1 Section 4.4 Exercises

Exercises with solutions from Section 4.4 of [UA].

Exercise 4.4.1. (a) Show that $f(x) = x^3$ is continuous on all of $\mathbf{R}$.

(b) Argue, using Theorem 4.4.5, that f is not uniformly continuous on $\mathbf{R}$.

(c) Show that f is uniformly continuous on any bounded subset of $\mathbf{R}$.

Solution. (a) As Example 4.3.5 shows, any polynomial is continuous on all of $\mathbf{R}$.

(b) Define sequences (x_n) and (y_n) by $x_n = n + \frac{1}{n}$ and $y_n = n$. Then

$$|x_n - y_n| = \tfrac{1}{n} \to 0 \quad \text{and} \quad |f(x_n) - f(y_n)| = 3n + \tfrac{3}{n} + \tfrac{1}{n^3} > 3.$$

Theorem 4.4.5 allows us to conclude that f is not uniformly continuous on $\mathbf{R}$.

(c) Suppose that $A \subseteq \mathbf{R}$ is a bounded subset of $\mathbf{R}$, so that there is an $M > 0$ such that $A \subseteq [-M, M]$. For any $x, y \in A$, it follows that

$$\left|x^2 + xy + y^2\right| \leq |x|^2 + |x||y| + |y|^2 \leq 3M^2.$$

Let $\epsilon > 0$ be given and set $\delta = \frac{\epsilon}{3M^2}$. For any $x, y \in A$ we then have

$$\left|x^3 - y^3\right| = |x - y|\left|x^2 + xy + y^2\right| \leq 3M^2\delta = \epsilon.$$

Thus f is uniformly continuous on A.

Exercise 4.4.2. (a) Is $f(x) = 1/x$ uniformly continuous on $(0, 1)$?

(b) Is $g(x) = \sqrt{x^2 + 1}$ uniformly continuous on $(0, 1)$?

(c) Is $h(x) = x \sin(1/x)$ uniformly continuous on $(0, 1)$?

Solution. (a) Define sequences (x_n) and (y_n) by $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. Then

$$|x_n - y_n| = \tfrac{1}{n} - \tfrac{1}{n+1} \to 0 \quad \text{and} \quad |f(x_n) - f(y_n)| = 1.$$

Theorem 4.4.5 allows us to conclude that f is not uniformly continuous on $\mathbf{R}$.

(b) If a function is uniformly continuous on some $B \subseteq \mathbf{R}$, then it is also uniformly continuous on any subset $A \subseteq B$. The function $g(x) = \sqrt{x^2 + 1}$ is continuous on all of $\mathbf{R}$, hence uniformly continuous on the compact set $[0, 1]$ (Theorem 4.4.7), and hence uniformly continuous on the subset $(0, 1)$.

(c) Define $h : \mathbf{R} \to \mathbf{R}$ by

$$h(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The continuity of h away from the origin is clear. As shown in Example 4.3.6, h is also continuous at the origin and thus continuous on all of $\mathbf{R}$. It follows that h is uniformly continuous on the compact set $[0, 1]$ (Theorem 4.4.7) and hence uniformly continuous on the subset $(0, 1)$.

Exercise 4.4.3. Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Solution. For any $x, y \in [1, \infty)$, we have

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = \frac{x+y}{x^2y^2}|x-y| = \left(\frac{1}{xy^2} + \frac{1}{x^2y}\right)|x-y| \leq 2|x-y|.$$

Let $\epsilon > 0$ be given and set $\delta = \frac{\epsilon}{2}$. For any $x, y \in [1, \infty)$ such that $|x - y| < \delta$, we then have

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \leq 2|x-y| < 2\delta = \epsilon.$$

Thus f is uniformly continuous on $[1, \infty)$.

Define the sequences (x_n) and (y_n) in $(0, 1]$ by $x_n = \frac{1}{\sqrt{n}}$ and $y_n = \frac{1}{\sqrt{n+1}}$. Then

$$|x_n - y_n| = \tfrac{1}{\sqrt{n}} - \tfrac{1}{\sqrt{n+1}} \to 0 \quad \text{and} \quad |f(x_n) - f(y_n)| = 1.$$

It follows from Theorem 4.4.5 that f is not uniformly continuous on $(0, 1]$.

