Understanding Analysis Solutions

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Contents

Chapter 1. The Real Numbers

1.2. Some Preliminaries

Exercise 1.2.1.

(a) Prove that $\sqrt{3}$ is irrational. Does the same argument work to show that $\sqrt{6}$ is irrational?

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Solution.

(a) Suppose there was a rational number $p = \frac{m}{n}$, which we may assume is in lowest terms, such that $p^2 = 3$, i.e. such that $m^2 = 3n^2$. It follows that m^2 is divisible by 3; we claim that this implies that m is divisible by 3. Indeed, for any $k \in \mathbf{Z}$ we have

$$(3k+1)^2 = 3(3k^2 + 2k) + 1 \quad \text{and} \quad (3k+2)^2 = 3(3k^2 + 4k + 1) + 1.$$

Since m is of the form $3k+1$ or $3k+2$ for some integer k if m is not divisible by 3, it follows that

$$\text{if } m \text{ is not divisible by } 3, \text{then } m^2 \text{ is not divisible by } 3;$$

the contrapositive of this statement proves our claim.

Thus we may write $m = 3k$ for some $k \in \mathbf{Z}$ and substitute this into the equation $m^2 = 3n^2$ to obtain the equation $n^2 = 3k^2$, from which it follows that n is also divisible by 3, contradicting our assumption that m and n had no common factors. We may conclude that there is no rational number whose square is 3.

The same argument works to show that there is no rational number whose square is 6; the crux of this argument is the implication

$$\text{if } m^2 \text{ is divisible by } 6, \text{then } m \text{ is divisible by } 6.$$

This can be seen using what we have already proved. If m^2 is divisible by $6 = 2 \cdot 3$, then m^2 is divisible by 2 and 3. It follows that m is divisible by 2 and 3 and hence that m is divisible by 6.

(b) The argument breaks down when we try to assert that

$$\text{if } m^2 \text{ is divisible by } 4, \text{then } m \text{ is divisible by } 4.$$

This implication is false. For example, $2^2 = 4$ is divisible by 4 but 2 is not divisible by 4.

Exercise 1.2.2. Show that there is no rational number r satisfying $2^r = 3$.

Solution. Suppose there was a rational number $r = \frac{m}{n}$, which we may assume is in lowest terms with $n > 0$, such that $2^r = 3$. This implies that $2^m = 3^n$. Since $n > 0$ gives $3^n \geq 3$ and $2^m < 2$ for $m \leq 0$, it must be the case that $m > 0$. It follows that the left-hand side of the equation $2^m = 3^n$ is a positive even integer whereas the right-hand side is a positive odd integer, which is a contradiction. We may conclude that there is no rational number r such that $2^r = 3$.

Exercise 1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution.

(a) This is false, as Example 1.2.2 shows.

(b) This is true and we can use the following lemma to prove it.

Lemma L.1. If $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of positive integers, i.e. $a_{n+1} \leq a_n$ and $a_n \geq 1$ for all $n \in \mathbf{N}$, then $(a_n)_{n=1}^{\infty}$ must be eventually constant. That is, there exists an $N \in \mathbf{N}$ such that $a_n = a_N$ for all $n \geq N$.

Proof. Let $A = \{a_n : n \in \mathbf{N}\}$, which is non-empty and bounded below by 1. It follows from the well-ordering principle that A has a least element, say $\min A = a_N$ for some $N \in \mathbf{N}$. Let $n > N$ be given. It cannot be the case that $a_n < a_N$, since this would contradict that a_N is the least element of A, so we must have $a_n \geq a_N$. By assumption $a_n \leq a_N$ and so we may conclude that $a_n = a_N$. $\square$

Consider the sequence $(|A_n|)_{n=1}^{\infty}$, where $|A_n|$ is the number of elements contained in A_n. Because each A_n is finite and non-empty, this is a sequence of positive integers. Furthermore, this sequence is decreasing since the sets $(A_n)_{n=1}^{\infty}$ are nested:

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots.$$

We may now invoke Lemma L.1 to obtain an $N \in \mathbf{N}$ such that $|A_n| = |A_N|$ for all $n \geq N$. Combining this equality with the inclusion $A_n \subseteq A_N$ for each $n \geq N$, we see that $A_n = A_N$ for all $n \geq N$. It follows that $\bigcap_{n=1}^{\infty} A_n = A_N$, which by assumption is finite and non-empty.

