# 1 Section 7.5 Exercises

Exercises with solutions from Section 7.5 of [UA].

**Exercise 7.5.1.** (a) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^{x} f$. Find a piecewise algebraic formula for $F(x)$ for all $x$. Where is $F$ continuous? Where is $F$ differentiable? Where does $F'(x) = f(x)$?

(b) Repeat part (a) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

*Solution.* (a) Some calculations reveal that $F : [-1, \infty) \to \mathbf{R}$ is given by

$$F(x) = \begin{cases} \frac{1}{2}(1 - x^2) & \text{if } -1 \leq x \leq 0, \\ \frac{1}{2}(1 + x^2) & \text{if } x > 0. \end{cases}$$

It is straightforward to manually check that $F$ is differentiable (and hence continuous) on its domain, with derivative given by $F'(x) = f(x)$. However, note that the Fundamental Theorem of Calculus part (ii) (FToC, Theorem 7.5.1 (ii)) immediately implies that $F$ is continuous on any interval of the form $[-1, b]$ for $b \in \mathbf{R}$ (in fact, Lipschitz on such intervals) and hence is continuous on its domain. Furthermore, as $f$ is continuous everywhere, the FToC also implies that $F$ is differentiable on its domain with derivative given by $F'(x) = f(x)$.

(b) In this case, the function $F : [-1, \infty) \to \mathbf{R}$ is given by

$$F(x) = \begin{cases} 1 + x & \text{if } -1 \leq x \leq 0, \\ 1 + 2x & \text{if } x > 0. \end{cases}$$

As in part (a), the FToC part (ii) implies that $F$ is continuous on its domain. Furthermore, since $f$ is continuous on $A = [-1, 0) \cup (0, \infty)$, the FToC implies that $F$ is differentiable on $A$ with derivative given by $F'(x) = f(x)$. However, because $f$ is not continuous at 0 the FToC does not allow us to conclude that $F$ is differentiable at 0. Indeed, $F$ fails to be differentiable here:

$$\lim_{x \to 0^-} \frac{F(x) - F(0)}{x} = 1 \neq 2 = \lim_{x \to 0^+} \frac{F(x) - F(0)}{x}.$$

**Exercise 7.5.2.** Decide whether each statement is true or false, providing a short justification for each conclusion.

(a) If $g = h'$ for some $h$ on $[a, b]$, then $g$ is continuous on $[a, b]$.

(b) If $g$ is continuous on $[a, b]$, then $g = h'$ for some $h$ on $[a, b]$.

(c) If $H(x) = \int_a^x h$ is differentiable at $c \in [a, b]$, then $h$ is continuous at $c$.

*Solution.* (a) This is false. For a counterexample, consider the function $h : [-1, 1] \to \mathbf{R}$ given by

$$h(x) = \begin{cases} x^{5/3} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then, as we showed in Exercise 5.2.7 (a), $h$ is differentiable but $h'$ is not continuous at 0.

(b) This is true. Theorem 7.2.9 implies that $g$ is integrable on $[a, b]$ and so we are justified in defining $h : [a, b] \to \mathbf{R}$ by $h(x) = \int_a^x g$; the continuity of $g$ on $[a, b]$ then allows us to use the FToC part (ii) to conclude that $g = h'$.

(c) This is false. For a counterexample, consider $h : [-1, 1] \to \mathbf{R}$ given by

$$h(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then $H : [-1, 1] \to \mathbf{R}$ defined by $H(x) = \int_{-1}^x h(t)\,dt$ is identically zero and hence differentiable at 0, but $h$ is not continuous at 0.

**Exercise 7.5.3.** The hypothesis in Theorem 7.5.1 (i) that $F'(x) = f(x)$ for all $x \in [a, b]$ is slightly stronger than it needs to be. Carefully read the proof and state exactly what needs to be assumed with regard to the relationship between $f$ and $F$ for the proof to be valid.

*Solution.* In light of Theorem 7.4.1 and the fact that the Mean Value Theorem only requires differentiability on an open interval, it would suffice for $F$ to be continuous on $[a, b]$ and $F'(x) = f(x)$ to hold for all but finitely many $x \in [a, b]$.

