1 Section 1.B Exercises

Exercises with solutions from Section 1.B of [LADR].

Exercise 1.B.1. Prove that $-(-v) = v$ for every $v \in V$.

Solution. Since $v + (-v) = 0$ and additive inverses are unique (1.26), it must be the case that $-(-v) = v$.

Exercise 1.B.2. Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution. It will suffice to show that if $av = 0$ and $a \neq 0$, then $v = 0$. Since $a \neq 0$, a^{-1} exists. Then

$$av = 0 \implies a^{-1}(av) = a^{-1}0 \implies (a^{-1}a)v = 0 \implies 1v = 0 \implies v = 0.$$

Exercise 1.B.3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution. If $v, w, x \in V$ are such that $v + 3x = w$, then

$$v + 3x = w \implies 3x = w - v \implies x = \tfrac{1}{3}(w - v);$$

conversely, it is easily verified that $x = \frac{1}{3}(w - v)$ satisfies the equation $v + 3x = w$.

Exercise 1.B.4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Solution. The empty set does not contain an additive identity.

Exercise 1.B.5. Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V. (The phrase "a condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.)

Solution. If V satisfies all of the conditions in (1.19), then as shown in (1.26) we have $0v = 0$ for all $v \in V$. Now suppose that V satisfies all of the conditions in (1.19), except we have replaced the additive inverse condition with the condition that $0v = 0$ for all $v \in V$. We want to show that for each $v \in V$, there exists an element w such that $v + w = 0$. Let $v \in V$ be given and set $w = (-1)v$. Then

$$v + w = 1v + (-1)v = (1 - 1)v = 0v = 0.$$

Exercise 1.B.6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\,\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty, \quad \infty + (-\infty) = 0.$$

Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution. This is not a vector space. By (1.25), the additive identity in a vector space must be unique. However, we have $\infty + t = \infty$ for all $t \in \mathbf{R}$.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edn.