1 Section 2.3 Exercises

Exercises with solutions from Section 2.3 of [UA].

Exercise 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Solution. (a) Let $\epsilon > 0$ be given. Then there exists an $N \in \mathbf{N}$ such that

$$n \geq N \implies |x_n - 0| = x_n < \epsilon^2 \iff \sqrt{x_n} < \epsilon.$$

It follows that $\lim(\sqrt{x_n}) = 0$.

(b) By Theorem 2.3.4, we must have $x \geq 0$. The case $x = 0$ was handled in part (a), so suppose that $x > 0 \iff \sqrt{x} > 0$. Let $\epsilon > 0$ be given. Then there exists an $N \in \mathbf{N}$ such that $n \geq N \implies |x_n - x| < \epsilon\sqrt{x}$. Observe that

$$\left|\sqrt{x_n} - \sqrt{x}\right|\left|\sqrt{x_n} + \sqrt{x}\right| = |x_n - x| \iff \left|\sqrt{x_n} - \sqrt{x}\right| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}.$$

So if we take $n \geq N$ we will have

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leq \frac{|x_n - x|}{\sqrt{x}} < \epsilon.$$

It follows that $\lim(\sqrt{x_n}) = \sqrt{x}$.

Exercise 2.3.2. Using only Definition 2.2.3, prove that if $(x_n) \to 2$ then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

Solution. (a) Let $\epsilon > 0$ be given. There exists an $N \in \mathbf{N}$ such that $n \geq N$ implies that $|x_n - 2| < \frac{3\epsilon}{2}$. Then for $n \geq N$ we have

$$\left|\frac{2x_n - 1}{3} - 1\right| = \left|\frac{2x_n - 4}{3}\right| = \tfrac{2}{3}|x_n - 2| < \epsilon.$$

It follows that $\left(\frac{2x_n - 1}{3}\right) \to 1$.

(b) There is an $N_1 \in \mathbf{N}$ such that $n \geq N_1 \implies |x_n - 2| < 1$. Then for $n \geq N_1$ we have

$$2 \leq |x_n - 2| + |x_n| < 1 + |x_n| \implies 1 < |x_n| \implies \frac{1}{|x_n|} < 1.$$

Let $\epsilon > 0$ be given. There is an $N_2 \in \mathbf{N}$ such that $n \geq N_2 \implies |x_n - 2| < 2\epsilon$. Set $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{2 - x_n}{2x_n}\right| = \frac{|x_n - 2|}{2|x_n|} < \frac{|x_n - 2|}{2} < \epsilon.$$

It follows that $(1/x_n) \to 1/2$.

Exercise 2.3.3 (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution. Let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |x_n - l| < \epsilon \iff -\epsilon < x_n - l < \epsilon,$$

$$n \geq N_2 \implies |z_n - l| < \epsilon \iff -\epsilon < z_n - l < \epsilon.$$

Let $N = \max\{N_1, N_2\}$. Then since $x_n - l \leq y_n - l \leq z_n - l$ for all $n \in \mathbf{N}$, for $n \geq N$ we have

$$-\epsilon < y_n - l < \epsilon \iff |y_n - l| < \epsilon.$$

It follows that $\lim y_n = l$.

Exercise 2.3.4. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$.

Solution. The manipulations of limits in these solutions are justified by the Algebraic Limit Theorem.

(a) We have

$$\lim \left(\frac{1 + 2a_n}{1 + 3a_n - 4a_n^2}\right) = \frac{1 + 2 \lim a_n}{1 + 3 \lim a_n - 4(\lim a_n)^2} = \frac{1}{1} = 1.$$

(b) We have

$$\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)=\lim\left(\frac{a_n^2+4a_n}{a_n}\right)=\lim(a_n+4)=\lim a_n+4=4.$$

(c) We have

$$\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)=\lim\left(\frac{2+3a_n}{1+5a_n}\right)=\frac{2+3\lim a_n}{1+5\lim a_n}=\frac{2}{1}=2.$$

Exercise 2.3.5. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution. (z_n) is the sequence given by

$$z_n=\begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd,}\\ y_{\frac{n}{2}} & \text{if } n \text{ is even.}\end{cases}$$

Suppose that (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n = l$ for some $l \in \mathbf{R}$. Then there are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |x_n - l| < \epsilon \qquad \text{and} \qquad n \geq N_2 \implies |y_n - l| < \epsilon.$$

