1 Section 6.3 Exercises

Exercises with solutions from Section 6.3 of [UA].

Exercise 6.3.1. Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show (g_n) converges uniformly on $[0, 1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0, 1]$.

(b) Now, show that (g'_n) converges on $[0, 1]$. Is the convergence uniform? Set $h = \lim g'_n$ and compare h and g'. Are they the same?

Solution. (a) The limit function $\lim g_n = g : [0, 1] \to \mathbf{R}$ is given by $g(x) = 0$. Note that for any $x \in [0, 1]$ we have

$$|g_n(x) - g(x)| = \frac{x^n}{n} \leq \frac{1}{n}.$$

Since this bound converges to zero and does not depend on x, the convergence $g_n \to g$ is uniform on $[0, 1]$. Evidently g is differentiable on $[0, 1]$ and satisfies $g'(x) = 0$ for all $x \in [0, 1]$.

(b) The sequence (g'_n) is given by $g'_n(x) = x^{n-1}$ for $x \in [0, 1]$. This sequence converges pointwise to the function $h : [0, 1] \to \mathbf{R}$ given by

$$h(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

The convergence cannot be uniform since each g'_n is continuous at 1 but h is not. Note that $h \neq g'$; this gives an alternative proof for showing that the convergence $g'_n \to h$ is not uniform, as uniform convergence $g'_n \to h$ would imply that $g' = h$ by Theorem 6.3.1/6.3.3.

Exercise 6.3.2. Consider the sequence of functions

$$h_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

(a) Compute the pointwise limit of (h_n) and then prove that the convergence is uniform on $\mathbf{R}$.

(b) Note that each h_n is differentiable. Show $g(x) = \lim h'_n(x)$ exists for all x, and explain how we can be certain that the convergence is *not* uniform on any neighborhood of zero.

Solution. (a) The pointwise limit is the function $h : \mathbf{R} \to \mathbf{R}$ given by $h(x) = \sqrt{x^2} = |x|$. Note that for any $x \in \mathbf{R}$ we have

$$|h_n(x) - h(x)| = \sqrt{x^2 + n^{-1}} - \sqrt{x^2} = \frac{n^{-1}}{\sqrt{x^2 + n^{-1}} + \sqrt{x^2}} \leq \frac{n^{-1}}{n^{-1/2}} = \frac{1}{\sqrt{n}}.$$

Since this bound converges to zero and does not depend on x, we see that the convergence $h_n \to h$ is uniform on $\mathbf{R}$.

(b) Note that $h_n' : \mathbf{R} \to \mathbf{R}$ is given by

$$h_n'(x) = \frac{x}{\sqrt{x^2 + n^{-1}}}.$$

This sequence converges pointwise to the function $g : \mathbf{R} \to \mathbf{R}$ given by

$$g(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

The convergence $h_n' \to g$ cannot be uniform on any neighbourhood of zero since each h_n' is continuous at zero but g is not. Alternatively, if the convergence $h_n' \to g$ was uniform, then Theorem 6.3.1/6.3.3 would imply that h was differentiable at zero; but h fails to be differentiable precisely at zero.

Exercise 6.3.3. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}.$$

(a) Find the points on $\mathbf{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on $\mathbf{R}$. What is the limit function?

(b) Let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim f_n'(x)$.

Solution. (a) From the observation

$$\frac{1}{2\sqrt{n}} - \frac{x}{1 + nx^2} = \frac{nx^2 - 2\sqrt{n}x + 1}{2\sqrt{n}(1 + nx^2)} = \frac{(\sqrt{n}x - 1)^2}{2\sqrt{n}(1 + nx^2)} \geq 0$$

we can see that $0 \leq f_n(x) \leq \frac{1}{2\sqrt{n}}$ for all $x \geq 0$ and also that $f_n(x) = \frac{1}{2\sqrt{n}}$ precisely when $x = \frac{1}{\sqrt{n}}$. Combining this with the fact that each f_n is an odd function, we see that

$$-\frac{1}{2\sqrt{n}} \leq f_n(x) \leq \frac{1}{2\sqrt{n}}$$

for all $x \in \mathbf{R}$ and furthermore that

$$f_n(x) = -\tfrac{1}{2\sqrt{n}} \iff x = -\tfrac{1}{\sqrt{n}} \quad \text{and} \quad f_n(x) = \tfrac{1}{2\sqrt{n}} \iff x = \tfrac{1}{\sqrt{n}}.$$

The bound $|f_n(x)| \leq \frac{1}{2\sqrt{n}}$ converges to zero and does not depend on x, demonstrating that f_n converges uniformly to the zero function.

