1 Section 5.B Exercises

Exercises with solutions from Section 5.B of [LADR].

Exercise 5.B.1. Suppose $T \in \mathcal{L}(V)$ and there exists a positive integer n such that $T^n = 0$.

(a) Prove that $I - T$ is invertible and that

$$(I - T)^{-1} = I + T + \cdots + T^{n-1}.$$

(b) Explain how you would guess the formula above.

Solution. (a) A computation gives

$$\begin{aligned}(I - T)(I + T + \cdots + T^{n-1}) &= I + T + \cdots + T^{n-1} - T - T^2 - \cdots - T^n = I - T^n = I,\\ (I + T + \cdots + T^{n-1})(I - T) &= I - T + T - T^2 + \cdots + T^{n-1} - T^n = I - T^n = I.\end{aligned}$$

(b) We might guess the formula above from the familiar formula

$$(1 - x)(1 + x + \cdots + x^{n-1}) = 1 - x^n$$

for the partial sum of a geometric series.

Exercise 5.B.2. Suppose $T \in \mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Solution. We have $Tv = \lambda v$ for some $v \neq 0$. Observe that

$$0 = (T - 2I)(T - 3I)(T - 4I)v = (\lambda - 2)(\lambda - 3)(\lambda - 4)v.$$

Since $v \neq 0$, it must be the case that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Exercise 5.B.3. Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and -1 is not an eigenvalue of T. Prove that $T = I$.

Solution. Note that $T^2 = I$ if and only if $(T + I)(T - I) = 0$. Let $v \in V$ be given. If $Tv - v \neq 0$, then $0 = (T + I)(T - I)v = (T + I)(Tv - v)$, demonstrating that -1 is an eigenvalue of T with corresponding eigenvector $Tv - v$. By assumption -1 is not an eigenvalue of T, so it must be the case that $Tv = v$ for every $v \in V$, i.e. $T = I$.

Exercise 5.B.4. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

Solution. Let $v \in V$ be given. Then $P^2v = Pv$, which gives $P(Pv-v) = 0$, so that $Pv-v \in \operatorname{null} P$; say $Pv - v = u$ where $u \in \operatorname{null} P$. It follows that $v = -u + Pv \in \operatorname{null} P + \operatorname{range} P$ and hence that $V = \operatorname{null} P + \operatorname{range} P$.

To see that this sum is direct, suppose $v \in \operatorname{null} P$ and $v = Pw$ for some $w \in V$. Then $Pv = P^2w = Pw$, so that $w - v \in \operatorname{null} P$. Combining this with $v \in \operatorname{null} P$, we see that $w \in \operatorname{null} P$ also, giving us $v = Pw = 0$. It follows that $\operatorname{null} P \cap \operatorname{range} P = \{0\}$ and hence that the sum $V = \operatorname{null} P \oplus \operatorname{range} P$ is direct.

Exercise 5.B.5. Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in \mathcal{P}(\mathbf{F})$ is a polynomial. Prove that
$$p(STS^{-1}) = Sp(T)S^{-1}.$$

Solution. Note that $(STS^{-1})^0 = I = SIS^{-1} = ST^0S^{-1}$, and for $k \geq 2$ we have
$$(STS^{-1})^k = STS^{-1}STS^{-1} \cdots STS^{-1} = ST^kS^{-1}.$$
Thus for each non-negative integer k we have $(STS^{-1})^k = ST^kS^{-1}$. Suppose $p = \sum_{k=0}^n a_k x^k$. Then
$$p(STS^{-1}) = \sum_{k=0}^n a_k (STS^{-1})^k = \sum_{k=0}^n a_k ST^kS^{-1} = S\left(\sum_{k=0}^n a_k T^k\right)S^{-1} = Sp(T)S^{-1}.$$

Exercise 5.B.6. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $\mathcal{P}(\mathbf{F})$.

