1 Section 2.B Exercises

Exercises with solutions from Section 2.B of [LADR].

Exercise 2.B.1. Find all vector spaces that have exactly one basis.

Solution. We will consider only finite-dimensional vector spaces over $\mathbf{R}$ or $\mathbf{C}$.

First consider the trivial vector space $\{0\}$. There are two possible lists of vectors: the empty list $(\,)$ and the list 0. Since any list containing the zero vector is linearly dependent, the list 0 cannot be a basis for $\{0\}$. By definition the empty list is linearly independent and satisfies $\operatorname{span}(\,) = \{0\}$, so we see that the empty list is a basis for $\{0\}$. Thus the trivial vector space has exactly one basis.

Now suppose that $V \neq \{0\}$. By (2.32), V has a basis $v_1, \ldots, v_m$. Since $V \neq \{0\}$, this basis is not the empty list, so v_1 exists and is non-zero. It follows that $B := 2v_1, \ldots, 2v_m$ is distinct from $v_1, \ldots, v_m$. By Exercise 2.A.8, B is linearly independent. Furthermore, we claim that $\operatorname{span} B = V$. To see this, let $v \in V$ be given. Since $v_1, \ldots, v_m$ is a basis, there are scalars $a_1, \ldots, a_m$ such that $v = \sum_{j=1}^m a_j v_j$. This is equivalent to

$$v = \sum_{j=1}^{m} \left(\tfrac{1}{2}a_j\right)(2v_j),$$

whence $v \in \operatorname{span} B$. It follows that $V = \operatorname{span} B$ and hence that B is a basis for V, distinct from the original basis $v_1, \ldots, v_m$. We may conclude that the trivial vector space is the only vector space which has exactly one basis.

Exercise 2.B.2. Verify all the assertions in Example 2.28.

Solution. (a) The assertion is that the list $B := (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is a basis of $\mathbf{F}^n$. Since any $(x_1, x_2, \ldots, x_n) \in \mathbf{F}^n$ can be written as

$$x_1(1, 0, \ldots, 0) + x_2(0, 1, 0, \ldots, 0) + \cdots + x_n(0, \ldots, 0, 1),$$

it is clear that B spans $\mathbf{F}^n$ and that B is linearly independent.

(b) The assertion is that the list $B := (1, 2), (3, 5)$ is a basis of $\mathbf{F}^2$. Solving the two equations $x + 3y = 0$ and $3x + 5y = 0$ gives $x = y = 0$, demonstrating that B is linearly independent. If $(a, b) \in \mathbf{F}^2$, then observe that

$$(-5a + 3b)(1, 2) + (2a - b)(3, 5) = (a, b).$$

Hence $\operatorname{span} B = V$ and we may conclude that B is a basis of $\mathbf{F}^2$.

(c) The assertion is that the list $B := (1, 2, -4), (7, -5, 6)$ is linearly independent in $\mathbf{F}^3$ but is not a basis of $\mathbf{F}^3$ because it does not span $\mathbf{F}^3$. Solving the equations $x + 7y = 0$ and $2x - 5y = 0$ gives $x = y = 0$, demonstrating that B is linearly independent. However, since the list $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ of length 3 is linearly independent in $\mathbf{F}^3$ (see (a)), (2.23) implies that B cannot span $\mathbf{F}^3$.

(d) The assertion is that the list $B := (1, 2), (3, 5), (4, 13)$ spans $\mathbf{F}^2$ but is not a basis of $\mathbf{F}^2$ because it is not linearly independent. Part (b) shows that B spans $\mathbf{F}^2$ and that $(4, 13)$ lies in the span of $(1, 2)$ and $(3, 5)$, so that B is linearly dependent.

(e) The assertion is that the list $B := (1, 1, 0), (0, 0, 1)$ is a basis of $U := \{(x, x, y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}$. $\operatorname{span} U = B$ since $x(1, 1, 0) + y(0, 0, 1) = (x, x, y)$, and B is linearly independent since $(x, x, y) = (0, 0, 0)$ forces $x = y = 0$.

