1 Section 7.2 Exercises

Exercises with solutions from Section 7.2 of [UA].

Exercise 7.2.1. Let f be a bounded function on $[a,b]$, and let P be an arbitrary partition of $[a,b]$. First, explain why $U(f) \geq L(f,P)$. Now, prove Lemma 7.2.6.

Solution. Lemma 7.2.4 implies that $L(f,P)$ is a lower bound of the set $\{U(f,Q) : Q \in \mathcal{P}\}$ and thus $U(f) \geq L(f,P)$. Since P was an arbitrary partition of $[a,b]$, we have now shown that $U(f)$ is an upper bound of the set $\{L(f,P) : P \in \mathcal{P}\}$ and thus $U(f) \geq L(f)$.

Exercise 7.2.2. Consider $f(x) = 1/x$ over the interval $[1,4]$. Let P be the partition consisting of the points $\{1, 3/2, 2, 4\}$.

(a) Compute $L(f,P), U(f,P)$, and $U(f,P) - L(f,P)$.

(b) What happens to the value of $U(f,P) - L(f,P)$ when we add the point 3 to the partition?

(c) Find a partition P' of $[1,4]$ for which $U(f,P') - L(f,P') < 2/5$.

Solution. (a) Since f is strictly decreasing over $[1,4]$, we have:

$$\begin{aligned} m_1 &= \inf\big\{f(x) : x \in \big[1, \tfrac{3}{2}\big]\big\} = f\big(\tfrac{3}{2}\big) = \tfrac{2}{3}, \quad M_1 = \sup\big\{f(x) : x \in \big[1, \tfrac{3}{2}\big]\big\} = f(1) = 1, \\ m_2 &= \inf\big\{f(x) : x \in \big[\tfrac{3}{2}, 2\big]\big\} = f(2) = \tfrac{1}{2}, \quad M_2 = \sup\big\{f(x) : x \in \big[\tfrac{3}{2}, 2\big]\big\} = f\big(\tfrac{3}{2}\big) = \tfrac{2}{3}, \\ m_3 &= \inf\{f(x) : x \in [2,4]\} = f(4) = \tfrac{1}{4}, \quad M_3 = \sup\{f(x) : x \in [2,4]\} = f(2) = \tfrac{1}{2}, \end{aligned}$$

and thus

$$\begin{aligned} L(f,P) &= m_1(x_1 - x_0) + m_2(x_2 - x_1) + m_3(x_3 - x_2) \\ &= \tfrac{2}{3}\big(\tfrac{3}{2} - 1\big) + \tfrac{1}{2}\big(2 - \tfrac{3}{2}\big) + \tfrac{1}{4}(4 - 2) = \tfrac{13}{12}, \end{aligned}$$

$$\begin{aligned} U(f,P) &= M_1(x_1 - x_0) + M_2(x_2 - x_1) + M_3(x_3 - x_2) \\ &= \big(\tfrac{3}{2} - 1\big) + \tfrac{2}{3}\big(2 - \tfrac{3}{2}\big) + \tfrac{1}{2}(4 - 2) = \tfrac{11}{6}, \end{aligned}$$

$$U(f,P) - L(f,P) = \tfrac{11}{6} - \tfrac{13}{12} = \tfrac{3}{4}.$$

(b) Letting $P = \big\{1, \frac{3}{2}, 2, 3, 4\big\}$, a similar calculation to part (a) shows that $U(f,P) - L(f,P) = \frac{1}{2}$.

(c) Letting $P' = \left\{1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, 3, 4\right\}$, a straightforward calculation shows that

$$U(f, P') - L(f, P') = \tfrac{3}{8} < \tfrac{2}{5}.$$

Exercise 7.2.3 (Sequential Criterion for Integrability). (a) Prove that a bounded function f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0,$$

and in this case $\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$.

(b) For each n, let P_n be the partition of $[0, 1]$ into n equal subintervals. Find formulas for $U(f, P_n)$ and $L(f, P_n)$ if $f(x) = x$. The formula $1 + 2 + 3 + \cdots + n = n(n + 1)/2$ will be useful.

(c) Use the sequential criterion for integrability from (a) to show directly that $f(x) = x$ is integrable on $[0, 1]$ and compute $\int_0^1 f$.

Solution. (a) In light of Theorem 7.2.8, it will suffice to show the equivalence of the following two statements.

(i) There exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

(ii) For every $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

This equivalence is clear. Now suppose that such a sequence of partitions exists, so that f is integrable on $[a, b]$. For each $n \in \mathbf{N}$, the inequalities $L(f, P_n) \le L(f), U(f) \le U(f, P_n)$, and $L(f, P_n) \le U(f, P_n)$ imply that

$$L(f, P_n) - U(f, P_n) \le L(f) - U(f, P_n) = U(f) - U(f, P_n) \le U(f, P_n) - L(f, P_n)$$

and the squeeze theorem then implies that $\lim_{n\to\infty} U(f, P_n) = U(f) = \int_a^b f$. A similar argument shows that $\lim_{n\to\infty} L(f, P_n) = L(f) = \int_a^b f$.