Exercise 4.4.4. Decide whether each of the following statements is true or false, justifying each conclusion.

(a) If f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$ (meaning $1/f$ has bounded range).

(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

(c) If f is defined on $\mathbf{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbf{R}$.

Solution. (a) This is true. Since f is continuous on the compact set $[a,b]$, Theorem 4.4.2 implies that there exist $x_0, x_1 \in [a,b]$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in [a,b]$. By assumption we have $f(x_0) > 0$ and so

$$0 < f(x_0) \leq f(x) \leq f(x_1) \iff 0 < \frac{1}{f(x_1)} \leq \frac{1}{f(x)} \leq \frac{1}{f(x_0)}$$

for all $x \in [a,b]$, i.e. $1/f$ is bounded on $[a,b]$.

(b) This is true. Since A is bounded, there is a $K > 0$ such that $A \subseteq [-K, K]$, and since f is uniformly continuous on A, there is a $\delta > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta \implies |f(x) - f(y)| < 1.$$

Let $N \in \mathbf{N}$ be such that $\frac{2K}{N} < \delta$ and for each $j \in \{1, 2, \ldots, N\}$ define

$$I_j = \left[-K + (j-1)\frac{2K}{N}, -K + j\frac{2K}{N}\right],$$

so that $I_1 \cup \cdots \cup I_N = [-K, K]$. For $j \in \{1, 2, \ldots, N\}$, if $I_j \cap A \neq \emptyset$, then there exists some $a_j \in I_j \cap A$. Let

$$M = \max\{1 + |f(a_j)| : j \in \{1, 2, \ldots, N\} \text{ and } I_j \cap A \neq \emptyset\};$$

we are justified in taking the maximum of this set as it is finite and must be non-empty, since if A is non-empty (which we may as well assume) there must be some j such that $I_j \cap A \neq \emptyset$.

Suppose $x \in A$. Then since $I_1 \cup \cdots \cup I_N = [-K, K]$ and $A \subseteq [-K, K]$, there must be some $j \in \{1, 2, \ldots, N\}$ such that $x \in I_j \cap A$. Since $x, a_j \in I_j \cap A$, we then have $|x - a_j| \leq |I_j| = \frac{2K}{N} < \delta$ and thus

$$|f(x) - f(a_j)| < 1 \implies |f(x)| < 1 + |f(a_j)| \leq M.$$

It follows that $f(A) \subseteq [-M, M]$, i.e. that $f(A)$ is bounded.

(c) This is false. Let $f : \mathbf{R} \to \mathbf{R}$ be Dirichlet's function, i.e.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Then for any subset $A \subseteq \mathbf{R}$, the only possibilities for $f(A)$ are $\emptyset, \{0\}, \{1\}$, and $\{0, 1\}$; each of these is compact. However, f is nowhere-continuous.

Exercise 4.4.5. Assume that g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that g is uniformly continuous on (a, c).

Solution. Let $\epsilon > 0$ be given. There exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} x, y \in (a, b] \text{ and } |x - y| < \delta_1 &\implies |g(x) - g(y)| < \tfrac{\epsilon}{2}, \\ x, y \in [b, c) \text{ and } |x - y| < \delta_2 &\implies |g(x) - g(y)| < \tfrac{\epsilon}{2}. \end{aligned}$$

Set $\delta = \min\{\delta_1, \delta_2\}$ and suppose that $x, y \in (a, c)$ are such that $|x - y| < \delta$. There are four cases.

Case 1. If $x, y \in (a, b]$, then since $|x - y| < \delta \leq \delta_1$, we have $|g(x) - g(y)| < \frac{\epsilon}{2} < \epsilon$.

Case 2. If $x, y \in [b, c)$, then since $|x - y| < \delta \leq \delta_2$, we have $|g(x) - g(y)| < \frac{\epsilon}{2} < \epsilon$.

