# 1 Section 3.D Exercises

Exercises with solutions from Section 3.D of [LADR].

**Exercise 3.D.1.** Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

*Solution.* Observe that for any $u \in U$ we have

$$(T^{-1}S^{-1}ST)(u) = T^{-1}(S^{-1}(S(Tu))) = T^{-1}(Tu) = u.$$

Thus $T^{-1}S^{-1}ST$ is the identity on $U$. Similarly, for any $w \in W$ we have

$$(STT^{-1}S^{-1})(w) = S(T(T^{-1}(S^{-1}w))) = S(S^{-1}w) = w.$$

Thus $STT^{-1}S^{-1}$ is the identity on $W$. It follows that $ST$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

**Exercise 3.D.2.** Suppose $V$ is finite-dimensional and $\dim V > 1$. Prove that the set of noninvertible operators on $V$ is not a subspace of $\mathcal{L}(V)$.

*Solution.* Let $X = \{T \in \mathcal{L}(V) : T \text{ is not invertible}\}$. Consider Exercise 3.B.7, taking $W = V, n = m$, and $w_j = v_j$. The linear maps $S$ and $T$ defined there fail to be injective and thus belong to $X$, but the map $S+T$ is simply the identity on $V$ and hence belongs to $X$. Thus $X$ is not closed under addition and so is not a subspace of $\mathcal{L}(V)$.

**Exercise 3.D.3.** Suppose $V$ is finite-dimensional, $U$ is a subspace of $V$, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$ if and only if $S$ is injective.

*Solution.* Suppose there exists such an operator $T$ and let $u \in U$ be such that $Su = 0$. Then $Tu = Su = 0$ and thus $u = 0$ since $T$ is injective. Hence $\operatorname{null} S = \{0\}$ and we see that $S$ is injective.

Now suppose that $S$ is injective. Let $u_1, \dots, u_m$ be a basis of $U$, which we extend to a basis $u_1, \dots, u_m, x_1, \dots, x_n$ of $V$. Since $S$ is injective, Exercise 3.B.9 implies that $Su_1, \dots, Su_m$ is linearly independent in $V$ and thus can be extended to a basis $Su_1, \dots, Su_m, y_1, \dots, y_n$ of $V$. Define a linear map $T : V \to V$ by

$$Tu_j = Su_j \text{ for } 1 \le j \le m \quad \text{and} \quad Tx_j = y_j \text{ for } 1 \le j \le n.$$

Evidently, $T$ extends $S$. Suppose $v \in V$ is such that $Tv = 0$. There are scalars $a_1, \dots, a_m, b_1, \dots, b_n$ such that $v = \sum_{j=1}^m a_j u_j + \sum_{k=1}^n b_k x_k$. Then:

$$0 = Tv = T\left(\sum_{j=1}^m a_j u_j + \sum_{k=1}^n b_k x_k\right) = \sum_{j=1}^m a_j Tu_j + \sum_{k=1}^n b_k Tx_k = \sum_{j=1}^m a_j Su_j + \sum_{k=1}^n b_k y_k.$$

The linear independence of $Su_1, \dots, Su_m, y_1, \dots, y_n$ implies that $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$ and thus that $v = 0$. Hence $T$ is injective and 3.69 allows us to conclude that $T$ is invertible.

**Exercise 3.D.4.** Suppose $W$ is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.

*Solution.* Suppose there exists such an operator $S$ and suppose $v \in \operatorname{null} T_2$. Then $T_1 v = S(T_2 v) = S(0) = 0$ and thus $\operatorname{null} T_2 \subseteq \operatorname{null} T_1$. Since $S$ is invertible, we have $T_2 = S^{-1} T_1$ and we may similarly derive that $\operatorname{null} T_1 \subseteq \operatorname{null} T_2$. Thus $\operatorname{null} T_1 = \operatorname{null} T_2$.

