Understanding Analysis Solutions

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Contents

Chapter 1. The Real Numbers

1.2. Some Preliminaries

Exercise 1.2.1.

(a) Prove that $\sqrt{3}$ is irrational. Does the same argument work to show that $\sqrt{6}$ is irrational?

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Solution.

(a) Suppose there was a rational number $p = \frac{m}{n}$, which we may assume is in lowest terms, such that $p^2 = 3$, i.e. such that $m^2 = 3n^2$. It follows that m^2 is divisible by 3; we claim that this implies that m is divisible by 3. Indeed, for any $k \in \mathbf{Z}$ we have

$$(3k+1)^2 = 3(3k^2 + 2k) + 1 \quad \text{and} \quad (3k+2)^2 = 3(3k^2 + 4k + 1) + 1.$$

Since m is of the form $3k+1$ or $3k+2$ for some integer k if m is not divisible by 3, it follows that

$$\text{if } m \text{ is not divisible by } 3, \text{then } m^2 \text{ is not divisible by } 3;$$

the contrapositive of this statement proves our claim.

Thus we may write $m = 3k$ for some $k \in \mathbf{Z}$ and substitute this into the equation $m^2 = 3n^2$ to obtain the equation $n^2 = 3k^2$, from which it follows that n is also divisible by 3, contradicting our assumption that m and n had no common factors. We may conclude that there is no rational number whose square is 3.

The same argument works to show that there is no rational number whose square is 6; the crux of this argument is the implication

$$\text{if } m^2 \text{ is divisible by } 6, \text{then } m \text{ is divisible by } 6.$$

This can be seen using what we have already proved. If m^2 is divisible by $6 = 2 \cdot 3$, then m^2 is divisible by 2 and 3. It follows that m is divisible by 2 and 3 and hence that m is divisible by 6.

(b) The argument breaks down when we try to assert that

$$\text{if } m^2 \text{ is divisible by } 4, \text{then } m \text{ is divisible by } 4.$$

This implication is false. For example, $2^2 = 4$ is divisible by 4 but 2 is not divisible by 4.

Exercise 1.2.2. Show that there is no rational number r satisfying $2^r = 3$.

Solution. Suppose there was a rational number $r = \frac{m}{n}$, which we may assume is in lowest terms with $n > 0$, such that $2^r = 3$. This implies that $2^m = 3^n$. Since $n > 0$ gives $3^n \geq 3$ and $2^m < 2$ for $m \leq 0$, it must be the case that $m > 0$. It follows that the left-hand side of the equation $2^m = 3^n$ is a positive even integer whereas the right-hand side is a positive odd integer, which is a contradiction. We may conclude that there is no rational number r such that $2^r = 3$.

Exercise 1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution.

(a) This is false, as Example 1.2.2 shows.

(b) This is true and we can use the following lemma to prove it.

Lemma L.1. If $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of positive integers, i.e. $a_{n+1} \le a_n$ and $a_n \ge 1$ for all $n \in \mathbf{N}$, then $(a_n)_{n=1}^{\infty}$ must be eventually constant. That is, there exists an $N \in \mathbf{N}$ such that $a_n = a_N$ for all $n \ge N$.

Proof. Let $A = \{a_n : n \in \mathbf{N}\}$, which is non-empty and bounded below by 1. It follows from the well-ordering principle that A has a least element, say $\min A = a_N$ for some $N \in \mathbf{N}$. Let $n > N$ be given. It cannot be the case that $a_n < a_N$, since this would contradict that a_N is the least element of A, so we must have $a_n \ge a_N$. By assumption $a_n \le a_N$ and so we may conclude that $a_n = a_N$. $\square$

Consider the sequence $(|A_n|)_{n=1}^{\infty}$, where $|A_n|$ is the number of elements contained in A_n. Because each A_n is finite and non-empty, this is a sequence of positive integers. Furthermore, this sequence is decreasing since the sets $(A_n)_{n=1}^{\infty}$ are nested:

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots.$$

We may now invoke Lemma L.1 to obtain an $N \in \mathbf{N}$ such that $|A_n| = |A_N|$ for all $n \ge N$. Combining this equality with the inclusion $A_n \subseteq A_N$ for each $n \ge N$, we see that $A_n = A_N$ for all $n \ge N$. It follows that $\bigcap_{n=1}^{\infty} A_n = A_N$, which by assumption is finite and non-empty.

(c) This is false: let $A = B = \emptyset$ and $C = \{0\}$ and observe that

$$A \cap (B \cup C) = \emptyset \neq \{0\} = (A \cap B) \cup C.$$

(d) This is true, since

$$x \in A \cap (B \cap C)) \iff x \in A \text{ and } x \in (B \cap C) \iff x \in A \text{ and } (x \in B \text{ and } x \in C)$$
$$\iff (x \in A \text{ and } x \in B) \text{ and } x \in C \iff x \in (A \cap B) \text{ and } x \in C \iff x \in (A \cap B) \cap C,$$

where we have used that logical conjunction ("and") is associative for the third equivalence. It follows that x belongs to $A \cap (B \cap C)$ if and only if x belongs to $(A \cap B) \cap C$, which is to say that $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) This is true, since

$$x \in A \cap (B \cup C) \iff x \in A \text{ and } x \in (B \cup C) \iff x \in A \text{ and } (x \in B \text{ or } x \in C)$$
$$\iff (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \iff x \in (A \cap B) \text{ or } x \in (A \cap C)$$
$$\iff x \in (A \cap B) \cup (A \cap C),$$

where we have used that logical conjunction ("and") distributes over logical disjunction ("or") for the third equivalence. It follows that x belongs to $A \cap (B \cup C)$ if and only if x belongs to $(A \cap B) \cup (A \cap C)$, which is to say that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1.2.4. Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$.

Solution. Arrange $\mathbf{N}$ in a grid like so:

A_1	A_2	A_3	A_4	$\cdots$
1	3	6	10	$\cdots$
2	5	9	14	$\cdots$
4	8	13	19	$\cdots$
7	12	18	25	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now take A_i to be the set of numbers appearing in the i^{th} column.

Exercise 1.2.5 (De Morgan's Laws). Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution.

