Understanding Analysis Solutions

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Contents

Notation

I will sometimes use notation and terminology which Abbott does not use, or which differ from Abbott's notation and terminology. I will try to collect these differences here; please refer back to this section if you are unfamiliar with a term I have used.

Intervals

By a proper interval I mean an interval with at least two elements; such an interval must in fact have uncountably many elements.

Functions

Suppose $f : A \to B$ is a function. If $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B, then I will refer to f as injective or as an injection. Abbott uses the term one-to-one (1-1); both injective and one-to-one are common terms for this property of a function.

If given any $b \in B$ there exists an $a \in A$ such that $f(a) = b$, then I will refer to f as surjective or as a surjection. Abbott uses the term onto; both surjective and onto are common terms for this property of a function.

If f is both injective and surjective, then I will refer to f as bijective, or a bijection. Abbott simply calls such a function 1-1 and onto.

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1. (a) Prove that $\sqrt{3}$ is irrational. Does the same argument work to show that $\sqrt{6}$ is irrational?

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

Solution. (a) Suppose there was a rational number $p = \frac{m}{n}$, which we may assume is in lowest terms, such that $p^2 = 3$. Then $m^2 = 3n^2$, so that m^2 is divisible by 3. This implies that m is divisible by 3. To see this, observe that for any $k \in \mathbf{Z}$ we have

$$(3k+1)^2 = 3(3k^2+2k)+1 \qquad \text{and} \qquad (3k+2)^2 = 3(3k^2+4k+1)+1.$$

Since m is of the form $3k+1$ or $3k+2$ for some integer k if m is not divisible by 3, it follows that

$$\text{if } m \text{ is not divisible by 3, then } m^2 \text{ is not divisible by 3;}$$

the contrapositive of this statement is what we wanted to see.

Thus we may write $m = 3k$ for some $k \in \mathbf{Z}$ and substitute this into the equation $m^2 = 3n^2$ to obtain the equation $n^2 = 3k^2$, which implies that n is also divisible by 3. So m and n share the factor 3; this is a contradiction since we assumed that m and n had no common factors. We may conclude that there is no rational number whose square is 3.

The same argument works to show that there is no rational number whose square is 6; the crux of this argument is the implication

$$\text{if } m^2 \text{ is divisible by 6, then } m \text{ is divisible by 6.}$$

This can be seen using what we have already proved. If m^2 is divisible by $6 = 2 \cdot 3$, then m^2 is divisible by 2 and 3. It follows that m is divisible by 2 and 3 and hence that m is divisible by 6.

(b) The argument breaks down when we try to assert that

$$\text{if } m^2 \text{ is divisible by 4, then } m \text{ is divisible by 4.}$$

This implication is false. For example, $2^2 = 4$ is divisible by 4 but 2 is not divisible by 4.

Exercise 1.2.2. Show that there is no rational number r satisfying $2^r = 3$.

Solution. Suppose there was a rational number $r = \frac{m}{n}$, which we may assume is in lowest terms with $n > 0$, such that $2^r = 3$. This implies that $2^m = 3^n$. Since $n > 0$ gives $3^n \geq 3$ and $2^m < 2$ for $m \leq 0$, it must be the case that $m > 0$. Then the left-hand side of the equation $2^m = 3^n$ is a positive even integer whereas the right-hand side is a positive odd integer, which is a contradiction. We may conclude that there is no rational number r such that $2^r = 3$.

Exercise 1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution. (a) This is false, as Example 1.2.2 shows.

(b) This is true and we can use the following lemma to prove it.

Lemma L.1. If $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of positive integers, i.e., $a_{n+1} \leq a_n$ and $a_n \geq 1$ for all $n \in \mathbf{N}$, then $(a_n)_{n=1}^{\infty}$ must be eventually constant. That is, there exists an $N \in \mathbf{N}$ such that $a_n = a_N$ for all $n > N$.

Proof. Let A be the set $\{a_n : n \in \mathbf{N}\}$, which is non-empty and bounded below by 1. It follows from the well-ordering principle that A has a least element, say $\min A = a_N$ for some $N \in \mathbf{N}$. Let $n > N$ be given. It cannot be the case that $a_n < a_N$, since this would contradict that a_N is the least element of A, so we must have $a_n \geq a_N$. By assumption $a_n \leq a_N$ and so we may conclude that $a_n = a_N$. □

Consider the sequence $(|A_n|)_{n=1}^{\infty}$, where $|A_n|$ is the number of elements contained in A_n. This is a sequence of positive integers, because each A_n is finite and non-empty, and furthermore this sequence is decreasing because the sets $(A_n)_{n=1}^{\infty}$ are nested:

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots .$$

We may now invoke Lemma L.1 to obtain an $N \in \mathbf{N}$ such that $|A_n| = |A_N|$ for all $n > N$. Combining this equality with the inclusion $A_n \subseteq A_N$ for each $n > N$, we see that $A_n = A_N$ for all $n > N$. It follows that $\bigcap_{n=1}^{\infty} A_n = A_N$, which by assumption is finite and non-empty.

(c) This is false. Consider $A = B = \emptyset$ and $C = \{0\}$. Then

$$A \cap (B \cup C) = \emptyset \neq \{0\} = (A \cap B) \cup C.$$

(d) This is true, since

$$x \in A \cap (B \cap C) \iff x \in A \text{ and } x \in (B \cap C) \iff x \in A \text{ and } (x \in B \text{ and } x \in C)$$

$$\iff (x \in A \text{ and } x \in B) \text{ and } x \in C \iff x \in (A \cap B) \text{ and } x \in C \iff x \in (A \cap B) \cap C,$$

where we have used that logical conjunction ("and") is associative for the third equivalence. It follows that x belongs to $A \cap (B \cap C)$ if and only if x belongs to $(A \cap B) \cap C$, which is to say that $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) This is true, since

$$x \in A \cap (B \cup C) \iff x \in A \text{ and } x \in (B \cup C) \iff x \in A \text{ and } (x \in B \text{ or } x \in C)$$
$$\iff (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \iff x \in (A \cap B) \text{ or } x \in (A \cap C)$$
$$\iff x \in (A \cap B) \cup (A \cap C),$$

where we have used that logical conjunction ("and") distributes over logical disjunction ("or") for the third equivalence. It follows that x belongs to $A \cap (B \cup C)$ if and only if x belongs to $(A \cap B) \cup (A \cap C)$, which is to say that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1.2.4. Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$.

Solution. Arrange $\mathbf{N}$ in a grid like so:

A_1	A_2	A_3	A_4	$\cdots$
1	3	6	10	$\cdots$
2	5	9	14	$\cdots$
4	8	13	19	$\cdots$
7	12	18	25	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now take A_i to be the set of numbers appearing in the i^{th} column.

Exercise 1.2.5 (De Morgan's Laws). Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^{\mathsf{c}}$, explain why $x \in A^{\mathsf{c}} \cup B^{\mathsf{c}}$. This shows that $(A \cap B)^{\mathsf{c}} \subseteq A^{\mathsf{c}} \cup B^{\mathsf{c}}$.

(b) Prove the reverse inclusion $(A \cap B)^{\mathsf{c}} \supseteq A^{\mathsf{c}} \cup B^{\mathsf{c}}$, and conclude that $(A \cap B)^{\mathsf{c}} = A^{\mathsf{c}} \cup B^{\mathsf{c}}$.

(c) Show $(A \cup B)^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}}$ by demonstrating inclusion both ways.

Solution. (a) Observe that

$$x \in (A \cap B)^{\mathsf{c}} \iff x \notin A \cap B \iff \text{not } (x \in A \text{ and } x \in B)$$
$$\iff x \notin A \text{ or } x \notin B \iff x \in A^{\mathsf{c}} \cup B^{\mathsf{c}}.$$

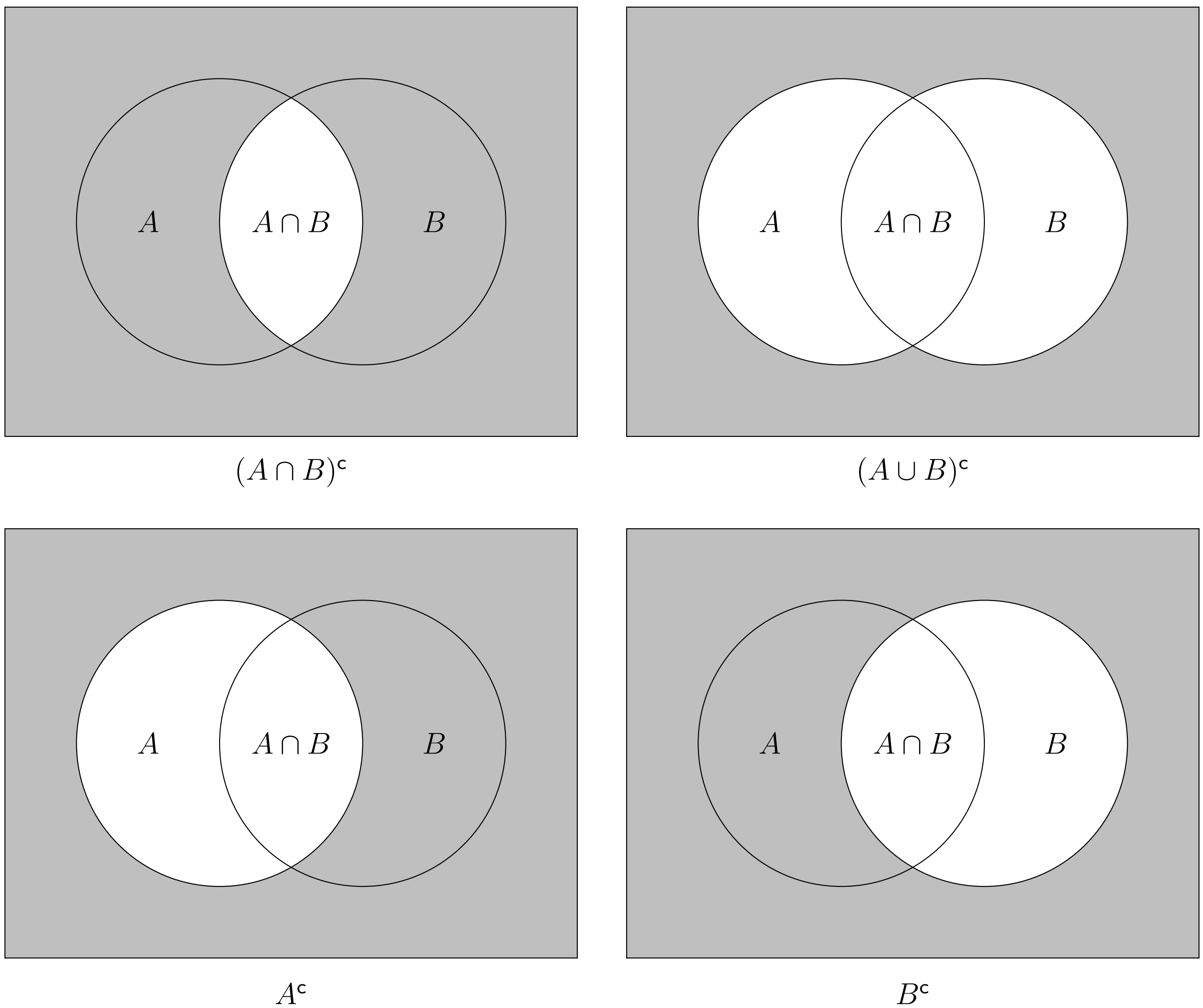


Figure F.1: Venn diagram for De Morgan's Laws; shaded regions are included, white regions are excluded

(b) See part (a). Figure F.1 shows some Venn diagrams which help to visualize De Morgan's Laws.

(c) The proof is similar to the one given in parts (a) and (b):

$$x \in (A \cup B)^{\mathsf{c}} \iff x \notin A \cup B \iff \text{not } (x \in A \text{ or } x \in B)$$
$$\iff x \notin A \text{ and } x \notin B \iff x \in A^{\mathsf{c}} \cap B^{\mathsf{c}}.$$

Exercise 1.2.6. (a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.

(d) Prove $||a| - |b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution. (a) First suppose that a and b are both non-negative, so that $a+b$ is also non-negative; it follows that $|a+b| = a+b$ and $|a|+|b| = a+b$. Thus the triangle inequality in this case reduces to the evidently true statement $a+b \leq a+b$.

Now suppose that a and b are both negative, so that $a+b$ is also negative; it follows that $|a+b| = -a-b$ and $|a|+|b| = -a-b$. Thus the triangle inequality in this case reduces to the evidently true statement $-a-b \leq -a-b$.

(b) Starting from the true statement $ab \leq |ab|$ and using that $a^2 = |a|^2$ and $|ab| = |a||b|$ for any real numbers a and b, observe that

$$2ab \leq 2|ab| \iff a^2 + 2ab + b^2 \leq |a|^2 + 2|a||b| + |b|^2$$
$$\iff (a+b)^2 \leq (|a|+|b|)^2 \iff |a+b|^2 \leq (|a|+|b|)^2.$$

Because both $|a+b|$ and $|a|+|b|$ are non-negative, the inequality $|a+b|^2 \leq (|a|+|b|)^2$ is equivalent to $|a+b| \leq |a|+|b|$, as desired.

(c) We apply the triangle inequality twice:

$$|a-b| = |a-c+c-b| \leq |a-c| + |c-b| \leq |a-c| + |c-d| + |d-b|.$$

(d) Using the triangle inequality and the fact that $|-a| = |a|$ for any $a \in \mathbf{R}$, we find that

$$|a| = |a - b + b| \le |a - b| + |b| \iff |a| - |b| \le |a - b|,$$

$$|b| = |b - a + a| \le |b - a| + |a| = |a - b| + |a| \iff |b| - |a| \le |a - b|.$$

Since $||a| - |b||$ equals either $|a| - |b|$ or $|b| - |a|$, it follows that $||a| - |b|| \le |a - b|$.

Exercise 1.2.7. Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbf{R} : 0 \le x \le 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution. (a) Some straightforward calculations reveal that

$$f(A) = [0, 4], \qquad f(A \cap B) = f([1, 2]) = [1, 4], \qquad f(A \cup B) = f([0, 4]) = [0, 16],$$

$$f(B) = [1, 16], \qquad f(A) \cap f(B) = [1, 4], \qquad f(A) \cup f(B) = [0, 16].$$

From this we see that $f(A \cap B) = f(A) \cap f(B)$ and $f(A \cup B) = f(A) \cup f(B)$.

(b) Let $A = \{-1\}$ and $B = \{1\}$. Then $f(A \cap B) = f(\emptyset) = \emptyset$ but

$$f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\} \neq \emptyset.$$

(c) Observe that

$$y \in g(A \cap B) \iff y = g(x) \text{ for some } x \in A \cap B$$

$$\implies (y = g(x_1) \text{ for some } x_1 \in A) \text{ and } (y = g(x_2) \text{ for some } x_2 \in B)$$

$$\iff y \in g(A) \text{ and } y \in g(B) \iff y \in g(A) \cap g(B).$$

It follows that y belongs to $g(A) \cap g(B)$ whenever y belongs to $g(A \cap B)$, which is to say that $g(A \cap B) \subseteq g(A) \cap g(B)$.

(d) We always have $g(A \cup B) = g(A) \cup g(B)$; indeed,

$$\begin{aligned} y \in g(A \cup B) &\iff y = g(x) \text{ for some } x \in A \cup B \\ &\iff y = g(x) \text{ for some } x \text{ such that } (x \in A \text{ or } x \in B) \\ &\iff (y = g(x_1) \text{ for some } x_1 \in A) \text{ or } (y = g(x_2) \text{ for some } x_2 \in B) \\ &\iff y \in g(A) \text{ or } y \in g(B) \iff y \in g(A) \cup g(B). \end{aligned}$$

It follows that $g(A \cup B) = g(A) \cup g(B)$.

Exercise 1.2.8. Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Give an example of each or state that the request is impossible:

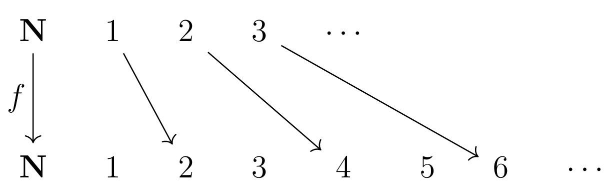
(a) $f : \mathbf{N} \to \mathbf{N}$ that is 1-1 but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not 1-1.

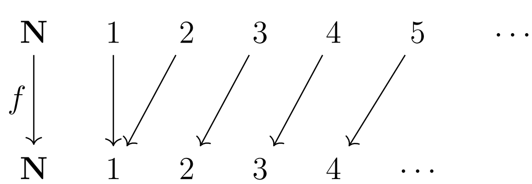
(c) $f : \mathbf{N} \to \mathbf{Z}$ that is 1-1 and onto.

Solution. (I prefer the terms injective/surjective/bijective rather than one-to-one and onto; see notation. I will use these terms throughout this document.)

(a) Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(n) = 2n$. Then f is injective since $n = m$ if and only if $2n = 2m$, but f is not surjective since the range of f contains only even numbers.



(b) Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(1) = 1$ and $f(n) = n - 1$ for $n \geq 2$. Then $f(n+1) = n$ for any $n \in \mathbf{N}$, so that f is surjective, but f is not injective since $f(1) = f(2) = 1$.



(c) Let $f : \mathbf{N} \to \mathbf{Z}$ be given by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\begin{array}{cccccccc} \mathbf{N} & 1 & 2 & 3 & 4 & 5 & \cdots \\ f\big\downarrow & \big\downarrow & \big\downarrow & \big\downarrow & \big\downarrow & \big\downarrow & \\ \mathbf{Z} & 0 & 1 & -1 & 2 & -2 & \cdots \end{array}$$

To see that f is injective, let $n \neq m$ be given and consider these cases.

Case 1. If n and m are both even, then $f(n) \neq f(m)$ since $n \neq m$ if and only if $\frac{n}{2} \neq \frac{m}{2}$.

Case 2. If n and m are both odd, then $f(n) \neq f(m)$ since $n \neq m$ if and only if $-\frac{n-1}{2} \neq -\frac{m-1}{2}$.

Case 3. If n and m have opposite signs, say n is even and m is odd, then $f(n) \neq f(m)$ since $f(n) > 0$ and $f(m) \leq 0$.

To see that f is surjective, let $n \in \mathbf{Z}$ be given. If $n > 0$, then $f(2n) = n$, and if $n \leq 0$ then $f(-2n+1) = n$.

Exercise 1.2.9. Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

Solution. (a) Some straightforward calculations reveal that

$$\begin{aligned} f^{-1}(A) &= [-2, 2], & f^{-1}(A \cap B) &= [-1, 1], & f^{-1}(A \cup B) &= [-2, 2], \\ f^{-1}(B) &= [-1, 1], & f^{-1}(A) \cap f^{-1}(B) &= [-1, 1], & f^{-1}(A) \cup f^{-1}(B) &= [-2, 2]. \end{aligned}$$

From this we see that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) Observe that

$$x \in g^{-1}(A \cap B) \iff g(x) \in A \cap B \iff (g(x) \in A) \text{ and } (g(x) \in B)$$
$$\iff (x \in g^{-1}(A)) \text{ and } (x \in g^{-1}(B)) \iff x \in g^{-1}(A) \cap g^{-1}(B).$$

Similarly,

$$x \in g^{-1}(A \cup B) \iff g(x) \in A \cup B \iff (g(x) \in A) \text{ or } (g(x) \in B)$$
$$\iff (x \in g^{-1}(A)) \text{ or } (x \in g^{-1}(B)) \iff x \in g^{-1}(A) \cup g^{-1}(B).$$

Exercise 1.2.10. Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution. (a) This is false; the implication

$$\text{if } a < b + \epsilon \text{ for every } \epsilon > 0, \text{ then } a < b$$

does not hold. The problem occurs when we consider the case where $a = b$. For example, we certainly have $1 < 1 + \epsilon$ for every $\epsilon > 0$ but of course $1 < 1$ is false.

(b) See part (a).

(c) This is true. The implication

$$\text{if } a \leq b, \text{ then } a < b + \epsilon \text{ for every } \epsilon > 0$$

follows since $a \leq b < b + \epsilon$ for every $\epsilon > 0$ and the implication

$$\text{if } a > b, \text{ then } a \geq b + \epsilon \text{ for some } \epsilon > 0$$

can be seen by taking $\epsilon = a - b > 0$, so that $b + \epsilon = a \leq a$.

Exercise 1.2.11. Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution. (a) The negated statement is:

there exist real numbers $a < b$ such that $a + \frac{1}{n} \geq b$ for all $n \in \mathbf{N}$.

The original statement is true and follows from the Archimedean Property (Theorem 1.4.2).

(b) The negated statement is:

for all $x > 0$, there exists an $n \in \mathbf{N}$ such that $\frac{1}{n} \leq x$.

The negated statement is true and again follows from the Archimedean Property (Theorem 1.4.2).

(c) The negated statement is:

there are two distinct real numbers with no rational number between them.

The original statement is true; this is the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3).

Exercise 1.2.12. Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution. (a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $y_n > -6$. Since $y_1 = 6$, the truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and observe that

$$y_{n+1} = \frac{2}{3}y_n - 2 > \frac{2}{3}(-6) - 2 = -6,$$

i.e., $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

(b) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $y_{n+1} \leq y_n$. Since $y_1 = 6$ and $y_2 = 2$, the truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and observe that

$$y_{n+2} = \frac{2}{3}y_{n+1} - 2 \leq \frac{2}{3}y_n - 2 = y_{n+1},$$

i.e., $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

Exercise 1.2.13. For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^{\mathsf{c}} = A_1^{\mathsf{c}} \cap A_2^{\mathsf{c}} \cap \cdots \cap A_n^{\mathsf{c}}$$

for any finite $n \in \mathbf{N}$.

(b) It is tempting to appeal to induction to conclude that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathsf{c}} = \bigcap_{i=1}^{\infty} A_i^{\mathsf{c}},$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution. (a) For $n \in \mathbf{N}$, let $P(n)$ be the statement that $(A_1 \cup \cdots \cup A_n)^{\mathsf{c}} = A_1^{\mathsf{c}} \cap \cdots \cap A_n^{\mathsf{c}}$ for any sets $A_1, \ldots, A_n$. The truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$, let $A_1, \ldots, A_n, A_{n+1}$ be given, and observe that

$$\begin{aligned}
(A_1 \cup \cdots \cup A_n \cup A_{n+1})^{\mathsf{c}} &= ((A_1 \cup \cdots \cup A_n) \cup (A_{n+1}))^{\mathsf{c}} \\
&= (A_1 \cup \cdots \cup A_n)^{\mathsf{c}} \cap A_{n+1}^{\mathsf{c}} && \text{(Exercise 1.2.5)} \\
&= A_1^{\mathsf{c}} \cap \cdots \cap A_n^{\mathsf{c}} \cap A_{n+1}^{\mathsf{c}}, && \text{(induction hypothesis)}
\end{aligned}$$

i.e., $P(n+1)$ holds. This completes the induction step and we may conclude that $P(n)$ holds for all $n \in \mathbf{N}$.

(b) Let $B_i = \{i, i+1, i+2, \ldots\}$, so that

$$B_1 = \{1,2,3,\ldots\}, \quad B_2 = \{2,3,4,\ldots\}, \quad B_3 = \{3,4,5,\ldots\}, \quad \text{etc.}$$

It is straightforward to verify that $\bigcap_{i=1}^{n} B_i = B_n \neq \emptyset$ for any $n \in \mathbf{N}$; however, as Example 1.2.2 shows, the intersection $\bigcap_{i=1}^{\infty} B_i$ is empty.

(c) Observe that

$$x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathsf{c}} \iff x \notin \bigcup_{i=1}^{\infty} A_i \iff x \notin A_i \text{ for every } i \in \mathbf{N} \iff x \in \bigcap_{i=1}^{\infty} A_i^{\mathsf{c}}.$$

It follows that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^{\mathsf{c}} = \bigcap_{i=1}^{\infty} A_i^{\mathsf{c}}.$$

1.3 The Axiom of Completeness

Exercise 1.3.1. (a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution. (a) A real number t is the *greatest lower bound* for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

(i) t is a lower bound for A;

(ii) if b is any lower bound for A, then $b \leq t$.

(b) Here is a version of Lemma 1.3.8 for greatest lower bounds.

Lemma L.2. Assume $t \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then $t = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $a < t + \epsilon$.

Proof. First, let us prove the implication

$$\text{if } t = \inf A, \text{ then for every } \epsilon > 0 \text{ there exists an } a \in A \text{ such that } a < t + \epsilon$$

by proving the contrapositive statement

$$\text{if there exists an } \epsilon > 0 \text{ such that } t + \epsilon \leq a \text{ for every } a \in A \text{ then } t \neq \inf A.$$

If such an $\epsilon > 0$ exists, then $t + \epsilon$ is a lower bound for A strictly greater than t; it follows that t is not the greatest lower bound for A, i.e., $t \neq \inf A$.

Now let us prove the converse:

$$\text{if for every } \epsilon > 0 \text{ there exists an } a \in A \text{ such that } a < t + \epsilon, \text{ then } t = \inf A.$$

Suppose $b \in \mathbf{R}$ is such that $b > t$. Taking $\epsilon = b - t > 0$, by assumption we are guaranteed the existence of an $a \in A$ such that $a < t + \epsilon = b$. Thus b is not a lower bound for A; this proves the contrapositive of criterion (ii) in part (a) and we may conclude that $t = \inf A$. $\square$

Exercise 1.3.2. Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution. (a) Take $B = \{0\}$, so that $\inf B = \sup B = 0$.

(b) This is impossible. To see this, let us first use induction to show that any non-empty finite subset of $\mathbf{R}$ contains a minimum and a maximum element.

Lemma L.3. If $E \subseteq \mathbf{R}$ is non-empty and finite, then E contains a minimum and a maximum element.

Proof. For $n \in \mathbf{N}$, let $P(n)$ be the statement that any subset of $\mathbf{R}$ containing n elements has a minimum and a maximum element. For the base case $P(1)$, simply observe that $\min\{x\} = \max\{x\} = x$ for any $x \in \mathbf{R}$.

Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and let $E \subseteq \mathbf{R}$ be a set containing $n + 1$ elements. Fix some $x \in E$ and consider the set $F = E \setminus \{x\}$, which contains n elements. Our induction hypothesis guarantees the existence of a minimum element $a := \min F$ and a maximum element $b := \max F$, which must satisfy $a \leq b$. There are then three cases; the conclusion in each case is straightforward to verify.

Case 1. If $x < a$, then $\min E = x$ and $\max E = b$.

Case 2. If $x > b$, then $\min E = a$ and $\max E = x$.

Case 3. If $a \leq x \leq b$, then $\min E = a$ and $\max E = b$.

In any case, the set E has a minimum and a maximum element, i.e., $P(n+1)$ holds. This completes the induction step and the proof. □

It is immediate from the definition of the supremum and the maximum of a set $E \subseteq \mathbf{R}$ that if $\max E$ exists then $\sup E = \max E$ (see Exercise 1.3.7); similarly, if $\min E$ exists then $\inf E = \min E$. It follows that the given request is impossible: if $E \subseteq \mathbf{R}$ is finite, then Lemma L.3 implies that $\min E = \inf E$ and $\max E = \sup E$ both exist and hence E contains both its infimum and its supremum.

(c) Consider the bounded set $E = \{p \in \mathbf{Q} : 0 < p \leq 1\}$, which satisfies $\sup E = 1 \in E$ and $\inf E = 0 \notin E$.

Exercise 1.3.3. (a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution. (a) B is non-empty since A is bounded below, and B is bounded above by any $x \in A$; there exists at least one such x since A is non-empty. It follows from the Axiom of Completeness that $\sup B$ exists. To see that $\sup B = \inf A$, we need to show that $\sup B$ satisfies criteria (i) and (ii) from Exercise 1.3.1 (a).

(i) First we need to prove that $\sup B$ is a lower bound for A, i.e., if $x \in A$, then $\sup B \leq x$. We will prove the contrapositive statement: if $x < \sup B$, then $x \notin A$. If x is strictly less than $\sup B$, then x cannot be an upper bound for B. Thus there exists some $b \in B$ such that $x < b$. Since b is a lower bound for A, it follows that $x \notin A$.

(ii) Suppose $y \in \mathbf{R}$ is a lower bound of A, so that y belongs to B; it follows that $y \leq \sup B$.

We may conclude that $\sup B = \inf A$.

(b) Part (a) shows that the existence of the greatest lower bound for non-empty bounded below subsets of $\mathbf{R}$ is implied by the Axiom of Completeness; adding this existence as part of the Axiom of Completeness would be redundant.

Exercise 1.3.4. Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^n A_k)$.

(b) Consider $\sup(\bigcup_{k=1}^\infty A_k)$. Does the formula in (a) extend to the infinite case?

Solution. (a) Let $n \in \mathbf{N}$ be given. For each $k \in \{1, \ldots, n\}$, the Axiom of Completeness guarantees that $\sup A_k$ exists. By Lemma L.3, the finite set $\{\sup A_1, \ldots, \sup A_k\}$ has a maximum element, say M; we claim that $\sup(\bigcup_{k=1}^n A_k) = M$. To prove this, we must verify criteria (i) and (ii) from Definition 1.3.2.

(i) If $x \in \bigcup_{k=1}^n A_k$, then $x \in A_k$ for some $k \in \{1, \ldots, n\}$; it follows that $x \leq \sup A_k \leq M$. Since x was arbitrary, we see that M is an upper bound for $\bigcup_{k=1}^n A_k$.

(ii) If $b \in \mathbf{R}$ is an upper bound for $\bigcup_{k=1}^n A_k$, then b must be an upper bound for each A_k. It follows that $\sup A_k \leq b$ for each $k \in \{1, \ldots, n\}$ and hence that $M \leq b$.

We may conclude that $\sup(\bigcup_{k=1}^n A_k) = M$.

(b) The proof given above does not extend to the infinite case, since in general the set $\{\sup A_1, \sup A_2, \ldots\}$ need not have a maximum. Indeed, it may be the case that $\sup(\bigcup_{k=1}^\infty A_k)$ does not exist. For example, take $A_k = [0, k]$. Then each A_k is non-empty and bounded above with $\sup A_k = k$, but $\bigcup_{k=1}^\infty A_k = [0, \infty)$, which does not have a supremum in $\mathbf{R}$.

Exercise 1.3.5. As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution. (a) If $c = 0$ then the result is clear, so suppose that $c > 0$. For any $x \in A$, notice that

$$x \leq \sup A \iff cx \leq c \sup A.$$

This demonstrates that $c \sup A$ is an upper bound for cA.

Suppose $b \in \mathbf{R}$ is an upper bound for cA, i.e., $cx \leq b$ for all $x \in A$. Then $x \leq c^{-1}b$ for all $x \in A$, i.e., $c^{-1}b$ is an upper bound for A. It follows that $\sup A \leq c^{-1}b$ and hence that $c \sup A \leq b$. We may conclude that $\sup(cA) = c \sup A$.

(b) If $c < 0$ and $\inf A$ exists then $\sup(cA) = c \inf A$. The proof is similar to part (a). For any $x \in A$, we have

$$\inf A \leq x \iff cx \leq c \inf A,$$

so that $c \inf A$ is an upper bound for cA.

Suppose $b \in \mathbf{R}$ is an upper bound for cA, i.e., $cx \leq b$ for all $x \in A$. Then $c^{-1}b \leq x$ for all $x \in A$, so that $c^{-1}b$ is a lower bound for A. It follows that $c^{-1}b \leq \inf A$ and hence that $c \inf A \leq b$. We may conclude that $\sup(cA) = c \inf A$.

If $\inf A$ doesn't exist then $\sup(cA)$ doesn't exist either, since for $c < 0$ the set A is bounded below if and only if cA is bounded above. For example, $A = (-\infty, 0)$ and $c = -1$ gives $cA = (0, \infty)$.

Exercise 1.3.6. Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A+B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution. (a) For any $a \in A$ and $b \in B$ we have $a \leq s$ and $b \leq t$. It follows that $a + b \leq s + t$ and thus $s + t$ is an upper bound for $A + B$.

(b) For any $b \in B$ we have $a + b \leq u$, which gives $b \leq u - a$. This demonstrates that $u - a$ is an upper bound for B and so it follows that $t \leq u - a$.

(c) Part (b) implies that for any $a \in A$ we have $t \leq u - a$, which gives $a \leq u - t$. This shows that $u - t$ is an upper bound for A and it follows that $s \leq u - t$, i.e., $s + t \leq u$. Since u was an arbitrary upper bound for $A + B$, we may conclude that

$$\sup(A + B) = s + t = \sup A + \sup B.$$

(d) Let $\epsilon > 0$ be given. By Lemma 1.3.8, there exist elements $a \in A$ and $b \in B$ such that $s - \frac{\epsilon}{2} < a$ and $t - \frac{\epsilon}{2} < b$, which implies that $s + t - \epsilon < a + b$. We showed in part (a) that $s + t$ is an upper bound for $A + B$, so we may invoke Lemma 1.3.8 to conclude that $\sup(A + B) = \sup A + \sup B$.

Exercise 1.3.7. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution. Let $b \in \mathbf{R}$ be an upper bound of A. Since $a \in A$, we must have $a \leq b$; it follows that $a = \sup A$.

Exercise 1.3.8. Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.

(c) $\{n/(3n+1) : n \in \mathbf{N}\}$.

(d) $\{m/(m+n) : m, n \in \mathbf{N}\}$.

Solution. (a) The supremum is 1 and the infimum is 0.

(b) The supremum is 1 and the infimum is -1.

(c) The supremum is $\frac{1}{3}$ and the infimum is $\frac{1}{4}$.

(d) The supremum is 1 and the infimum is 0.

Exercise 1.3.9. (a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Solution. (a) Let $\epsilon = \sup B - \sup A > 0$. By Lemma 1.3.8, there exists a $b \in B$ such that $\sup B - \epsilon = \sup A < b$. It follows that b is an upper bound for A.

(b) Take $A = B = (0, 1)$. Then $\sup A = \sup B = 1$, but no element of B is an upper bound for A (the interval $(0, 1)$ has no maximum element).

Exercise 1.3.10 (Cut Property). The *Cut Property* of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution. (a) Suppose that A and B are non-empty disjoint subsets of $\mathbf{R}$ such that $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$. Notice that A is non-empty (by assumption) and bounded above (because B is non-empty); the Axiom of Completeness then implies that $c := \sup A$ exists. It follows that $x \leq c$ for all $x \in A$ and, since each element of B is an upper bound for A, we also have $x \geq c$ for all $x \in B$.

(b) Suppose that $E \subseteq \mathbf{R}$ is non-empty and bounded above. Define

$$A = \{a \in \mathbf{R} : a \text{ is not an upper bound of } E\},$$

$$B = A^{\mathsf{c}} = \{b \in \mathbf{R} : b \text{ is an upper bound of } E\}.$$

Notice that B is non-empty as E is bounded above and A is non-empty because $x - 1 \in A$ for any $x \in E$; we are guaranteed the existence of at least one $x \in E$ as E is non-empty. Furthermore, A and B are evidently disjoint and satisfy $A \cup B = \mathbf{R}$.

Let $a \in A$ and $b \in B$ be given. Since a is not an upper bound for E there exists some $x \in E$ such that $a < x$ and since b is an upper bound for E, we must then have $x \leq b$; it follows that $a < b$. We may now invoke the Cut Property to obtain a $c \in \mathbf{R}$ such that $x \leq c$ for all $x \in A$ and $x \geq c$ for all $x \in B$.

We claim that $c = \sup E$. Since $A \cup B = \mathbf{R}$ and $A \cap B = \emptyset$, exactly one of $c \in A$ or $c \in B$ holds. Suppose that $c \in A$, i.e., c is not an upper bound of E, which is the case if and only if there is some $z \in E$ such that $c < z$. Observe that $y := \frac{c+z}{2}$ satisfies $c < y < z$, so that $y \in A$; but this contradicts the fact that $x \leq c$ for all $x \in A$.

So it must be the case that $c \in B$, i.e., c is an upper bound for E. The Cut Property says that $c \leq x$ for all $x \in B$; in other words, c is less than all other upper bounds of E. We may conclude that $c = \sup E$.

(c) A concrete example is given in the following lemma, which will also be useful for later exercises.

Lemma L.4. Let

$$A = \{p \in \mathbf{Q} : p < 0 \text{ or } p^2 < 2\} \quad \text{and} \quad B = \{p \in \mathbf{Q} : p > 0 \text{ and } p^2 > 2\}.$$

Then:

(i) A and B are non-empty, $A \cup B = \mathbf{Q}$, and $A \cap B = \emptyset$.

(ii) $p < q$ for all $p \in A$ and $q \in B$.

(iii) A has no maximum element and B has no minimum element.

Proof. (i) It is clear that A and B are non-empty. The negation of the statement "$p < 0$ or $p^2 < 2$" is "$p > 0$ and $p^2 \geq 2$"; by Theorem 1.1.1, this negated statement is equivalent to "$p > 0$ and $p^2 > 2$" for $p \in \mathbf{Q}$. Thus $B = \mathbf{Q} \setminus A$, from which it follows that $A \cup B = \mathbf{Q}$ and $A \cap B = \emptyset$.

(ii) Let $p \in A$ and $q \in B$ be given. If $p \leq 0$ then evidently $p < q$, so suppose that $p > 0$. It must then be the case that $p^2 < 2$, whence $p^2 < q^2$. Since p and q are positive, this implies that $p < q$.

(iii) Let $p \in A$ be given. We need to show that there exists some $q \in A$ such that $p < q$. If $p \leq 0$, we can take $q = 1$; if $p > 0$, so that $p^2 < 2$, then define

$$q = p + \frac{2 - p^2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{1}$$

Notice that $0 < \frac{2-p^2}{p+2}$, since $p^2 < 2$, from which it follows that $p < q$. A straightforward calculation yields

$$2 - q^2 = \frac{2(2 - p^2)}{(p + 2)^2};$$

using again that $p^2 < 2$, we see that $2 - q^2 > 0$ and thus $q \in A$.

Now let $p \in B$ be given. We need to show that there exists some $q \in B$ such that $q < p$. In fact, we can define q by equation (1) again; an argument similar to the one just given shows that $q < p$ and $q \in B$. □

Parts (i) and (ii) of Lemma L.4 show that the sets A and B satisfy the hypotheses of the Cut Property. If the Cut Property held for $\mathbf{Q}$, then we would be able to obtain a $c \in \mathbf{Q}$

such that $p \le c$ for all $p \in A$ and $c \le q$ for all $q \in B$. Since $A \cup B = \mathbf{Q}$ and $A \cap B = \emptyset$, this implies that c is either the maximum of A or the minimum of B—but this contradicts part (iii) of Lemma L.4. We may conclude that the Cut Property does not hold for $\mathbf{Q}$.

Exercise 1.3.11. Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \le \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution. (a) This is true. The Axiom of Completeness guarantees that $\sup A$ and $\sup B$ both exist. Furthermore, since each element of A is an element of B, any upper bound of B must be an upper bound of A also. In particular, $\sup B$ must be an upper bound of A; it follows that $\sup A \le \sup B$.

(b) This is true. Let $c = \frac{\sup A + \inf B}{2}$, so that $\sup A < c < \inf B$, and notice that for any $a \in A$ and $b \in B$ we have

$$a \le \sup A < c < \inf B \le b.$$

(c) This is false. Consider $A = (-1, 0)$ and $B = (0, 1)$, and notice that $c = 0$ satisfies $a < c < b$ for all $a \in A$ and $b \in B$, but $\sup A = \inf B = 0$.

1.4 Consequences of Completeness

Exercise 1.4.1. Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a + b$ are elements of $\mathbf{Q}$ as well.

(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a + t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Solution. (a) Suppose $a = \frac{m}{n}$ and $b = \frac{p}{q}$. Then

$$ab = \frac{mp}{nq} \qquad \text{and} \qquad a + b = \frac{mq + np}{nq},$$

which are rational numbers.

(b) Let $a \in \mathbf{Q}$ be fixed. We want to prove that

$$\text{if } t \in \mathbf{I}, \text{ then } a + t \in \mathbf{I}.$$

To do this, we will prove the contrapositive statement

$$\text{if } a + t \in \mathbf{Q}, \text{ then } t \in \mathbf{Q}.$$

Simply observe that $t = (a + t) - a$; it follows from part (a) that $t \in \mathbf{Q}$.

Similarly, let $a \in \mathbf{Q}$ be non-zero. We can show that

$$\text{if } at \in \mathbf{Q}, \text{ then } t \in \mathbf{Q}$$

by observing that $t = a^{-1}(at)$ and appealing to part (a) to conclude that $t \in \mathbf{Q}$.

(c) $\mathbf{I}$ is not closed under addition or multiplication. For example, $-\sqrt{2}$ and $\sqrt{2}$ are irrational numbers, but their sum is the rational number 0 and their product is the rational number -2. The sum or product of two irrational numbers may be irrational; for example, it can be shown that $\sqrt{2} + \sqrt{3}$ and $\sqrt{2}\sqrt{3} = \sqrt{6}$ are irrational:

- For the irrationality of $\sqrt{6}$, see Exercise 1.2.1 (a).
- For the irrationality of $\sqrt{2} + \sqrt{3}$, observe that $\sqrt{2} + \sqrt{3}$ is a root of the polynomial $x^4 - 10x^2 + 1$; the rational root theorem says that the only possible rational roots of this polynomial are ± 1—but neither of these solve the equation $x^4 - 10x^2 + 1 = 0$.

So in general, we cannot say anything about the sum or product of two irrational numbers without more information.

Exercise 1.4.2. Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution. If s is not an upper bound for A then there must exist some $x \in A$ such that $s < x$. By the Archimedean Property (Theorem 1.4.2), there exists a natural number n such that $s + \frac{1}{n} < x$, which implies that $s + \frac{1}{n}$ is not an upper bound for A. Given our hypothesis that $s + \frac{1}{n}$ is an upper bound for A for all $n \in \mathbf{N}$, we see that s must be an upper bound for A.

Now let $\epsilon > 0$ be given and using the Archimedean Property (Theorem 1.4.2), pick a natural number n such that $\frac{1}{n} < \epsilon$. By assumption $s - \frac{1}{n}$ is not an upper bound for A, so there must exist some $x \in A$ such that $s - \frac{1}{n} < x$; this implies that $s - \epsilon < x$ since $\frac{1}{n} < \epsilon$. Because $\epsilon > 0$ was arbitrary, we may invoke Lemma 1.3.8 to conclude that $s = \sup A$.

Exercise 1.4.3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution. It is clear that any $x \leq 0$ does not belong to $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$. Let $x > 0$ be given and use the Archimedean Property (Theorem 1.4.2) to choose an $N \in \mathbf{N}$ such that $\frac{1}{N} < x$. It follows that $x \notin \left(0, \frac{1}{N}\right)$ and hence that $x \notin \bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$. We may conclude that $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right) = \emptyset$.

Exercise 1.4.4. Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$.

Solution. It is clear that b is an upper bound for T. Let $\epsilon > 0$ be given. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), there exists a rational number p satisfying

$$\max\{a, b - \epsilon\} < p < b.$$

It follows that $p \in T$ and $b - \epsilon < p$ and hence, by Lemma 1.3.8, we may conclude that $\sup T = b$.

Exercise 1.4.5. Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), there exists a rational number p satisfying $a - \sqrt{2} < p < b - \sqrt{2}$, which gives $a < p + \sqrt{2} < b$. Since $p + \sqrt{2}$ is irrational (Exercise 1.4.1 (b)), the corollary is proved.

Exercise 1.4.6. Recall that a set B is *dense* in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2.

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution. (a) This set is not dense in $\mathbf{R}$. For $1 \leq q \leq 10$, observe that if $p \geq 1$ then $\frac{p}{q} \geq \frac{1}{10}$, if $p \leq -1$ then $\frac{p}{q} \leq -\frac{1}{10}$, and if $p = 0$ then $\frac{p}{q} = 0$. So there is no element of this set between the real numbers $\frac{1}{1000}$ and $\frac{1}{100}$, for example.

(b) This set is dense in $\mathbf{R}$. Let $a < b$ be given real numbers. Using the Archimedean Property (Theorem 1.4.2), let $n \in \mathbf{N}$ be such that $\frac{1}{n} < b - a$, which implies that $\frac{1}{2^n} < b - a$. Now let p be the smallest integer greater than $2^n a$, so that $p - 1 \leq 2^n a < p$, and observe that

$$2^n a < p \leq 1 + 2^n a < 2^n b;$$

it follows that $\frac{p}{2^n}$ lies between a and b.

(c) This set is not dense in $\mathbf{R}$. If $p > 0$ then

$$10|p| \geq q \iff 10p \geq q \iff \frac{p}{q} \geq \frac{1}{10},$$

and if $p < 0$ then

$$10|p| \geq q \iff -10p \geq q \iff \frac{p}{q} \leq -\frac{1}{10}.$$

We cannot have $p = 0$ since q is a positive integer. Thus there is no element of this set between the real numbers 0 and $\frac{1}{100}$, for example.

Exercise 1.4.7. Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Solution. Assuming that $\alpha^2 - 2 > 0$, the Archimedean Property (Theorem 1.4.2) implies that there is an $n \in \mathbf{N}$ such that

$$\frac{2\alpha}{n} < \alpha^2 - 2 \iff 2 < \alpha^2 - \frac{2\alpha}{n}.$$

Let $\beta = \alpha - \frac{1}{n}$ and note that since $1 \in T$ we have $\alpha \geq 1$ and hence $\beta \geq 0$; it follows that $t \leq \beta$ for all $t \in T$ such that $t < 0$. Now observe that

$$\beta^2 = \left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n} > 2,$$

so that for any $t \in T$ we have $t^2 < 2 < \beta^2$. If $t \in T$ is such that $t \geq 0$ then the inequality $t^2 < \beta^2$ implies that $t < \beta$, as β is also non-negative.