Now suppose that $\operatorname{null} T_1 = \operatorname{null} T_2$. Since $W$ is finite-dimensional, $\operatorname{range} T_2$ is also finite-dimensional; let $T_2 v_1, \dots, T_2 v_m$ be a basis of $\operatorname{range} T_2$, for some vectors $v_1, \dots, v_m$ in $V$. Define a linear map $S' : \operatorname{range} T_2 \to W$ by $S'(T_2 v_j) = T_1 v_j$. For any $v \in V$, there are scalars $a_1, \dots, a_m$ such that $T_2 v = a_1 T_2 v_1 + \cdots + a_m T_2 v_m$. This gives

$$\begin{aligned} T_2 v = T_2(a_1 v_1 + \cdots + a_m v_m) &\iff v - (a_1 v_1 + \cdots + a_m v_m) \in \operatorname{null} T_2 \\ &\iff v - (a_1 v_1 + \cdots + a_m v_m) \in \operatorname{null} T_1, \end{aligned}$$

where we have used the assumption that $\operatorname{null} T_1 = \operatorname{null} T_2$. Since $v - (a_1 v_1 + \cdots + a_m v_m) \in \operatorname{null} T_1$, we have that $T_1 v = a_1 T_1 v_1 + \cdots + a_m T_1 v_m$. It follows that

$$S'(T_2 v) = a_1 S'(T_2 v_1) + \cdots + a_m S'(T_2 v_m) = a_1 T_1 v_1 + \cdots + a_m T_m v_m = T_1 v.$$

Hence $T_1 = S' T_2$.

Now we claim that $S'$ is injective. Suppose that $T_2 v \in \operatorname{range} T_2$ is such that $S'(T_2 v) = 0$. Then $T_1 v = 0$, so that $v \in \operatorname{null} T_1 = \operatorname{null} T_2$. Thus $T_2 v = 0$ and we see that $\operatorname{null} S' = \{0\}$, i.e. that $S'$ is injective. Exercise 3.D.3 now implies that there is an invertible operator $S \in \mathcal{L}(W)$ which extends $S'$ and hence satisfies $T_1 = ST_2$.

**Exercise 3.D.5.** Suppose $V$ is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} T_1 = \operatorname{range} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$.

*Solution.* Suppose there exists such an operator $S$. If $T_1 v \in \operatorname{range} T_1$ for some $v \in V$, then $T_1 v = T_2(Sv)$, so that $T_1 v \in \operatorname{range} T_2$ also. If $T_2 v \in \operatorname{range} T_2$ for some $v \in V$, then $T_2 v = T_1(S^{-1} v)$, so that $T_2 v \in \operatorname{range} T_1$ also. Thus $\operatorname{range} T_1 = \operatorname{range} T_2$.

Now suppose that $\operatorname{range} T_1 = \operatorname{range} T_2$. Let $u_1, \dots, u_m$ be a basis of $\operatorname{null} T_1$, which we extend to a basis $u_1, \dots, u_m, x_1, \dots, x_n$ of $V$. If we let $X := \operatorname{span}(x_1, \dots, x_n)$, we then have $V = \operatorname{null} T_1 \oplus X$.

The restriction of $T_1$ to $X$ is injective since $\operatorname{null} T_1 \cap X = \{0\}$, so Exercise 3.B.9 implies that the list $T_1 x_1, \dots, T_1 x_n$ is linearly independent. Furthermore, Exercise 3.B.10 implies that the list

$$T_1 u_1, \dots, T_1 u_m, T_1 x_1, \dots, T_1 x_n$$

spans range $T_1$. Since each $T_1 u_j = 0$, we can discard these vectors to see that the list $T_1 x_1, \ldots, T_1 x_n$ spans range $T_1$. We have now shown that $T_1 x_1, \ldots, T_1 x_n$ is a basis of range $T_1$.

By assumption, we have range $T_1 =$ range $T_2$, and thus there are vectors $y_1, \ldots, y_n$ in $V$ such that $T_1 x_j = T_2 y_j$. Since the list $T_2 y_1, \ldots, T_2 y_n$ is linearly independent, Exercise 3.A.4 shows that the list $y_1, \ldots, y_n$ is linearly independent. Let $v_1, \ldots, v_m$ be a basis of null $T_2$ (since range $T_1 =$ range $T_2$, the Fundamental Theorem of Linear Maps (3.22) implies that $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$, so that this basis is also of length $m$). As the proof of 3.22 shows, $v_1, \ldots, v_m, y_1, \ldots, y_n$ must be a basis of $V$.