**Exercise 7.5.4.** Show that if $f : [a, b] \to \mathbf{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a, b]$, then $f(x) = 0$ everywhere on $[a, b]$. Provide an example to show that this conclusion does not follow if $f$ is not continuous.

*Solution.* Define $F : [a, b] \to \mathbf{R}$ by $F(x) = \int_a^x f$. On one hand, since by assumption $F$ is identically zero on $[a, b]$, we have that $F$ is differentiable on $[a, b]$ and satisfies $F'(x) = 0$ for all $x \in [a, b]$. On the other hand, because $f$ is continuous on $[a, b]$, the FToC part (ii) implies that $F'(x) = f(x)$ for all $x \in [a, b]$. Thus $f$ is identically zero on $[a, b]$.

For an example demonstrating that this conclusion does not follow if $f$ is not continuous, consider $f : [0, 1] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$$

Then $\int_0^x f = 0$ for all $x \in [0, 1]$, but $f$ is not identically zero.

**Exercise 7.5.5.** The Fundamental Theorem of Calculus can be used to supply a shorter argument for Theorem 6.3.1 under the additional assumption that the sequence of derivatives is continuous.

Assume $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a, b]$. Assuming each $f_n'$ is continuous, we can apply Theorem 7.5.1 (i) to get

$$\int_a^x f_n' = f_n(x) - f_n(a)$$

for all $x \in [a, b]$. Show that $g(x) = f'(x)$.

*Solution.* Let $x \in [a, b]$ be given. Because $f_n' \to g$ uniformly on $[a, x]$, Theorem 7.4.4 shows that

$$\lim_{n \to \infty} \int_a^x f_n' = \int_a^x g.$$

We can then take the limit as $n \to \infty$ on both sides of the equation $\int_a^x f_n' = f_n(x) - f_n(a)$ and use the pointwise convergence $f_n \to f$ to see that

$$f(x) = f(a) + \int_a^x g$$

for all $x \in [a, b]$. Since $g$ is the uniform limit of a sequence of continuous functions it is itself continuous (Theorem 6.2.6) and so we may invoke the FToC part (ii) to conclude that $f'(x) = g(x)$ for all $x \in [a, b]$.

**Exercise 7.5.6 (Integration-by-parts).** (a) Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$ and derive the familiar integration-by-parts formula

$$\int_a^b h(t)k'(t)\, dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)\, dt.$$

(b) Explain how the result in Exercise 7.4.6 can be used to slightly weaken the hypothesis in part (a).

*Solution.* (a) By assumption the functions $h, h', k$, and $k'$ are continuous on $[a,b]$; it follows that $(hk)' = hk' + h'k$ is continuous on $[a,b]$. Theorem 7.2.9 then implies that $(hk)'$ is integrable on $[a,b]$ and so we may use the FToC part (i) to see that

$$\int_a^b h(t)k'(t) + h'(t)k(t)\,dt = \int_a^b (h(t)k(t))'\,dt = h(b)k(b) - h(a)k(a).$$

(b) In light of Exercise 7.4.6, we need only assume that $h'$ and $k'$ are integrable on $[a,b]$.

**Exercise 7.5.7.** Use part (ii) of Theorem 7.5.1 to construct another proof of part (i) of Theorem 7.5.1 under the stronger hypothesis that $f$ is continuous. (To get started, set $G(x) = \int_a^x f$.)

*Solution.* It will suffice to show that $G(b) = F(b) - F(a)$. Because $f$ is continuous on $[a,b]$, the FToC part (ii) implies that $G'(x) = f(x) = F'(x)$ for all $x \in [a,b]$; it follows from Corollary 5.3.4 that $G(x) = F(x) + k$ for some constant $k$. Substituting $x = a$, we see that $k = -F(a)$ and thus $G(b) = F(b) - F(a)$, as desired.

**Exercise 7.5.8 (Natural Logarithm and Euler's Constant).** Let

$$L(x) = \int_1^x \frac{1}{t}\,dt,$$

where we consider only $x > 0$.

(a) What is $L(1)$? Explain why $L$ is differentiable and find $L'(x)$.

(b) Show that $L(xy) = L(x) + L(y)$. (Think of $y$ as a constant and differentiate $g(x) = L(xy)$.)