We have now shown that $t \leq \beta$ for all $t \in T$, i.e., β is an upper bound for T—but this contradicts the fact that α is the supremum of T since $\beta < \alpha$.

Exercise 1.4.8. Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbf{R} : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution. (a) Let

$$A = \left\{-\frac{1}{2n} : n \in \mathbf{N}\right\} = \left\{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots\right\}$$

$$\text{and} \qquad B = \left\{-\frac{1}{2n-1} : n \in \mathbf{N}\right\} = \left\{-1, -\frac{1}{3}, -\frac{1}{5}, \ldots\right\}.$$

Then $A \cap B = \emptyset$ and $\sup A = \sup B = 0$, which belongs to neither A nor B.

(b) If we let $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbf{N}$, then $\bigcap_{n=1}^{\infty} J_n = \{0\}$.

(c) For $n \in \mathbf{N}$, let $L_n = [n, \infty)$.

(d) This is impossible. To see this, let $(I_n)_{n=1}^{\infty}$ be a sequence of closed bounded intervals satisfying $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for every $N \in \mathbf{N}$. Define $J_N = \bigcap_{n=1}^{N} I_n$ for $N \in \mathbf{N}$ and note that any finite intersection of closed bounded intervals is a (possibly empty) closed bounded interval. Thus:

- each J_N is a closed bounded interval;
- these intervals are non-empty and nested, i.e., $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$;

- $\bigcap_{n=1}^{\infty} I_n = \bigcap_{N=1}^{\infty} J_N$.

It then follows from the Nested Interval Property (Theorem 1.4.1) that $\bigcap_{n=1}^{\infty} I_n = \bigcap_{N=1}^{\infty} J_N$ is non-empty.

1.5 Cardinality

Exercise 1.5.1. Finish the following proof for Theorem 1.5.7.

Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$ which is 1-1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbf{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbf{N}$ onto A.

Solution. Given $n_1 = \min f^{-1}(A) = \min\{n \in \mathbf{N} : f(n) \in A\}$, we can construct a sequence $(n_k)_{k=1}^{\infty}$ of natural numbers recursively by defining

$$n_k = \min\big(f^{-1}(A) \setminus \{n_1, \dots, n_{k-1}\}\big) = \min(\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, \dots, n_{k-1}\})$$

for $k \geq 2$. Because A is infinite and f is surjective, the set $\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, \dots, n_{k-1}\}$ is non-empty (indeed, it must be infinite) for each $k \geq 2$; it follows that each n_k is well-defined. See Figure F.2 for an example construction of the sequence $(n_k)_{k=1}^{\infty}$, for some bijection $f : \mathbf{N} \to B$.

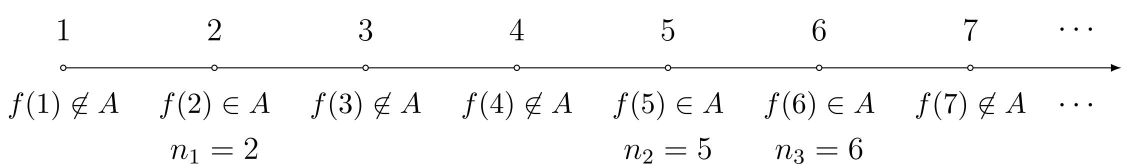


Figure F.2: Example construction of $(n_k)_{k=1}^{\infty}$

It is clear from this construction that $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence.

Define $g : \mathbf{N} \to A$ by $g(k) = f(n_k)$; we claim that g is a bijection. For injectivity, observe that

$$g(\ell) = g(k) \iff f(n_\ell) = f(n_k) \iff n_\ell = n_k \iff \ell = k,$$

where we have used the injectivity of f for the second equivalence and the strict monotonicity of the sequence $(n_k)_{k=1}^{\infty}$ for the third equivalence.

For the surjectivity of g, let $a \in A$ be given. Since f is surjective, there is a positive integer N such that $f(N) = a$; we need to find some $k \in \mathbf{N}$ such that $n_k = N$. It cannot be the case that $N < n_1$, otherwise n_1 would not be the minimum of $\{n \in \mathbf{N} : f(n) \in A\}$, so we must have $n_1 \leq N$. Given this, and the fact that $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers, there must exist a $k \in \mathbf{N}$ such that $n_k \leq N < n_{k+1}$. In fact, it must be the case that $n_k = N$, otherwise n_{k+1} would not be the minimum of $\{n \in \mathbf{N} : f(n) \in A\} \setminus \{n_1, \ldots, n_k\}$. Thus $g(k) = f(n_k) = f(N) = a$.

Exercise 1.5.2. Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbf{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbf{Q}$ is uncountable:

Assume, for contradiction, that $\mathbf{Q}$ is countable. Thus we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbf{Q}$ must therefore be uncountable.

Solution. The construction does not imply that $\bigcap_{n=1}^{\infty} I_n = \emptyset$; it only guarantees that this intersection does not contain any rational numbers.

Exercise 1.5.3. Use the following outline to supply proofs for the statements in Theorem 1.5.8.

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Now, explain how the more general statement in (i) follows.

(b) Explain why induction *cannot* be used to prove part (ii) of Theorem 1.5.8 from part (i).

(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof of Theorem 1.5.8 (ii).

Solution. (a) As noted, it will suffice to show that $A_1 \cup B_2$ is countable, where $B_2 = A_2 \setminus A_1$. Since A_1 is countable, there exists a bijection $f : \mathbf{N} \to A_1$. Consider the following cases.

Case 1. If B_2 is empty, then $A_1 \cup B_2 = A_1$, which is countable by assumption.

Case 2. Suppose that B_2 is non-empty and finite, say $B_2 = \{x_1, \ldots, x_k\}$ for some $k \in \mathbf{N}$. Define $g : \mathbf{N} \to A_1 \cup B_2$ by

$$g(n) = \begin{cases} x_n & \text{if } 1 \leq n \leq k, \\ f(n-k) & \text{if } k < n. \end{cases}$$

$$\begin{array}{cccc}
\begin{array}{c} \mathbf{N} \\ {\scriptstyle g}\big\downarrow \\ A_1 \cup B_2 \end{array} &
\underbrace{\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ x_1 & x_2 & x_3 \end{array}}_{B_2} &
\underbrace{\begin{array}{ccccc} 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ f(1) & f(2) & f(3) & f(4) & f(5) \end{array}}_{A_1} &
\begin{array}{c} \cdots \\ \\ \cdots \end{array}
\end{array}$$

Figure F.3: Example construction of g with $B_2 = \{x_1, x_2, x_3\}$

The injectivity of g follows as A_1 and B_2 are disjoint and f is injective. For the surjectivity of g, it is clear that every element of B_2 belongs to the range of g; the surjectivity of f implies that the elements of A_1 belong to the range of g also.

Case 3. Suppose that B_2 is infinite. Since B_2 is a subset of the countable set A_2, Exercise 1.5.1 implies that B_2 is countable, i.e., there exists a bijection $h : \mathbf{N} \to B_2$. Define $g : \mathbf{N} \to A_1 \cup B_2$ by

$$g(n) = \begin{cases} f\left(\frac{n}{2}\right) & \text{if } n \text{ is even}, \\ h\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd}. \end{cases}$$

$$\begin{array}{ccccccc}
\mathbf{N} & 1 & 2 & 3 & 4 & 5 & \cdots \\
{\scriptstyle g}\big\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
A_1 \cup B_2 & h(1) & f(1) & h(2) & f(2) & h(3) & \cdots
\end{array}$$

To see that g is injective, suppose that m and n are distinct positive integers.

Case 3.1 If both of m and n are even then $g(m) \neq g(n)$ since f is injective.

Case 3.2 If both of m and n are odd then $g(m) \neq g(n)$ since h is injective.

Case 3.3 If one of m and n is even and the other is odd then $g(m) \neq g(n)$ since f maps into A_1, h maps into B_2, and $A_1 \cap B_2 = \emptyset$.

To see that g is surjective, let $x \in A_1 \cup B_2$ be given. Since $A_1 \cap B_2 = \emptyset$, exactly one of the statements $x \in A_1$ or $x \in B_2$ holds. Suppose $x \in A_1$. Because f is surjective, there is a positive integer n such that $f(n) = x$; it follows that $g(2n) = f(n) = x$. If $x \in B_2$, then the surjectivity of h implies that there is a positive integer n such that $h(n) = x$; it follows that $g(2n-1) = h(n) = x$. We may conclude that g is a bijection and hence that $A_1 \cup B_2$ is countable.

A simple induction argument proves the more general statement in Theorem 1.5.8 (i). Let $P(n)$ be the statement that for countable sets $A_1, \ldots, A_n$, the union $A_1 \cup \cdots \cup A_n$ is countable. The truth of $P(1)$ is clear. Suppose that $P(n)$ holds for some $n \in \mathbf{N}$ and suppose we have countable sets $A_1, \ldots, A_n, A_{n+1}$. Let $A' = A_1 \cup \cdots \cup A_n$; the induction hypothesis guarantees that A' is countable. Observe that

$$A_1 \cup \cdots \cup A_n \cup A_{n+1} = A' \cup A_{n+1}.$$

Since A' and A_{n+1} are countable, the union $A' \cup A_{n+1}$ is also countable by our previous proof, i.e., $P(n+1)$ holds. This completes the induction step and the proof.

(b) Induction can only be used to show that a particular statement $P(n)$ holds for each value of $n \in \mathbf{N}$.

(c) For each $n \in \mathbf{N}$ there exists a bijection $f_n : \mathbf{N} \to A_n$. Let $a_{mn} = f_n(m)$ and arrange these into another two-dimensional array like so:

$$\begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & A_5 & \cdots \\ \hline a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \ddots & \\ a_{31} & a_{32} & a_{33} & \ddots & & \\ a_{41} & a_{42} & \ddots & & & \\ a_{51} & \ddots & & & & \\ \vdots & & & & & \end{array} \qquad \begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \ddots & \\ 4 & 8 & 13 & \ddots & & \\ 7 & 12 & \ddots & & & \\ 11 & \ddots & & & & \\ \vdots & & & & & \end{array}$$

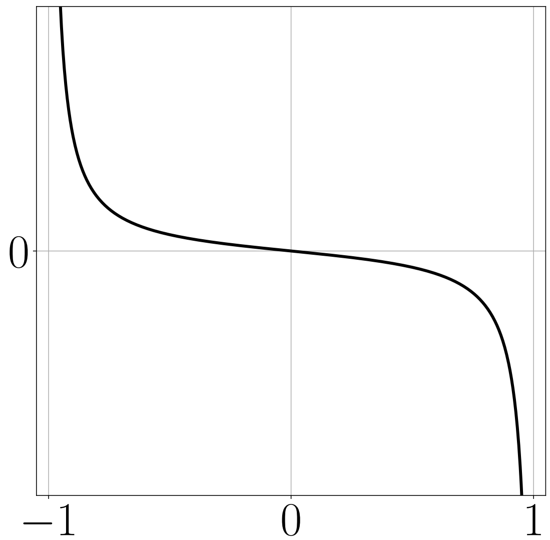
Since each f_n is surjective, each element of $\bigcup_{n=1}^{\infty} A_n$ appears somewhere in the left array. We define a function $g : \bigcup_{n=1}^{\infty} A_n \to \mathbf{N}$ by working through the grid along the diagonals (first a_{11}, then a_{21}, then a_{12}, then a_{31}, and so on), mapping an element a_{mn} to the natural number appearing in the corresponding position in the right array. The A_n's may have elements in common; if we encounter an element a_{mn} that we have already seen before, we simply skip this element and move on to the next one. In this way, we obtain an injective function g. If we denote the range of g by $B \subseteq \mathbf{N}$, then $g : \bigcup_{n=1}^{\infty} A_n \to B$ is a bijection. Since the infinite set A_1 is contained in the union $\bigcup_{n=1}^{\infty} A_n$ and g is injective, it must be the case that B is infinite; Exercise 1.5.1 then implies that B is countable, i.e., there is a bijection $h : \mathbf{N} \to B$. It follows that the function $g^{-1} \circ h : \mathbf{N} \to \bigcup_{n=1}^{\infty} A_n$ is a bijection and we may conclude that $\bigcup_{n=1}^{\infty} A_n$ is countable.

Exercise 1.5.4. (a) Show $(a, b) \sim \mathbf{R}$ for any interval (a, b).

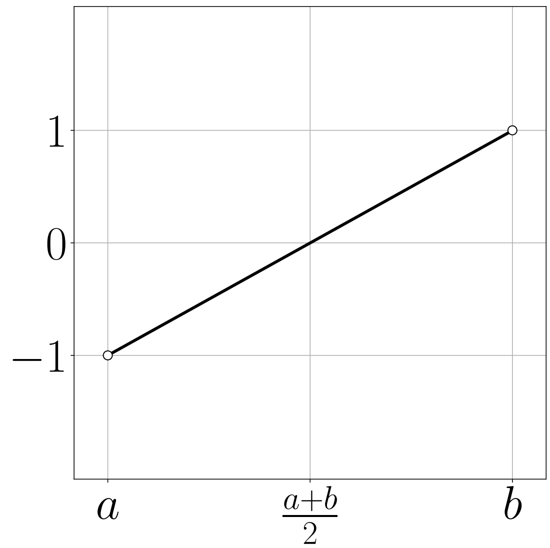
(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbf{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Solution. (a) Let $f : (-1, 1) \to \mathbf{R}$ be the bijection given by $f(x) = \frac{x}{x^2-1}$ (see Example 1.5.4 and Figure F.4) and let $g : (a, b) \to (-1, 1)$ be given by $g(x) = \frac{2(x-a)}{b-a} - 1$ (see Figure F.4); it is straightforward to verify that g is a bijection. Thus $(a, b) \sim (-1, 1) \sim \mathbf{R}$ and it follows that $(a, b) \sim \mathbf{R}$ (Exercise 1.5.5).



(a) $f(x) = \frac{x}{x^2-1}$ on $(-1,1)$



(b) $g(x) = \frac{2(x-a)}{b-a} - 1$ on (a,b)

Figure F.4: Bijections $f : (-1,1) \to \mathbf{R}$ and $g : (a,b) \to (-1,1)$

(b) Let $f : (a,\infty) \to (0,1)$ be the bijection given by $f(x) = \frac{1}{x+1-a}$ (see Figure F.5). Thus $(a,\infty) \sim (0,1)$ and, by part (a), $(0,1) \sim \mathbf{R}$; it follows from Exercise 1.5.5 that $(a,\infty) \sim \mathbf{R}$.

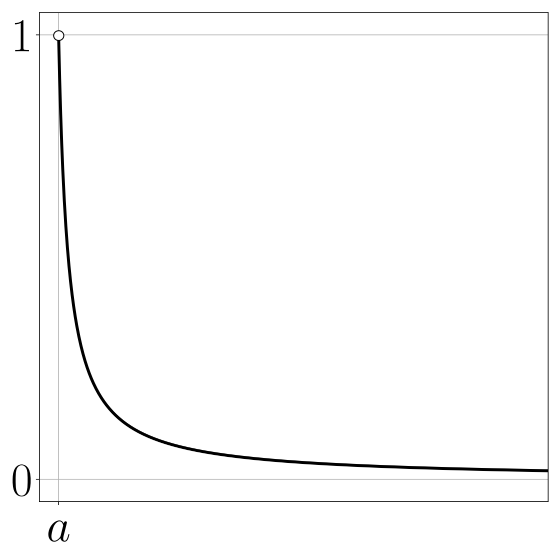


Figure F.5: Bijection $f : (a,\infty) \to (0,1)$ given by $f(x) = \frac{1}{x+1-a}$

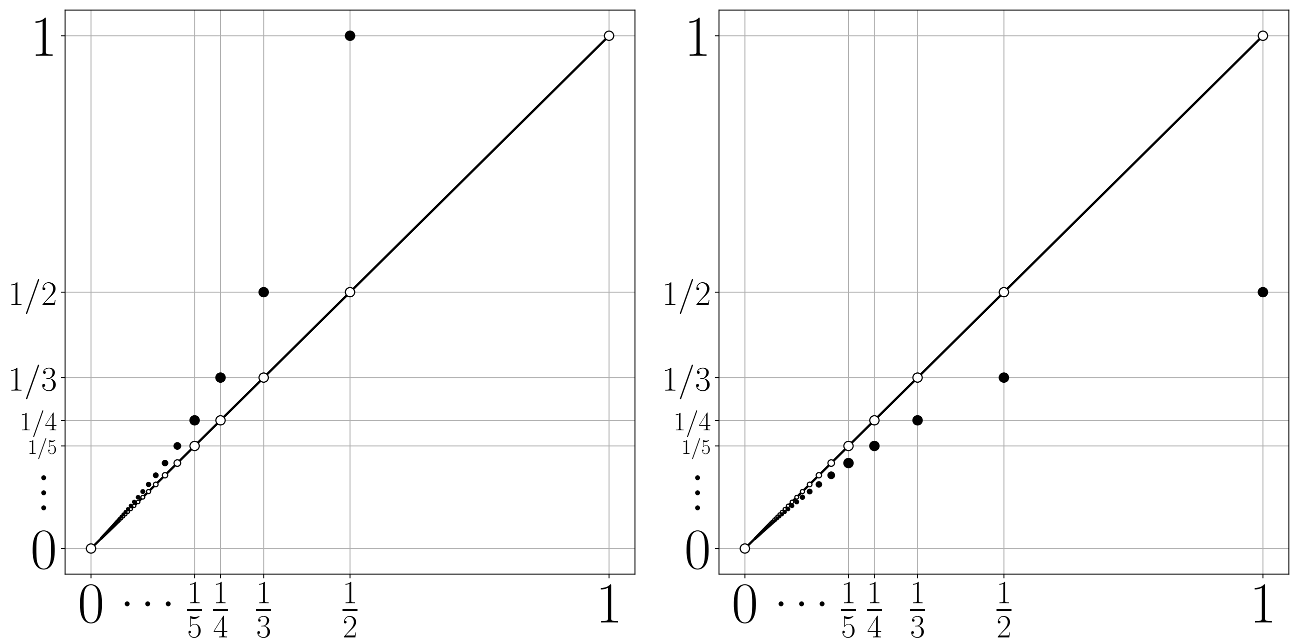
(c) It is clear that $[0,1) \sim (0,1]$ via the map $x \mapsto 1-x$ and so, by Exercise 1.5.5, it will suffice to show that $(0,1) \sim (0,1]$. Define a function $f : (0,1) \to (0,1]$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n+1} \text{ for some } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

This function is a bijection since it has an inverse $f^{-1} : (0,1] \to (0,1)$ given by

$$f^{-1}(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

See Figure F.6 for a graph of f and f^{-1}.



(a) f on $(0,1)$ (b) f^{-1} on $(0,1]$

Figure F.6: Bijections $f : (0,1) \to (0,1]$ and $f^{-1} : (0,1] \to (0,1)$

Exercise 1.5.5. (a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

Solution. (a) The identity function $f : A \to A$ given by $f(x) = x$ is a bijection.

(b) Since $A \sim B$, there is a bijection $f : A \to B$. A function is bijective if and only if it has an inverse function $f^{-1} : B \to A$, which must also be bijective.

(c) There are bijections $f : A \to B$ and $g : B \to C$. It follows that the composition $g \circ f : A \to C$ is also a bijection.

Exercise 1.5.6. (a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Solution. (a) Take $A_n = (n, n+1)$ for $n \in \mathbf{N}$.

(b) No such collection exists. To see this, suppose there was such a collection $\{I_a : a \in A\}$ for some uncountable set A. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exists a rational number $r_a \in I_a$ for each $a \in A$. Since the intervals are disjoint, each r_a must be distinct and hence the collection $\{r_a : a \in A\}$ must be an uncountable subset of $\mathbf{Q}$—but this contradicts Exercise 1.5.1.

Exercise 1.5.7. Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to produce a 1-1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999 . . . .)

The Schröder-Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

Solution. (a) Take $f : (0, 1) \to S$ given by $f(x) = \left(x, \frac{1}{2}\right)$.

(b) For $(x, y) \in S$, suppose x has decimal representation $0.x_1x_2x_3\ldots$ and y has decimal representation $0.y_1y_2y_3\ldots$, where if necessary we choose the decimal representation terminating in 0's. To define $g : S \to (0, 1)$, let $g(x, y) = 0.x_1y_1x_2y_2x_3y_3\ldots$

For the injectivity of g, suppose we have $(x, y) \neq (a, b)$ in S, so that at least one of $x \neq a$ or $y \neq b$ holds. Assuming $x \neq a$ (the case where $y \neq b$ is handled similarly), let $0.x_1x_2x_3\ldots$ be the decimal representation of x and let $0.a_1a_2a_3\ldots$ be the decimal representation of a. Since $x \neq a$, there must be some index n such that $x_n \neq a_n$. If $g(x, y)$ has decimal representation $0.s_1s_2s_3\ldots$ and $g(a, b)$ has decimal representation $0.t_1t_2t_3\ldots$, then

$$s_{2n-1} = x_n \neq a_n = t_{2n-1}.$$

This implies that $g(x, y) \neq g(a, b)$, provided it is not the case that $g(x, y)$ terminates in 0's and $g(a, b)$ terminates in 9's, or vice versa. To rule this out, note that $g(a, b)$ terminates in 9's only if both a and b terminate in 9's—but our construction specifically chooses the decimal representations for a and b terminating in 0's if necessary. The case where $g(x, y)$ terminates in 9's is handled similarly.

This function g is not surjective since 0.1 does not belong to the range of g. Indeed,

$$g(x, y) = 0.x_1y_1x_2y_2\ldots = 0.1000\ldots$$

implies that $y = 0$, but $(x, 0) \notin S$ for any $x \in (0, 1)$.

Exercise 1.5.8. Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

Solution. Suppose $a \in (0, 1]$; we claim that $B \cap (a, 2]$ must be a (possibly empty) finite set. By the Archimedean Property (Theorem 1.4.2), there is an $n \in \mathbf{N}$ such that $na > 2$. If $B \cap (a, 2]$ contains at least n elements, say $\{b_1, \ldots, b_n\}$, then since each $b_i > a$ we have

$$b_1 + \cdots + b_n > na > 2.$$

This contradicts our hypotheses, so it must be the case that $B \cap (a, 2]$ contains less than n elements and our claim follows.

Any element of B must be less than or equal to 2, so $B \subseteq (0, 2]$ and it follows that

$$B = \bigcup_{n=1}^{\infty} \big(B \cap \big(\tfrac{1}{n}, 2\big]\big).$$

This expresses B as a countable union of finite sets and thus B is either finite or countable (Theorem 1.5.8).

Exercise 1.5.9. A real number $x \in \mathbf{R}$ is called *algebraic* if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbf{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbf{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution. (a) $\sqrt{2}$ is a root of the polynomial $x^2 - 2$, $\sqrt[3]{2}$ is a root of the polynomial $x^3 - 2$, and $\sqrt{3} + \sqrt{2}$ is a root of the polynomial $x^4 - 10x^2 + 1$.

(b) We will use the following useful corollary of Theorem 1.5.8 (ii).

Lemma L.5. If $A_1, \ldots, A_n$ are countable sets, then $A_1 \times \cdots \times A_n$ is also countable.

Proof. Suppose that A and B are countable sets, so that $B = \{b_1, b_2, b_3, \ldots\}$. For each $n \in \mathbf{N}$, it is clear that the set $A \times \{b_n\}$ is countable. Now observe that

$$A \times B = \bigcup_{n=1}^{\infty} (A \times \{b_n\}).$$

It follows from Theorem 1.5.8 (ii) that $A \times B$ is countable. A straightforward induction argument proves the general case. $\square$

Let P_n be the collection of polynomials with integer coefficients that have degree n, i.e. $P_n = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 : a_n, \ldots, a_0 \in \mathbf{Z}, a_n \neq 0\}$. Notice that

$$P_n \sim (\mathbf{Z} \setminus \{0\}) \times \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{n \text{ times}}$$

via the map

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mapsto (a_n, a_{n-1}, \ldots, a_1, a_0).$$

It then follows from Lemma L.5 that P_n is countable. For a polynomial $p \in P_n$, let R_p be the set of its roots, i.e., $R_p = \{x \in \mathbf{R} : p(x) = 0\}$, and note that R_p is always a finite set. Now observe that

$$A_n = \bigcup_{p \in P_n} R_p,$$

demonstrating that A_n is a countable union of finite sets; it follows from Theorem 1.5.8 that A_n is either finite or countable. Since $\sqrt[n]{k} \in A_n$ for each $k \in \mathbf{N}$ (it is a root of the polynomial $x^n - k$), we see that A_n must be infinite and hence countable.

(c) If we let A be the set of all algebraic numbers then $A = \bigcup_{n=1}^{\infty} A_n$, i.e., A is a countable union of countable sets. It follows from Theorem 1.5.8 (ii) that A is countable.

A consequence of this is that the set of transcendental numbers A^{c} must be uncountable. To see this, note that $\mathbf{R} = A \cup A^{\mathsf{c}}$, the union of two countable sets is countable, and $\mathbf{R}$ is not countable.

Exercise 1.5.10. (a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable, and let $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

Solution. (a) If we suppose that for each $a \in (0,1)$ the set $C \cap [a,1]$ is countable, then we can express C as a countable union of countable sets:

$$C = \bigcup_{n=2}^{\infty} \left(C \cap \left[\frac{1}{n}, 1 \right] \right).$$

This implies that C is countable (Theorem 1.5.8 (ii)). Thus, given that C is uncountable, there must exist some $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Not necessarily. Suppose $C = [0,1]$. Then for all $a \in (0,1)$, we have $C \cap [a,1] = [a,1]$, which is uncountable. Thus $A = (0,1)$ and it follows that $\alpha = \sup A = 1$, but $C \cap [\alpha,1] = \{1\}$ is not uncountable.

(c) The statement is no longer true in general. If we let $C = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ then no matter which $a \in (0,1)$ we choose, the intersection $C \cap [a,1]$ is a finite set (since there are only finitely many positive integers less than or equal to a^{-1}, there are only finitely many reciprocals of positive integers greater than or equal to a).

Exercise 1.5.11 (Schröder-Bernstein Theorem). Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$.

The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbf{N}\}$ is a similar collection in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

Solution. (a) Abusing notation slightly, we have bijections $f : A \to B$ and $g : B' \to A'$, and their inverses $f^{-1} : B \to A$ and $g^{-1} : A' \to B'$. Since $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, the functions $h : X \to Y$ and $h' : Y \to X$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g^{-1}(x) & \text{if } x \in A', \end{cases} \qquad h'(y) = \begin{cases} f^{-1}(y) & \text{if } y \in B, \\ g(y) & \text{if } y \in B' \end{cases}$$

are well-defined. It is straightforward to verify that h and h' are mutual inverses and thus $X \sim Y$.

(b) If A_1 is empty then $X = g(Y)$, i.e., g is surjective. Since g is injective by assumption, it immediately follows that $X \sim Y$ via g.

Let $P(n)$ be the statement that $\{A_1, \ldots, A_n\}$ is a pairwise disjoint collection of sets; to prove that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection, we will first use induction to prove that $P(n)$ holds for all $n \in \mathbf{N}$. The truth of $P(1)$ is clear, so suppose that $P(n)$ holds for some $n \in \mathbf{N}$. To demonstrate the truth of $P(n+1)$, we need to show that $A_k \cap A_{n+1} = \emptyset$ for all $1 \leq k \leq n$. Because $A_{n+1} = g(f(A_n)) \subseteq g(Y)$ and $A_1 = X \setminus g(Y)$, we see that $A_1 \cap A_{n+1} = \emptyset$. If $n \geq 2$, suppose that $2 \leq k \leq n$ and observe that

$$\begin{aligned}
A_k \cap A_{n+1} &= g(f(A_{k-1})) \cap g(f(A_n)) \\
&= g(f(A_{k-1} \cap A_n)) && (f \text{ and } g \text{ are injective}) \\
&= g(f(\emptyset)) && (\text{induction hypothesis}) \\
&= \emptyset.
\end{aligned}$$

Hence $P(n+1)$ holds; this completes the induction step and it follows that $P(n)$ holds for all $n \in \mathbf{N}$.

We can now show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of sets. Let A_m and A_n be given and suppose without loss of generality that $m < n$. By the previous paragraph the collection $\{A_1, \ldots, A_m, \ldots A_n\}$ is pairwise disjoint and thus $A_m \cap A_n = \emptyset$.

That $\{f(A_n) : n \in \mathbf{N}\}$ is a pairwise disjoint collection now follows immediately from the injectivity of f.

(c) Observe that

$$f(A) = f\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f(A_n) = B,$$

where we have used that the image of a union is the union of the images; the proof of this is similar to the proof of the special case given in Exercise 1.2.7 (d).

(d) Notice that

$$\begin{aligned}
b \in B' &\iff b \notin f(A_n) \text{ for all } n \in \mathbf{N} \\
&\iff g(b) \notin g(f(A_n)) \text{ for all } n \in \mathbf{N} && (g \text{ is injective}) \\
&\iff g(b) \notin A_{n+1} \text{ for all } n \in \mathbf{N}
\end{aligned}$$

$$\iff g(b) \notin A_n \text{ for all } n \geq 2.$$

Notice further that $g(y) \notin X \setminus g(Y) = A_1$ for any $y \in Y$. It follows that

$$b \in B' \iff g(b) \notin A_n \text{ for all } n \in \mathbf{N} \iff g(b) \in A'. \tag{$*$}$$

Thus g maps B' into A'. To see that $g : B' \to A'$ is surjective, observe that for any $a \in A'$ we have, in particular, $a \notin A_1 = X \setminus g(Y)$, so that $a \in g(Y)$, i.e., $a = g(y)$ for some $y \in Y$. It then follows from $(*)$ that $y \in B'$.

1.6 Cantor's Theorem

Exercise 1.6.1. Show that $(0, 1)$ is uncountable if and only if $\mathbf{R}$ is uncountable. This shows that Theorem 1.6.1 is equivalent to Theorem 1.5.6.

Solution. We have $(0, 1) \sim \mathbf{R}$ by Exercise 1.5.4 (a).

Exercise 1.6.2. (a) Explain why the real number $x = .b_1b_2b_3b_4\ldots$ cannot be $f(1)$.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbf{N}$.

(c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

Solution. (a) We have decimal expansions

$$f(1) = .a_{11}a_{12}a_{13}a_{14}\ldots \qquad \text{and} \qquad x = .b_1b_2b_3b_4\ldots.$$

By construction, $b_1 \neq a_{11}$. This implies that $f(1) \neq x$, provided these decimal expansions are not two different representations of the same real number (for example, .3 and .2999 . . .). However, since the only way this can occur is when one decimal expansion terminates in repeating 0's and the other terminates in repeating 9's, and the digits b_n are always either 2 or 3, we see that $.b_1b_2b_3b_4\ldots$ must be the unique decimal representation of a real number.

(b) Since $.b_1b_2b_3b_4\ldots$ is the unique decimal expansion of the real number x (see part (a)) and $b_n \neq a_{nn}$, we have $x \neq f(n)$ for every $n \in \mathbf{N}$. Here is an example construction of x given some function $f : \mathbf{N} \to (0,1)$:

$$\begin{array}{rclllllllll} f(1) & = & 0 & . & 9 & 2 & 8 & 4 & 7 & 6 & \dots \\ f(2) & = & 0 & . & 2 & 2 & 8 & 4 & 9 & 1 & \dots \\ f(3) & = & 0 & . & 9 & 9 & 1 & 0 & 2 & 5 & \dots \\ f(4) & = & 0 & . & 2 & 1 & 1 & 9 & 2 & 1 & \dots \\ f(5) & = & 0 & . & 1 & 2 & 5 & 7 & 2 & 3 & \dots \\ f(6) & = & 0 & . & 9 & 7 & 7 & 5 & 1 & 8 & \dots \\ \vdots & & & & & & & & & & \\ x & = & 0 & . & 2 & 3 & 2 & 2 & 3 & 2 & \dots \end{array}$$

Notice how the first digit (after the decimal point) of x differs from the first digit of $f(1)$, the second digit of x differs from the second digit of $f(2)$, and so on.

(c) The real number x belongs to $(0,1)$ but not to the image of f, which contradicts our assumption that f was surjective. It follows that there cannot exist a bijection between $\mathbf{N}$ and $(0,1)$. Since $(0,1)$ is clearly infinite, we may conclude that $(0,1)$ is uncountable.

Exercise 1.6.3. Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbf{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.

(b) Some numbers have *two* different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, 1/2 can also be written as .5 or as .4999 . . . . Doesn't this cause some problems?

Solution. (a) The problem with this reasoning is that the real number

$$x = .b_1 b_2 b_3 b_4 \dots$$

that we construct may not be rational. For example, consider the function $f : \mathbf{N} \to (0,1) \cap \mathbf{Q}$

given by

$$\begin{aligned} f(1) &= .3, & f(6) &= .000003, \\ f(2) &= .02, & f(7) &= .0000003, \\ f(3) &= .003, & f(8) &= .00000003, & \cdots \\ f(4) &= .0003, & f(9) &= .000000002, \\ f(5) &= .00002, & f(10) &= .0000000003, \end{aligned}$$

This results in $x = .2322322232\ldots$, which is not rational since its decimal expansion does not repeat. So while x does not belong to the image of f, this is not a problem because x does not belong to $(0,1) \cap \mathbf{Q}$ either.

(b) We addressed this issue in Exercise 1.6.2 (a).

Exercise 1.6.4. Let S be the set consisting of all sequences of 0's and 1's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}.$$

As an example, the sequence $(1,0,1,0,1,0,1,0,\ldots)$ is an element of S, as is the sequence $(1,1,1,1,1,1,\ldots)$.

Give a rigorous argument showing that S is uncountable.

Solution. Suppose we have a function $f : \mathbf{N} \to S$. For each $m \in \mathbf{N}$, let a_{mn} be the element in the n^{th} position of $f(m)$, so that

$$f(m) = (a_{m1}, a_{m2}, a_{m3}, a_{m4}, \ldots) \in S.$$

Let $b = (b_1, b_2, b_3, b_4, \ldots)$ be the sequence given by

$$b_n = \begin{cases} 0 & \text{if } a_{nn} = 1, \\ 1 & \text{if } a_{nn} = 0. \end{cases}$$

Notice that $b \in S$ but $b \neq f(n)$ for any $n \in \mathbf{N}$, since b differs from $f(n)$ in the n^{th} position. Here is an example construction of the sequence b, given some $f : \mathbf{N} \to S$:

$$\begin{array}{rcllllllll}
f(1) & = & (1, & 0, & 0, & 1, & 0, & 1, & \dots \\
f(2) & = & (0, & 0, & 1, & 1, & 1, & 0, & \dots \\
f(3) & = & (0, & 1, & 1, & 0, & 0, & 0, & \dots \\
f(4) & = & (1, & 1, & 1, & 1, & 0, & 0, & \dots \\
f(5) & = & (0, & 0, & 1, & 0, & 0, & 1, & \dots \\
f(6) & = & (1, & 0, & 0, & 1, & 0, & 1, & \dots \\
\vdots & & & & & & & & \\
b & = & (0, & 1, & 0, & 0, & 1, & 0, & \dots
\end{array}$$

Notice that b differs from $f(1)$ in the first position, from $f(2)$ in the second position, and so on.

Thus $b \notin f(\mathbf{N})$, so that f is not a surjection. Since f was arbitrary, it follows that there can be no bijection between $\mathbf{N}$ and S. It is clear that S is infinite, so we may conclude that S is uncountable.

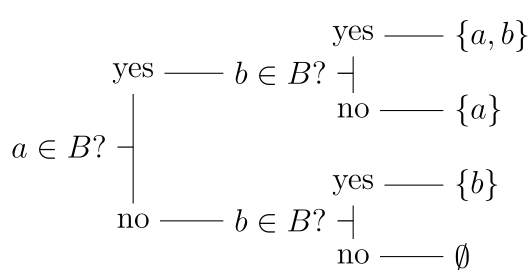
Exercise 1.6.5. (a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution. (a) We have

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

(b) To form a subset B of A, for each element $a \in A$, we must decide whether to include a in B or not. This is a binary choice to be made for each of the n elements of A; it follows that there are 2^n subsets of A. For example, here is a tree listing all $2^2 = 4$ subsets of $\{a, b\}$:



Exercise 1.6.6. (a) Using the particular set $A = \{a, b, c\}$, exhibit two different 1-1 mappings from A into $P(A)$.

(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a 1-1 map $g : C \to P(C)$.

(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are *onto*.

Solution. (a) Here are two injections $f : A \to P(A)$ and $g : A \to P(A)$:

$$\begin{aligned} f(a) &= \{a\}, \qquad & g(a) &= \{a, b\}, \\ f(b) &= \{b\}, \qquad & g(b) &= \{b, c\}, \\ f(c) &= \{c\}, \qquad & g(c) &= \{a, c\}. \end{aligned}$$

(b) Let g be given by

$$\begin{aligned} g(1) &= \{1\}, \qquad & g(3) &= \{3\}, \\ g(2) &= \{2\}, \qquad & g(4) &= \{4\}. \end{aligned}$$

(c) The power set of a finite set A always contains strictly more elements than A (Exercise 1.6.5 (b)). For finite sets, it is impossible to construct a surjective function from a set A to a set B if B contains strictly more elements than A.

Exercise 1.6.7. Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule. In each case, note that B is not in the range of the function used.

Solution. For all three functions from Exercise 1.6.6 we have $B = \emptyset$, which does not belong to the range of any of the functions.

Exercise 1.6.8. (a) First, show that the case $a' \in B$ leads to a contradiction.

(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

Solution. (a) and (b). We have $a' \in B$ if and only if $a' \notin f(a') = B$, which is clearly a contradiction since a' either does or does not belong to B.

Exercise 1.6.9. Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $P(\mathbf{N}) \sim \mathbf{R}$.

Solution. First, let us show that $P(\mathbf{N}) \sim S$, where S is the set of all binary sequences defined in Exercise 1.6.4. Consider the function $f : P(\mathbf{N}) \to S$ given by $f(E) = (a_1, a_2, a_3, \ldots)$ where

$$a_n = \begin{cases} 1 & \text{if } n \in E, \\ 0 & \text{if } n \notin E. \end{cases}$$

For example,

$$f(\{1, 3, 4, 6, 7, 10, \ldots\}) = (1, 0, 1, 1, 0, 1, 1, 0, 0, 1, \ldots).$$

This function is a bijection since it has an inverse $f^{-1} : S \to P(\mathbf{N})$ given by

$$f^{-1}(a_1, a_2, a_3, \ldots) = \{n \in \mathbf{N} : a_n = 1\}.$$

Now let us show that $S \sim (0, 1)$. Consider the function $g : S \to (0, 1)$ given by

$$g(a_1, a_2, a_3, \ldots) = 0.5a_1a_2a_3 \ldots,$$

where $0.5a_1a_2a_3\ldots$ is a decimal expansion (for example, $g(1, 0, 1, 0, 0, 0, \ldots) = 0.5101$). This function is injective since if $a = (a_1, a_2, a_3, \ldots) \neq b = (b_1, b_2, b_3, \ldots)$, there must exist some $n \in \mathbf{N}$ such that $a_n \neq b_n$. It follows that $g(a) \neq g(b)$, provided $g(a) = 0.5a_1a_2a_3\ldots$ and $g(b) = 0.5b_1b_2b_3\ldots$ are not two different decimal expansions of the same real number. This cannot be the case since each a_i and b_i is either 0 or 1, and never 9.

Now consider the function $h : (0, 1) \to S$ given by

$$h(x) = h(0.a_1a_2a_3 \ldots) = (a_1, a_2, a_3, \ldots),$$

where $0.a_1a_2a_3\ldots$ is the **binary** expansion of $x \in (0, 1)$, choosing that expansion which terminates in 0's if x has two different binary expansions. This function is injective since if $x = 0.a_1a_2a_3\ldots \neq y = 0.b_1b_2b_3\ldots$, then there must be some $n \in \mathbf{N}$ such that $a_n \neq b_n$. It follows that $h(x) \neq h(y)$.

The Schröder-Bernstein Theorem (Exercise 1.5.11) now implies that $S \sim (0, 1)$. We showed in Exercise 1.5.4 that $(0, 1) \sim \mathbf{R}$ and thus

$$P(\mathbf{N}) \sim S \sim (0, 1) \sim \mathbf{R}.$$

In Exercise 1.5.5 we showed that $\sim$ is an equivalence relation, so the chain of equivalences above allows us to conclude that $P(\mathbf{N}) \sim \mathbf{R}$.

Exercise 1.6.10. As a final exercise, answer each of the following by establishing a 1-1 correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0, 1\}$ to $\mathbf{N}$ countable or uncountable?

(b) Is the set of all functions from $\mathbf{N}$ to $\{0, 1\}$ countable of uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an *antichain* if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbf{N})$ contain an uncountable antichain?

Solution. (a) Let $\mathbf{N}^{\{0,1\}}$ be the set of all functions from $\{0, 1\}$ to $\mathbf{N}$. Consider the function $F : \mathbf{N}^{\{0,1\}} \to \mathbf{N} \times \mathbf{N}$ given by $F(f) = (f(0), f(1))$. This function is a bijection since it has an inverse $F^{-1} : \mathbf{N} \times \mathbf{N} \to \mathbf{N}^{\{0,1\}}$ given by $F^{-1}(a, b) = f$, where $f : \{0, 1\} \to \mathbf{N}$ is the function satisfying $f(0) = a, f(1) = b$. Thus

$$\mathbf{N}^{\{0,1\}} \sim \mathbf{N} \times \mathbf{N} \sim \mathbf{N},$$

where we have used Lemma L.5 for the second equivalence. We may conclude that $\mathbf{N}^{\{0,1\}}$ is countable.

(b) The set of all functions from $\mathbf{N}$ to $\{0, 1\}$ is nothing but the set of all binary sequences S defined in Exercise 1.6.4, since a function $f : \mathbf{N} \to \{0, 1\}$ can be identified with the sequence $(f(0), f(1), f(2), \ldots)$. Thus the set of all functions from $\mathbf{N}$ to $\{0, 1\}$ is uncountable, since we showed that S is uncountable in Exercise 1.6.4.

(c) Consider the following collection of subsets of $P(\mathbf{Q})$:

$$\mathcal{A} := \{(a, a + 1) \cap \mathbf{Q} : a \in \mathbf{R}\}.$$

For real numbers $a < b$, it follows from the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3) that there exist rational numbers p and q such that $a < p < b$ and $a < q < a + 1$. Let $r = \min\{p, q\}$ and notice that $a < r < b$ and $a < r < a+1$. It follows that $r \in (a, a+1)$ and $r \notin (b, b+1)$, whence

$$(a, a + 1) \cap \mathbf{Q} \not\subseteq (b, b + 1) \cap \mathbf{Q}.$$

A similar argument shows that this non-inclusion still holds if $b < a$ and so it follows that for any real numbers $a \neq b$ we have

$$(a, a + 1) \cap \mathbf{Q} \not\subseteq (b, b + 1) \cap \mathbf{Q},$$

i.e., $\mathcal{A}$ is an antichain.

Another consequence of the previous paragraph is that if a and b are distinct real numbers, then

$$(a, a+1) \cap \mathbf{Q} \neq (b, b+1) \cap \mathbf{Q}.$$

It follows that the map $g : \mathbf{R} \to \mathcal{A}$ defined by $g(a) = (a, a+1) \cap \mathbf{Q}$ is injective. Since g is evidently surjective, we have that $\mathbf{R} \sim \mathcal{A}$.

To finish the exercise, we will need the following two lemmas.

Lemma L.6. Suppose A and B are sets and $f : A \to B$ is a bijection. Define $F : P(A) \to P(B)$

$$F(X) = f(X) = \{f(x) : x \in X\}.$$

Then F is a bijection.

Proof. Suppose $X, Y \in P(A)$ are such that $X \neq Y$. Without loss of generality suppose that $X \not\subseteq Y$, so that there is some $x \in X$ such that $x \notin Y$. The injectivity of f then implies that $f(x) \notin f(Y)$, whence $F(X) \neq F(Y)$. Thus F is injective.

Now let $Y \in P(B)$ be given. For each $y \in Y$, the surjectivity of f implies that there is some $x \in A$ such that $f(x) = y$; let X be the collection of these x. It follows that $F(X) = Y$ and hence that F is surjective. □

Lemma L.7. Suppose A and B are sets and $f : A \to B$ is injective. Then if $\mathcal{A} \subseteq P(A)$ is an antichain, so is $\mathcal{A}' := \{f(X) : X \in \mathcal{A}\} \subseteq P(B)$.

Proof. Suppose we have two elements $f(X)$ and $f(Y)$ in $\mathcal{A}'$, where X and Y belong to $\mathcal{A}$. Since $\mathcal{A}$ is an antichain, we have $X \not\subseteq Y$, which can be the case if and only if there is some $x \in X$ such that $x \notin Y$. The injectivity of f then implies that $f(x) \in f(X)$ but $f(x) \notin f(Y)$. It follows that $f(X)$ is not a subset of $f(Y)$ and we may conclude that $\mathcal{A}'$ is an antichain. □

Returning to the exercise, let $f : \mathbf{Q} \to \mathbf{N}$ be a bijection (such a function exists by Theorem 1.5.6 (i)). By Lemma L.6, the function $F : P(\mathbf{Q}) \to P(\mathbf{N})$ defined by $F(X) = f(X)$ is also a bijection, which restricts to a bijection $F : \mathcal{A} \to F(\mathcal{A})$. Thus $F(\mathcal{A}) \sim \mathcal{A} \sim \mathbf{R}$, so that $F(\mathcal{A})$ is uncountable. We may now use Lemma L.7 to conclude that $F(\mathcal{A}) \subseteq P(\mathbf{N})$ is an uncountable antichain.