Let $N = \max\{N_1, N_2\}$ and suppose $n \in \mathbf{N}$ is such that $n \geq 2N$. If n is odd then $\frac{n+1}{2} \in \mathbf{N}$ and

$$n \geq 2N > 2N-1 \implies \tfrac{n+1}{2} > N \geq N_1 \implies \left|x_{\frac{n+1}{2}} - l\right| < \epsilon.$$

Hence

$$|z_n - l| = \left|x_{\frac{n+1}{2}} - l\right| < \epsilon.$$

If n is even then $\frac{n}{2} \in \mathbf{N}$ and

$$n \geq 2N \implies \tfrac{n}{2} \geq N \geq N_2 \implies \left|y_{\frac{n}{2}} - l\right| < \epsilon.$$

Hence

$$|z_n - l| = \left|y_{\frac{n}{2}} - l\right| < \epsilon.$$

In either case we have $|z_n - l| < \epsilon$, i.e.

$$n \geq 2N \implies |z_n - l| < \epsilon.$$

It follows that $\lim z_n = l$.

Now suppose that (z_n) is convergent with $\lim z_n = l$ for some $l \in \mathbf{R}$. Let $\epsilon > 0$ be given. Then there exists $N \in \mathbf{N}$ such that $n \geq N \implies |z_n - l| < \epsilon$. Suppose $n \in \mathbf{N}$ is such that $n \geq N$. Then $2n > 2n-1 \geq N$, so

$$|x_n - l| = |z_{2n-1} - l| < \epsilon \qquad \text{and} \qquad |y_n - l| = |z_{2n} - l| < \epsilon.$$

It follows that $\lim x_n = \lim y_n = l$.

Exercise 2.3.6. Consider the sequence given by $b_n = n - \sqrt{n^2+2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Solution. Observe that

$$b_n = n - \sqrt{n^2+2n} = \frac{(n-\sqrt{n^2+2n})(n+\sqrt{n^2+2n})}{n+\sqrt{n^2+2n}} = \frac{-2n}{n+\sqrt{n^2+2n}} = \frac{-2}{1+\sqrt{1+(2/n)}}.$$

Hence

$$\lim b_n = \lim\left(\frac{-2}{1+\sqrt{1+(2/n)}}\right) = \frac{-2}{1+\sqrt{1+2\lim(1/n)}} = \frac{-2}{1+\sqrt{1}} = -1.$$

Exercise 2.3.7. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Solution. (a) Take $x_n = n$ and $y_n = -n$.

(b) This is impossible. If (x_n) and $(x_n + y_n)$ both converge, then by the Algebraic Limit Theorem, (y_n) must be convergent with limit $\lim y_n = \lim(x_n + y_n) - \lim x_n$.

(c) Take $b_n = 1/n$.

(d) This is impossible; $(a_n - b_n)$ must be unbounded. Since (b_n) is convergent, it must be bounded (Theorem 2.3.2), i.e. there exists some $m \geq 0$ such that $|b_n| \leq m$ for all $n \in \mathbf{N}$. Let $M \geq 0$ be given. Since (a_n) is unbounded, there exists some $N \in \mathbf{N}$ such that $|a_N| \geq M + m$. Then observe that

$$|a_N - b_N| \geq ||a_N| - |b_N|| \geq |a_N| - |b_N| \geq M + m - m = M.$$

(e) Take $a_n = 1/n^2$ and $b_n = n$.

Exercise 2.3.8. Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Solution. (a) Suppose $p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$. The Algebraic Limit Theorem and some simple induction arguments allow us to make the following manipulations:

$$\begin{aligned}
\lim(p(x_n)) &= \lim(a_m x_n^m + a_{m-1}x_n^{m-1} + \cdots + a_1 x_n + a_0) \\
&= a_m(\lim x_n)^m + a_{m-1}(\lim x_n)^{m-1} + \cdots + a_1 \lim x_n + a_0 \\
&= a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \\
&= p(x).
\end{aligned}$$

(b) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise} \end{cases}$$

and the convergent sequence $(x_n) = (1/n) \to 0$. Then the sequence $(f(x_n)) = (1, 1, 1, \ldots)$ converges to $1 \neq 0 = f(0)$.

