1 Banach fixed-point theorem

The following theorem is known as the *Banach fixed-point theorem*, *contractive mapping theorem*, or some variant thereof.

Theorem 1. Let (X, d) be a metric space and let $f : X \to X$ be a contraction on X, i.e. f is Lipschitz with Lipschitz constant $0 \leq L < 1$. Then if f has a fixed point, this fixed point is unique. Furthermore, if X is non-empty and complete then f has a fixed point x and this fixed point is given by $x = \lim_{n\to\infty} x_n$, where $x_n = f(x_{n-1})$ for $n \geq 1$ and x_0 is any point in X.

Proof. Suppose that x and y are fixed points of f. Since f is a contraction, we must have

$$d(x, y) = d(f(x), f(y)) \leq L\, d(x, y),$$

where $0 \leq L < 1$. This can only be satisfied if $d(x, y) = 0$, i.e. if $x = y$. So any fixed point of f must be unique.

Now suppose that X is non-empty and complete. Let $x_0 \in X$ be arbitrary and set $x_n = f(x_{n-1})$ for $n \geq 1$. For any $n \geq 0$ we have the inequality

$$\boxed{d(x_{n+1}, x_n) \leq L^n\, d(x_1, x_0)} \tag{1}$$

which can be seen by induction on n:

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \\ &\leq L\, d(x_n, x_{n-1}) \\ &\cdots \\ &\leq L^n\, d(x_1, x_0). \end{aligned}$$

Then for any $n > m \geq 0$ we apply inequality (1) as follows:

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \\ &\leq (L^{n-1} + \cdots + L^m)\, d(x_1, x_0) \\ &= L^m\, (L^{n-m-1} + \cdots + 1)\, d(x_1, x_0) \\ &\leq L^m \left(\sum_{i=0}^{\infty} L^i \right) d(x_1, x_0) \\ &= L^m\, \frac{d(x_1, x_0)}{1 - L}, \end{aligned}$$

where we have used that $0 \leq L < 1$. So for any $n > m \geq 0$ we have the inequality

$$\boxed{d(x_n, x_m) \leq L^m \frac{d(x_1, x_0)}{1 - L}.} \tag{2}$$

Now let $\varepsilon > 0$ be given. Since $0 \leq L < 1$, there exists a positive integer M such that

$$m \geq M \implies L^m \frac{d(x_1, x_0)}{1 - L} < \varepsilon.$$

Then provided we take $n > m \geq M$, inequality (2) gives us $d(x_n, x_m) < \varepsilon$, demonstrating that $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence. By the completeness of X, there then exists some $x \in X$ such that $\lim_{n\to\infty} x_n = x$. This x is the fixed point of f:

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x. \qquad \square$$

2 A corollary

Theorem 2. Let (X, d) be a non-empty and complete metric space and let $f : X \to X$ be such that f^N is a contraction for some $N \geq 1$. Then f has a unique fixed point.

Proof. By Theorem 1, there exists a unique $x \in X$ such that $f^N(x) = x$. Observe that

$$d(f(x), x) = d(f^{N+1}(x), f^N(x)) \leq L\, d(f(x), x),$$

where $0 \leq L < 1$ is the Lipschitz constant of f^N. This inequality can only be satisfied if $d(f(x), x) = 0$, i.e. if $f(x) = x$. So x is also a fixed point of f.

For the uniqueness of x as a fixed point of f, suppose that y is a fixed point of f. Then y must also be a fixed point of f^N:

$$f^N(y) = f^{N-1}(f(y)) = f^{N-1}(y) = \cdots = f(y) = y.$$

The uniqueness of x as a fixed point of f^N then implies $x = y$. $\square$