(c) This is false: let $A = B = \emptyset$ and $C = \{0\}$ and observe that

$$A \cap (B \cup C) = \emptyset \neq \{0\} = (A \cap B) \cup C.$$

(d) This is true, since

$$x \in A \cap (B \cap C)) \;\Leftrightarrow\; x \in A \text{ and } x \in (B \cap C) \;\Leftrightarrow\; x \in A \text{ and } (x \in B \text{ and } x \in C)$$

$$\Leftrightarrow\; (x \in A \text{ and } x \in B) \text{ and } x \in C \;\Leftrightarrow\; x \in (A \cap B) \text{ and } x \in C \;\Leftrightarrow\; x \in (A \cap B) \cap C,$$

where we have used that logical conjunction ("and") is associative for the third equivalence. It follows that x belongs to $A \cap (B \cap C)$ if and only if x belongs to $(A \cap B) \cap C$, which is to say that $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) This is true, since

$$x \in A \cap (B \cup C) \;\Leftrightarrow\; x \in A \text{ and } x \in (B \cup C) \;\Leftrightarrow\; x \in A \text{ and } (x \in B \text{ or } x \in C)$$

$$\Leftrightarrow\; (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \;\Leftrightarrow\; x \in (A \cap B) \text{ or } x \in (A \cap C)$$

$$\Leftrightarrow\; x \in (A \cap B) \cup (A \cap C),$$

where we have used that logical conjunction ("and") distributes over logical disjunction ("or") for the third equivalence. It follows that x belongs to $A \cap (B \cup C)$ if and only if x belongs to $(A \cap B) \cup (A \cap C)$, which is to say that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1.2.4. Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$.

Solution. Arrange $\mathbf{N}$ in a grid like so:

A_1	A_2	A_3	A_4	$\cdots$
1	3	6	10	$\cdots$
2	5	9	14	$\cdots$
4	8	13	19	$\cdots$
7	12	18	25	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now take A_i to be the set of numbers appearing in the i^{th} column.

Exercise 1.2.5 (De Morgan's Laws). Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution.

(a) Observe that

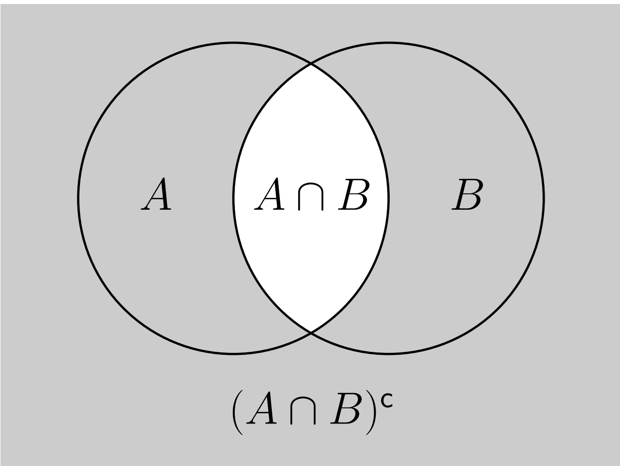
$$x \in (A \cap B)^c \quad\Leftrightarrow\quad x \notin A \cap B \quad\Leftrightarrow\quad \text{not } (x \in A \text{ and } x \in B)$$
$$\Leftrightarrow\quad x \notin A \text{ or } x \notin B \quad\Leftrightarrow\quad x \in A^c \cup B^c$$