(a) Observe that

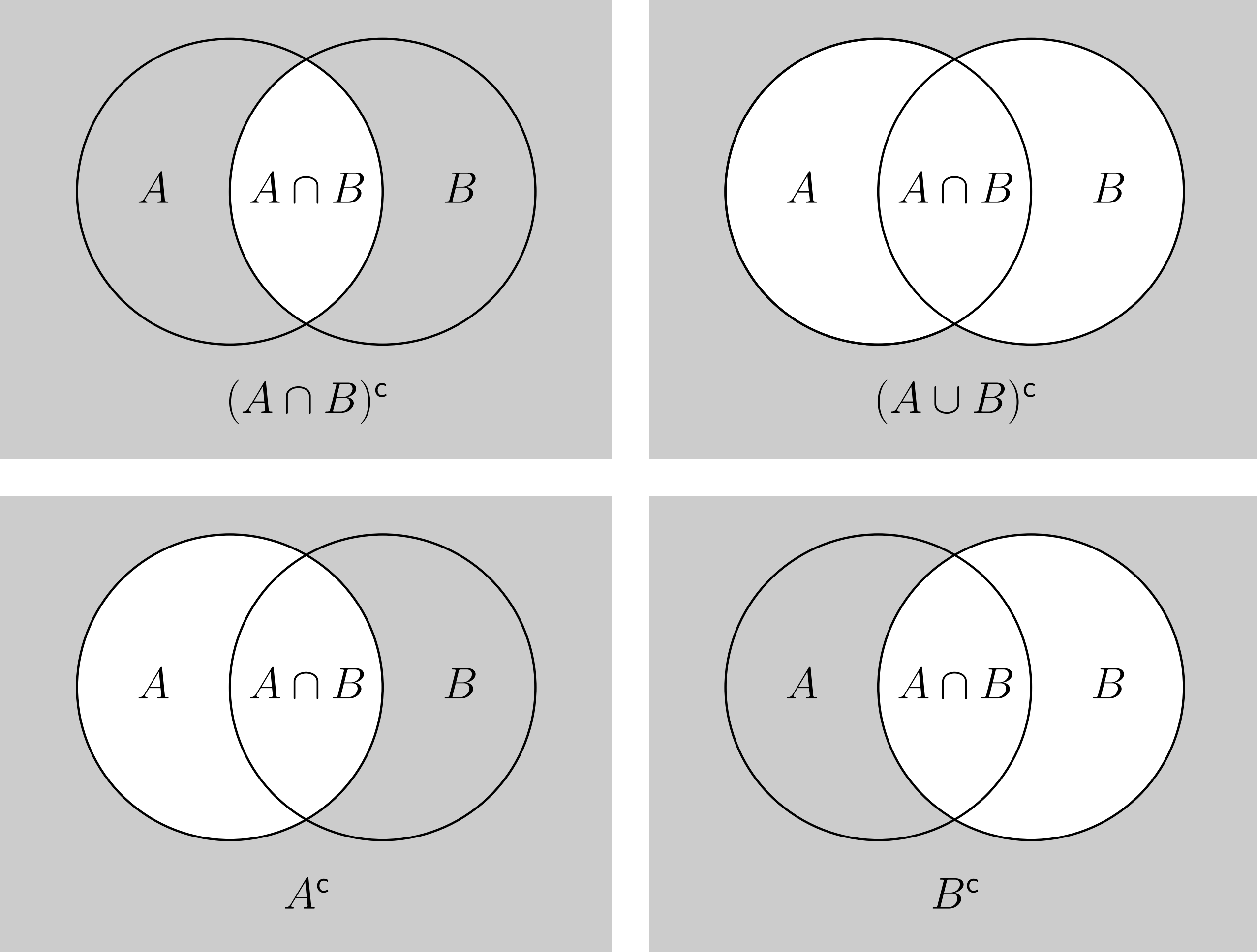
$$\begin{aligned} x \in (A \cap B)^c \quad \Leftrightarrow \quad x \notin A \cap B \quad \Leftrightarrow \quad \text{not } (x \in A \text{ and } x \in B) \\ \Leftrightarrow \quad x \notin A \text{ or } x \notin B \quad \Leftrightarrow \quad x \in A^c \cup B^c \end{aligned}$$

(b) See part (a).

(c) The proof is similar to the one given in part (a).

$$\begin{aligned} x \in (A \cup B)^c \quad \Leftrightarrow \quad x \notin A \cup B \quad \Leftrightarrow \quad \text{not } (x \in A \text{ or } x \in B) \\ \Leftrightarrow \quad x \notin A \text{ and } x \notin B \quad \Leftrightarrow \quad x \in A^c \cap B^c \end{aligned}$$

The following Venn diagrams help to visualize De Morgan's Laws. The shaded regions are included and the unshaded regions are excluded.



Exercise 1.2.6.

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution.

(a) First suppose that a and b are both non-negative, so that $a+b$ is also non-negative; it follows that $|a+b| = a+b$ and $|a|+|b| = a+b$. Thus the triangle inequality in this case reduces to the evidently true statement $a+b \leq a+b$.

Now suppose that a and b are both negative, so that $a+b$ is also negative; it follows that $|a+b| = -a-b$ and $|a|+|b| = -a-b$. Thus the triangle inequality in this case reduces to the evidently true statement $-a-b \leq -a-b$.

(b) Starting from the true statement $ab \leq |ab|$ and using that $a^2 = |a|^2$ and $|ab| = |a||b|$ for any real numbers a and b, observe that

$$2ab \leq 2|ab| \quad \Leftrightarrow \quad a^2 + 2ab + b^2 \leq |a|^2 + 2|a||b| + |b|^2$$
$$\Leftrightarrow \quad (a+b)^2 \leq (|a| + |b|)^2 \quad \Leftrightarrow \quad |a+b|^2 \leq (|a| + |b|)^2.$$

Because both $|a + b|$ and $|a| + |b|$ are non-negative, the inequality $|a+b|^2 \leq (|a| + |b|)^2$ is equivalent to $|a + b| \leq |a| + |b|$, as desired.

