# 1    Section 1.5 Exercises

Exercises with solutions from Section 1.5 of [UA].

**Exercise 1.5.1.** Finish the following proof for Theorem 1.5.7.

Assume $B$ is a countable set. Thus, there exists $f : \mathbf{N} \to B$ which is 1-1 and onto. Let $A \subseteq B$ be an infinite subset of $B$. We must show that $A$ is countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbf{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function $g$ from $\mathbf{N}$ onto $A$.

*Solution.* Set $n_2 = \min(\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1\}), n_3 = \min(\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, n_2\})$, and in general for $k \geq 2$, $n_k = \min(\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, \ldots, n_{k-1}\})$. Since $A$ is infinite and $f$ is onto, the set $\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, \ldots, n_{k-1}\}$ is always non-empty (indeed, it must be infinite), so the minimum exists and each $n_k$ is well-defined. It is also clear from the definition of this sequence that $n_1 < n_2 < n_3 < \cdots$. Now set $g(k) = f(n_k)$ for each $k \in \mathbf{N}$. Then $g$ is 1-1 since

$$
\begin{aligned}
g(l) = g(k) &\iff f(n_l) = f(n_k) \\
&\iff n_l = n_k && \text{(f is 1-1)} \\
&\iff l = k. && \text{(sequence is increasing)}
\end{aligned}
$$

To see that $g$ is onto, let $a \in A$ be given. Since $f$ is onto, there is a positive integer $N$ such that $f(N) = a$. Suppose that for all $k \in \mathbf{N}$ we have $N \neq n_k$. It cannot be the case that $N < n_1$, else $n_1$ would not be the minimum of $\{n \in \mathbf{N} : f(n) \in A\}$, so we must have $n_1 < N$. Then there must exist some $l \in \mathbf{N}$ such that $n_l < N < n_{l+1}$; but this contradicts $n_{l+1}$ being the minimum of the set $\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, \ldots, n_l\}$. So in fact there must exist a $k \in \mathbf{N}$ such that $n_k = N$ and it follows that $g(k) = f(n_k) = f(N) = a$.

**Exercise 1.5.2.** Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbf{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbf{Q}$ is uncountable:

Assume, for contradiction, that $\mathbf{Q}$ is countable. Thus we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbf{Q}$ must therefore be uncountable.

*Solution.* The problem with this "proof" is that $\mathbf{Q}$ does not have the nested interval property. It can be shown that for an ordered field, the NIP and the least-upper-bound property are equivalent, so that $\mathbf{R}$ is the only ordered field with the NIP.

**Exercise 1.5.3.** Use the following outline to supply proofs for the statements in Theorem 1.5.8.

(a) First, prove statement (i) for two countable sets, $A_1$ and $A_2$. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing $A_2$ with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets $A_1$ and $B_2$ are disjoint. (What happens if $B_2$ is finite?)

Now, explain how the more general statement in (i) follows.

(b) Explain why induction *cannot* be used to prove part (ii) of Theorem 1.5.8 from part (i).

(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$
\begin{array}{ccccc}
1 & 3 & 6 & 10 & 15 \quad \cdots \\
2 & 5 & 9 & 14 & \cdots \\
4 & 8 & 13 & \cdots \\
7 & 12 & \cdots \\
11 & \cdots \\
\vdots
\end{array}
$$

leads to a proof of Theorem 1.5.8 (ii).

*Solution.* (a) Since $A_1$ is countable, there exists a 1-1 and onto function $f : \mathbf{N} \to A_1$. It will suffice to show that $A_1 \cup B_2$ is countable. First, suppose that $B_2$ is empty. Then $A_1 \cup B_2 = A_1$ which is countable by assumption. Next, suppose that $B_2$ is nonempty and finite, say $B_2 = \{x_1, \ldots, x_k\}$ for some $k \in \mathbf{N}$. Define $g : \mathbf{N} \to A_1 \cup B_2$ by

$$
g(n) = \begin{cases} x_n & \text{if } 1 \le n \le k, \\ f(n-k) & \text{if } k < n. \end{cases}
$$

That $g$ is 1-1 follows since $A_1$ and $B_2$ are disjoint and $f$ is 1-1. It is clear that $g$ is onto since $f$ is onto.