(c) Show $L(x/y) = L(x) - L(y)$.

(d) Let

$$\gamma_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - L(n).$$

Prove that $(\gamma_n)$ converges. The constant $\gamma = \lim \gamma_n$ is called Euler's constant.

(e) Show how consideration of the sequence $\gamma_{2n} - \gamma_n$ leads to the interesting identity

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

*Solution.* (a) We have $L(1) = 0$. Because $t^{-1}$ is continuous on $(0, \infty)$, the FToC part (ii) shows that $L$ is differentiable on $(0, \infty)$ and satisfies $L'(x) = x^{-1}$.

(b) Note that, by part (a),

$$\frac{d}{dx}L(xy) = yL'(xy) = \frac{y}{xy} = \frac{1}{x} = L'(x).$$

Corollary 5.3.4 then implies that $L(xy) = L(x) + k$ for some constant $k$. Substituting $x = 1$, we see that $k = L(y)$ and thus $L(xy) = L(x) + L(y)$, as desired.

(c) Observe that, by parts (a) and (b),

$$0 = L(1) = L\left(\frac{y}{y}\right) = L(y) + L\left(\frac{1}{y}\right),$$

so that $L\left(\frac{1}{y}\right) = -L(y)$ for any $y > 0$. Combining this with part (b) shows that $L\left(\frac{x}{y}\right) = L(x) - L(y)$.

(d) Let $n \geq 2$ be given and consider the partition $P = \{1, \ldots, n\}$ of $[1, n]$. Then

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} > 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} = U\left(\frac{1}{t}, P\right) \geq U\left(\frac{1}{t}\right) = L(n).$$

Thus $\gamma_n \geq 0$ for each $n \in \mathbf{N}$, so that $(\gamma_n)$ is bounded below.

Again, let $n \in \mathbf{N}$ be given and observe that

$$\gamma_n - \gamma_{n+1} = L\left(1 + \frac{1}{n}\right) - \frac{1}{n+1}.$$

Since $\frac{1}{t} \geq \frac{n}{n+1}$ on $\left[1, 1 + \frac{1}{n}\right]$, Theorem 7.4.2 (iii) shows that

$$L\left(1 + \frac{1}{n}\right) \geq \frac{1}{n+1}$$

and hence $\gamma_n \geq \gamma_{n+1}$ for each $n \in \mathbf{N}$, so that $(\gamma_n)$ is decreasing; we can now appeal to the Monotone Convergence Theorem (Theorem 2.4.2) to conclude that $(\gamma_n)$ converges.

(e) For $n \in \mathbf{N}$, observe that

$$\begin{aligned}
\gamma_{2n} - \gamma_n &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - L(2n) + L(n) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - \left(\frac{2}{2} + \frac{2}{4} + \cdots + \frac{2}{2n}\right) - L(2) - L(n) + L(n) \\
&= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}\right) - L(2).
\end{aligned}$$

Taking the limit as $n \to \infty$ on both sides gives the desired equality.

**Exercise 7.5.9.** Given a function $f$ on $[a,b]$, define the *total variation* of $f$ to be

$$Vf = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\right\},$$

where the supremum is taken over all partitions $P$ of $[a,b]$.

(a) If $f$ is continuously differentiable ($f'$ exists as a continuous function), use the Fundamental Theorem of Calculus to show $Vf \le \int_a^b |f'|$.

(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $Vf = \int_a^b |f'|$.

*Solution.* (a) Let $P = \{x_0, \ldots, x_n\}$ be an arbitrary partition of $[a,b]$. Because $f'$ is continuous on $[a,b]$, it is integrable on $[a,b]$ and so we may use the FToC part (i) and Theorem 7.4.2 (v) to see that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f' \right| \le \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'| = \int_a^b |f'|.$$

As $P$ was arbitrary, it follows that $Vf \le \int_a^b |f'|$.