Case 3. If $x \in (a, b]$ and $y \in [b, c)$, then note that $|x - b| \leq |x - y| < \delta \leq \delta_1$ and $|b - y| \leq |x - y| < \delta \leq \delta_2$. It follows that $|g(x) - g(b)| < \frac{\epsilon}{2}$ and that $|g(b) - g(y)| < \frac{\epsilon}{2}$, which gives us

$$|g(x) - g(y)| \leq |g(x) - g(b)| + |g(b) - g(y)| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

Case 4. The case where $x \in [b, c)$ and $y \in (a, b]$ is handled similarly to Case 3.

In any case, we have $|g(x) - g(y)| < \epsilon$. It follows that g is uniformly continuous on (a, c).

Exercise 4.4.6. Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case.

(a) A continuous function $f : (0, 1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

(b) A uniformly continuous function $f : (0, 1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

(c) A continuous function $f : [0, \infty) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Solution. (a) Let $f : (0, 1) \to \mathbf{R}$ be given by $f(x) = \frac{1}{x}$ and consider the Cauchy sequence $(x_n) \subseteq (0, 1)$ given by $x_n = \frac{1}{n+1}$. Then $f(x_n) = n + 1$ which is not convergent and hence not Cauchy.

(b) This is impossible, as we will show in Exercise 4.4.13 (a).

(c) This is impossible. Let $f : [0, \infty) \to \mathbf{R}$ be continuous on its domain and let $(x_n) \subseteq [0, \infty)$ be a Cauchy sequence; equivalently, (x_n) is a convergent sequence, say $\lim x_n = x$. Since $[0, \infty)$ is a closed set, we must have $x \in [0, \infty)$, and then since f is continuous on its domain, it must be continuous at x. It follows that $\lim f(x_n) = f(x)$ and hence that $(f(x_n))$ is a Cauchy sequence.

Exercise 4.4.7. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Solution. f is continuous on the compact set $[0, 1]$ and hence is uniformly continuous on $[0, 1]$ (Theorem 4.4.7). Note that for any $x, y \in [1, \infty)$ we have

$$\left|\sqrt{x} - \sqrt{y}\right| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \tfrac{1}{2}|x - y|.$$

It is now straightforward to show that f is uniformly continuous on $[1, \infty)$ (see Exercise 4.4.9). By an argument analogous to the one given in Exercise 4.4.5, we may now conclude that f is uniformly continuous on $[0, \infty)$.

Exercise 4.4.8. Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0, 1]$ with range $(0, 1)$.

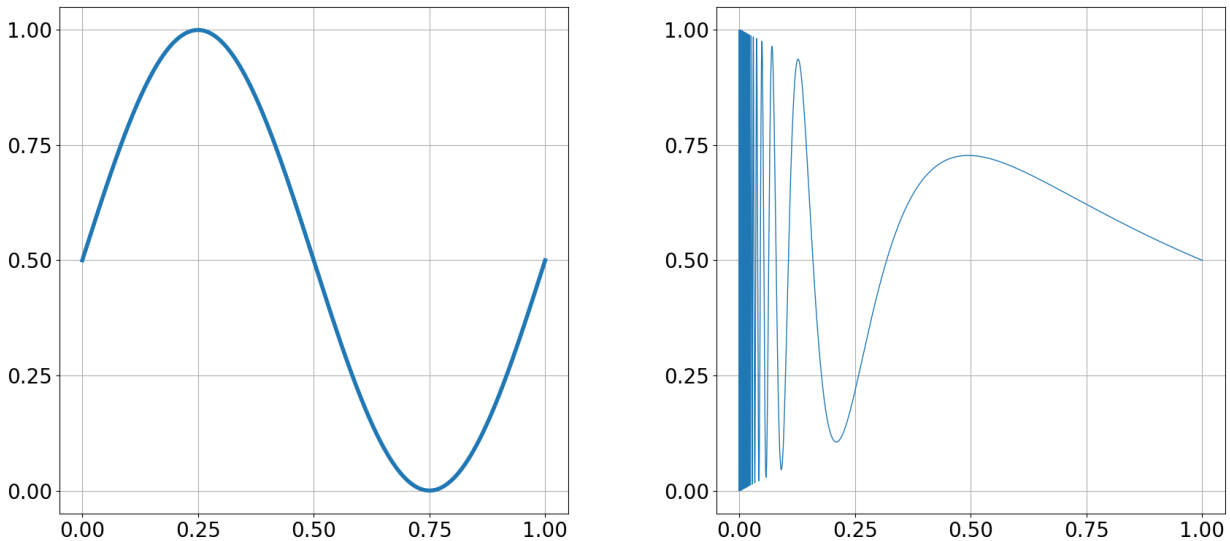
(b) A continuous function defined on $(0, 1)$ with range $[0, 1]$.