(c) We apply the triangle inequality twice:

$$|a - b| = |a - c + c - b| \leq |a - c| + |c - b| \leq |a - c| + |c - d| + |d - b|.$$

(d) Using the triangle inequality and the fact that $|-a| = |a|$ for any $a \in \mathbf{R}$, we find that

$$|a| = |a - b + b| \leq |a - b| + |b| \quad \Leftrightarrow \quad |a| - |b| \leq |a - b|,$$
$$|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a| \quad \Leftrightarrow \quad |b| - |a| \leq |a - b|.$$

Because $||a| - |b||$ equals either $|a| - |b|$ or $|b| - |a|$, it follows that $||a| - |b|| \leq |a - b|$.

Exercise 1.2.7. Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbf{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution.

(a) Some straightforward calculations reveal that

$$f(A) = [0, 4], \qquad f(A \cap B) = [1, 4], \qquad f(A \cup B) = [0, 16],$$
$$f(B) = [1, 16], \qquad f(A) \cap f(B) = [1, 4], \qquad f(A) \cup f(B) = [0, 16].$$

From this we see that $f(A \cap B) = f(A) \cap f(B)$ and $f(A \cup B) = f(A) \cup f(B)$.

(b) Let $A = \{-1\}$ and $B = \{1\}$ and note that $f(A \cap B) = f(\emptyset) = \emptyset$, but

$$f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\} \neq \emptyset.$$

(c) Observe that

$$
\begin{aligned}
y \in g(A \cap B) \quad &\Leftrightarrow \quad y = g(x) \text{ for some } x \in A \cap B \\
\Rightarrow \quad &(y = g(x_1) \text{ for some } x_1 \in A) \text{ and } (y = g(x_2) \text{ for some } x_2 \in B) \\
\Leftrightarrow \quad &y \in g(A) \text{ and } y \in g(B) \quad \Leftrightarrow \quad y \in g(A) \cap g(B).
\end{aligned}
$$

It follows that y belongs to $g(A) \cap g(B)$ whenever y belongs to $g(A \cap B)$, which is to say that $g(A \cap B) \subseteq g(A) \cap g(B)$.

(d) We always have $g(A \cup B) = g(A) \cup g(B)$; indeed,

$$
\begin{aligned}
y \in g(A \cup B) \quad &\Leftrightarrow \quad y = g(x) \text{ for some } x \in A \cup B \\
\Leftrightarrow \quad &y = g(x) \text{ for some } x \text{ such that } (x \in A \text{ or } x \in B) \\
\Leftrightarrow \quad &(y = g(x_1) \text{ for some } x_1 \in A) \text{ or } (y = g(x_2) \text{ for some } x_2 \in B) \\
\Leftrightarrow \quad &y \in g(A) \text{ or } y \in g(B) \quad \Leftrightarrow \quad y \in g(A) \cup g(B).
\end{aligned}
$$

It follows that $g(A \cup B) = g(A) \cup g(B)$.

Exercise 1.2.8. Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:

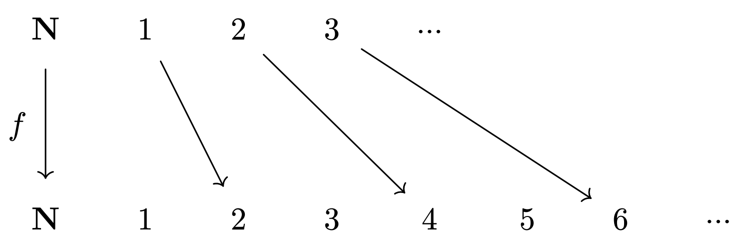
(a) $f : \mathbf{N} \to \mathbf{N}$ that is 1-1 but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not 1-1.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is 1-1 and onto.

Solution. (I prefer the terms injective/surjective/bijective rather than one-to-one and onto. I will use these terms throughout this document.)

(a) Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(n) = 2n$. Notice that f is injective since $n = m$ if and only if $2n = 2m$, but f is not surjective since the range of f contains only even numbers.



(b) Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(1) = 1$ and $f(n) = n - 1$ for $n \geq 2$. Notice that $f(n+1) = n$ for any $n \in \mathbf{N}$, so that f is surjective, but f is not injective since $f(1) = f(2) = 1$.

$$\begin{tikzcd}
\mathbf{N} \arrow[d, "f"'] & 1 \arrow[d] & 2 \arrow[dl] & 3 \arrow[dl] & 4 \arrow[dl] & 5 \arrow[dl] & \cdots \\
\mathbf{N} & 1 & 2 & 3 & 4 & \cdots &
\end{tikzcd}$$

(c) Let $f : \mathbf{N} \to \mathbf{Z}$ be given by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\begin{tikzcd}
\mathbf{N} \arrow[d, "f"'] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & 4 \arrow[d] & 5 \arrow[d] & \cdots \\
\mathbf{Z} & 0 & 1 & -1 & 2 & -2 & \cdots
\end{tikzcd}$$

To see that f is injective, let $n \neq m$ be given and consider these cases.

Case 1. If n and m are both even, then $f(n) \neq f(m)$ since $n \neq m$ if and only if $\frac{n}{2} \neq \frac{m}{2}$.

Case 2. If n and m are both odd, then $f(n) \neq f(m)$ since $n \neq m$ if and only if $-\frac{n-1}{2} \neq -\frac{m-1}{2}$.

Case 3. If n and m have opposite signs, say n is even and m is odd, then $f(n) \neq f(m)$ since $f(n) > 0$ and $f(m) \leq 0$.

To see that f is surjective, let $n \in \mathbf{Z}$ be given. If $n > 0$ then $f(2n) = n$, and if $n \leq 0$ then $f(-2n+1) = n$.

Exercise 1.2.9. Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

Solution.