(b) For any $\epsilon > 0$, there exists a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that

$$\left(\int_a^b |f'|\right) - \epsilon = L(|f'|) - \epsilon < L(|f'|, P).$$

For $k \in \{1, \ldots, n\}$, apply the Mean Value Theorem on the interval $[x_{k-1}, x_k]$ to obtain some $t_k \in (x_{k-1}, x_k)$ such that

$$|f'(t_k)|(x_k - x_{k-1}) = |f(x_k) - f(x_{k-1})|.$$

It follows that

$$\begin{aligned} L(|f'|, P) &= \sum_{k=1}^{n} \inf\{|f'(t)| : t \in [x_{k-1}, x_k]\}(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} |f'(t_k)|(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \\ &\leq Vf. \end{aligned}$$

We have now shown that for every $\epsilon > 0$ it holds that

$$\int_a^b |f'| \leq Vf + \epsilon$$

and thus we obtain the inequality $\int_a^b |f'| \leq Vf$. Given part (a), we may conclude that $Vf = \int_a^b |f'|$.

**Exercise 7.5.10 (Change-of-variable Formula).** Let $g : [a, b] \to \mathbf{R}$ be differentiable and assume $g'$ is continuous. Let $f : [c, d] \to \mathbf{R}$ be continuous, and assume that the range of $g$ is contained in $[c, d]$ so that the composition $f \circ g$ is properly defined.

(a) Why are we sure $f$ is the derivative of some function? How about $(f \circ g)g'$?

(b) Prove the change-of-variable formula

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(t)\, dt.$$

*Solution.* (a) $f$ is integrable on $[c, d]$ because it is continuous on $[c, d]$ and so if we let $F(x) = \int_c^x f$ then the FToC part (ii) implies that $F'(x) = f(x)$ for each $x \in [c, d]$. Similarly, note that $f \circ g$ is continuous on $[a, b]$, being a composition of continuous functions, and hence is integrable on $[a, b]$. By assumption $g'$ is continuous on $[a, b]$ and so is also integrable on $[a, b]$. We can now use Exercise 7.4.6 to see that $(f \circ g)g'$ is integrable on $[a, b]$, so that we can define $G(x) = \int_a^x (f \circ g)g'$ and use the FToC part (ii) to see that $G'(x) = f(g(x))g'(x)$ for each $x \in [a, b]$.

(b) Define $F : [c, d] \to \mathbf{R}$ and $G : [a, b] \to \mathbf{R}$ by

$$F(t) = \int_{g(a)}^{t} f(x)\, dx \quad \text{and} \quad G(t) = \int_{a}^{t} f(g(x))g'(x)\, dx.$$

Then $F'(t) = f(t)$, so that $[F(g(t))]' = f(g(t))g'(t)$, and $G'(t) = f(g(t))g'(t)$. It follows that $F(g(t)) = G(t) + k$ on $[a, b]$ for some constant $k$. Substituting $t = a$, we see that $k = 0$ and thus $F(g(b)) = G(b)$, i.e.

$$\int_{g(a)}^{g(b)} f(x)\, dx = \int_{a}^{b} f(g(x))g'(x)\, dx.$$

**Exercise 7.5.11.** Assume $f$ is integrable on $[a, b]$ and has a "jump discontinuity" at $c \in (a, b)$. This means that both one-sided limits exist as $x$ approaches $c$ from the left and from the right, but that

$$\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x).$$

(This phenomenon is discussed in more detail in Section 4.6.)

(a) Show that, in this case, $F(x) = \int_a^x f$ is not differentiable at $x = c$.

(b) The discussion in Section 5.5 mentions the existence of a continuous monotone function that fails to be differentiable on a dense subset of $\mathbf{R}$. Combine the results of part (a) with Exercise 6.4.10 to show how to construct such a function.

*Solution.* (a) Let $A = \lim_{x\to c^-} f(x)$ and $B = \lim_{x\to c^+} f(x)$. A small modification of the proof of the FToC part (ii) shows that

$$\lim_{x\to c^-} \frac{F(x) - F(c)}{x - c} = A \quad \text{and} \quad \lim_{x\to c^+} \frac{F(x) - F(c)}{x - c} = B.$$

Since $A \neq B$, we see that $\lim_{x\to c} \frac{F(x)-F(c)}{x-c}$ does not exist, i.e. $F$ is not differentiable at $c$.