Finally, suppose that $B_2$ is infinite. Since $B_2$ is a subset of the countable set $A_2$, Exercise 1.5.1 implies that $B_2$ is countable, i.e. there exists a function $h : \mathbf{N} \to B_2$ which is 1-1 and onto. Define $g : \mathbf{N} \to A_1 \cup B_2$ by

$$
g(n) = \begin{cases} f\left(\frac{n}{2}\right) & \text{if } n \text{ is even}, \\ h\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd}. \end{cases}
$$

To see that $g$ is 1-1, suppose that $m \ne n$ are positive integers. If both of $m, n$ are even then $g(m) \ne g(n)$ since $f$ is 1-1, if both of $m, n$ are odd then $g(m) \ne g(n)$ since $h$ is 1-1, and if

one of $m, n$ is even and the other is odd then $g(m) \neq g(n)$ since $f$ maps into $A_1$, $h$ maps into $B_2$, and $A_1 \cap B_2 = \emptyset$.

To see that $g$ is onto, let $x \in A_1 \cup B_2$ be given. Since $A_1 \cap B_2 = \emptyset$, exactly one of $x \in A_1$ or $x \in B_2$ holds. Suppose $x \in A_1$. Then since $f$ is onto, there is a positive integer $N$ such that $f(N) = x$; it follows that $g(2N) = f(N) = x$. If $x \in B_2$, then since $h$ is onto there exists a positive integer $N$ such that $h(N) = x$. Then $g(2N - 1) = h(N) = x$.

We may conclude that $g$ is 1-1 and onto and hence that $A_1 \cup B_2$ is countable.

A simple induction argument proves the more general statement in Theorem 1.5.8 (i). Let $P(n)$ be the statement that for countable sets $A_1, \ldots, A_n$, the union $A_1 \cup \cdots \cup A_n$ is countable. The truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and suppose we have countable sets $A_1, \ldots, A_{n+1}$. Let $A' = A_1 \cup \cdots \cup A_n$. Then by the induction hypothesis, $A'$ is countable. Observe that

$$A_1 \cup \cdots \cup A_n \cup A_{n+1} = A' \cup A_{n+1}.$$

Since $A'$ and $A_{n+1}$ are countable, the union $A' \cup A_{n+1}$ is also countable by the previous discussion, i.e. $P(n+1)$ holds. Hence we may conclude that, by induction, $P(n)$ is true for all $n \in \mathbf{N}$.

(b) Induction can only be used to show that a particular statement $P(n)$ holds for each value of $n \in \mathbf{N}$.

(c) Since each $A_n$ is countable, for each $n \in \mathbf{N}$ there exists a function $f_n : \mathbf{N} \to A_n$ which is 1-1 and onto. Let $a_{mn} = f_n(m)$ and arrange these into another two-dimensional array like so:

| $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $\cdots$ | | 1 | 3 | 6 | 10 | 15 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_{21}$ | $a_{22}$ | $a_{23}$ | $a_{24}$ | $\cdots$ | | | 2 | 5 | 9 | 14 | $\cdots$ | |
| $a_{31}$ | $a_{32}$ | $a_{33}$ | $\cdots$ | | | | 4 | 8 | 13 | $\cdots$ | | |
| $a_{41}$ | $a_{42}$ | $\cdots$ | | | | | 7 | 12 | $\cdots$ | | | |
| $a_{51}$ | $\cdots$ | | | | | | 11 | $\cdots$ | | | | |
| $\vdots$ | | | | | | | $\vdots$ | | | | | |

Since each $f_n$ is onto, each element of $\bigcup_{n=1}^{\infty} A_n$ appears somewhere in the left array. We define a function $g : \bigcup_{n=1}^{\infty} A_n \to \mathbf{N}$ by working through the grid along the diagonals (first $a_{11}$, then $a_{21}$, then $a_{12}$, then $a_{31}$, and so on), mapping an element $a_{mn}$ to the natural number appearing in the corresponding position in the right array. Since the $A_n$'s may have elements in common, if we encounter an element $a_{mn}$ that we have already seen before, we simply skip this element and move on to the next one. In this way, we obtain a 1-1 function $g$. If we denote the range of $g$ by $B \subseteq \mathbf{N}$, then $g : \bigcup_{n=1}^{\infty} A_n \to B$ is both 1-1 and onto. Since

$A_1 \subseteq \bigcup_{n=1}^{\infty} A_n$, $A_1$ is infinite, and $g$ is 1-1, it follows that $B$ is infinite. Then by Exercise 1.5.1, $B$ must be countable, i.e. there is a function $h : \mathbf{N} \to B$ which is 1-1 and onto. Then the function $g^{-1} \circ h : \mathbf{N} \to \bigcup_{n=1}^{\infty} A_n$ is 1-1 and onto; we may conclude that $\bigcup_{n=1}^{\infty} A_n$ is countable.

