1 Section 3.A Exercises

Exercises with solutions from Section 3.A of [LADR].

Exercise 3.A.1. Suppose $b, c \in \mathbf{R}$. Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

Solution. Suppose that $b = c = 0$, so that T is the map

$$T(x, y, z) = (2x - 4y + 3z, 6x).$$

Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbf{R}^3$ and $\lambda \in \mathbf{R}$ be given. Then

$$\begin{aligned} T(x_1 + x_2, y_1 + y_2, z_1 + z_2) &= (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2), 6(x_1 + x_2)) \\ &= (2x_1 + 2x_2 - 4y_1 - 4y_2 + 3z_1 + 3z_2, 6x_1 + 6x_2) \\ &= (2x_1 - 4y_1 + 3z_1, 6x_1) + (2x_2 - 4y_2 + 3z_2, 6x_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2). \end{aligned}$$

$$\begin{aligned} T(\lambda x_1, \lambda y_1, \lambda z_1) &= (2\lambda x_1 - 4\lambda y_1 + 3\lambda z_1, 6\lambda x_1) \\ &= (\lambda(2x_1 - 4y_1 + 3z_1), \lambda(6x_1)) \\ &= \lambda(2x_1 - 4y_1 + 3z_1, 6x_1) \\ &= \lambda T(x_1, y_1, z_1). \end{aligned}$$

Thus T is linear.

Suppose that $b \neq 0$. Then $T(0, 0, 0) = (b, 0) \neq (0, 0)$, so T cannot be linear by 3.11. Now suppose that $c \neq 0$. Then

$$T(1, 1, 1) = (1 + b, 6 + c) \quad \text{and} \quad T(2, 2, 2) = (2 + b, 12 + 8c).$$

Since $2(6 + c) = 12 + 2c \neq 12 + 8c$ for $c \neq 0$, we see that $2T(1, 1, 1) \neq T(2, 2, 2)$. Thus T is not linear.

Exercise 3.A.2. Suppose $b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c \sin p(0)\right).$$

Show that T is linear if and only if $b = c = 0$.

Solution. Suppose that $b = c = 0$, so that T is the map

$$Tp = \left(3p(4) + 5p'(6), \int_{-1}^{2} x^3 p(x)dx\right).$$

Let $p, q \in \mathcal{P}(\mathbf{R})$ and $\lambda \in \mathbf{R}$ be given. Then

$$\begin{aligned}
T(p+q) &= \left(3(p+q)(4) + 5(p+q)'(6), \int_{-1}^{2} x^3 (p+q)(x)dx\right) \\
&= \left(3(p(4)+q(4)) + 5(p'(6)+q'(6)), \int_{-1}^{2} x^3 (p(x)+q(x))dx\right) \\
&= \left(3p(4) + 3q(4) + 5p'(6) + 5q'(6), \int_{-1}^{2} x^3 p(x)dx + \int_{-1}^{2} x^3 q(x)dx\right) \\
&= \left(3p(4) + 5p'(6), \int_{-1}^{2} x^3 p(x)dx\right) + \left(3q(4) + 5q'(6), \int_{-1}^{2} x^3 q(x)dx\right) \\
&= Tp + Tq.
\end{aligned}$$

$$\begin{aligned}
T(\lambda p) &= \left(3(\lambda p)(4) + 5(\lambda p)'(6), \int_{-1}^{2} x^3 (\lambda p)(x)dx\right) \\
&= \left(3(\lambda p(4)) + 5(\lambda p'(6)), \int_{-1}^{2} x^3 (\lambda p(x))dx\right) \\
&= \left(\lambda(3p(4) + 5p'(6)), \lambda \int_{-1}^{2} x^3 p(x)dx\right) \\
&= \lambda \left(3p(4) + 5p'(6), \int_{-1}^{2} x^3 p(x)dx\right) \\
&= \lambda Tp.
\end{aligned}$$

Thus T is linear.