Solution. Suppose $u \in U$. Then $T^0u = Iu = u \in U$. If $T^ku \in U$ for some non-negative integer k, then $T^{k+1}u = T(T^ku) \in U$ since U is invariant under T. Hence by induction $T^ku \in U$ for every non-negative integer k.

Suppose $p = \sum_{k=0}^n a_k x^k$ and $u \in U$. Then
$$p(T)(u) = \left(\sum_{k=0}^n a_k T^k\right)(u) = \sum_{k=0}^n a_k T^k u,$$
which belongs to U since we showed that each $T^ku \in U$ and U is closed under linear combinations. Thus U is invariant under $p(T)$.

Exercise 5.B.7. Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

Solution. The forward implication is proved in the solution to Exercise 5.A.22. For the converse implication, suppose that there is a non-zero $v \in V$ such that $Tv = \pm 3v$. Then $T^2v = (\pm 3)^2 v = 9v$ and we see that 9 is an eigenvalue of T^2.

Exercise 5.B.8. Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$ (see errata).

Solution. Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$T(1,0) = \left(\cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}}(1,1) \quad \text{and} \quad T(0,1) = \left(\cos\tfrac{3\pi}{4}, \sin\tfrac{3\pi}{4}\right) = \tfrac{1}{\sqrt{2}}(-1,1).$$

Note that T is a counterclockwise rotation about the origin by 45 degrees; it follows that T^4 is a counterclockwise rotation about the origin by 180 degrees, i.e.

$$T^4(1,0) = (\cos\pi, \sin\pi) = (-1,0) \quad \text{and} \quad T(0,1) = \left(\cos\tfrac{3\pi}{2}, \sin\tfrac{3\pi}{2}\right) = (0,-1).$$

Thus $T^4 = -I$.

Exercise 5.B.9. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$ with $v \neq 0$. Let p be a nonzero polynomial of smallest degree such that $p(T)v = 0$. Prove that every zero of p is an eigenvalue of T.

Solution. Note that if p is a nonzero constant polynomial, then $p(T)v \neq 0$, so we may assume that $\deg p \geq 1$. Suppose λ is a zero of p. Then $p(x) = (x-\lambda)q(x)$ for some polynomial q satisfying $\deg q = \deg p - 1$. Since p is of smallest degree, it must be the case that $v' := q(T)v \neq 0$. Thus

$$p(T)v = (T-\lambda)q(T)v = (T-\lambda)v' = 0 \iff Tv' = \lambda v',$$

demonstrating that λ is an eigenvalue of T with a corresponding eigenvector v'.

Exercise 5.B.10. Suppose $T \in \mathcal{L}(V)$ and v is an eigenvector of T with eigenvalue λ. Suppose $p \in \mathcal{P}(\mathbf{F})$. Prove that $p(T)v = p(\lambda)v$.

Solution. If $p = 0$ this is clear, so suppose that $p(x) = \sum_{k=0}^{n} a_k x^k$, where $n = \deg p \geq 0$. It is straightforward to verify that $T^k v = \lambda^k v$ for all non-negative integers k. Then

$$p(T)v = \left(\sum_{k=0}^{n} a_k T^k\right) v = \sum_{k=0}^{n} a_k T^k v = \sum_{k=0}^{n} a_k \lambda^k v = \left(\sum_{k=0}^{n} a_k \lambda^k\right) v = p(\lambda)v.$$

Exercise 5.B.11. Suppose $\mathbf{F} = \mathbf{C}, T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbf{C})$ is a nonconstant (see errata) polynomial, and $\alpha \in \mathbf{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of T.

Solution. If $\alpha = p(\lambda)$ for some eigenvalue λ of T, then Exercise 5.B.10 shows that α is an eigenvalue of $p(T)$.