**Exercise 1.5.4.** (a) Show $(a, b) \sim \mathbf{R}$ for any interval $(a, b)$.

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbf{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

*Solution.* (a) Let $f : (-1, 1) \to \mathbf{R}$ be given by $f(x) = \frac{x}{x^2-1}$. Then $f$ is 1-1 and onto (see Example 1.5.4 in [UA]). Now let $g : (a, b) \to (-1, 1)$ be given by $g(x) = \frac{2(x-a)}{b-a} - 1$. It is easily verified that $g$ is 1-1 and onto. Then $f \circ g : (a, b) \to \mathbf{R}$ is 1-1 and onto (see Exercise 1.5.5), so that $(a, b) \sim \mathbf{R}$.

(b) Let $f : (a, \infty) \to (0, 1)$ be given by $f(x) = \frac{1}{x+1-a}$. It is easily verified that $f$ is 1-1 and onto, so that $(a, \infty) \sim (0, 1)$. By part (a) we have $(0, 1) \sim \mathbf{R}$; it follows that $(a, \infty) \sim \mathbf{R}$ (see Exercise 1.5.5).

(c) It is clear that $[0, 1) \sim (0, 1]$ via the map $x \mapsto 1 - x$. Define a function $f : (0, 1) \to (0, 1]$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n+1} \text{ for some } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

This function is 1-1 and onto since it has an inverse $f^{-1} : (0, 1] \to (0, 1)$ given by

$$f^{-1}(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

It follows that $(0, 1) \sim (0, 1]$ and hence that $(0, 1) \sim [0, 1)$ (see Exercise 1.5.5).

**Exercise 1.5.5.** (a) Why is $A \sim A$ for every set $A$?

(b) Given sets $A$ and $B$, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets $A, B,$ and $C$, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

*Solution.* (a) The function $f : A \to A$ given by $f(x) = x$ is clearly 1-1 and onto.

(b) Since $A \sim B$, there is a 1-1 and onto function $f : A \to B$. A function is 1-1 and onto if and only if it has an inverse function $f^{-1} : B \to A$ which must also be 1-1 and onto. It follows that $B \sim A$. The equivalence is given by swapping the roles of $A$ and $B$.

(c) There are 1-1 and onto functions $f : A \to B$ and $g : B \to C$. It follows that the composite function $g \circ f : A \to C$ is also 1-1 and onto.

**Exercise 1.5.6.** (a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

*Solution.* (a) Take $A_n = (n, n+1)$ for $n \in \mathbf{N}$.

(b) No such collection exists. To see this, suppose there was such a collection $\{I_a : a \in A\}$ for some uncountable set $A$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exists a rational number $r_a \in I_a$ for each $a \in A$. Since the intervals are disjoint, each $r_a$ must be distinct and hence the collection $\{r_a : a \in A\}$ must be an uncountable subset of $\mathbf{Q}$; but this contradicts Exercise 1.5.1.

**Exercise 1.5.7.** Consider the open interval $(0, 1)$, and let $S$ be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0, 1)$ into, but not necessarily onto, $S$. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to produce a 1-1 function that maps $S$ into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999.... .)

The Schröder-Bernstein theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

*Solution.* (a) Take $f : (0, 1) \to S$ given by $f(x) = (x, 1/2)$.

(b) For $(x, y) \in S$, suppose $x$ has decimal representation $0.x_1 x_2 x_3 \ldots$ and $y$ has decimal representation $0.y_1 y_2 y_3 \ldots$, where if necessary we choose the decimal representation terminating in 0s. To define $g : S \to (0, 1)$, let $g(x, y) = 0.x_1 y_1 x_2 y_2 x_3 y_3 \ldots$ . To see that $g$ is 1-1, suppose we have $(x, y) \neq (a, b)$ in $S$. Then at least one of $x \neq a$ or $y \neq b$ holds. Suppose that $x \neq a$ (the case where $y \neq b$ is similar), so that there is some index $n$ such that $x_n \neq a_n$.