(a) Some straightforward calculations reveal that

$$f^{-1}(A) = [-2, 2], \qquad f^{-1}(A \cap B) = [-1, 1], \qquad f^{-1}(A \cup B) = [-2, 2],$$
$$f^{-1}(B) = [-1, 1], \qquad f^{-1}(A) \cap f^{-1}(B) = [-1, 1], \qquad f^{-1}(A) \cup f^{-1}(B) = [-2, 2].$$

From this we see that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \quad \text{and} \quad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

(b) Observe that

$$\begin{aligned} x \in g^{-1}(A \cap B) \quad &\Leftrightarrow \quad g(x) \in A \cap B \quad \Leftrightarrow \quad (g(x) \in A) \text{ and } (g(x) \in B) \\ &\Leftrightarrow \quad (x \in g^{-1}(A)) \text{ and } (x \in g^{-1}(B)) \quad \Leftrightarrow \quad x \in g^{-1}(A) \cap g^{-1}(B). \end{aligned}$$

Similarly,

$$\begin{aligned} x \in g^{-1}(A \cup B) \quad &\Leftrightarrow \quad g(x) \in A \cup B \quad \Leftrightarrow \quad (g(x) \in A) \text{ or } (g(x) \in B) \\ &\Leftrightarrow \quad (x \in g^{-1}(A)) \text{ or } (x \in g^{-1}(B)) \quad \Leftrightarrow \quad x \in g^{-1}(A) \cup g^{-1}(B). \end{aligned}$$

Exercise 1.2.10. Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \varepsilon$ for every $\varepsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

Solution.

(a) This is false; the implication

$$\text{if } a < b + \varepsilon \text{ for every } \varepsilon > 0, \text{then } a < b$$

does not hold. The problem occurs when we consider the case where $a = b$. For example, we certainly have $1 < 1 + \varepsilon$ for every $\varepsilon > 0$ but of course $1 < 1$ is false.

(b) See part (a).

(c) This is true. The implication

$$\text{if } a \leq b, \text{then } a < b + \varepsilon \text{ for every } \varepsilon > 0$$

follows since $a \leq b < b + \varepsilon$ for every $\varepsilon > 0$ and the implication

$$\text{if } a > b, \text{then } a \geq b + \varepsilon \text{ for some } \varepsilon > 0$$

can be seen by taking $\varepsilon = a - b > 0$, so that $b + \varepsilon = a \leq a$.

Exercise 1.2.11. Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that…" in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution.

(a) The negated statement is:

$$\text{there exist real numbers } a < b \text{ such that } a + \frac{1}{n} \geq b \quad \text{for all } n \in \mathbf{N}.$$

The original statement is true and follows from the Archimedean Property (Theorem 1.4.2).

(b) The negated statement is:

$$\text{for all } x > 0, \text{there exists an } n \in \mathbf{N} \text{ such that } \frac{1}{n} \leq x.$$

The negated statement is true and again follows from the Archimedean Property (Theorem 1.4.2).

(c) The negated statement is:

there are two distinct real numbers with no rational number between them.

The original statement is true; this is the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3).

Exercise 1.2.12. Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show that the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution.

(a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $y_n > -6$. Since $y_1 = 6$, the truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and observe that

$$y_{n+1} = \tfrac{2}{3}y_n - 2 > \tfrac{2}{3}(-6) - 2 = -6,$$

i.e. $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

(b) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $y_{n+1} \leq y_n$. Since $y_1 = 6$ and $y_2 = 2$, the truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and observe that

$$y_{n+2} = \tfrac{2}{3}y_{n+1} - 2 \leq \tfrac{2}{3}y_n - 2 = y_{n+1},$$

i.e. $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

Exercise 1.2.13. For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \dots \cup A_n)^{\mathrm{c}} = A_1^{\mathrm{c}} \cap A_2^{\mathrm{c}} \cap \dots \cap A_n^{\mathrm{c}}$$

for any finite $n \in \mathbf{N}$.

(b) It is tempting to appeal to induction to conclude that

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^{\mathrm{c}} = \bigcap_{i=1}^{\infty} A_i^{\mathrm{c}},$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \dots$ where $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution.

(a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $(A_1 \cup \dots \cup A_n)^{\mathrm{c}} = A_1^{\mathrm{c}} \cap \dots \cap A_n^{\mathrm{c}}$ for any sets $A_1, \dots, A_n$. The truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$, let $A_1, \dots, A_n, A_{n+1}$ be given, and observe that

$$\begin{aligned}
(A_1 \cup \dots \cup A_n \cup A_{n+1})^{\mathrm{c}} &= ((A_1 \cup \dots \cup A_n) \cup (A_{n+1}))^{\mathrm{c}} \\
&= (A_1 \cup \dots \cup A_n)^{\mathrm{c}} \cap A_{n+1}^{\mathrm{c}} && \text{(Exercise 1.2.5)} \\
&= A_1^{\mathrm{c}} \cap \dots \cap A_n^{\mathrm{c}} \cap A_{n+1}^{\mathrm{c}}, && \text{(induction hypothesis)}
\end{aligned}$$

i.e. $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

(b) Let $B_i = \{i, i+1, i+2, \dots\}$, so that

$$B_1 = \{1, 2, 3, \dots\}, \quad B_2 = \{2, 3, 4, \dots\}, \quad B_3 = \{3, 4, 5, \dots\}, \quad \text{etc.}$$

It is straightforward to verify that $\bigcap_{i=1}^{n} B_i = B_n \neq \emptyset$ for any $n \in \mathbf{N}$. However, as Example 1.2.2 shows, the intersection $\bigcap_{i=1}^{\infty} B_i$ is empty.

(c) Observe that

$$x \in \left(\bigcup_{i=1}^{\infty} A_i \right)^{\mathrm{c}} \quad \Leftrightarrow \quad x \notin \bigcup_{i=1}^{\infty} A_i \quad \Leftrightarrow \quad x \notin A_i \text{ for every } i \in \mathbf{N} \quad \Leftrightarrow \quad x \in \bigcap_{i=1}^{\infty} A_i^{\mathrm{c}}.$$

It follows that

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^{\mathrm{c}} = \bigcap_{i=1}^{\infty} A_i^{\mathrm{c}}.$$

1.3. The Axiom of Completeness

Exercise 1.3.1.

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution.

(a) A real number t is the *greatest lower bound* for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

   (i) t is a lower bound for A;

   (ii) if b is any lower bound for A, then $b \leq t$.

(b) Here is a version of Lemma 1.3.8 for greatest lower bounds.