(b) For each $0 \leq k \leq n-1$, let $x_k = \frac{k}{n-1}$, and let $P_n = \{x_0, x_1, \ldots, x_{n-1}\}$. Since f is strictly increasing on $[0,1]$, we then have

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = x_{k-1} = \frac{k-1}{n-1},$$
$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = x_k = \frac{k}{n-1}$$

for each $1 \leq k \leq n-1$. It follows that

$$U(f, P_n) = \sum_{k=1}^{n-1} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n-1} \frac{k}{(n-1)^2} = \frac{n}{2(n-1)},$$
$$L(f, P_n) = \sum_{k=1}^{n-1} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n-1} \frac{k-1}{(n-1)^2} = \frac{n}{2(n-1)} - \frac{1}{n-1}.$$

(c) From part (b) we have

$$U(f, P_n) - L(f, P_n) = \frac{1}{n-1} \to 0.$$

It then follows from part (a) that f is integrable on $[0,1]$ and that

$$\int_0^1 f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{n}{2(n-1)} = \frac{1}{2}.$$

Exercise 7.2.4. Let g be bounded on $[a,b]$ and assume there exists a partition P with $L(g,P) = U(g,P)$. Describe g. Is it integrable? If so, what is the value of $\int_a^b g$?

Solution. Suppose $P = \{x_0, x_1, \ldots, x_n\}$ is such that $L(g,P) = U(g,P)$. Given that $m_k \leq M_k$ for all $1 \leq k \leq n$, we have the implication

$$m_k < M_k \text{ for some } k \in \{1, \ldots, n\} \quad \Longrightarrow \quad L(g,P) < U(g,P).$$

Since $L(g,P) \leq U(g,P)$, the contrapositive of the above result is

$$L(g,P) = U(g,P) \quad \Longrightarrow \quad m_k = M_k \text{ for all } k \in \{1, \ldots, n\}.$$

Consider a subinterval $[x_{k-1}, x_k]$ for some $k \in \{1, \ldots, n\}$. Since $m_k = M_k$, it must be the case that g is constant on this subinterval, say $g(x) = c_k$ for all $x \in [x_{k-1}, x_k]$. In fact, since $g(x_k) = c_k = c_{k+1}$, we see that $c_1 = \cdots = c_n$. Denoting this common value by c, we then have $g(x) = c$ for all $x \in [a,b]$.

Since $U(g,P) - L(g,P) = 0$, Theorem 7.2.8 implies that g is integrable. Let $S = U(g,P) = L(g,P)$. On one hand, $S = L(g,P)$ is a lower bound of the set $\{U(g,Q) : Q \in \mathcal{P}\}$, as we noted in Exercise 7.2.1. On the other hand, $S = U(g,P)$ belongs to the set $\{U(g,Q) : Q \in \mathcal{P}\}$ and hence must be the minimum of this set. Since the minimum and the infimum of a set necessarily coincide when they both exist, we see that

$$\int_a^b g = U(g) = U(g,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = c\sum_{k=1}^{n}(x_k - x_{k-1}) = c(x_n - x_0) = c(b-a).$$

Exercise 7.2.5. Assume that, for each n, f_n is an integrable function on $[a,b]$. If $(f_n) \to f$ uniformly on $[a,b]$, prove that f is also integrable on this set. (We will see that this conclusion does not necessarily follow if the convergence is pointwise.)

Solution. Let $\epsilon > 0$ be given. Because $f_n \to f$ uniformly on $[a,b]$, there exists an $N \in \mathbf{N}$ such that

$$n \geq N \text{ and } x \in [a,b] \quad \Longrightarrow \quad |f_n(x) - f(x)| < \frac{\epsilon}{3(b-a)}. \tag{1}$$

By hypothesis the function f_N is integrable on $[a,b]$ and thus by Theorem 7.2.8 there exists a partition $P = \{x_0, \ldots, x_m\}$ of $[a,b]$ such that $U(f_N, P) - L(f_N, P) < \frac{\epsilon}{3}$. Consider a subinterval $[x_{k-1}, x_k]$ for some $k \in \{1, \ldots, m\}$, and let

$$M_k^N = \sup\{f_N(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

Inequality (1) implies that

$$\left|M_k^N - M_k\right| \leq \frac{\epsilon}{3(b-a)},$$

which gives us

$$|U(f_N, P) - U(f,P)| \leq \sum_{k=1}^{m}\left|M_k^N - M_k\right|(x_{k-1} - x_k) \leq \frac{\epsilon}{3(b-a)}\sum_{k=1}^{m}(x_{k-1} - x_k) = \frac{\epsilon}{3}.$$

Similarly, we can show that $|L(f_N, P) - L(f,P)| \leq \frac{\epsilon}{3}$. It follows that

$$\begin{aligned} U(f,P) - L(f,P) &\leq |U(f_N, P) - U(f,P)| + |L(f_N,P) - L(f,P)| \\ &\qquad + |U(f_N,P) - L(f_N,P)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

and an appeal to Theorem 7.2.8 allows us to conclude that f is integrable on $[a,b]$.