(b) The quotient rule gives us

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}.$$

For $x \neq 0$, we have

$$f_n'(x) = \frac{\frac{1}{n^2x^4} - \frac{1}{nx^2}}{\left(\frac{1}{nx^2} + 1\right)^2} \to 0 \text{ as } n \to \infty,$$

and for $x = 0$ we have $f_n'(0) = 1$. In part (a) we showed that $\lim f_n = f : \mathbf{R} \to \mathbf{R}$ was given by $f(x) = 0$. Thus $f'(x) = \lim f_n'(x) = 0$ for all $x \neq 0$, and $f'(0) = 0 \neq 1 = \lim f_n'(0)$.

Exercise 6.3.4. Let

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Show that $h_n \to 0$ uniformly on $\mathbf{R}$ but that the sequence of derivatives (h_n') diverges for every $x \in \mathbf{R}$.

Solution. Observe that

$$|h_n(x)| \leq \frac{1}{\sqrt{n}}$$

for any $x \in \mathbf{R}$. Since this bound converges to zero and does not depend on x, we see that $h_n \to 0$ uniformly on $\mathbf{R}$. The sequence of derivatives (h_n') is given by

$$a_n := h_n'(x) = \sqrt{n}\cos(nx).$$

We claim that (a_n) does not converge for any $x \in \mathbf{R}$; to see this, we will consider three cases.

Case 1. Suppose $x = k\pi$, where k is an even integer. In this case, we have $a_n = \sqrt{n}$, which clearly diverges.

Case 2. Suppose $x = k\pi$, where k is an odd integer. In this case, we have $a_n = (-1)^n\sqrt{n}$, which clearly diverges.

Case 3. Suppose x is not of the form $k\pi$ for any integer k and suppose by way of contradiction that $a_n \to L$ for some $L \in \mathbf{R}$. It follows that

$$\frac{a_n}{\sqrt{n}} = \cos(nx) \to 0,$$

which also implies that $\cos((n+1)x) \to 0$. Consider the trigonometric identity

$$\sin(nx) = \frac{\cos(nx)\cos(x) - \cos((n+1)x)}{\sin(x)};$$

the fact that $x \neq k\pi$ for any integer k means we are not dividing by zero here. Since both $\cos(nx) \to 0$ and $\cos((n+1)x) \to 0$, we see that $\sin(nx) \to 0$, which in turn implies that

$$\sin^2(nx) + \cos^2(nx) \to 0.$$

This is a contradiction since $\sin^2(nx) + \cos^2(nx) = 1$ for all $n \in \mathbf{N}$.

Exercise 6.3.5. Let

$$g_n(x) = \frac{nx + x^2}{2n},$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

(b) Compute $g_n'(x)$ for each $n \in \mathbf{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g_n'(x)$.

(c) Repeat parts (a) and (b) for the sequence $f_n(x) = (nx^2 + 1)/(2n + x)$.

Solution. (a) For a fixed $x \in \mathbf{R}$ we have

$$g_n(x) = \frac{x}{2} + \frac{x^2}{2n} \to \frac{x}{2} \text{ as } n \to \infty.$$

It follows that $g'(x) = \frac{1}{2}$ for any $x \in \mathbf{R}$.

(b) The sequence of derivatives (g_n') is given by

$$g_n'(x) = \frac{1}{2} + \frac{x}{n}.$$

For $x \in [-M, M]$ we have

$$\left| g_n'(x) - \frac{1}{2} \right| = \frac{|x|}{n} \leq \frac{M}{n}.$$

Since this bound converges to zero as $n \to \infty$ and does not depend on x, we see that $g_n' \to \frac{1}{2}$ uniformly on any interval of the form $[-M, M]$. Observe that $0 \in [-M, M]$ and $g_n(0) = 0$ is convergent. Theorem 6.3.3 implies that $g_n \to g$ uniformly on $[-M, M]$ and furthermore that $g'(x) = \lim g_n'(x) = \frac{1}{2}$ for any $x \in [-M, M]$. By taking M sufficiently large, this shows that $g'(x) = \frac{1}{2}$ for all $x \in \mathbf{R}$.

(c) The sequence (f_n) is given by

$$f_n(x) = \frac{nx^2 + 1}{2n + x}.$$

(Strictly speaking this is only defined on $\mathbf{R} \setminus \{-2n\}$, but since we are only interested in the limit as $n \to \infty$, this isn't a problem; eventually the sequence is defined on any interval of the form $[-M, M]$.)

Note that

$$f_n(x) = \frac{x^2 + \frac{1}{n}}{2 + \frac{x}{n}} \to \frac{x^2}{2} \text{ as } n \to \infty,$$

so that the pointwise limit function is $f(x) = \frac{x^2}{2}$, which satisfies $f'(x) = x$.