Then if $g(x, y)$ has decimal representation $0.s_1 s_2 s_3 \ldots$ and $g(a, b)$ has decimal representation $0.t_1 t_2 t_3 \ldots$, we have $s_{2n-1} = x_n \neq a_n = t_{2n-1}$. This implies that $g(x, y) \neq g(a, b)$, provided $g(x, y)$ does not terminate in 0s and $g(a, b)$ does not terminate in 9s, or vice versa. To rule this out, suppose that $g(a, b)$ does terminate in 9s (the case where $g(x, y)$ terminates in 9s is similarly handled). Simply observe that this implies that both $a$ and $b$ terminate in 9s; but our construction specifically chooses the decimal representation terminating in 0s if necessary.

This function $g$ is not onto; $g(x, y) = 0.1000 \ldots$ implies that $y = 0.000 \ldots$, but $(x, 0) \notin S$ for any $x \in (0, 1)$.

**Exercise 1.5.8.** Let $B$ be a set of positive real numbers with the property that adding together any finite subset of elements from $B$ always gives a sum of 2 or less. Show $B$ must be finite or countable.

*Solution.* Suppose $a \in (0, 1]$. We claim that $B \cap (a, 2]$ must be a (possibly empty) finite set. By the Archimedean property of $\mathbf{R}$, there is an $n \in \mathbf{N}$ such that $na > 2$. Suppose that $B \cap (a, 2]$ contains at least $n$ elements, say $\{b_1, \ldots, b_n\}$. Then since each $b_i > a$, we have

$$b_1 + \cdots + b_n > na > 2.$$

This contradicts our hypotheses, so it must be the case that $B \cap (a, 2]$ contains less than $n$ elements, so that $B \cap (a, 2]$ is finite.

Any element of $B$ must be less than or equal to 2, so $B \subseteq (0, 2]$. It follows that

$$B = \bigcup_{n=1}^{\infty} (B \cap (1/n, 2]).$$

So we have expressed $B$ as a countable union of finite sets; this implies that $B$ is either finite or countable.

**Exercise 1.5.9.** A real number $x \in \mathbf{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbf{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbf{N}$, and let $A_n$ be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree $n$. Using the fact that every polynomial has a finite number of roots, show that $A_n$ is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

*Solution.* (a) One can verify that $\sqrt{2}$ is a root of the polynomial $x^2 - 2$, $\sqrt[3]{2}$ is a root of the polynomial $x^3 - 2$, and $\sqrt{3} + \sqrt{2}$ is a root of the polynomial $x^4 - 10x^2 + 1$.

(b) Let $P_n$ be the collection of polynomials with integer coefficients that have degree $n$, i.e.

$$P_n = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : a_n, \ldots, a_0 \in \mathbf{Z}, a_n \neq 0\}.$$

It is not hard to see that

$$P_n \sim (\mathbf{Z} \setminus \{0\}) \times \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{n \text{ times}}.$$

A corollary of Theorem 1.5.8 (ii) is that finite Cartesian products of countable sets are themselves countable (see Corollary 8 here); it follows that $P_n$ is countable. For a polynomial $p \in P_n$, let $R_p$ be the set of its roots, i.e. $R_p = \{x \in \mathbf{R} : p(x) = 0\}$, and note that $R_p$ is always a finite set. Now observe that

$$A_n = \bigcup_{p \in P_n} R_p.$$

So we have expressed $A_n$ as a countable union of finite sets. It follows that $A_n$ is either finite or countable. Since $k^{1/n} \in A_n$ for each $k \in \mathbf{N}$ (it is a root of the polynomial $x^n - k$), we see that $A_n$ must be infinite and hence countable.

(c) If we let $A$ be the set of all algebraic numbers, then $A = \bigcup_{n=1}^{\infty} A_n$, a countable union of countable sets. It follows that $A$ is countable.

A consequence of this is that the set of transcendental numbers $A^{\complement}$ must be uncountable. To see this, simply note that: $\mathbf{R} = A \cup A^{\complement}$; the union of two countable sets is countable; and $\mathbf{R}$ is not countable.