Lemma L.2. If $t \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$, then $t = \inf A$ if and only if for every choice of $\varepsilon > 0$, there exists an element $a \in A$ satisfying $a < t + \varepsilon$.

Proof. First, let us prove the implication

if $t = \inf A$, then for every $\varepsilon > 0$ there exists an $a \in A$ such that $a < t + \varepsilon$

by proving the contrapositive statement

if there exists an $\varepsilon > 0$ such that $t + \varepsilon \leq a$ for every $a \in A$, then $t \neq \inf A$.

If such an $\varepsilon > 0$ exists, then $t + \varepsilon$ is a lower bound for A strictly greater than t; it follows that t is not the greatest lower bound for A, i.e. $t \neq \inf A$.

Now let us prove the converse:

if for every $\varepsilon > 0$ there exists an $a \in A$ such that $a < t + \varepsilon$, then $t = \inf A$.

Suppose $b \in \mathbf{R}$ is such that $b > t$. Letting $\varepsilon = b - t > 0$, we are guaranteed the existence of an $a \in A$ such that $a < t + \varepsilon = b$; it follows that b is not a lower bound for A. This proves the contrapositive of criterion (ii) in part (a) and we may conclude that $t = \inf A$. □

Exercise 1.3.2. Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution.

(a) Let $B = \{0\}$ and notice that $\inf B = \sup B = 0$.

(b) This is impossible. To see this, let us first use induction to show that any non-empty finite subset of $\mathbf{R}$ contains a minimum and a maximum element.

Lemma L.3. If $E \subseteq \mathbf{R}$ is non-empty and finite, then E contains a minimum and a maximum element.

Proof. For $n \in \mathbf{N}$, let $P(n)$ be the statement that any subset of $\mathbf{R}$ containing n elements has a minimum and a maximum element. For the base case $P(1)$, simply observe that $\min\{x\} = \max\{x\} = x$ for any $x \in \mathbf{R}$.

Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and let $E \subseteq \mathbf{R}$ be a set containing $n+1$ elements. Fix some $x \in E$ and consider the set $F = E \setminus \{x\}$, which contains n elements. Our induction hypothesis guarantees the existence of a minimum element $a = \min F$ and a maximum element $b = \max F$, which must satisfy $a \leq b$. There are now three cases; the conclusion is each case is straightforward to verify.

Case 1. If $x < a$, then $\min E = x$ and $\max E = b$.

Case 2. If $x > b$, then $\min E = a$ and $\max E = x$.

Case 3. If $a \leq x \leq b$, then $\min E = a$ and $\max E = b$.

In any case, the set E has a minimum and a maximum element, i.e. $P(n+1)$ holds. This completes the induction step and the proof. $\square$

It is immediate from the definition of the supremum and the maximum of a set $E \subseteq \mathbf{R}$ that if $\max E$ exists then $\sup E = \max E$ (see Exercise 1.3.7); similarly, if $\min E$ exists then $\inf E = \min E$. It follows that the given request is impossible: if $E \subseteq \mathbf{R}$ is finite, then Lemma L.3 implies that $\min E = \inf E$ and $\max E = \sup E$ both exist and hence E contains both its infimum and its supremum.

(c) Consider the bounded set $E = \{p \in \mathbf{Q} : 0 < p \leq 1\}$, which satisfies $\sup E = 1 \in E$ and $\inf E = 0 \notin E$.

Exercise 1.3.3.

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution.

(a) B is non-empty since A is bounded below, and B is bounded above by any $x \in A$; there exists at least one such x since A is non-empty. It follows from the Axiom of Completeness that $\sup B$ exists. To see that $\sup B = \inf A$, we need to show that $\sup B$ satisfies criteria (i) and (ii) from Exercise 1.3.1 (a).

(i) First we need to prove that $\sup B$ is a lower bound of A, i.e. if $x \in A$ then $\sup B \leq x$. We will prove the contrapositive statement: if $x < \sup B$ then $x \notin A$. If x is strictly less than $\sup B$, then x cannot be an upper bound of B. Thus there exists some $b \in B$ such that $x < b$. Since b is a lower bound of A, it follows that $x \notin A$.

(ii) Now we need to show that $\sup B$ is the greatest lower bound of A. Indeed, suppose $y \in \mathbf{R}$ is a lower bound of A, so that $y \in B$; it follows that $y \leq \sup B$.

We may conclude that $\sup B = \inf A$.

(b) Part (a) shows that the existence of the greatest lower bound for non-empty bounded below subsets of $\mathbf{R}$ is implied by the Axiom of Completeness; adding this existence as part of the Axiom of Completeness would be redundant.

Exercise 1.3.4. Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

Solution.

(a) Let $n \in \mathbf{N}$ be given. For each $k \in \{1, \ldots, n\}$, the Axiom of Completeness guarantees that $\sup A_k$ exists. By Lemma L.3, the finite set $\{\sup A_1, \ldots, \sup A_k\}$ has a maximum element, say M; we claim that $\sup\left(\bigcup_{k=1}^{n} A_k\right) = M$. To prove this, we must verify criteria (i) and (ii) from Definition 1.3.2.

(i) If $x \in \bigcup_{k=1}^{n} A_k$, then $x \in A_k$ for some $k \in \{1, \ldots, n\}$; it follows that $x \leq \sup A_k \leq M$. Since x was arbitrary, we see that M is an upper bound for $\bigcup_{k=1}^{n} A_k$.

(ii) If $b \in \mathbf{R}$ is an upper bound for $\bigcup_{k=1}^{n} A_k$, then b must be an upper bound for each A_k. It follows that $\sup A_k \leq b$ for each $k \in \{1, \ldots, n\}$ and thus $M \leq b$.

We may conclude that $\sup\left(\bigcup_{k=1}^{n} A_k\right) = M$.

(b) The proof given above does not extend to the infinite case, since the set $\{\sup A_1, \sup A_2, \ldots\}$ need not have a maximum. Indeed, it may be the case that $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$ does not exist. For example, let $A_k = \{k\}$, which is non-empty and bounded above with $\sup A_k = k$, but $\bigcup_{k=1}^{\infty} A_k = \mathbf{N}$, which does not have a supremum in $\mathbf{R}$.