(c) A continuous function defined on $(0, 1]$ with range $(0, 1)$.

Solution. (a) This is impossible. If $f : [0, 1] \to \mathbf{R}$ is continuous, then since $[0, 1]$ is compact, the image of f must be compact (Theorem 4.4.1). However, $(0, 1)$ is not a compact set.

(b) Consider $f : (0, 1) \to \mathbf{R}$ given by $f(x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{2}$; the image of f is then $[0, 1]$ (see Figure 1).

(c) Consider $f : (0, 1] \to \mathbf{R}$ given by $f(x) = \frac{1}{2}(1 - x)\sin\left(\frac{1}{x}\right) + \frac{1}{2}$; the image of f is then $(0, 1)$ (see Figure 1).



(a) $\frac{1}{2}\sin(2\pi x) + \frac{1}{2}$ on $(0, 1)$

(b) $\frac{1}{2}(1 - x)\sin\left(\frac{1}{x}\right) + \frac{1}{2}$ on $(0, 1]$

Figure 1: Exercise 4.4.8 function graphs

Exercise 4.4.9 (Lipschitz Functions). A function $f : A \to \mathbf{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

for all $x \neq y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbf{R}$ is Lipschitz, then it is uniformly continuous on A.

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Solution. (a) Since f is Lipschitz, there is an $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in A$. Let $\epsilon > 0$ be given and set $\delta = \frac{\epsilon}{M}$. Then for any $x, y \in A$ satisfying $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq M|x - y| < M\delta = \epsilon.$$

It follows that f is uniformly continuous on A.

(b) The converse statement is not true. Consider $f : [0, \infty) \to \mathbf{R}$ given by $f(x) = \sqrt{x}$. As we showed in Exercise 4.4.7, this function is uniformly continuous on $[0, \infty)$. However, we claim that f is not Lipschitz on $[0, \infty)$. To show this, for each $M > 0$ we need to find some $x \neq y \in [0, \infty)$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| > M.$$

So, for $M > 0$, let $x = \frac{1}{4M^2}$ and $y = 0$. Then

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \left| \frac{\frac{1}{2M}}{\frac{1}{4M^2}} \right| = 2M > M.$$

Exercise 4.4.10. Assume that f and g are uniformly continuous functions defined on a common domain A. Which of the following combinations are necessarily uniformly continuous on A:

$$f(x) + g(x), \quad f(x)g(x), \quad \frac{f(x)}{g(x)}, \quad f(g(x))\,?$$

(Assume that the quotient and the composition are properly defined and thus at least continuous.)

Solution. We claim that $f + g$ is uniformly continuous on A. To see this, let $\epsilon > 0$ be given. There exist $\delta_1, \delta_2 > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta_1 \implies |f(x) - f(y)| < \tfrac{\epsilon}{2},$$
$$x, y \in A \text{ and } |x - y| < \delta_2 \implies |g(x) - g(y)| < \tfrac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and observe that for any $x, y \in A$ satisfying $|x - y| < \delta$, we then have

$$|f(x) + g(x) - f(y) - g(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

Thus $f + g$ is uniformly continuous on A.

The product fg need not be uniformly continuous. For a counterexample, consider $f, g : \mathbf{R} \to \mathbf{R}$ given by $f(x) = g(x) = x$. This function is clearly Lipschitz and hence uniformly continuous on all of $\mathbf{R}$ (Exercise 4.4.9). However, the product $f(x)g(x) = x^2$ is not uniformly continuous on $\mathbf{R}$; this can be seen using the same sequences as in Exercise 4.4.1 (b) and appealing to Theorem 4.4.5.