Define a linear map $S : V \to V$ by

$$Su_j = v_j \text{ for } 1 \le j \le m \quad \text{and} \quad Sx_j = y_j \text{ for } 1 \le j \le n.$$

If $v = a_1 u_1 + \cdots + a_{m+n} x_n$ is such that $Sv = 0$, then

$$0 = a_1 S u_1 + \cdots + a_{m+n} S x_n = a_1 v_1 + \cdots + a_{m+n} y_n.$$

The linear independence of the basis $v_1, \ldots, y_n$ then implies that $a_1 = \cdots = a_{m+n} = 0$ and hence that $v = 0$. Thus null $S = \{0\}$ and we see that $S$ is injective; 3.69 allows us to conclude that $S$ is an invertible operator. Furthermore, we have $T_1 = T_2 S$. Indeed, for any $v = \sum_{j=1}^m a_j u_j + \sum_{k=1}^n b_j x_j$ in $V$, we have

$$\begin{aligned}(T_2 S)(v) &= \sum_{j=1}^m a_j T_2(Su_j) + \sum_{k=1}^n b_j T_2(Sx_j) = \sum_{j=1}^m a_j T_2 v_j + \sum_{k=1}^n b_j T_2 y_j \\ &= \sum_{j=1}^m a_j T_1 u_j + \sum_{k=1}^n b_j T_1 x_j = T_1\left(\sum_{j=1}^m a_j u_j + \sum_{k=1}^n b_j x_j\right) = T_1 v.\end{aligned}$$

**Exercise 3.D.6.** Suppose $V$ and $W$ are finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that there exist invertible operators $R \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$ such that $T_1 = ST_2R$ if and only if $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$.

*Solution.* Suppose there exist such operators $R$ and $S$, so that $T_1 = ST_2R$. Notice that this gives us $T_2 = S^{-1} T_1 R^{-1}$. Exercise 3.B.22 now implies that

$$\dim \operatorname{null} T_1 = \dim \operatorname{null} ST_2R \le \dim \operatorname{null} S + \dim \operatorname{null} T_2 + \dim \operatorname{null} R = \dim \operatorname{null} T_2,$$

$$\dim \operatorname{null} T_2 = \dim \operatorname{null} S^{-1} T_1 R^{-1} \le \dim \operatorname{null} S^{-1} + \dim \operatorname{null} T_1 + \dim \operatorname{null} R^{-1} = \dim \operatorname{null} T_1,$$

where we have used that each invertible linear map $R, S, R^{-1}$, and $S^{-1}$ are injective and hence have trivial null space. These two inequalities combine to give us $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$.

Now suppose that $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$. Let $u_1, \dots, u_m$ be a basis of $\operatorname{null} T_1$, which we extend to a basis $u_1, \dots, u_m, x_1, \dots, x_n$ of $V$, and let $v_1, \dots, v_m$ be a basis of $\operatorname{null} T_2$, which we extend to a basis $v_1, \dots, v_m, y_1, \dots, y_n$ of $V$. Define an operator $R : V \to V$ by

$$Ru_j = v_j \text{ for } 1 \leq j \leq m \quad \text{and} \quad Rx_j = y_j \text{ for } 1 \leq j \leq n.$$

As in the solution to Exercise 3.D.5, this operator must be invertible since it maps a basis to a basis. We claim that $\operatorname{null} T_1 = \operatorname{null} T_2 R$. Suppose that $u \in \operatorname{null} T_1$, so that $u = a_1 u_1 + \cdots + a_m u_m$ for some scalars $a_1, \dots, a_m$. Then

$$T_2(Ru) = T_2(a_1 R u_1 + \cdots + a_m R u_m) = T_2(a_1 v_1 + \cdots + a_m v_m) = 0.$$

Thus $\operatorname{null} T_1 \subseteq \operatorname{null} T_2 R$. Suppose that $v \in \operatorname{null} T_2 R$, i.e. $T_2(Rv) = 0$. Then $Rv \in \operatorname{null} T_2$, so that $Rv = a_1 v_1 + \cdots + a_m v_m$ for some scalars $a_1, \dots, a_m$. This gives us

$$Rv = a_1 R u_1 + \cdots + a_m R u_m = R(a_1 u_1 + \cdots + a_m u_m),$$

which implies that $v = a_1 u_1 + \cdots + a_m u_m \in \operatorname{null} T_1$ since $R$ is injective. Thus $\operatorname{null} T_1 = \operatorname{null} T_2 R$, as claimed.