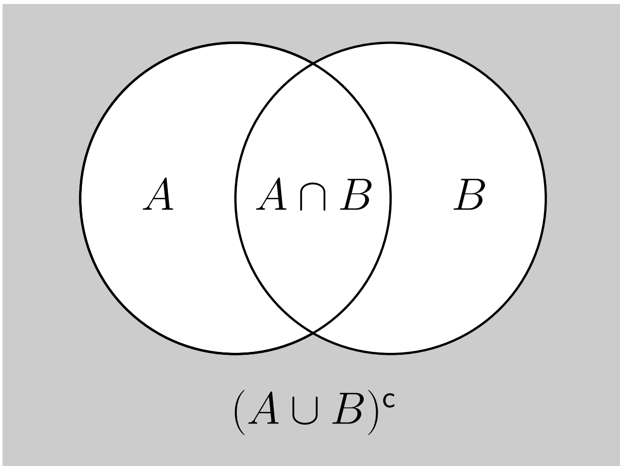
(b) See part (a).

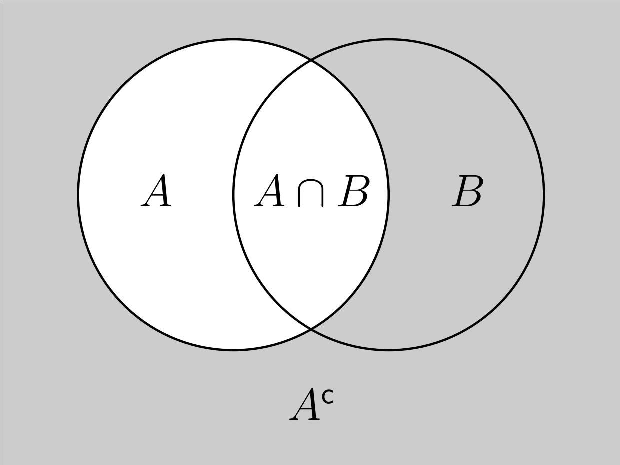
(c) The proof is similar to the one given in part (a).

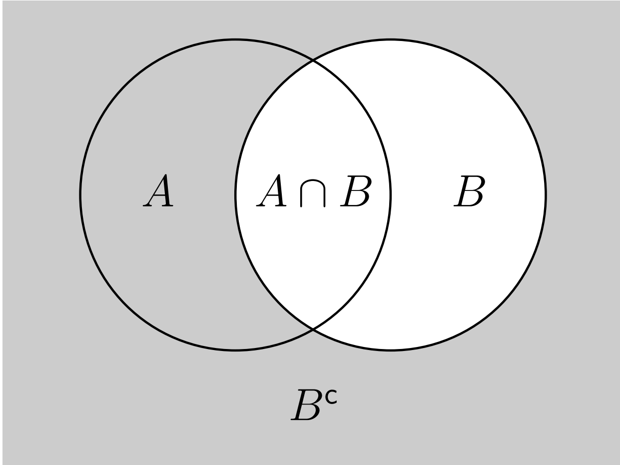
$$x \in (A \cup B)^c \quad\Leftrightarrow\quad x \notin A \cup B \quad\Leftrightarrow\quad \text{not } (x \in A \text{ or } x \in B)$$
$$\Leftrightarrow\quad x \notin A \text{ and } x \notin B \quad\Leftrightarrow\quad x \in A^c \cap B^c$$

The following Venn diagrams help to visualize De Morgan's Laws. The shaded regions are included and the unshaded regions are excluded.









Exercise 1.2.6.

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution.

(a) First suppose that a and b are both non-negative, so that $a+b$ is also non-negative; it follows that $|a+b| = a+b$ and $|a|+|b| = a+b$. Thus the triangle inequality in this case reduces to the evidently true statement $a+b \leq a+b$.

Now suppose that a and b are both negative, so that $a+b$ is also negative; it follows that $|a+b| = -a-b$ and $|a|+|b| = -a-b$. Thus the triangle inequality in this case reduces to the evidently true statement $-a-b \leq -a-b$.

