The following is paraphrased from Chapter 1 of [PMA].

1 $\mathbb{Q}$ does not have the least-upper-bound property

Let A be the set of rational numbers which are either negative or have square less than 2 and let B be the set of positive rational numbers with square greater than 2, i.e.

$$A = \{p \in \mathbb{Q} : p < 0 \text{ or } p^2 < 2\}, \quad B = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 > 2\}.$$

Note that A and B partition $\mathbb{Q}$ since there is no rational number whose square is 2.

Lemma 1. A contains no greatest element and B contains no least element. That is, for any $p \in A$ there exists a $q \in A$ with $q > p$ and for any $p \in B$ there exists a $q \in B$ with $q < p$.

Proof. If $p \in A$ and $p \leq 0$, then take $q = 1$. Otherwise, for a positive rational number p, define

$$q = p + \frac{2 - p^2}{p + 2} = \frac{2p + 2}{p + 2}.$$

Then

$$2 - q^2 = 2 - \frac{(2p + 2)^2}{(p + 2)^2} = \frac{2(2 - p^2)}{(p + 2)^2}.$$

For $p \in A$ with $p > 0$, we have $2 - p^2 > 0$, so that $q > p$ and $q \in A$; for $p \in B$ we have $2 - p^2 < 0$, so that $q < p$ and $q \in B$. $\square$

Lemma 2. The upper bounds of A are exactly the elements of B.

Proof. Suppose $r \in \mathbb{Q}$ is an upper bound for A. Then certainly r is positive, since $1 \in A$. Furthermore, exactly one of the following is true: $r^2 < 2$, $r^2 = 2$, or $r^2 > 2$. If $r^2 < 2$, then $r \in A$; but this implies that r is the greatest element of A, contradicting Lemma 1. So $r^2 \geq 2$ and since $r^2 = 2$ is impossible for rational r, we must have $r^2 > 2$, i.e. $r \in B$.

Now suppose $r \in B$ and let p be any element of A. If $r < p$, then $r^2 < p^2 < 2$ since r is positive. This contradicts $r \in B$, so in fact we must have $r \geq p$, so that r is an upper bound for A. $\square$

Since A and B partition $\mathbb{Q}$, Lemma 1 and Lemma 2 give us the following corollary.

Corollary 3. The lower bounds of B are exactly the elements of A.

Theorem 4. The set of rational numbers $\mathbb{Q}$ does not have the least-upper-bound property.

Proof. A is non-empty and bounded above; the upper bounds of A are exactly the elements of B. Since B has no least element, it follows that A has no least upper bound. $\square$

[PMA] Rudin, W. (1976) *Principles of Mathematical Analysis.* 3rd edn.