1 Section 7.A Exercises

Exercises with solutions from Section 7.A of [LADR].

Exercise 7.A.1. Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by

$$T(z_1, \dots, z_n) = (0, z_1, \dots, z_{n-1}).$$

Find a formula for $T^*(z_1, \dots, z_n)$.

Solution. Observe that

$$\begin{aligned} \langle (w_1, \dots, w_n), T^*(z_1, \dots, z_n) \rangle &= \langle T(w_1, \dots, w_n), (z_1, \dots, z_n) \rangle \\ &= \langle (0, w_1, \dots, w_{n-1}), (z_1, \dots, z_n) \rangle \\ &= w_1 z_2 + \cdots + w_{n-1} z_n \\ &= \langle (w_1, \dots, w_n), (z_2, \dots, z_n, 0) \rangle. \end{aligned}$$

Thus $T^*(z_1, \dots, z_n) = (z_2, \dots, z_n, 0)$.

Exercise 7.A.2. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. Prove that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

Solution. Observe that

$$\begin{aligned} \lambda \text{ is an eigenvalue of } T &\iff T - \lambda I \text{ is not surjective} && (5.6\text{ (c)}) \\ &\iff \operatorname{range}(T - \lambda I) \neq V \\ &\iff (\operatorname{range}(T - \lambda I))^\perp \neq \{0\} && (\text{Exercise 6.C.2}) \\ &\iff \operatorname{null}(T - \lambda I)^* \neq \{0\} && (7.7\text{ (a)}) \\ &\iff \operatorname{null}\left(T^* - \overline{\lambda} I\right) \neq \{0\} && (7.6\text{ (a), (b), (d)}) \\ &\iff T^* - \overline{\lambda} I \text{ is not injective} \\ &\iff \overline{\lambda} \text{ is an eigenvalue of } T^*. && (5.6\text{ (b)}) \end{aligned}$$

Exercise 7.A.3. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

Solution. Suppose that U is invariant under T and let $v \in U^\perp$ be given. Observe that

$$\langle u, T^*v \rangle = \langle Tu, v \rangle = 0$$

for any $u \in U$, where the last equality follows since $Tu \in U$ and $v \in U^\perp$. Thus $T^*v \in U^\perp$ and we see that $U^\perp$ is invariant under T^*.

Now suppose that $U^\perp$ is invariant under T^*. The previous paragraph shows that $(U^\perp)^\perp$ is invariant under $(T^*)^*$, which by 6.51 and 7.6 (c) is exactly the statement that U is invariant under T.

Exercise 7.A.4. Suppose $T \in \mathcal{L}(V, W)$. Prove that

(a) T is injective if and only if T^* is surjective;

(b) T is surjective if and only if T^* is injective.

Solution. (a) Observe that

$$\begin{aligned} T \text{ is injective} &\iff \operatorname{null} T = \{0\} \\ &\iff (\operatorname{range} T^*)^\perp = \{0\} && \text{(7.7 (c))} \\ &\iff \operatorname{range} T^* = V && \text{(Exercise 6.C.2)} \\ &\iff T^* \text{ is surjective.} \end{aligned}$$

(b) Part (a) shows that T^* is injective if and only if $(T^*)^*$ is surjective, which by 7.6 (c) is equivalent to T^* being injective if and only if T is surjective.

Exercise 7.A.5. Prove that

$$\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$$

and

$$\dim \operatorname{range} T^* = \dim \operatorname{range} T$$

for every $T \in \mathcal{L}(V, W)$.

Solution. We have

$$\begin{aligned} \dim \operatorname{null} T^* &= \dim(\operatorname{range} T)^\perp && \text{(7.7 (a))} \\ &= \dim W - \dim \operatorname{range} T && \text{(6.50)} \\ &= \dim \operatorname{null} T + \dim W - \dim V. && \text{(3.22)} \end{aligned}$$

Similarly,

$$\begin{aligned}
\dim \operatorname{range} T^* &= \dim(\operatorname{null} T)^\perp && \text{(7.7 (b))} \\
&= \dim V - \dim \operatorname{null} T && \text{(6.50)} \\
&= \dim \operatorname{range} T. && \text{(3.22)}
\end{aligned}$$

Exercise 7.A.6. Make $\mathcal{P}_2(\mathbf{R})$ into an inner product space by defining

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,\mathrm{d}x.$$

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbf{R}))$ by $T(a_0 + a_1 x + a_2 x^2) = a_1 x$.

(a) Show that T is not self-adjoint.

(b) The matrix of T with respect to the basis $(1, x, x^2)$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

Solution. (a) Let $p, q \in \mathcal{P}_2(\mathbf{R})$ be given by $p(x) = 2x$ and $q(x) = 1$, so that $Tp = p$ and $Tq = 0$. Then

$$\langle Tp, q \rangle = \int_0^1 2x\,\mathrm{d}x = 1 \neq 0 = \langle p, Tq \rangle.$$

Thus T is not self-adjoint.