Chapter 2

Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.1. What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution. First observe that the statement

$$\text{for all } N \in \mathbf{N},\, n \geq N \implies |x_n - x| < \epsilon$$

is equivalent to

$$\text{for all } n \in \mathbf{N},\, |x_n - x| < \epsilon.$$

So a sequence verconges to x if there exists an $\epsilon > 0$ such that $|x_n - x| < \epsilon$, or equivalently such that $x_n \in (x - \epsilon, x + \epsilon)$, for all $n \in \mathbf{N}$.

For an example of a vercongent sequence that diverges, consider $(x_n) = (1, 0, 1, 0, \ldots)$. This sequence verconges to $\frac{1}{2}$ since $\left|x_n - \frac{1}{2}\right| = \frac{1}{2} < 1$ for all $n \in \mathbf{N}$. To see that this sequence diverges, suppose there was some $x \in \mathbf{R}$ such that $\lim x_n = x$. Then there must exist some $N \in \mathbf{N}$ such that $n \geq N$ implies that $|x_n - x| < \frac{1}{2}$. Observe that

$$1 = |x_N - x_{N+1}| \leq |x_N - x| + |x_{N+1} - x| < \frac{1}{2} + \frac{1}{2} = 1,$$

i.e., $1 < 1$, which is a contradiction.

A sequence can verconge to two different values. The sequence $(x_n) = (1, 1, 1, 1, \ldots)$ verconges to 1:

$$|x_n - 1| = 0 < 1 \text{ for all } n \in \mathbf{N},$$

and also to 0:

$$|x_n| = 1 < 2 \text{ for all } n \in \mathbf{N}.$$

This definition describes the bounded sequences (see Definition 2.3.1); a sequence which verconges to some $x \in \mathbf{R}$ must be bounded, and conversely any bounded sequence verconges to some $x \in \mathbf{R}$.

Exercise 2.2.2. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution. See Figure F.7 for graphs of the first thirty terms of each sequence. The graphs of the sequences in parts (a) and (b) give us a good idea of the limiting value of each sequence; the graph for part (c) is not so clear.

(a) Let $\epsilon > 0$ be given. Choose $N \in \mathbf{N}$ such that $N > \frac{3}{25\epsilon}$ and observe that for $n \geq N$ we have

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \frac{3}{25n+20} < \frac{3}{25n} \leq \frac{3}{25N} < \epsilon.$$

It follows that $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) Let $\epsilon > 0$ be given. Choose $N \in \mathbf{N}$ such that $N > \frac{2}{\epsilon}$ and observe that for $n \geq N$ we have

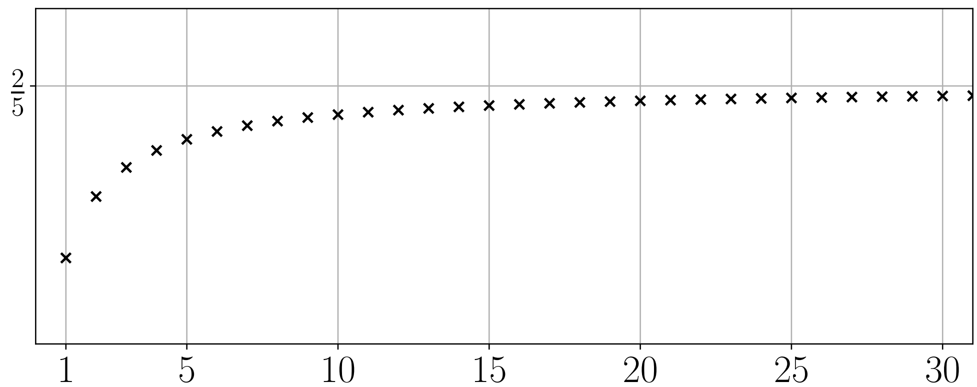
$$\left|\frac{2n^2}{n^3+3}\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

It follows that $\lim \frac{2n^2}{n^3+3} = 0$.

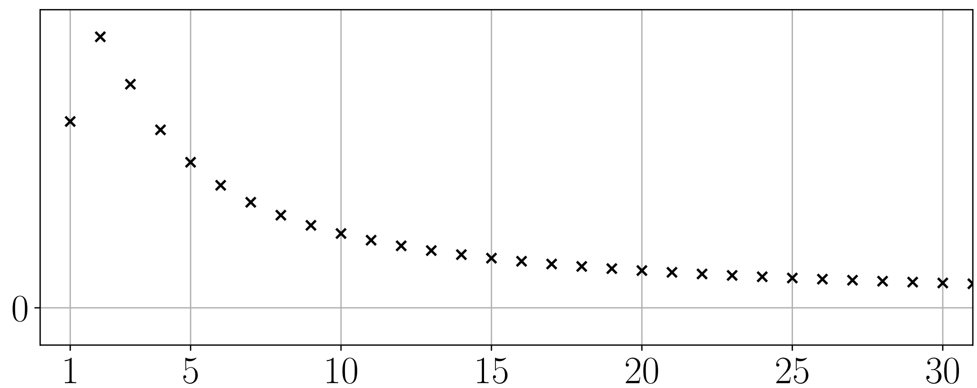
(c) Let $\epsilon > 0$ be given. Choose $N \in \mathbf{N}$ such that $N > \frac{1}{\epsilon^3}$ and observe that for $n \geq N$ we have

$$\left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| = \frac{|\sin(n^2)|}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \epsilon.$$

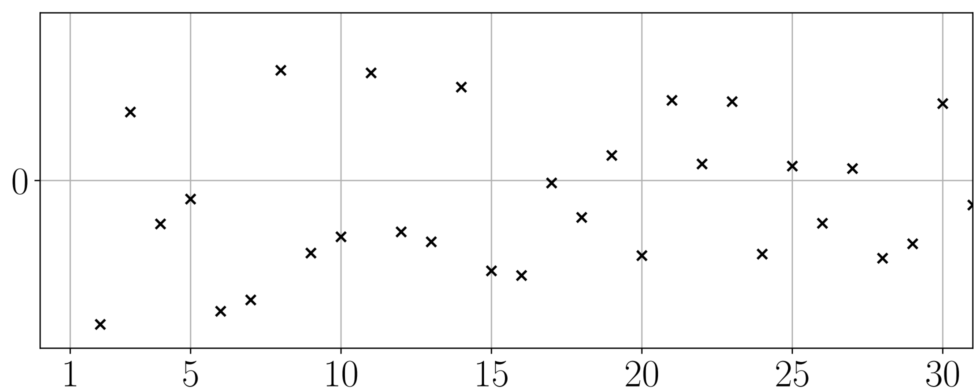
It follows that $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.



(a) $\frac{2n+1}{5n+4}$ for $1 \leq n \leq 30$



(b) $\frac{2n^2}{n^3+3}$ for $1 \leq n \leq 30$



(c) $\frac{\sin(n^2)}{\sqrt[3]{n}}$ for $1 \leq n \leq 30$

Figure F.7: Exercise 2.2.2 sequences for $1 \leq n \leq 30$

Exercise 2.2.3. Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution. (a) We would have to find a college in the United States where every student is less than seven feet tall.

(b) We would have to find a college in the United States where each professor gives at least one student a grade of C or worse.

(c) We would have to show that every college in the United States has a student who is less than six feet tall.

Exercise 2.2.4. Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution. (a) Consider $(x_n) = (1, 0, 1, 0, \ldots)$. This sequence has an infinite number of ones but, as shown in Exercise 2.2.1, diverges.

(b) This is impossible. Suppose (x_n) is such a sequence with $\lim x_n = x \neq 1$. Then there must exist some $N \in \mathbf{N}$ such that for all $n \geq N$ we have $|x_n - x| < |1 - x|$. Since this sequence contains infinitely many ones, it must be the case that there is some $m \geq N$ such that $x_m = 1$. This implies that $|x_m - x| = |1 - x| < |1 - x|$, which is a contradiction.

(c) Consider the sequence

$$(x_n) = (1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \ldots).$$

For each $n \in \mathbf{N}$ we can find n consecutive ones starting at the m^{th} position and, for $n \geq 2$, we can find a zero at the $(m-1)^{\text{th}}$ position, where $m = \frac{n(n+1)}{2}$. Furthermore, the sequence is divergent. To see this, suppose there was some $x \in \mathbf{R}$ such that $\lim x_n = x$. It follows that there is an $N \in \mathbf{N}$ such that $n \geq N$ implies that $|x_n - x| < \frac{1}{2}$. Since the sequence contains infinitely many ones and zeros, we can find indices $k, \ell \geq N$ such that $x_k = 1$ and $x_\ell = 0$. Then

$$1 = |x_k - x_\ell| \leq |x_k - x| + |x_\ell - x| < \frac{1}{2} + \frac{1}{2} = 1,$$

i.e., $1 < 1$, which is a contradiction.

Exercise 2.2.5. Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$,

(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution. (a) We claim that $\lim a_n = 0$. Let $\epsilon > 0$ be given and observe that if $n \geq 6$, then

$$0 < \frac{5}{n} < 1 \implies \left[\left[\frac{5}{n}\right]\right] = 0.$$

So if we take $N = 6$, then $n \geq N$ implies that $\left|\left[\left[\frac{5}{n}\right]\right]\right| = 0 < \epsilon$.

(b) We claim that $\lim a_n = 1$. Let $\epsilon > 0$ be given and observe that if $n \geq 7$, then

$$\frac{1}{n} < \frac{1}{6} \iff \frac{4}{n} < \frac{2}{3} \iff \frac{4}{n} + \frac{1}{3} < 1.$$

Hence for $n \geq 7$ we have

$$0 < \frac{4}{n} + \frac{1}{3} < 1 \implies \left[\left[\frac{4}{n} + \frac{1}{3}\right]\right] = 0.$$

So if we take $N = 7$, then $n \geq N$ implies that

$$\left[\left[\frac{12 + 4n}{3n} - 1\right]\right] = \left[\left[\frac{4}{n} + \frac{1}{3}\right]\right] = 0 < \epsilon.$$

These examples demonstrate that taking smaller ϵ-neighbourhoods may not require us to take larger values of N; the same value of N in each example works for every ϵ-neighbourhood that we choose.

Exercise 2.2.6. Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and $(a_n) \to b$. Now argue $a = b$.

Solution. Let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |a_n - a| < \frac{\epsilon}{2} \qquad \text{and} \qquad n \geq N_2 \implies |a_n - b| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$|a - b| = |a - a_n + a_n - b| \leq |a_n - a| + |a_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So we have shown that $|a - b| < \epsilon$ for any $\epsilon > 0$; it follows from Theorem 1.2.6 that $a = b$.

Exercise 2.2.7. Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution. (a) The sequence $(-1)^n$ is frequently but not eventually in the set $\{1\}$. To see this, let $N \in \mathbf{N}$ be given. If N is even, then $(-1)^N \in \{1\}$ and $(-1)^{N+1} \notin \{1\}$, and if N is odd then $(-1)^N \notin \{1\}$ and $(-1)^{N+1} \in \{1\}$. In any case, we can always find indices $m, n \geq N$ such that $(-1)^m \notin \{1\}$ (this shows that the sequence is not eventually in $\{1\}$) and such that $(-1)^n \in \{1\}$ (this shows that the sequence is frequently in $\{1\}$).

(b) Eventually is the stronger definition. Frequently does not imply eventually, as part (a) shows, but eventually does imply frequently. To see this, suppose that (a_n) is eventually in a set A, i.e., there is an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$. Let $M \in \mathbf{N}$ be given. Set $n = \max\{M, N\}$ and observe that $n \geq M$ and $a_n \in A$. Hence (a_n) is frequently in A.

(c) The term we want is eventually. Here is a rephrasing of Definition 2.2.3B: a sequence (a_n) converges to a if, given any $\epsilon > 0$, the sequence (a_n) is eventually in the ϵ-neighbourhood $V_\epsilon(a)$ of a.

(d) Such a sequence is not necessarily eventually in $(1.9, 2.1)$; consider the sequence $(x_n) = (2, 0, 2, 0, 2, \ldots)$ for example. For any $N \in \mathbf{N}$, we can always find an index $n \geq N$ (either $n = N$ or $n = N + 1$) such that $x_n = 0 \notin (1.9, 2.1)$. However, such a sequence must be frequently in $(1.9, 2.1)$. To see this, let $N \in \mathbf{N}$ be given. Then there must exist an index $n \geq N$ such that $x_n = 2 \in (1.9, 2.1)$ (otherwise there would be only finitely many twos in the sequence).

Exercise 2.2.8. For some additional practice with nested quantifiers, consider the following invented defintion:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero-heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counterexample.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if ....

Solution. (a) This sequence is zero-heavy; $M = 1$ works. Indeed, let $N \in \mathbf{N}$ be given. If N is odd then let $n = N$ and if N is even then let $n = N + 1$. In either case, we have $N \leq n \leq N + 1$ and $x_n = 0$.

(b) A zero-heavy sequence must contain an infinite number of zeros. To see this, suppose (x_n) is a sequence with a finite number of zeros, i.e. there is an $N \in \mathbf{N}$ such that $x_n \neq 0$ for all $n \geq N$. Then no matter which M we choose, we will never be able to find $n \in \mathbf{N}$ with $N \leq n \leq N + M$ and $x_n = 0$. Thus the sequence (x_n) is not zero-heavy.

(c) A sequence with an infinite number of zeros is not necessarily zero-heavy. For a counterexample, consider the sequence

$$(x_n) = (1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \ldots).$$

This sequence contains infinitely many zeros, but is not zero-heavy. To see this, let $M \in \mathbf{N}$ be given. It is always possible to find M consecutive ones in the sequence (x_n) (see Exercise 2.2.4 (c)); suppose this string of ones starts at $x_N = 1$. Then for each $n \in \mathbf{N}$ satisfying $N \leq n \leq N + M$, we have $x_n = 1 \neq 0$.

(d) A sequence is not zero-heavy if for every $M \in \mathbf{N}$ there exists an $N \in \mathbf{N}$ such that $x_n \neq 0$ for each $n \in \mathbf{N}$ satisfying $N \leq n \leq N + M$.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Solution. (a) Let $\epsilon > 0$ be given. Since $x_n \to 0$, there exists an $N \in \mathbf{N}$ such that

$$n \geq N \implies |x_n| = x_n < \epsilon^2 \iff \sqrt{x_n} < \epsilon.$$

It follows that $\lim\left(\sqrt{x_n}\right) = 0$.

(b) By Theorem 2.3.4, we must have $x \geq 0$. The case $x = 0$ was handled in part (a), so suppose that $x > 0$, which gives $\sqrt{x} > 0$. For each $n \in \mathbf{N}$, observe that

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \frac{\left|\sqrt{x_n} - \sqrt{x}\right|\left(\sqrt{x_n} + \sqrt{x}\right)}{\sqrt{x_n} + \sqrt{x}} = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}.$$

Let $\epsilon > 0$ be given. Since $x_n \to x$, there exists an $N \in \mathbf{N}$ such that $|x_n - x| < \epsilon\sqrt{x}$ whenever $n \geq N$. For $n \geq N$, it follows that

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leq \frac{|x_n - x|}{\sqrt{x}} < \epsilon.$$

Thus $\lim(\sqrt{x_n}) = \sqrt{x}$.

Exercise 2.3.2. Using only Definition 2.2.3, prove that if $(x_n) \to 2$ then

(a) $\left(\frac{2x_n-1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

Solution. (a) Let $\epsilon > 0$ be given. Since $x_n \to 2$, there exists an $N \in \mathbf{N}$ such that $n \geq N$ implies that $|x_n - 2| < \frac{3\epsilon}{2}$. For $n \geq N$ we then have

$$\left|\frac{2x_n - 1}{3} - 1\right| = \left|\frac{2x_n - 4}{3}\right| = \frac{2}{3}|x_n - 2| < \epsilon.$$

It follows that $\left(\frac{2x_n-1}{3}\right) \to 1$.

(b) Since $x_n \to 2$, there is an $N_1 \in \mathbf{N}$ such that $n \geq N_1 \implies |x_n - 2| < 1$. For $n \geq N_1$ we then have

$$2 \leq |x_n - 2| + |x_n| < 1 + |x_n| \implies 1 < |x_n| \implies \frac{1}{|x_n|} < 1.$$

Let $\epsilon > 0$ be given. Since $x_n \to 2$, there is an $N_2 \in \mathbf{N}$ such that $|x_n - 2| < 2\epsilon$ whenever $n \geq N_2$. Set $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{2 - x_n}{2x_n}\right| = \frac{|x_n - 2|}{2|x_n|} < \frac{|x_n - 2|}{2} < \epsilon.$$

It follows that $\frac{1}{x_n} \to \frac{1}{2}$.

Exercise 2.3.3 (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution. Let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |x_n - l| < \epsilon \iff -\epsilon < x_n - l < \epsilon,$$

$$n \geq N_2 \implies |z_n - l| < \epsilon \iff -\epsilon < z_n - l < \epsilon.$$

Let $N = \max\{N_1, N_2\}$. Then since $x_n - l \leq y_n - l \leq z_n - l$ for all $n \in \mathbf{N}$, for $n \geq N$ we have

$$-\epsilon < y_n - l < \epsilon \iff |y_n - l| < \epsilon.$$

It follows that $\lim y_n = l$.

Exercise 2.3.4. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$.

Solution. The manipulations of limits in these solutions are justified by the Algebraic Limit Theorem (Theorem 2.3.3).

(a) We have
$$\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) = \frac{1+2\lim a_n}{1+3\lim a_n - 4(\lim a_n)^2} = \frac{1}{1} = 1.$$

(b) We have
$$\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) = \lim\left(\frac{a_n^2+4a_n}{a_n}\right) = \lim(a_n+4) = \lim a_n + 4 = 4.$$

(c) We have
$$\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) = \lim\left(\frac{2+3a_n}{1+5a_n}\right) = \frac{2+3\lim a_n}{1+5\lim a_n} = \frac{2}{1} = 2.$$

Exercise 2.3.5. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution. (z_n) is the sequence given by
$$z_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd,} \\ y_{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$$

Suppose that (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n = L$ for some $L \in \mathbf{R}$ and let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that
$$n \geq N_1 \implies |x_n - L| < \epsilon \qquad \text{and} \qquad n \geq N_2 \implies |y_n - L| < \epsilon.$$

Let $N = \max\{N_1, N_2\}$ and suppose $n \in \mathbf{N}$ is such that $n \geq 2N$. If n is odd then $\frac{n+1}{2} \in \mathbf{N}$ and

$$n \geq 2N > 2N - 1 \implies \frac{n+1}{2} > N \geq N_1 \implies \left| x_{\frac{n+1}{2}} - L \right| < \epsilon.$$

Hence

$$|z_n - L| = \left| x_{\frac{n+1}{2}} - L \right| < \epsilon.$$

If n is even then $\frac{n}{2} \in \mathbf{N}$ and

$$n \geq 2N \implies \frac{n}{2} \geq N \geq N_2 \implies \left| y_{\frac{n}{2}} - L \right| < \epsilon.$$

Hence

$$|z_n - L| = \left| y_{\frac{n}{2}} - L \right| < \epsilon.$$

In either case we have $|z_n - L| < \epsilon$, i.e.,

$$n \geq 2N \implies |z_n - L| < \epsilon.$$

It follows that $\lim z_n = L$.

Now suppose that (z_n) is convergent with $\lim z_n = L$ for some $L \in \mathbf{R}$. Let $\epsilon > 0$ be given. Since $z_n \to L$, there exists an $N \in \mathbf{N}$ such that $|z_n - L| < \epsilon$ whenever $n \geq N$. For such n, we have $2n > 2n - 1 \geq n \geq N$ and so

$$|x_n - L| = |z_{2n-1} - L| < \epsilon \qquad \text{and} \qquad |y_n - L| = |z_{2n} - L| < \epsilon.$$

It follows that $\lim x_n = \lim y_n = L$.

Exercise 2.3.6. Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Solution. Observe that

$$b_n = n - \sqrt{n^2 + 2n} = \frac{(n - \sqrt{n^2 + 2n})(n + \sqrt{n^2 + 2n})}{n + \sqrt{n^2 + 2n}} = \frac{-2n}{n + \sqrt{n^2 + 2n}} = \frac{-2}{1 + \sqrt{1 + \frac{2}{n}}}.$$

Hence, using Exercise 2.3.1,

$$\lim b_n = \lim \left(\frac{-2}{1 + \sqrt{1 + \frac{2}{n}}} \right) = \frac{-2}{1 + \sqrt{1 + 2 \lim \frac{1}{n}}} = \frac{-2}{1 + \sqrt{1}} = -1.$$

Figure F.8 shows a graph of the first thirty terms of this sequence.

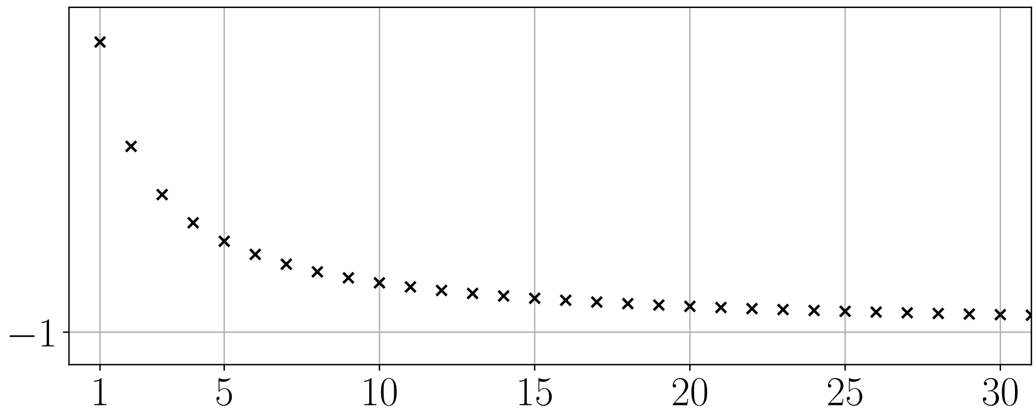


Figure F.8: $b_n = n - \sqrt{n^2 + 2n}$ for $1 \leq n \leq 30$

Exercise 2.3.7. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Solution. (a) Take $x_n = n$ and $y_n = -n$.

(b) This is impossible. If (x_n) and $(x_n + y_n)$ both converge, then by the Algebraic Limit Theorem (Theorem 2.3.3) (y_n) must be convergent with limit $\lim y_n = \lim(x_n + y_n) - \lim x_n$.

(c) Take $b_n = \frac{1}{n}$.

(d) This is impossible; $(a_n - b_n)$ must be unbounded. Since (b_n) is convergent, it must be bounded (Theorem 2.3.2), i.e., there exists some $B \geq 0$ such that $|b_n| \leq B$ for all $n \in \mathbf{N}$. Let $M \geq 0$ be given. Since (a_n) is unbounded, there exists some $N \in \mathbf{N}$ such that $|a_N| \geq M + B$. Then observe that

$$|a_N - b_N| \geq ||a_N| - |b_N|| \geq |a_N| - |b_N| \geq M + B - B = M,$$

where we have used Exercise 1.2.6 (d) for the first inequality. Since M was arbitrary, we see that the sequence $(a_n - b_n)$ is unbounded.

(e) Take $a_n = \frac{1}{n^2}$ and $b_n = n$.

Exercise 2.3.8. Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Solution. (a) Suppose $p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$. The Algebraic Limit Theorem (Theorem 2.3.3) and some simple induction arguments allow us to make the following manipulations:

$$\begin{aligned}
\lim(p(x_n)) &= \lim(a_m x_n^m + a_{m-1}x_n^{m-1} + \cdots + a_1 x_n + a_0) \\
&= a_m(\lim x_n)^m + a_{m-1}(\lim x_n)^{m-1} + \cdots + a_1 \lim x_n + a_0 \\
&= a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \\
&= p(x).
\end{aligned}$$

(b) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and the convergent sequence $x_n = \frac{1}{n} \to 0$. We then have $(f(x_n)) = (1, 1, 1, \ldots)$, which converges to $1 \neq 0 = f(0)$.

Exercise 2.3.9. (a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution. (a) There is an $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$. Let $\epsilon > 0$ be given. Since $b_n \to 0$, there is an $N \in \mathbf{N}$ such that

$$n \geq N \implies |b_n| < \frac{\epsilon}{M}.$$

Observe that for $n \geq N$ we have

$$|a_n b_n| = |a_n||b_n| \leq M|b_n| < \frac{M\epsilon}{M} = \epsilon.$$

It follows that $\lim(a_n b_n) = 0$. We may not use the Algebraic Limit Theorem here since the sequence (a_n) is not necessarily convergent; the hypotheses of that theorem require both sequences (a_n) and (b_n) to be convergent.

(b) If the sequence (a_n) converges to some a then we may use the Algebraic Limit Theorem to conclude that $\lim(a_n b_n) = ab$. If the sequence (a_n) is divergent, then $(a_n b_n)$ must also be divergent. To see this, we will prove the contrapositive, i.e., if $(a_n b_n)$ converges to some $x \in \mathbf{R}$ then (a_n) is convergent. Indeed, since $b \neq 0$, the Algebraic Limit Theorem implies that

$$\lim a_n = \lim\left(\frac{a_n b_n}{b_n}\right) = \frac{x}{b}.$$

(c) Since (b_n) is convergent, it is bounded (Theorem 2.3.2). So we may apply part (a) (we have swapped the roles of (a_n) and (b_n)) to conclude that

$$\lim(a_n b_n) = 0 = 0b = ab.$$

Exercise 2.3.10. Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Solution. (a) This is false; consider $a_n = b_n = (-1)^n$.

(b) This is true. Let $\epsilon > 0$ be given. Since $b_n \to b$, there is an $N \in \mathbf{N}$ such that $|b_n - b| < \epsilon$ whenever $n \geq N$. For such n, the reverse triangle inequality (Exercise 1.2.6 (d)) gives

$$||b_n| - |b|| \leq |b_n - b| < \epsilon.$$

It follows that $\lim|b_n| = |b|$.

(c) This is true. Using the Algebraic Limit Theorem (Theorem 2.3.3), we have

$$\lim b_n = \lim(b_n - a_n + a_n) = \lim(b_n - a_n) + \lim a_n = 0 + a = a.$$

(d) This is true. Since $0 \leq |b_n - b| \leq a_n$ for every $n \in \mathbf{N}$, the Squeeze Theorem (Exercise 2.3.3) implies that $\lim|b_n - b| = 0$, i.e., for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n \geq N \implies ||b_n - b| - 0| = |b_n - b| < \epsilon,$$

which is exactly the statement $\lim b_n = b$.

Exercise 2.3.11 (Cesaro Means). (a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution. (a) Suppose $\lim x_n = x$ and let $\epsilon > 0$ be given. Since $x_n \to x$, there is a positive integer $N_1 \in \mathbf{N}$ such that

$$n \geq N_1 \implies |x_n - x| < \frac{\epsilon}{2}.$$

Given this N_1, notice that the sequence

$$\left(\frac{|x_1 - x| + \cdots + |x_{N_1} - x|}{n}\right)_{n=1}^{\infty}$$

has non-negative terms and converges to zero (the numerator is a constant); it follows that there is an $N_2 \in \mathbf{N}$ such that

$$n \geq N_2 \implies \frac{|x_1 - x| + \cdots + |x_{N_1} - x|}{n} < \frac{\epsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$ and observe that for $n \geq N+1$ we have

$$\begin{aligned}
|y_n - x| &= \left| \frac{x_1 + \cdots + x_n}{n} - \frac{nx}{n} \right| \\
&= \left| \frac{(x_1 - x) + \cdots + (x_n - x)}{n} \right| \\
&\leq \frac{|x_1 - x| + \cdots + |x_{N_1} - x|}{n} + \frac{|x_{N_1+1} - x| + \cdots + |x_n - x|}{n} \\
&< \frac{\epsilon}{2} + \frac{n - N_1}{n} \cdot \frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

It follows that $\lim y_n = x$.

(b) Consider the divergent sequence $x_n = (-1)^{n+1}$. The sequence of averages (y_n) is then

$$y_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

which satisfies $\lim y_n = 0$.

Exercise 2.3.12. A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0,1)$, then a is also in the complement of $(0,1)$.

(c) If every a_n is rational, then a is rational.

Solution. (a) This is true. For any $b \in B$ we have $b \leq a_n$ for all $n \in \mathbf{N}$; the Order Limit Theorem (Theorem 2.3.4) then implies that $b \leq a$ and it follows that a is an upper bound for B.

(b) This is true. Observe that for a real number x we have

$$x \notin (0,1) \iff x \leq 0 \text{ or } x \geq 1 \iff \left|x - \frac{1}{2}\right| \geq \frac{1}{2}.$$

So for each $n \in \mathbf{N}$ we have $\left|a_n - \frac{1}{2}\right| \geq \frac{1}{2}$. The Algebraic Limit Theorem (Theorem 2.3.3) and Exercise 2.3.10 (b) imply that $\lim\left|a_n - \frac{1}{2}\right| = \left|a - \frac{1}{2}\right|$, and hence the Order Limit Theorem (Theorem 2.3.4) implies that $\left|a - \frac{1}{2}\right| \geq \frac{1}{2}$. It follows that a belongs to the complement of $(0,1)$.

(c) This is false. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), for each $n \in \mathbf{N}$ we may pick a rational number a_n satisfying $\sqrt{2} < a_n < \sqrt{2} + \frac{1}{n}$. The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim a_n = \sqrt{2}$, which is an irrational number.

Exercise 2.3.13 (Iterated Limits). Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the *iterated* limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \qquad \text{and} \qquad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right).$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2+n^2)$.

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) = b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

Solution. (a) We apply the Algebraic Limit Theorem (Theorem 2.3.3):

$$\lim_{m\to\infty} a_{mn} = \lim_{m\to\infty}\left(\frac{m}{m+n}\right) = \lim_{m\to\infty}\left(\frac{1}{1+\frac{n}{m}}\right) = \frac{1}{1 + n\lim_{m\to\infty}\left(\frac{1}{m}\right)} = \frac{1}{1} = 1.$$

Hence $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn}) = \lim_{n\to\infty}(1) = 1$. Similarly,

$$\lim_{n\to\infty} a_{mn} = \lim_{n\to\infty}\left(\frac{m}{m+n}\right) = \lim_{n\to\infty}\left(\frac{\frac{m}{n}}{1+\frac{m}{n}}\right) = \frac{m\lim_{n\to\infty}\left(\frac{1}{n}\right)}{1+m\lim_{n\to\infty}\left(\frac{1}{n}\right)} = \frac{0}{1} = 0.$$

Thus $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn}) = \lim_{m\to\infty}(0) = 0$.

(b) For $a_{mn} = \frac{1}{m+n}$, we have $\lim_{m,n\to\infty} a_{mn} = 0$. To see this, let $\epsilon > 0$ be given. There is an $N \in \mathbf{N}$ such that $\frac{1}{n} < \epsilon$ whenever $n \geq N$, so that for $m, n \geq N$ we have

$$|a_{mn}| = \frac{1}{m+n} < \frac{1}{n} < \epsilon.$$

Thus $\lim_{m,n\to\infty} a_{mn} = 0$. The two iterated limits also exist and are equal to 0. Indeed, observe that for all $m, n \in \mathbf{N}$ we have $0 < \frac{1}{m+n} < \frac{1}{m}$. The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim_{m\to\infty} a_{mn} = \lim_{m\to\infty}\frac{1}{m+n} = 0$ and it follows that $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn}) = \lim_{n\to\infty}(0) = 0$. A similar argument shows that $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn}) = 0$.

Now let $a_{mn} = \frac{mn}{m^2+n^2}$; we claim that $\lim_{m,n\to\infty} a_{mn}$ does not exist. To see this, let us seek a contradiction and suppose that $\lim_{m,n\to\infty} a_{mn} = x$ for some $x \in \mathbf{R}$. There then exists an $N \in \mathbf{N}$ such that $|a_{mn} - x| < \frac{1}{20}$ whenever $m, n \geq N$. In particular, taking $n = m$,

$$m \geq N \implies \left|\frac{m^2}{m^2+m^2} - x\right| = \left|\frac{1}{2} - x\right| < \frac{1}{20} \iff x \in \left(\frac{9}{20}, \frac{11}{20}\right).$$

Similarly, taking $n = 2m$,

$$m \geq N \implies \left|\frac{2m^2}{m^2+4m^2} - x\right| = \left|\frac{2}{5} - x\right| < \frac{1}{20} \iff x \in \left(\frac{7}{20}, \frac{9}{20}\right).$$

So assuming that $\lim_{m,n\to\infty} a_{mn} = x$ for some $x \in \mathbf{R}$ leads us to the contradiction that $x < \frac{9}{20}$ and $x > \frac{9}{20}$; it follows that $\lim_{m,n\to\infty} a_{mn}$ does not exist. However, the two iterated limits do exist and are equal to 0. Using the Algebraic Limit Theorem (Theorem 2.3.3), for any $n \in \mathbf{N}$ we have

$$\lim_{m\to\infty}\left(\frac{mn}{m^2+n^2}\right) = \lim_{m\to\infty}\left(\frac{\frac{n}{m}}{1+\frac{n^2}{m^2}}\right) = \frac{n\lim_{m\to\infty}\left(\frac{1}{m}\right)}{1+n^2\lim_{m\to\infty}\left(\frac{1}{m^2}\right)} = \frac{0}{1} = 0.$$

It follows that $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn}) = 0$ and we can use a similar argument to show that $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn}) = 0$.

(c) Let $a_{mn} = (-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right)$; we claim that $\lim_{m,n\to\infty} a_{mn} = 0$. To see this, let $\epsilon > 0$ be given. There is an $N \in \mathbf{N}$ such that $\frac{1}{n} < \frac{\epsilon}{2}$ whenever $n \geq N$. For $m, n \geq N$ we then have

$$|a_{mn}| = \left|(-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right)\right| = \frac{1}{m} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\lim_{m,n\to\infty} a_{mn} = 0$. However, neither iterated limit exists. Fix $n \in \mathbf{N}$ and observe that

$$\begin{aligned}
|a_{mn} - a_{m+1,n}| &= \left|(-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right) - (-1)^{m+n+1}\left(\frac{1}{m+1} + \frac{1}{n}\right)\right| \\
&= \left|(-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n} + \frac{1}{m+1} + \frac{1}{n}\right)\right| \\
&= \left|\frac{1}{m} + \frac{1}{n} + \frac{1}{m+1} + \frac{1}{n}\right| \\
&= \frac{1}{m} + \frac{1}{m+1} + \frac{2}{n} \\
&\geq \frac{2}{n}.
\end{aligned}$$

Since $n \in \mathbf{N}$ is fixed, this implies that the sequence $(a_{mn} - a_{m+1,n})_{m=1}^{\infty}$ cannot converge to 0. Now observe that for any sequence (b_m), the Algebraic Limit Theorem (Theorem 2.3.3) implies that

$$\lim_{m\to\infty} b_m \text{ exists} \implies \lim_{m\to\infty}(b_m - b_{m+1}) = 0.$$

The contrapositive of this statement then implies that the limit $\lim_{m\to\infty} a_{mn}$ does not exist for any $n \in \mathbf{N}$ and it follows that the iterated limit $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn})$ does not exist. Using the symmetry of a_{mn} and swapping the roles of m and n in our argument shows that the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn})$ does not exist either.

(d) Seeking a contradiction, suppose that (b_m) does not converge to a, i.e., there is some $\epsilon > 0$ such that for all $N \in \mathbf{N}$ there is an $M \geq N$ such that $|b_M - a| \geq \epsilon$. Since $\lim_{m,n\to\infty} a_{mn} = a$, there exists some $N_1 \in \mathbf{N}$ such that

$$m, n \geq N_1 \implies |a_{mn} - a| < \frac{\epsilon}{2}. \tag{1}$$

By the previous discussion, there is an $M \geq N_1$ such that $|b_M - a| \geq \epsilon$. By assumption we have $\lim_{n\to\infty} a_{Mn} = b_M$, so there is an $N_2 \in \mathbf{N}$ such that $|a_{Mn} - b_M| < \frac{\epsilon}{2}$ whenever

$n \geq N_2$. Let $N = \max\{N_1, N_2\}$ and observe that $|a_{MN} - a| < \frac{\epsilon}{2}$ by (1). However, the reverse triangle inequality (Exercise 1.2.6 (d)) gives us

$$\begin{aligned} |a_{MN} - a| &= |a_{MN} - b_M + b_M - a| \\ &\geq ||b_M - a| - |a_{MN} - b_M|| \\ &\geq |b_M - a| - |a_{MN} - b_M| \\ &> \epsilon - \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

So assuming that (b_m) does not converge to a leads us the contradiction that there exist positive integers M and N such that $|a_{MN} - a|$ is both less than and greater than $\frac{\epsilon}{2}$. We may conclude that $\lim_{m\to\infty} b_m = a$.

(e) If the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn})$ exists, then it must be the case that for each fixed $m \in \mathbf{N}$, the limit $\lim_{n\to\infty} a_{mn}$ exists. Part (d) then implies that

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = \lim_{m,n\to\infty} a_{mn}.$$

Swapping the roles of m and n and repeating the above argument shows that

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{m,n\to\infty} a_{mn}.$$

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1. (a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Solution. See Figure F.9 for a graph of the first thirty terms of (x_n).

(a) Let $P(n)$ be the statement that $x_{n+1} \le x_n$ and $x_n \ge -1$; we will use strong induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. Since $x_1 = 3$ and $x_2 = 1$, we see that $P(1)$ holds. Now suppose that $P(1), \dots, P(n)$ all hold for some $n \in \mathbf{N}$ and observe that

$$x_{n+1} \le x_n \le 3 \implies 1 \le 4 - x_n \le 4 - x_{n+1} \implies \frac{1}{4 - x_{n+1}} \le \frac{1}{4 - x_n},$$

i.e., $x_{n+2} \le x_{n+1}$. Furthermore,

$$-1 \le x_n \le 3 \implies 1 \le 4 - x_n \le 5 \implies x_{n+1} = \frac{1}{4 - x_n} \ge \frac{1}{5} > -1.$$

Thus $P(n+1)$ holds. This completes the induction step and it follows that $P(n)$ holds for all $n \in \mathbf{N}$.

We have now shown that the sequence (x_n) is bounded below and decreasing and hence by the Monotone Convergence Theorem (Theorem 2.4.2) we may conclude that the sequence converges.

(b) If (x_n) is any convergent sequence with $\lim x_n = x$, then the sequence (y_n) given by $y_n = x_{n+k}$ for any $k \in \mathbf{N}$ is also convergent with $\lim y_n = x$. To see this, let $\epsilon > 0$ be given. Since $x_n \to x$, there exists an $N \in \mathbf{N}$ such that $|x_n - x| < \epsilon$ whenever $n \ge N$. Suppose $n \ge \max\{N - k, 1\}$, so that $n + k \ge N$. It follows that

$$|y_n - x| = |x_{n+k} - x| < \epsilon.$$

Thus $\lim y_n = x$.

(c) By parts (a) and (b) we have $\lim x_n = \lim x_{n+1} = x$ for some $x \in \mathbf{R}$. It then follows from the Algebraic Limit Theorem (Theorem 2.3.3) that

$$x_{n+1} = \frac{1}{4 - x_n} \implies \lim x_{n+1} = \frac{1}{4 - \lim x_n} \iff x = \frac{1}{4 - x} \iff x^2 - 4x + 1 = 0.$$

This quadratic equation has solutions $x = 2 \pm \sqrt{3}$. Since (x_n) is decreasing and $x_2 = 1$, the Order Limit Theorem (Theorem 2.3.4) implies that $\lim x_n = x \le 1 < 2 + \sqrt{3}$ and so we may discard the solution $x = 2 + \sqrt{3}$ to conclude that $\lim x_n = 2 - \sqrt{3}$.

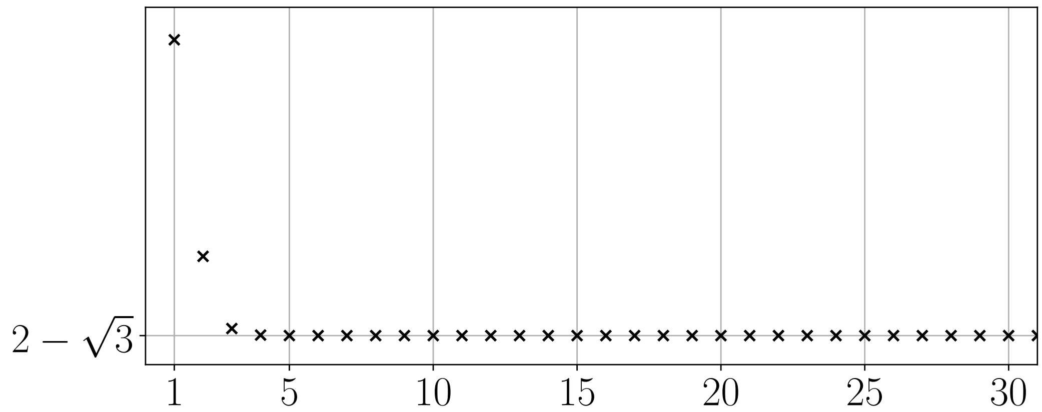


Figure F.9: x_n for $1 \leq n \leq 30$

Exercise 2.4.2. (a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$.

What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Solution. (a) The problem is we have assumed that $\lim y_n$ exists. Looking at the first few terms of the sequence $y_1 = 1, y_2 = 2, y_3 = 1, y_4 = 2, \ldots$, we see that in fact the sequence oscillates and does not converge.

(b) The strategy works this time. Let $P(n)$ be the statement that $y_{n+1} \geq y_n$ and $y_n \leq 3$; we will use strong induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. Since $y_1 = 1$ and $y_2 = 2$, we see that $P(1)$ holds. Suppose that $P(1), \ldots, P(n)$ all hold for some $n \in \mathbf{N}$ and observe that

$$y_{n+1} \geq y_n \geq 1 \implies \frac{1}{y_{n+1}} \leq \frac{1}{y_n} \implies 3 - \frac{1}{y_{n+1}} \geq 3 - \frac{1}{y_n},$$

i.e., $y_{n+2} \geq y_{n+1}$. Furthermore,

$$1 \leq y_n \leq 3 \implies \frac{1}{3} \leq \frac{1}{y_n} \implies y_{n+1} = 3 - \frac{1}{y_n} \leq \frac{8}{3} < 3.$$

Thus $P(n+1)$ holds. This completes the induction step and it follows that $P(n)$ holds for all $n \in \mathbf{N}$.

We have now shown that (y_n) is bounded above and increasing, so by the Monotone Convergence Theorem (Theorem 2.4.2) we have $\lim y_n = y$ for some $y \in \mathbf{R}$. Given this, the following manipulations are valid:

$$y_{n+1} = 3 - \frac{1}{y_n} \implies y = 3 - \frac{1}{y} \iff y^2 - 3y + 1 = 0.$$

This quadratic equation has solutions $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$. Since (y_n) is increasing and $y_2 = 2$, we must have $y \geq 2 > \frac{3}{2} - \frac{1}{2}\sqrt{5}$ and so we may discard the solution $y = \frac{3}{2} - \frac{1}{2}\sqrt{5}$ to conclude that $\lim y_n = \frac{3}{2} + \frac{1}{2}\sqrt{5}$. See Figure F.10 for a graph of the first thirty terms of (y_n).

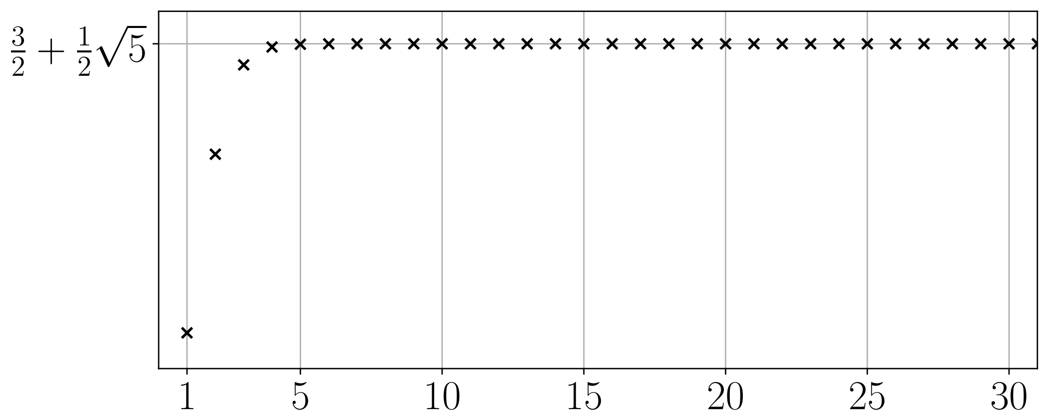


Figure F.10: y_n for $1 \leq n \leq 30$

Exercise 2.4.3. (a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

Solution. (a) Let $x_1 = \sqrt{2}, x_{n+1} = \sqrt{2 + x_n}$, and let $P(n)$ be the statement that $x_{n+1} \geq x_n$ and $x_n \leq 2$; we will use strong induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. Since $x_1 = \sqrt{2}$ and $x_2 = \sqrt{2 + \sqrt{2}}$, we see that $P(1)$ holds. Suppose that $P(1), \ldots, P(n)$ all hold for some $n \in \mathbf{N}$ and observe that

$$x_{n+1} \geq x_n \geq \sqrt{2} \implies \sqrt{2 + x_{n+1}} \geq \sqrt{2 + x_n},$$

i.e., $x_{n+2} \geq x_{n+1}$. Furthermore,

$$\sqrt{2} \leq x_n \leq 2 \implies \sqrt{2 + x_n} \leq \sqrt{4} = 2.$$

Thus $P(n+1)$ holds. This completes the induction step and it follows that $P(n)$ holds for all $n \in \mathbf{N}$.