Exercise 2.3.9. (a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution. (a) There is an $M > 0$ such that $|a_n| \le M$ for all $n \in \mathbf{N}$. Let $\epsilon > 0$ be given. Then there is an $N \in \mathbf{N}$ such that

$$n \ge N \implies |b_n| < \frac{\epsilon}{M}.$$

Observe that for $n \ge N$ we have

$$|a_n b_n| = |a_n||b_n| \le M|b_n| < \frac{M\epsilon}{M} = \epsilon.$$

It follows that $\lim(a_n b_n) = 0$. We may not use the Algebraic Limit Theorem here since the sequence (a_n) is not necessarily convergent; the hypotheses of that theorem require both sequences (a_n) and (b_n) to be convergent.

(b) If the sequence (a_n) is convergent to some a then we may use the Algebraic Limit Theorem to conclude that $\lim(a_n b_n) = ab$. If the sequence (a_n) is divergent, then $(a_n b_n)$ must also be divergent. To see this, we will prove the contrapositive, i.e. if $(a_n b_n)$ is convergent to some x then (a_n) is convergent. Indeed, since $b \ne 0$, the Algebraic Limit Theorem implies

$$\lim a_n = \lim \left(\frac{a_n b_n}{b_n} \right) = \frac{x}{b}.$$

(c) Since (b_n) is convergent, it is bounded (Theorem 2.3.2). So we may apply part (a) (we have swapped the roles of (a_n) and (b_n)) to conclude that

$$\lim(a_n b_n) = 0 = 0b = ab.$$

Exercise 2.3.10. Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \le a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Solution. (a) This is true if (a_n) and (b_n) are convergent sequences (Algebraic Limit Theorem), however it may be the case that they are both divergent and $\lim(a_n - b_n) = 0$; for example, $a_n = b_n = n$. So the conjecture is false in general.

(b) This is true. Let $\epsilon > 0$ be given. Then there is an $N \in \mathbf{N}$ such that $n \geq N \implies |b_n - b| < \epsilon$. For $n \geq N$ the reverse triangle inequality gives

$$||b_n| - |b|| \leq |b_n - b| < \epsilon.$$

It follows that $\lim|b_n| = |b|$.

(c) This is true. Using the Algebraic Limit Theorem, we have

$$\lim b_n = \lim(b_n - a_n + a_n) = \lim(b_n - a_n) + \lim a_n = 0 + a = a.$$

(d) This is true. Since $0 \leq |b_n - b| \leq a_n$ for every $n \in \mathbf{N}$, the Squeeze Theorem (Exercise 2.3.3) implies that $\lim|b_n - b| = 0$, i.e. for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n \geq N \implies ||b_n - b| - 0| = |b_n - b| < \epsilon,$$

which is exactly the statement $\lim b_n = b$.

Exercise 2.3.11 (Cesaro Means). (a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution. (a) Suppose $\lim x_n = x$. Let $\epsilon > 0$ be given. There is a positive integer $N_1 \geq 2$ such that

$$n \geq N_1 \implies |x_n - x| < \frac{\epsilon}{2}.$$

Given this N_1, there is an $N_2 \in \mathbf{N}$ such that

$$n \geq N_2 \implies \frac{|x_1 - x| + \cdots + |x_{N_1-1} - x|}{n} < \frac{\epsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$\begin{aligned}
|y_n - x| &= \left| \frac{x_1 + \cdots + x_n}{n} - x \right| \\
&= \left| \frac{x_1 + \cdots + x_n}{n} - \frac{nx}{n} \right| \\
&= \left| \frac{(x_1 - x) + \cdots + (x_n - x)}{n} \right| \\
&\leq \frac{|x_1 - x| + \cdots + |x_{N_1 - 1} - x|}{n} + \frac{|x_{N_1} - x| + \cdots + |x_n - x|}{n} \\
&< \frac{\epsilon}{2} + \frac{n - N_1 + 1}{n} \cdot \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

It follows that $\lim y_n = x$.

(b) Consider the divergent sequence $x_n = (-1)^{n+1}$. The sequence of averages (y_n) is then

$$y_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

which satisfies $\lim y_n = 0$.

Exercise 2.3.12. A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

Solution. (a) This is true. Let $b \in B$ be given. Then $b \leq a_n$ for all $n \in \mathbf{N}$, so by the Order Limit Theorem we have $b \leq a$. It follows that a is an upper bound for B.