Exercise 1.3.5. As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution.

(a) If $c = 0$ then the result is clear, so suppose that $c > 0$. For any $x \in A$, notice that

$$x \leq \sup A \quad \Leftrightarrow \quad cx \leq c \sup A.$$

This demonstrates that $c \sup A$ is an upper bound of cA.

Now observe that

$$\begin{gathered} b \in \mathbf{R} \text{ is an upper bound of } cA \quad \Leftrightarrow \quad cx \leq b \text{ for all } x \in A \\ \Leftrightarrow \quad x \leq c^{-1}b \text{ for all } x \in A \quad \Leftrightarrow \quad c^{-1}b \text{ is an upper bound of } A. \end{gathered}$$

It follows that $\sup A \leq c^{-1}b$ and hence that $c \sup A \leq b$. We may conclude that $\sup(cA) = c \sup A$.

(b) If $c < 0$ and $\inf A$ exists then $\sup(cA) = c \inf A$. The proof is similar to part (a). For any $x \in A$, we have

$$\inf A \leq x \quad \Leftrightarrow \quad cx \leq c \inf A,$$

so that $c \inf A$ is an upper bound of cA.

Observe that

$$b \in \mathbf{R} \text{ is an upper bound of } cA \quad \Leftrightarrow \quad cx \le b \text{ for all } x \in A$$
$$\Leftrightarrow \quad c^{-1}b \le x \text{ for all } x \in A \quad \Leftrightarrow \quad c^{-1}b \text{ is an lower bound of } A.$$

It follows that $c^{-1}b \le \inf A$ and hence that $c \inf A \le b$. We may conclude that $\sup(cA) = c \inf A$.

If $\inf A$ doesn't exist then $\sup(cA)$ doesn't exist either, since for $c < 0$ the set A is bounded below if and only if cA is bounded above. For example, $A = (-\infty, 0)$ and $c = -1$ gives $cA = (0, \infty)$.

Exercise 1.3.6. Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \le u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution.

(a) For any $a \in A$ and $b \in B$ we have $a \le s$ and $b \le t$. It follows that $a + b \le s + t$ and thus $s + t$ is an upper bound of $A + B$.

(b) For any $b \in B$ we have $a + b \le u$, which gives $b \le u - a$. This demonstrates that $u - a$ is an upper bound for B and so it follows that $t \le u - a$.

(c) Part (b) implies that for any $a \in A$ we have $t \le u - a$, which gives $a \le u - t$. This shows that $u - t$ is an upper bound of A and it follows that $s \le u - t$, i.e. $s + t \le u$. Since u was an arbitrary upper bound of $A + B$, we may conclude that

$$\sup(A + B) = s + t = \sup A + \sup B.$$

(d) Let $\varepsilon > 0$ be given. By Lemma 1.3.8, there exist elements $a \in A$ and $b \in B$ such that $s - \frac{\varepsilon}{2} < a$ and $t - \frac{\varepsilon}{2} < b$, which implies that $s + t - \varepsilon < a + b$. We showed in part (a) that $s + t$ is an upper bound of $A + B$, so we may invoke Lemma 1.3.8 to conclude that $\sup(A + B) = \sup A + \sup B$.

Exercise 1.3.7. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution. Let $b \in \mathbf{R}$ be an upper bound of A. Since $a \in A$, we must have $a \leq b$; it follows that $a = \sup A$.

Exercise 1.3.8. Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.

(c) $\{n/(3n+1) : n \in \mathbf{N}\}$.

(d) $\{m/(m+n) : m, n \in \mathbf{N}\}$.

Solution.

(a) The supremum is 1 and the infimum is 0.

(b) The supremum is 1 and the infimum is -1.

(c) The supremum is $\frac{1}{3}$ and the infimum is $\frac{1}{4}$.

(d) The supremum is 1 and the infimum is 0.

Exercise 1.3.9.

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Solution.

(a) Let $\varepsilon = \sup B - \sup A > 0$. By Lemma 1.3.8, there exists some $b \in B$ such that $\sup B - \varepsilon = \sup A < b$. It follows that b is an upper bound of A.

(b) If we let $A = B = (0, 1)$ then $\sup A = \sup B = 1$, but no element of B is an upper bound of A.

Exercise 1.3.10 (Cut Property). The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution.

(a) Suppose that A and B are non-empty disjoint subsets of $\mathbf{R}$ such that $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$. Notice that A is non-empty (by assumption) and bounded above (because B is non-empty); the Axiom of Completeness then implies that $c = \sup A$ exists. It follows that $x \leq c$ for all $x \in A$ and, since each element of B is an upper bound of A, we also have $x \geq c$ for all $x \in B$.

(b) Suppose that $E \subseteq \mathbf{R}$ is non-empty and bounded above. Define

$$A = \{a \in \mathbf{R} : a \text{ is not an upper bound of } E\}$$
$$\text{and} \quad B = A^{c} = \{b \in \mathbf{R} : b \text{ is an upper bound of } E\}.$$

Notice that B is non-empty as E is bounded above and A is non-empty because $x - 1 \in A$ for any $x \in E$; we are guaranteed the existence of at least one $x \in E$ as E is non-empty. Furthermore, A and B are evidently disjoint and satisfy $A \cup B = \mathbf{R}$.

Let $a \in A$ and $b \in B$ be given. Since a is not an upper bound of E there exists some $x \in E$ such that $a < x$ and since b is an upper bound of E, we must then have $x \leq b$; it follows that $a < b$. We may now invoke the Cut Property to obtain a $c \in \mathbf{R}$ such that $x \leq c$ for all $x \in A$ and $x \geq c$ for all $x \in B$.

We claim that $c = \sup E$. Since $A \cup B = \mathbf{R}$ and $A \cap B = \emptyset$, exactly one of $c \in A$ or $c \in B$ holds. Suppose that $c \in A$, i.e. c is not an upper bound of E, which is the case if and only if there is some $t \in E$ such that $c < t$. Observe that $y = \frac{c+t}{2}$ satisfies $c < y < t$, so that $y \in A$—but this contradicts the fact that $x \leq c$ for all $x \in A$.