We have now shown that the sequence (x_n) is bounded above and increasing, so by the Monotone Convergence Theorem (Theorem 2.4.2) we have $\lim x_n = x$ for some $x \in \mathbf{R}$. We may now take the limit on both sides of the recursive equation:

$$x_{n+1} = \sqrt{2 + x_n} \implies x = \sqrt{2 + x} \implies x^2 - x - 2 = 0 \iff (x-2)(x+1) = 0.$$

So $x = 2$ or $x = -1$. Since the sequence is increasing and $x_1 = \sqrt{2}$, we must have $x \geq \sqrt{2} > -1$ and thus $\lim x_n = 2$. See Figure F.11 for a graph of the first thirty terms of the sequence (x_n).

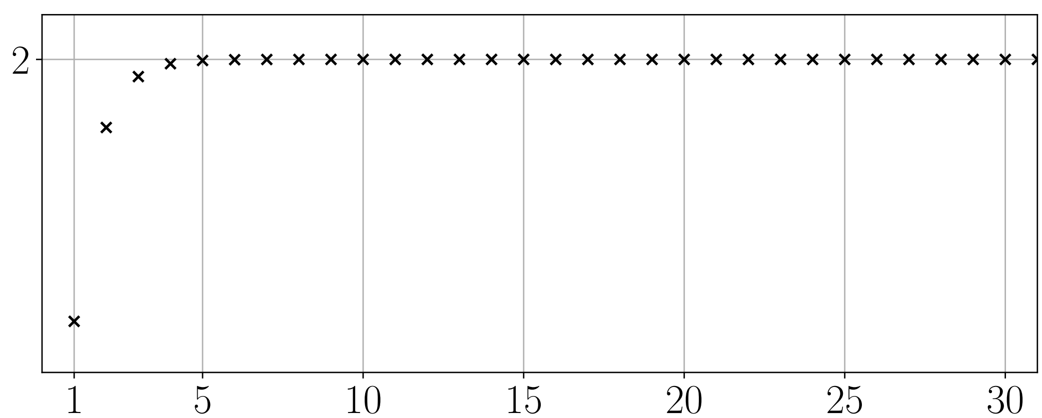


Figure F.11: x_n for $1 \leq n \leq 30$

(b) The sequence does converge. Let $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$, and let $P(n)$ be the statement that $x_{n+1} \geq x_n$ and $x_n \leq 2$. We will use strong induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. Since $x_1 = \sqrt{2}$ and $x_2 = \sqrt{2\sqrt{2}}$, we see that $P(1)$ holds. Suppose that $P(1), \ldots, P(n)$ all hold for some $n \in \mathbf{N}$ and observe that

$$x_{n+1} \geq x_n \geq \sqrt{2} \implies \sqrt{2x_{n+1}} \geq \sqrt{2x_n},$$

i.e., $x_{n+2} \geq x_{n+1}$. Furthermore,

$$\sqrt{2} \leq x_n \leq 2 \implies \sqrt{2x_n} \leq \sqrt{4} = 2.$$

Thus $P(n+1)$ holds. This completes the induction step and it follows that $P(n)$ holds for all $n \in \mathbf{N}$.

We have shown that the sequence (x_n) is bounded above and increasing, so by the Monotone Convergence Theorem (Theorem 2.4.2) we have $\lim x_n = x$ for some $x \in \mathbf{R}$. We may now take the limit on both sides of the recursive equation:

$$x_{n+1} = \sqrt{2x_n} \implies x = \sqrt{2x} \implies x^2 - 2x = 0 \iff x(x-2) = 0.$$

So $x = 2$ or $x = 0$. Since the sequence is increasing and $x_1 = \sqrt{2}$, we must have $x \geq \sqrt{2} > 0$ and thus $\lim x_n = 2$. See Figure F.12 for a graph of the first thirty terms of (x_n).

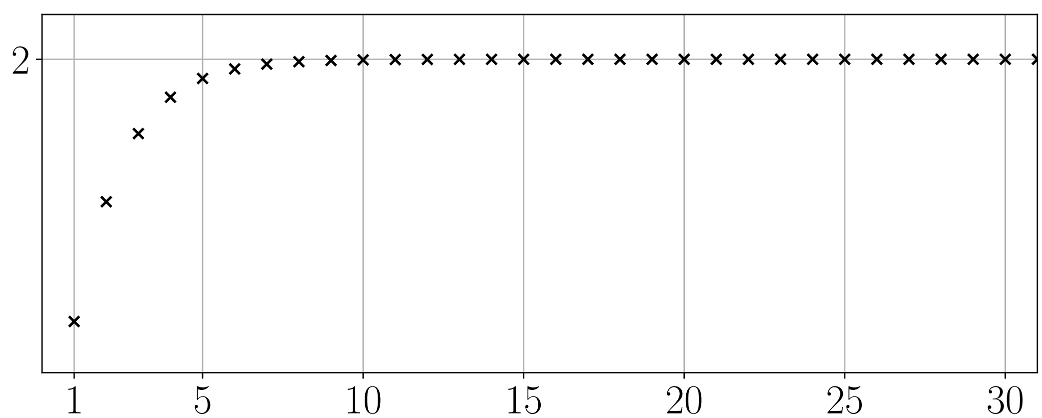


Figure F.12: x_n for $1 \leq n \leq 30$

Exercise 2.4.4. (a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbf{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.

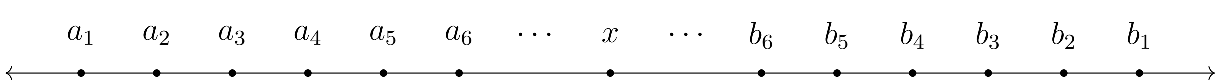
These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

Solution. (a) Assuming that any bounded monotone sequence converges, we want to prove part (i) of Theorem 1.4.2: for any $x \in \mathbf{R}$, there exists an $n \in \mathbf{N}$ satisfying $n > x$. Part (ii) of Theorem 1.4.2 will then follow by taking $x = \frac{1}{y}$ in part (i). Let x be given. Seeking a contradiction, suppose that $n \leq x$ for each $n \in \mathbf{N}$. Then the sequence (n) is bounded above and clearly monotone increasing, so by assumption this sequence converges, say $\lim n = y$ for some $y \in \mathbf{R}$. There then exists an $N \in \mathbf{N}$ such that $|n - y| < \frac{1}{2}$ whenever $n \geq N$. Observe that

$$1 = |N + 1 - y + y - N| \leq |N + 1 - y| + |N - y| < \frac{1}{2} + \frac{1}{2} = 1,$$

i.e., $1 < 1$, which is a contradiction. We may conclude that there exists some $n \in \mathbf{N}$ such that $n > x$.

(b) Assuming that any bounded monotone sequence converges, we want to prove that any sequence of nested intervals $I_n = [a_n, b_n]$ has non-empty intersection. Consider the sequence (a_n) of left-hand endpoints. Because the intervals are nested, this is an increasing sequence which is bounded above by any right-hand endpoint, so by assumption this sequence converges, say $\lim a_n = x$ for some $x \in \mathbf{R}$. For any $n \in \mathbf{N}$ we have $a_n \leq a_m \leq b_m \leq b_n$ for all $m \geq n$. The Order Limit Theorem (Theorem 2.3.4) then implies that $x = \lim_{m\to\infty} a_m \leq b_n$ and $a_n \leq \lim_{m\to\infty} a_m = x$; it follows that $a_n \leq x \leq b_n$ for all $n \in \mathbf{N}$, i.e., $x \in \bigcap_{n=1}^{\infty} I_n$.



Exercise 2.4.5 (Calculating Square Roots). Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution. (a) Let $P(n)$ be the statement that $x_n \geq \sqrt{2}$. We will use induction to show that $P(n)$ holds for all $n \in \mathbf{N}$. The truth of $P(1)$ is clear, so suppose that $P(n)$ holds for some $n \in \mathbf{N}$. Observe that

$$\left(x_n - \sqrt{2}\right)^2 = x_n^2 - 2\sqrt{2}x_n + 2 \geq 0.$$

Our induction hypothesis guarantees that $x_n \geq \sqrt{2} > 0$ and so we may divide by x_n to obtain the inequality

$$x_n - 2\sqrt{2} + \frac{2}{x_n} \geq 0 \iff \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq \sqrt{2},$$

i.e., $x_{n+1} \geq \sqrt{2}$. This completes the induction step and thus $P(n)$ holds for all $n \in \mathbf{N}$; in particular, we have $x_n^2 \geq 2$ for each $n \in \mathbf{N}$. Given this, for any $n \in \mathbf{N}$ we have

$$\begin{aligned} x_n^2 - 2 \geq 0 \iff x_n - \frac{2}{x_n} \geq 0 \iff \frac{x_n}{2} - \frac{1}{x_n} \geq 0 \\ \iff x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq 0 \iff x_n - x_{n+1} \geq 0. \end{aligned}$$

It follows that the sequence (x_n) satisfies $x_{n+1} \leq x_n$ for all $n \in \mathbf{N}$.

We have now shown that the sequence (x_n) is decreasing and bounded below. The Monotone Convergence Theorem (Theorem 2.4.2) then implies that $\lim x_n = x$ for some $x \in \mathbf{R}$. Since $x_n \geq \sqrt{2}$ for all $n \in \mathbf{N}$, the Order Limit Theorem (Theorem 2.3.4) implies that $x \geq \sqrt{2} > 0$ and so we can use the Algebraic Limit Theorem (Theorem 2.3.3) to take the limit across the recursive equation:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \implies x = \frac{1}{2}\left(x + \frac{2}{x}\right) \implies x^2 = 2.$$

Since $x \geq \sqrt{2}$, we may conclude that $x = \sqrt{2}$. See Figure F.13 for a graph of the first thirty terms of the sequence (x_n).

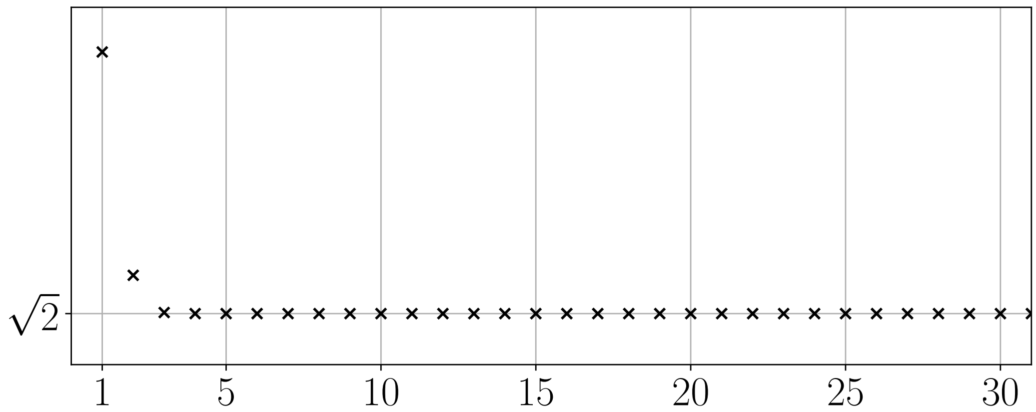


Figure F.13: x_n for $1 \le n \le 30$

(b) For $c \ge 0$, let $x_1 = 1 + c$ and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right).$$

Repeating the argument given in part (a), replacing 2 with c where appropriate, shows that $\lim x_n = \sqrt{c}$. For the base case of the induction argument, note that

$$x_1 = 1 + c > 1 \implies x_1 = 1 + c > \sqrt{1+c} > \sqrt{c}.$$

Exercise 2.4.6 (Arithmetic-Geometric Mean). (a) Explain why $\sqrt{xy} \le (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(b) Now let $0 \le x_1 \le y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \qquad \text{and} \qquad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

Solution. (a) Observe that

$$0 \le (x-y)^2 \iff 0 \le x^2 - 2xy + y^2 \iff 4xy \le x^2 + 2xy + y^2 \iff 4xy \le (x+y)^2.$$

Since x and y are both positive, this implies that $\sqrt{xy} \le \frac{x+y}{2}$.

(b) By part (a), we have $x_n \le y_n$ for all $n \in \mathbf{N}$. It follows that

$$y_{n+1} = \frac{x_n + y_n}{2} \le \frac{y_n + y_n}{2} = y_n \qquad \text{and} \qquad x_{n+1} = \sqrt{x_n y_n} \ge \sqrt{x_n^2} = x_n.$$

Thus (x_n) is increasing and (y_n) is decreasing. Furthermore, (y_n) is bounded below: for any $n \in \mathbf{N}$, we have

$$y_n \ge x_n \ge \cdots \ge x_1.$$

It follows from the Monotone Convergence Theorem (Theorem 2.4.2) that $\lim y_n = y$ for some $y \in \mathbf{R}$. The Algebraic Limit Theorem (Theorem 2.3.3) then gives

$$x_n = 2y_{n+1} - y_n \implies \lim x_n = 2\lim y_{n+1} - \lim y_n = 2y - y = y.$$

Exercise 2.4.7 (Limit Superior). Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \ge n\}$ converges.

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \le \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Solution. (a) Suppose $M > 0$ is the bound for (a_n), i.e., $|a_n| \le M$ for all $n \in \mathbf{N}$. It follows that $y_n \ge a_n \ge -M$ for any $n \in \mathbf{N}$, so that the sequence (y_n) is bounded below. Furthermore, for any $n \in \mathbf{N}$ we have

$$\{a_k : k \ge n+1\} \subseteq \{a_k : k \ge n\} \implies \sup\{a_k : k \ge n+1\} \le \sup\{a_k : k \ge n\}$$

$$\iff y_{n+1} \le y_n,$$

i.e., the sequence (y_n) is decreasing. We may now invoke the Monotone Convergence Theorem (Theorem 2.4.2) to conclude that (y_n) converges.

(b) Let $z_n = \inf\{a_k : k \geq n\}$. Similarly to part (a), we can show that this sequence is bounded above, increasing, and hence convergent. We then define the limit inferior as $\liminf a_n = \lim z_n$.

(c) The infimum of a bounded set is always less than or equal to the supremum of that set, so we have $z_n \leq y_n$ for each $n \in \mathbf{N}$. The Order Limit Theorem (Theorem 2.3.4) then implies that $\lim z_n \leq \lim y_n$, i.e., $\liminf a_n \leq \limsup a_n$.

For an example of a bounded sequence where this inequality is strict, consider the sequence $a_n = (-1)^n$. For this sequence we have $y_n = (1, 1, 1, \ldots)$ and $z_n = (-1, -1, -1, \ldots)$, so that $\liminf a_n = -1 < 1 = \limsup a_n$.

(d) Suppose $\liminf a_n = \limsup a_n$. Since $z_n \leq a_n \leq y_n$ for all $n \in \mathbf{N}$, the Squeeze Theorem (Exercise 2.3.3) implies that (a_n) converges and that $\liminf a_n = \limsup a_n = \lim a_n$.

Now suppose that $\lim a_n = a$ for some $a \in \mathbf{R}$ and let $\epsilon > 0$ be given. Since $a_n \to a$, there is an $N \in \mathbf{N}$ such that

$$n \geq N \implies a - \frac{\epsilon}{2} < a_n < a + \frac{\epsilon}{2}.$$

This implies that $a - \frac{\epsilon}{2}$ is a lower bound for $\{a_k : k \geq N\}$ and that $a + \frac{\epsilon}{2}$ is an upper bound for $\{a_k : k \geq N\}$. It follows that $a - \frac{\epsilon}{2} \leq z_N \leq a_N \leq y_N \leq a + \frac{\epsilon}{2}$. Since (z_n) is increasing and (y_n) is decreasing, we then have

$$n \geq N \implies a - \epsilon < a - \frac{\epsilon}{2} \leq z_N \leq z_n \leq a_n \leq y_n \leq y_N \leq a + \frac{\epsilon}{2} < a + \epsilon,$$

i.e., $|z_n - a| < \epsilon$ and $|y_n - a| < \epsilon$ for all $n \geq N$. It follows that $\liminf a_n = \limsup a_n = \lim a_n = a$.

Exercise 2.4.8. For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \qquad \text{(c)} \sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$$

(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

Solution. For each series, let (s_m) be its sequence of partial sums; see Figure F.14 for graphs of the first thirty terms of these partial sum sequences.

(a) Here we have

$$\begin{aligned} s_m &= \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-1}} + \frac{1}{2^m} \\ \implies 2s_m &= 1 + \frac{1}{2} + \cdots + \frac{1}{2^m} + \frac{1}{2^{m+1}} \\ \implies 2s_m &= \frac{1 - 2^{-(m+2)}}{1 - \frac{1}{2}} \\ \implies s_m &= 1 - \frac{1}{2^{m+2}}, \end{aligned}$$

where we have used the formula $(1-x)(1+x+x^2+\cdots+x^n) = 1 - x^{n+1}$. It follows that $\lim s_m = 1$.

(b) For this series,

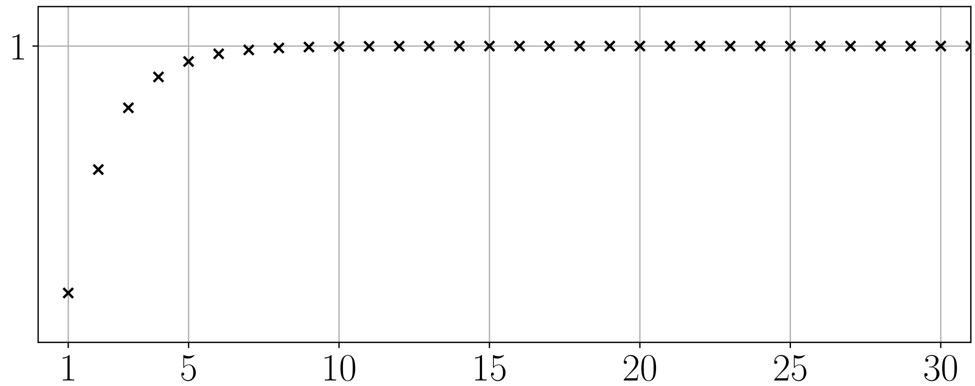
$$\begin{aligned} s_m = \sum_{n=1}^{m} \frac{1}{n(n+1)} &= \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m} - \frac{1}{m+1}\right) = 1 - \frac{1}{m+1}. \end{aligned}$$

It follows that $\lim s_m = 1$.

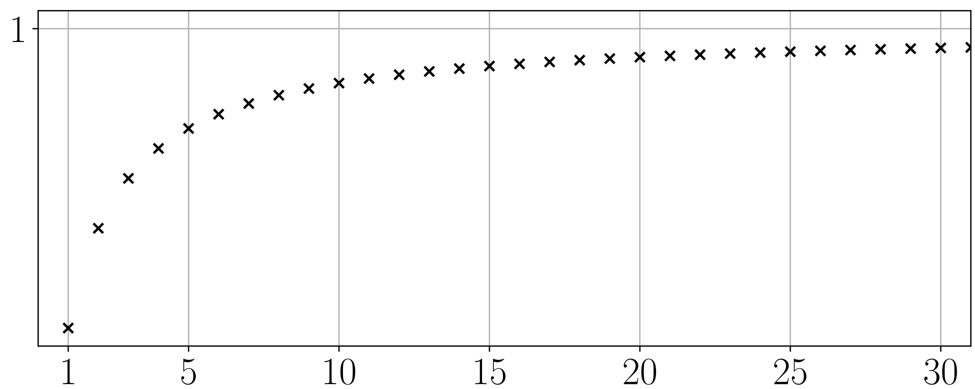
(c) We have

$$\begin{aligned} s_m &= \sum_{n=1}^{m} \log\left(\frac{n+1}{n}\right) \\ &= \sum_{n=1}^{m} (\log(n+1) - \log(n)) \\ &= (\log(2) - \log(1)) + (\log(3) - \log(2)) + \cdots + (\log(m+1) - \log(m)) \\ &= \log(m+1). \end{aligned}$$

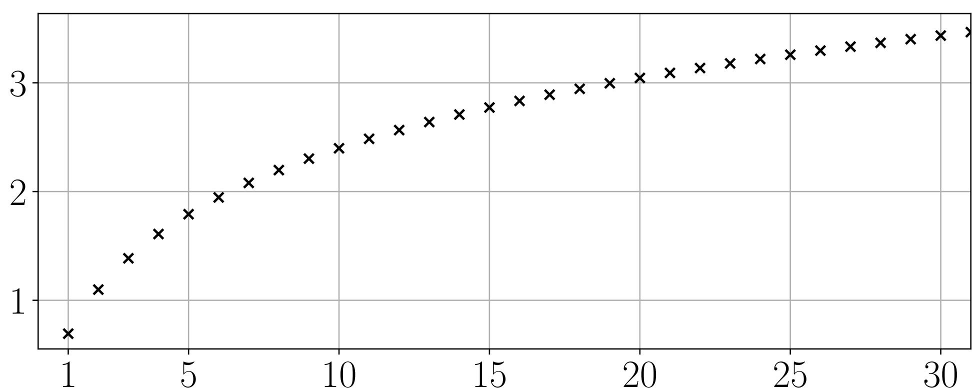
So $s_m = \log(m+1)$, which is unbounded and hence not convergent.



(a) $\sum_{n=1}^{m} \frac{1}{2^n}$ for $1 \leq m \leq 30$



(b) $\sum_{n=1}^{m} \frac{1}{n(n+1)}$ for $1 \leq m \leq 30$



(c) $\sum_{n=1}^{m} \log\left(\frac{n+1}{n}\right)$ for $1 \leq m \leq 30$

Figure F.14: Exercise 2.4.8 partial sums for $1 \leq m \leq 30$

Exercise 2.4.9. Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Solution. Define the sequences of partial sums

$$s_m = b_1 + b_2 + \cdots + b_m \qquad \text{and} \qquad t_m = b_1 + 2b_2 + \cdots + 2^m b_{2^m}.$$

We will use induction to show that $t_m \leq 2s_{2^m}$ for each $m \in \mathbf{N}$. For the base case $m = 1$ we have

$$t_1 = b_1 + 2b_2 \leq 2b_1 + 2b_2 = 2s_2,$$

where we have used that b_1 is non-negative. Suppose that the inequality holds for some $m \in \mathbf{N}$. Because the sequence (b_n) is decreasing, we have $b_{2^{m+1}} \leq b_{2^m+j}$ for each $1 \leq j \leq 2^m$. It follows that $2^m b_{2^{m+1}} \leq \sum_{j=1}^{2^m} b_{2^m+j}$. Combining this inequality with our induction hypothesis, we see that

$$t_{m+1} = t^m + 2^{m+1} b_{2^{m+1}} \leq 2s_{2^m} + 2\sum_{j=1}^{2^m} b_{2^m+j} = 2s_{2^{m+1}}.$$

This completes the induction step.

Since each b_n is non-negative, both sequences of partial sums (s_m) and (t_m) are increasing. It follows from the Monotone Convergence Theorem (Theorem 2.4.2) that the convergence of each series is equivalent to the boundedness of the respective sequence of partial sums. Given this, we want to show that if (t_m) is unbounded, then so is (s_m); this follows immediately from the inequality $t_m \leq 2s_{2^m}$.

Exercise 2.4.10 (Infinite Products). A close relative of infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots, \quad \text{where } a_n \geq 0.$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

Solution. (a) For $a_n = \frac{1}{n}$, observe that

$$p_m = \prod_{n=1}^{m}\left(1 + \frac{1}{n}\right) = \prod_{n=1}^{m}\left(\frac{n+1}{n}\right) = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{m}{m-1} \cdot \frac{m+1}{m}$$

$$= \frac{2}{2} \cdot \frac{3}{3} \cdot \frac{4}{4} \cdots \frac{m}{m} \cdot (m+1) = m+1.$$

It follows that (p_m) does not converge.

For $a_n = \frac{1}{n^2}$, the first few partial products are

$$\begin{aligned} p_1 &= 2, \\ p_2 &= 2(1 + 1/4) = 5/2 = 2.5, \\ p_3 &= (5/2)(1 + 1/9) = 25/9 \approx 2.778, \\ p_4 &= (25/9)(1 + 1/16) = 425/144 \approx 2.951. \end{aligned}$$

It looks like the partial products could be bounded; we conjecture that this infinite product converges. Indeed, part (b) proves our conjecture, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series; the Weierstrass factorization theorem can be used to show that

$$\prod_{n=1}^{\infty}\left(1 + \frac{1}{n^2}\right) = \frac{\sinh(\pi)}{\pi}.$$

See Figure F.15 for a graph of the first thirty terms of the partial product sequence.

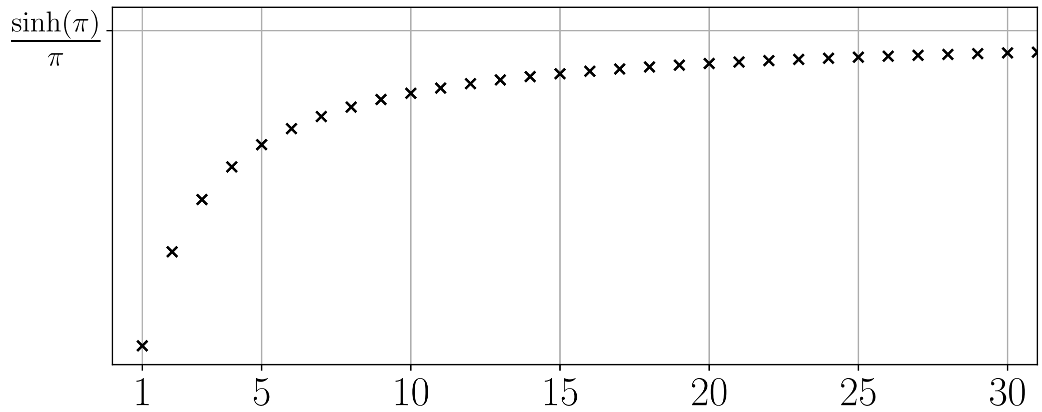


Figure F.15: $\prod_{n=1}^{m}\left(1+\frac{1}{n^2}\right)$ for $1 \le m \le 30$

(b) Let

$$s_m = \sum_{n=1}^{m} a_n \qquad \text{and} \qquad p_m = \prod_{n=1}^{m}(1+a_n).$$

Since $a_n \ge 0$ for all $n \in \mathbf{N}$, the sequence of partial sums and the sequence of partial products are both non-negative and increasing. It follows from the Monotone Convergence Theorem (Theorem 2.4.2) that the convergence of each sequence is equivalent to the boundedness of that sequence. By multiplying out the terms in the partial product p_m, we would obtain the sum s_m and some other non-negative terms; it follows that $s_m \le p_m$. The hint gives us

$$p_m = \prod_{n=1}^{m}(1+a_n) \le \prod_{n=1}^{m} 3^{a_n} = 3^{\sum_{n=1}^{m} a_n} = 3^{s_m}.$$

So we have the inequalities $s_m \le p_m \le 3^{s_m}$. It is then clear that (s_m) is bounded if (p_m) is bounded, and furthermore if (s_m) is bounded by some $M > 0$, then (p_m) is bounded by 3^M. It follows that for this special case, $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

2.5 Subsequences and the Bolzano-Weierstrass Theorem

Exercise 2.5.1. Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set

$$\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}.$$

(d) A sequence that contains subsequences converging to every point in the infinite set

$$\{1, 1/2, 1/3, 1/4, 1/5, \ldots\},$$

and no subsequences converging to points outside of this set.

Solution. (a) This is impossible. If a sequence (a_n) has a bounded subsequence (a_{n_k}), then by the Bolzano-Weierstrass Theorem (Theorem 2.5.5) there must be a convergent sub-subsequence $\left(a_{n_{k_\ell}}\right)$; this is also a convergent subsequence of the original sequence (a_n).

(b) Consider the sequence

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}, \ldots\right),$$

i.e., the sequence (a_n) given by

$$a_n = \begin{cases} \frac{1}{n+1} & \text{if } n \text{ is odd,} \\ 1 - \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

This sequence does not contain 0 or 1 as a term, the subsequence (a_{2n-1}) converges to 0, and the subsequence (a_{2n}) converges to 1.

(c) Consider the following infinite array:

$$\begin{array}{cccccc}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Let (a_n) be the sequence obtained by following the diagonals of this array, i.e.,

$$(a_n) = \left(1, 1, \tfrac{1}{2}, 1, \tfrac{1}{2}, \tfrac{1}{3}, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, 1, \ldots\right).$$

The subsequence given by the n^{th} column is $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots\right)$, which converges to $\frac{1}{n}$.

(d) This is impossible. Suppose that (a_n) is a sequence that contains subsequences converging to every point in the infinite set

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}.$$

We claim that (a_n) must have a subsequence converging to 0. We will construct this subsequence recursively as follows. Since there is a subsequence converging to 1, there must be some index n_1 such that

$$|a_{n_1} - 1| < 1 \iff 0 < a_{n_1} < 2.$$

Since there is a subsequence converging to $\frac{1}{2}$, there must be some index $n_2 > n_1$ such that

$$\left|a_{n_2} - \frac{1}{2}\right| < \frac{1}{2} \iff 0 < a_{n_2} < 1.$$

We continue in this manner, obtaining a subsequence (a_{n_k}) satisfying $0 < a_{n_k} < \frac{2}{k}$. The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim_{k\to\infty} a_{n_k} = 0$.

Exercise 2.5.2. Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Solution. (a) This is true. By assumption, the subsequence $(x_2, x_3, x_4, \ldots)$ converges; it follows that (x_n) also converges to the same limit (see Exercise 2.4.1 (b)).

(b) This is true. Consider the contrapositive statement: if (x_n) converges, then all subsequences of (x_n) converge. This is implied by Theorem 2.5.2.

(c) This is true. Consider the sequences

$$y_n = \sup\{x_m : m \geq n\} \qquad \text{and} \qquad z_n = \inf\{x_m : m \geq n\}.$$

As shown in Exercise 2.4.7, these sequences both converge since (x_n) is bounded and their limits are denoted by

$$\limsup x_n = \lim y_n \qquad \text{and} \qquad \liminf x_n = \lim z_n.$$

We claim that there are subsequences of (x_n) converging to $\limsup x_n$ and $\liminf x_n$. First, we will recursively construct a subsequence converging to $\limsup x_n$. Let $n_0 = 0$. By Lemma 1.3.8, there exists an $n_1 \geq 1$ such that $y_1 - 1 < x_{n_1} \leq y_1$. There then exists an $n_2 \geq n_1 + 1$ such that $y_{n_1+1} - \frac{1}{2} < x_{n_2} \leq y_{n_1+1}$. Continuing in this fashion, we obtain indices $n_1 < \cdots < n_k < \cdots$ such that

$$y_{n_{k-1}+1} - \frac{1}{k} < x_{n_k} \leq y_{n_{k-1}+1}$$

for each $k \in \mathbf{N}$. The subsequence $(y_{n_{k-1}+1})$ converges to $\limsup x_n$ (Theorem 2.5.2) and hence by the Squeeze Theorem (Exercise 2.3.3) the subsequence (x_{n_k}) converges to $\limsup x_n$. Similarly, we can obtain another subsequence converging to $\liminf x_n$. As we showed in Exercise 2.4.7, the fact that (x_n) diverges implies that $\liminf x_n < \limsup x_n$ and so we have found two subsequences of (x_n) that converge to different limits.

(d) This is true. Suppose that (x_n) is decreasing; the case where (x_n) is increasing is handled similarly. By assumption, there is a subsequence (x_{n_k}), which must also be decreasing, converging to some $x \in \mathbf{R}$. By the Monotone Convergence Theorem (Theorem 2.4.2) and the uniqueness of limits (Theorem 2.2.7/Exercise 2.2.6), we have

$$\lim_{k\to\infty} x_{n_k} = x = \inf\{x_{n_k} : k \in \mathbf{N}\}.$$

Let $\epsilon > 0$ be given. Since $x_{n_k} \to x$, there is a $K \in \mathbf{N}$ such that $|x_{n_K} - x| < \epsilon$. Suppose that $n \in \mathbf{N}$ is such that $n \geq n_K$. Because (x_{n_k}) is a subsequence, there exists some $k \in \mathbf{N}$ such that $n_k \geq n$. Since (x_n) is decreasing, we then have

$$x \leq x_{n_k} \leq x_n \leq x_{n_K} < x + \epsilon \implies |x_n - x| < \epsilon.$$

Thus $\lim_{n\to\infty} x_n = x$.

Exercise 2.5.3. (a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

leads to a series that also converges to L.

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

Solution. (a) We have indices $n_1 < \cdots < n_k < \cdots$ and we want to show that $\sum_{k=1}^{\infty} b_k = L$, where $b_1 = a_1 + \cdots + a_{n_1} = s_{n_1}$ and

$$b_k = a_{n_{k-1}+1} + \cdots + a_{n_k} = s_{n_k} - s_{n_{k-1}}$$

for $k \geq 2$. Observe that for $m \geq 2$, the partial sums are

$$t_m = \sum_{k=1}^{m} b_k = s_{n_1} + \sum_{k=2}^{m} (s_{n_k} - s_{n_{k-1}}) = s_{n_1} + (s_{n_2} - s_{n_1}) + \cdots + (s_{n_m} - s_{n_{m-1}}) = s_{n_m}.$$

It follows from Theorem 2.5.2 that $\sum_{k=1}^{\infty} b_k = \lim_{m\to\infty} t_m = \lim_{m\to\infty} s_{n_m} = L$.

(b) Our proof does not apply to the series $\sum_{n=1}^{\infty}(-1)^n$ since this series does not converge: the sequence of partial sums is $(-1, 0, -1, 0, \ldots)$.

Exercise 2.5.4. The Bolzano-Weierstrass Theorem is extremely important, and so is the strategy employed in the proof. To gain some more experience with this technique, assume the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

Solution. Let $E \subseteq \mathbf{R}$ be non-empty and bounded above by some $b_1 \in \mathbf{R}$; we will show that $\sup E$ exists. If E has a maximum x, then $\sup E = x$ and we are done. Otherwise, we shall use a recursive argument to construct a sequence $(I_n)_{n=1}^{\infty}$ of nested intervals. Pick some $a_1 \in E$; it must be the case that a_1 is not an upper bound of E since E has no maximum. Let $I_1 = [a_1, b_1]$ and note that:

- a_1 is not an upper bound of E;
- b_1 is an upper bound of E;
- $|I_1| = 2^0(b_1 - a_1)$.

Suppose that after N steps we have chosen intervals $I_n = [a_n, b_n], 1 \leq n \leq N$, such that

- $a_1 \leq \cdots \leq a_N$ are not upper bounds of E;
- $b_N \leq \cdots \leq b_1$ are upper bounds of E;
- $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for $1 \leq n \leq N$.

Let $m = \frac{a_N + b_N}{2}$, the midpoint of the interval I_N. If m is not an upper bound of E, let

$$a_{N+1} = m, \quad b_{N+1} = b_N, \qquad \text{and} \qquad I_{N+1} = [a_{N+1}, b_{N+1}].$$

If m is an upper bound of E, let

$$a_{N+1} = a_N, \quad b_{N+1} = m, \qquad \text{and} \qquad I_{N+1} = [a_{N+1}, b_{N+1}].$$

In either case, we have chosen intervals $I_n = [a_n, b_n], 1 \leq n \leq N+1$, such that

- $a_1 \leq \cdots \leq a_{N+1}$ are not upper bounds of E;
- $b_{N+1} \leq \cdots \leq b_1$ are upper bounds of E;
- $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for $1 \leq n \leq N+1$.

In this way we obtain a sequence $(I_n)_{n=1}^{\infty}$ of intervals $I_n = [a_n, b_n]$ such that

- $a_1 \leq \cdots \leq a_n \leq \cdots$ are not upper bounds of E;
- $\cdots \leq b_n \leq \cdots \leq b_1$ are upper bounds of E;
- $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for all $n \in \mathbf{N}$.

Hence $(I_n)_{n=1}^{\infty}$ is a sequence of nested intervals. By assumption $\mathbf{R}$ has the Nested Interval Property (Theorem 1.4.1), so there exists an $x \in \mathbf{R}$ such that $x \in \bigcap_{n=1}^{\infty} I_n$; we claim that $x = \sup E$. For $y \in E$, let us seek a contradiction and suppose that $x < y$. Since $|I_n| = 2^{-(n-1)}(b_1 - a_1)$ for all $n \in \mathbf{N}$ and $(2^{-n}) \to 0$ (by assumption), there must exist an $N \in \mathbf{N}$ such that

$$|I_N| = b_N - a_N < y - x \implies x + (b_N - a_N) < y.$$

Since $x \in \bigcap_{n=1}^{\infty} I_n$, we have

$$a_N \leq x \implies 0 \leq x - a_N \implies b_N \leq x + (b_N - a_N) \implies b_N < y.$$

This is a contradiction since b_N is an upper bound of E. It follows that $y \leq x$ and thus x is an upper bound of E.

Now suppose that $z \in \mathbf{R}$ is such that $z < x$. Since $(|I_n|) \to 0$, there must be an $N \in \mathbf{N}$ such that

$$|I_N| = b_N - a_N < x - z \implies z < x - (b_N - a_N).$$

Since $x \in \bigcap_{n=1}^{\infty} I_n$, we have

$$x \leq b_N \implies x - b_N \leq 0 \implies x - (b_N - a_N) \leq a_N \implies z < a_N.$$

It follows that z is not an upper bound of E since a_N is not an upper bound of E. We may conclude that x is the least upper bound of E, i.e., $x = \sup E$.

We had to assume that $(2^{-n}) \to 0$ since the usual proof of this would involve the Archimedean Property (Theorem 1.4.2), which we proved using the Axiom of Completeness.

Exercise 2.5.5. Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbf{R}$. Show that (a_n) must converge to a.

Solution. Since (a_n) is bounded, $\limsup a_n$ and $\liminf a_n$ both exist. In the solution to Exercise 2.5.2 (c), we showed that there are subsequences (a_{n_k}) and (a_{n_ℓ}) such that

$$\lim_{k\to\infty} a_{n_k} = \limsup a_n \qquad \text{and} \qquad \lim_{\ell\to\infty} a_{n_\ell} = \liminf a_n.$$

By assumption we have $\lim_{k\to\infty} a_{n_k} = \lim_{\ell\to\infty} a_{n_\ell} = a$ and so by the uniqueness of limits (Theorem 2.2.7/Exercise 2.2.6) it follows that $\limsup a_n = \liminf a_n = a$; Exercise 2.4.7 then implies that $\lim a_n = a$.

Exercise 2.5.6. Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

Solution. If $b = 0$ then $b^{1/n} = 0$ for any $n \in \mathbf{N}$, so $\lim_{n\to\infty} b^{1/n} = 0$. Suppose that $b > 0$. If $0 < b < 1$, then

$$b < b^{1/2} < b^{1/3} < \cdots < 1.$$

If $b \geq 1$, then

$$b \geq b^{1/2} \geq b^{1/3} \geq \cdots \geq 1.$$

In either case, $(b^{1/n})$ is bounded and monotone and hence convergent by the Monotone Convergence Theorem (Theorem 2.4.2), say $\lim b^{1/n} = L \in \mathbf{R}$. It follows from Theorem 2.5.2 that $\lim b^{1/2n} = L$ also. Note that

$$\lim b^{1/2n} = \lim \sqrt{b^{1/n}} = \sqrt{\lim b^{1/n}} = \sqrt{L}$$

by Exercise 2.3.1. Since limits are unique (Theorem 2.2.7/Exercise 2.2.6), we must have $L = \sqrt{L}$, which implies that $L = 0$ or $L = 1$. If $0 < b < 1$ then the Order Limit Theorem (Theorem 2.3.4) gives $0 < b < L \leq 1$, whence $L = 1$, and if $b \geq 1$ then the Order Limit Theorem (Theorem 2.3.4) gives $L \geq 1$ so again we must have $L = 1$.

We may conclude that $\lim b^{1/n} = 0$ if $b = 0$ and $\lim b^{1/n} = 1$ if $b > 0$. See Figure F.16 for a graph of the first thirty terms of the sequence $(b^{1/n})$ for $b = 0, \frac{1}{2}, 1, 2$, demonstrating the behaviour of the sequence as b varies.

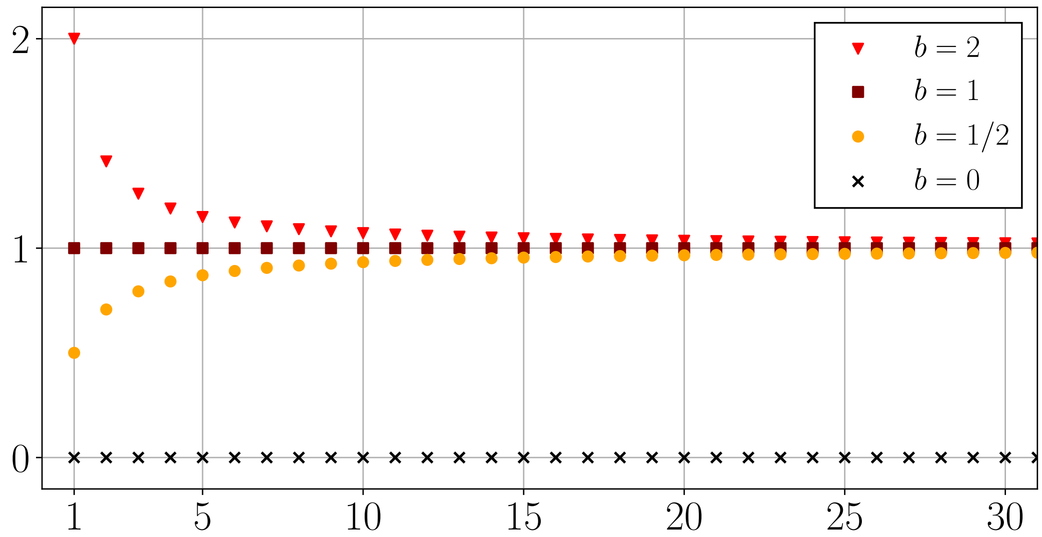


Figure F.16: $b^{1/n}$ for $1 \leq n \leq 30$ and $b = 0, \frac{1}{2}, 1, 2$

Exercise 2.5.7. Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Solution. We will consider the following cases.

Case 1. $b > 1$. In this case, (b^n) is unbounded and hence divergent.

Case 2. $b = 1$. In this case, $(b^n) = (1, 1, 1, \ldots)$ and hence $\lim b^n = 1$.

Case 3. $0 < b < 1$. Example 2.5.3 shows that in this case we have $\lim b^n = 0$.

Case 4. $b = 0$. In this case, $(b^n) = (0, 0, 0, \ldots)$ and hence $\lim b^n = 0$.

Case 5. $-1 < b < 0$. Observe that $b = (-1)|b|$, so that $b^n = (-1)^n|b|^n$. Since $0 < |b| < 1$, we have $\lim|b|^n = 0$ by the $0 < b < 1$ case. Given this, and the boundedness of $(-1)^n$, it follows from Exercise 2.3.9 (a) that

$$\lim b^n = \lim[(-1)^n|b|^n] = 0.$$

Case 6. $b = -1$. In this case $b^n = (-1)^n$, which is divergent since it has two convergent subsequences with different limits:

$$\lim[(-1)^{2n}] = 1 \neq -1 = \lim[(-1)^{2n+1}].$$

Case 7. $b < -1$. We have $b^n = (-1)^n|b|^n$ with $|b| > 1$. Observe that the subsequence $(b^{2n}) = \left(|b|^{2n}\right)$ is divergent by the $b > 1$ case; it then follows from Exercise 2.5.2 (b) that the sequence (b^n) is divergent.

We may conclude that $\lim b^n = 0$ if and only if $-1 < b < 1$.

Exercise 2.5.8. Another way to prove the Bolzano-Weierstrass Theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of a *peak term*. Given a sequence (x_n), a particular term x_m is a peak term if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.

(a) Find examples of sequences with zero, one, and two peak terms. Find an example of a sequence with infinitely many peak terms that is not monotone.

(b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano-Weierstrass Theorem.