We may now appeal to Exercise 3.D.4 to obtain an invertible operator $S$ such that $T_1 = S T_2 R$.

**Exercise 3.D.7.** Suppose $V$ and $W$ are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

(a) Show that $E$ is a subspace of $\mathcal{L}(V, W)$.

(b) Suppose $v \neq 0$. What is $\dim E$?

*Solution.* (a) Suppose $S, T \in E$ and $\lambda \in \mathbf{F}$. Then

$$(\lambda S + T)(v) = \lambda S v + T v = 0.$$

Thus $\lambda S + T \in E$ and so $E$ is a subspace of $\mathcal{L}(V, W)$.

(b) Set $v_1 := v$. Since $v_1 \neq 0$, we can extend this list to a basis $v_1, \dots, v_m$ of $V$. Let $w_1, \dots, w_n$ be any basis of $W$. By 3.60, the linear map $\mathcal{M} : \mathcal{L}(V, W) \to \mathbf{F}^{n,m}$ is an isomorphism, which restricts to an isomorphism $\mathcal{M} : E \to \mathcal{E}$, where $\mathcal{E} = \{\mathcal{M}(T) : T \in E\}$; we use this notation in place of $\mathcal{M}(E)$ for obvious reasons. We claim that $\mathcal{E}$ is the subspace of matrices whose entries in the first column are all 0. Indeed, if $T \in E$, then $Tv_1 = 0$ and thus the entries in the first column of $\mathcal{M}(T)$ are all 0. Conversely, if $A$ is an $n$-by-$m$ matrix with entries $A_{j,k}$ such that $A_{\cdot,1} = 0$, then the linear map $T : V \to W$ defined by

$$Tv_k = A_{1,k} w_1 + \cdots + A_{n,k} w_n$$

satisfies $Tv_1 = 0$ and hence belongs to $E$; evidently we have $\mathcal{M}(T) = A$. Our claim follows. Using the same logic as in the proof of 3.40, which shows that $\dim F^{n,m} = nm$, it is easily verified that $\mathcal{E}$, and hence $E$, has dimension $(m-1)n$. In conclusion, we have

$$\dim E = (\dim V - 1)(\dim W).$$

**Exercise 3.D.8.** Suppose $V$ is finite-dimensional and $T : V \to W$ is a surjective linear map of $V$ onto $W$. Prove that there is a subspace $U$ of $V$ such that $T|_U$ is an isomorphism of $U$ onto $W$. (Here $T|_U$ means the function $T$ restricted to $U$. In other words, $T|_U$ is the function whose domain is $U$, with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.)

*Solution.* By Exercise 3.B.12, there exists a subspace $U$ of $V$ such that $V = U \oplus \operatorname{null} T$ and $W = \operatorname{range} T = \{Tu : u \in U\}$. Thus the restriction $T|_U : U \to W$ is surjective. Furthermore, since $U \cap \operatorname{null} T = \{0\}$, we have $\operatorname{null} T|_U = \{0\}$ and hence $T|_U$ is also injective. We may conclude that $T|_U$ is an invertible linear map, i.e. an isomorphism.

**Exercise 3.D.9.** Suppose $V$ is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST$ is invertible if and only if $S$ and $T$ are both invertible.

*Solution.* If $S$ and $T$ are both invertible, then $ST$ is invertible by Exercise 3.D.1. If either $S$ or $T$ fails to be invertible, then 3.69 implies that at least one of these maps is not surjective. Thus

$$\min\{\dim \operatorname{range} S, \dim \operatorname{range} T\} < \dim V.$$

Exercise 3.B.23 then implies that $\dim \operatorname{range} ST < \dim V$ and hence that $ST$ is not surjective. We may now apply 3.69 again to see that $ST$ is not invertible.

**Exercise 3.D.10.** Suppose $V$ is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

*Solution.* Suppose that $ST = I$. Then $ST$ is invertible, so Exercise 3.D.9 implies that $S$ and $T$ are both invertible. Applying $S^{-1}$ on the left to both sides of $ST = I$ shows that $T = S^{-1}$; it follows that $TS = S^{-1}S = I$. Reversing the roles of $S$ and $T$ in the preceding argument gives us the converse implication.