So it must be the case that $c \in B$, i.e. c is an upper bound of E. The Cut Property guarantees that $c \leq x$ for all $x \in B$. In other words, c is less than all other upper bounds of E; we may conclude that $c = \sup E$.

(c) A concrete example is given in the following lemma.

Lemma L.4. The sets

$$A = \{p \in \mathbf{Q} : p < 0 \text{ or } p^2 < 2\} \quad \text{and} \quad B = \{p \in \mathbf{Q} : p > 0 \text{ and } p^2 > 2\}$$

satisfy the following:

(i) A and B are non-empty, $A \cup B = \mathbf{Q}$, and $A \cap B = \emptyset$;

(ii) $p < q$ for all $p \in A$ and $q \in B$;

(iii) A has no maximum element and B has no minimum element.

Proof.

(i) Certainly A and B are non-empty. The negation of the statement "$p < 0$ or $p^2 < 2$" is "$p > 0$ and $p^2 \geq 2$"; by Theorem 1.1.1, this negated statement is equivalent to "$p > 0$ and $p^2 > 2$" for $p \in \mathbf{Q}$. Thus $B = \mathbf{Q} \setminus A$, from which it follows that $A \cup B = \mathbf{Q}$ and $A \cap B = \emptyset$.

(ii) Let $p \in A$ and $q \in B$ be given. If $p \leq 0$ then certainly $p < q$, so suppose that $p > 0$. It must then be the case that $p^2 < 2$, whence $p^2 < q^2$. Since p and q are positive, this implies that $p < q$.

(iii) Let $p \in A$ be given. We need to show that there exists some $q \in A$ such that $p < q$. If $p \leq 0$, we can take $q = 1$; if $p > 0$, so that $p^2 < 2$, then define

$$q = p + \frac{2 - p^2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{1}$$

Notice that $0 < \frac{2-p^2}{p+2}$, since $p^2 < 2$, from which it follows that $p < q$. A straightforward calculation yields

$$2 - q^2 = \frac{2(2 - p^2)}{(p + 2)^2};$$

again using that $p^2 < 2$, we see that $2 - q^2 > 0$ and thus $q \in A$.

Now let $p \in B$ be given. We need to show that there exists some $q \in B$ such that $q < p$. In fact, we can define q by equation (1) again; an argument similar to the one just given shows that $q < p$ and $q \in B$. $\square$

Parts (i) and (ii) of Lemma L.4 show that the sets A and B satisfy the hypotheses of the Cut Property. If the Cut Property held for $\mathbf{Q}$, then we would be able to obtain a $c \in \mathbf{Q}$ such that $p \leq c$ for all $p \in A$ and $c \leq q$ for all $q \in B$. Since $A \cup B = \mathbf{Q}$ and $A \cap B = \emptyset$, this implies that c is either the maximum of A or the minimum of B—but this contradicts part (iii) of Lemma L.4. We may conclude that the Cut Property does not hold for $\mathbf{Q}$.

Exercise 1.3.11. Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution.

(a) This is true. The Axiom of Completeness guarantees that $\sup A$ and $\sup B$ both exist. Furthermore, since each element of A is an element of B, any upper bound of B must be an upper bound of A also. In particular, $\sup B$ must be an upper bound of A; it follows that $\sup A \leq \sup B$.

(b) This is true. Let $c = \frac{\sup A + \inf B}{2}$, so that $\sup A < c < \inf B$, and notice that for any $a \in A$ and $b \in B$ we have

$$a \leq \sup A < c < \inf B \leq b.$$

(c) This is false. Consider $A = (-1, 0)$ and $B = (0, 1)$, and notice that $c = 0$ satisfies $a < c < b$ for all $a \in A$ and $b \in B$, but $\sup A = \inf B = 0$.

1.4. Consequences of Completeness

Exercise 1.4.1. Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a+b$ are elements of $\mathbf{Q}$ as well.

(b) Show tha if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a+t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

Solution.

(a) Suppose $a = \frac{m}{n}$ and $b = \frac{p}{q}$ and observe that

$$ab = \frac{mp}{nq} \quad \text{and} \quad a+b = \frac{mq+np}{nq},$$

which are rational numbers.

(b) Let $a \in \mathbf{Q}$ be fixed. We want to prove that

$$t \in \mathbf{I} \quad \Rightarrow \quad a+t \in \mathbf{I}.$$

To do this, we will prove the contrapositive statement

$$a+t \in \mathbf{Q} \quad \Rightarrow \quad t \in \mathbf{Q}.$$

Simply observe that $t = (a+t) - a$; it follows from part (a) that $t \in \mathbf{Q}$.

Similarly, let $a \in \mathbf{Q}$ be non-zero. We can show that

$$at \in \mathbf{Q} \quad \Rightarrow \quad t \in \mathbf{Q}$$

by observing that $t = a^{-1}(at)$ and appealing to part (a) to conclude that $t \in \mathbf{Q}$.

(c) $\mathbf{I}$ is not closed under addition or multiplication. For example, $-\sqrt{2}$ and $\sqrt{2}$ are irrational numbers, but their sum is the rational number 0 and their product is the rational number -2. The sum or product of two irrational numbers may be irrational. For example, it can be shown that $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}\sqrt{3} = \sqrt{6}$ are irrational:

- For the irrationality of $\sqrt{6}$, see Exercise 1.2.1 (a).
- For the irrationality of $\sqrt{2}+\sqrt{3}$, observe that $\sqrt{2}+\sqrt{3}$ is a root of the polynomial $x^4 - 10x^2 + 1$. The rational root theorem says that the only possible rational roots of this polynomial are ± 1—but neither of these solve the equation $x^4 - 10x^2 + 1 = 0$.

So in general, we cannot say anything about the sum or product of two irrational numbers without more information.

Exercise 1.4.2. Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution. If s is not an upper of A then there must exist some $x \in A$ such that $s < x$. By the Archimedean Property (Theorem 1.4.2), there then exists a natural number n such that $s + \frac{1}{n} < x$, which implies that $s + \frac{1}{n}$ is not an upper bound of A. Given our hypothesis that $s + \frac{1}{n}$ is an upper bound of A for all $n \in \mathbf{N}$, we see that s must be an upper bound of A.