Now suppose that T is linear. Observe that

$$\begin{aligned}
T(\pi) &= \left(3\pi + b\pi^2, \frac{15\pi}{4} + c\right), \\
2T(\pi) &= \left(6\pi + 2b\pi^2, \frac{15\pi}{2} + 2c\right), \\
T(2\pi) &= \left(6\pi + 4b\pi^2, \frac{15\pi}{2}\right).
\end{aligned}$$

Since T is linear, we must have $2T(\pi) = T(2\pi)$, i.e.

$$\left(6\pi + 2b\pi^2, \frac{15\pi}{2} + 2c\right) = \left(6\pi + 4b\pi^2, \frac{15\pi}{2}\right) \iff (2b\pi^2, 2c) = (4b\pi^2, 0) \iff b = c = 0.$$

Exercise 3.A.3. Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbf{F}^n$.
[*The exercise above shows that T has the form promised in the last item of Example 3.4.*]

Solution. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbf{F}^n$ and let $f_1, \ldots, f_m$ be the standard basis of $\mathbf{F}^m$. For any $k \in \{1, \ldots, n\}$, we have $Te_k \in \mathbf{F}^m$. Thus there exist scalars $A_{1,k}, \ldots, A_{m,k}$ such that

$$Te_k = \sum_{j=1}^{m} A_{j,k} f_j.$$

Let $x = (x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k e_k$ be given. Then by linearity,

$$\begin{aligned}
Tx &= T\left(\sum_{k=1}^{n} x_k e_k\right) \\
&= \sum_{k=1}^{n} x_k T e_k \\
&= \sum_{k=1}^{n} x_k \sum_{j=1}^{m} A_{j,k} f_j \\
&= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} A_{j,k} x_k\right) f_j \\
&= \left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k\right) \\
&= (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n).
\end{aligned}$$

Exercise 3.A.4. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Solution. Suppose we have scalars $a_1, \ldots, a_m$ such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Applying T to both sides of this equation and using linearity, we obtain

$$T(a_1 v_1 + \cdots + a_m v_m) = T(0) \iff a_1 T v_1 + \cdots + a_m T v_m = 0.$$

Since the list $Tv_1, \ldots, Tv_m$ is linearly independent, this implies that $a_1 = \cdots = a_m = 0$. Thus the list $v_1, \ldots, v_m$ is linearly independent.

Exercise 3.A.5. Prove the assertion in 3.7.

Solution. The assertion is that, if V and W are vector spaces, then $\mathcal{L}(V, W)$ is a vector space. First, let us show that $\mathcal{L}(V, W)$ is closed under addition and scalar multiplication, i.e. if S and T are linear maps from V to W and $\lambda \in \mathbf{F}$ is a scalar, then $S + T$ and λS are linear maps from V to W. Let $u, v \in V$ and $\alpha \in \mathbf{F}$ be given. Then

$$\begin{aligned}(S+T)(u+v) = S(u+v) + T(u+v) &= Su + Sv + Tu + Tv \\ &= Su + Tu + Sv + Tv = (S+T)(u) + (S+T)(v).\end{aligned}$$

$$(S+T)(\alpha u) = S(\alpha u) + T(\alpha u) = \alpha Su + \alpha Tu = \alpha(Su + Tu) = \alpha(S+T)(u).$$

Thus $S + T \in \mathcal{L}(V, W)$. Similarly,

$$(\lambda S)(u+v) = \lambda S(u+v) = \lambda(Su + Sv) = \lambda Su + \lambda Sv = (\lambda S)(u) + (\lambda S)(v).$$

$$(\lambda S)(\alpha u) = \lambda S(\alpha u) = \lambda(\alpha Su) = \alpha(\lambda Su) = \alpha(\lambda S)(u).$$

Thus $\lambda S \in \mathcal{L}(V, W)$. To prove that $\mathcal{L}(V, W)$ is a vector space with these operations, we will verify each of the requirements from 1.19.

Commutativity. Suppose $S, T \in \mathcal{L}(V, W)$ and $u \in V$. Then

$$(S+T)(u) = Su + Tu = Tu + Su = (T+S)(u).$$

Thus $S + T = T + S$.