Solution. (a) Any strictly increasing sequence will have zero peak terms; the sequence (n) for example. For sequences with one and two peak terms, consider (respectively)

$$\left(2, 0, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5} \ldots\right) \qquad \text{and} \qquad \left(3, 2, 0, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5} \ldots\right).$$

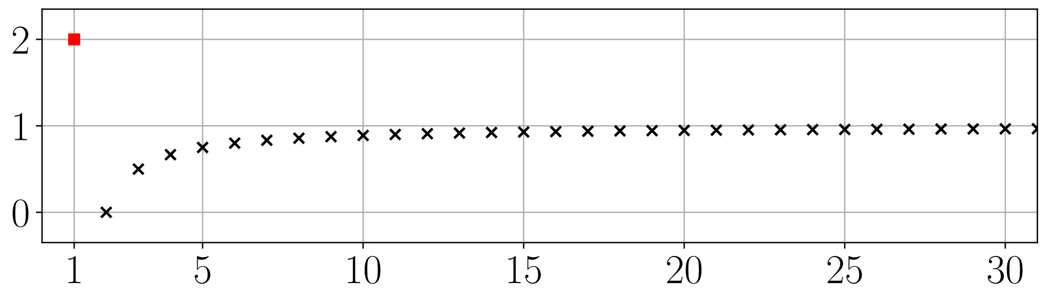
For a sequence with infinitely many peak terms but which is not monotone, consider

$$(0, 1, -2, -1, -4, -3, -6, -5, \ldots).$$

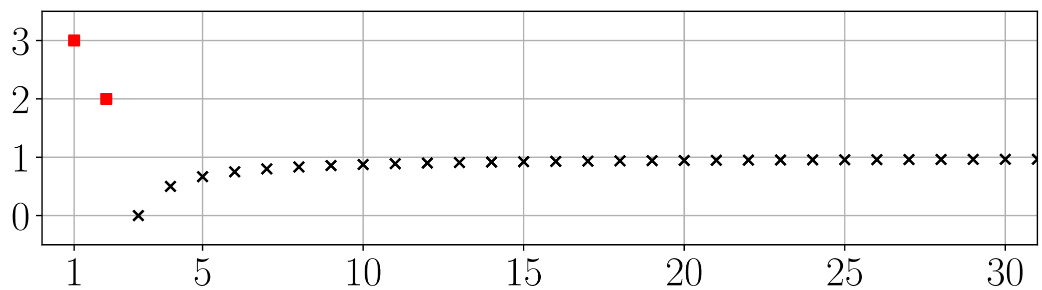
See Figure F.17 for a graph of the first thirty terms of these sequences.

(b) Let (x_n) be a sequence; we will show that (x_n) contains a monotone subsequence. First, suppose that (x_n) contains infinitely many peak terms, say $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$, where we may assume that $n_1 < n_2 < \cdots < n_k < \cdots$; the subsequence (x_{n_k}) is then a decreasing subsequence of (x_n). Next, suppose that (x_n) contains only finitely many peak terms. In this case, we are guaranteed the existence of a term x_{n_1} which is not a peak term and after which there are no peak terms. Since x_{n_1} is not a peak term, there exists an $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$ and x_{n_2} is not a peak term. Since x_{n_2} is not a peak term, there exists an $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$ and x_{n_3} is not a peak term. Continuing in this way, we recursively obtain an increasing subsequence (x_{n_k}) of (x_n). In either case, we have shown that (x_n) must contain a monotone subsequence.

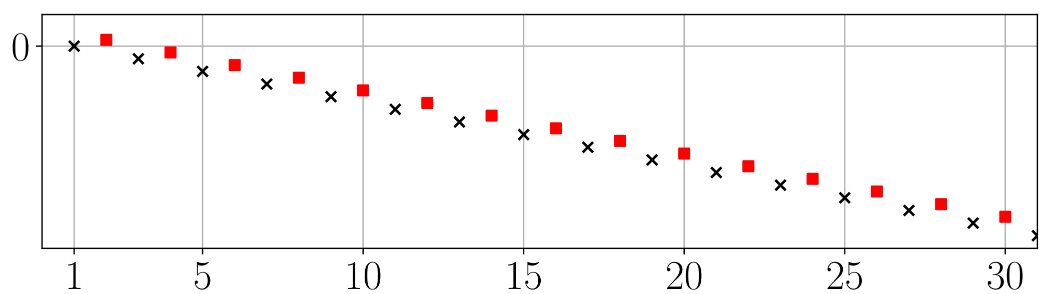
Now suppose that (x_n) is a bounded sequence. By the previous paragraph, there exists a monotone subsequence (x_{n_k}), which must also be bounded. The Monotone Convergence Theorem (Theorem 2.4.2) then implies that (x_{n_k}) is convergent; this provides another proof of the Bolzano-Weierstrass Theorem (Theorem 2.5.5).



(a) Sequence with one peak term



(b) Sequence with two peak terms



(c) Non-monotone sequence with infinitely many peak terms

Figure F.17: Exercise 2.5.8 (a) sequences with one, two, and infinitely many peak terms; the red squares indicate peak terms

Exercise 2.5.9. Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbf{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)

Solution. Since (a_n) is bounded, there is an $M > 0$ such that $-M \leq a_n \leq M$ for all $n \in \mathbf{N}$. It follows that $(-\infty, -M) \subseteq S$, so that S is non-empty, and for any $x \in S$ we have $x < a_n \leq M$ for some $n \in \mathbf{N}$, so that S is bounded above by M. The Axiom of Completeness then implies that $s := \sup S$ exists in $\mathbf{R}$.

Let k be a positive integer. We claim that the set

$$C_k = \left\{n \in \mathbf{N} : s - \tfrac{1}{k} < a_n \leq s + \tfrac{1}{k}\right\}$$

is infinite. By Lemma 1.3.8, there exists an $x \in S$ such that $s - \frac{1}{k} < x \leq s$. Define the sets

$$E = \{n \in \mathbf{N} : x < a_n\}, \quad A_k = \left\{n \in \mathbf{N} : s + \tfrac{1}{k} < a_n\right\}$$
$$\text{and} \quad B_k = \left\{n \in \mathbf{N} : x < a_n \leq s + \tfrac{1}{k}\right\}.$$

Observe that E is the disjoint union of A_k and B_k and that E is infinite since $x \in S$. Furthermore, A_k must be finite, otherwise we would have $s + \frac{1}{k} \in S$. It follows that B_k is infinite and hence that C_k is infinite, since $B_k \subseteq C_k$.

Since C_1 is infinite, there exists some $n_1 \in \mathbf{N}$ such that $s - 1 < a_{n_1} \leq s + 1$. Since C_2 is infinite, there exists some $n_2 > n_1$ such that $s - \frac{1}{2} < a_{n_2} \leq s + \frac{1}{2}$. We continue this process recursively to obtain a subsequence (a_{n_k}) satisfying $s - \frac{1}{k} < a_{n_k} \leq s + \frac{1}{k}$. The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim_{k\to\infty} a_{n_k} = s$.

2.6 The Cauchy Criterion

Exercise 2.6.1. Supply a proof for Theorem 2.6.2.

Solution. Suppose $x_n \to x$ for some $x \in \mathbf{R}$; we will show that (x_n) is Cauchy. Let $\epsilon > 0$ be given. There is an $N \in \mathbf{N}$ such that $n \geq N$ implies that $|x_n - x| < \frac{\epsilon}{2}$. For $m, n \geq N$ we then have

$$|x_n - x_m| \leq |x_n - x| + |x_m - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows that (x_n) is a Cauchy sequence.

Exercise 2.6.2. Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

(c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

Solution. (a) Consider the sequence (x_n) given by $x_n = \frac{(-1)^n}{n}$. The sequence is convergent ($\lim x_n = 0$) and hence Cauchy (Theorem 2.6.4), but is certainly not monotone.

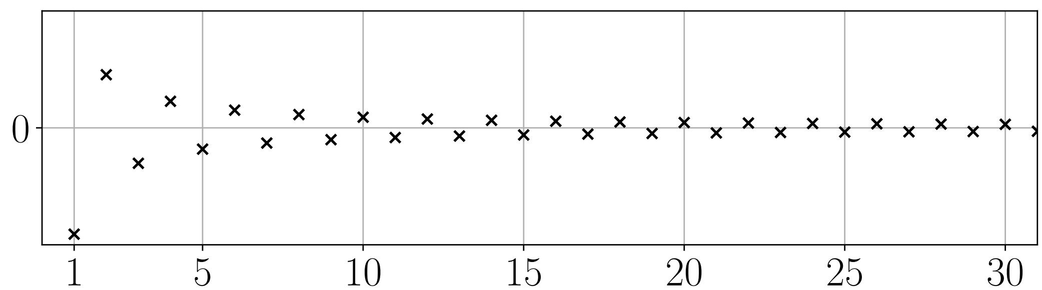


Figure F.18: $\frac{(-1)^n}{n}$ for $1 \leq n \leq 30$

(b) This is impossible. A Cauchy sequence (x_n) is necessarily convergent (Theorem 2.6.4) and hence all subsequences of (x_n) must be convergent (Theorem 2.5.2); each subsequence must then be bounded (Theorem 2.3.2).

(c) First, let us prove the following result.

Lemma L.8. If (x_n) is an unbounded monotone sequence, then all subsequences of (x_n) are also unbounded and monotone.

Proof. Suppose (x_n) is increasing (the case where (x_n) is decreasing is handled similarly) and let (x_{n_k}) be a subsequence of (x_n). If $k > \ell$, then $n_k > n_\ell$ and so $x_{n_k} \geq x_{n_\ell}$ since (x_n) is increasing; it follows that (x_{n_k}) is an increasing sequence. Now let $M > 0$ be given. Since

(x_n) is unbounded, there is an $N \in \mathbf{N}$ such that $x_N > M$, and since (x_{n_k}) is a subsequence of (x_n) we are guaranteed the existence of a $K \in \mathbf{N}$ such that $n_K > N$; it follows that $x_{n_K} \geq x_N > M$ since (x_n) is increasing. We may conclude that (x_{n_k}) is unbounded. $\square$

We can now show that the given request is impossible. If (x_n) is a divergent monotone sequence, then by the Monotone Convergence Theorem (Theorem 2.4.2) the sequence (x_n) must be unbounded. It follows from Lemma L.8 that all subsequences of (x_n) are unbounded, hence divergent (Theorem 2.3.2), and hence not Cauchy (Theorem 2.6.4).

(d) Consider the unbounded sequence $(0, 1, 0, 2, 0, 3, \ldots)$; the subsequence $(0, 0, 0, \ldots)$ is convergent and hence Cauchy (Theorem 2.6.4).

Exercise 2.6.3. If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

Solution. (a) Let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$m, n \geq N_1 \implies |x_n - x_m| < \frac{\epsilon}{2} \qquad \text{and} \qquad m, n \geq N_2 \implies |y_n - y_m| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$ and observe that for $m, n \geq N$ we have

$$|x_n + y_n - x_m - y_m| \leq |x_n - x_m| + |y_n - y_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows that $(x_n + y_n)$ is a Cauchy sequence.

(b) Because Cauchy sequences are bounded (Lemma 2.6.3), there are positive real numbers M_1 and M_2 such that $|x_n| \leq M_1$ and $|y_n| \leq M_2$ for all $n \in \mathbf{N}$. Let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$m, n \geq N_1 \implies |x_n - x_m| < \frac{\epsilon}{2M_2} \qquad \text{and} \qquad m, n \geq N_2 \implies |y_n - y_m| < \frac{\epsilon}{2M_1}.$$

Let $N = \max\{N_1, N_2\}$ and observe that for $m, n \geq N$ we have

$$|x_n y_n - x_m y_m| = |x_n y_n - x_m y_n + x_m y_n - x_m y_m| \leq |y_n||x_n - x_m| + |x_m||y_n - y_m|$$
$$< M_2 \frac{\epsilon}{2M_2} + M_1 \frac{\epsilon}{2M_1} = \epsilon.$$

It follows that $(x_n y_n)$ is a Cauchy sequence.

Exercise 2.6.4. Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

(a) $c_n = |a_n - b_n|$

(b) $c_n = (-1)^n a_n$

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

Solution. By the Cauchy Criterion (Theorem 2.6.4), we have $\lim a_n = a$ and $\lim b_n = b$ for some real numbers a and b. Again by the Cauchy Criterion, it will suffice to consider convergence of the given sequences (c_n).

(a) By Exercise 2.3.10 (b) and the Algebraic Limit Theorem (Theorem 2.3.3), we have

$$\lim c_n = \lim |a_n - b_n| = |\lim a_n - \lim b_n| = |a - b|.$$

So (c_n) is convergent and hence Cauchy.

(b) Suppose that $a = 0$. By Exercise 2.3.9 (a) we then have $\lim c_n = 0$ and it follows that (c_n) is Cauchy. If $a \neq 0$, then observe that

$$\lim c_{2n} = \lim a_{2n} = a \neq -a = \lim(-a_{2n-1}) = \lim c_{2n-1}.$$

So (c_n) has two subsequences which converge to different limits. It follows that (c_n) is not convergent (Theorem 2.5.2) and hence not Cauchy.

(c) Suppose that a is not an integer, so that $[[a]] < a < [[a]] + 1$. Let

$$\delta = \min\{a - [[a]], [[a]] + 1 - a\}.$$

Since $\lim a_n = a$, there is a positive integer N such that $n \geq N$ implies that $a_n \in (a-\delta, a+\delta)$. Observe that $[[a]] \leq a-\delta$ and $a+\delta \leq [[a]]+1$. For $n \geq N$ we then have $[[a]] < a_n < [[a]]+1$, which gives us $[[a_n]] = [[a]]$. Thus the sequence $[[a_n]]$ is eventually constant with value $[[a]]$; it follows that $[[a_n]]$ is convergent with limit $[[a]]$ and hence Cauchy.

If a is an integer, then the sequence $([[a_n]])$ may or may not be convergent (and so may or may not be Cauchy). For example, if (a_n) is the sequence $(0, 0, 0, \ldots)$ then clearly $\lim[[a_n]] = 0$. However, consider the sequence $a_n = \frac{(-1)^n}{n}$, which also satisfies $\lim a_n = 0$. This gives

$$([[a_n]]) = (-1, 0, -1, 0, -1, 0, \ldots),$$

which is divergent.

Exercise 2.6.5. Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$.

Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Psuedo-Cauchy sequences are bounded.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

Solution. (i) This statement is false: consider the sequence (s_n) given by $s_n = \sum_{m=1}^{n} \frac{1}{m}$. This sequence satisfies $s_{n+1} - s_n = \frac{1}{n+1} \to 0$, so that (s_n) is pseudo-Cauchy. However, as shown in Example 2.4.5, (s_n) is unbounded.

(ii) This statement is true. Let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |x_{n+1} - x_n| < \frac{\epsilon}{2} \qquad \text{and} \qquad n \geq N_2 \implies |y_{n+1} - y_n| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$ and observe that for $n \geq N$ we have

$$|x_{n+1} + y_{n+1} - x_n - y_n| \leq |x_{n+1} - x_n| + |y_{n+1} - y_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows that $(x_n + y_n)$ is pseudo-Cauchy.

Exercise 2.6.6. Let's call a sequence (a_n) *quasi-increasing* if for all $\epsilon > 0$ there exists an N such that whenever $n > m \geq N$ it follows that $a_n > a_m - \epsilon$.

(a) Give an example of a sequence that is quasi-increasing but not monotone or eventually monotone.

(b) Give an example of a quasi-increasing sequence that is divergent and not monotone or eventually monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasi-increasing sequences? Give an example of a bounded, quasi-increasing sequence that doesn't converge, or prove that no such sequence exists.

Solution. (a) Consider the sequence (a_n) given by

$$a_n = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} - \frac{2}{n} & \text{if } n \text{ is even.} \end{cases}$$

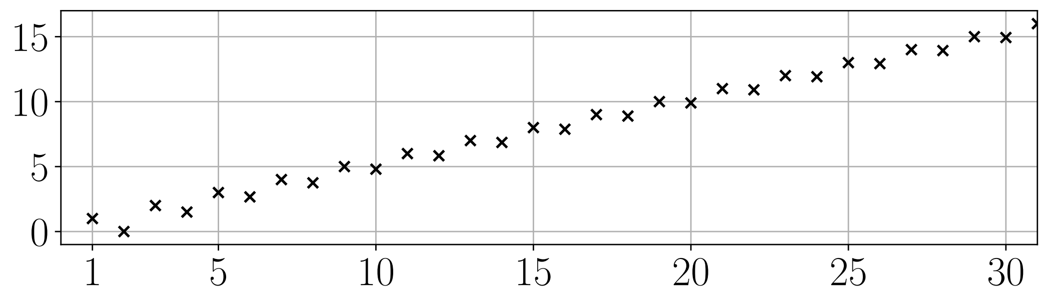


Figure F.19: a_n for $1 \leq n \leq 30$

Some calculations reveal that this sequence has the following properties.

(i) If $m \in \mathbf{N}$ is even, then $a_n > a_m$ for all $n > m$.

(ii) If $m \in \mathbf{N}$ is odd, then $a_n > a_m$ for all $n > m+1$ and $a_m - a_{m+1} = \frac{2}{m+1} > 0$.

It follows that (a_n) is not eventually monotone, for if N is a positive integer, choose an odd integer m such that $m > N$; by property (ii) we then have $a_m > a_{m+1}$ and $a_m < a_{m+2}$. Furthermore, (a_n) is quasi-increasing. To see this, let $\epsilon > 0$ be given. Choose a positive integer N such that $\frac{2}{N+1} < \epsilon$ and suppose that $n > m \geq N$. By properties (i) and (ii), we have

$$a_m - a_n < 0 < \epsilon \implies a_n > a_m - \epsilon$$

unless m is odd and $n = m + 1$. In that case we have

$$a_m - a_{m+1} = \frac{2}{m+1} \leq \frac{2}{N+1} < \epsilon \implies a_n > a_m - \epsilon.$$

(b) The sequence (a_n) given in part (a) is unbounded and hence divergent.

(c) There is an analogue of the Monotone Convergence Theorem (Theorem 2.4.2) for bounded quasi-increasing sequences. Let (a_n) be such a sequence; we will show that (a_n) converges to $\limsup a_n$.

Let $s = \limsup a_n$ and $y_n = \sup\{a_\ell : \ell \geq n\}$, so that $\lim y_n = s$. By Exercise 2.5.2 (c), there is a subsequence (a_{n_k}) converging to s. Let $\epsilon > 0$ be given. There is an $N_1 \in \mathbf{N}$ such that $|y_n - s| < \epsilon$ whenever $n \geq N_1$. Since $a_n \leq y_n$ for all $n \in \mathbf{N}$, we have

$$n \geq N_1 \implies a_n < s + \epsilon. \tag{1}$$

Since (a_n) is quasi-increasing, there is an $N_2 \in \mathbf{N}$ such that

$$n > m \geq N_2 \implies a_m - \frac{\epsilon}{2} < a_n, \tag{2}$$

and since $(a_{n_k}) \to s$, there is a $K \in \mathbf{N}$ such that

$$k \geq K \implies |a_{n_k} - s| < \frac{\epsilon}{2}. \tag{3}$$

Because (a_{n_k}) is a subsequence, there must be some $k' \in \mathbf{N}$ such that both $k' \geq K$ and $n_{k'} \geq N_2$. It follows that

$$n > n_{k'} \implies a_{n_{k'}} - \frac{\epsilon}{2} < a_n \qquad \text{by (2)},$$

and $s - \epsilon < a_{n_{k'}} - \frac{\epsilon}{2}$ by (3). Combining these gives

$$n > n_{k'} \implies s - \epsilon < a_n. \tag{4}$$

Let $N = \max\{N_1, n_{k'}\}$. By (1) and (4), we then have

$$n > N \implies s - \epsilon < a_n < s + \epsilon.$$

It follows that $\lim a_n = s$.

Exercise 2.6.7. Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show that the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano-Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC, and MCT are all equivalent.

(b) Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

Solution. (a) Suppose (x_n) is bounded and increasing (the case where (x_n) is decreasing is handled similarly). By assumption, there is a convergent subsequence (x_{n_k}), say $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in \mathbf{R}$. Let $\epsilon > 0$ be given. There is a $K \in \mathbf{N}$ such that

$$k \geq K \implies |x_{n_k} - x| < \epsilon. \tag{1}$$

Suppose $n \in \mathbf{N}$ is such that $n \geq n_K$. Since (x_n) is increasing, we then have $x - \epsilon < x_{n_K} \leq x_n$. Furthermore, it must be the case that $x_n < x + \epsilon$. Indeed, if $x_n \geq x + \epsilon$, then since (x_{n_k}) is a subsequence there must be some $k \in \mathbf{N}$ such that $n_k \geq n \geq n_K$. This implies that $k \geq K$ and, since (x_n) is increasing, that $x_{n_k} \geq x_n \geq x + \epsilon$; this contradicts (1). So we have shown that

$$n \geq n_K \implies x - \epsilon < x_n < x + \epsilon.$$

It follows that $\lim x_n = x$.

(b) Let (x_n) be a sequence bounded by some $M > 0$. As in the proof of the Bolzano-Weierstrass Theorem (Theorem 2.5.5) given in the textbook, construct a sequence of nested intervals (I_k) with length $M \cdot 2^{-k+1}$ and a subsequence (x_{n_k}) such that $x_{n_k} \in I_k$. Let $\epsilon > 0$ be given. Assuming that $2^{-k} \to 0$ (this is equivalent to assuming the Archimedean Property (Theorem 1.4.2)), there is a $K \in \mathbf{N}$ such that $M \cdot 2^{-K+1} < \epsilon$. Suppose that $k > \ell \geq K$. Since the intervals are nested, both x_{n_k} and x_{n_ℓ} belong to I_K. It follows that x_{n_k} and x_{n_ℓ} are no further apart than the width of I_K, i.e.,

$$|x_{n_k} - x_{n_\ell}| \leq \frac{M}{2^{K-1}} < \epsilon.$$

This demonstrates that (x_{n_k}) is a Cauchy sequence. By assumption, this is equivalent to (x_{n_k}) being convergent.

(c) The ordered field $\mathbf{Q}$ has the Archimedean Property but does not satisfy the Axiom of Completeness (see Lemma L.4; the subset $A \subseteq \mathbf{Q}$ given there is non-empty and bounded above but has no supremum in $\mathbf{Q}$).

2.7 Properties of Infinite Series

Exercise 2.7.1. Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Solution. First note that since (a_n) is decreasing and converges to zero, $a_n \geq 0$ and $a_n - a_{n+1} \geq 0$ for all $n \in \mathbf{N}$.

(a) Suppose $n > m$ are positive integers. If $n - m$ is even, then

$$s_n - s_m = \underbrace{a_{m+1} - a_{m+2}}_{\geq 0} + \underbrace{a_{m+3} - a_{m+4}}_{\geq 0} + \cdots + \underbrace{a_{n-1} - a_n}_{\geq 0} \geq 0,$$

and if $n - m$ is odd, then

$$s_n - s_m = \underbrace{a_{m+1} - a_{m+2}}_{\geq 0} + \underbrace{a_{m+3} - a_{m+4}}_{\geq 0} + \cdots + \underbrace{a_{n-2} - a_{n-1}}_{\geq 0} + \underbrace{a_n}_{\geq 0} \geq 0.$$

It follows that $|s_n - s_m| = s_n - s_m = a_{m+1} - a_{m+2} + \cdots \pm a_n$. If $n - m$ is even, then

$$|s_n - s_m| = a_{m+1} + \underbrace{(-a_{m+2} + a_{m+3})}_{\leq 0} + \cdots + \underbrace{(-a_{n-2} + a_{n-1})}_{\leq 0} + \underbrace{(-a_n)}_{\leq 0} \leq a_{m+1},$$

and if $n - m$ is odd, then

$$|s_n - s_m| = a_{m+1} + \underbrace{(-a_{m+2} + a_{m+3})}_{\le 0} + \cdots + \underbrace{(-a_{n-1} + a_n)}_{\le 0} \le a_{m+1}.$$

It follows that $|s_n - s_m| \le a_{m+1}$. Let $\epsilon > 0$ be given. Since $a_n \to 0$, there is an $N \in \mathbf{N}$ such that $|a_n| = a_n < \epsilon$ for all $n \ge N$. For $n > m \ge N$ we then have

$$|s_n - s_m| \le a_{m+1} < \epsilon.$$

It follows that (s_n) is a Cauchy sequence.

(b) Let n be a positive integer. Observe that

$$s_{2n-1} - s_{2n} = a_{2n} \ge 0 \implies s_{2n} \le s_{2n-1},$$

$$s_{2n-1} - s_{2n-3} = a_{2n-1} - a_{2n-2} \le 0 \implies s_{2n-1} \le s_{2n-3},$$

$$s_{2n} - s_{2n-2} = a_{2n-1} - a_{2n} \ge 0 \implies s_{2n-2} \le s_{2n}.$$

Thus $(I_n = [s_{2n}, s_{2n-1}])_{n=1}^{\infty}$ is a sequence of nested intervals. It follows from the Nested Interval Property (Theorem 1.4.1) that there exists some $x \in \bigcap_{n=1}^{\infty} I_n$; we claim that $\lim s_n = x$. To see this, suppose that $n \in \mathbf{N}$. If n is even, then $s_n \in I_{n/2} = [s_n, s_{n-1}]$ and so

$$|s_n - x| \le \left|I_{n/2}\right| = s_{n-1} - s_n = a_n.$$

If n is odd, then $s_n \in I_{(n+1)/2} = [s_{n+1}, s_n]$ and so

$$|s_n - x| \le \left|I_{(n+1)/2}\right| = s_n - s_{n+1} = a_{n+1} \le a_n.$$

It follows that for all $n \in \mathbf{N}$ we have $|s_n - x| \le a_n$; since $a_n \to 0$, an application of the Squeeze Theorem (Exercise 2.3.3) then yields $\lim s_n = x$.

(c) As shown in (b), the sequence (s_{2n}) is increasing and bounded above by s_1, and the sequence (s_{2n+1}) is decreasing and bounded below by s_2. The Monotone Convergence Theorem (Theorem 2.4.2) then implies that $\lim s_{2n}$ and $\lim s_{2n+1}$ both exist. The relationship $s_{2n+1} - s_{2n} = a_{2n+1}$ gives

$$\lim(s_{2n+1} - s_{2n}) = \lim a_{2n+1} = 0,$$

so that (s_{2n}) and (s_{2n+1}) both converge to the same limit $x \in \mathbf{R}$ (Exercise 2.3.10 (c)). It follows that $\lim s_n = x$, as the next lemma shows.

Lemma L.9. If (x_n) is a sequence of real numbers such that

$$\lim x_{2n} = \lim x_{2n+1} = x$$

for some $x \in \mathbf{R}$, then $\lim x_n = x$.

Proof. Let $\epsilon > 0$ be given. There are positive integers N_1 and N_2 such that

$$n \geq N_1 \implies |x_{2n} - x| < \epsilon, \tag{1}$$

$$n \geq N_2 \implies |x_{2n+1} - x| < \epsilon. \tag{2}$$

Let $N = \max\{N_1, N_2\}$ and suppose that $n \in \mathbf{N}$ is such that $n \geq 2N + 1$. If n is even, then $\frac{n}{2} > N \geq N_1$ and so $|x_n - x| < \epsilon$ by (1). If n is odd, then $\frac{n-1}{2} \geq N \geq N_2$ and so $|x_n - x| < \epsilon$ by (2). Thus

$$n \geq 2N + 1 \implies |x_n - x| < \epsilon.$$

It follows that $\lim x_n = x$. $\square$

Exercise 2.7.2. Decide whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ (b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

Solution. See Figure F.20 for graphs of the first thirty terms of relevant partial sums for each series.

(a) Observe that for each $n \in \mathbf{N}$ we have

$$0 < \frac{1}{2^n + n} < \frac{1}{2^n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ (Example 2.7.5), the Comparison Test (Theorem 2.7.4) implies that $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ is convergent.

(b) Observe that for each $n \in \mathbf{N}$ we have

$$0 < \frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (Example 2.4.4), the Comparison Test (Theorem 2.7.4) implies that $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ is absolutely convergent and hence convergent (Theorem 2.7.6).

(c) This is the series $\sum_{n=1}^{\infty} a_n$, where

$$a_n = (-1)^{n+1}\frac{n+1}{2n} = (-1)^{n+1}\left(\frac{1}{2} + \frac{1}{2n}\right).$$

The sequence (a_n) is divergent by Theorem 2.5.2:

$$\lim a_{2n} = -\frac{1}{2} \neq \frac{1}{2} = \lim a_{2n+1}.$$

It follows from Theorem 2.7.3 that $\sum_{n=1}^{\infty} a_n$ is divergent.

(d) For the series $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$, let (s_n) be the sequence of partial sums and consider the subsequence (s_{3n}). Observe that

$$\begin{aligned}
s_{3n} &= \left(1 + \frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n}\right) \\
&\geq \left(1 + \frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{5} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n-1}\right) \\
&= 1 + \frac{1}{4} + \cdots + \frac{1}{3n-2} \\
&= \frac{1}{3}\sum_{k=1}^{n} \frac{1}{k - \frac{2}{3}} \\
&\geq \frac{1}{3}\sum_{k=1}^{n} \frac{1}{k}.
\end{aligned}$$

So we have shown that $s_{3n} \geq \frac{1}{3}\sum_{k=1}^{n} \frac{1}{k}$ for all $n \in \mathbf{N}$. Since $\sum_{k=1}^{n} \frac{1}{k}$ is unbounded in n (Example 2.4.5), it follows that (s_{3n}) is unbounded. This implies that (s_n) is unbounded and hence divergent (Theorem 2.3.2).

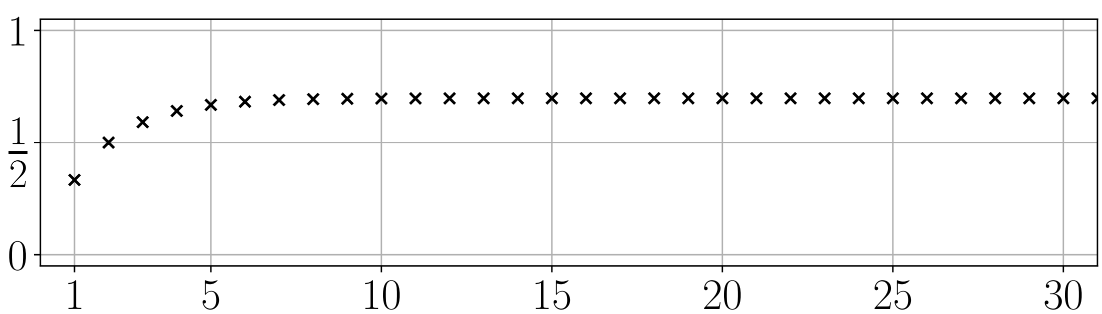
(e) For the series $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$, let (s_n) be the sequence of partial sums and consider the subsequence (s_{2n}). For any $m \geq 2$, we have

$$\frac{1}{m^2} \leq \frac{1}{m(m-1)} = \frac{1}{m-1} - \frac{1}{m} \implies -\frac{1}{m^2} \geq -\frac{1}{m-1} + \frac{1}{m}.$$

It follows that

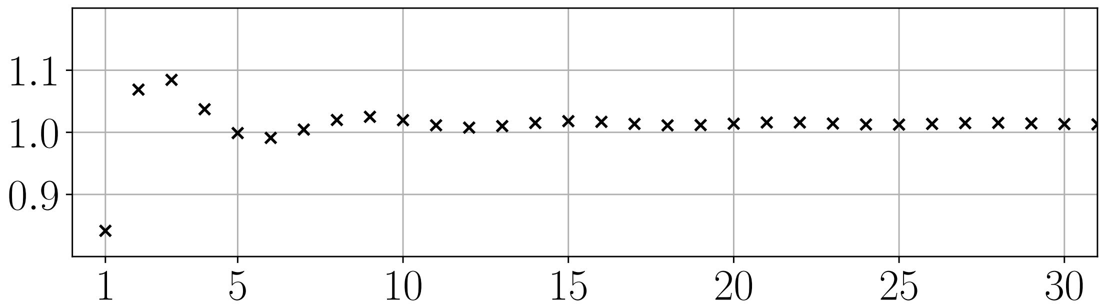
$$\begin{aligned} s_{2n} &= \left(1 - \frac{1}{2^2}\right) + \left(\frac{1}{3} - \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{(2n)^2}\right) \\ &\geq \left(1 - 1 + \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n-1} + \frac{1}{2n}\right) \\ &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} \\ &= \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}. \end{aligned}$$

So we have shown that $s_{2n} \geq \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}$ for all $n \in \mathbf{N}$. Since $\sum_{k=1}^{n}\frac{1}{k}$ is unbounded in n (Example 2.4.5), it follows that (s_{2n}) is unbounded. This implies that (s_n) is unbounded and hence divergent (Theorem 2.3.2).

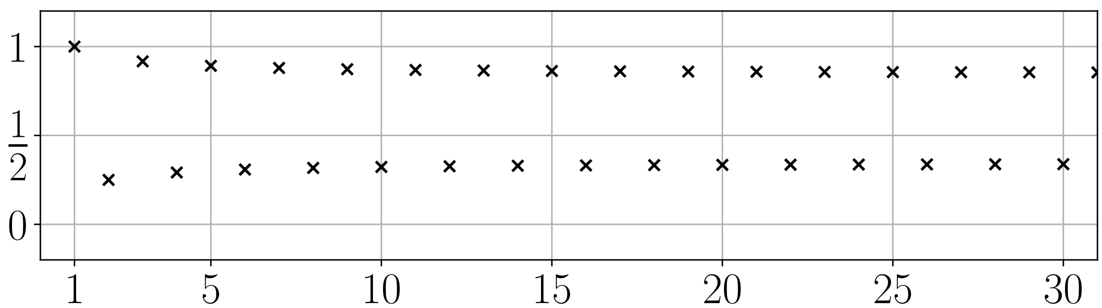


(a) $\sum_{k=1}^{n} \frac{1}{2^k+k}$ for $1 \leq n \leq 30$

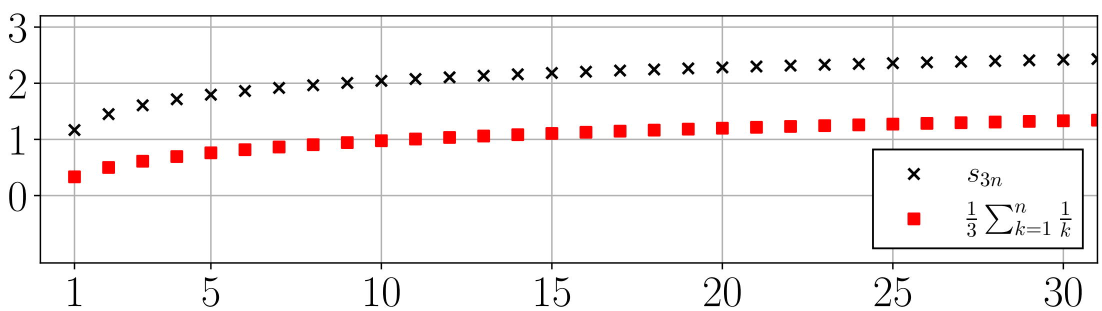
Figure F.20: Exercise 2.7.2 partial sums for $1 \leq n \leq 30$



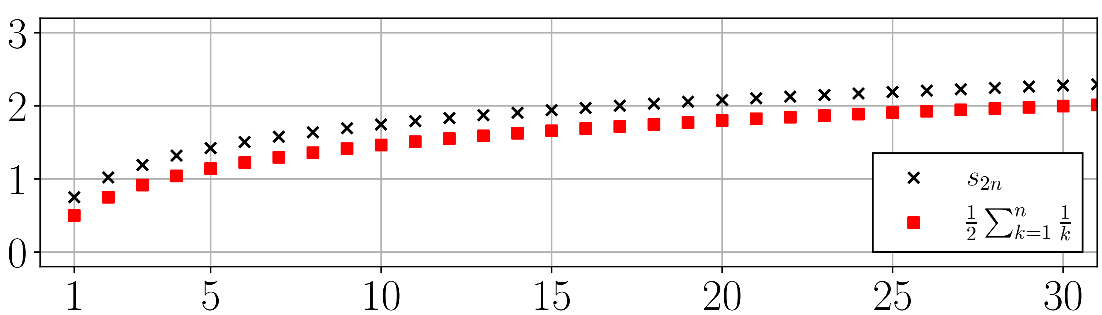
(b) $\sum_{k=1}^{n} \frac{\sin(k)}{k^2}$ for $1 \leq n \leq 30$



(c) $\sum_{k=1}^{n} (-1)^{k+1} \frac{k+1}{2k}$ for $1 \leq n \leq 30$



(d) s_{3n} and $\frac{1}{3} \sum_{k=1}^{n} \frac{1}{k}$ for $1 \leq n \leq 30$



(e) s_{2n} and $\frac{1}{2} \sum_{k=1}^{n} \frac{1}{k}$ for $1 \leq n \leq 30$

Figure F.20: Exercise 2.7.2 partial sums for $1 \leq n \leq 30$ (cont.)

Exercise 2.7.3. (a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

Solution. (a) Since $0 \le a_k \le b_k$ for all $k \in \mathbf{N}$, for any $n > m$ we have

$$|a_{m+1} + \cdots + a_n| = a_{m+1} + \cdots + a_n \le b_{m+1} + \cdots + b_n = |b_{m+1} + \cdots + b_n|. \tag{1}$$

Suppose that $\sum_{k=1}^{\infty} b_k$ is convergent and let $\epsilon > 0$ be given. By the Cauchy Criterion for Series (Theorem 2.7.2), there exists an $N \in \mathbf{N}$ such that

$$n > m \ge N \implies |b_{m+1} + \cdots + b_n| < \epsilon.$$

It then follows from inequality (1) that $|a_{m+1} + \cdots + a_n| < \epsilon$ for all $n > m \ge N$. The Cauchy Criterion for Series (Theorem 2.7.2) allows us to conclude that $\sum_{k=1}^{\infty} a_k$ is convergent.

Now suppose that $\sum_{k=1}^{\infty} a_k$ is divergent. By the Cauchy Criterion for Series (Theorem 2.7.2), there must exist an $\epsilon > 0$ such that for all $N \in \mathbf{N}$ there are positive integers n and m such that

$$n > m \ge N \qquad \text{and} \qquad |a_{m+1} + \cdots + a_n| \ge \epsilon.$$

Let $N \in \mathbf{N}$ be given and let n and m be the positive integers obtained above. Inequality (1) then gives us $|b_{m+1} + \cdots + b_n| \ge \epsilon$; it follows from the Cauchy Criterion for Series (Theorem 2.7.2) that $\sum_{k=1}^{\infty} b_k$ is divergent.

(b) Define the sequences of partial sums

$$s_n = a_1 + \cdots + a_n \qquad \text{and} \qquad t_n = b_1 + \cdots + b_n.$$

Since $0 \le a_k \le b_k$ for all $k \in \mathbf{N}$, both sequences of partial sums are increasing and satisfy $0 \le s_n \le t_n$ for all $n \in \mathbf{N}$. It follows from the Monotone Convergence Theorem (Theorem 2.4.2) that the convergence of each sequence is equivalent to the boundedness of that sequence. From the inequality $0 \le s_n \le t_n$, it is clear that (s_n) is bounded if (t_n) is bounded and that (t_n) is unbounded if (s_n) is unbounded.

Exercise 2.7.4. Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge but $\sum y_n$ diverges.

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

Solution. (a) If we let (x_n) and (y_n) be the sequences given by $x_n = y_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series (Example 2.4.5), but $\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (Example 2.4.4).

(b) Let (x_n) be the sequence given by $x_n = \frac{(-1)^{n+1}}{n}$ and (y_n) be the bounded sequence given by $y_n = (-1)^{n+1}$. It then follows from the Alternating Series Test (Theorem 2.7.7) that $\sum_{n=1}^{\infty} x_n$ is convergent, but $\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series (Example 2.4.5).

(c) This is impossible; by Theorem 2.7.1 we must have

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} (x_n + y_n) - \sum_{n=1}^{\infty} x_n.$$

(d) Let (x_n) be the sequence given by

$$x_n = \begin{cases} \frac{1}{2(n+1)} & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even,} \end{cases} \qquad \text{i.e., } (x_n) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{4}, \frac{1}{12}, \frac{1}{6}, \dots \right),$$

so that $0 \leq x_n \leq \frac{1}{n}$ for all $n \in \mathbf{N}$, and let (s_n) be the sequence of partial sums for the series $\sum_{n=1}^{\infty} (-1)^n x_n$. Observe that

$$\begin{aligned} s_{2n} &= \left(-\frac{1}{4} + \frac{1}{2} \right) + \left(-\frac{1}{8} + \frac{1}{4} \right) + \cdots + \left(-\frac{1}{4n} + \frac{1}{2n} \right) \\ &= \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{4n} \\ &= \frac{1}{4} \sum_{k=1}^{n} \frac{1}{k}. \end{aligned}$$

It follows that (s_{2n}) is unbounded (Example 2.4.5) and hence that $\sum_{n=1}^{\infty}(-1)^n x_n$ is divergent.

Exercise 2.7.5. Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7.

Solution. We want to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. If $p \leq 0$, then $\frac{1}{n^p}$ does not converge to zero; it follows that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges (Theorem 2.7.3). Suppose that $p > 0$ and notice that the sequence $\frac{1}{n^p}$ is positive and decreasing. The Cauchy Condensation Test (Theorem 2.4.6) then implies that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if the series

$$\sum_{n=0}^{\infty} \frac{2^n}{(2^n)^p} = \sum_{n=0}^{\infty} \left(2^{1-p}\right)^n$$

is convergent. This is a geometric series with common ratio 2^{1-p}, so by Example 2.7.5 this series is convergent if and only if

$$\left|2^{1-p}\right| < 1 \iff 1 - p < 0 \iff p > 1.$$

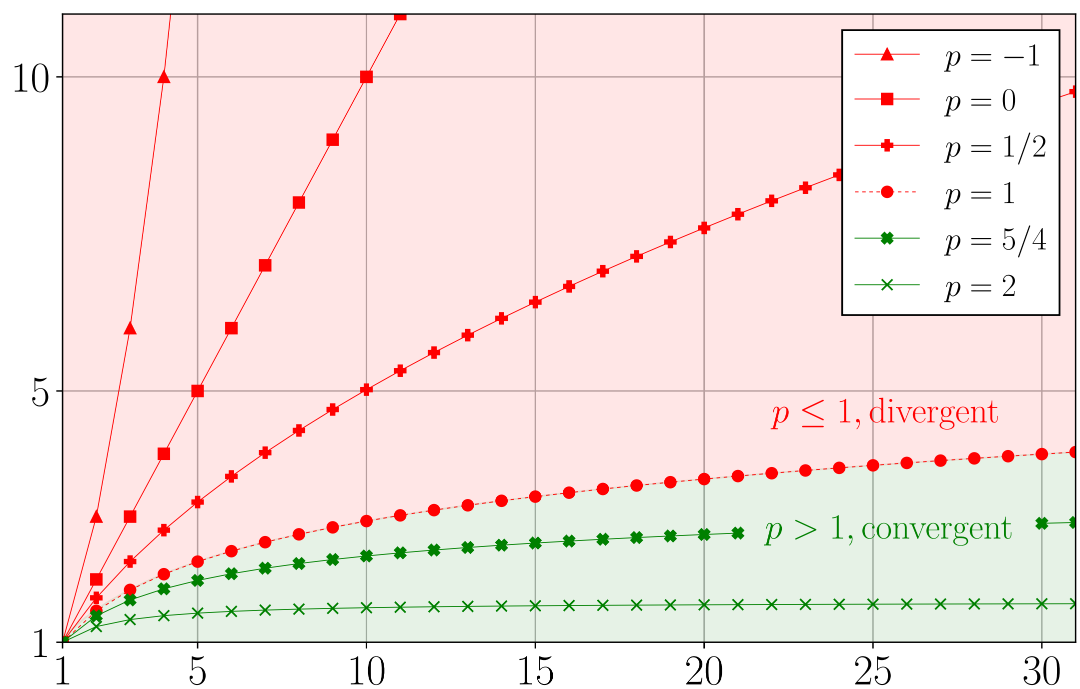


Figure F.21: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ partial sums for various values of p

Exercise 2.7.6. Let's say that a series subverges if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about subvergent series:

(a) If (a_n) is bounded, then $\sum a_n$ subverges.

(b) All convergent series are subvergent.

(c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.

(d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

Solution. (a) This is false in general. For the bounded sequence $(a_n) = (1, 1, 1, \ldots)$, the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is $(1, 2, 3, \ldots)$. This sequence is unbounded and monotone and hence contains no convergent subsequence (Lemma L.8).

(b) This is true. If the sequence of partial sums (s_n) is convergent then any subsequence of (s_n) is convergent; (s_n) itself, for example.

(c) This is true; we will prove the contrapositive statement. Define the sequences of partial sums

$$s_n = |a_1| + \cdots + |a_n| \quad \text{and} \quad t_n = a_1 + \cdots + a_n.$$

We want to show that if (t_n) has no convergent subsequence, then neither does (s_n). By the Bolzano-Weierstrass Theorem (Theorem 2.5.5) it must be the case that (t_n) is unbounded and, since $t_n \leq s_n$ for all $n \in \mathbf{N}$, it follows that (s_n) is unbounded. Thus (s_n) is an increasing unbounded sequence; such sequences do not have convergent subsequences, as shown in Lemma L.8.

(d) This is false in general. Consider the sequence $(a_n) = (1, -1, 2, -2, 3, -3, \ldots)$. The sequence of partial sums is $(s_n) = (1, 0, 2, 0, 3, 0, \ldots)$, which has the convergent subsequence $(0, 0, 0, \ldots)$; it follows that $\sum_{n=1}^{\infty} a_n$ subverges. However, (a_n) has no convergent subsequence. To see this, observe that for any sequence (x_n) we have

$$(x_n) \text{ has a convergent subsequence} \implies (|x_n|) \text{ has a convergent subsequence},$$

since if $\lim_k x_{n_k} = x$ then $\lim_k |x_{n_k}| = |x|$ (Exercise 2.3.10 (b)). Because $(|a_n|) = (1, 1, 2, 2, 3, 3, \ldots)$ has no convergent subsequence (see Lemma L.8), it follows that (a_n) has no convergent subsequence.

Exercise 2.7.7. (a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Solution. The condition that $a_n > 0$ can be relaxed to $a_n \geq 0$ for both parts of this exercise.

(a) Because $na_n \geq 0$ for all $n \in \mathbf{N}$, the Order Limit Theorem (Theorem 2.3.4) and the assumption $l \neq 0$ imply that $l > 0$. Since $na_n \to l$, there exists an $N \in \mathbf{N}$ such that

$$n \geq N \implies 0 < \frac{l}{2} < na_n \implies 0 < \frac{l}{2n} < a_n.$$

Thus the series $\sum_{n=1}^{\infty} a_n$ diverges by comparison (Theorem 2.7.4) with the divergent series $\sum_{n=1}^{\infty} \frac{l}{2n}$ (Example 2.4.5).