(b) This is true. Observe that for a real number x we have

$$x \notin (0,1) \iff x \leq 0 \text{ or } x \geq 1 \iff \left|x - \tfrac{1}{2}\right| \geq \tfrac{1}{2}.$$

So for each $n \in \mathbf{N}$ we have $\left|a_n - \frac{1}{2}\right| \geq \frac{1}{2}$. The Algebraic Limit Theorem and Exercise 2.3.10 (b) imply that $\lim\left|a_n - \frac{1}{2}\right| = \left|a - \frac{1}{2}\right|$, and hence the Order Limit Theorem implies that $\left|a - \frac{1}{2}\right| \geq \frac{1}{2}$. It follows that a belongs to the complement of $(0,1)$.

(c) This is false. By the density of $\mathbf{Q}$ in $\mathbf{R}$, for each $n \in \mathbf{N}$ we may pick a rational number a_n satisfying $\sqrt{2} < a_n < \sqrt{2} + \frac{1}{n}$. The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim a_n = \sqrt{2}$, an irrational number.

Exercise 2.3.13 (Iterated Limits). Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the *iterated* limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \qquad \text{and} \qquad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right).$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2+n^2)$.

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) = b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

Solution. (a) We apply the Algebraic Limit Theorem.

$$\lim_{m\to\infty} a_{mn} = \lim_{m\to\infty}\left(\frac{m}{m+n}\right) = \lim_{m\to\infty}\left(\frac{1}{1+\frac{n}{m}}\right) = \frac{1}{1 + n\lim_{m\to\infty}\left(\frac{1}{m}\right)} = \frac{1}{1} = 1.$$

Hence $\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{n\to\infty}(1) = 1$. Similarly,

$$\lim_{n\to\infty} a_{mn} = \lim_{n\to\infty}\left(\frac{m}{m+n}\right) = \lim_{n\to\infty}\left(\frac{\frac{m}{n}}{1+\frac{m}{n}}\right) = \frac{m\lim_{n\to\infty}\left(\frac{1}{n}\right)}{1 + m\lim_{n\to\infty}\left(\frac{1}{n}\right)} = \frac{0}{1} = 0.$$

Hence $\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = \lim_{m\to\infty}(0) = 0$.

(b) For $a_{mn} = 1/(m+n)$, we have $\lim_{m,n\to\infty} a_{mn} = 0$. To see this, let $\epsilon > 0$ be given. There is an $N \in \mathbf{N}$ such that $n \geq N \implies \frac{1}{n} < \epsilon$. Then for $m, n \geq N$ we have

$$|a_{mn} - 0| = \frac{1}{m+n} < \frac{1}{n} < \epsilon.$$

The two iterated limits also exist and are equal to 0. To see this, observe that for all $m \in \mathbf{N}$ we have $0 < \frac{1}{m+n} < \frac{1}{m}$. Then by the Squeeze Theorem, $\lim_{m\to\infty} a_{mn} = \lim_{m\to\infty} \frac{1}{m+n} = 0$. It follows that $\lim_{n\to\infty} (\lim_{m\to\infty} a_{mn}) = \lim_{n\to\infty}(0) = 0$. Similarly, $\lim_{m\to\infty} (\lim_{n\to\infty} a_{mn}) = 0$.

Now let $a_{mn} = mn/(m^2+n^2)$. We claim that $\lim_{m,n\to\infty} a_{mn}$ does not exist. To see this, suppose that $\lim_{m,n\to\infty} a_{mn} = x$ for some $x \in \mathbf{R}$. Then there exists some $N \in \mathbf{N}$ such that $m, n \geq N \implies |a_{mn} - x| < \frac{1}{20}$. In particular, taking $n = m$,

$$m \geq N \implies \left|\frac{m^2}{m^2+m^2} - x\right| = \left|\tfrac{1}{2} - x\right| < \tfrac{1}{20} \iff x \in \left(\tfrac{9}{20}, \tfrac{11}{20}\right).$$

Furthermore, taking $n = 2m$,

$$m \geq N \implies \left|\frac{2m^2}{m^2+4m^2} - x\right| = \left|\tfrac{2}{5} - x\right| < \tfrac{1}{20} \iff x \in \left(\tfrac{7}{20}, \tfrac{9}{20}\right).$$

So assuming that $\lim_{m,n\to\infty} a_{mn} = x$ leads us to the contradiction that $x < \frac{9}{20}$ and $x > \frac{9}{20}$. It follows that $\lim_{m,n\to\infty} a_{mn}$ does not exist. However, the two iterated limits do exist and are equal to 0. Using the Algebraic Limit Theorem, we have

$$\lim_{m\to\infty}\left(\frac{mn}{m^2+n^2}\right) = \lim_{m\to\infty}\left(\frac{\frac{n}{m}}{1+\frac{n^2}{m^2}}\right) = \frac{n\lim_{m\to\infty}\left(\frac{1}{m}\right)}{1+n^2\lim_{m\to\infty}\left(\frac{1}{m^2}\right)} = \frac{0}{1} = 0.$$

It follows that $\lim_{n\to\infty} (\lim_{m\to\infty} a_{mn}) = 0$ and similarly that $\lim_{m\to\infty} (\lim_{n\to\infty} a_{mn}) = 0$.