**Exercise 3.D.11.** Suppose $V$ is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that $T$ is invertible and that $T^{-1} = US$.

*Solution.* Applying Exercise 3.D.9 twice shows that each of $S, T$, and $U$ are invertible operators. It follows that $T = S^{-1}U^{-1}$ and hence by Exercise 3.D.1 we have $T^{-1} = US$.

**Exercise 3.D.12.** Show that the result in the previous exercise can fail without the hypothesis that $V$ is finite-dimensional.

*Solution.* Let $V$ be the infinite-dimensional vector space $\mathbf{R}^\infty$ and set $S = I$. Take $T$ to be the left-shift operator and $U$ to be the right-shift operator, i.e.

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots) \quad \text{and} \quad U(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots).$$

Then $STU = I$, but $T$ is not invertible. Indeed, $T$ fails to be injective, since $T(1, 0, 0, \ldots) = 0$.

**Exercise 3.D.13.** Suppose $V$ is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that $RST$ is surjective. Prove that $S$ is injective.

*Solution.* By 3.69, $RST$ must be invertible. Applying Exercise 3.D.9 twice shows that each of $R, S$, and $T$ are invertible operators. Thus $S$ is injective.

**Exercise 3.D.14.** Suppose $v_1, \ldots, v_n$ is a basis of $V$. Prove that the map $T : V \to \mathbf{F}^{n,1}$ defined by

$$Tv = \mathcal{M}(v)$$

is an isomorphism of $V$ onto $\mathbf{F}^{n,1}$; here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \ldots, v_n$.

*Solution.* First, let us show that $T$ is linear. Suppose $u, v \in V$, so that there are scalars $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that

$$u = a_1 v_1 + \cdots + a_n v_n \quad \text{and} \quad v = b_1 v_1 + \cdots + b_n v_n,$$

and let $\lambda \in \mathbf{F}$ be given. Then

$$u + v = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n \quad \text{and} \quad \lambda u = (\lambda a_1)v_1 + \cdots + (\lambda a_n)v_n.$$

Thus

$$Tu = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad Tv = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad T(u+v) = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} \quad \text{and} \quad T(\lambda u) = \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix}.$$

Hence $T(u + v) = Tu + Tv$ and $T(\lambda u) = \lambda T u$ and thus $T$ is linear. Furthermore, $T$ is surjective, since for any scalars $a_1, \ldots, a_n$ we have

$$T(a_1 v_1 + \cdots + a_n v_n) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Since $V$ and $\mathbf{F}^{n,1}$ are both of dimension $n$, 3.69 allows us to conclude that $T$ is an isomorphism.

**Exercise 3.D.15.** Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then there exists an $m$-by-$n$ matrix $A$ such that $Tx = Ax$ for every $x \in \mathbf{F}^{n,1}$.

*Solution.* Let $e_j$ be the column vector with a 1 in the $j^{\text{th}}$ row and a 0 in each other row. Take $e_1, \ldots, e_n$ as a basis of $\mathbf{F}^{n,1}$ and $e_1, \ldots, e_m$ as a basis of $\mathbf{F}^{m,1}$. Then for each $x \in \mathbf{F}^{n,1}$ we have $\mathcal{M}(x) = x$ and $\mathcal{M}(Tx) = Tx$. By 3.65, the desired matrix $A$ is $\mathcal{M}(T)$.

**Exercise 3.D.16.** Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that $T$ is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

*Solution.* Suppose that $T$ is a scalar multiple of the identity, say $T = \lambda I$ for some $\lambda \in \mathbf{F}$. Let $S \in \mathcal{L}(V)$ and $v \in V$ be given. Then

$$S(Tv) = S((\lambda I)(v)) = S(\lambda v) = \lambda(Sv) = (\lambda I)(Sv) = T(Sv).$$

Thus $ST = TS$.