Associativity. Suppose $R, S, T \in \mathcal{L}(V, W)$, $a, b \in \mathbf{F}$, and $u \in V$. Then

$$\begin{aligned}((R+S)+T)(u) = (R+S)(u) + Tu = (Ru + Su) + Tu &= Ru + (Su + Tu) \\ &= Ru + (S+T)(u) = (R + (S+T))(u).\end{aligned}$$

$$((ab)R)(u) = (ab)Ru = a(bRu) = a((bR)(u)) = (a(bR))(u).$$

Thus $(R+S)+T = R + (S+T)$ and $(ab)R = a(bR)$.

Additive identity. It is easily verified that the map $0 : V \to W$ given by $v \mapsto 0$ belongs to $\mathcal{L}(V,W)$. We claim that this map is the additive identity in $\mathcal{L}(V,W)$. Let $S \in \mathcal{L}(V,W)$ and $u \in V$ be given. Then

$$(S+0)(u) = Su + 0u = Su + 0 = Su.$$

Thus $S + 0 = S$.

Additive inverse. Suppose that $S \in \mathcal{L}(V,W)$. Define $T : V \to W$ by $Tu = -Su$; it is not hard to see that T is linear. We claim that T is the additive inverse to S. Indeed, for any $u \in V$,

$$(S+T)(u) = Su + Tu = Su + (-Su) = 0.$$

Thus $S + T = 0$.

Multiplicative identity. Let $S \in \mathcal{L}(V,W)$ and $u \in V$ be given. Then

$$(1S)(u) = 1Su = Su.$$

Thus $1S = S$.

Distributive properties. Let $S, T \in \mathcal{L}(V,W)$, $a, b \in \mathbf{F}$, and $u \in V$ be given. Then

$$(a(S+T))(u) = a(S+T)(u) = a(Su+Tu) = aSu + aTu = (aS)(u) + (aT)(u).$$

$$((a+b)S)(u) = (a+b)Su = aSu + bSu = (aS)(u) + (bS)(u).$$

Thus $a(S+T) = aS + aT$ and $(a+b)S = aS + bS$.

Exercise 3.A.6. Prove the assertions in 3.9.

Solution. The first assertion is that if the products make sense, then $(T_1T_2)T_3 = T_1(T_2T_3)$. This is certainly the case, since the composition of functions is associative.

The second assertion is that if $T \in \mathcal{L}(V,W)$, I_V is the identity map on V, and I_W is the identity map on W, then $TI_V = I_WT = T$. Indeed, let $v \in V$ be given. Then

$$(TI_V)(v) = T(I_Vv) = Tv \quad \text{and} \quad (I_WT)(v) = I_W(Tv) = Tv.$$

The third assertion is that if $T, T_1, T_2 \in \mathcal{L}(U,V)$ and $S, S_1, S_2 \in \mathcal{L}(V,W)$, then

$$(S_1+S_2)T = S_1T + S_2T \quad \text{and} \quad S(T_1+T_2) = ST_1 + ST_2.$$

Let $u \in U$ be given. Then

$$((S_1+S_2)T)(u) = (S_1+S_2)(Tu) = S_1(Tu) + S_2(Tu) = (S_1T)(u) + (S_2T)(u).$$

$$(S(T_1+T_2))(u) = S((T_1+T_2)(u)) = S(T_1u + T_2u) = S(T_1u) + S(T_2u) = (ST_1)(u) + (ST_2)(u).$$

Exercise 3.A.7. Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution. Since $\dim V = 1$, there exists a basis u for V. Then since $Tu \in V$, it must be of the form λu for some $\lambda \in \mathbf{F}$. Let $v = \alpha u \in V$ be given. Then

$$Tv = T(\alpha u) = \alpha Tu = \alpha(\lambda u) = \lambda(\alpha u) = \lambda v.$$

Exercise 3.A.8. Give an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.
[*The exercise above and the next exercise show that neither homogeneity nor additivity alone is enough to imply that a function is a linear map.*]

Solution. Let $\varphi : \mathbf{R}^2 \to \mathbf{R}$ be given by $\varphi(x, y) = (x^3 + y^3)^{\frac{1}{3}}$. Then for any $a \in \mathbf{R}$ and $(x, y) \in \mathbf{R}^2$, we have

$$\varphi(ax, ay) = \left((ax)^3 + (ay)^3\right)^{\frac{1}{3}} = \left(a^3\right)^{\frac{1}{3}} \left(x^3 + y^3\right)^{\frac{1}{3}} = a\left(x^3 + y^3\right)^{\frac{1}{3}} = a\varphi(x, y).$$

However, observe that

$$\varphi(1, 0) + \varphi(0, 1) = 1 + 1 = 2 \neq 2^{\frac{1}{3}} = \varphi(1, 1).$$

Thus φ is not linear.

