1 Section 8.6 Exercises

Exercises with solutions from Section 8.6 of [UA].

Exercise 8.6.1. (a) Fix $r \in \mathbf{Q}$. Show that the set $C_r = \{t \in \mathbf{Q} : t < r\}$ is a cut.

The temptation to think of all cuts as being of this form should be avoided. Which of the following subsets of $\mathbf{Q}$ are cuts?

(b) $S = \{t \in \mathbf{Q} : t \leq 2\}$

(c) $T = \{t \in \mathbf{Q} : t^2 < 2 \text{ or } t < 0\}$

(d) $U = \{t \in \mathbf{Q} : t^2 \leq 2 \text{ or } t < 0\}$

Solution. (a) It is clear that C_r satisfies (c1) and (c2). To see that C_r satisfies (c3), observe that if $t \in C_r$, then $t < \frac{t+r}{2}$ and $\frac{t+r}{2} \in C_r$.

(b) This is not a cut, since it has 2 as a maximum element.

(c) This is a cut. T satisfies (c1) since $0 \in T$ and $2 \notin T$.

Suppose $t \in T$ and r is a rational such that $r < t$. If $r < 0$ then certainly $r \in T$, so suppose that $r \geq 0$, which implies that $t > 0$. It follows that $r^2 < t^2 < 2$ and so $r \in T$. Thus T satisfies (c2).

Suppose $t \in T$. If $t \leq 0$ then let $r = 1$ and if $t > 0$ then let $r = \frac{2t+2}{t+2}$. In either case, one can verify that $t < r$ and $r \in T$. Thus T satisfies (c3).

(d) By Theorem 1.1.1 we have $U = T$ and hence by part (c) U is a cut.

Exercise 8.6.2. Let A be a cut. Show that if $r \in A$ and $s \notin A$, then $r < s$.

Solution. Given that $r \in A$, the implication $s \notin A \implies r < s$ is the contrapositive of (c2).

Exercise 8.6.3. Using the usual definitions of addition and multiplication, determine which of these properties are possessed by $\mathbf{N}, \mathbf{Z}$, and $\mathbf{Q}$, respectively.

Solution. $\mathbf{N}$ satisfies (f1), (f2), and (f5). It fails (f3) since there is no additive identity and it fails (f4) since no element has an additive inverse and only 1 has a multiplicative inverse (1 is its own inverse).

$\mathbf{Z}$ satisfies (f1), (f2), (f3), and (f5). It fails (f4) since, while each element has an additive inverse, only 1 and -1 have multiplicative inverses (they are their own inverses).

$\mathbf{Q}$ satisfies each property (f1) - (f5).

Exercise 8.6.4. Show that this defines an ordering on $\mathbf{R}$ by verifying properties (o1), (o2), and (o3) from Definition 8.6.5.

Solution. Properties (o2) and (o3) are clear, so let us verify property (o1). It will suffice to show that if $B \not\subseteq A$, then $A \subseteq B$. Since B is not a subset of A, there exists some $s \in B$ such that $s \notin A$. Let $r \in A$ be given. By Exercise 8.6.2 we must have $r < s$ and so by (c2) we have $r \in B$. Thus $A \subseteq B$.

Exercise 8.6.5. (a) Show that (c1) and (c3) also hold for $A + B$. Conclude that $A + B$ is a cut.

(b) Check that addition in $\mathbf{R}$ is commutative (f1) and associative (f2).

(c) Show that property (o4) holds.

(d) Show that the cut

$$O = \{p \in \mathbf{Q} : p < 0\}$$

successfully plays the role of the additive identity (f3). (Showing $A + O = A$ amounts to proving that these two sets are the same. The standard way to prove such a thing is to show two inclusions: $A + O \subseteq A$ and $A \subseteq A + O$.)

Solution. (a) Since A and B are non-empty, $A + B$ must also be non-empty. Since neither A nor B contains every rational number, there exist rationals $r \notin A$ and $s \notin B$. It follows from Exercise 8.6.2 that $a + b < r + s$ for every $a \in A$ and $b \in B$, so that $r + s \notin A + B$. Thus $A + B \neq \mathbf{Q}$ and we have now shown that $A + B$ satisfies (c1).

Let $a + b \in A + B$ be given. By (c3), there exist rationals $r \in A$ and $s \in B$ such that $a < r$ and $b < s$. It follows that $a + b < r + s$ and $r + s \in A + B$. Thus $A + B$ satisfies (c3).