Now suppose that $ST = TS$ for every $S \in \mathcal{L}(V)$. Let $v_1, \ldots, v_m$ be a basis of $V$ and for $1 \leq j, k \leq m$, define $S_{j,k} : V \to V$ by

$$S_{j,k} v_i = \begin{cases} v_k & \text{if } i = j, \\ v_j & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \mathcal{M}(T)$, so that

$$Tv_j = A_{1,j} v_1 + \cdots + A_{m,j} v_m.$$

For any $1 \leq j \leq m$, observe that

$$S_{j,j}(Tv_j) = S_{j,j}(A_{1,j} v_1 + \cdots + A_{m,j} v_m) = A_{1,j} S_{j,j} v_1 + \cdots + A_{m,j} S_{j,j} v_m = A_{j,j} v_j,$$
$$T(S_{j,j} v_j) = Tv_j = A_{1,j} v_1 + \cdots + A_{m,j} v_m.$$

By assumption we have $S_{j,j}(Tv_j) = T(S_{j,j} v_j)$, so

$$A_{j,j} v_j = A_{1,j} v_1 + \cdots + A_{m,j} v_m \iff A_{1,j} v_1 + \cdots + A_{j-1,j} v_{j-1} + A_{j+1,j} v_{j+1} + \cdots + A_{m,j} v_m = 0.$$

Thus by linear independence we have

$$A_{1,j} = \cdots = A_{j-1,j} = A_{j+1,j} = \cdots = A_{m,j} = 0.$$

In other words, each non-diagonal entry of $A$ is 0, so that $Tv_j = A_{j,j}v_j$ for each $1 \leq j \leq m$. Now suppose that $1 \leq j < k \leq m$. Then

$$S_{j,k}(Tv_j) = S_{j,k}(A_{j,j}v_j) = A_{j,j}S_{j,k}v_j = A_{j,j}v_k \quad \text{and} \quad T(S_{j,k}v_j) = Tv_k = A_{k,k}v_k.$$

By assumption these two must be equal, i.e. $A_{j,j}v_k = A_{k,k}v_k$, and so $A_{j,j} = A_{k,k}$ since $v_k \neq 0$. Thus, letting $\lambda = A_{1,1}$, we have shown that $Tv_j = \lambda v_j$ for each basis vector $v_j$. It follows that $T = \lambda I$.

**Exercise 3.D.17.** Suppose $V$ is finite-dimensional and $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ and $TS \in \mathcal{E}$ for all $S \in \mathcal{L}(V)$ and all $T \in \mathcal{E}$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = \mathcal{L}(V)$.

*Solution.* If $\mathcal{E} = \{0\}$ then evidently $\mathcal{E}$ has the desired properties, so it will suffice to show that if there is some $T \in \mathcal{E}$ with $T \neq 0$, then $\mathcal{E} = \mathcal{L}(V)$. Let $v_1, \ldots, v_m$ be a basis of $V$. Define a linear map $E_{i,j} \in \mathcal{L}(V)$ by

$$E_{i,j}v_k = \delta^i_k v_j,$$

where $\delta^i_k$ is the Kronecker delta, i.e. $\delta^i_k = 1$ if $i = k$ and $\delta^i_k = 0$ otherwise. In other words, $E_{i,j}$ sends $v_i$ to $v_j$ and each other basis vector to 0.

Since $T \neq 0$, there must be some $Tv_p \neq 0$. Suppose that

$$Tv_p = a_1v_1 + \cdots + a_mv_m.$$

Since $Tv_p \neq 0$, there is some $a_r \neq 0$. For each $1 \leq i \leq m$, we have

$$a_r^{-1}E_{r,i}TE_{i,p}v_k = a_r^{-1}E_{r,i}T(\delta^i_k v_p) = a_r^{-1}\delta^i_k E_{r,i}(a_1v_1 + \cdots + a_mv_m) = \delta^i_k v_i.$$

Set $L := a_r^{-1}(E_{r,1}TE_{1,p} + \cdots + E_{r,m}TE_{m,p})$. By the equality above, we then have $Lv_k = v_k$, i.e. $L$ is the identity map on $V$. By assumption, we have $E_{r,i}TE_{i,p} \in \mathcal{E}$ for each $1 \leq i \leq m$. Since $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$, it follows that $L \in \mathcal{E}$. Then for any $S \in \mathcal{L}(V)$, we have $SL = S \in \mathcal{E}$ and thus $\mathcal{E} = \mathcal{L}(V)$.