Exercise 7.2.6. A *tagged partition* $(P, \{c_k\})$ is one where in addition to a partition P we choose a sampling point c_k in each of the subintervals $[x_{k-1}, x_k]$. The corresponding *Riemann sum*,

$$R(f, P) = \sum_{k=1}^{n} f(c_k)\Delta x_k,$$

is discussed in Section 7.1, where the following definition is alluded to.
Riemann's Original Definition of the Integral: A bounded function f is *integrable* on $[a, b]$ with $\int_a^b f = A$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition $(P, \{c_k\})$ satisfying $\Delta x_k < \delta$ for all k, it follows that

$$|R(f, P) - A| < \epsilon.$$

Show that if f satisfies Riemann's definition above, then f is integrable in the sense of Definition 7.2.7. (The full equivalence of these two characterizations of integrability is proved in Section 8.1.)

Solution. Let $\epsilon > 0$ be given. Since f satisfies Riemann's definition of integrability, there exists a $\delta > 0$ such that for any tagged partition $(P, \{c_k\})$ satisfying $\Delta x_k < \delta$ for all k, it follows that

$$|R(f, P) - A| < \frac{\epsilon}{2}.$$

Let $N \in \mathbf{N}$ be such that $\frac{b-a}{N} < \delta$, for each $k \in \{0, \ldots, N\}$ set $y_k = a + k\frac{b-a}{N}$, and let Q_1 be the partition $\{y_0, \ldots, y_N\}$ of $[a, b]$; note that $\Delta y_k = \frac{b-a}{N} < \delta$. Since $U(f)$ is the infimum of the set $\{U(f, Q) : Q \in \mathcal{P}\}$, there exists a partition Q_2 of $[a, b]$ such that $U(f) \le U(f, Q_2) < U(f) + \frac{\epsilon}{4}$. Let P be the common refinement of Q_1 and Q_2, say

$$P = Q_1 \cup Q_2 = \{x_0, \ldots, x_n\}.$$

Note that $\Delta x_k \le \Delta y_k = \frac{b-a}{N} < \delta$, so that for any choice of sampling points we have

$$|R(f, P) - A| < \frac{\epsilon}{2}. \tag{1}$$

Note further that since $Q_2 \subseteq P$, Lemma 7.2.3 gives us

$$U(f) \le U(f, P) \le U(f, Q_2) < U(f) + \frac{\epsilon}{4}. \tag{2}$$

For each $k \in \{1, \ldots, n\}$, since $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$, there exists some $c_k \in [x_{k-1}, x_k]$ such that

$$M_k - \frac{\epsilon}{4(b-a)} < f(c_k) \le M_k.$$

Take the collection $\{c_k\}$ as the sampling points for the partition P. It follows that

$$0 \leq U(f,P) - R(f,P) = \sum_{k=1}^{n}(M_k - f(c_k))\Delta x_k < \frac{\epsilon}{4(b-a)}\sum_{k=1}^{n}\Delta x_k = \frac{\epsilon}{4}. \tag{3}$$

Now observe that by (1), (2), and (3), we have

$$\begin{aligned}|U(f) - A| &\leq |U(f) - R(f,P)| + |R(f,P) - A| \\ &\leq |U(f) - U(f,P)| + |U(f,P) - R(f,P)| + |R(f,P) - A| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we see that $U(f) = A$. An analogous argument shows that $L(f) = A$ and thus $U(f) = L(f)$, i.e. f is integrable in the sense of Definition 7.2.7.

Exercise 7.2.7. Let $f : [a,b] \to \mathbf{R}$ be increasing on the set $[a,b]$ (i.e., $f(x) \leq f(y)$ whenever $x < y$). Show that f is integrable on $[a,b]$.

Solution. Let $\epsilon > 0$ be given and let $n \in \mathbf{N}$ be such that

$$\frac{(b-a)(f(b) - f(a))}{n} < \epsilon.$$

For $k \in \{0,\dots,n\}$ let $x_k = a + k\frac{b-a}{n}$ and let P be the partition $\{x_0,\dots,x_n\}$ of $[a,b]$. Note that, since f is increasing on $[a,b]$, we have

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = f(x_{k-1}) \quad \text{and} \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = f(x_k)$$

for each $k \in \{1,\dots,n\}$. Hence

$$\begin{aligned}U(f,P) - L(f,P) &= \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) \\ &= \frac{b-a}{n}\sum_{k=1}^{n}(f(x_k) - f(x_{k-1})) = \frac{(b-a)(f(b) - f(a))}{n} < \epsilon\end{aligned}$$

and it follows from Theorem 7.2.8 that f is integrable on $[a,b]$.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.