The quotient $\frac{f}{g}$ need not be uniformly continuous. For a counterexample, consider $f, g : (0, 1] \to \mathbf{R}$ given by $f(x) = 1$ and $g(x) = x$. Both are uniformly continuous, but the quotient $\frac{f(x)}{g(x)} = \frac{1}{x}$ is not (Exercise 4.4.2 (a)).

Suppose that $g(A) \subseteq A$, so that the composition $f \circ g : A \to \mathbf{R}$ is well-defined. We claim that this composition is also uniformly continuous. To see this, let $\epsilon > 0$ be given. There exists a $\delta_2 > 0$ such that

$$s, t \in A \text{ and } |s - t| < \delta_2 \implies |f(s) - f(t)| < \epsilon.$$

There then exists a $\delta_1 > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta_1 \implies |g(x) - g(y)| < \delta_2.$$

By assumption, if $x, y \in A$ then $g(x), g(y) \in A$. Thus

$$\begin{aligned} x, y \in A \text{ and } |x - y| < \delta_1 \implies g(x), g(y) \in A \text{ and } |g(x) - g(y)| < \delta_2 \\ \implies |f(g(x)) - f(g(y))| < \epsilon. \end{aligned}$$

Thus $f \circ g$ is uniformly continuous on A.

Exercise 4.4.11 (Topological Characterization of Continuity). Let g be defined on all of $\mathbf{R}$. If B is a subset of $\mathbf{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbf{R} : g(x) \in B\}.$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbf{R}$ is an open set.

Solution. Suppose g is continuous and $O \subseteq \mathbf{R}$ is an open set. Fix $c \in g^{-1}(O)$, so that $g(c) \in O$. Since O is open, there exists an $\epsilon > 0$ such that $V_\epsilon(g(c)) \subseteq O$, and since g is continuous at c, there is a $\delta > 0$ such that $x \in V_\delta(c)$ implies that $g(x) \in V_\epsilon(g(c)) \subseteq O$ (Theorem 4.3.2 (ii)). In other words, any $x \in V_\delta(c)$ also belongs to $g^{-1}(O)$, so that $V_\delta(c) \subseteq g^{-1}(O)$. It follows that $g^{-1}(O)$ is an open set.

Now suppose that $g^{-1}(O)$ is open whenever $O \subseteq \mathbf{R}$ is an open set. Fix $c \in \mathbf{R}$ and let $\epsilon > 0$ be given. The set $V_\epsilon(g(c))$ is open, so by assumption the set $g^{-1}[V_\epsilon(g(c))]$ is also open. Certainly we have $c \in g^{-1}[V_\epsilon(g(c))]$, so there exists a $\delta > 0$ such that $V_\delta(c) \subseteq g^{-1}[V_\epsilon(g(c))]$. It follows that if $x \in V_\delta(c)$, then $g(x) \in V_\epsilon(g(c))$ and so Theorem 4.3.2 (ii) allows us to conclude that g is continuous at each $c \in \mathbf{R}$.

Exercise 4.4.12. Review Exercise 4.4.11, and then determine which of the following statements is true about a continuous function defined on $\mathbf{R}$:

(a) $f^{-1}(B)$ is finite whenever B is finite.

(b) $f^{-1}(K)$ is compact whenever K is compact.

(c) $f^{-1}(A)$ is bounded whenever A is bounded.

(d) $f^{-1}(F)$ is closed whenever F is closed.

Solution. (a) This is false. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 0$. Then $f^{-1}(\{0\}) = \mathbf{R}$.

(b) This is false; see part (a) for a counterexample.

(c) This is false; see part (a) for a counterexample.

(d) This is true. If F is closed, then F^{c} is open. Since f is continuous, we have that $f^{-1}(F^{\mathsf{c}})$ is also open (Exercise 4.4.11). It follows that $(f^{-1}(F^{\mathsf{c}}))^{\mathsf{c}}$ is closed. This set is nothing but $f^{-1}(F)$:

$$x \in (f^{-1}(F^{\mathsf{c}}))^{\mathsf{c}} \iff x \notin f^{-1}(F^{\mathsf{c}}) \iff f(x) \notin F^{\mathsf{c}} \iff f(x) \in F \iff x \in f^{-1}(F).$$

Exercise 4.4.13 (Continuous Extension Theorem). (a) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbf{R}$ is uniformly continuous and $(x_n) \subseteq A$ is a Cauchy sequence, then show $f(x_n)$ is a Cauchy sequence.