**Exercise 3.D.18.** Show that $V$ and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces.

*Solution.* Given $v \in V$, define a linear map $T_v : \mathbf{F} \to V$ by $T_v(1) = v$ and a map $\Phi : V \to \mathcal{L}(\mathbf{F}, V)$ by $\Phi(v) = T_v$. Showing that $\Phi$ is linear amounts to showing that for $u, v \in V$ and $\lambda \in \mathbf{F}$, one has $T_{u+\lambda v} = T_u + \lambda T_v$. Observe that

$$T_{u+\lambda v}(1) = u + \lambda v = T_u(1) + \lambda T_v(1) = (T_u + \lambda T_v)(1).$$

The uniqueness part of 3.5 now implies that $T_{u+\lambda v} = T_u + \lambda T_v$ and thus $\Phi$ is linear.

The map $\Phi$ is injective. Indeed, if $v \in V$ is such that $\Phi(v) = T_v = 0$, then in particular $T_v(1) = v = 0$ and so $\operatorname{null} \Phi = \{0\}$. By 3.61 we have $\dim V = \dim \mathcal{L}(\mathbf{F}, V)$ and so 3.69 allows us to conclude that $\Phi$ is an isomorphism.

**Exercise 3.D.19.** Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is such that $T$ is injective and $\deg Tp \leq \deg p$ for every nonzero polynomial $p \in \mathcal{P}(\mathbf{R})$.

(a) Prove that $T$ is surjective.

(b) Prove that $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}(\mathbf{R})$.

*Solution.* (a) For each positive integer $m$, consider the restriction $T_m : \mathcal{P}_m(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ given by $T_m p = Tp$; since $T$ is injective, each $T_m$ is also injective. The hypothesis $\deg Tp \leq \deg p$ shows that this restriction actually maps into $\mathcal{P}_m(\mathbf{R})$ and so 3.69 implies that each $T_m$ is an isomorphism. Then for any $p \in \mathcal{P}(\mathbf{R})$, let $m = \deg p$. As we just showed, there exists some $q \in \mathcal{P}_m(\mathbf{R})$ such that $T_m q = Tq = p$.

(b) We will prove this by induction on the degree of $p$. Let $P(n)$ be the statement that for all polynomials $p$ of degree $n$, we have $\deg Tp = n$. For the base case $P(0)$, let $p$ be a non-zero constant polynomial. Then since $T$ is injective, we have $Tp \neq 0$ and thus $\deg Tp = 0$.

Now suppose that $P(n)$ is true for some $n$ and let $p$ be a polynomial of degree $n+1$. Suppose by way of contradiction that $\deg Tp < n + 1$. Then as we showed in part (a), there exists a polynomial $q$ with $\deg q \leq n$ such that $Tq = Tp$. The injectivity of $T$ implies that $p = q$, but this is a contradiction since $p$ and $q$ have different degrees. Thus $\deg Tp = n + 1$ and so $P(n + 1)$ holds. This completes the induction step and the proof.

**Exercise 3.D.20.** Suppose $n$ is a positive integer and $A_{i,j} \in \mathbf{F}$ for $i, j = 1, \ldots, n$. Prove that the following are equivalent (note that in both parts below, the number of equations equals the number of variables):

(a) The trivial solution $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations

$$\begin{gathered} \sum_{k=1}^{n} A_{1,k} x_k = 0 \\ \vdots \\ \sum_{k=1}^{n} A_{n,k} x_k = 0. \end{gathered}$$

(b) For every $c_1, \ldots, c_n \in \mathbf{F}$, there exists a solution to the system of equations

$$\sum_{k=1}^{n} A_{1,k} x_k = c_1$$
$$\vdots$$
$$\sum_{k=1}^{n} A_{n,k} x_k = c_n.$$

*Solution.* As in Example 3.25, we can rephrase this question in terms of a linear map. Define $T : \mathbf{F}^n \to \mathbf{F}^n$ by

$$T(x_1, \ldots, x_n) = \left( \sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{n,k} x_k \right).$$

Then (a) is equivalent to the injectivity of $T$ and (b) is equivalent to the surjectivity of $T$. The equivalence of (a) and (b) then follows from 3.69.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edition.