Exercise 3.A.9. Give an example of a function $\varphi : \mathbf{C} \to \mathbf{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbf{C}$ but φ is not linear. (Here $\mathbf{C}$ is thought of as a complex vector space.)
[*There also exists a function $\varphi : \mathbf{R} \to \mathbf{R}$ such that φ satisfies the additivity condition above but φ is not linear. However, showing the existence of such a function involves considerably more advanced tools.*]

Solution. Let $\varphi : \mathbf{C} \to \mathbf{C}$ be given by $\varphi(x + iy) = x$, i.e. φ takes a complex number to its real part. Then

$$\varphi((x + iy) + (u + iv)) = \varphi((x + u) + i(y + v)) = x + u = \varphi(x + iy) + \varphi(u + iv).$$

However, $\varphi(i) = 0$ and $\varphi(i^2) = \varphi(-1) = -1 \neq i\varphi(i)$. Thus φ is not linear.

Exercise 3.A.10. Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T : V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

Solution. There is some $u \in U$ such that $Su \neq 0$, and since $U \neq V$ there is some $v \in V$ such that $v \notin U$. This implies that $u - v \notin U$, otherwise $v = -(u - v) + u \in U$. Then we have

$$Tv + T(u - v) = 0 + 0 = 0 \neq Su = Tu = T(v + u - v).$$

Thus T is not linear.

Exercise 3.A.11. Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Solution. By 2.34 there is a subspace X of V such that $V = U \oplus X$. Then for $v \in V$, we have $v = u + x$ for unique vectors $u \in U$ and $x \in X$. Given this, the map

$$\begin{aligned} T : V &\to W \\ v &\mapsto Su \end{aligned}$$

is well-defined. For $u \in U$, it is clear that the unique representation of u as a sum in $U \oplus X$ is $u + 0$, so that $Tu = Su$. Thus T extends S.

Suppose $v_1, v_2 \in V$. Then there are unique vectors $u_1, u_2 \in U$ and $x_1, x_2 \in X$ such that $v_1 = u_1 + x_1$ and $v_2 = u_2 + x_2$. Since $v_1 + v_2 = (u_1 + u_2) + (x_1 + x_2)$, this must be the unique representation of $v_1 + v_2$ as a sum in $U \oplus X$. Then

$$T(v_1 + v_2) = S(u_1 + u_2) = Su_1 + Su_2 = Tv_1 + Tv_2.$$

Suppose $v \in V$ and $\lambda \in \mathbf{F}$. Then there are unique vectors $u \in U$ and $x \in X$ such that $v = u + x$. Since $\lambda v = \lambda u + \lambda x$, this must be the unique representation of λv as a sum in $U \oplus X$. Then

$$T(\lambda v) = S(\lambda u) = \lambda Su = \lambda Tv.$$

Thus T is linear.

Exercise 3.A.12. Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Solution. For a finite-dimensional vector space V with $\dim V > 0$, we wish to prove that

$$W \text{ is infinite-dimensional} \implies \mathcal{L}(V,W) \text{ is infinite-dimensional.}$$

We will prove the contrapositive statement:

$$\mathcal{L}(V,W) \text{ is finite-dimensional} \implies W \text{ is finite-dimensional.}$$