(b) Suppose that $\lim(n^2 a_n) = L$; the Order Limit Theorem (Theorem 2.3.4) implies that $L \geq 0$. There is an $N \in \mathbf{N}$ such that

$$n \geq N \implies 0 \leq n^2 a_n < L + 1 \implies 0 \leq a_n < \frac{L+1}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{L+1}{n^2}$ is convergent (Corollary 2.4.7), the Comparison Test (Theorem 2.7.4) implies that $\sum_{n=1}^{\infty} a_n$ is also convergent.

Exercise 2.7.8. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

Solution. (a) This is true. Since the series $\sum_{n=1}^{\infty} |a_n|$ converges, we must have $\lim |a_n| = 0$ (Theorem 2.7.3). There is then an $N \in \mathbf{N}$ such that $0 \leq |a_n| \leq 1$ for $n \geq N$; it follows that $0 \leq |a_n|^2 = a_n^2 \leq |a_n|$ for $n \geq N$. We may now apply the Comparison Test (Theorem 2.7.4) to conclude that $\sum_{n=1}^{\infty} a_n^2$ converges absolutely.

(b) This is false. Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$, so that $\lim b_n = 0$. Notice that $\sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test (Theorem 2.7.7), but $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent (Example 2.4.5).

(c) This is true; we will prove that

$$\sum_{n=1}^{\infty} |a_n| \text{ diverges} \implies \sum_{n=1}^{\infty} n^2 a_n \text{ diverges},$$

by proving the contrapositive statement

$$\sum_{n=1}^{\infty} n^2 a_n \text{ converges} \implies \sum_{n=1}^{\infty} |a_n| \text{ converges}.$$

By Theorem 2.7.3 we have $\lim(n^2 a_n) = 0$, which implies that $\lim(n^2|a_n|) = 0$. We may now apply Exercise 2.7.7 (b) to conclude that $\sum_{n=1}^{\infty}|a_n|$ is convergent.

Exercise 2.7.9 (Ratio Test). Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

(b) Why does $|a_N| \sum (r')^n$ converge?

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Solution. (a) Since $\lim\left|\frac{a_{n+1}}{a_n}\right| = r$ and $r' - r > 0$, there is an $N \in \mathbf{N}$ such that

$$n \geq N \implies \left| \left| \frac{a_{n+1}}{a_n} \right| - r \right| < r' - r \implies \frac{|a_{n+1}|}{|a_n|} < r' \implies |a_{n+1}| < |a_n| r'.$$

(b) Since $0 < r' < 1$, the geometric series $\sum_{n=0}^{\infty} (r')^n$ converges (Example 2.7.5).

(c) By part (a) we have

$$|a_{N+n}| < |a_{N+n-1}| r' < |a_{N+n-2}| (r')^2 < \cdots < |a_N| (r')^n$$

for any $n \in \mathbf{N}$. It then follows from part (b) and the Comparison Test (Theorem 2.7.4) that the series

$$\sum_{n=0}^{\infty} |a_{N+n}| = \sum_{n=N}^{\infty} |a_n|$$

is convergent. Since a finite number of terms do not affect convergence, we see that the series $\sum_{n=1}^{\infty}|a_n|$ is convergent; the convergence of $\sum_{n=1}^{\infty} a_n$ is then given by Theorem 2.7.6.

Exercise 2.7.10 (Infinite Products). Review Exercise 2.4.10 about infinite products and then answer the following questions:

(a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$ converge?

(b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots$ certainly converges. (Why?) Does it converge to zero?

(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}.$$

Show that the left side of this identity at least converges to something. (A complete proof of this result is taken up in Section 8.3.)

Solution. (a) This is the infinite product

$$\prod_{n=0}^{\infty} \frac{2^n + 1}{2^n} = \prod_{n=0}^{\infty}\left(1 + \frac{1}{2^n}\right).$$

By Exercise 2.4.10, this infinite product converges if and only if the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges. This series is geometric with common ratio $r = \frac{1}{2}$ and hence convergent by Example 2.7.5; it follows that the infinite product converges.

(b) This is the infinite product

$$\prod_{n=1}^{\infty} \frac{2n - 1}{2n} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{2n}\right).$$

The sequence of partial products is positive and decreasing, since each term in the partial product satisfies $0 < 1 - \frac{1}{2n} < 1$; the Monotone Convergence Theorem (Theorem 2.4.2) then implies that the infinite product converges.

Indeed, this infinite product converges to zero. To see this, let (p_m) be the sequence of partial products:

$$p_m = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2m - 1}{2m}.$$

As stated above, (p_m) is decreasing and satisfies $0 < p_m < 1$ for all $m \in \mathbf{N}$, so we can look at the sequence of reciprocals (p_m^{-1}):

$$\begin{aligned}
\frac{1}{p_m} &= \frac{2}{1} \cdot \frac{4}{3} \cdots \frac{2m}{2m-1} \\
&= \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{2m-1}\right) \\
&\geq \sum_{n=1}^{m} \frac{1}{2n-1} \\
&\geq \frac{1}{2} \sum_{n=1}^{m} \frac{1}{n}.
\end{aligned}$$

It follows from Example 2.4.5 that (p_m^{-1}) is unbounded above. Thus, for any $\epsilon > 0$, there is an $M \in \mathbf{N}$ such that $p_M^{-1} > \epsilon^{-1}$, and since (p_m) is decreasing we then have

$$m \geq M \implies |p_m| = p_m \leq p_M < \epsilon.$$

Hence $\lim p_m = 0$.

(c) This is the infinite product

$$\prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{(2n-1)(2n+1)}\right) = \prod_{n=1}^{\infty} \left(1 + \frac{1}{4n^2-1}\right).$$

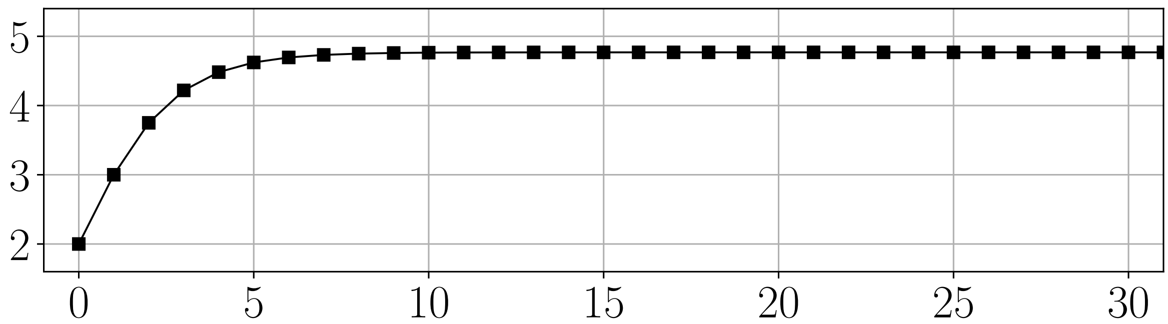
By Exercise 2.4.10, this infinite product converges if and only if the series $\sum_{n=0}^{\infty} \frac{1}{4n^2-1}$ converges. Observe that for all $n \in \mathbf{N}$ we have

$$n^2 - 1 \geq 0 \implies 4n^2 - 1 \geq 3n^2 \implies \frac{1}{4n^2-1} \leq \frac{1}{3n^2}.$$

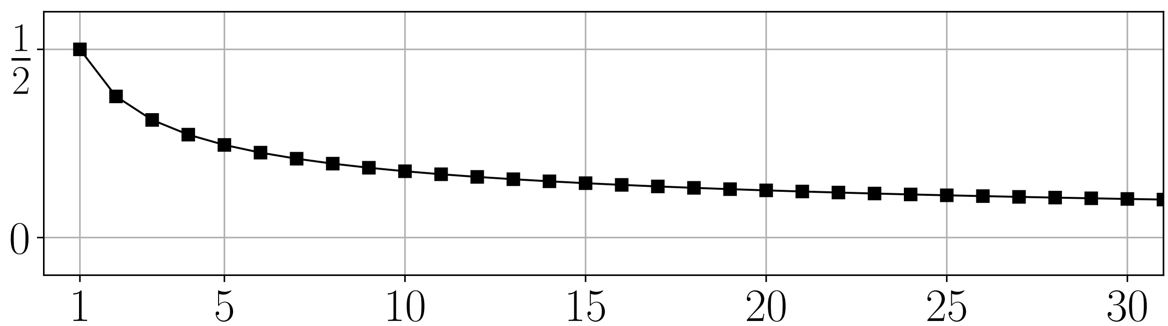
The series $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ is convergent (Corollary 2.4.7), so the Comparison Test (Theorem 2.7.4) implies that the series $\sum_{n=0}^{\infty} \frac{1}{4n^2-1}$ is also convergent; it follows that the infinite product

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots$$

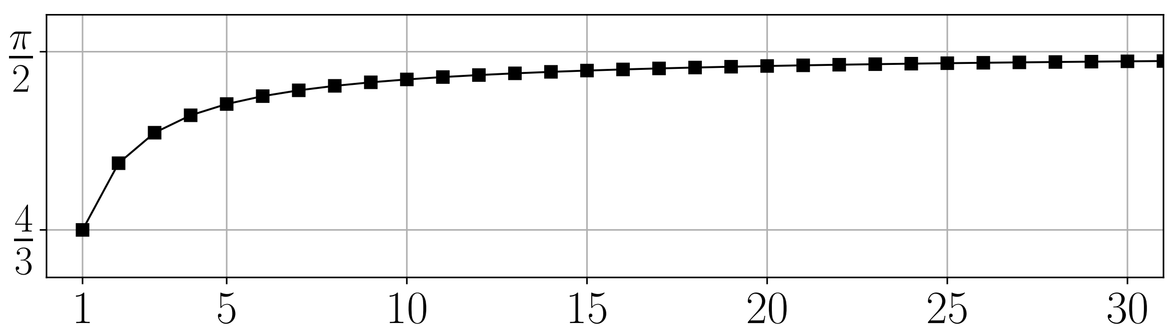
converges.



(a) $\prod_{n=0}^{m}\left(1+\frac{1}{2^n}\right)$ for $0 \leq m \leq 30$



(b) $\prod_{n=1}^{m}\left(1-\frac{1}{2n}\right)$ for $1 \leq m \leq 30$



(c) $\prod_{n=1}^{m}\left(1+\frac{1}{4n^2-1}\right)$ for $1 \leq m \leq 30$

Figure F.22: Exercise 2.7.10 partial products

Exercise 2.7.11. Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

Solution. Consider the series

$$\sum_{n=1}^{\infty} a_n = \underbrace{\frac{1}{1^2}}_{\substack{\text{1 term} \\ \text{sum} = 1}} + \frac{1}{2^2} + \cdots + \frac{1}{5^2} + \underbrace{\frac{1}{6^2} + \cdots + \frac{1}{6^2}}_{\substack{6^2 \text{ terms} \\ \text{sum} = 1}} + \frac{1}{42^2} + \cdots + \frac{1}{1805^2} + \cdots$$

$$\sum_{n=1}^{\infty} b_n = \frac{1}{1^2} + \underbrace{\frac{1}{2^2} + \cdots + \frac{1}{2^2}}_{\substack{2^2 \text{ terms} \\ \text{sum} = 1}} + \frac{1}{6^2} + \cdots + \frac{1}{41^2} + \underbrace{\frac{1}{42^2} + \cdots + \frac{1}{42^2}}_{\substack{42^2 \text{ terms} \\ \text{sum} = 1}} + \cdots$$

Both (a_n) and (b_n) are strictly positive and decreasing and

$$\sum_{n=1}^{\infty} \min\{a_n, b_n\} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a convergent series. Furthermore, both $\sum a_n$ and $\sum b_n$ diverge since their respective sequences of partial sums are unbounded; we can find arbitrarily many groupings of terms which sum to 1 as shown above.

Exercise 2.7.12 (Summation by parts). Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).$$

Solution. For positive integers $n > m$,

$$\begin{aligned}
\sum_{j=m}^{n} x_j y_j &= \sum_{j=m}^{n} (s_j - s_{j-1}) y_j \\
&= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_{j-1} y_j \\
&= \sum_{j=m}^{n} s_j y_j - \sum_{j=m-1}^{n-1} s_j y_{j+1}
\end{aligned}$$

$$= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_j y_{j+1} + s_n y_{n+1} - s_{m-1} y_m$$
$$= s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).$$

Exercise 2.7.13 (Abel's Test). Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0,$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}),$$

where $s_n = x_1 + x_2 + \cdots + x_n$.

(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_k (y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

Solution. (a) This follows immediately from Exercise 2.7.12, taking $m = 1$ and remembering that $s_0 := 0$.

(b) By assumption the sequence (s_k) is convergent and hence, by Theorem 2.3.2, bounded by some $M > 0$, so that for each $k \in \mathbf{N}$ we have the inequality

$$0 \leq |s_k (y_k - y_{k+1})| = |s_k| (y_k - y_{k+1}) \leq M (y_k - y_{k+1}). \tag{1}$$

Notice that since (y_k) is decreasing and bounded below, the limit $y := \lim_{k\to\infty} y_k$ exists by the Monotone Convergence Theorem (Theorem 2.4.2). It follows that the series $\sum_{k=1}^{\infty} (y_k - y_{k+1})$ is convergent since, letting t_m be the m^{th} partial sum, we have

$$t_m = (y_1 - y_2) + (y_2 - y_3) + \cdots + (y_m - y_{m+1}) = y_1 - y_{m+1} \to y_1 - y \text{ as } m \to \infty.$$

Inequality (1) and the Comparison Test (Theorem 2.7.4) then imply that $\sum_{k=1}^{\infty} s_k (y_k - y_{k+1})$ is absolutely convergent and hence convergent (Theorem 2.7.6). From part (a) we have $\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1})$; it follows that

$$\sum_{k=1}^{\infty} x_k y_k = \lim_{n\to\infty} \left(s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}) \right) = y \sum_{k=1}^{\infty} x_k + \sum_{k=1}^{\infty} s_k (y_k - y_{k+1}).$$

Exercise 2.7.14 (Dirichlet's Test). Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if (y_k) is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$ with $\lim y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test in Exercise 2.7.13, but show that essentially the same strategy can be used to provide a proof.

(b) Show how the Alternating Series Test (Theorem 2.7.7) can be derived as a special case of Dirichlet's Test.

Solution. (a) Abel's Test has the stronger hypothesis that the sequence of partial sums of $\sum_{k=1}^{\infty} x_k$ is convergent (and hence bounded), but the weaker hypothesis that (y_k) only satisfies $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$ without necessarily converging to zero.

The proof of Dirichlet's Test is almost identical to the proof of Abel's Test given in Exercise 2.7.13 (b). Letting (s_k) be the k^{th} partial sum of $\sum_{n=1}^{\infty} x_n$, we are given that (s_k) is bounded by some $M > 0$. It follows that

$$0 \leq |s_k(y_k - y_{k+1})| = |s_k|(y_k - y_{k+1}) \leq M(y_k - y_{k+1}) \tag{1}$$

for each $k \in \mathbf{N}$. The series $\sum_{k=1}^{\infty}(y_k - y_{k+1})$ is convergent since it has m^{th} partial sum

$$(y_1 - y_2) + (y_2 - y_3) + \cdots + (y_m - y_{m+1}) = y_1 - y_{m+1} \to y_1 \text{ as } m \to \infty.$$

Inequality (1) and the Comparison Test (Theorem 2.7.4) then imply that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ is absolutely convergent and hence convergent. Since (s_k) is bounded and $\lim y_k = 0$, we have $\lim(s_k y_{k+1}) = 0$ also (Exercise 2.3.9 (b)). It follows that

$$\sum_{k=1}^{\infty} x_k y_k = \lim_{n\to\infty}\left(s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1})\right) = \sum_{k=1}^{\infty} s_k(y_k - y_{k+1}).$$

(b) The Alternating Series Test (Theorem 2.7.7) can be recovered from Dirichlet's Test by taking $x_k = (-1)^{k+1}$; the sequence of partial sums of $\sum_{k=1}^{\infty} x_k$ is then $(1, 0, 1, 0, \ldots)$, which is certainly bounded.

2.8 Double Summations and Products of Infinite Series

Exercise 2.8.1. Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{nn}$. How does this value compare to the two iterated values for the sum already computed?

Solution. The array in question is

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & -1 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $a_{ij} = 2^{i-j}$ if $j > i, a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$. If we let $f(j)$ be the sum of the first row up to the j^{th} column, then using the formula for the partial sums of a geometric series, we find that

$$\begin{aligned} f(j) &= \begin{cases} -1 & \text{if } j = 1, \\ -1 + \frac{1}{2} + \cdots + \frac{1}{2^{j-1}} = -\frac{1}{2^{j-1}} & \text{if } j \geq 2 \end{cases} \\ &= -\frac{1}{2^{j-1}}. \end{aligned}$$

Since subsequent rows are simply the first row shifted along, it is clear that $s_{11} = f(1), s_{22} = f(1) + f(2), s_{33} = f(1) + f(2) + f(3)$, and in general

$$s_{nn} = \sum_{j=1}^{n} f(j) = \sum_{j=1}^{n} \frac{-1}{2^{j-1}} = -\sum_{j=0}^{n-1} \frac{1}{2^j}.$$

It follows that

$$\lim_{n\to\infty} s_{nn} = -\sum_{j=0}^{\infty} \frac{1}{2^j} = -2.$$

At the beginning of Section 2.1, we found that summing along the rows first gave a value of 0 for the double sum, whereas summing down the columns first gave a value of -2.

Exercise 2.8.2. Show that if the iterated series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges (meaning for each fixed $i \in \mathbf{N}$ the series $\sum_{j=1}^{\infty} |a_{ij}|$ converges to some real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$$

converges.

Solution. For each $i \in \mathbf{N}$, Theorem 2.7.6 implies that the series $\sum_{j=1}^{\infty} a_{ij}$ converges to some real number c_i. Observe that

$$0 \leq |c_i| = \left| \sum_{j=1}^{\infty} a_{ij} \right| \leq \sum_{j=1}^{\infty} |a_{ij}| = b_i.$$

Since $\sum_{i=1}^{\infty} b_i$ converges, the Comparison Test (Theorem 2.7.4) implies that the series $\sum_{i=1}^{\infty} c_i$ is absolutely convergent and hence convergent (Theorem 2.7.6).

Exercise 2.8.3. (a) Prove that (t_{nn}) converges.

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges.

Solution. (a) Since $|a_{ij}| \geq 0$ for all positive integers i and j, the sequence (t_{nn}) is increasing and bounded above by the real number $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$. Thus (t_{nn}) converges by the Monotone Convergence Theorem (Theorem 2.4.2).

(b) Suppose $n > m$ are positive integers. By examining the following array, which has the special case $n = 6$ and $m = 3$,

$$\begin{array}{ccc|ccc}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
\hline
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{array}$$

we see that
$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}-\sum_{i=1}^{m}\sum_{j=1}^{m}a_{ij}=\sum_{i=1}^{m}\sum_{j=m+1}^{n}a_{ij}+\sum_{i=m+1}^{n}\sum_{j=1}^{n}a_{ij}.$$
(The sum of the top right "square" (in red) and the bottom "rectangle" (in blue) of the array.) Let $\epsilon > 0$ be given. Since (t_{nn}) is a Cauchy sequence, there exists an $N \in \mathbf{N}$ such that $n > m \geq N$ implies
$$|t_{nn}-t_{mm}| = t_{nn}-t_{mm} < \epsilon.$$
For such n and m, observe that
$$\begin{aligned}
|s_{nn}-s_{mm}| &= \left|\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}-\sum_{i=1}^{m}\sum_{j=1}^{m}a_{ij}\right| \\
&= \left|\sum_{i=1}^{m}\sum_{j=m+1}^{n}a_{ij}+\sum_{i=m+1}^{n}\sum_{j=1}^{n}a_{ij}\right| \\
&\leq \sum_{i=1}^{m}\sum_{j=m+1}^{n}|a_{ij}|+\sum_{i=m+1}^{n}\sum_{j=1}^{n}|a_{ij}| \\
&= t_{nn}-t_{mm} \\
&< \epsilon.
\end{aligned}$$
It follows that (s_{nn}) is a Cauchy sequence and hence convergent.

Exercise 2.8.4. (a) Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbf{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

(b) Now, show that there exists an N such that
$$|s_{mn} - S| < \epsilon$$
for all $m, n \geq N$.

Solution. (a) By Lemma 1.3.8, there exist positive integers m', n' such that $B - \frac{\epsilon}{2} < t_{m'n'} \leq B$. Set $N_1 = \max\{m', n'\}$. Since each $|a_{ij}|$ is positive, (t_{mn}) is increasing in both m and n; it follows that for $m, n \geq N_1$ we have $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

(b) Since $\lim_{n\to\infty} s_{nn} = S$, there is an $N_2 \in \mathbf{N}$ such that $|s_{nn} - S| < \frac{\epsilon}{2}$ for all $n \geq N_2$. Set $N = \max\{N_1, N_2\}$ and suppose that $m, n > N$. Similarly to Exercise 2.8.3 (b), we have

$$\begin{aligned}
|s_{mn} - s_{NN}| &= \left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} - \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\right| \\
&= \left|\sum_{i=1}^{N}\sum_{j=N+1}^{n} a_{ij} + \sum_{i=N+1}^{m}\sum_{j=1}^{n} a_{ij}\right| \\
&\leq \sum_{i=1}^{N}\sum_{j=N+1}^{n} |a_{ij}| + \sum_{i=N+1}^{m}\sum_{j=1}^{n} |a_{ij}| \\
&= t_{mn} - t_{NN} \\
&\leq B - t_{NN} \\
&< \frac{\epsilon}{2}.
\end{aligned}$$

It follows that

$$|s_{mn} - S| \leq |s_{mn} - s_{NN}| + |s_{NN} - S| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Exercise 2.8.5. (a) Show that for all $m \geq N$

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \epsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges to S.

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$, converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

Solution. (a) Suppose that $n \geq N$. Then

$$\begin{aligned}
|(r_1 + \cdots + r_m) - S| &\leq |(r_1 + \cdots + r_m) - s_{mn}| + |s_{mn} - S| \\
&< \left|(r_1 + \cdots + r_m) - \left(\sum_{j=1}^{n} a_{1j} + \cdots + \sum_{j=1}^{n} a_{mj}\right)\right| + \epsilon \\
&\leq \left|r_1 - \sum_{j=1}^{n} a_{1j}\right| + \cdots + \left|r_m - \sum_{j=1}^{n} a_{mj}\right| + \epsilon.
\end{aligned}$$

Since this is true for any $n \geq N$ and for any given i we have $\sum_{j=1}^{\infty} a_{ij} = r_i$, taking the limit in n on both sides of the inequality

$$|(r_1 + r_2 + \cdots + r_m) - S| < \left| r_1 - \sum_{j=1}^{n} a_{1j} \right| + \left| r_2 - \sum_{j=1}^{n} a_{2j} \right| + \cdots + \left| r_m - \sum_{j=1}^{n} a_{mj} \right| + \epsilon$$

gives us

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \epsilon.$$

It follows that $\lim_{m\to\infty}(\sum_{i=1}^{m} r_i) = S$, i.e., $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = S$.

(b) Fix $j \in \mathbf{N}$ and let (x_n) be the sequence of partial sums of the series $\sum_{i=1}^{\infty}|a_{ij}|$, i.e.,

$$x_n = |a_{1j}| + |a_{2j}| + \cdots + |a_{nj}|.$$

Since each $|a_{ij}|$ is a term of the convergent series $\sum_{j=1}^{\infty}|a_{ij}| = r_i$, which has only non-negative terms, we see that $|a_{ij}| \leq r_i$, so that

$$x_n \leq r_1 + r_2 + \cdots + r_n \leq \sum_{i=1}^{\infty} r_i,$$

where the last inequality follows since each r_i is non-negative. So (x_n) is an increasing and bounded sequence and hence converges by the Monotone Convergence Theorem (Theorem 2.4.2). It follows that $\sum_{i=1}^{\infty} a_{ij}$ converges to some (non-negative) real number c_j.

Let $\epsilon > 0$ be given. As in Exercise 2.8.4, there is an $N \in \mathbf{N}$ such that $|s_{mn} - S| < \epsilon$ for all $m, n \geq N$. We can write s_{mn} as

$$s_{mn} = \sum_{i=1}^{m} a_{i1} + \sum_{i=1}^{m} a_{i2} + \cdots + \sum_{i=1}^{m} a_{in}.$$

Suppose that $m, n \geq N$. Then

$$\begin{aligned} |(c_1 + \cdots + c_n) - S| &\leq |(c_1 + \cdots + c_n) - s_{mn}| + |s_{mn} - S| \\ &< \left| (c_1 + \cdots + c_n) - \left(\sum_{i=1}^{m} a_{i1} + \cdots + \sum_{i=1}^{m} a_{in} \right) \right| + \epsilon \\ &\leq \left| c_1 - \sum_{i=1}^{m} a_{i1} \right| + \cdots + \left| c_n - \sum_{i=1}^{m} a_{in} \right| + \epsilon. \end{aligned}$$

Since this is true for any $m \geq N$ and for any given j we have $\sum_{i=1}^{\infty} a_{ij} = c_j$, taking the limit in m on both sides of the inequality

$$|(c_1 + c_2 + \cdots + c_n) - S| < \left|c_1 - \sum_{i=1}^{m} a_{i1}\right| + \left|c_2 - \sum_{i=1}^{m} a_{i2}\right| + \cdots + \left|c_n - \sum_{i=1}^{m} a_{in}\right| + \epsilon$$

gives us

$$|(c_1 + c_2 + \cdots + c_n) - S| \leq \epsilon.$$

It follows that $\lim_{n\to\infty}\left(\sum_{j=1}^{n} c_j\right) = S$, i.e., $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = S$.

Exercise 2.8.6. (a) Assuming the hypothesis—and hence the conclusion—of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

(b) Imitate the strategy in the proof of Theorem 2.8.1 to show that $\sum_{k=2}^{\infty} d_k$ converges to $S = \lim_{n\to\infty} s_{nn}$.

Solution. (a) Observe that

$$|d_2| = |a_{11}| = \sum_{i=1}^{1}\sum_{j=1}^{2-i}|a_{ij}|,$$

$$|d_2| + |d_3| = |a_{11}| + |a_{12} + a_{21}| \leq (|a_{11}| + |a_{12}|) + |a_{21}| = \sum_{i=1}^{2}\sum_{j=1}^{3-i}|a_{ij}|,$$

$$\begin{aligned}|d_2| + |d_3| + |d_4| &= |a_{11}| + |a_{12} + a_{21}| + |a_{13} + a_{22} + a_{31}| \\ &\leq (|a_{11}| + |a_{12}| + |a_{13}|) + (|a_{21}| + |a_{22}|) + |a_{31}| = \sum_{i=1}^{3}\sum_{j=1}^{4-i}|a_{ij}|,\end{aligned}$$

and in general for $n \geq 2$,

$$\sum_{k=2}^{n}|d_k| \leq \sum_{i=1}^{n-1}\sum_{j=1}^{n-i}|a_{ij}| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|,$$

where the last inequality follows since each $|a_{ij}|$ is non-negative. By assumption $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$ is finite, so the sequence $\sum_{k=2}^{n}|d_k|$ is increasing and bounded above and hence converges by the Monotone Convergence Theorem (Theorem 2.4.2).

(b) By considering Figure F.23, which shows the special case $n = 6$, we see that for each $n \geq 2$,

$$s_{nn} - \sum_{k=2}^{n} d_k = \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} a_{ij}.$$

a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}
a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}
a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}
a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}
a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}

Figure F.23: $s_{66} - \sum_{k=2}^{6} d_k = \sum_{i=1}^{6} \sum_{j=7-i}^{6} a_{ij}$

Similarly, letting

$$e_k = |a_{1,k-1}| + |a_{2,k-2}| + \cdots + |a_{k-1,1}|$$

for $k \geq 2$, we find that

$$t_{nn} - \sum_{k=2}^{n} e_k = \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} |a_{ij}|$$

for each $n \geq 2$. It follows that

$$\left| s_{nn} - \sum_{k=2}^{n} d_k \right| = \left| \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} a_{ij} \right| \leq \sum_{i=1}^{n} \sum_{j=n+1-i}^{n} |a_{ij}| = t_{nn} - \sum_{k=2}^{n} e_k. \tag{1}$$

Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} s_{nn} = S$ and (t_{nn}) is an increasing Cauchy sequence, there are positive integers N_1, N_2 such that

$$n \geq N_1 \implies |s_{nn} - S| < \frac{\epsilon}{2} \qquad \text{and} \qquad n > m \geq N_2 \implies t_{nn} - t_{mm} < \frac{\epsilon}{2}. \tag{2}$$

Set $N = \max\{N_1, 2N_2\}$ and suppose $n \geq N$. Since $n \geq 2N_2$, each term of $t_{N_2N_2}$ appears in $\sum_{k=2}^{n} e_k$ (see Figure F.24, which has the special case $n = 6$ and $N_2 = 3$).

$\lvert a_{11}\rvert$	$\lvert a_{12}\rvert$	$\lvert a_{13}\rvert$	$\lvert a_{14}\rvert$	$\lvert a_{15}\rvert$	$\lvert a_{16}\rvert$
$\lvert a_{21}\rvert$	$\lvert a_{22}\rvert$	$\lvert a_{23}\rvert$	$\lvert a_{24}\rvert$	$\lvert a_{25}\rvert$	$\lvert a_{26}\rvert$
$\lvert a_{31}\rvert$	$\lvert a_{32}\rvert$	$\lvert a_{33}\rvert$	$\lvert a_{34}\rvert$	$\lvert a_{35}\rvert$	$\lvert a_{36}\rvert$
$\lvert a_{41}\rvert$	$\lvert a_{42}\rvert$	$\lvert a_{43}\rvert$	$\lvert a_{44}\rvert$	$\lvert a_{45}\rvert$	$\lvert a_{46}\rvert$
$\lvert a_{51}\rvert$	$\lvert a_{52}\rvert$	$\lvert a_{53}\rvert$	$\lvert a_{54}\rvert$	$\lvert a_{55}\rvert$	$\lvert a_{56}\rvert$
$\lvert a_{61}\rvert$	$\lvert a_{62}\rvert$	$\lvert a_{63}\rvert$	$\lvert a_{64}\rvert$	$\lvert a_{65}\rvert$	$\lvert a_{66}\rvert$

Figure F.24: $t_{33} \leq \sum_{k=2}^{6} e_k$

It follows that $t_{N_2N_2} \leq \sum_{k=2}^{n} e_k$ and thus by (1) and (2) we have

$$\left| s_{nn} - \sum_{k=2}^{n} d_k \right| \leq t_{nn} - t_{N_2N_2} < \frac{\epsilon}{2},$$

which implies

$$\left| \sum_{k=2}^{n} d_k - S \right| \leq |s_{nn} - S| + \left| s_{nn} - \sum_{k=2}^{n} d_k \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We may conclude that $\lim_{n\to\infty} \sum_{k=2}^{n} d_k = S$.

Exercise 2.8.7. Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges so that we may apply Theorem 2.8.1.

(b) Let $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1$.

Solution. (a) Let $A' = \sum_{i=1}^{\infty} |a_i|$ and $B' = \sum_{j=1}^{\infty} |b_j|$. Notice that for a fixed $i \in \mathbf{N}$ we have

$$\sum_{j=1}^{n} |a_i b_j| = |a_i| \sum_{j=1}^{n} |b_j| \to |a_i| B' \text{ as } n \to \infty.$$

It follows that

$$\sum_{i=1}^{n} |a_i| B' = B' \sum_{i=1}^{n} |a_i| \to A'B' \text{ as } n \to \infty,$$

i.e.,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_i b_j| = A'B'.$$

(b) For each $n \in \mathbf{N}$ we have

$$s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j = \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right).$$

The Algebraic Limit Theorem (Theorem 2.3.3) now implies that $\lim_{n\to\infty} s_{nn} = AB$, and Theorem 2.8.1 then gives the desired result.

Chapter 3

Basic Topology of R

3.2 Open and Closed Sets

Exercise 3.2.1. (a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be *finite* get used?

(b) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, \ldots\}$ whose intersection $\bigcap_{n=1}^{\infty} O_n$ is closed, not empty and not all of $\mathbf{R}$.

Solution. (a) This assumption is used when we let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$; this minimum is guaranteed to exist because the set $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$ is finite (see Lemma L.3). An infinite subset of $\mathbf{R}$ does not necessarily have a minimum. For example, $\{n^{-1} : n \in \mathbf{N}\}$ has no minimum.

(b) If we let $O_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbf{N}$, then each O_n is open by Example 3.2.2 (ii), the collection $\{O_1, O_2, O_3, \ldots\}$ is countable, and $\bigcap_{n=1}^{\infty} O_n = \{0\} = [0, 0]$, which is non-empty, not equal to $\mathbf{R}$, and closed by Example 3.2.9 (ii).

Exercise 3.2.2. Let

$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots\right\} \qquad \text{and} \qquad B = \{x \in \mathbf{Q} : 0 < x < 1\}.$$

Answer the following questions for each set:

(a) What are the limit points?

(b) Is the set open? Closed?

(c) Does the set contain any isolated points?

(d) Find the closure of the set.

Solution. Let us consider the set A first.

(a) Let L_A be the set of limit points of A. We claim that $L_A = \{-1, 1\}$. To see this, first let (x_n) be the sequence given by $x_n = (-1)^n + \frac{2}{n}$ and notice that:

- $A = \{x_n : n \in \mathbf{N}\}$;
- $\lim_{n\to\infty} x_{2n-1} = -1$;
- $x_{2n-1} \neq -1$ for each $n \in \mathbf{N}$;
- $\lim_{n\to\infty} x_{2n} = 1$;
- $x_{2n} \neq 1$ for each $n \in \mathbf{N}$.

It follows from Theorem 3.2.5 that -1 and 1 are limit points of A, i.e., $\{-1, 1\} \subseteq L_A$.

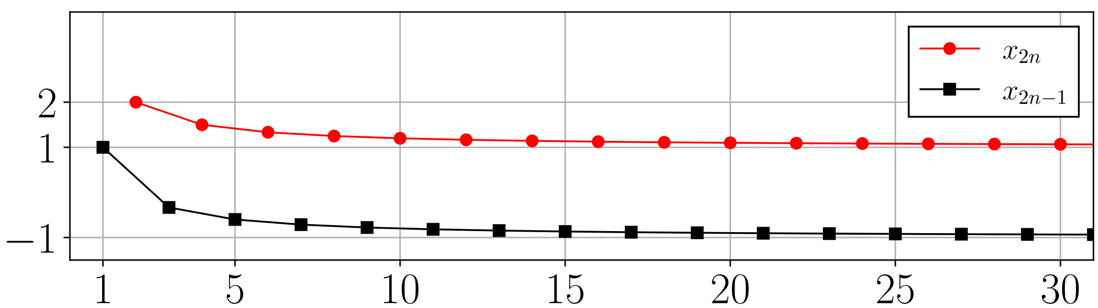


Figure F.25: x_{2n} and x_{2n-1}; A is the union of the red circles and the black squares

Now suppose that $x \in \mathbf{R}$ is such that $x \notin \{-1, 1\}$; we claim that x is not a limit point of A. First note that the distance from x to each of -1 and 1 is strictly positive, so that

$$\epsilon := \min\{|x+1|, |x-1|\} > 0.$$

Since $\lim x_{2n-1} = -1$ and $\lim x_{2n} = 1$, the terms of (x_n) (i.e., the elements of A) must eventually be contained inside

$$V_{\epsilon/2}(-1) \cup V_{\epsilon/2}(1) = \left(-1 - \frac{\epsilon}{2}, -1 + \frac{\epsilon}{2}\right) \cup \left(1 - \frac{\epsilon}{2}, 1 + \frac{\epsilon}{2}\right).$$

Our choice of ϵ is such that

$$\big[V_{\epsilon/2}(-1) \cup V_{\epsilon/2}(1)\big] \cap V_{\epsilon/2}(x) = \emptyset.$$

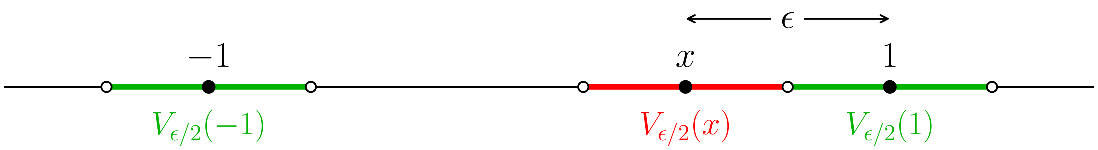


Figure F.26: Special case where x is closer to 1 than to -1

Thus there can be only finitely many elements of A in $V_{\epsilon/2}(x)$; it follows that x cannot possibly be the limit of any sequence of elements of A distinct from x, which by Theorem 3.2.5 is to say that x cannot be a limit point of A. We may conclude that $L_A = \{-1, 1\}$.

(b) A is not open. To see this, consider the point $2 \in A$. We claim that for any $\epsilon > 0$, the neighbourhood $V_\epsilon(2)$ contains some $x \notin A$, so that $V_\epsilon(2) \not\subseteq A$. It is straightforward to verify that every element $a \in A$ satisfies $a \leq 2$ (see Figure F.25). Given this, if we let $x = 2 + \frac{\epsilon}{2}$ then $x \in V_\epsilon(2)$ and $x \notin A$.

A is not closed either since it does not contain the limit point -1: for any $n \in \mathbf{N}$ we have $(-1)^n + \frac{2}{n} > -1$ (see Figure F.25).

(c) Since $L_A = \{-1, 1\}, 1 \in A$ and $-1 \notin A$, every point of A other than 1 is an isolated point of A.

(d) The closure is

$$\overline{A} = A \cup L_A = \{-1\} \cup \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots\right\}.$$

Now let us consider the set B.

(a) Let L_B be the set of limit points of B. We claim that $L_B = [0, 1]$. To see this, first suppose that $x \in [0, 1]$ and let $\epsilon > 0$ be given. Observe that

$$V_\epsilon(x) \cap (0, 1) = (\max\{x - \epsilon, 0\}, \min\{x + \epsilon, 1\}).$$

This is a proper interval contained in $(0,1)$ and hence, by the density of $\mathbf{Q}$ in $\mathbf{R}$, contains infinitely many elements of B. It follows that x is a limit point of B and hence that $[0,1] \subseteq L_B$.

If x is a limit point of B then by Theorem 3.2.5 it must be the case that x is the limit of a sequence of elements of B. The Order Limit Theorem (Theorem 2.3.4) then implies that $0 \le x \le 1$, so that $L_B \subseteq [0,1]$. We may conclude that $L_B = [0,1]$.

(b) B is not open, since for any $x \in B$ and $\epsilon > 0$, the set $V_\epsilon(x)$ will contain irrational numbers (Corollary 1.4.4) and hence cannot be contained in B. B is also not closed, since it does not contain the limit point 0.

(c) B does not contain any isolated points, since $B \subseteq L_B = [0,1]$.

(d) We have $\overline{B} = B \cup L_B = L_B = [0,1]$.

Exercise 3.2.3. Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbf{Q}$.

(b) $\mathbf{N}$.

(c) $\{x \in \mathbf{R} : x \neq 0\}$.

(d) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbf{N}\}$.

(e) $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbf{N}\}$.

Solution. (a) $\mathbf{Q}$ is neither open nor closed. To see that $\mathbf{Q}$ fails to be open, observe that by Corollary 1.4.4, for any $\epsilon > 0$ there are infinitely many irrational numbers in $V_\epsilon(0) = (-\epsilon, \epsilon)$; it follows that $V_\epsilon(0)$ cannot be contained in $\mathbf{Q}$. To see that $\mathbf{Q}$ fails to be closed, observe that $\sqrt{2} \notin \mathbf{Q}$ is a limit point of $\mathbf{Q}$ (Theorems 3.2.5 and 3.2.10).

(b) $\mathbf{N}$ is closed but not open. To see that $\mathbf{N}$ is not open, observe that for any $\epsilon > 0$ there are infinitely many non-integers in $V_\epsilon(1) = (1-\epsilon, 1+\epsilon)$; it follows that $V_\epsilon(1)$ cannot be contained in $\mathbf{N}$. To see that $\mathbf{N}$ is closed, we will show that $\mathbf{N}^{\mathsf{c}}$ is open and appeal to Theorem 3.2.13. Note that

$$\mathbf{N}^{\mathsf{c}} = (-\infty, 1) \cup \bigcup_{n=1}^{\infty} (n, n+1) = \bigcup_{n=1}^{\infty} ((-n, -n+2) \cup (n, n+1)).$$

Since open intervals are open (Example 3.2.2 (ii)), we see that $\mathbf{N}^c$ is a union of open sets and hence is itself open (Theorem 3.2.3 (i)). (It is also straightforward to argue directly from Definition 3.2.1 that intervals of the form $(-\infty, a)$ or (a, ∞) for $a \in \mathbf{R}$ are open. Indeed, if $x \in (-\infty, a)$, then observe that $V_\epsilon(x) \subseteq (-\infty, a)$, where $\epsilon = a - x$; a similar argument holds for (a, ∞).)

(c) If we let E be the set in question, then E is open since it is the union of two open sets, $E = (-\infty, 0) \cup (0, \infty)$, but E is not closed: notice that $\frac{1}{n} \in E$ for each $n \in \mathbf{N}$ and $\frac{1}{n} \to 0$, so that 0 is a limit point of E (Theorem 3.2.5), but $0 \notin E$.

(d) Let E be the set in question; we claim that E is not open. To see this, let $\epsilon > 0$ be given and note that each element $x \in E$ satisfies $x \geq 1$. It follows that $V_\epsilon(1)$ cannot be contained in E, since it contains infinitely many real numbers $x < 1$.

Now we claim that E is not closed. From Example 2.4.4, we know that $1 + \frac{1}{4} + \frac{1}{9} + \cdots$ converges to some $L \in \mathbf{R}$. Observe that for any $n \in \mathbf{N}$

$$L - \sum_{j=1}^{n} \frac{1}{j^2} = \sum_{j=n+1}^{\infty} \frac{1}{j^2} > \frac{1}{(n+1)^2} > 0,$$

so that $L \neq \sum_{j=1}^{n} \frac{1}{j^2}$ for any $n \in \mathbf{N}$. This implies that L is a limit point of E (Theorem 3.2.5), and also that $L \notin E$; it follows that E is not closed.

(e) Let E be the set in question. As in part (d), we have $V_\epsilon(1) \nsubseteq E$ for any $\epsilon > 0$ and thus E is not open. Unlike part (d), we claim that E is closed. Let $s_n = \sum_{j=1}^{n} \frac{1}{n}$, so that $E = \{s_n : n \in \mathbf{N}\}$, and notice that if E had a limit point then the sequence (s_n) would contain a convergent subsequence (Theorem 3.2.5). However, from Example 2.4.5 we know that (s_n) is strictly increasing and unbounded. Since such sequences have no convergent subsequences (Lemma L.8), we see that E has no limit points and hence is closed.

Exercise 3.2.4. Let A be nonempty and bounded above so that $s = \sup A$ exists.

(a) Show that $s \in \overline{A}$.

(b) Can an open set contain its supremum?

Solution. (a) If $s \in A$ then certainly $s \in \overline{A}$, so suppose that $s \notin A$. For each $n \in \mathbf{N}$ we may use Lemma 1.3.8 to choose some $a_n \in A$ satisfying $s - \frac{1}{n} < a_n < s$ (the last inequality is strict as $s \notin A$). The Squeeze Theorem (Exercise 2.3.3) then implies that $\lim a_n = s$ and thus s is a limit point of A by Theorem 3.2.5, whence $s \in \overline{A}$.

(b) An open set cannot contain its supremum. To see this, suppose that A is open and x belongs to A. There then exists an $\epsilon > 0$ such that $V_\epsilon(x) \subseteq A$, which implies that $x + \frac{\epsilon}{2} \in A$; it follows that x is not the supremum of A.

Exercise 3.2.5. Prove Theorem 3.2.8.

Solution. Theorem 3.2.8 states that a set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

First suppose that every Cauchy sequence contained in F has a limit that is also an element of F and let $x \in \mathbf{R}$ be a limit point of F. By Theorem 3.2.5 there is a sequence (x_n) contained in F such that $\lim x_n = x$; since convergent sequences are also Cauchy sequences (Theorem 2.6.4), our hypothesis then guarantees that $x \in F$. Thus F contains each of its limit points, i.e., F is closed.

Now suppose that there exists a Cauchy sequence (x_n) contained in F satisfying $x := \lim x_n \notin F$. As (x_n) is entirely contained in F and $x \notin F$, it must be the case that $x_n \neq x$ for each $n \in \mathbf{N}$; it follows from Theorem 3.2.5 that x is a limit point of F. Thus F fails to contain one of its limit points, i.e., F is not closed.

Exercise 3.2.6. Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) An open set that contains every rational number must necessarily be all of $\mathbf{R}$.

(b) The Nested Interval Property remains true if the "closed interval" is replaced by "closed set".

(c) Every nonempty open set contains a rational number.

(d) Every bounded infinite closed set contains a rational number.

(e) The Cantor set is closed.

Solution. (a) This is false. Consider the set $\mathbf{R} \setminus \{\sqrt{2}\} = (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$. This contains every rational number since $\sqrt{2}$ is irrational and is an open set since it is the union of two open sets (Theorem 3.2.3 (i)).