Suppose that α is an eigenvalue of $p(T)$, i.e. there exists some $v \neq 0$ such that $p(T)v = \alpha v$. Let $q \in \mathcal{P}(\mathbf{C})$ be given by $q(z) = p(z) - \alpha$. Since q is a polynomial over $\mathbf{C}$, there is a factorization

$$q(z) = c(z - \lambda_1) \cdots (z - \lambda_m),$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbf{C}$. Since $\deg q = \deg p \geq 1$, it must be the case that $c \neq 0$ and $m \geq 1$. We have $q(T)v = 0$ since $p(T)v = \alpha v$, so

$$c(T - \lambda_1 I) \cdots (T - \lambda_m I)v = 0.$$

It follows that there is a $k \in \{1, \ldots, m\}$ such that $T - \lambda_k I$ is not injective, or equivalently such that λ_k is an eigenvalue of T. Furthermore, $p(\lambda_k) = q(\lambda_k) + \alpha = \alpha$ since λ_k is a zero of q.

Exercise 5.B.12. Show that the result in the previous exercise does not hold if $\mathbf{C}$ is replaced with $\mathbf{R}$.

Solution. Consider the linear operator $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x, y) = (-y, x)$, i.e. a counterclockwise rotation about the origin by 90 degrees. As shown in Example 5.8 (a), T has no eigenvalues. However, letting $p(x) = x^2$, we have $p(T) = T^2 = -I$, since T^2 is a counterclockwise rotation about the origin by 180 degrees. Thus $p(T)$ has the eigenvalue -1, but we cannot possibly express -1 as $p(\lambda)$ for some eigenvalue λ of T, since T has no eigenvalues.

Exercise 5.B.13. Suppose W is a complex vector space and $T \in \mathcal{L}(W)$ has no eigenvalues. Prove that every subspace of W invariant under T is either $\{0\}$ or infinite-dimensional.

Solution. Suppose U is a non-zero subspace of W invariant under T and consider the restriction operator $T|_U$. If U is finite-dimensional, then 5.21 implies that $T|_U$ has an eigenvalue, which must also be an eigenvalue of T. Since T has no eigenvalues, it must be the case that U is infinite-dimensional.

Exercise 5.B.14. Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible.
[*The exercise above and the exercise below show that 5.30 fails without the hypothesis that an upper-triangular matrix is under consideration.*]

Solution. Consider the invertible operator $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x, y) = (y, x)$, which is its own inverse. The matrix of this operator with respect to the standard basis of $\mathbf{R}^2$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Exercise 5.B.15. Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

Solution. Consider the operator $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x, y) = (x + y, x + y)$, which is not injective, and hence not invertible, since $T(1, -1) = (0, 0)$. The matrix of this operator with respect to the standard basis of $\mathbf{R}^2$ is

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Exercise 5.B.16. Rewrite the proof of 5.21 using the linear map that sends $p \in \mathcal{P}_n(\mathbf{C})$ to $(p(T))v \in V$ (and use 3.23).

Solution. Suppose V is a complex vector space with dimension $n > 0$ and $T \in \mathcal{L}(V)$. Choose $v \in V$ with $v \neq 0$ and define a map $\Psi : \mathcal{P}_n(\mathbf{C}) \to V$ by $\Psi(p) = p(T)v$. It is straightforward to verify that Ψ is linear. Since $\dim \mathcal{P}_n(\mathbf{C}) = n + 1 > n = \dim V$, it must be the case that Ψ is not injective (3.23). Thus there exists some non-zero $p \in \mathcal{P}_n(\mathbf{C})$ such that $\Psi(p) = p(T)v = 0$. Observe that since v is non-zero, p cannot be a non-zero constant polynomial either. Thus $\deg p \geq 1$.

Since p is a polynomial over $\mathbf{C}$, there is a factorization

$$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m),$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbf{C}$. Note that $c \neq 0$ and $m \geq 1$ since $\deg p \geq 1$. Thus

$$p(T)v = c(T - \lambda_1 I) \cdots (T - \lambda_m I)v = 0.$$

Since $v \neq 0$, it follows that there is a $k \in \{1, \ldots, m\}$ such that $T - \lambda_k I$ is not injective, or equivalently such that λ_k is an eigenvalue of T.