(b) The result in 7.10 requires that the basis of $\mathcal{P}_2(\mathbf{R})$ is orthonormal, but $(1, x, x^2)$ is not an orthonormal basis:

$$\langle 1, x \rangle = \int_0^1 x\,\mathrm{d}x = \tfrac{1}{2} \neq 0.$$

Exercise 7.A.7. Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

Solution. Observe that

$$(ST)^* = T^* S^* = TS,$$

where the first equality is 7.6 (e). It follows that $(ST)^* = ST$ if and only if $TS = ST$.

Exercise 7.A.8. Suppose V is a real inner product space. Show that the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.

Solution. Clearly the zero operator is self-adjoint. Closure under additivity and scalar multiplication follows from 7.6 (a) and (b).

Exercise 7.A.9. Suppose V is a complex inner product space with $V \neq \{0\}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Solution. 7.6 (d) shows that the identity operator I is self-adjoint. Since $V \neq \{0\}$, there is some non-zero $v \in V$. Observe that

$$(iI)(v) = iv \neq -iv = \left(\bar{i}I\right)(v) = (iI)^*(v),$$

where we have used 7.6 (b) and (d). It follows that iI is not self-adjoint and hence that the set of self-adjoint operators on V is not closed under scalar multiplication.

Exercise 7.A.10. Suppose $\dim V \geq 2$. Show that the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.

Solution. Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of V and let $S, T \in \mathcal{L}(V)$ be the operators whose matrices with respect to this basis are

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

respectively. Note that S is self-adjoint and hence normal. Note further that T satisfies $T^* = -T$, so that $TT^* = T^*T = -T^2$; it follows that T is also normal. However, some calculations reveal that

$$(A+B)(A+B)^* = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad (A+B)^*(A+B) = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus $S + T$ is not normal.

Exercise 7.A.11. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that there is a subspace U of V such that $P = P_U$ if and only if P is self-adjoint.

Solution. Suppose that $P = P_U$ for some subspace U of V. Let $v = u + x$ and $w = u' + x'$ be given, where $u, u' \in U$ and $x, x' \in U^\perp$. Then

$$\langle P_U v, w\rangle = \langle u, u' + x'\rangle = \langle u, u'\rangle \quad \text{and} \quad \langle v, P_U w\rangle = \langle u + x, u'\rangle = \langle u, u'\rangle.$$

It follows that P_U is self-adjoint.

Now suppose that P is self-adjoint and let $U = \operatorname{range} P$. We claim that $P = P_U$. Let $v = Px + w$ be given, where $Px \in \operatorname{range} P$ and $w \in (\operatorname{range} P)^\perp$. Note that

$$(\operatorname{range} P)^\perp = \operatorname{null} P^* = \operatorname{null} P,$$

where the first equality follows from 7.7 (a) and the second equality follows since P is self-adjoint. Hence $w \in \operatorname{null} P$ and we see that

$$Pv = P^2 x = Px = P_U v.$$

Exercise 7.A.12. Suppose that T is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

Solution. There are eigenvectors $u, w \in V$ satisfying

$$Tu = 3u, \quad Tw = 4w, \quad \text{and} \quad \langle u, w\rangle = 0;$$

this last equality follows from 7.22. Define

$$v := \frac{u}{\|u\|} + \frac{w}{\|w\|}$$

and note that

$$\|v\|^2 = \left\| \frac{u}{\|u\|} \right\|^2 + \left\| \frac{w}{\|w\|} \right\|^2 = 2 \quad \Longrightarrow \quad \|v\| = \sqrt{2},$$

where we have used the Pythagorean theorem (6.13). Furthermore,

$$\|Tv\|^2 = \left\| T\frac{u}{\|u\|} + T\frac{w}{\|w\|} \right\|^2 = \left\| \frac{3u}{\|u\|} + \frac{4w}{\|w\|} \right\|^2 = \left\| \frac{3u}{\|u\|} \right\| + \left\| \frac{4w}{\|w\|} \right\|^2 = 3^2 + 4^2,$$

where we have again used the Pythagorean theorem (6.13). It follows that $\|Tv\| = 5$.

Exercise 7.A.13. Give an example of an operator $T \in \mathcal{L}(\mathbf{C}^4)$ such that T is normal but not self-adjoint.

Solution. Let $T \in \mathcal{L}(\mathbf{C}^4)$ be the operator whose matrix with respect to the standard orthonormal basis is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As we showed in Exercise 7.A.10, this operator satisfies $T^* = -T$, which implies that $TT^* = T^*T = -T^2$. It follows that T is normal but not self-adjoint.

