.5.10 (Change-of-variable Formula). Let $g : [a, b] \to \mathbf{R}$ be differentiable and assume g' is continuous. Let $f : [c, d] \to \mathbf{R}$ be continuous, and assume that the range of g is contained in $[c, d]$ so that the composition $f \circ g$ is properly defined.

(a) Why are we sure f is the derivative of some function? How about $(f \circ g)g'$?

(b) Prove the change-of-variable formula

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt.$$

Solution. (a) f is integrable on $[c, d]$ because it is continuous on $[c, d]$ and so if we let $F(x) = \int_c^x f$ then the FToC part (ii) implies that $F'(x) = f(x)$ for each $x \in [c, d]$. Similarly, note that $f \circ g$ is continuous on $[a, b]$, being a composition of continuous functions, and hence is integrable on $[a, b]$. By assumption g' is continuous on $[a, b]$ and so is also integrable on $[a, b]$. We can now use Exercise 7.4.6 to see that $(f \circ g)g'$ is integrable on $[a, b]$, so that we can define $G(x) = \int_a^x (f \circ g)g'$ and use the FToC part (ii) to see that $G'(x) = f(g(x))g'(x)$ for each $x \in [a, b]$.

(b) Define $F : [c, d] \to \mathbf{R}$ and $G : [a, b] \to \mathbf{R}$ by

$$F(t) = \int_{g(a)}^{t} f(x)\,dx \quad \text{and} \quad G(t) = \int_{a}^{t} f(g(x))g'(x)\,dx.$$

Then $F'(t) = f(t)$, so that $[F(g(t))]' = f(g(t))g'(t)$, and $G'(t) = f(g(t))g'(t)$. It follows that $F(g(t)) = G(t) + k$ on $[a, b]$ for some constant k. Substituting $t = a$, we see that $k = 0$ and thus $F(g(b)) = G(b)$, i.e.

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_{a}^{b} f(g(x))g'(x)\,dx.$$

Exercise 7.5.11. Assume f is integrable on $[a, b]$ and has a "jump discontinuity" at $c \in (a, b)$. This means that both one-sided limits exist as x approaches c from the left and from the right, but that

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x).$$

(This phenomenon is discussed in more detail in Section 4.6.)

(a) Show that, in this case, $F(x) = \int_a^x f$ is not differentiable at $x = c$.

(b) The discussion in Section 5.5 mentions the existence of a continuous monotone function that fails to be differentiable on a dense subset of $\mathbf{R}$. Combine the results of part (a) with Exercise 6.4.10 to show how to construct such a function.

Solution. (a) Let $A = \lim_{x \to c^-} f(x)$ and $B = \lim_{x \to c^+} f(x)$. A small modification of the proof of the FToC part (ii) shows that

$$\lim_{x \to c^-} \frac{F(x) - F(c)}{x - c} = A \quad \text{and} \quad \lim_{x \to c^+} \frac{F(x) - F(c)}{x - c} = B.$$

Since $A \neq B$, we see that $\lim_{x \to c} \frac{F(x) - F(c)}{x - c}$ does not exist, i.e. F is not differentiable at c.

(b) As in Exercise 6.4.10, let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rationals and for each $n \in \mathbf{N}$ define $u_n : \mathbf{R} \to \mathbf{R}$ by

$$u_n(x) = \begin{cases} 2^{-n} & \text{if } r_n < x, \\ 0 & \text{if } x \leq r_n. \end{cases}$$

Now define $h : \mathbf{R} \to \mathbf{R}$ by $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Let $[a, b]$ be a given interval and note that for each $N \in \mathbf{N}$ the partial sum function $h_N(x) = \sum_{n=1}^{N} u_n(x)$ has at most N jump discontinuities on $[a,b]$; it follows from Theorem 7.4.1 that h_N is integrable on $[a,b]$. In Exercise 6.4.10 we showed that $h_N \to h$ uniformly on $\mathbf{R}$ and hence by Theorem 7.4.4 we see that h is integrable on $[a,b]$. We can now define $H : \mathbf{R} \to \mathbf{R}$ by $H(x) = \int_0^x h$. The FToC part (ii) shows that H is continuous, and we can use Theorem 7.4.1 and the fact that h is non-negative to see that H is monotone increasing.

Now we will prove that h has a jump discontinuity at each rational number. Let $r_m \in \mathbf{Q}$ be given; we have two claims.

(i) Our first claim is that $\lim_{x\to r_m^-} h(x) = h(r_m)$. To see this, let $\epsilon > 0$ be given and choose $N \in \mathbf{N}$ such that $2^{-N} < \epsilon$. Because the set $\{r_1, \ldots, r_N\}$ is finite, we can choose a $\delta > 0$ such that the intersection $(r_m - \delta, r_m) \cap \{r_1, \ldots, r_N\}$ is empty, i.e. if $r_n \in (r_m - \delta, r_m)$, then $n > N$.

Now suppose that $x \in (r_m - \delta, r_m)$ and enumerate the rationals in $[x, r_m)$ as a subsequence $\{r_{n_1}, r_{n_2}, r_{n_3}, \ldots\}$ of the sequence $\{r_1, r_2, r_3, \ldots\}$; by our previous discussion, we must have $n_k > N$ for each $k \in \mathbf{N}$. As we showed in Exercise 6.4.10, h is strictly increasing and $h(r_m) - h(x) = \sum_{k=1}^{\infty} 2^{-n_k}$. Thus

$$|h(r_m) - h(x)| = 2^{-N} \sum_{k=1}^{\infty} 2^{-n_k+N} \le 2^{-N} \sum_{n=1}^{\infty} 2^{-n} = 2^{-N} < \epsilon$$

and our claim follows.

(ii) Our second claim is that $\lim_{x\to r_m^+} h(x) = h(r_m) + 2^{-m}$. Again, let $\epsilon > 0$ be given and choose $N \in \mathbf{N}$ such that $2^{-N} < \epsilon$. Similarly to before, we can choose a $\delta > 0$ such that if $r_n \in (r_m, r_m + \delta)$ then $n > N$. For $x \in (r_m, r_m + \delta)$, enumerate the rationals in (r_m, x) as a subsequence $\{r_{n_1}, r_{n_2}, r_{n_3}, \ldots\}$ of the sequence $\{r_1, r_2, r_3, \ldots\}$, so that

$$[r_m, x) = \{r_m, r_{n_1}, r_{n_2}, r_{n_3}, \ldots\};$$

by our previous discussion, we must have $n_k > N$ for each $k \in \mathbf{N}$. Thus $h(x) - h(r_m) = 2^{-m} + \sum_{k=1}^{\infty} 2^{-n_k}$ and, arguing as in our first claim, it follows that

$$\left|h(x) - h(r_m) - 2^{-m}\right| = \sum_{k=1}^{\infty} 2^{-n_k} \le 2^{-N} < \epsilon.$$

This proves our second claim.

We have now shown that if $r_m \in \mathbf{Q}$, then

$$\lim_{x\to r_m^-} h(x) = h(r_m) < h(r_m) + 2^{-m} = \lim_{x\to r_m^+} h(x),$$

so that h has a jump discontinuity at each rational number; it follows from part (a) that H fails to be differentiable at each rational number.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.