Exercise 5.B.17. Rewrite the proof of 5.21 using the linear map that sends $p \in \mathcal{P}_{n^2}(\mathbf{C})$ to $p(T) \in \mathcal{L}(V)$ (and use 3.23).

Solution. Suppose V is a complex vector space with dimension $n > 0$ and $T \in \mathcal{L}(V)$. Choose $v \in V$ with $v \neq 0$ and define a map $\Psi : \mathcal{P}_{n^2}(\mathbf{C}) \to \mathcal{L}(V)$ by $\Psi(p) = p(T)$. It is straightforward to verify that Ψ is linear. Since $\dim \mathcal{P}_{n^2}(\mathbf{C}) = n^2 + 1 > n^2 = \dim \mathcal{L}(V)$, it must be the case that Ψ is not injective (3.23). Thus there exists some non-zero $p \in \mathcal{P}_{n^2}(\mathbf{C})$ such that $\Psi(p) = p(T) = 0$. Since non-zero scalar multiples of the identity operator are not the zero map (provided $V \neq \{0\}$, which is the case here), we see that $\deg p \geq 1$.

Since p is a polynomial over $\mathbf{C}$, there is a factorization

$$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m),$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbf{C}$. Note that $c \neq 0$ and $m \geq 1$ since $\deg p \geq 1$. Thus

$$p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0.$$

Since the zero map is not injective, there must exist a $k \in \{1, \ldots, m\}$ such that $T - \lambda_k I$ is not injective, or equivalently such that λ_k is an eigenvalue of T.

Exercise 5.B.18. Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Define a function $f : \mathbf{C} \to \mathbf{R}$ by

$$f(\lambda) = \dim \operatorname{range}(T - \lambda I).$$

Prove that f is not a continuous function.

Solution. (If $V = \{0\}$, then f is the constant function $\lambda \mapsto 0$ and hence is continuous.)

Suppose that $\dim V = m > 0$. By 5.21, there exists an eigenvalue $\mu \in \mathbf{C}$ of T. By 5.6, the operator $T - \mu I$ must fail to be surjective; equivalently, we have $f(\mu) < m$. Consider the sequence $(\lambda_n)_{n \in \mathbf{N}}$ of distinct complex numbers given by $\lambda_n = \mu + \frac{1}{n}$, which satisfies $\lim_{n \to \infty} \lambda_n = \mu$. By 5.13, T can have at most m distinct eigenvalues, and so we may choose a subsequence $(\lambda_{n_k})_{k \in \mathbf{N}}$ such that each λ_{n_k} is not an eigenvalue of T. By 5.6, each operator $T - \lambda_{n_k} I$ must be surjective; equivalently, we have $f(\lambda_{n_k}) = m$. It follows that $\lim_{k \to \infty} \lambda_{n_k} = \mu$, however

$$\lim_{k \to \infty} f(\lambda_{n_k}) = m > f(\mu).$$

Thus f is not continuous at μ.

Exercise 5.B.19. Suppose V is finite-dimensional with $\dim V > 1$ and $T \in \mathcal{L}(V)$. Prove that

$$\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V).$$

Solution. If every linear map in $\mathcal{L}(V)$ could be realised as $p(T)$ for some $p \in \mathcal{P}(\mathbf{F})$, then each pair of linear maps in $\mathcal{L}(V)$ would commute with each other (5.20). However, by Exercise 3.A.14, there exist two linear maps in $\mathcal{L}(V)$ which do not commute with each other.

Exercise 5.B.20. Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T has an invariant subspace of dimension k for each $k = 1, \ldots, \dim V$.

Solution. This is immediate from 5.27 and the equivalence of (a) and (c) in 5.26.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edition.