(b) As in Exercise 6.4.10, let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rationals and for each $n \in \mathbf{N}$ define $u_n : \mathbf{R} \to \mathbf{R}$ by

$$u_n(x) = \begin{cases} 2^{-n} & \text{if } r_n < x, \\ 0 & \text{if } x \leq r_n. \end{cases}$$

Now define $h : \mathbf{R} \to \mathbf{R}$ by $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Let $[a, b]$ be a given interval and note that for each $N \in \mathbf{N}$ the partial sum function $h_N(x) = \sum_{n=1}^{N} u_n(x)$ has at most $N$ jump discontinuities on $[a, b]$; it follows from Theorem 7.4.1 that $h_N$ is integrable on $[a, b]$. In Exercise 6.4.10 we showed that $h_N \to h$ uniformly on $\mathbf{R}$ and hence by Theorem 7.4.4 we see that $h$ is integrable on $[a, b]$. We can now define $H : \mathbf{R} \to \mathbf{R}$ by $H(x) = \int_0^x h$. The FToC part (ii) shows that $H$ is continuous, and we can use Theorem 7.4.1 and the fact that $h$ is non-negative to see that $H$ is monotone increasing.

Now we will prove that $h$ has a jump discontinuity at each rational number. Let $r_m \in \mathbf{Q}$ be given; we have two claims.

(i) Our first claim is that $\lim_{x \to r_m^-} h(x) = h(r_m)$. To see this, let $\epsilon > 0$ be given and choose $N \in \mathbf{N}$ such that $2^{-N} < \epsilon$. Because the set $\{r_1, \ldots, r_N\}$ is finite, we can choose a $\delta > 0$ such that the intersection $(r_m - \delta, r_m) \cap \{r_1, \ldots, r_N\}$ is empty, i.e. if $r_n \in (r_m - \delta, r_m)$, then $n > N$.

Now suppose that $x \in (r_m - \delta, r_m)$ and enumerate the rationals in $[x, r_m)$ as a subsequence $\{r_{n_1}, r_{n_2}, r_{n_3}, \ldots\}$ of the sequence $\{r_1, r_2, r_3, \ldots\}$; by our previous discussion, we must have $n_k > N$ for each $k \in \mathbf{N}$. As we showed in Exercise 6.4.10, $h$ is strictly increasing and $h(r_m) - h(x) = \sum_{k=1}^{\infty} 2^{-n_k}$. Thus

$$|h(r_m) - h(x)| = 2^{-N} \sum_{k=1}^{\infty} 2^{-n_k + N} \leq 2^{-N} \sum_{n=1}^{\infty} 2^{-n} = 2^{-N} < \epsilon$$

and our claim follows.

(ii) Our second claim is that $\lim_{x \to r_m^+} h(x) = h(r_m) + 2^{-m}$. Again, let $\epsilon > 0$ be given and choose $N \in \mathbf{N}$ such that $2^{-N} < \epsilon$. Similarly to before, we can choose a $\delta > 0$ such that if $r_n \in (r_m, r_m + \delta)$ then $n > N$. For $x \in (r_m, r_m + \delta)$, enumerate the rationals in $(r_m, x)$ as a subsequence $\{r_{n_1}, r_{n_2}, r_{n_3}, \ldots\}$ of the sequence $\{r_1, r_2, r_3, \ldots\}$, so that

$$[r_m, x) = \{r_m, r_{n_1}, r_{n_2}, r_{n_3}, \ldots\};$$

by our previous discussion, we must have $n_k > N$ for each $k \in \mathbf{N}$. Thus $h(x) - h(r_m) = 2^{-m} + \sum_{k=1}^{\infty} 2^{-n_k}$ and, arguing as in our first claim, it follows that

$$\left|h(x) - h(r_m) - 2^{-m}\right| = \sum_{k=1}^{\infty} 2^{-n_k} \leq 2^{-N} < \epsilon.$$

This proves our second claim.

We have now shown that if $r_m \in \mathbf{Q}$, then

$$\lim_{x \to r_m^-} h(x) = h(r_m) < h(r_m) + 2^{-m} = \lim_{x \to r_m^+} h(x),$$

so that $h$ has a jump discontinuity at each rational number; it follows from part (a) that $H$ fails to be differentiable at each rational number.

[UA] Abbott, S. (2015) *Understanding Analysis.* $2^{\text{nd}}$ edition.