(b) Starting from the true statement $ab \leq |ab|$ and using that $a^2 = |a|^2$ and $|ab| = |a||b|$ for any real numbers a and b, observe that

$$2ab \leq 2|ab| \quad \Leftrightarrow \quad a^2 + 2ab + b^2 \leq |a|^2 + 2|a||b| + |b|^2$$
$$\Leftrightarrow \quad (a+b)^2 \leq (|a|+|b|)^2 \quad \Leftrightarrow \quad |a+b|^2 \leq (|a|+|b|)^2.$$

Because both $|a+b|$ and $|a|+|b|$ are non-negative, the inequality $|a+b|^2 \leq (|a|+|b|)^2$ is equivalent to $|a+b| \leq |a|+|b|$, as desired.

(c) We apply the triangle inequality twice:

$$|a-b| = |a-c+c-b| \leq |a-c| + |c-b| \leq |a-c| + |c-d| + |d-b|.$$

(d) Using the triangle inequality and the fact that $|-a| = |a|$ for any $a \in \mathbf{R}$, we find that

$$|a| = |a - b + b| \le |a - b| + |b| \quad \Leftrightarrow \quad |a| - |b| \le |a - b|,$$

$$|b| = |b - a + a| \le |b - a| + |a| = |a - b| + |a| \quad \Leftrightarrow \quad |b| - |a| \le |a - b|.$$

Because $||a| - |b||$ equals either $|a| - |b|$ or $|b| - |a|$, it follows that $||a| - |b|| \le |a - b|$.

Exercise 1.2.7. Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbf{R} : 0 \le x \le 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution.

(a) Some straightforward calculations reveal that

$$f(A) = [0, 4], \qquad f(A \cap B) = [1, 4], \qquad f(A \cup B) = [0, 16],$$

$$f(B) = [1, 16], \qquad f(A) \cap f(B) = [1, 4], \qquad f(A) \cup f(B) = [0, 16].$$

From this we see that $f(A \cap B) = f(A) \cap f(B)$ and $f(A \cup B) = f(A) \cup f(B)$.

(b) Let $A = \{-1\}$ and $B = \{1\}$ and note that $f(A \cap B) = f(\emptyset) = \emptyset$, but

$$f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\} \neq \emptyset.$$

(c) Observe that

$$y \in g(A \cap B) \quad \Leftrightarrow \quad y = g(x) \text{ for some } x \in A \cap B$$

$$\Rightarrow \quad (y = g(x_1) \text{ for some } x_1 \in A) \text{ and } (y = g(x_2) \text{ for some } x_2 \in B)$$

$$\Leftrightarrow \quad y \in g(A) \text{ and } y \in g(B) \quad \Leftrightarrow \quad y \in g(A) \cap g(B).$$

It follows that y belongs to $g(A) \cap g(B)$ whenever y belongs to $g(A \cap B)$, which is to say that $g(A \cap B) \subseteq g(A) \cap g(B)$.

(d) We always have $g(A \cup B) = g(A) \cup g(B)$; indeed,

$$
\begin{aligned}
y \in g(A \cup B) \quad &\Leftrightarrow \quad y = g(x) \text{ for some } x \in A \cup B \\
&\Leftrightarrow \quad y = g(x) \text{ for some } x \text{ such that } (x \in A \text{ or } x \in B) \\
&\Leftrightarrow \quad (y = g(x_1) \text{ for some } x_1 \in A) \text{ or } (y = g(x_2) \text{ for some } x_2 \in B) \\
&\Leftrightarrow \quad y \in g(A) \text{ or } y \in g(B) \quad \Leftrightarrow \quad y \in g(A) \cup g(B).
\end{aligned}
$$

It follows that $g(A \cup B) = g(A) \cup g(B)$.

Exercise 1.2.8. Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:

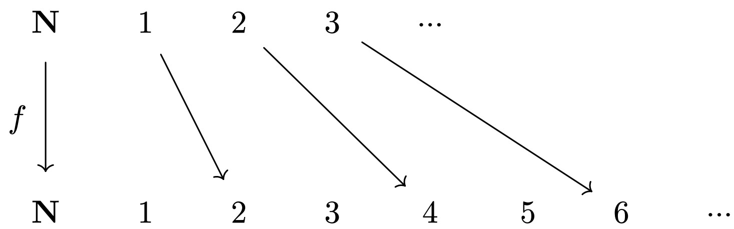
(a) $f : \mathbf{N} \to \mathbf{N}$ that is 1-1 but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not 1-1.