(b) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended g is continuous.)

Solution. (a) Let $\epsilon > 0$ be given. Since f is uniformly continuous, there is a $\delta > 0$ such that for any $x, y \in A$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Since $(x_n) \subseteq A$ is a Cauchy sequence, there is an $N \in \mathbf{N}$ such that for all $n > m \geq N$, we have $|x_n - x_m| < \delta$, which implies that $|f(x_n) - f(x_m)| < \epsilon$. Thus $(f(x_n))$ is also a Cauchy sequence.

(b) Suppose that g is uniformly continuous on (a, b). Define a sequence $a_n = a + \frac{b-a}{2n}$, so that $(a_n) \subseteq (a, b)$ and $\lim a_n = a$. Then since (a_n) is Cauchy, part (a) implies that the sequence $(g(a_n))$ is also Cauchy and hence convergent, say $\lim g(a_n) = y \in \mathbf{R}$. Define $g(a) := y$.

We claim that this extended g is continuous at a. Let $(x_n) \subseteq (a, b)$ be a sequence such that $\lim x_n = a$ and let $\epsilon > 0$ be given. Since g is uniformly continuous on (a, b), there is a $\delta > 0$ such that for any $x, y \in (a, b)$ satisfying $|x - y| < \delta$, we have $|g(x) - g(y)| < \epsilon$. Note that $\lim |x_n - a_n| = 0$ since $\lim x_n = \lim a_n = a$, so there is an $N \in \mathbf{N}$ such that $n \geq N$ implies that $|x_n - a_n| < \delta$, which gives us $|g(x_n) - g(a_n)| < \epsilon$. Thus $\lim |g(x_n) - g(a_n)| = 0$. Combining this with $\lim g(a_n) = g(a)$, we see that $\lim g(x_n) = g(a)$ also and hence g is continuous at a.

An analogous argument shows that we can also continuously extend g to be defined at b by considering the sequence $b_n = b - \frac{b-a}{2n}$.

For the converse implication, we apply Theorem 4.4.7 to see that g is uniformly continuous on the compact set $[a, b]$ and hence uniformly continuous on the subset (a, b).

Exercise 4.4.14. Construct an alternate proof of Theorem 4.4.7 using the open cover characterization of compactness from the Heine-Borel Theorem (Theorem 3.3.8 (iii)).

Solution. Suppose $f : K \to \mathbf{R}$ is continuous, where K is compact. Let $\epsilon > 0$ be given. Since f is continuous on K, for each $t \in K$ there exists a $\delta_t > 0$ such that

$$x \in K \text{ and } |x - t| < \delta_t \implies |f(x) - f(t)| < \tfrac{\epsilon}{2}.$$

Observe that the collection $\{V_{\delta_t/2}(t) : t \in K\}$ forms an open cover of K. Since K is compact, there exists a finite subcover $\{V_{\delta_{t_1}/2}(t_1), \ldots, V_{\delta_{t_n}/2}(t_n)\}$. Let $\delta = \min\{\delta_{t_1}/2, \ldots, \delta_{t_n}/2\}$ and suppose that $x, y \in K$ are such that $|x - y| < \delta$. There is a $j \in \{1, \ldots, n\}$ such that $x \in V_{\delta_{t_j}/2}(t_j)$, so that $|x - t_j| < \delta_{t_j}/2 < \delta_{t_j}$ and thus $|f(x) - f(t_j)| < \frac{\epsilon}{2}$. Note that

$$|y - t_j| \leq |x - y| + |x - t_j| < \delta + \delta_{t_j}/2 \leq \delta_{t_j}/2 + \delta_{t_j}/2 = \delta_{t_j}.$$

It follows that $|f(y) - f(t_j)| < \frac{\epsilon}{2}$ and hence that

$$|f(x) - f(y)| \leq |f(x) - f(t_j)| + |f(y) - f(t_j)| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

Thus f is uniformly continuous on K.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.