(f) The assertion is that the list $B := (1, -1, 0), (1, 0, -1)$ is a basis of

$$U := \{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\} = \{(-y - z, y, z) \in \mathbf{F}^3 : y, z \in \mathbf{F}\}.$$

B is linearly independent since

$$y(1, -1, 0) + z(1, 0, -1) = (y + z, -y, -z) = (0, 0, 0)$$

gives $y = z = 0$, and B spans U since

$$(-y - z, y, z) = (-y)(1, -1, 0) + (-z)(1, 0, -1).$$

(g) The assertion is that the list $B := 1, z, \ldots, z^m$ is a basis of $\mathcal{P}_m(\mathbf{F})$. B is linearly independent by Exercise 2.A.2 (d), and B spans $\mathcal{P}_m(\mathbf{F})$ since any polynomial in $\mathcal{P}_m(\mathbf{F})$ is of the form

$$a_0 + a_1 z + \cdots + a_m z^m$$

for scalars $a_0, a_1, \ldots, a_m$.

Exercise 2.B.3. (a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbf{R}^5$.

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

Solution. (a) U is the subspace

$$U = \{(3x_2, x_2, 7x_4, x_4, x_5) \in \mathbf{R}^5 : x_2, x_4, x_5 \in \mathbf{R}\}.$$

Let $v_1 = (3, 1, 0, 0, 0), v_2 = (0, 0, 7, 1, 0), v_3 = (0, 0, 0, 0, 1)$, and $B = v_1, v_2, v_3$. Then since

$$x_2 v_1 + x_4 v_2 + x_5 v_3 = (3x_2, x_2, 7x_4, x_4, x_5),$$

it is clear that B spans U and that B is linearly independent.

(b) Denote the i^{th} standard basis vector of $\mathbf{R}^5$ by e_i. Following the procedure outlined in (2.31) and (2.33), we adjoin the five standard basis vectors to B to obtain the spanning list

$$v_1, v_2, v_3, e_1, e_2, e_3, e_4, e_5.$$

- e_1 does not belong to $\operatorname{span}(v_1, v_2, v_3)$, so we do not delete it.
- Note that $e_2 = v_1 - 3e_1$, so we delete e_2 from the list.
- e_3 does not belong to $\operatorname{span}(v_1, v_2, v_3, e_1)$, so we do not delete it.
- Note that $e_4 = v_2 - 7e_3$, so we delete e_4 from the list.
- Since $e_5 = v_3$, we delete e_5 from the list.

We are left with the list v_1, v_2, v_3, e_1, e_3; as the proof of (2.33) shows, this must be a basis of $\mathbf{R}^5$.

(c) As shown in the proof of (2.34), if we let

$$W = \operatorname{span}(e_1, e_3) = \{(x_1, 0, x_3, 0, 0) \in \mathbf{R}^5 : x_1, x_3 \in \mathbf{R}\},$$

then $\mathbf{R}^5 = U \oplus W$.

Exercise 2.B.4. (a) Let U be the subspace of $\mathbf{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbf{C}^5$.

(c) Find a subspace W of $\mathbf{C}^5$ such that $\mathbf{C}^5 = U \oplus W$.

Solution. (a) U is the subspace

$$\{(z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5) \in \mathbf{C}^5 : z_1, z_4, z_5 \in \mathbf{C}\}.$$

Let $v_1 = (1, 6, 0, 0, 0), v_2 = (0, 0, -2, 1, 0), v_3 = (0, 0, -3, 0, 1)$, and $B = v_1, v_2, v_3$. Then since

$$z_1 v_1 + z_4 v_2 + z_5 v_3 = (z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5),$$

it is clear that B spans U and that B is linearly independent.

(b) Denote the i^{th} standard basis vector of $\mathbf{C}^5$ by e_i. Following the procedure outlined in (2.31) and (2.33), we adjoin the five standard basis vectors to B to obtain the spanning list

$$v_1, v_2, v_3, e_1, e_2, e_3, e_4, e_5.$$

- e_1 does not belong to $\operatorname{span}(v_1, v_2, v_3)$, so we do not delete it.
- Note that $e_2 = \frac{1}{6}(v_1 - e_1)$, so we delete e_2 from the list.
- e_3 does not belong to $\operatorname{span}(v_1, v_2, v_3, e_1)$, so we do not delete it.
- Note that $e_4 = v_2 + 2e_3$, so we delete e_4 from the list.
- Note that $e_5 = v_3 + 3e_3$, so we delete e_5 from the list.

We are left with the list v_1, v_2, v_3, e_1, e_3; as the proof of (2.33) shows, this must be a basis of $\mathbf{C}^5$.