(b) Commutativity and associativity of addition in $\mathbf{R}$ follow immediately from commutativity and associativity of addition in $\mathbf{Q}$.

(c) Let A, B, and C be cuts such that $B \subseteq C$. If $a + b \in A + B$, then $a + b \in A + C$ also since $B \subseteq C$. Thus $A + B \subseteq A + C$.

(d) Let $a + p \in A + O$ be given. Then $p < 0$, so $a + p < a$ and it follows from (c2) that $a + p \in A$; thus $A + O \subseteq A$.

Now let $a \in A$ be given. By (c3) there exists some $b \in A$ such that $a < b$. Notice that $a = b + (a - b) \in A + O$, since $a - b < 0$. It follows that $A \subseteq A + O$ and we may conclude that $A + O = A$.

Exercise 8.6.6. (a) Prove that $-A$ defines a cut.

(b) What goes wrong if we set $-A = \{r \in \mathbf{Q} : -r \notin A\}$?

(c) If $a \in A$ and $r \in -A$, show $a + r \in O$. This shows $A + (-A) \subseteq O$. Now, finish the proof of property (f4) for addition in Definition 8.6.4.

Solution. (a) Since $A \neq \mathbf{Q}$, there exists a $t \notin A$. Then $-t-1 \in -A$, since $t < -(-t-1) = t+1$. Thus $-A$ is non-empty. Since A is non-empty, there exists some $r \in A$. Then $-r \notin -A$, since if $t \notin A$ then $t > -(-r) = r$ by Exercise 8.6.2. Thus $-A \neq \mathbf{Q}$ and we see that $-A$ satisfies (c1).

Suppose that $r \in -A$, so that there is some $t \notin A$ such that $t < -r$, and suppose that $s < r$. Then $t < -r < -s$, demonstrating that $s \in -A$ also. Thus $-A$ satisfies (c2).

Suppose that $r \in -A$, so that there is some $t \notin A$ such that $t < -r$. Define $s = r - \frac{r+t}{2}$ and notice that $r < s$ since $0 < -r - t$. Furthermore, $s \in -A$ since

$$t \notin A \quad \text{and} \quad t < \frac{t-r}{2} = -s.$$

Thus $-A$ satisfies (c3) and we may conclude that $-A$ is a cut.

(b) This does not necessarily define a cut. For example, let C_2 be the cut $\{r \in \mathbf{Q} : r < 2\}$. Then using this definition, we find that $-C_2 = \{r \in \mathbf{Q} : r \leq -2\}$, which fails property (c3).

(c) There exists a $t \notin A$ such that $t < -r$. By Exercise 8.6.2 it must be the case that $a < t < -r$ and thus $a + r < 0$, i.e. $a + r \in O$. Thus $A + (-A) \subseteq O$.

For the reverse inclusion, let $p < 0$ be a given rational number in O. We claim that there must exist some $r \in A$ such that $r - \frac{p}{2} \notin A$, and we will prove this by contradiction. Suppose that $r - \frac{p}{2} \in A$ for all $r \in A$. Since A is a cut, there is some $r_0 \in A$. An induction argument shows that $r_0 - \frac{np}{2} \in A$ for all $n \in \mathbf{N}$. Let $q \in \mathbf{Q}$ be given and use the Archimedean property of $\mathbf{Q}$ to obtain an $n \in \mathbf{N}$ such that $r_0 - \frac{np}{2} > q$; it follows from (c2) that $q \in A$. The conclusion is that $A = \mathbf{Q}$, contradicting (c1).

Thus there is some $r \in A$ such that $r - \frac{p}{2} \notin A$. Since $r - \frac{p}{2} < r - p$, it follows that $p - r \in -A$. Then $p = r + p - r \in A + (-A)$, demonstrating that $O \subseteq A + (-A)$. We may conclude that $A + (-A) = O$.

Exercise 8.6.7. (a) Show that AB is a cut and that property (o5) holds.

(b) Propose a good candidate for the multiplicative identity (1) on $\mathbf{R}$ and show that this works for all cuts $A \geq O$.

(c) Show the distributive property (f5) holds for non-negative cuts.