(b) This is false. For a counterexample, consider the closed sets $[n, \infty)$ for $n \in \mathbf{N}$. These sets are nested, however the Archimedean Property (Theorem 1.4.2) shows that

$$\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset.$$

(c) This is true. Suppose that A is open and non-empty, so that there exists some $x \in A$ and some $\epsilon > 0$ such that $V_\epsilon(x) \subseteq A$. By the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3), there are infinitely many rational numbers contained in $V_\epsilon(x)$ and hence in A.

(d) This is false. Consider the set

$$E = \left\{\sqrt{2}\right\} \cup \left\{\sqrt{2} + \frac{\sqrt{2}}{n} : n \in \mathbf{N}\right\}.$$

This is a bounded infinite set which contains only irrational numbers (notice that

$$\sqrt{2} + \frac{\sqrt{2}}{n} = \sqrt{2}\left(\frac{n+1}{n}\right) \in \mathbf{Q} \implies \sqrt{2} \in \mathbf{Q},$$

which contradicts Theorem 1.1.1). An argument similar to the one given in Exercise 3.2.2 (a) shows that $\sqrt{2}$ is the only limit point of E and thus E is closed.

(e) This is true. Because each C_n is the union of 2^n closed intervals, Theorem 3.2.14 (i) implies that each C_n is closed. It follows that $C = \bigcap_{n=1}^{\infty} C_n$ is an intersection of closed sets and hence is itself closed (Theorem 3.2.14 (ii)).

Exercise 3.2.7. Given $A \subseteq \mathbf{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof for Theorem 3.2.12.

Solution. (a) Suppose that $x \in \mathbf{R}$ is a limit point of L; we will show that x is a limit point of A also. Let $\epsilon > 0$ be given. Because x is a limit point of L, there exists some $y \in L$ such that $0 < |x - y| < \frac{\epsilon}{2}$, and then since y is a limit point of A, there exists some $a \in A$ such that $|y - a| < |x - y|$. Notice that:

- $|x - a| \leq |x - y| + |y - a| < 2|x - y| < \epsilon$, so that $a \in V_\epsilon(x)$;

- $|x-a| \geq |x-y| - |y-a| > 0$, so that $a \neq x$.

Thus x is a limit point of A, i.e., $x \in L$. We may conclude that L is closed.

(b) Let $\epsilon > 0$ be given. Since x is a limit point of $A \cup L$, the neighbourhood $V_{\epsilon/2}(x)$ contains some $y \in A \cup L$ such that $y \neq x$. If $y \in A$ then $V_\epsilon(x)$ contains a point of A other than x, and if $y \in L$ then the argument given in part (a) shows that $V_\epsilon(x)$ again contains a point of A other than x; it follows that x is a limit point of A.

This shows that $\overline{A} = A \cup L$ contains all of its limit points and hence is closed.

Exercise 3.2.8. Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.

(a) $\overline{A \cup B}$

(b) $A \setminus B = \{x \in A : x \notin B\}$

(c) $(A^{\mathsf{c}} \cup B)^{\mathsf{c}}$

(d) $(A \cap B) \cup (A^{\mathsf{c}} \cap B)$

(e) $\overline{A}^{\mathsf{c}} \cap \overline{A^{\mathsf{c}}}$

Solution. (a) $\overline{A \cup B}$ is definitely closed, by Theorem 3.2.12. It may or may not be open. For example, if $A = B = \mathbf{R}$, then $\overline{A \cup B} = \mathbf{R}$ is open. If $A = (0,1)$ and $B = [0,1]$, then $\overline{A \cup B} = [0,1]$ is not open.

(b) Since $A \setminus B = A \cap B^{\mathsf{c}}$ is the intersection of two open sets, $A \setminus B$ is definitely open (Theorem 3.2.3 (ii)). It may or may not be closed. For example, if $A = (0,1)$ and $B = [0,1]$, then $A \setminus B = \emptyset$ is closed. If $A = (0,1)$ and $B = [2,3]$, then $A \setminus B = (0,1)$ is not closed.

(c) $A^{\mathsf{c}} \cup B$ is the union of two closed sets and hence is closed (Theorem 3.2.14 (i)). The complement $(A^{\mathsf{c}} \cup B)^{\mathsf{c}}$ is then definitely open (Theorem 3.2.13). It may or may not be closed. For example, if $A = B = \mathbf{R}$, then $(A^{\mathsf{c}} \cup B)^{\mathsf{c}} = (\emptyset \cup \mathbf{R})^{\mathsf{c}} = \mathbf{R}^{\mathsf{c}} = \emptyset$ is closed. If $A = (0,1)$ and $B = A^{\mathsf{c}} = (-\infty, 0] \cup [1, \infty)$, then

$$(A^{\mathsf{c}} \cup B)^{\mathsf{c}} = (A^{\mathsf{c}} \cup A^{\mathsf{c}})^{\mathsf{c}} = (A^{\mathsf{c}})^{\mathsf{c}} = A$$

is not closed.

(d) This is simply the set B, which is given as definitely closed. It may or may not be open; $B = \mathbf{R}$ is closed and open, whereas $B = [0,1]$ is closed but not open.

(e) We claim that $\overline{A}^{\mathsf{c}}$ is a subset of $\overline{A^{\mathsf{c}}}$. To see this, let L_A be the set of limit points of A and let $L_{A^{\mathsf{c}}}$ be the set of limit points of A^{c}. Notice that

$$\overline{A}^{\mathsf{c}} = (A \cup L_A)^{\mathsf{c}} = A^{\mathsf{c}} \cap L_A^{\mathsf{c}} \qquad \text{and} \qquad \overline{A^{\mathsf{c}}} = A^{\mathsf{c}} \cup L_{A^{\mathsf{c}}}.$$

Our claim now follows since $\overline{A}^{\mathsf{c}} \subseteq A^{\mathsf{c}} \subseteq \overline{A^{\mathsf{c}}}$. Given this, we have $\overline{A}^{\mathsf{c}} \cap \overline{A^{\mathsf{c}}} = \overline{A}^{\mathsf{c}}$, which is the complement of a closed set and hence is definitely open (Theorem 3.2.13). It may or may not be closed. For example, if $A = \emptyset$ then $\overline{A}^{\mathsf{c}} = \emptyset^{\mathsf{c}} = \mathbf{R}$ is closed. If $A = (-\infty, 0)$, then $\overline{A}^{\mathsf{c}} = (-\infty, 0]^{\mathsf{c}} = (0, \infty)$ is not closed.

Exercise 3.2.9 (De Morgan's Laws). A proof for De Morgan's Laws in the case of two sets is outlined in Exercise 1.2.5. The general argument is similar.

(a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{c}} = \bigcap_{\lambda\in\Lambda} E_\lambda^{\mathsf{c}} \qquad \text{and} \qquad \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{c}} = \bigcup_{\lambda\in\Lambda} E_\lambda^{\mathsf{c}}.$$

(b) Now, provide the details for the proof of Theorem 3.2.14.

Solution. (a) We have

$$\begin{aligned}
x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{c}} &\iff x \notin \bigcup_{\lambda\in\Lambda} E_\lambda \\
&\iff x \notin E_\lambda \text{ for all } \lambda \in \Lambda \\
&\iff x \in E_\lambda^{\mathsf{c}} \text{ for all } \lambda \in \Lambda \\
&\iff x \in \bigcap_{\lambda\in\Lambda} E_\lambda^{\mathsf{c}}.
\end{aligned}$$

The equality $\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{c}} = \bigcap_{\lambda\in\Lambda} E_\lambda^{\mathsf{c}}$ follows. Similarly,

$$x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{c}} \iff x \notin \bigcap_{\lambda\in\Lambda} E_\lambda$$

$$\iff x \notin E_{\lambda'} \text{ for some } \lambda' \in \Lambda$$
$$\iff x \in E_{\lambda'}^{\mathsf{c}} \text{ for some } \lambda' \in \Lambda$$
$$\iff x \in \bigcup_{\lambda \in \Lambda} E_\lambda^{\mathsf{c}}.$$

Thus $\left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{c}} = \bigcup_{\lambda\in\Lambda} E_\lambda^{\mathsf{c}}$.

(b) Suppose we have finitely many closed sets $E_1, \ldots, E_n$ and let $E = E_1 \cup \cdots \cup E_n$. It follows from part (a) that

$$E^{\mathsf{c}} = (E_1 \cup \cdots \cup E_n)^{\mathsf{c}} = E_1^{\mathsf{c}} \cap \cdots \cap E_n^{\mathsf{c}}.$$

Each E_i^{c} is open, so Theorem 3.2.3 (ii) implies that E^{c}, which is a finite intersection of open sets, is also open; it then follows from Theorem 3.2.13 that E is closed.

Now suppose that we have an arbitrary collection $\{E_\lambda : \lambda \in \Lambda\}$ of closed sets and let $E = \bigcap_{\lambda\in\Lambda} E_\lambda$. By part (a), we have

$$E^{\mathsf{c}} = \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^{\mathsf{c}} = \bigcup_{\lambda\in\Lambda} E_\lambda^{\mathsf{c}}.$$

Each E_λ^{c} is open, so Theorem 3.2.3 (i) implies that E^{c}, which is an arbitrary union of open sets, is also open. Thus E is closed (Theorem 3.2.13).

Exercise 3.2.10. Only one of the following three descriptions can be realized. Provide an example that illustrates the viable description, and explain why the other two cannot exist.

(a) A countable set contained in $[0, 1]$ with no limit points.

(b) A countable set contained in $[0, 1]$ with no isolated points.

(c) A set with an uncountable number of isolated points.

Solution. (a) This is impossible. Suppose that $E \subseteq [0, 1]$ is countable, i.e., there is a bijection $f : \mathbf{N} \to E$; for $n \in \mathbf{N}$, let $x_n = f(n)$. The sequence (x_n) is certainly bounded, so the Bolzano-Weierstrass Theorem (Theorem 2.5.5) implies that there is a convergent subsequence $(x_{n_k}) \to x$ for some $x \in [0, 1]$. It then follows from Theorem 3.2.5 that x is a limit point of E. (If $x_{n_k} = x$ for some $k \in \mathbf{N}$, simply remove this term from the sequence; there can be at most one such k as f is injective, so this will not affect the convergence of the subsequence.)

(b) This is possible. Consider the countable set $B = (0,1) \cap \mathbf{Q}$ from Exercise 3.2.2: we showed there that B has no isolated points.

(c) This is impossible. Suppose that E is a subset of $\mathbf{R}$ and let A be the set of isolated points of E. If $x \in A$, then there is an $\epsilon > 0$ such that $V_\epsilon(x) \cap E = \{x\}$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exist rational numbers p, q such that $x - \epsilon < p < x < q < x + \epsilon$. Thus, letting $U_x = (p, q)$, we have $U_x \cap E = \{x\}$. Define $f : A \to B$ by $f(x) = U_x$, where

$$B = \bigcup_{\substack{p,q \in \mathbf{Q}, \\ p<q}} \{(p,q)\};$$

Theorem 1.5.8 (ii) shows that B is a countable set. If we assume that A is uncountable, then the function f cannot possibly be injective. Therefore there must exist $x \neq y$ in A such that $f(x) = f(y)$, i.e., $U_x = U_y$. This implies that

$$\{x\} = U_x \cap E = U_y \cap E = \{y\} \implies x = y,$$

contradicting $x \neq y$. It follows that A cannot be uncountable.

Exercise 3.2.11. (a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Does this result about closures extend to infinite unions of sets?

Solution. (a) First, let us prove the following lemma.

Lemma L.10. If A and B are subsets of $\mathbf{R}$, then $x \in \mathbf{R}$ is a limit point of $A \cup B$ if and only if x is a limit point of A or x is a limit point of B, i.e.,

$$L_{A \cup B} = L_A \cup L_B,$$

where $L_{A\cup B}, L_A$, and L_B are the collections of limit points of $A \cup B, A$, and B.

Proof. Suppose that $x \in \mathbf{R}$ is a limit point of A and let $\epsilon > 0$ be given. Because $x \in L_A$, there exists some $a \in A \subseteq A \cup B$ such that $a \in V_\epsilon(x)$ and $a \neq x$. It follows that there is an element of $A \cup B$ distinct from x and contained in $V_\epsilon(x)$; as ϵ was arbitrary, we see that x is also a limit point of $A \cup B$. A similar argument shows that x is a limit point of $A \cup B$ if x is a limit point of B and thus $L_A \cup L_B \subseteq L_{A \cup B}$.

Now suppose that x is not a limit point of A and not a limit point of B, i.e., there exist positive real numbers ϵ_1 and ϵ_2 such that $V_{\epsilon_1}(x) \cap A \subseteq \{x\}$ and $V_{\epsilon_2}(x) \cap B \subseteq \{x\}$. If we let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, then $V_\epsilon(x) \cap (A \cup B) \subseteq \{x\}$; it follows that x is not a limit point of $A \cup B$. Thus

$$x \notin L_A \text{ and } x \notin L_B \implies x \notin L_{A\cup B}.$$

The contrapositive of this implication gives us the inclusion $L_{A\cup B} \subseteq L_A \cup L_B$. □

Now let us show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. If $x \in \overline{A \cup B}$, then either $x \in A \cup B$ or x is a limit point of $A \cup B$. If $x \in A \cup B$ then certainly $x \in \overline{A} \cup \overline{B}$, and if x is a limit point of $A \cup B$ then by Lemma L.10 x is a limit point of A or a limit point of B; in either case, $x \in \overline{A} \cup \overline{B}$ and thus $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

If $x \in \overline{A} \cup \overline{B}$, then either $x \in \overline{A}$ or $x \in \overline{B}$. If $x \in \overline{A}$, then either $x \in A$ or x is a limit point of A. If $x \in A$, then certainly $x \in \overline{A \cup B}$, and if x is a limit point of A then by Lemma L.10 x is a limit point of $A \cup B$ and hence belongs to $\overline{A \cup B}$. Similarly, if $x \in \overline{B}$ then $x \in \overline{A \cup B}$. Thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$ and we may conclude that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) The result does not extend to the infinite case. For a counterexample, consider the closed sets $A_n := \left[\frac{1}{n}, 1\right]$ for $n \in \mathbf{N}$:

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \overline{(0,1]} = [0,1] \quad \text{but} \quad \bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} A_n = (0,1].$$

Exercise 3.2.12. Let A be an uncountable set and let B be the set of real numbers that divides A into two uncountable sets; that is, $s \in B$ if both $\{x : x \in A \text{ and } x < s\}$ and $\{x : x \in A \text{ and } x > s\}$ are uncountable. Show B is nonempty and open.

Solution. Define the sets

$$B_1 = \{x \in \mathbf{R} : (-\infty, x) \cap A \text{ is uncountable}\}$$
$$\text{and} \quad B_2 = \{x \in \mathbf{R} : (x, \infty) \cap A \text{ is uncountable}\}.$$

We claim that B_1 is non-empty. To see this, suppose that $B_1 = \emptyset$, i.e., for all $x \in \mathbf{R}$ the intersection $(-\infty, x) \cap A$ is either countable or finite, and observe that

$$A = \mathbf{R} \cap A = \left(\bigcup_{n=1}^{\infty} (-\infty, n)\right) \cap A = \bigcup_{n=1}^{\infty} \big((-\infty, n) \cap A\big).$$

So we have expressed A as a countable union of countable or finite sets; it follows from Theorem 1.5.8 that A is countable or finite. Given that A is uncountable, it must be the case that B_1 is non-empty.

Next we claim that B_1 is open. Let $x \in B_1$ be given, so that $(-\infty, x) \cap A$ is uncountable. Note that for any $y \in \mathbf{R}$ with $y > x$ we must have $y \in B_1$ also, since

$$\big((-\infty, x) \cap A\big) \subseteq \big((-\infty, y) \cap A\big).$$

Given this, we would like to find an $\epsilon > 0$ such that $x - \epsilon \in B_1$; it will follow that $(x - \epsilon, \infty) \subseteq B_1$, so that $V_\epsilon(x) \subseteq B_1$. Seeking a contradiction, suppose that for every $\epsilon > 0$ it holds that $x - \epsilon \notin B_1$. In particular we have $x - \frac{1}{n} \notin B_1$ for each $n \in \mathbf{N}$, so that $\big(-\infty, x - \frac{1}{n}\big) \cap A$ is either countable or finite for each $n \in \mathbf{N}$. Notice that

$$(-\infty, x) \cap A = \bigcup_{n=1}^{\infty} \Big(\big(-\infty, x - \tfrac{1}{n}\big) \cap A\Big);$$

it then follows from Theorem 1.5.8 that $(-\infty, x) \cap A$ is countable or finite, which is a contradiction since $x \in B_1$. Thus there must exist an $\epsilon > 0$ such that $x - \epsilon \in B_1$. As noted above we then have $V_\epsilon(x) \subseteq B_1$ and so we may conclude that B_1 is open. Similar arguments show that B_2 is also non-empty and open.

Now let us show that $B_1 \cup B_2 = \mathbf{R}$. If $x \in \mathbf{R}$ is such that $x \notin B_1$ and $x \notin B_2$, i.e., both $(-\infty, x) \cap A$ and $(x, \infty) \cap A$ are either countable or finite, then observe that

$$A = \mathbf{R} \cap A = \big((-\infty, x) \cap A\big) \cup \big(\{x\} \cap A\big) \cup \big((x, \infty) \cap A\big).$$

Thus A is a union of three countable or finite sets and it follows from Theorem 1.5.8 that A is either countable or finite. Since A is given as uncountable, it must be the case that there is no such x; that is, $B_1 \cup B_2 = \mathbf{R}$.

Finally, observe that $B = B_1 \cap B_2$. To see that B is non-empty, suppose otherwise, so that $B_1^{\mathsf{c}} = B_2$. This demonstrates that B_1 is closed as well as open (Theorem 3.2.13). However, since B_1 is non-empty and not equal to $\mathbf{R}$ (since B_2 is non-empty), and these are the only sets which are both closed and open (see Exercise 3.2.13), this is a contradiction; it follows that B is non-empty. Furthermore, B is open since it is the union of two open sets (Theorem 3.2.3 (i)).

Exercise 3.2.13. Prove that the only sets that are both open and closed are $\mathbf{R}$ and the empty set $\emptyset$.

Solution. It will suffice to show that if $E \subseteq \mathbf{R}$ is non-empty, open, and closed, then $E = \mathbf{R}$. Since $E \neq \emptyset$, there exists some $x \in E$. Let

$$S = \{t \in \mathbf{R} : t \geq x \text{ and } [x, t] \subseteq E\}.$$

Note that S is non-empty since $x \in S$. We claim that S is unbounded above. To see this, suppose otherwise, so that $s := \sup S$ exists. If $s \in S$, then $s \in E$. If $s \notin S$, then for any $\epsilon > 0$ there exists some $t \in S$ such that $s - \epsilon < t < s$ (Lemma 1.3.8; the second inequality is strict because $s \notin S$). Since $t \neq s$ and $t \in S$ implies $t \in E$, we see that for any $\epsilon > 0, V_\epsilon(s) \cap E$ contains a point of E other than s; that is, s is a limit point of E. Since E is closed it follows that $s \in E$.

In either case, we have $s \in E$. Since E is open, there exists an $\epsilon > 0$ such that $(s-\epsilon, s+\epsilon) \subseteq E$. This implies that $\left[x, s + \frac{\epsilon}{2}\right] \subseteq E$, so that $s + \frac{\epsilon}{2} \in S$, contradicting the fact that s is the supremum of S. Hence S must be unbounded above and it follows that if $t \geq x$, then $t \in E$. A similar argument with the infimum of the set $\{t \in \mathbf{R} : t \leq x \text{ and } [t, x] \subseteq E\}$ shows that if $t \leq x$, then $t \in E$. Thus $E = \mathbf{R}$.

Exercise 3.2.14. A dual notion to the closure of a set is the interior of a set. The *interior* of E is denoted E^{o} and is defined as

$$E^{\mathrm{o}} = \{x \in E : \text{there exists } V_\epsilon(x) \subseteq E\}.$$

Results about closures and interiors possess a useful symmetry.

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^{\mathrm{o}} = E$.

(b) Show that $\overline{E}^{\mathsf{c}} = (E^{\mathsf{c}})^{\mathrm{o}}$, and similarly that $(E^{\mathrm{o}})^{\mathsf{c}} = \overline{E^{\mathsf{c}}}$.

Solution. (a) Let L be the set of limit points of E and observe that $E \cup L = E$ if and only if $L \subseteq E$. This is exactly the statement that $\overline{E} = E$ if and only if E is closed.

Since $E^{\mathrm{o}} \subseteq E$, it will suffice to show that E is open if and only if $E \subseteq E^{\mathrm{o}}$. This is clear once we note that $E \subseteq E^{\mathrm{o}}$ if and only if, for each $x \in E$, there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subseteq E$.

(b) Let L be the set of limit points of E and observe that

$$\begin{aligned} x \in \overline{E}^{\mathsf{c}} &\iff x \in (E \cup L)^{\mathsf{c}} \\ &\iff x \in E^{\mathsf{c}} \cap L^{\mathsf{c}} \end{aligned}$$

$$
\begin{aligned}
&\iff x \notin E \text{ and } x \text{ is not a limit point of } E \\
&\iff \text{there exists an } \epsilon > 0 \text{ such that } V_\epsilon(x) \cap E = \emptyset \\
&\iff \text{there exists an } \epsilon > 0 \text{ such that } V_\epsilon(x) \subseteq E^{\mathsf{c}} \\
&\iff x \in (E^{\mathsf{c}})^{\circ}.
\end{aligned}
$$

Thus $\overline{E}^{\mathsf{c}} = (E^{\mathsf{c}})^{\circ}$. Similarly,

$$
\begin{aligned}
x \in (E^{\circ})^{\mathsf{c}} &\iff x \notin E^{\circ} \\
&\iff \text{for all } \epsilon > 0, V_\epsilon(x) \not\subseteq E \\
&\iff \text{for all } \epsilon > 0, V_\epsilon(x) \cap E^{\mathsf{c}} \neq \emptyset \\
&\iff (\text{for all } \epsilon > 0)(x \in E^{\mathsf{c}} \text{ or there exists } y \in V_\epsilon(x) \cap E^{\mathsf{c}} \text{ with } y \neq x) \\
&\iff x \in E^{\mathsf{c}} \text{ or for all } \epsilon > 0 \text{ there exists } y \in V_\epsilon(x) \cap E^{\mathsf{c}} \text{ with } y \neq x \\
&\iff x \in E^{\mathsf{c}} \text{ or } x \text{ is a limit point of } E^{\mathsf{c}} \\
&\iff x \in \overline{E^{\mathsf{c}}}.
\end{aligned}
$$

Thus $(E^{\circ})^{\mathsf{c}} = \overline{E^{\mathsf{c}}}$.

Exercise 3.2.15. A set A is called an F_σ set if it can be written as the countable union of closed sets. A set B is called a G_δ set if it can be written as the countable intersection of open sets.

(a) Show that a closed interval $[a, b]$ is a G_δ set.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.

(c) Show that $\mathbf{Q}$ is an F_σ set, and the set of irrationals $\mathbf{I}$ forms a G_δ set. (We will see in Section 3.5 that $\mathbf{Q}$ is *not* a G_δ set, nor is $\mathbf{I}$ an F_σ set.)

Solution. (a) Observe that

$$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right).$$

(b) For any $n \in \mathbf{N}$ the set $\left(a - \frac{1}{n}, b + \frac{1}{n}\right) \setminus \{a\} = \left(a - \frac{1}{n}, b + \frac{1}{n}\right) \cap \{a\}^{\mathsf{c}}$ is the intersection of two open sets and hence is open. Observe that

$$(a, b] = [a, b] \setminus \{a\} = \left(\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)\right) \setminus \{a\} = \bigcap_{n=1}^{\infty} \left(\left(a - \frac{1}{n}, b + \frac{1}{n}\right) \setminus \{a\}\right).$$

Thus $(a, b]$ is a G_δ set. Next, observe that for any $n \in \mathbf{N}$ the set $\left[a + \frac{1}{n}, b - \frac{1}{n}\right] \cup \{b\}$ is the union of two closed sets and hence is closed. Notice that

$$(a, b] = \bigcup_{n=1}^{\infty} \left(\left[a + \frac{1}{n}, b - \frac{1}{n}\right] \cup \{b\} \right).$$

Thus $(a, b]$ is an F_σ set.

(c) Observe that

$$\mathbf{Q} = \bigcup_{r \in \mathbf{Q}} \{r\}.$$

Since $\mathbf{Q}$ is countable, this demonstrates that $\mathbf{Q}$ is an F_σ set. De Morgan's Laws (Exercise 3.2.9) imply that the complement of an F_σ set is a G_δ set (and vice versa), so we have also shown that $\mathbf{I}$ is a G_δ set.

3.3 Compact Sets

Exercise 3.3.1. Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

Solution. K is given as non-empty and must be bounded by Theorem 3.3.8, so the Axiom of Completeness guarantees that $\sup K$ and $\inf K$ both exist. Suppose that $\sup K \notin K$. By Lemma 1.3.8, for each $n \in \mathbf{N}$ there exists an $x_n \in K$ satisfying

$$\sup K - \frac{1}{n} < x_n < \sup K;$$

the last inequality is strict as $\sup K \notin K$. The Squeeze Theorem (Exercise 2.3.3) and Theorem 3.2.5 now imply that $\sup K$ is a limit point of K; since $\sup K \notin K$, it follows that K is not closed, which contradicts Theorem 3.3.8. Thus $\sup K \in K$, and a similar argument with the infimum and Exercise 1.3.1 (b) shows that $\inf K \in K$.

Exercise 3.3.2. Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

(a) $\mathbf{N}$.

(b) $\mathbf{Q} \cap [0,1]$.

(c) The Cantor set.

(d) $\{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in \mathbf{N}\}$.

(e) $\{1, 1/2, 2/3, 3/4, 4/5, \ldots\}$.

Solution. (a) $\mathbf{N}$ is not compact. Consider the sequence $(1, 2, 3, \ldots)$, which is increasing and unbounded. As shown in Lemma L.8, such sequences do not have convergent subsequences.

(b) $\mathbf{Q} \cap [0,1]$ is not compact. Let $x = \frac{\sqrt{2}}{2} \in (0,1)$. By Theorem 3.2.10, there is a sequence of rational numbers (x_n) converging to x. Since $0 < x < 1$, this sequence must eventually be contained in $(0,1)$. By removing a finite number of terms from the sequence if necessary, which will not affect convergence, we may assume that the sequence is entirely contained in $\mathbf{Q} \cap [0,1]$. It follows from Theorem 2.5.2 that every subsequence of (x_n) also converges to x, which does not belong to $\mathbf{Q} \cap [0,1]$.

(c) The Cantor set C is compact by Theorem 3.3.8: C is closed by Exercise 3.2.6 (e) and bounded as $C \subseteq [0,1]$.

(d) Let E be the set in question and let $s_n = \sum_{j=1}^{n} \frac{1}{j^2}$. Certainly (s_n) is contained in E and from Example 2.4.4 we know that $\lim s_n = L$ for some $L \in \mathbf{R}$. As shown in Exercise 3.2.3 (d), L does not belong to E, and since all subsequences of s_n also converge to L (Theorem 2.5.2), we see that E is not compact.

(e) Let E be the set in question, i.e.,

$$E = \{1\} \cup \left\{1 - \tfrac{1}{n} : n \in \mathbf{N}\right\}.$$

Using a similar argument to the one given in Exercise 3.2.2, we see that 1 is the only limit point of E. It follows that E is closed and bounded, and hence compact (Theorem 3.3.8).

Exercise 3.3.3. Prove the converse of Theorem 3.3.4 by showing that if a set $K \subseteq \mathbf{R}$ is closed and bounded, then it is compact.

Solution. Suppose that $K \subseteq \mathbf{R}$ is closed and bounded. If (x_n) is an arbitrary sequence contained in K, then (x_n) must be bounded and so the Bolzano-Weierstrass Theorem (Theorem 2.5.5) implies that there exists a subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in \mathbf{R}$; our aim is to show that $x \in K$.

If $x \notin K$ then $x_{n_k} \neq x$ for each $k \in \mathbf{N}$. It follows from Theorem 3.2.5 that x is a limit point of K which does not belong to K, so that K is not closed. Thus

$$x \notin K \implies K \text{ not closed.}$$

The contrapositive of this implication and the hypothesis that K is closed gives us $x \in K$, as desired.

Exercise 3.3.4. Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$

(b) $\overline{F^{\mathsf{c}} \cup K^{\mathsf{c}}}$

(c) $K \setminus F = \{x \in K : x \notin F\}$

(d) $\overline{K \cap F^{\mathsf{c}}}$

Solution. Throughout this exercise, we will repeatedly use that a subset of $\mathbf{R}$ is compact if and only if it is closed and bounded (Theorem 3.3.8).

(a) K is closed since it is compact, so $K \cap F$ is the intersection of two closed sets and hence is definitely closed (Theorem 3.2.14 (ii)). Certainly the intersection of a bounded set with any other set is again bounded, so since K is bounded by virtue of being compact, we see that $K \cap F$ is bounded as well as closed. It follows that $K \cap F$ is definitely compact .

(b) The closure of any set is closed (Theorem 3.2.12), so $\overline{F^{\mathsf{c}} \cup K^{\mathsf{c}}}$ is definitely closed. However, $\overline{F^{\mathsf{c}} \cup K^{\mathsf{c}}}$ cannot be compact since it is unbounded. To see this, let us first show that if $E \subseteq \mathbf{R}$ is bounded, then E^{c} is unbounded. Since E is bounded, there exists an $M > 0$ such that $E \subseteq [-M, M]$. It follows that $((-\infty, M) \cup (M, \infty)) \subseteq E^{\mathsf{c}}$, whence E^{c} is unbounded. Returning to the set $\overline{F^{\mathsf{c}} \cup K^{\mathsf{c}}}$, we have that K is bounded since it is compact and thus K^{c} is unbounded. Since

$$K^{\mathsf{c}} \subseteq F^{\mathsf{c}} \cup K^{\mathsf{c}} \subseteq \overline{F^{\mathsf{c}} \cup K^{\mathsf{c}}},$$

we see that $\overline{F^{\mathsf{c}} \cup K^{\mathsf{c}}}$ must also be unbounded. It follows that $\overline{F^{\mathsf{c}} \cup K^{\mathsf{c}}}$ cannot be compact.

(c) Since K is bounded, $K \setminus F$ must also be bounded, and hence $K \setminus F$ is compact if and only if it is closed. $K \setminus F$ could be compact/closed: for example, taking $F = \emptyset$. $K \setminus F$ could also fail to be compact/closed. For example, if we take $K = [-2, 2]$ and $F = [-1, 1]$, then $K \setminus F = [-2, 1) \cup (1, 2]$, which is not compact/closed.

(d) First, notice that if $E \subseteq \mathbf{R}$ is bounded by some $M > 0$, i.e., $E \subseteq [-M, M]$, then since $[-M, M]$ is closed it follows from Theorem 3.2.12 that $\overline{E} \subseteq [-M, M]$ also, i.e., $\overline{E}$ is also bounded by M.

Returning to the question, note that since K is bounded, $K \cap F^{\mathsf{c}}$ must also be bounded. By the previous paragraph, it follows that $\overline{K \cap F^{\mathsf{c}}}$ is bounded. Thus $\overline{K \cap F^{\mathsf{c}}}$ is compact since it is closed and bounded.

Exercise 3.3.5. Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) The arbitrary intersection of compact sets is compact.

(b) The arbitrary union of compact sets is compact.

(c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Solution. Throughout this exercise, we will repeatedly use that a subset of $\mathbf{R}$ is compact if and only if it is closed and bounded (Theorem 3.3.8).

(a) This is true. Suppose we have some collection $\{K_a : a \in A\}$ of compact sets. Each K_a must be closed and bounded and so the intersection $K := \bigcap_{a \in A} K_a$ is also closed (Theorem 3.2.14 (ii)) and bounded. Thus K is compact.

(b) This is false. For each $n \in \mathbf{N}$, let $K_n = [0, n]$; each K_n is closed and bounded and hence compact. However, $\bigcup_{n=1}^{\infty} K_n = [0, \infty)$, which is unbounded and hence not compact.

(c) This is false. If we let $A = (0, 1)$ and $K = [0, 1]$, then K is compact since it is closed and bounded but $A \cap K = (0, 1)$, which is not closed and hence not compact.

(d) This is false; see Exercise 3.2.6 (b) for a counterexample.

Exercise 3.3.6. This exercise is meant to illustrate the point made in the opening paragraph to Section 3.3. Verify that the following three statements are true if every blank is filled in with the word "finite." Which are true if every blank is filled in with the word "compact"? Which are true if every blank is filled in with the word "closed"?

(a) Every _____ set has a maximum.

(b) If A and B are _____, then $A+B=\{a+b : a \in A, b \in B\}$ is also _____.

(c) If $\{A_n : n \in \mathbf{N}\}$ is a collection of _____ sets with the property that every finite subcollection has a nonempty intersection, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty as well.

Solution. (a) Every non-empty finite set has a maximum by Lemma L.3, and every non-empty compact set has a maximum by Exercise 3.3.1. However, not every closed set has a maximum: $\mathbf{R}$ is closed but has no maximum element.

(b) If A is finite with m elements and B is finite with n elements, then $A+B$ can have at most mn elements since the map $A \times B \to A+B$; $(a,b) \mapsto a+b$ is a surjection. Thus $A+B$ is also finite.

If A and B are compact, then so is $A+B$. To see this, let (x_n) be a sequence contained in $A+B$, so that there are sequences (a_n) contained in A and (b_n) contained in B such that $x_n = a_n + b_n$ for each $n \in \mathbf{N}$. Since A is compact, the sequence (a_n) has a subsequence (a_{n_k}) such that $\lim_{k\to\infty} a_{n_k} = a$ for some $a \in A$. Since B is compact, the sequence (b_{n_k}) has a subsequence $\left(b_{n_{k_\ell}}\right)$ such that $\lim_{\ell\to\infty} b_{n_{k_\ell}} = b$ for some $b \in B$. Observe that

$$\lim_{\ell\to\infty} x_{n_{k_\ell}} = \lim_{\ell\to\infty}\left(a_{n_{k_\ell}} + b_{n_{k_\ell}}\right) = \lim_{\ell\to\infty} a_{n_{k_\ell}} + \lim_{\ell\to\infty} b_{n_{k_\ell}} = a+b \in A+B.$$

It follows that $A+B$ is compact.

It is not necessarily the case that $A+B$ is closed for closed sets A and B. For a counterexample, let $A = \mathbf{N}$ and let $B = \left\{-n + \frac{1}{n} : n \in \mathbf{N}\right\}$. For each $n \in \mathbf{N}$, we have $n + \left(-n + \frac{1}{n}\right) = \frac{1}{n} \in A+B$; it follows from Theorem 3.2.5 that 0 is a limit point of $A+B$.

Notice that, for $n, k \in \mathbf{Z}$,

$$n - k + \frac{1}{k} = 0 \iff k = 1 \text{ and } n = 0.$$

Since any element of $A+B$ is of the form $n - k + \frac{1}{k}$ for some positive integers n, k, we see that the limit point 0 fails to belong to $A+B$. Thus $A+B$ is not closed.

(c) Suppose $\{A_n : n \in \mathbf{N}\}$ is a collection of finite sets with the property that every finite subcollection has a non-empty intersection. For each $k \in \mathbf{N}$, let m_k be the number of elements in $\bigcap_{n=1}^{k} A_n$. By assumption, $m_k \geq 1$ for all $k \in \mathbf{N}$. It is also clear that

$$m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_k \geq \cdots.$$

By Lemma L.1, this sequence must eventually be constant, i.e., there exists a positive integer K such that

$$k \geq K \implies m_k = m_K \iff \bigcap_{n=1}^{k} A_n = \bigcap_{n=1}^{K} A_n.$$

Since $m_K \geq 1$ there exists some $x \in \bigcap_{n=1}^{K} A_n$, and since $x \in \bigcap_{n=1}^{k} A_n$ for all $k \geq K$ we then have $x \in \bigcap_{n=1}^{\infty} A_n$.

Suppose $\{A_n : n \in \mathbf{N}\}$ is a collection of compact sets with the property that every finite subcollection has a non-empty intersection. For $m \in \mathbf{N}$ define $K_m = \bigcap_{n=1}^{m} A_n$ and observe that each K_m is non-empty by assumption, each K_m is compact by Exercise 3.3.5 (a), and the sequence (K_m) satisfies

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots \supseteq K_m \supseteq \cdots .$$

It then follows from Theorem 3.3.5 that the intersection $\bigcap_{m=1}^{\infty} K_m = \bigcap_{n=1}^{\infty} A_n$ is non-empty.

For each $n \in \mathbf{N}$, let A_n be the closed set $[n, \infty)$. For a finite subcollection $\{A_{n_1}, \ldots, A_{n_m}\}$, we have

$$\bigcap_{i=1}^{m} A_{n_i} = [N, \infty) \neq \emptyset,$$

where $N = \max_{1 \leq i \leq m} n_i$. However, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Exercise 3.3.7. As some more evidence of the surprising nature of the Cantor set, follow these steps to show that the sum $C + C = \{x + y : x, y \in C\}$ is equal to the closed interval $[0, 2]$. (Keep in mind that C has zero length and contains no intervals.)

Because $C \subseteq [0, 1], C + C \subseteq [0, 2]$, so we only need to prove the reverse inclusion $[0, 2] \subseteq \{x + y : x, y \in C\}$. Thus, given $s \in [0, 2]$, we must find two elements $x, y \in C$ satisfying $x + y = s$.

(a) Show that there exist $x_1, y_1 \in C_1$ for which $x_1 + y_1 = s$. Show in general that, for an arbitrary $n \in \mathbf{N}$, we can always find $x_n, y_n \in C_n$ for which $x_n + y_n = s$.

(b) Keeping in mind that the sequences (x_n) and (y_n) do not necessarily converge, show how they can nevertheless be used to produce the desired x and y in C satisfying $x + y = s$.

Solution. (a) If $\frac{s}{2} \in C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ then take $x_1 = y_1 = \frac{s}{2}$, and if $\frac{s}{2} \in \left(\frac{1}{3}, \frac{2}{3}\right)$ then take $x_1 = \frac{s}{2} - \frac{1}{3}$ and $y_1 = \frac{s}{2} + \frac{1}{3}$. In either case, we have $x_1, y_1 \in C_1$ and $x_1 + y_1 = s$. Geometrically, we have shown that for any $s \in [0, 2]$, the line given by $x + y = s$ must intersect the set $C_1 \times C_1 \subseteq C_0 \times C_0 = [0, 1]^2$; see Figure F.27, which shows the set $C_1 \times C_1$ as well as the lines $x + y = s$ for several values of s.

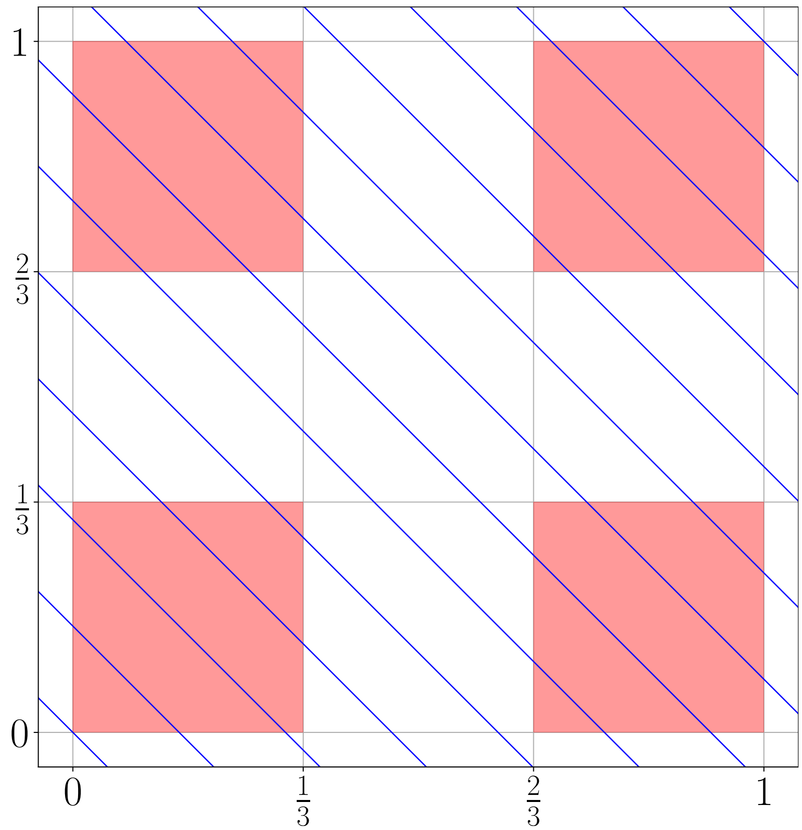


Figure F.27: $C_1 \times C_1$ and $x + y = s$ for various values of $s \in [0, 2]$

Let us argue inductively that for any $n \in \mathbf{N}$ we can find $x_n, y_n \in C_n$ such that $x_n + y_n = s$. The base case $n = 1$ was handled above, so suppose that for some $n \in \mathbf{N}$ we have $x_n, y_n \in C_n$ such that $x_n + y_n = s$. Since C_n consists of 2^n closed intervals each of length 3^{-n}, the set $C_n \times C_n$ consists of $(2^n)^2$ closed squares each with side length 3^{-n}. Geometrically, the induction hypothesis guarantees that the line $x + y = s$ intersects the set $C_n \times C_n$ and thus must intersect one of the $(2^n)^2$ closed squares. Moving from C_n to C_{n+1}, the middle third of each of the 2^n intervals is removed. This has the effect of splitting each of the $(2^n)^2$ squares of $C_n \times C_n$ into four subsquares; see Figure F.28. $C_{n+1} \times C_{n+1}$ then consists of the collection of these subsquares.

Now we make the observation that this situation is essentially the same as in the base case: given that the line $x+y=s$ intersects one of the squares of $C_n \times C_n$, it must intersect at least one of the four subsquares after we remove the middle third of the sides of the square; see Figure F.28 again. We are then guaranteed the existence of some $x_{n+1}, y_{n+1} \in C_{n+1}$ such that $x_{n+1} + y_{n+1} = s$. This completes the induction step.

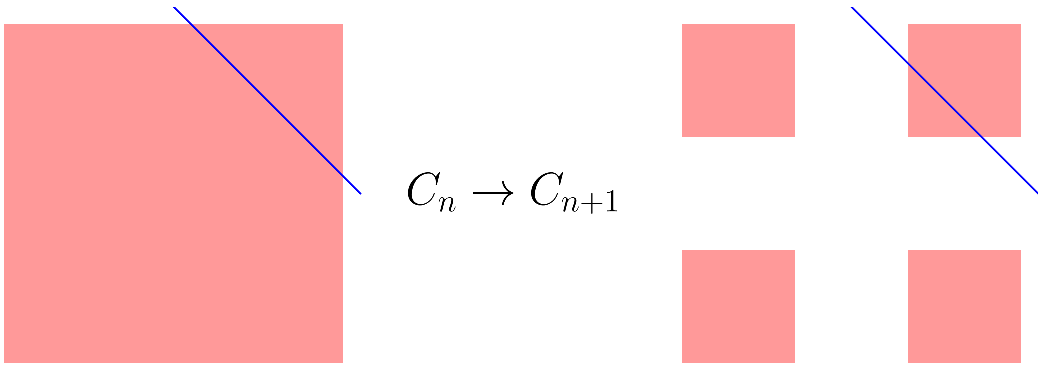


Figure F.28: Subsquares of $C_n \times C_n$ and $C_{n+1} \times C_{n+1}$ intersecting the line $x + y = s$

(b) The sequence (x_n) is certainly bounded, so by the Bolzano-Weierstrass Theorem (Theorem 2.5.5) it has a convergent subsequence $(x_{n_k}) \to x$ for some $x \in \mathbf{R}$. Similarly, the sequence (y_{n_k}) is bounded and hence has a convergent subsequence $\left(y_{n_{k_\ell}}\right) \to y$ for some $y \in \mathbf{R}$. Since the sequence (C_n) is nested, we have $x_{n_{k_\ell}} \in C_1$ for all $\ell \in \mathbf{N}$; it follows that $x \in C_1$ since C_1 is closed. The terms $x_{n_{k_\ell}}$ belong to C_2 provided $n_{k_\ell} \geq 2$, i.e., all but a finite number of terms of $\left(x_{n_{k_\ell}}\right)$ belong to C_2. Since C_2 is closed, it must then be the case that $x \in C_2$. Continuing in this fashion, we see that $x \in C_n$ for all $n \in \mathbf{N}$, i.e., $x \in C$. Similarly, we obtain $y \in C$. Now observe that on one hand,

$$\lim_{\ell \to \infty} \left(x_{n_{k_\ell}} + y_{n_{k_\ell}}\right) = x + y.$$

On the other hand,

$$\lim_{\ell \to \infty} \left(x_{n_{k_\ell}} + y_{n_{k_\ell}}\right) = \lim_{\ell \to \infty} s = s.$$

Since limits are unique (Theorem 2.2.7), we may conclude that $x + y = s$.

Exercise 3.3.8. Let K and L be nonempty compact sets, and define

$$d = \inf\{|x-y| : x \in K \text{ and } y \in L\}.$$

This turns out to be a reasonable definition for the *distance* between K and L.

(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

(b) Show that it's possible to have $d = 0$ if we assume only that the disjoint sets K and L are closed.