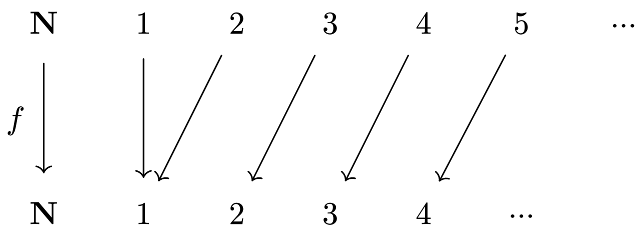
(c) $f : \mathbf{N} \to \mathbf{Z}$ that is 1-1 and onto.

Solution. (I prefer the terms injective/surjective/bijective rather than one-to-one and onto. I will use these terms throughout this document.)

(a) Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(n) = 2n$. Notice that f is injective since $n = m$ if and only if $2n = 2m$, but f is not surjective since the range of f contains only even numbers.



(b) Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(1) = 1$ and $f(n) = n - 1$ for $n \geq 2$. Notice that $f(n+1) = n$ for any $n \in \mathbf{N}$, so that f is surjective, but f is not injective since $f(1) = f(2) = 1$.



(c) Let $f : \mathbf{N} \to \mathbf{Z}$ be given by

$$
f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}
$$

$$
\begin{array}{ccccccc}
\mathbf{N} & 1 & 2 & 3 & 4 & 5 & \cdots \\
f \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\mathbf{Z} & 0 & 1 & -1 & 2 & -2 & \cdots
\end{array}
$$

To see that f is injective, let $n \neq m$ be given and consider these cases.

Case 1. If n and m are both even, then $f(n) \neq f(m)$ since $n \neq m$ if and only if $\frac{n}{2} \neq \frac{m}{2}$.

Case 2. If n and m are both odd, then $f(n) \neq f(m)$ since $n \neq m$ if and only if $-\frac{n-1}{2} \neq -\frac{m-1}{2}$.

Case 3. If n and m have opposite signs, say n is even and m is odd, then $f(n) \neq f(m)$ since $f(n) > 0$ and $f(m) \leq 0$.

To see that f is surjective, let $n \in \mathbf{Z}$ be given. If $n > 0$ then $f(2n) = n$, and if $n \leq 0$ then $f(-2n+1) = n$.

Exercise 1.2.9. Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

Solution.

(a) Some straightforward calculations reveal that

$$
\begin{aligned}
f^{-1}(A) &= [-2, 2], & f^{-1}(A \cap B) &= [-1, 1], & f^{-1}(A \cup B) &= [-2, 2], \\
f^{-1}(B) &= [-1, 1], & f^{-1}(A) \cap f^{-1}(B) &= [-1, 1], & f^{-1}(A) \cup f^{-1}(B) &= [-2, 2].
\end{aligned}
$$

From this we see that

$$
f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \quad \text{and} \quad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).
$$

(b) Observe that

$$
\begin{aligned}
x \in g^{-1}(A \cap B) \quad &\Leftrightarrow \quad g(x) \in A \cap B \quad \Leftrightarrow \quad (g(x) \in A) \text{ and } (g(x) \in B) \\
&\Leftrightarrow \quad (x \in g^{-1}(A)) \text{ and } (x \in g^{-1}(B)) \quad \Leftrightarrow \quad x \in g^{-1}(A) \cap g^{-1}(B).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
x \in g^{-1}(A \cup B) \quad &\Leftrightarrow \quad g(x) \in A \cup B \quad \Leftrightarrow \quad (g(x) \in A) \text{ or } (g(x) \in B) \\
&\Leftrightarrow \quad (x \in g^{-1}(A)) \text{ or } (x \in g^{-1}(B)) \quad \Leftrightarrow \quad x \in g^{-1}(A) \cup g^{-1}(B).
\end{aligned}
$$

Exercise 1.2.10. Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \varepsilon$ for every $\varepsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

Solution.