Exercise 7.A.14. Suppose T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations

$$\|v\| = \|w\| = 2, \quad Tv = 3v, \quad Tw = 4w.$$

Show that $\|T(v + w)\| = 10$.

Solution. Since v and w are eigenvectors of T corresponding to distinct eigenvalues, they must be orthogonal (7.22). The Pythagorean theorem (6.13) then implies that

$$\|T(v + w)\|^2 = \|3v + 4w\|^2 = \|3v\|^2 + \|4w\|^2 = 9\|v\|^2 + 16\|w\|^2 = 36 + 64 = 100;$$

it follows that $\|T(v + w)\| = 10$.

Exercise 7.A.15. Fix $u, x \in V$. Define $T \in \mathcal{L}(V)$ by

$$Tv = \langle v, u\rangle x$$

for every $v \in V$.

(a) Suppose $\mathbf{F} = \mathbf{R}$. Prove that T is self-adjoint if and only if u, x is linearly dependent.

(b) Prove that T is normal if and only if u, x is linearly dependent.

Solution. Note that Example 7.4 gives us the formula

$$T^*v = \langle v, x\rangle u.$$

(a) Suppose that u, x is linearly dependent, say $x = \lambda u$ for some $\lambda \in \mathbf{R}$. Then for any $v \in V$

$$Tv = \langle v, u\rangle x = \lambda\langle v, u\rangle u = \langle v, \lambda u\rangle u = \langle v, x\rangle u = T^*v.$$

Now suppose that T is self-adjoint. If $u = 0$ we are done, so suppose that $u \neq 0$. Since T is self-adjoint, we must have

$$Tv = \langle v, u\rangle x = \langle v, x\rangle u = T^*v$$

for every $v \in V$. In particular,

$$\langle u, u\rangle x = \langle u, x\rangle u \quad \Longrightarrow \quad x = \frac{\langle u, x\rangle}{\langle u, u\rangle} u,$$

demonstrating that u, x is linearly dependent.

(b) Note that

$$(TT^* - T^*T)(v) = \langle v, x\rangle\langle u, u\rangle x - \langle v, u\rangle\langle x, x\rangle u$$

for any $v \in V$. Suppose that u, x is linearly dependent, say $x = \lambda u$ for some $\lambda \in \mathbf{F}$. Then

$$\begin{aligned}(TT^* - T^*T)(v) &= \langle v, \lambda u\rangle\langle u, u\rangle\lambda u - \langle v, u\rangle\langle \lambda u, \lambda u\rangle u \\ &= |\lambda|^2\langle v, u\rangle\langle u, u\rangle u - |\lambda|^2\langle v, u\rangle\langle u, u\rangle u = 0.\end{aligned}$$

Thus T is normal. Conversely, suppose that T is normal. If $u = 0$ we are done, so suppose that $u \neq 0$. Then

$$(TT^* - T^*T)(x) = \langle x, x\rangle\langle u, u\rangle x - \langle x, u\rangle\langle x, x\rangle u = 0 \quad \Longrightarrow \quad x = \frac{\langle x, u\rangle}{\langle u, u\rangle} u,$$

demonstrating that u, x is linearly dependent.

Exercise 7.A.16. Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\operatorname{range} T = \operatorname{range} T^*.$$

Solution. Observe that 7.20 implies $\operatorname{null} T = \operatorname{null} T^*$. It then follows from 7.7 that

$$\operatorname{range} T^* = (\operatorname{null} T)^\perp = (\operatorname{null} T^*)^\perp = \operatorname{range} T.$$

Exercise 7.A.17. Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\operatorname{null} T^k = \operatorname{null} T \quad \text{and} \quad \operatorname{range} T^k = \operatorname{range} T$$

for every positive integer k.

Solution. Let us prove by induction that $\operatorname{null} T^k = \operatorname{null} T$ for every positive integer k. The base case $k = 1$ is clear, so suppose that the result is true for some positive integer k. The containment $\operatorname{null} T^k \subseteq \operatorname{null} T^{k+1}$ is evident. Suppose that $v \in \operatorname{null} T^{k+1}$. By 7.20 we then have $T^*T^k v = 0$ and it follows that

$$\langle T^*T^k v, T^{k-1}v\rangle = 0 \quad \Longleftrightarrow \quad \langle T^k v, T^k v\rangle = 0 \quad \Longleftrightarrow \quad T^k v = 0.$$

Thus $\operatorname{null} T^{k+1} = \operatorname{null} T^k = \operatorname{null} T$; the last equality is our induction hypothesis. This completes the induction step and the proof.