The sequence of derivatives (f_n') is given by

$$f_n'(x) = \frac{nx^2 + 4n^2x - 1}{x^2 + 4nx + 4n^2} = \frac{\frac{x^2}{n} + 4x - \frac{1}{n^2}}{\frac{x^2}{n^2} + \frac{4x}{n} + 4} \to x \text{ as } n \to \infty.$$

For any $x \in [-M, M]$, observe that

$$|f_n'(x) - x| = \left| \frac{x^3 + 3nx^2 + 1}{4n^2 + 4nx + x^2} \right| \leq \frac{M^3 + 3M^2 + 1}{|x + n|} \leq \frac{M^3 + 3M^2 + 1}{n - M}$$

provided $n > M$. Since this bound converges to zero as $n \to \infty$ and does not depend on x, we see that $f_n' \to x$ uniformly on $[-M, M]$. Observe that $0 \in [-M, M]$ and $f_n(0) = \frac{1}{2n} \to 0$ as $n \to \infty$. Theorem 6.3.3 implies that $f_n \to f$ uniformly on $[-M, M]$ and furthermore that $f'(x) = \lim f_n'(x) = x$ for any $x \in [-M, M]$. By taking M sufficiently large, this shows that $f'(x) = x$ for all $x \in \mathbf{R}$.

Exercise 6.3.6. Provide an example or explain why the request is impossible. Let's take the domain of the functions to be all of $\mathbf{R}$.

(a) A sequence (f_n) of nowhere differentiable functions with $f_n \to f$ uniformly and f everywhere differentiable.

(b) A sequence (f_n) of differentiable functions such that (f_n') converges uniformly but the original sequence (f_n) does not converge for any $x \in \mathbf{R}$.

(c) A sequence (f_n) of differentiable functions such that both (f_n) and (f_n') converge uniformly but $f = \lim f_n$ is not differentiable at some point.

Solution. (a) Define a sequence $(f_n : \mathbf{R} \to \mathbf{R})$ by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Then $|f_n(x)| \leq \frac{1}{n}$ for any $x \in \mathbf{R}$, demonstrating that $f_n \to 0$ uniformly on $\mathbf{R}$. Clearly the zero function is differentiable everywhere, but each f_n is nowhere continuous and hence nowhere differentiable.

(b) Define a sequence $(f_n : \mathbf{R} \to \mathbf{R})$ by

$$f_n(x) = n$$

for all $x \in \mathbf{R}$. Then each f_n is differentiable and the sequence (f_n') is given by $f_n'(x) = 0$, which converges uniformly to the zero function. However, $(f_n(x))$ is divergent for every $x \in \mathbf{R}$.

(c) This is impossible. Any point $x \in \mathbf{R}$ is contained in some interval of the form $[-M, M]$; applying Theorem 6.3.3 to this interval shows that f is differentiable at x.

Exercise 6.3.7. Use the Mean Value Theorem to supply a proof for Theorem 6.3.2. To get started, observe that the triangle inequality implies that, for any $x \in [a, b]$ and $m, n \in \mathbf{N}$,

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|.$$

Solution. Let $\epsilon > 0$ be given. Since the sequence $(f_n(x_0))$ is convergent, there exists an $N_1 \in \mathbf{N}$ such that

$$n, m \geq N_1 \implies |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2},$$

and since the sequence (f_n') converges uniformly on $[a, b]$, there exists an $N_2 \in \mathbf{N}$ such that

$$x \in [a, b] \text{ and } n, m \geq N_2 \implies |f_n'(x) - f_m'(x)| < \frac{\epsilon}{2(b-a)}.$$

Set $N = \max\{N_1, N_2\}$ and suppose that $n, m \geq N$ and $x \in (x_0, b]$ (the argument is easily modified if $x \in [a, x_0)$). Note that $f_n - f_m$ is differentiable on the interval $[x_0, x]$; the Mean Value Theorem then implies that there is some $c \in (x_0, x)$ such that

$$|x - x_0||f_n'(c) - f_m'(c)| = |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))|.$$

It follows that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\
&= |x - x_0||f_n'(c) - f_m'(c)| + |f_n(x_0) - f_m(x_0)| \\
&\leq (b - a)|f_n'(c) - f_m'(c)| + |f_n(x_0) - f_m(x_0)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

We have now shown that for any $n, m \geq N$ and $x \in [a, b]$ it holds that

$$|f_n(x) - f_m(x)| < \epsilon;$$

it follows from Theorem 6.2.5 that the sequence (f_n) is uniformly convergent on $[a, b]$.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.