(c) As shown in the proof of (2.34), if we let

$$W = \operatorname{span}(e_1, e_3) = \{(z_1, 0, z_3, 0, 0) \in \mathbf{C}^5 : z_1, z_3 \in \mathbf{C}\},$$

then $\mathbf{C}^5 = U \oplus W$.

Exercise 2.B.5. Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Solution. This is true. Consider $B = 1, x, x^2 + x^3, x^3$; none of the polynomials in this list has degree 2. Suppose a_0, a_1, a_2, a_3 are scalars such that

$$a_0 + a_1 x + a_2(x^2 + x^3) + a_3 x^3 = a_0 + a_1 x + a_2 x^2 + (a_2 + a_3)x^3 = 0$$

for all $x \in \mathbf{F}$. This implies that $a_0 = a_1 = a_2 = a_2 + a_3 = 0$, which in turn gives $a_3 = 0$. It follows that B is linearly independent. Now suppose that $p = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in \mathcal{P}_3(\mathbf{F})$ is given. Observe that

$$a_0 + a_1 x + a_2(x^2 + x^3) + (a_3 - a_2)x^3 = p,$$

so that $p \in \operatorname{span} B$. Thus B is a basis for $\mathcal{P}_3(\mathbf{F})$.

Exercise 2.B.6. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Solution. Suppose there are scalars a_1, a_2, a_3, a_4 such that

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4v_4 = a_1v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 + (a_3 + a_4)v_4 = 0.$$

Since v_1, v_2, v_3, v_4 is a basis, this implies that $a_1 = a_1 + a_2 = a_2 + a_3 = a_3 + a_4 = 0$, which in turn gives $a_1 = a_2 = a_3 = a_4 = 0$. Hence the list $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent. Now let $v \in V$ be given. Since v_1, v_2, v_3, v_4 is a basis, there are scalars a_1, a_2, a_3, a_4 such that $v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$. Then observe that

$$\begin{aligned} a_1(v_1 + v_2) + (a_2 - a_1)(v_2 + v_3) + (a_3 - a_2 + a_1)(v_3 + v_4) + (a_4 - a_3 + a_2 - a_1)v_4 \\ = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = v. \end{aligned}$$

It follows that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ spans V and hence that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is a basis for V.

Exercise 2.B.7. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Solution. For a counterexample, consider $V = \mathbf{R}^4$, let $v_1 = (1, 0, 0, 0), v_2 = (0, 1, 0, 0), v_3 = (0, 0, 1, 1), v_4 = (1, 0, 0, 1)$, and $U = \operatorname{span}(v_1, v_2, (0, 0, 1, 0))$. Suppose we have scalars a_1, a_2, a_3, a_4 such that

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = (a_1 + a_4, a_2, a_3, a_3 + a_4) = (0, 0, 0, 0).$$

It is easy to see that this gives $a_1 = a_2 = a_3 = a_4 = 0$; it follows that the list v_1, v_2, v_3, v_4 is linearly independent. Suppose that $(a_1, a_2, a_3, a_4) \in \mathbf{R}^4$ is given. Then

$$(a_1 + a_3 - a_4)v_1 + a_2v_2 + a_3v_3 + (a_4 - a_3)v_4 = (a_1, a_2, a_3, a_4).$$

Thus $\mathbf{R}^4 = \operatorname{span}(v_1, v_2, v_3, v_4)$, and so v_1, v_2, v_3, v_4 is a basis for $\mathbf{R}^4$. Clearly, $v_1, v_2 \in U$. Since each vector (a_1, a_2, a_3, a_4) in U must satisfy $a_4 = 0$, we have $v_3, v_4 \notin U$. However, v_1, v_2 is not a basis for U: since v_1, v_2, and $(0, 0, 1, 0)$ are evidently linearly independent, any spanning list for U must contain at least three vectors.

Exercise 2.B.8. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Solution. Let $v \in V$ be given. Since $V = U \oplus W$, there are unique vectors $u \in U$ and $w \in W$ such that $v = u + w$. Since $u_1, \ldots, u_m$ is a basis for U, by (2.29) there are unique scalars $a_1, \ldots, a_m$ such that $u = a_1 u_1 + \cdots + a_m u_m$. Similarly, there are unique scalars $b_1, \ldots, b_n$ such that $w = b_1 w_1 + \cdots + b_n w_n$. It follows that v can be uniquely represented as

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots b_n w_n.$$

Hence by (2.29), $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis for V.

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