**Exercise 1.5.10.** (a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(b) Now let $A$ be the set of all $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable, and let $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

*Solution.* (a) Suppose that for each $a \in (0, 1)$, the set $C \cap [a, 1]$ is countable. Then we can express $C$ as a countable union of countable sets:

$$C = \bigcup_{n=2}^{\infty} (C \cap [1/n, 1]),$$

This implies that $C$ is countable. So if $C$ is uncountable, there must exist some $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(b) Not necessarily. Suppose $C = [0, 1]$. Then for all $a \in (0, 1)$, we have that $C \cap [a, 1] = [a, 1]$ is uncountable, so that $A = (0, 1)$. Then $\alpha = \sup A = 1$, but $C \cap [\alpha, 1] = \{1\}$ is not uncountable.

(c) The statement is no longer true in general. Consider $C = \{1/n : n \in \mathbf{N}\}$. Then no matter which $a \in (0, 1)$ we choose, $C \cap [a, 1]$ is a finite set (since there are finitely many positive integers less than or equal to $1/a$).

**Exercise 1.5.11 (Schröder-Bernstein Theorem).** Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$.

The strategy is to partition $X$ and $Y$ into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that $f$ maps $A$ onto $B$, and $g$ maps $B'$ onto $A'$.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of subsets of $X$, while $\{f(A_n) : n \in \mathbf{N}\}$ is a similar collection in $Y$.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that $f$ maps $A$ onto $B$.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show $g$ maps $B'$ onto $A'$.

*Solution.* (a) Abusing notation slightly, we have 1-1 and onto functions $f : A \to B$ and $g : B' \to A'$, and their inverses $f^{-1} : B \to A$ and $g^{-1} : A' \to B'$. Since $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, the functions $h : X \to Y$ and $h' : Y \to X$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g^{-1}(x) & \text{if } x \in A', \end{cases} \qquad h'(y) = \begin{cases} f^{-1}(y) & \text{if } y \in B, \\ g(y) & \text{if } y \in B' \end{cases}$$

are well-defined and mutual inverses. It follows that $X \sim Y$.

(b) If $A_1 = \emptyset$, then $X = g(Y)$ i.e. $g$ is onto. Then since $g$ is assumed to be 1-1, we could immediately conclude that $X \sim Y$ via $g$.

Let $P(n)$ be the statement that $\{A_1, \ldots, A_n\}$ is a pairwise disjoint collection of sets. The truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$. Then to show that $P(n+1)$ is true, we need to show that for all $1 \le k \le n$, $A_k \cap A_{n+1} = \emptyset$. It is clear that $A_1 \cap A_{n+1} = \emptyset$ since $A_{n+1} \subseteq g(Y)$. Suppose that $2 \le k \le n$. Then observe that

$$
\begin{aligned}
A_k \cap A_{n+1} &= g(f(A_{k-1})) \cap g(f(A_n)) \\
&= g(f(A_{k-1} \cap A_n)) & (f \text{ and } g \text{ are 1-1}) \\
&= g(f(\emptyset)) & (\text{induction hypothesis}) \\
&= \emptyset.
\end{aligned}
$$

Hence $P(n+1)$ holds. Then by induction, $P(n)$ holds for all $n \in \mathbf{N}$. This implies that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of sets. Since $f$ is 1-1, we immediately have that $\{f(A_n) : n \in \mathbf{N}\}$ is also a pairwise disjoint collection of sets.

(c) $f : A \to B$ is 1-1 since $f : X \to Y$ is 1-1 and it is clear from the definition of $A$ and $B$ that $f : A \to B$ really does map into $B$ and in fact is onto.

(d) Again, it is clear that $g : B' \to A'$ is 1-1. That $g$ maps $B'$ into $A'$ follows since

$$
\begin{aligned}
b \in B' &\iff \forall n \in \mathbf{N}, b \notin f(A_n) \\
&\iff \forall n \in \mathbf{N}, g(b) \notin g(f(A_n)) & (g \text{ is 1-1}) \\
&\iff \forall n \in \mathbf{N}, g(b) \notin A_{n+1} \\
&\iff \forall n \ge 2, g(b) \notin A_n.
\end{aligned}
$$

Since $A_1$ is the complement of the image of $Y$ under $g$, it follows that $g(y) \notin A_1$ for any $y \in Y$. Hence

$$
b \in B' \iff \forall n \in \mathbf{N}, g(b) \notin A_n \iff g(b) \in A'.
$$

Furthermore, $g : B' \to A'$ is onto since for any $a \in A'$ we have $a \notin A_1 \iff a \in g(Y)$), so that $a = g(y)$ for some $y \in Y$. The chain of biconditionals above then shows that $y \in B'$.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edn.