(c) Let $a_{mn} = (-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right)$. We claim that $\lim_{m,n\to\infty} a_{mn} = 0$. To see this, let $\epsilon > 0$ be given. There is an $N \in \mathbf{N}$ such that $n \geq N \implies \frac{1}{n} < \frac{\epsilon}{2}$. Then for $m, n \geq N$ we have

$$|a_{mn}| = \left|(-1)^{m+n}\left(\frac{1}{m}+\frac{1}{n}\right)\right| = \frac{1}{m} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

However, neither iterated limit exists. Fix $n \in \mathbf{N}$ and observe that

$$\begin{aligned}
|a_{mn} - a_{m+1,n}| &= \left|(-1)^{m+n}\left(\frac{1}{m}+\frac{1}{n}\right) - (-1)^{m+n+1}\left(\frac{1}{m+1}+\frac{1}{n}\right)\right| \\
&= \left|\frac{1}{m}+\frac{1}{n}+\frac{1}{m+1}+\frac{1}{n}\right| \\
&= \frac{1}{m}+\frac{1}{m+1}+\frac{2}{n} \\
&\geq \frac{2}{n}.
\end{aligned}$$

Since $n \in \mathbf{N}$ is fixed, this implies that the sequence $(a_{mn} - a_{m+1,n})_{m\in\mathbf{N}}$ cannot converge to 0. Now observe that for any sequence (b_m), the Algebraic Limit Theorem implies that

$$\lim_{m\to\infty} b_m = x \text{ for some } x \in \mathbf{R} \implies \lim_{m\to\infty}(b_m - b_{m+1}) = 0.$$

The contrapositive of this statement then implies that the limit $\lim_{m\to\infty} a_{mn}$ does not exist for any $n \in \mathbf{N}$. It follows that the iterated limit $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn})$ does not exist. Swapping the roles of m and n in our argument shows that the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn})$ does not exist either.

(d) Seeking a contradiction, suppose that (b_m) does not converge to a, i.e. there exists some $\epsilon > 0$ such that for all $N \in \mathbf{N}$ there is an $M \geq N$ such that $|b_M - a| \geq \epsilon$. Since $\lim_{m,n\to\infty} a_{mn} = a$, there exists some $N_1 \in \mathbf{N}$ such that

$$m, n \geq N_1 \implies |a_{mn} - a| < \tfrac{\epsilon}{2}. \tag{1}$$

Then by the previous sentence, there exists $M \geq N_1$ such that $|b_M - a| \geq \epsilon$. By assumption, we have $\lim_{n\to\infty} a_{Mn} = b_M$, so there is an $N_2 \in \mathbf{N}$ such that $n \geq N_2 \implies |a_{Mn} - b_M| < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$ and observe that $|a_{MN} - a| < \frac{\epsilon}{2}$ by (1). However, the reverse triangle inequality gives us

$$\begin{aligned}
|a_{MN} - a| &= |a_{MN} - b_M + b_M - a| \\
&\geq ||b_M - a| - |a_{MN} - b_M|| \\
&\geq |b_M - a| - |a_{MN} - b_M| \\
&> \epsilon - \tfrac{\epsilon}{2} \\
&= \tfrac{\epsilon}{2}.
\end{aligned}$$

So assuming that (b_m) does not converge to a leads us the contradiction that there exist positive integers M and N such that $|a_{MN} - a|$ is both less than and greater than $\frac{\epsilon}{2}$. Hence it must be the case that $\lim_{m\to\infty} b_m = a$.

(e) If the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn})$ exists, then it must be the case that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty} a_{mn}$ exists. Then by part (d), we must have

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = \lim_{m,n\to\infty} a_{mn}.$$

Swapping the roles of m and n and repeating the above argument shows that

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{m,n\to\infty} a_{mn}.$$

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edn.