1 Section 6.5 Exercises

Exercises with solutions from Section 6.5 of [UA].

Exercise 6.5.1. Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots .$$

(a) Is g defined on $(-1, 1)$? Is it continuous on this set? Is g defined on $(-1, 1]$? Is it continuous on this set? What happens on $[-1, 1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.

(b) For what values of x is $g'(x)$ defined? Find a formula for g'.

Solution. (a) Observe that

$$g(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent by the Alternating Series Test (Theorem 2.7.7). It follows from Theorem 6.5.1 that g converges absolutely on $(-1, 1)$ and hence g is defined on $(-1, 1]$. Theorem 6.5.7 then implies that g is continuous on $(-1, 1]$. Note that

$$g(-1) = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$$

is the (negated) divergent harmonic series. We claim that $g(x)$ cannot possibly converge for any other points $|x| > 1$. To see this, suppose that $g(x)$ does converge for some $x \in \mathbf{R}$ such that $|x| > 1$ and let $r \in \mathbf{R}$ be such that $|x| > r > 1$. It follows from Theorem 6.5.1 that $g(r)$ converges absolutely; but

$$r + \frac{r^2}{2} + \frac{r^3}{3} + \frac{r^4}{4} + \cdots$$

diverges by comparison with the harmonic series.

(b) Theorem 6.5.7 guarantees that g is differentable on $(-1, 1)$ and the derivative is given by

$$g'(x) = 1 - x + x^2 - x^3 + x^4 - \cdots .$$

Note that this series does not converge at $x = 1$, despite $g(1)$ converging.

Exercise 6.5.2. Find suitable coefficients (a_n) so that the resulting power series $\sum a_n x^n$ has the given properties, or explain why such a request is impossible.

(a) Converges for every value of $x \in \mathbf{R}$.

(b) Diverges for every value of $x \in \mathbf{R}$.

(c) Converges absolutely for all $x \in [-1, 1]$ and diverges off of this set.

(d) Converges conditionally at $x = -1$ and converges absolutely at $x = 1$.

(e) Converges conditionally at both $x = -1$ and $x = 1$.

Solution. (a) Take $a_n = 0$ for every $n \geq 0$.

(b) This is impossible; any power series converges to zero at $x = 0$.

(c) Take $a_0 = 0$ and $a_n = \frac{1}{n^2}$ for each $n \in \mathbf{N}$. Then for $|x| \leq 1$ we have

$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$$

and thus $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. If $|x| > 1$ then

$$\frac{x^n}{n^2} \not\to 0$$

and thus $\sum_{n=0}^{\infty} a_n x^n$ diverges.

(d) This is impossible. Note that

$$\sum_{n=0}^{\infty} |a_n(-1)^n| = \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} |a_n 1^n|,$$

so that a power series converges absolutely at $x = 1$ if and only if it converges absolutely at $x = -1$.

(e) Take

$$a_n = \begin{cases} 0 & \text{if } n = 0 \text{ or } n \text{ is odd,} \\ \frac{2(-1)^{1+n/2}}{n} & \text{if } n \text{ is even,} \end{cases}$$

so that

$$\sum_{n=0}^{\infty} a_n x^n = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots .$$

Then

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n(-1)^n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

are both conditionally convergent series.

Exercise 6.5.3. Use the Weierstrass M-Test to prove Theorem 6.5.2.

Solution. Note that for any $x \in \mathbf{R}$ such that $|x| \leq |x_0|$, we have

$$|a_n x^n| = |a_n||x|^n \leq |a_n||x_0|^n.$$

The series $\sum_{n=0}^{\infty}|a_n||x_0|^n$ is convergent by assumption so the Weierstrass M-Test implies that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$, where $c = |x_0|$.

Exercise 6.5.4 (Term-by-term Antidifferentiation). Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is defined on $(-R, R)$ and satisfies $F'(x) = f(x)$.

(b) Antiderivatives are not unique. If g is a arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.

Solution. (a) Let $x \in (-R, R)$ be given. Theorem 6.5.1 implies that the series $\sum_{n=0}^{\infty}|a_n||x|^n$ is convergent, which implies that the series $\sum_{n=0}^{\infty}|a_n||x|^{n+1}$ is convergent. Observe that

$$\left|\frac{a_n}{n+1} x^{n+1}\right| = \frac{|a_n|}{n+1}|x|^{n+1} \leq |a_n||x|^{n+1}$$

for each $n \geq 0$ and thus $F(x)$ is absolutely convergent by the Comparison Test (Theorem 2.7.4). It follows that F is defined on the open interval $(-R, R)$ and it is then immediate from Theorem 6.5.7 that $F'(x) = f(x)$ on this interval.

(b) Corollary 5.3.4 implies that $g(x) = k + F(x)$ on $(-R, R)$ for some constant $k \in \mathbf{R}$. Thus

$$g(x) = k + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = k + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots .$$

Exercise 6.5.5. (a) If s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $n \geq 1$.