The second half of the exercise is a quick corollary of the first half. Evidently, T normal implies T^* normal. Furthermore, 7.6 (e) implies that $(T^k)^* = (T^*)^k$ for all positive integers k. It follows that

$$\begin{aligned}
\operatorname{range} T^k &= \left(\operatorname{null}\left(T^k\right)^*\right)^{\perp} && \text{(7.7 (d))}\\
&= \left(\operatorname{null}\,(T^*)^k\right)^{\perp} \\
&= (\operatorname{null} T^*)^{\perp} && (T^* \text{ is normal})\\
&= \operatorname{range} T. && \text{(7.7 (d))}
\end{aligned}$$

Exercise 7.A.18. Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $\|Te_j\| = \|T^*e_j\|$ for each j, then T is normal.

Solution. This is false. Let T be the operator on $\mathbf{F}^2$ whose matrix with respect to the standard orthonormal basis e_1, e_2 is

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix};$$

as we showed in Exercise 7.A.10, T is not normal. However,

$$\|Te_1\| = \|T^*e_1\| = \sqrt{2} \quad \text{and} \quad \|Te_2\| = \|T^*e_2\| = 1.$$

Exercise 7.A.19. Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is normal and $T(1,1,1) = (2,2,2)$. Suppose $(z_1, z_2, z_3) \in \operatorname{null} T$. Prove that $z_1 + z_2 + z_3 = 0$.

Solution. If $u := (z_1, z_2, z_3) = (0,0,0)$ then we are done, so suppose that $u \neq 0$. It follows that u is an eigenvector of T corresponding to the eigenvalue 0. Note that $v := (1,1,1)$ is an eigenvector of T corresponding to the eigenvalue 2. Since these are eigenvectors of a normal operator corresponding to distinct eigenvalues, they must be orthogonal (7.22). That is,

$$\langle u, v \rangle = z_1 + z_2 + z_3 = 0.$$

Exercise 7.A.20. Suppose $T \in \mathcal{L}(V, W)$ and $\mathbf{F} = \mathbf{R}$. Let Φ_V be the isomorphism from V onto the dual space V' given by Exercise 17 in Section 6.B, and let Φ_W be the corresponding isomorphism from W onto W'. Show that if Φ_V and Φ_W are used to identify V and W with V' and W', then T^* is identified with the dual map T'. More precisely, show that $\Phi_V \circ T^* = T' \circ \Phi_W$.

Solution. Let $w \in W$ be given. Then
$$(\Phi_V \circ T^*)(w) = \Phi_V T^* w$$
is the map $V \to \mathbf{R}$ given by
$$(\Phi_V T^* w)(v) = \langle v, T^* w\rangle.$$
On the other hand,
$$(T' \circ \Phi_W)(w) = T'\Phi_W w = \Phi_W w \circ T$$
is the map $V \to \mathbf{R}$ given by
$$(\Phi_W w \circ T)(v) = (\Phi_W w)(Tv) = \langle Tv, w\rangle.$$
Since $\langle v, T^* w\rangle = \langle Tv, w\rangle$ for all $v \in V$, we see that $(\Phi_V \circ T^*)(w) = (T' \circ \Phi_W)(w)$ for all $w \in W$.

Exercise 7.A.21. Fix a positive integer n. In the inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product
$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x,$$
let
$$V = \operatorname{span}(1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx).$$

(a) Define $D \in \mathcal{L}(V)$ by $Df = f'$. Show that $D^* = -D$. Conclude that D is normal but not self-adjoint.

(b) Define $T \in \mathcal{L}(V)$ by $Tf = f''$. Show that T is self-adjoint.

Solution. (a) Let
$$v = \frac{1}{\sqrt{2\pi}}, \quad e_j = \frac{\cos jx}{\sqrt{\pi}}, \quad \text{and} \quad f_j = \frac{\sin jx}{\sqrt{\pi}}$$
for each $1 \le j \le n$, and let $B := v, e_1, \ldots, e_n, f_1, \ldots, f_n$. Then B is an orthonormal basis of V (Exercise 6.B.4). Observe that $Dv = 0, De_j = -jf_j$, and $Df_j = je_j$ for each $1 \le j \le n$. It follows that the matrix of D with respect to B is
$$\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 2 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & n\\
0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & -2 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \cdots & -n & 0 & 0 & \cdots & 0
\end{pmatrix},$$

from which we see that $D^* = -D$. Such operators are normal but not self-adjoint, as we showed in Exercise 7.A.10.

(b) Observe that $Tv = 0, Te_j = -j^2 e_j$, and $Tf_j = -j^2 f_j$ for each $1 \leq j \leq n$. It follows that the matrix of T with respect to B is

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -4 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -n^2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -n^2 \end{pmatrix},$$

from which we see that T is self-adjoint (indeed, diagonal).

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edition.