Now let $\varepsilon > 0$ be given and usng the Archimedean Property (Theorem 1.4.2), pick a natural number n such that $\frac{1}{n} < \varepsilon$. By assumption $s - \frac{1}{n}$ is not an upper bound of A, so there must exist some $x \in A$ such that $s - \frac{1}{n} < x$, which implies that $s - \varepsilon < x$ since $\frac{1}{n} < \varepsilon$. Because $\varepsilon > 0$ was arbitrary, we may invoke Lemma 1.3.8 to conclude that $s = \sup A$.

Exercise 1.4.3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closeed for the conclusion of the theorem to hold.

Solution. Certainly $x \notin \bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$ if $x \leq 0$, so suppose that $x > 0$. Use the Archimedean Property (Theorem 1.4.2) to choose an $N \in \mathbf{N}$ such that $\frac{1}{N} < x$; it follows that $x \notin \left(0, \frac{1}{N}\right)$ and hence that $x \notin \bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$. We may conclude that $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right) = \emptyset$.

Exercise 1.4.4. Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$.

Solution. Certainly b is an upper bound of T. Let $\varepsilon > 0$ be given. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), there exists a rational number p satisfying

$$\max\{a, b - \varepsilon\} < p < b.$$

It follows that $p \in T$ and $b - \varepsilon < p$ and hence, by Lemma 1.3.8, we may conclude that $\sup T = b$.

Exercise 1.4.5. Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), there exists a rational number p satisfying $a - \sqrt{2} < p < b - \sqrt{2}$, which gives $a < p + \sqrt{2} < b$. Since $p + \sqrt{2}$ is irrational (Exercise 1.4.1 (b)), the corollary is proved.

Exercise 1.4.6. Recall that a set B is *dense* in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2.

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution.

(a) This set is not dense in $\mathbf{R}$. For $1 \leq q \leq 10$, observe that if $p \geq 1$ then $\frac{p}{q} \geq \frac{1}{10}$, if $p \leq -1$ then $\frac{p}{q} \leq -\frac{1}{10}$, and if $p = 0$ then $\frac{p}{q} = 0$. So there is no element of this set between the real numbers $\frac{1}{1000}$ and $\frac{1}{100}$, for example.

(b) This set is dense in $\mathbf{R}$. Let $a < b$ be given real numbers. Using the Archimedean Property (Theorem 1.4.2), let $n \in \mathbf{N}$ be such that $\frac{1}{n} < b - a$, which implies that $\frac{1}{2^n} < b - a$. Now let p be the smallest integer greater than $2^n a$, so that $p - 1 \leq 2^n a < p$, and observe that

$$2^n a < p \leq 1 + 2^n a < 2^n b;$$

it follows that $\frac{p}{2^n}$ lies between a and b.

(c) This set is not dense in $\mathbf{R}$. If $p > 0$ then

$$10|p| \geq q \quad \Leftrightarrow \quad 10p \geq q \quad \Leftrightarrow \quad \frac{p}{q} \geq \frac{1}{10},$$

and if $p < 0$ then

$$10|p| \geq q \quad \Leftrightarrow \quad -10p \geq q \quad \Leftrightarrow \quad \frac{p}{q} \leq -\frac{1}{10}.$$

We cannot have $p = 0$ since q is a positive integer. Thus there is no element of this set between the real numbers 0 and $\frac{1}{100}$, for example.

Exercise 1.4.7. Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Solution. Assuming that $\alpha^2 - 2 > 0$, the Archimedean Property (Theorem 1.4.2) implies that there is an $n \in \mathbf{N}$ such that

$$\frac{2\alpha}{n} < \alpha^2 - 2 \quad \Leftrightarrow \quad 2 < \alpha^2 - \frac{2\alpha}{n}.$$

Let $\beta = \alpha - \frac{1}{n}$ and note that since $1 \in T$ we have $\alpha \geq 1$ and hence $\beta \geq 0$; it follows that $t \leq \beta$ for all $t \in T$ such that $t < 0$. Now observe that

$$\beta^2 = \left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n} > 2,$$

so that for any $t \in T$ we have $t^2 < 2 < \beta^2$. If $t \in T$ is such that $t \geq 0$ then the inequality $t^2 < \beta^2$ implies that $t < \beta$, as β is also non-negative.

We have now shown that $t \leq \beta$ for all $t \in T$, i.e. β is an upper bound for T—but this contradicts the fact that α is the supremum of T since $\beta < \alpha$.

Exercise 1.4.8. Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbf{R} : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution.

(a) Let

$$A = \left\{-\frac{1}{2n} : n \in \mathbf{N}\right\} = \left\{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots\right\}$$

$$\text{and} \quad B = \left\{-\frac{1}{2n-1} : n \in \mathbf{N}\right\} = \left\{-1, -\frac{1}{3}, -\frac{1}{5}, \ldots\right\}.$$

Notice that $A \cap B = \emptyset$ and $\sup A = \sup B = 0$, which belongs to neither A nor B.

(b) If we let $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbf{N}$, then $\bigcap_{n=1}^{\infty} J_n = \{0\}$.

(c) For $n \in \mathbf{N}$, let $L_n = [n, \infty)$.

(d) This is impossible. To see this, let $(I_n)_{n=1}^{\infty}$ be a sequence of closed bounded intervals satisfying $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for every $N \in \mathbf{N}$. Define $J_N = \bigcap_{n=1}^{N} I_n$ for $N \in \mathbf{N}$ and note that any finite intersection of closed bounded intervals is a (possibly empty) closed bounded interval. Thus:

- each J_N is a closed bounded interval;
- these intervals are non-empty and nested, i.e. $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$;
- $\bigcap_{n=1}^{\infty} I_n = \bigcap_{N=1}^{\infty} J_N$.

It then follows from the Nested Interval Property (Theorem 1.4.1) that $\bigcap_{n=1}^{\infty} I_n = \bigcap_{N=1}^{\infty} J_N$ is non-empty.