(a) This is false; the implication

$$\text{if } a < b + \varepsilon \text{ for every } \varepsilon > 0, \text{then } a < b$$

does not hold. The problem occurs when we consider the case where $a = b$. For example, we certainly have $1 < 1 + \varepsilon$ for every $\varepsilon > 0$ but of course $1 < 1$ is false.

(b) See part (a).

(c) This is true. The implication

$$\text{if } a \leq b, \text{then } a < b + \varepsilon \text{ for every } \varepsilon > 0$$

follows since $a \leq b < b + \varepsilon$ for every $\varepsilon > 0$ and the implication

$$\text{if } a > b, \text{then } a \geq b + \varepsilon \text{ for some } \varepsilon > 0$$

can be seen by taking $\varepsilon = a - b > 0$, so that $b + \varepsilon = a \leq a$.

Exercise 1.2.11. Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that…" in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution.

(a) The negated statement is:

$$\text{there exist real numbers } a < b \text{ such that } a + \tfrac{1}{n} \geq b \;\text{ for all } n \in \mathbf{N}.$$

The original statement is true and follows from the Archimedean Property (Theorem 1.4.2).

(b) The negated statement is:

$$\text{for all } x > 0, \text{there exists an } n \in \mathbf{N} \text{ such that } \frac{1}{n} \leq x.$$

The negated statement is true and again follows from the Archimedean Property (Theorem 1.4.2).

(c) The negated statement is:

there are two distinct real numbers with no rational number between them.

The original statement is true; this is the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3).

Exercise 1.2.12. Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show that the sequence $(y_1, y_2, y_3, ...)$ is decreasing.

Solution.

(a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $y_n > -6$. Since $y_1 = 6$, the truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and observe that

$$y_{n+1} = \tfrac{2}{3}y_n - 2 > \tfrac{2}{3}(-6) - 2 = -6,$$

i.e. $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

(b) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $y_{n+1} \leq y_n$. Since $y_1 = 6$ and $y_2 = 2$, the truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and observe that

$$y_{n+2} = \tfrac{2}{3}y_{n+1} - 2 \leq \tfrac{2}{3}y_n - 2 = y_{n+1},$$

i.e. $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

Exercise 1.2.13. For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbf{N}$.

(b) It is tempting to appeal to induction to conclude that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution.

(a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$ for any sets $A_1, \ldots, A_n$. The truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$, let $A_1, \ldots, A_n, A_{n+1}$ be given, and observe that

$$\begin{aligned}
(A_1 \cup \cdots \cup A_n \cup A_{n+1})^c &= ((A_1 \cup \cdots \cup A_n) \cup (A_{n+1}))^c \\
&= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c && \text{(Exercise 1.2.5)} \\
&= A_1^c \cap \cdots \cap A_n^c \cap A_{n+1}^c, && \text{(induction hypothesis)}
\end{aligned}$$

i.e. $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

(b) Let $B_i = \{i, i+1, i+2, \ldots\}$, so that

$$B_1 = \{1, 2, 3, \ldots\}, \quad B_2 = \{2, 3, 4, \ldots\}, \quad B_3 = \{3, 4, 5, \ldots\}, \quad \text{etc.}$$

It is straightforward to verify that $\bigcap_{i=1}^{n} B_i = B_n \neq \emptyset$ for any $n \in \mathbf{N}$. However, as Example 1.2.2 shows, the intersection $\bigcap_{i=1}^{\infty} B_i$ is empty.

(c) Observe that

$$x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c \quad \Leftrightarrow \quad x \notin \bigcup_{i=1}^{\infty} A_i \quad \Leftrightarrow \quad x \notin A_i \text{ for every } i \in \mathbf{N} \quad \Leftrightarrow \quad x \in \bigcap_{i=1}^{\infty} A_i^c.$$

It follows that $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$.