(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use this start to construct a proof for Theorem 6.5.6.

Solution. (a) Clearly $0 < ns^{n-1}$ for each $n \geq 1$. Let $N \in \mathbf{N}$ be such that $s \leq 1 - \frac{1}{N+1}$. For $n \geq N$ we then have

$$s \leq 1 - \frac{1}{n+1} \quad \iff \quad (n+1)s \leq n \quad \iff \quad (n+1)s^n \leq ns^{n-1}.$$

Thus the sequence $(ns^{n-1})_{n=1}^{\infty}$ is bounded below and eventually decreasing. It follows from the Monotone Convergence Theorem that this sequence is convergent and hence bounded.

(b) From part (a), there is an $M > 0$ such that

$$n\left|\frac{x}{t}\right|^{n-1} \leq M$$

for all $n \in \mathbf{N}$. Since $t \in (-R, R)$, Theorem 6.5.1 implies that the series $\sum_{n=0}^{\infty} a_n t^n$ is absolutely convergent. It follows that the series

$$\sum_{n=1}^{\infty} M|a_n|t^{n-1}$$

is convergent. Now observe that

$$\left|na_n x^{n-1}\right| = n|a_n||x|^{n-1} = n\left|\frac{x}{t}\right|^{n-1}|a_n|t^{n-1} \leq M|a_n|t^{n-1}$$

for each $n \in \mathbf{N}$. By comparison with the convergent series $\sum_{n=1}^{\infty} M|a_n|t^{n-1}$ we see that the series $\sum_{n=1}^{\infty} na_n x^{n-1}$ is absolutely convergent. It follows that the power series $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges on the open interval $(-R, R)$.

Exercise 6.5.6. Previous work on geometric series (Example 2.7.5) justifies the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \quad \text{for all } |x| < 1.$$

Use the results about power series proved in this section to find values for $\sum_{n=1}^{\infty} n/2^n$ and $\sum_{n=1}^{\infty} n^2/2^n$. The discussion in Section 6.1 may be helpful.

Solution. The power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

has radius of convergence $R = 1$. Theorem 6.5.6 then implies that the formula

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

is valid on $(-1,1)$, from which we obtain

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n \tag{1}$$

for all $x \in (-1,1)$. Substituting $x = \frac{1}{2}$ gives us

$$2 = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Differentiating the power series (1) term-by-term gives us

$$\frac{1+x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^{n-1},$$

valid on $(-1,1)$, from which we obtain

$$\frac{x(1+x)}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n,$$

valid on $(-1,1)$. Substituting $x = \frac{1}{2}$ gives us

$$6 = \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Exercise 6.5.7. Let $\sum a_n x^n$ be a power series with $a_n \neq 0$, and assume

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists.

(a) Show that if $L \neq 0$, then the series converges for all $x \in (-1/L, 1/L)$. (The advice in Exercise 2.7.9 may be helpful.)

(b) Show that if $L = 0$, then the series converges for all $x \in \mathbf{R}$.

(c) Show that (a) and (b) continue to hold if L is replaced by the limit

$$L' = \lim_{n\to\infty} s_n \quad \text{where} \quad s_n = \sup\left\{ \left| \frac{a_{k+1}}{a_k} \right| : k \geq n \right\}.$$

(General properties of the *limit superior* are discusseed in Exercise 2.4.7.)

Solution. (a) Clearly the power series converges if $x = 0$, so suppose that $0 < |x| < \frac{1}{L}$. It follows that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = L|x| < 1$$

and hence the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent by the Ratio Test (Exercise 2.7.9).

(b) Clearly the power series converges if $x = 0$, so suppose that $x \neq 0$. It follows that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = 0$$

and hence the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent by the Ratio Test (Exercise 2.7.9).

(c) Let us refine the result in Exercise 2.7.9 as follows. Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, if

$$\lim_{n\to\infty} s_n = r < 1 \quad \text{where} \quad s_n = \sup\left\{\left|\frac{a_{k+1}}{a_k}\right| : k \geq n\right\},$$

then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. To see this, let r' be such that $r < r' < 1$. Since $\lim_{n\to\infty} s_n = r$, there is an $N \in \mathbf{N}$ such that

$$|s_N - r| = s_N - r < r' - r \quad \Longrightarrow \quad s_N < r';$$

for the first equality we have used that the sequence (s_n) decreases to r (see Exercise 2.4.7). It follows from this inequality that

$$n \geq N \quad \Longrightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| \leq s_N < r' \quad \Longrightarrow \quad |a_{n+1}| < |a_n| r'.$$

We may now argue as in Exercise 2.7.9 to conclude the proof of this refined ratio test. Using this refined test, the desired results about power series follow as in parts (a) and (b).

Exercise 6.5.8. (a) Show that power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $x \in (-R, R)$ and $f(0) = 1$. Deduce the values of a_n.