Solution. (a) It is clear that AB is non-empty. If either $A = O$ or $B = O$, then it is straightforward to verify that $AB = O \neq \mathbf{Q}$. Suppose that $A > O$ and $B > O$. There exist rationals $r \notin A$ and $s \notin B$; clearly, $r, s > 0$. If $q \in AB$, then either $q < 0$ or $q = ab$ for $a \in A, b \in B$ and $a, b \geq 0$. By Exercise 8.6.2 we must have $a < r$ and $b < s$, so that $ab < rs$. In either case, we have $q < rs$ and thus $rs \notin AB$, demonstrating that $AB \neq \mathbf{Q}$. Thus AB satisfies (c1).

Suppose $r \in AB$ and $q < r$. If $q < 0$ then $q \in AB$, so suppose that $q \geq 0$, which implies that $r > 0$. We must then have $r = ab$ for some $a \in A, b \in B$ with $a, b > 0$. Notice that $\frac{q}{b} < a$; (c2) then implies that $\frac{q}{b} \in A$ and hence $q = \frac{q}{b} \cdot b \in AB$. Thus AB satisfies (c2).

If $A = O$ or $B = O$ then $AB = O$, which has no maximum element. Suppose that $A > O$ and $B > O$ and let $r \in AB$ be given. If $r \leq 0$ then let q be any positive rational in AB. If $r > 0$, then $r = ab$ for some $a \in A, b \in B$ with $a, b > 0$. By (c3), there exist rationals $s \in A, t \in B$ such that $a < s$ and $b < t$. Let $q = st$ and notice that $q \in AB$ and $r = ab < st = q$. In either case, there exists a $q \in AB$ with $r < q$. Thus AB satisfies (c3) and we may conclude that AB is a cut.

Property (o5) is clear from the definition of AB.

(b) Define $I = \{p \in \mathbf{Q} : p < 1\}$ and let $A \geq O$ be given. We claim that $AI = A$. Suppose that $r \in AI$. If $r < 0$, then $r \in A$, so suppose that $r \geq 0$. Thus $r = ab$ for some $a \in A$ such that $a \geq 0$ and some $0 \leq b < 1$. It follows that $ab < a$ and so by (c2) we have $r = ab \in A$. Thus $AI \subseteq A$.

Now suppose that $a \in A$. If $a \leq 0$, then (c2) implies that $2a \in A$ and thus $a = (2a) \cdot \frac{1}{2} \in AI$. If $a > 0$, then (c3) implies there is some $r \in A$ with $a < r$. Thus $\frac{a}{r} \in I$ and we see that $a = r \cdot \frac{a}{r} \in AI$. Hence $A \subseteq AI$ and we may conclude that $AI = A$.

(c) Let $A, B, C \geq O$ be cuts. If $ABC = O$ then the equality $A(B + C) = AB + AC$ is clear, so suppose that $A, B, C > O$ and suppose that $q \in A(B + C)$. If $q < 0$ then $q = \frac{q}{2} + \frac{q}{2} \in AB + AC$. Suppose that $q \geq 0$. Then $q = a(b + c) = ab + ac$, where $a \in A, b \in B, c \in C$ and $a, b + c \geq 0$. There are three cases: $b, c \geq 0$, $b \geq 0$ and $c < 0$, or $b < 0$ and $c \geq 0$. In any of these cases, it is straightforward to verify that $ab + ac \in AB + AC$. Thus $A(B + C) \subseteq AB + AC$.

Now suppose that $p + q \in AB + AC$. If $p + q < 0$, then $p + q \in A(B + C)$, so suppose that $p + q \geq 0$. If $p, q \geq 0$, then $p = a_1 b$ and $q = a_2 c$, for some $a_1, a_2 \in A, b \in B$, and $c \in C$ such that $a_1, a_2, b, c \geq 0$. Let $a = \max\{a_1, a_2\}$ and notice that $a(b + c) \in A(B + C)$. Furthermore, $p + q = a_1 b + a_2 c \leq ab + ac = a(b + c)$. It follows from (c2) that $p + q \in A(B + C)$.

Next, suppose that $p < 0$ and $q \geq 0$, so that $q = ac$ for some $a \in A, c \in C$ with $a, c \geq 0$.

Let $b \in B$ be such that $b \geq 0$; such a b exists since $B > O$. Now notice that

$$p + q = p + ac < ac \leq a(b + c) \in A(B + C).$$

It follows from (c2) that $p + q \in A(B + C)$. The case where $p \geq 0$ and $q < 0$ is handled similarly. Thus $AB + AC \subseteq A(B + C)$ and we may conclude that $A(B + C) = AB + AC$.