Suppose therefore that $\mathcal{L}(V,W) = \operatorname{span}(S_1,\ldots,S_m)$ for some (possibly empty) list $S_1,\ldots,S_m$ in $\mathcal{L}(V,W)$. Since V is finite-dimensional with $\dim V > 0$, there is a non-empty basis $v_1,\ldots,v_n$ for V. Consider the (possibly empty) list $w_1,\ldots,w_m$ where $w_j := S_j v_1$. We claim that this list spans W. To see this, let $w \in W$ be given. By 3.5, there is a (unique) linear map $T : \operatorname{span}(v_1) \to W$ such that $Tv_1 = w$. Then by Exercise 3.A.11, T can be extended to a linear map $S : V \to W$. Since $\mathcal{L}(V,W) = \operatorname{span}(S_1,\ldots,S_m)$, there are scalars $a_1,\ldots,a_m$ such that $S = a_1S_1 + \cdots + a_mS_m$ (this is to be understood as the "empty linear combination" if $\mathcal{L}(V,W) = \{0\}$, so that $S = 0$). Then observe that

$$w = Sv_1 = (a_1S_1 + \cdots + a_mS_m)(v_1) = a_1S_1v_1 + \cdots + a_mS_mv_1 = a_1w_1 + \cdots + a_mw_m.$$

Thus $W = \operatorname{span}(w_1,\ldots,w_m)$. This shows that W is finite-dimensional and moreover that $\dim W \le \dim \mathcal{L}(V,W)$.

Exercise 3.A.13. Suppose $v_1,\ldots,v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1,\ldots,w_m \in W$ such that no $T \in \mathcal{L}(V,W)$ satisfies $Tv_k = w_k$ for each $k = 1,\ldots,m$.

Solution. Since $W \neq \{0\}$, there is a $w \in W$ such that $w \neq 0$, and by the Linear Dependence Lemma, there is a $j \in \{1,\ldots,m\}$ such that $v_j \in \operatorname{span}(v_1,\ldots,v_{j-1})$. If $j = 1$ then $v_1 = 0$. Consider the list $w,\ldots,w$ of length m in W. If $T \in \mathcal{L}(V,W)$, then $Tv_1 = T(0) = 0 \neq w$, and so we have found the desired list.

If $j > 1$, then there are scalars $a_1,\ldots,a_{j-1}$ such that $v_j = a_1v_1 + \cdots + a_{j-1}v_{j-1}$. Consider the list $w_1,\ldots,w_m$ where $w_k = w$ if $k \neq j$ and $w_j = (a_1 + \cdots + a_{j-1} + 1)w$. Suppose we have some $T \in \mathcal{L}(V,W)$ such that $Tv_k = w_k$ for $1 \le k < j$. Then observe that

$$\begin{aligned} Tv_j = T(a_1v_1 + \cdots + a_{j-1}v_{j-1}) &= a_1Tv_1 + \cdots + a_{j-1}Tv_{j-1} \\ &= a_1w + \cdots + a_{j-1}w = (a_1 + \cdots + a_{j-1})w. \end{aligned}$$

Since $w \neq 0$, we have $Tv_j = (a_1+\cdots+a_{j-1})w \neq (a_1+\cdots+a_{j-1}+1)w = w_j$. Thus no $T \in \mathcal{L}(V,W)$ can possibly satisfy $Tv_k = w_k$ for each $k = 1,\ldots,m$.

Exercise 3.A.14. Suppose V is finite-dimensional with $\dim V \ge 2$. Prove that there exist $S, T \in \mathcal{L}(V,V)$ such that $ST \neq TS$.

Solution. There is a basis $v_1, v_2, \ldots, v_n$ for V with $n \geq 2$. By 3.5, to define $S, T \in \mathcal{L}(V, V)$, it is enough to specify where the basis vectors $v_1, v_2, \ldots, v_n$ get mapped to. So let S be the linear map defined by $Sv_1 = v_2, Sv_2 = v_1$, and $Sv_j = v_j$ for $j \geq 2$, and let T be the linear map defined by $Tv_1 = 2v_2, Tv_2 = v_1$, and $Tv_j = v_j$ for $j \geq 2$. Then observe that

$$(ST - TS)(v_1) = S(Tv_1) - T(Sv_1) = S(2v_2) - Tv_2 = 2v_1 - v_1 = v_1 \neq 0.$$

Thus $ST \neq TS$.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edition.