Solution. (a) Let us show that if a power series $h(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies $h(x) = 0$ for all $x \in (-R, R)$, then $a_n = 0$ for all $n \geq 0$. Theorem 6.5.7 implies that

$$h^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n x^{n-k}$$

for all $x \in (-R, R)$ and all $k \geq 0$, where $h^{(k)}$ is the k^{th} derivative of h. Since h is identically zero on $(-R, R)$ it must be the case that $h^{(k)}$ is identically zero on $(-R, R)$ and thus

$$0 = h^{(k)}(0) = k!a_k \quad \iff \quad a_k = 0$$

for each $k \geq 0$.

Now suppose that

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an interval $(-R, R)$. Then

$$\sum_{n=0}^{\infty} (a_n - b_n)x^n = 0$$

for all $x \in (-R, R)$ and our previous discussion shows that we must have $a_n - b_n = 0$ for each $n \geq 0$.

(b) Theorem 6.5.7 gives us

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = f'(x)$$

for all $x \in (-R, R)$. It follows from part (a) that

$$a_{n+1} = \frac{a_n}{n+1}$$

for all $n \geq 0$. From $f(0) = 1$ we obtain $a_0 = 1$ and hence $a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}$, and, in general,

$$a_n = \frac{1}{n!}.$$

Exercise 6.5.9. Review the definitions and results from Section 2.8 concerning products of series and Cauchy products in particular. At the end of Section 2.9, we mentioned the following result: If both $\sum a_n$ and $\sum b_n$ converge conditionally to A and B respectively, then it is possible for the Cauchy product,

$$\sum d_n \quad \text{where} \quad d_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0,$$

to diverge. However, if $\sum d_n$ does converge, then it must converge to AB. To prove this, set

$$f(x) = \sum a_n x^n, \quad g(x) = \sum b_n x^n, \quad \text{and} \quad h(x) = \sum d_n x^n.$$

Use Abel's Theorem and the result in Exercise 2.8.7 to establish this result.

Solution. Our hypothesis is that f, g, and h all converge at $x = 1$. It follows from Theorem 6.5.1 that f and g converge absolutely for any $x \in (-1, 1)$ and hence by Exercise 2.8.7 we have

$$h(x) = \sum_{n=0}^{\infty} d_n x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = f(x)g(x) \quad \text{for all } x \in (-1,1). \tag{1}$$

Abel's Theorem (Theorem 6.5.4) implies that f, g, and h converge uniformly and hence are continuous on $[0, 1]$. The continuity at $x = 1$ allows us to extend the equality in (1) to all $x \in (-1, 1]$, which gives us

$$h(1) = \sum_{n=0}^{\infty} d_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = f(1)g(1) = AB.$$

Exercise 6.5.10. Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge on $(-R, R)$, and assume $(x_n) \to 0$ with $x_n \neq 0$. If $g(x_n) = 0$ for all $n \in \mathbf{N}$, show that $g(x)$ must be identically zero on all of $(-R, R)$.

Solution. Theorem 6.5.7 implies that g is continuous at zero. It follows that

$$b_0 = g(0) = g\left(\lim_{k\to\infty} x_k\right) = \lim_{k\to\infty} g(x_k) = 0.$$

Theorem 6.5.7 also allows us to differentiate g term-by-term, obtaining the power series

$$g'(x) = \sum_{n=1}^{\infty} n b_n x^{n-1},$$

valid on $(-R, R)$. It follows that

$$b_1 = g'(0) = \lim_{k\to\infty} \frac{g(x_k) - g(0)}{x_k} = 0.$$

We can continue in this manner to see that $b_0 = b_1 = \cdots = 0$, which by Exercise 6.5.8 implies that g is identically zero on $(-R, R)$.

Exercise 6.5.11. A series $\sum_{n=0}^{\infty} a_n$ is said to be *Abel-summable to L* if the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in [0,1)$ and $L = \lim_{x\to 1^-} f(x)$.

(a) Show that any series that converges to a limit L is also Abel-summable to L.

(b) Show that $\sum_{n=0}^{\infty}(-1)^n$ is Abel-summable and find the sum.

Solution. (a) Suppose $\sum_{n=0}^{\infty} a_n$ converges to L. In other words, the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges to L at $x = 1$; Abel's Theorem then implies that the power series is uniformly convergent and hence continuous on $[0,1]$. It follows that

$$\lim_{x\to 1^-} f(x) = f\left(\lim_{x\to 1^-} x\right) = f(1) = \sum_{n=0}^{\infty} a_n = L.$$

(b) Let

$$f(x) = \sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{1+x};$$

this is valid for $|x| < 1$ (see Exercise 6.5.6). It follows that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{1+x} = \frac{1}{2}$$

and hence $\sum_{n=0}^{\infty}(-1)^n$ is Abel-summable to $\frac{1}{2}$.

[UA] Abbott, S. (2015) *Understanding Analysis.* 2nd edition.