Exercise 8.6.8. Let $\mathcal{A} \subseteq \mathbf{R}$ be nonempty and bounded above, and let S be the *union* of all $A \in \mathcal{A}$.

(a) First, prove that $S \in \mathbf{R}$ by showing that it is a cut.

(b) Now, show that S is the least upper bound for $\mathcal{A}$.

Solution. (a) Since $\mathcal{A}$ is non-empty, it contains some cut A, so that $A \subseteq S$. It follows that S is non-empty as A is non-empty. Since $\mathcal{A}$ is bounded above, there exists some cut B such that $A \subseteq B$ for all $A \in \mathcal{A}$. It follows that $S \subseteq B$ and hence that $S \neq \mathbf{Q}$ since $B \neq \mathbf{Q}$. Thus S satisfies (c1).

Suppose $r \in S$, so that $r \in A$ for some $A \in \mathcal{A}$, and suppose $q < r$. Since A is a cut we must have $q \in A$, which gives $q \in S$. Thus S satisfies (c2).

Suppose $r \in S$, so that $r \in A$ for some $A \in \mathcal{A}$. Since A is a cut there exists some $q \in A$ such that $r < q$; note that $q \in S$ also. Thus S satisfies (c3). We may conclude that S is a cut.

(b) It is clear that S is an upper bound for $\mathcal{A}$. If B is any upper bound for $\mathcal{A}$, then B contains every $A \in \mathcal{A}$ and hence must contain the union of all $A \in \mathcal{A}$, i.e. $S \subseteq B$. It follows that S is the least upper bound for $\mathcal{A}$.

Exercise 8.6.9. Consider the collection of so-called "rational" cuts of the form

$$C_r = \{t \in \mathbf{Q} : t < r\}$$

where $r \in \mathbf{Q}$. (See Exercise 8.6.1.)

(a) Show that $C_r + C_s = C_{r+s}$ for all $r, s \in \mathbf{Q}$. Verify $C_r C_s = C_{rs}$ for the case when $r, s \geq 0$.

(b) Show that $C_r \leq C_s$ if and only if $r \leq s$ in $\mathbf{Q}$.

Solution. (a) Let $r, s \in \mathbf{Q}$ be given and suppose $a + b \in C_r + C_s$, i.e. $a < r$ and $b < s$. It follows that $a + b < r + s$ and hence that $a + b \in C_{r+s}$. Thus $C_r + C_s \subseteq C_{r+s}$. Now suppose that $t \in C_{r+s}$, so that $t < r + s$. Choose a positive integer $n \in \mathbf{N}$ such that $t + \frac{1}{n} < r + s$ and note that:

- $s - \frac{1}{n} < s$, so that $s - \frac{1}{n} \in C_s$;
- $t + \frac{1}{n} - s < r$, so that $t + \frac{1}{n} - s \in C_r$;
- $t = \left(t + \frac{1}{n} - s\right) + \left(s - \frac{1}{n}\right) \in C_r + C_s$.

Thus $C_{r+s} \subseteq C_r + C_s$ and we may conclude that $C_r + C_s = C_{r+s}$.

It is clear that $C_r C_s = C_{rs}$ if $rs = 0$, so suppose that $r, s > 0$ and let $q \in C_r C_s$ be given. If $q \leq 0$ then $q < rs$, i.e. $q \in C_{rs}$. If $q > 0$ then $q = ab$ for some $0 < a < r$ and $0 < b < s$. It follows that $0 < ab < rs$ and thus $q = ab \in C_{rs}$. Hence $C_r C_s \subseteq C_{rs}$.

Now let $q \in C_{rs}$ be given. If $q \leq 0$ then certainly $q \in C_r C_s$, so suppose that $q > 0$ and define $p = \frac{1}{2}\left(\frac{q}{s} + r\right)$. Notice that:

- $0 < \frac{q}{s} < p < r$, so that $p \in C_r$;
- $0 < \frac{q}{p} < s$, so that $\frac{q}{p} \in C_s$;
- $q = p \cdot \frac{q}{p} \in C_r C_s$.

Thus $C_{rs} \subseteq C_r C_s$ and we may conclude that $C_r C_s = C_{rs}$.

(b) If $r \leq s$ then it is clear that $C_r \subseteq C_s$. If $s < r$, then it is again clear that $C_s \subseteq C_r$. Furthermore, notice that $C_s \neq C_r$ since $\frac{s+r}{2}$ belongs to C_r but not to C_s. Thus $C_s \subsetneq C_r$.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.