Solution. (a) Let $E = \{|x-y| : x \in K \text{ and } y \in L\}$ and notice that E is non-empty (since K and L are non-empty) and bounded below by 0; it follows that $d = \inf E$ exists. By Exercise 1.3.1 (b), for each $n \in \mathbf{N}$ there exist elements $x_n \in K$ and $y_n \in L$ such that

$$d \leq |x_n - y_n| < d + \frac{1}{n}. \tag{1}$$

Since (x_n) is entirely contained in the compact set K, we are guaranteed the existence of a convergent subsequence $(x_{n_k}) \to x_0$ for some $x_0 \in K$. Similarly, since the sequence (y_{n_k}) is entirely contained in the compact set L, we are guaranteed the existence of a convergent subsequence $\left(y_{n_{k_\ell}}\right) \to y_0$ for some $y_0 \in L$. We then have, by Theorem 2.5.2,

$$\lim_{\ell\to\infty}\left|x_{n_{k_\ell}} - y_{n_{k_\ell}}\right| = |x_0 - y_0|.$$

However, inequality (1) and the Squeeze Theorem (Exercise 2.3.3) imply that

$$\lim_{\ell\to\infty}\left|x_{n_{k_\ell}} - y_{n_{k_\ell}}\right| = d.$$

It follows from the uniqueness of limits (Theorem 2.2.7) that $|x_0 - y_0| = d$. Since K and L are disjoint, it must be the case that $x_0 \neq y_0$ and hence $d > 0$.

(b) Let $K = \mathbf{N}$ and $L = \left\{n + \frac{1}{n} : n \geq 2\right\}$ and observe that K and L are non-empty and disjoint. Furthermore, since

$$K^{\mathsf{c}} = (-\infty, 1) \cup \bigcup_{n=1}^{\infty}(n, n+1) \qquad \text{and} \qquad L^{\mathsf{c}} = \left(-\infty, \frac{5}{2}\right) \cup \bigcup_{n=2}^{\infty}\left(n + \frac{1}{n}, n+1+\frac{1}{n+1}\right),$$

we see that K^{c} and L^{c} are both open (Theorem 3.2.3 (i)) and hence that K and L are both closed (Theorem 3.2.13). Letting $E = \{|x-y| : x \in K \text{ and } y \in L\}$ again, note that for each $n \geq 2$, by taking $n \in K$ and $n + \frac{1}{n} \in L$, we have $\frac{1}{n} \in E$. It follows that $d = \inf E = 0$.

Exercise 3.3.9. Follow these steps to prove the final implication in Theorem 3.3.8.

Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim|I_n| = 0$.

(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

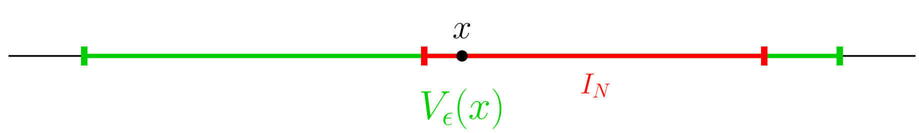
(c) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

Solution. (a) Let us proceed by induction. For the base case, $I_0 \cap K = K$ cannot be covered by any finite subcollection of $\{O_\lambda : \lambda \in \Lambda\}$ and we have $|I_0| = 2^0|I_0|$.

Suppose that after n steps we have chosen nested closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_{n-1}$ such that, for each $0 \leq m \leq n-1$, $I_m \cap K$ cannot be covered by any finite subcollection of $\{O_\lambda : \lambda \in \Lambda\}$ and $|I_m| = 2^{-m}|I_0|$. Suppose that $I_{n-1} = [a, c]$ and let $b = \frac{a+c}{2}$. Note that if both of the sets $[a, b] \cap K$ and $[b, c] \cap K$ could be covered by a finite subcollection of $\{O_\lambda : \lambda \in \Lambda\}$, then $I_{n-1} \cap K$ could also be finitely covered. By assumption this is not the case, so at least one of the intervals $[a, b]$ or $[b, c]$ must have the property that its intersection with K cannot be finitely covered. Let I_n be this interval and note that $I_n \subseteq I_{n-1}$. Furthermore, since $|I_{n-1}| = 2^{-n+1}|I_0|$, we have $|I_n| = 2^{-n}|I_0|$. This completes the induction step and we obtain the desired sequence of nested closed intervals.

(b) For each $n \in \mathbf{N}$, $I_n \cap K$ is the intersection of two compact sets and hence is itself compact (Exercise 3.3.5 (a)). Furthermore, since the sequence (I_n) is nested, the sequence $(I_n \cap K)$ is also nested. It follows from Theorem 3.3.5 that there exists some $x \in \bigcap_{n=1}^{\infty}(I_n \cap K) = K \cap \bigcap_{n=1}^{\infty} I_n$.

(c) Because x belongs to the open set O_{λ_0}, there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subseteq O_{\lambda_0}$, and since $\lim|I_n| = 0$ there exists an $N \in \mathbf{N}$ such that $|I_N| < \frac{\epsilon}{2}$. Thus, since $x \in I_N$, we must have $I_N \subseteq V_\epsilon(x)$ and hence $(I_N \cap K) \subseteq V_\epsilon(x)$. This implies that $I_N \cap K \subseteq O_{\lambda_0}$, contradicting the fact that $I_N \cap K$ cannot be covered by any finite subcollection of $\{O_\lambda : \lambda \in \Lambda\}$.



Exercise 3.3.10. Here is an alternate proof to the one given in Exercise 3.3.9 for the final implication in the Heine-Borel Theorem.

Consider the special case where K is a closed interval. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define S to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$.

(a) Argue that S is nonempty and bounded, and thus $s = \sup S$ exists.

(b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.

(c) Finally, prove the theorem for an arbitrary closed and bounded set K.

Solution. (a) Since $a \in [a, b]$ there must be some O_{λ_0} such that $a \in O_{\lambda_0}$, so that $[a, a]$ is finitely covered; it follows that $a \in S$. Evidently, S is bounded above by b. Thus $s = \sup S$ exists.

(b) Seeking a contradiction, suppose that $s < b$, so that $\epsilon_1 := \frac{b-s}{2} > 0$. Since $s \in [a, b]$, there exists some O_{λ_0} such that $s \in O_{\lambda_0}$ and thus there is an $\epsilon_2 > 0$ such that $V_{\epsilon_2}(s) \subseteq O_{\lambda_0}$. Let $\epsilon := \min\{\epsilon_1, \epsilon_2\} > 0$. By Lemma 1.3.8 there exists an $x \in S$ such that $s - \epsilon < x \le s$, so that $x \in V_\epsilon(s)$ and

$$[a, x] \subseteq O_{\lambda_1} \cup \cdots \cup O_{\lambda_n}$$

for some finite subcollection $\{O_{\lambda_1}, \ldots, O_{\lambda_n}\}$. Observe that $s + \frac{\epsilon}{2} \le s + \frac{\epsilon_1}{2} = \frac{s+b}{2} \in [a, b]$ and

$$\left[a, s + \tfrac{\epsilon}{2}\right] \subseteq V_\epsilon(s) \cup [a, x] \subseteq V_{\epsilon_2}(s) \cup [a, x] \subseteq O_{\lambda_0} \cup O_{\lambda_1} \cup \cdots \cup O_{\lambda_n}.$$

It follows that $s + \frac{\epsilon}{2} \in S$, contradicting the fact that s is the supremum of S. Hence it must be the case that $s = b$.

This implies that $[a, b]$ has a finite subcover: since $b \in [a, b]$ there must be some O_{λ_0} such that $b \in O_{\lambda_0}$ and hence some $\epsilon > 0$ such that $V_\epsilon(b) \subseteq O_{\lambda_0}$, and since $\sup S = b$ there is some $x \in S$ such that $b - \epsilon < x \le b$ and

$$[a, x] \subseteq O_{\lambda_1} \cup \cdots \cup O_{\lambda_n}$$

for some finite subcollection $\{O_{\lambda_1}, \ldots, O_{\lambda_n}\}$. It follows that

$$[a, b] \subseteq V_\epsilon(b) \cup [a, x] \subseteq O_{\lambda_0} \cup O_{\lambda_1} \cup \cdots \cup O_{\lambda_n}.$$

(c) Let $\{O_\lambda : \lambda \in \Lambda\}$ be an arbitrary open cover of K. Since K is bounded, it is contained in some closed interval $[a, b]$. Note that since K is closed, the collection $\{K^{\mathsf{c}}\} \cup \{O_\lambda : \lambda \in \Lambda\}$

is an open cover of $\mathbf{R}$ and hence of $[a,b]$; by part (b), there then exists a finite subcover of $[a,b]$. Since K is contained in $[a,b]$, this finite subcover must also cover K, and since K^{c} evidently does not cover K, this finite subcover must contain some sets $O_{\lambda_1},\dots,O_{\lambda_n}$. It follows that K is covered by the finite collection $O_{\lambda_1},\dots,O_{\lambda_n}$.

Exercise 3.3.11. Consider each of the sets listed in Exercise 3.3.2. For each one that is not compact, find an open cover for which there is no finite subcover.

Solution. The sets from Exercise 3.3.2 which are not compact are $\mathbf{N}$, $\mathbf{Q}\cap[0,1]$, and

$$E=\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{k^2}:k\in\mathbf{N}\right\}.$$

Let us consider $\mathbf{N}$ first. For each $n\in\mathbf{N}$, let $O_n=(n-1,n+1)$, so that the collection $\{O_n:n\in\mathbf{N}\}$ covers $\mathbf{N}$. Since each $n\in\mathbf{N}$ belongs to exactly the set O_n and no others, there are in fact no proper subcovers, finite or otherwise.

Next, let us consider $\mathbf{Q}\cap[0,1]$. Let y be the irrational number $\frac{\sqrt{2}}{2}\in(0,1)$. For each $n\in\mathbf{N}$, define

$$O_n=\left(-\infty,y-\frac{1}{n}\right)\cup\left(y+\frac{1}{n},\infty\right)$$

and notice that $\bigcup_{n=1}^{\infty}O_n=\mathbf{R}\setminus\{y\}$; it follows that the collection $\{O_n:n\in\mathbf{N}\}$ covers $\mathbf{Q}\cap[0,1]$ since y is irrational. We claim that there can be no finite subcover. If $\{O_{n_1},\dots,O_{n_m}\}$ is some finite subcollection, then let $N=\max\{n_1,\dots,n_m\}$ and observe that

$$\bigcup_{i=1}^{m}O_{n_i}=\left(-\infty,y-\frac{1}{N}\right)\cup\left(y+\frac{1}{N},\infty\right).$$

Since

$$\left[y-\frac{1}{N},y+\frac{1}{N}\right]\cap[0,1]=\left[\max\left\{0,y-\tfrac{1}{N}\right\},\min\left\{1,y+\tfrac{1}{N}\right\}\right],$$

which is a proper interval, we are guaranteed by the density of $\mathbf{Q}$ in $\mathbf{R}$ (Theorem 1.4.3) the existence of a rational number $p\in\left[y-\frac{1}{N},y+\frac{1}{N}\right]\cap[0,1]$. It follows that $\mathbf{Q}\cap[0,1]\not\subseteq\bigcup_{i=1}^{m}O_{n_i}$.

Now let us consider the set $E=\{s_k:k\in\mathbf{N}\}$, where $s_k=\sum_{j=1}^{k}\frac{1}{j^2}$. We know by the Monotone Convergence Theorem (Theorem 2.4.2) that $L:=\lim s_k$ is the supremum of E. Furthermore, as noted in Exercise 3.2.2, L does not belong to E. For each $n\in\mathbf{N}$, let $O_n=\left(-\infty,L-\frac{1}{n}\right)$ and note that

$$\bigcup_{n=1}^{\infty}O_n=(-\infty,L),$$

which must cover E since L is the supremum of E but does not belong to E. We claim that there cannot exist a finite subcover. If $\{O_{n_1}, \ldots, O_{n_m}\}$ is some finite subcollection, then let $N = \max\{n_1, \ldots, n_m\}$ and observe that

$$\bigcup_{i=1}^{m} O_{n_i} = \left(-\infty, L - \tfrac{1}{N}\right).$$

Since $\lim s_k = L$, the sequence (s_k) must eventually be contained in the interval $\left(L - \frac{1}{N}, L + \frac{1}{N}\right)$ and it follows that $\{O_{n_1}, \ldots, O_{n_m}\}$ cannot cover E.

Exercise 3.3.12. Using the concept of open covers (and explicitly avoiding the Bolzano-Weierstrass Theorem), prove that every bounded infinite set has a limit point.

Solution. We will prove the contrapositive statement. That is, if $E \subseteq \mathbf{R}$ is bounded, then

$$E \text{ has no limit points} \implies E \text{ is finite.}$$

If E is empty, we are done. Otherwise, each $x \in E$ must be an isolated point, i.e., there exists some $\epsilon_x > 0$ such that $V_{\epsilon_x}(x) \cap E = \{x\}$. Notice that the collection $\{V_{\epsilon_x}(x) : x \in E\}$ is an open cover of E. Since E has no limit points, E must be closed; the Heine-Borel Theorem (Theorem 3.3.8) then implies that there exist finitely many points $\{x_1, \ldots, x_n\}$ such that

$$E \subseteq V_{\epsilon_{x_1}}(x_1) \cup \cdots \cup V_{\epsilon_{x_n}}(x_n).$$

This implies that

$$\begin{aligned} E = E \cap (V_{\epsilon_{x_1}}(x_1) \cup \cdots \cup V_{\epsilon_{x_n}}(x_n)) = (V_{\epsilon_{x_1}}(x_1) \cap E) \cup \cdots \cup (V_{\epsilon_{x_n}}(x_n) \cap E) \\ = \{x_1\} \cup \cdots \cup \{x_n\} = \{x_1, \ldots, x_n\}. \end{aligned}$$

Thus E is finite.

Exercise 3.3.13. Let's call a set *clompact* if it has the property that every *closed* cover (i.e., a cover consisting of closed sets) admits a finite subcover. Describe all of the clompact subsets of $\mathbf{R}$.

Solution. Let E be a subset of $\mathbf{R}$. Suppose that E is finite. If E is empty then certainly E is clompact, so suppose that $E = \{x_1, \ldots, x_n\}$ and let $\{F_\lambda : \lambda \in \Lambda\}$ be a closed cover of E. For each $x_i \in E$, there is some F_{λ_i} such that $x_i \in F_{\lambda_i}$; it follows that $\{F_{\lambda_1}, \ldots, F_{\lambda_n}\}$ is a finite subcover of E and hence that E is clompact.

Now suppose that E is infinite and consider the closed cover $\{\{x\} : x \in E\}$ of E. Since E is infinite, finitely many singletons cannot possibly cover E. So we have found a closed cover of E which cannot have a finite subcover and hence E is not clompact.

To conclude, the clompact subsets of $\mathbf{R}$ are precisely the finite subsets of $\mathbf{R}$.

3.4 Perfect Sets and Connected Sets

Exercise 3.4.1. If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Solution. P is closed, so $P \cap K$ must be compact (Exercise 3.3.4 (a)). However, $P \cap K$ need not be perfect. For a counterexample, consider $P = [0, 1]$ and $K = \{0\}$.

Exercise 3.4.2. Does there exist a perfect set consisting of only rational numbers?

Solution. No. By Theorem 3.4.3, a non-empty perfect set must be uncountable, but any subset of $\mathbf{Q}$ is either finite or countably infinite (Theorem 1.5.6 (i) and Theorem 1.5.7). (Strictly speaking, the empty set is both perfect and a subset of the rationals; I suspect this is not what Abbott had in mind.)

Exercise 3.4.3. Review the portion of the proof given in Example 3.4.2 and follow these steps to complete the argument.

(a) Because $x \in C_1$, argue that there exists an $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| \leq 1/3$.

(b) Finish the proof by showing that for each $n \in \mathbf{N}$, there exists $x_n \in C \cap C_n$, different from x satisfying $|x - x_n| \leq 1/3^n$.

Solution. (a) We have $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. The endpoints of these intervals are never removed at any subsequent stage of the construction of the Cantor set, so they belong to C. Since $x \in C_1$, it must belong to one of these intervals, say the interval $\left[0, \frac{1}{3}\right]$. If $0 \leq x < \frac{1}{3}$, then take $x_1 = \frac{1}{3}$, and if $x = \frac{1}{3}$, then take $x_1 = 0$. We can make similar choices if $x \in \left[\frac{2}{3}, 1\right]$. In any case, we have chosen an $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| \leq \frac{1}{3}$.

(b) Let $n \in \mathbf{N}$ be given. The set C_n consists of 2^n disjoint closed intervals each of length $\frac{1}{3^n}$. The endpoints of these intervals are never removed at any subsequent stage of the construction of the Cantor set, so they belong to C. Since $x \in C$, we have $x \in C_n$ and hence x must belong to one of the disjoint closed intervals, say $I = [a, b]$ where $b - a = \frac{1}{3^n}$. If $a \leq x < b$, then let $x_n = b$, and if $x = b$ then let $x_n = a$. In either case, we have chosen an $x_n \in C \cap C_n$ such that $x \neq x_n$ and $|x - x_n| \leq b - a = \frac{1}{3^n}$.

Thus, by the Squeeze Theorem (Exercise 2.3.3), x is the limit of a sequence (x_n) contained in C such that $x_n \neq x$ for all $n \in \mathbf{N}$. It follows from Theorem 3.2.5 that x is a limit point of C and hence that C contains no isolated points.

Exercise 3.4.4. Repeat the Cantor construction from Section 3.1 starting with the interval $[0, 1]$. This time, however, remove the open middle *fourth* from each component.

(a) Is the resulting set compact? Perfect?

(b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

Solution. We begin with $B_0 := [0, 1]$ and remove the open middle fourth to obtain $B_1 = \left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]$. Notice that each interval has length $\frac{3}{8}$. Next we remove the open middle fourth from each of the two intervals of B_1 to obtain

$$B_2 = \left(\left[0, \frac{9}{64}\right] \cup \left[\frac{15}{64}, \frac{24}{64}\right]\right) \cup \left(\left[\frac{40}{64}, \frac{49}{64}\right] \cup \left[\frac{55}{64}, 1\right]\right).$$

Notice that each interval has length $\left(\frac{3}{8}\right)^2$. We continue in this fashion, obtaining sets B_n consisting of 2^n disjoint closed intervals each of length $\left(\frac{3}{8}\right)^n$, and define our Cantor-like set $B := \bigcap_{n=0}^{\infty} B_n$.

(a) The set B is compact and perfect; the arguments used for the Cantor set work equally well for B. Each B_n is closed, being a finite union of closed intervals, and thus B is an intersection of closed sets and hence is itself closed. Certainly B is bounded and so it follows from the Heine-Borel Theorem (Theorem 3.3.8) that B is compact.

As in Exercise 3.4.3, given any $x \in B$ we can find a sequence of endpoints (x_n) such that $x_n \in B, x_n \neq x$, and $|x - x_n| \leq \left(\frac{3}{8}\right)^n$ for each $n \in \mathbf{N}$. It follows from the Squeeze Theorem (Exercise 2.3.3) and Theorem 3.2.5 that x is a limit point of B and hence that B has no isolated points. Since B is also closed, we see that B is a perfect set.

(b) At the first stage, we remove an interval of length $\frac{1}{4}$. At the n^{th} stage ($n = 2, 3, 4 \ldots$), we remove 2^{n-1} intervals each of length $\frac{1}{4}\left(\frac{3}{8}\right)^{n-1}$. Thus the length of B is

$$1 - \left(\frac{1}{4} + 2 \cdot \frac{1}{4} \cdot \frac{3}{8} + 2^2 \cdot \frac{1}{4} \cdot \left(\frac{3}{8}\right)^2 + \cdots\right)$$

$$= 1 - \frac{1}{4}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right) = 1 - \frac{\frac{1}{4}}{1 - \frac{3}{4}} = 0.$$

To calculate the dimension of B, we magnify the set by a factor of $\frac{8}{3}$, so that B_0 becomes the closed interval $\left[0, \frac{8}{3}\right]$. When we remove the open middle fourth of this interval, we are left with two intervals of length 1:

$$B_1 = [0, 1] \cup \left[\frac{5}{3}, \frac{8}{3}\right].$$

Thus we will obtain two copies of B. The dimension x of B is then given by solving $2 = \left(\frac{8}{3}\right)^x$, which gives

$$x = \frac{\log(2)}{\log(8) - \log(3)} \approx 0.7067.$$

Exercise 3.4.5. Let A and B be nonempty subsets of $\mathbf{R}$. Show that if there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.

Solution. Observe that V^{c} is a closed set which contains A (since $U \cap V = \emptyset$ implies that $A \cap V = \emptyset$). Since $\overline{A}$ is the smallest closed set containing A (Theorem 3.2.12), we must have $\overline{A} \subseteq V^{\mathsf{c}}$, which gives

$$\overline{A} \subseteq V^{\mathsf{c}} \implies \overline{A} \cap V = \emptyset \implies \overline{A} \cap B = \emptyset.$$

Similarly, $A \cap \overline{B} = \emptyset$. Thus A and B are separated.

Exercise 3.4.6. Prove Theorem 3.4.6.

Solution. Suppose we have non-empty subsets $A, B \subseteq \mathbf{R}$ such that every convergent sequence contained in one of the subsets has a limit which does not belong to the other subset. Since a limit point of A is the limit of a sequence contained in A (Theorem 3.2.5) and an element of A is the limit of a constant sequence contained in A, and by assumption these limits do not belong to B, we see that $\overline{A} \cap B = \emptyset$. Similarly, $A \cap \overline{B} = \emptyset$. Thus A and B are separated.

Conversely, suppose that A and B are separated. If $(x_n) \to x$ is a convergent sequence contained in A, then $x \in \overline{A}$. It follows that $x \notin B$ since $\overline{A} \cap B = \emptyset$. Similarly, the limit of any convergent sequence contained in B must not belong to A.

We have now shown that for non-empty subsets $A, B \subseteq \mathbf{R}$, A and B being separated is equivalent to the condition that every convergent sequence contained in one of the subsets has a limit which does not belong to the other subset.

Proving Theorem 3.4.6 is equivalent to showing that a subset $E \subseteq \mathbf{R}$ is disconnected if and only if there exist non-empty subsets $A, B \subseteq E$ such that $E = A \cup B$ and every convergent sequence contained in one of the subsets has a limit which does not belong to the other subset. By the previous discussion, such subsets are separated. So the theorem follows by the definition of disconnectedness.

Exercise 3.4.7. A set E is *totally disconnected* if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A, y \in B$, and $E = A \cup B$.

(a) Show that $\mathbf{Q}$ is totally disconnected.

(b) Is the set of irrational numbers totally disconnected?

Solution. (a) Suppose that $p < q$ are rational numbers. By the density of $\mathbf{I}$ in $\mathbf{R}$, there exists an irrational number y such that $p < y < q$. Define the sets

$$A = (-\infty, y) \cap \mathbf{Q} \qquad \text{and} \qquad B = (y, \infty) \cap \mathbf{Q}.$$

Notice that $p \in A, q \in B$, and since $y \notin \mathbf{Q}$, we have $A \cup B = \mathbf{Q}$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, we have $\overline{A} = (-\infty, y]$ and $\overline{B} = [y, \infty)$. It follows that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ and hence that A and B are separated. Thus $\mathbf{Q}$ is totally disconnected.

(b) $\mathbf{I}$ is also totally disconnected. To see this, reverse the roles of $\mathbf{Q}$ and $\mathbf{I}$ in the solution to part (a).

Exercise 3.4.8. Follow these steps to show that the Cantor set is totally disconnected in the sense described in Exercise 3.4.7.

Let $C = \bigcap_{n=0}^{\infty} C_n$, as defined in Section 3.1.

(a) Given $x, y \in C$, with $x < y$, set $\epsilon = y - x$. For each $n = 0, 1, 2, \ldots$, the set C_n consists of a finite number of closed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval of C_N.

(b) Show that C is totally disconnected.

Solution. (a) If I is an interval of length δ, then any $a, b \in I$ must satisfy $|a - b| \leq \delta$. In the construction of C, each C_n consists of 2^n disjoint closed intervals each of length 3^{-n}. Thus we can find an N large enough so that C_N consists of closed intervals each of length $3^{-N} < \epsilon = y - x$, i.e., whose length is smaller than the distance between x and y. It follows that x and y cannot possibly belong to the same interval of C_N.

(b) Let $[a, b]$ be the closed interval of C_N which contains x and note that the open interval $\left(b, b + \frac{1}{3^N}\right)$ was either removed at the N^{th} stage of construction or is a subset of an open interval which was removed at some previous stage of construction. It follows that $t := b + \frac{1}{2 \cdot 3^N} \notin C$. Since $y \notin [a, b]$ and $y > x$, we must have $y > t$. Define

$$A = (-\infty, t) \cap C \qquad \text{and} \qquad B = (t, \infty) \cap C.$$

Notice that $x \in A, y \in B$, and since $t \notin C$, we have $A \cup B = C$. If $(z_n) \to z$ is a convergent sequence contained in A, then the Order Limit Theorem (Theorem 2.3.4) implies that $z \leq t$ and hence $z \notin B$. Similarly, the limit of any convergent sequence contained in B cannot belong to A. Thus A and B are separated by Theorem 3.4.6 (see Exercise 3.4.6) and it follows that C is totally disconnected.

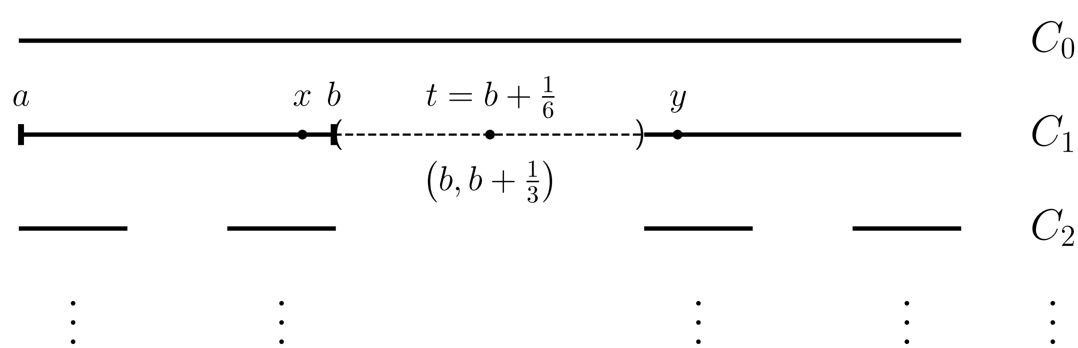


Figure F.29: Example construction of t; here $N = 1$, but notice that any $N \geq 1$ would also work

Exercise 3.4.9. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rational numbers, and for each $n \in \mathbf{N}$ set $\epsilon_n = 1/2^n$. Define $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ and let $F = O^{\mathsf{c}}$.

(a) Argue that F is a closed, nonempty set consisting only of irrational numbers.

(b) Does F contain any nonempty open intervals? Is F totally disconnected? (See Exercise 3.4.7 for the definition.)

(c) Is it possible to know whether F is perfect? If not, can we modify this construction to produce a nonempty perfect set of irrational numbers?

Solution. (a) O is an open set since it is a union of open intervals, so $F = O^{\mathsf{c}}$ must be closed. To see that F is non-empty, suppose otherwise, so that $O = \mathbf{R}$. It follows that the collection $\{V_{\epsilon_n}(r_n) : n \in \mathbf{N}\}$ is an open cover of the compact set $[0, 10]$. Thus, by Theorem 3.3.8, there exist finitely many indices $n_1 < \cdots < n_K$ such that

$$[0, 10] \subseteq V_{\epsilon_{n_1}}(r_{n_1}) \cup \cdots \cup V_{\epsilon_{n_K}}(r_{n_K}).$$

However, the interval $[0, 10]$ has length 10, whereas the set $V_{\epsilon_{n_1}}(r_{n_1}) \cup \cdots \cup V_{\epsilon_{n_K}}(r_{n_K})$ has total length at most

$$\sum_{k=1}^{K} \frac{1}{2^{n_k - 1}} \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = 2,$$

since $\left|V_{\epsilon_{n_k}}(r_{n_k})\right| = 2\epsilon_{n_k} = 2^{-n_k+1}$. So we have a set of length 10 contained inside a set of length 2, which is a contradiction; it follows that F is non-empty. Finally, since $\mathbf{Q} \subseteq O$, we see that $F = O^{\mathsf{c}}$ can contain only irrational numbers.

(b) F cannot contain any non-empty open intervals, since this would imply that F contains a rational number (indeed, infinitely many rational numbers), but by part (a) F contains only irrational numbers.

To see that F is totally disconnected, let us prove the following lemma.

Lemma L.11. Suppose $G \subseteq \mathbf{R}$ is totally disconnected. If E is a non-empty subset of G, then E is also totally disconnected.

Proof. Let $x, y \in E$ be given. Since x and y belong to the totally disconnected set G, there exist separated sets A and B such that $x \in A, y \in B$, and $G = A \cup B$. Set $A' = A \cap E$ and $B' = B \cap E$ and note that $x \in A'$ and $y \in B'$. Furthermore, $A' \subseteq A$ and $B' \subseteq B$, so

$$\overline{A'} \subseteq \overline{A} \implies \left(\overline{A'} \cap B'\right) \subseteq \left(\overline{A} \cap B'\right) \subseteq \left(\overline{A} \cap B\right) = \emptyset.$$

Thus $\overline{A'} \cap B' = \emptyset$, and similarly $A' \cap \overline{B'} = \emptyset$; it follows that A' and B' are separated. Finally,

$$E = E \cap G = E \cap (A \cup B) = (A \cap E) \cup (B \cap E) = A' \cup B'.$$

Thus E is totally disconnected. □

Since F is a subset of $\mathbf{I}$, which we showed was totally disconnected in Exercise 3.4.7, it follows from Lemma L.11 that F is totally disconnected.

(c) There are enumerations of $\mathbf{Q}$ which, when used in this construction, will result in an F which is not perfect, i.e., an F with at least one isolated point. We will construct such an enumeration (r_n), which gives an F with $\sqrt{2}$ as an isolated point, via the following four step process. (Any irrational number would also work in place of $\sqrt{2}$. This construction is due to math.SE user Ingix, however the exposition given here is my own; what follows is lengthy and involved.)

Step 1. We will first construct a strictly increasing sequence (p_n) of distinct rational numbers such that:

(1.1) $p_1 < p_2 < p_3 < \cdots < \sqrt{2}$;

(1.2) $\left(\sqrt{2} - \frac{1}{16}, \sqrt{2}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(p_n)$;

(1.3) $\sqrt{2} \notin \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(p_n)$.

This sequence will be placed in the final enumeration (r_n) as $r_{4n} = p_n$, so that

$$r_4 = p_1, r_8 = p_2, r_{12} = p_3, \ldots$$

Step 2. Mirroring Step 1, we will construct a strictly decreasing sequence (q_n) of distinct rational numbers such that:

(2.1) $\sqrt{2} < \cdots < q_3 < q_2 < q_1$;

(2.2) $\left(\sqrt{2}, \sqrt{2} + \frac{1}{16}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(q_n)$;

(2.3) $\sqrt{2} \notin \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(q_n)$.

This sequence will be placed in the final enumeration (r_n) as $r_{4n-2} = q_n$, so that

$$r_2 = q_1, r_6 = q_2, r_{10} = q_3, \ldots$$

Step 3. There are infinitely many rational numbers which belong to neither of the sequences (p_n) nor (q_n) from Steps 1 and 2. We will construct a sequence (a_n) which enumerates these remaining rational numbers in such a way that $\sqrt{2}$ will not be excluded from F in the final construction, i.e., a sequence (a_n) such that:

(3.1) $a_m \neq a_n$ for $m \neq n$;

(3.2) for each rational $r \in (\{p_1, p_2, \ldots\} \cup \{q_1, q_2, \ldots\})^{\mathsf{c}}$, there exists an $n \in \mathbf{N}$ such that $a_n = r$.

(3.3) $\sqrt{2} \notin \bigcup_{n=1}^{\infty} V_{\epsilon_{2n-1}}(a_n)$.

This sequence will be placed in the final enumeration (r_n) as $r_{2n-1} = a_n$, so that

$$r_1 = a_1, r_3 = a_2, r_5 = a_3, \ldots$$

Step 4. We will combine the sequences $(p_n), (q_n)$, and (a_n) to obtain an enumeration (r_n) of $\mathbf{Q}$ given by

$$a_1, q_1, a_2, p_1, a_3, q_2, a_4, p_2, \ldots$$

Letting $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ and $F = O^{\mathsf{c}}$, we will have

$$\left(\sqrt{2} - \tfrac{1}{16}, \sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{2} + \tfrac{1}{16}\right) \subseteq O \qquad \text{and} \qquad \sqrt{2} \notin O,$$

so that $\left(\sqrt{2} - \frac{1}{16}, \sqrt{2} + \frac{1}{16}\right) \cap F = \left\{\sqrt{2}\right\}$. Thus $\sqrt{2}$ will be an isolated point of F.

Step 1.

For each $n \in \mathbf{N}$, let p_n be a rational number satisfying

$$\sqrt{2} - \frac{1}{2^{4n}} - \frac{1}{2^{4n+4}} < p_n < \sqrt{2} - \frac{1}{2^{4n}} < \sqrt{2};$$

the existence of such a rational number is guaranteed by the density of $\mathbf{Q}$ in $\mathbf{R}$. From this definition, it is straightforward to verify that the sequence (p_n) satisfies condition (1.1). Observe that for any $n \in \mathbf{N}$ we have

$$p_n < p_{n+1} < p_n + \frac{1}{2^{4n}} < \sqrt{2};$$

it follows that $\sqrt{2} \notin V_{\epsilon_{4n}}(p_n)$ for each $n \in \mathbf{N}$, so that the sequence (p_n) satisfies condition (1.3). Furthermore, $p_{n+1} \in (p_n, p_n + 2^{-4n}) \subseteq V_{\epsilon_{4n}}(p_n)$, i.e., the centre of $V_{\epsilon_{4n+4}}(p_{n+1})$ is contained in $V_{\epsilon_{4n}}(p_n)$. Thus, for any $N \in \mathbf{N}$, the union $\bigcup_{n=1}^{N} V_{\epsilon_{4n}}(p_n)$ must be an open interval:

$$\bigcup_{n=1}^{N} V_{\epsilon_{4n}}(p_n) = \left(p_1 - \frac{1}{16}, B\right),$$

where $B = \max\{p_n + 2^{-4n} : 1 \leq n \leq N\}$ (the exact value of B is not important, but note that it must be strictly less than $\sqrt{2}$). Since

$$\sqrt{2} - \frac{1}{16} \in \left(p_1 - \frac{1}{16}, p_1 + \frac{1}{16}\right) = V_{\epsilon_4}(p_1),$$

we have $\sqrt{2} - \frac{1}{16} \in \bigcup_{n=1}^{N} V_{\epsilon_{4n}}(p_n)$ for any $N \in \mathbf{N}$. Let $y \in \mathbf{R}$ be such that $\sqrt{2} - \frac{1}{16} < y < \sqrt{2}$. Because (p_n) converges to $\sqrt{2}$ (by the Squeeze Theorem), we can find an $N \in \mathbf{N}$ such that $y < p_N < \sqrt{2}$. It follows that $\sqrt{2} - \frac{1}{16}$ and p_N both belong to the open interval $\bigcup_{n=1}^{N} V_{\epsilon_{4n}}(p_n)$; since y lies between these two values, we must then have $y \in \bigcup_{n=1}^{N} V_{\epsilon_{4n}}(p_n) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(p_n)$. Thus the sequence (p_n) satisfies condition (1.2).

Step 2.

The construction of the sequence (q_n) is analogous to the construction given in Step 1; for each $n \in \mathbf{N}$, let q_n be a rational number satisfying

$$\sqrt{2} + \frac{1}{2^{4n-2}} < q_n < \sqrt{2} + \frac{1}{2^{4n-2}} + \frac{1}{2^{4n+2}}.$$

Arguing as in Step 1, we see that the sequence (q_n) satisfies condition (2.1), and furthermore that $\left(\sqrt{2}, \sqrt{2} + \frac{1}{4}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(q_n)$, which gives us condition (2.2). Condition (2.3) follows since $\sqrt{2} < q_n - \frac{1}{2^{4n-2}}$ for all $n \in \mathbf{N}$.

Step 3.

Since the sequences (p_n) and (q_n) constructed in Steps 1 and 2 are entirely contained inside the interval $[p_1, q_1]$, we still have infinitely many rational numbers left to enumerate. That is, letting

$$E = \mathbf{Q} \cap (\{p_1, p_2, \ldots\} \cup \{q_1, q_2, \ldots\})^{\mathsf{c}},$$

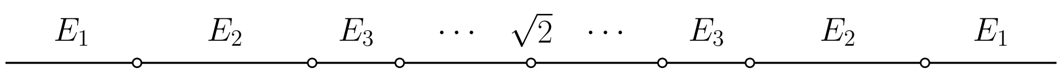
we have that E is countably infinite. However, enumerating E carelessly might exclude $\sqrt{2}$ from F in Step 4, since there are rational numbers in E arbitrarily close to $\sqrt{2}$; placing one of these rational numbers "too early" in the sequence (r_n) will include $\sqrt{2}$ in some $V_{\epsilon_n}(r_n)$. To surmount this problem, we will first partition E as follows. Let

$$A_n = \begin{cases} \left\{x \in \mathbf{R} : \epsilon_1 < \left|x - \sqrt{2}\right|\right\} & \text{if } n = 1, \\ \left\{x \in \mathbf{R} : \epsilon_{2n-1} < \left|x - \sqrt{2}\right| < \epsilon_{2n-3}\right\} & \text{if } n \geq 2. \end{cases}$$

Equivalently,

$$A_n = \begin{cases} \left(-\infty, \sqrt{2} - \epsilon_1\right) \cup \left(\sqrt{2} + \epsilon_1, \infty\right) & \text{if } n = 1, \\ \left(\sqrt{2} - \epsilon_{2n-3}, \sqrt{2} - \epsilon_{2n-1}\right) \cup \left(\sqrt{2} + \epsilon_{2n-1}, \sqrt{2} + \epsilon_{2n-3}\right) & \text{if } n \geq 2. \end{cases}$$

Now set $E_n = E \cap A_n$ for each $n \in \mathbf{N}$.



We have $\bigcup_{n=1}^{\infty} E_n = E$ since the only real numbers not contained in $\bigcup_{n=1}^{\infty} A_n$ are $\sqrt{2}$ and those of the form $\sqrt{2} \pm \epsilon_{2n-1}$ for some $n \in \mathbf{N}$, none of which are rational, and the collection $\{E_n : n \in \mathbf{N}\}$ is evidently pairwise disjoint; it follows that this collection is a partition of E.

Since $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \sqrt{2}$ and $\sqrt{2} \notin \overline{A_n}$ for any $n \in \mathbf{N}$, we see that there can be only finitely many terms of the sequences (p_n) and (q_n) contained in each A_n; it follows that each E_n is countably infinite. We can then enumerate each E_n:

$$E_n = \{e_{1,n}, e_{2,n}, e_{3,n}, \ldots\}.$$

These enumerations can be combined to form an enumeration (a_n) of E using the same method of the proof that a countable union of countable sets is itself countable (see, for example, Exercise 1.5.3 (c)). To be precise, consider the following “infinite arrays”.

$$\begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & \cdots \\ \hline e_{1,1} & e_{1,2} & e_{1,3} & e_{1,4} & e_{1,5} & \cdots \\ e_{2,1} & e_{2,2} & e_{2,3} & e_{2,4} & e_{2,5} & \cdots \\ e_{3,1} & e_{3,2} & e_{3,3} & e_{3,4} & e_{3,5} & \cdots \\ e_{4,1} & e_{4,2} & e_{4,3} & e_{4,4} & e_{4,5} & \cdots \\ e_{5,1} & e_{5,2} & e_{5,3} & e_{5,4} & e_{5,5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \qquad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \\ \hline a_1 & a_3 & a_6 & a_{10} & a_{15} & \cdots \\ a_2 & a_5 & a_9 & a_{14} & a_{20} & \cdots \\ a_4 & a_8 & a_{13} & a_{19} & a_{26} & \cdots \\ a_7 & a_{12} & a_{18} & a_{25} & a_{33} & \cdots \\ a_{11} & a_{17} & a_{24} & a_{32} & a_{41} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The enumeration of E_n is the n^{th} column of the left-hand array. The enumeration of E is obtained by letting a_N in the right-hand array be the element $e_{m,n}$ in the corresponding position of the left-hand array, so that

$$a_1 = e_{1,1}, a_2 = e_{2,1}, a_3 = e_{1,2}, a_4 = e_{3,1}, \ldots$$

This mapping is bijective because the collection $\{E_n : n \in \mathbf{N}\}$ is a partition of E and thus the sequence (a_n) satisfies conditions (3.1) and (3.2).

To show that the sequence (a_n) satisfies condition (3.3), we need to show that for all $n \in \mathbf{N}, \sqrt{2} \notin V_{\epsilon_{2n-1}}(a_n)$. Let $n \in \mathbf{N}$ be given. The element a_n belongs to some column of the right-hand array above, say the N^{th} column. From the definition of our enumeration (a_n), we have $a_n = e_{m,N}$ for some $m \in \mathbf{N}$. It follows that $a_n \in E_N$ and hence that $\left|a_n - \sqrt{2}\right| > \epsilon_{2N-1}$, which gives $\sqrt{2} \notin V_{\epsilon_{2N-1}}(a_n)$.

If we examine the right-hand array, we see that the element at the top of the N^{th} column is $a_{N(N+1)/2}$ (the N^{th} triangular number), and furthermore that $n \geq N(N+1)/2$. Thus

$$2n - 1 \geq 2N - 1 \implies \epsilon_{2n-1} \leq \epsilon_{2N-1} \implies V_{\epsilon_{2n-1}}(a_n) \subseteq V_{\epsilon_{2N-1}}(a_n).$$

Combining this with $\sqrt{2} \notin V_{\epsilon_{2N-1}}(a_n)$, we see that $\sqrt{2} \notin V_{\epsilon_{2n-1}}(a_n)$; it follows that the sequence (a_n) satisfies condition (3.3).

Step 4.

We can now form our final enumeration (r_n), by letting

$$r_{2n-1} = a_n, \quad r_{4n-2} = q_n, \qquad \text{and} \qquad r_{4n} = p_n,$$

so that (r_n) is the sequence

$$a_1, q_1, a_2, p_1, a_3, q_2, a_4, p_2, \ldots$$

Let $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ and $F = O^{\mathsf{c}}$. By condition (1.2), we have

$$\left(\sqrt{2} - \frac{1}{16}, \sqrt{2}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(p_n) = \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(r_{4n}) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n) = O,$$

and by condition (2.2), we have

$$\left(\sqrt{2}, \sqrt{2} + \frac{1}{16}\right) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(q_n) = \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(r_{4n-2}) \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n) = O.$$

Thus $\left(\sqrt{2} - \frac{1}{16}, \sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{2} + \frac{1}{16}\right) \subseteq O$. Furthermore, since

$$\begin{aligned} O &= \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n) \\ &= \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(r_{4n}) \cup \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(r_{4n-2}) \cup \bigcup_{n=1}^{\infty} V_{\epsilon_{2n-1}}(r_{2n-1}) \\ &= \bigcup_{n=1}^{\infty} V_{\epsilon_{4n}}(p_n) \cup \bigcup_{n=1}^{\infty} V_{\epsilon_{4n-2}}(q_n) \cup \bigcup_{n=1}^{\infty} V_{\epsilon_{2n-1}}(a_n), \end{aligned}$$

conditions (1.3), (2.3), and (3.3) imply that $\sqrt{2} \notin O$. It follows that

$$\left(\sqrt{2} - \frac{1}{16}, \sqrt{2} + \frac{1}{16}\right) \cap F = \left\{\sqrt{2}\right\},$$

so that $\sqrt{2}$ is an isolated point of F. We may conclude that F is not a perfect set.

Regarding the second half of the question, it is possible to modify the construction to produce a non-empty perfect set consisting of only irrational numbers. To do this, we start with any enumeration (r_n) of $\mathbf{Q}$ and recursively define a sequence of non-negative real numbers (ϵ_n) in such a way that if we set

$$O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n) \qquad \text{and} \qquad F = O^{\mathsf{c}},$$

then F will be a non-empty perfect of irrational numbers. Intuitively, we will recursively construct O as a union of disjoint open intervals, with no pair of these intervals sharing an endpoint. (In what follows, we adopt the convention that $V_\epsilon(x) = \emptyset$ if $\epsilon = 0$.)

Suppose that after N steps we have chosen $\epsilon_1, \ldots, \epsilon_N$ such that:

(IH1) $\{r_1, \ldots, r_N\} \subseteq \bigcup_{n=1}^{N} V_{\epsilon_n}(r_n)$;

(IH2) for all $1 \leq n \leq N$, either $\epsilon_n = 0$ or ϵ_n is positive, irrational, and satisfies $\epsilon_n \leq \frac{\sqrt{2}}{2^n}$;

(IH3) $\overline{V_{\epsilon_m}(r_m)} \cap \overline{V_{\epsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $1 \leq m < n \leq N$.

Let $U = \bigcup_{n=1}^{N} V_{\epsilon_n}(r_n)$. There are two cases.

Case 1. This is the easier case. If $r_{N+1} \in U$, then let $\epsilon_{N+1} = 0$, so that $V_{\epsilon_{N+1}}(r_{N+1}) = \emptyset$.

(IH1) combined with $r_{N+1} \in U$ gives us

$$\{r_1, \ldots, r_N, r_{N+1}\} \subseteq U = \bigcup_{n=1}^{N} V_{\epsilon_n}(r_n) = \bigcup_{n=1}^{N+1} V_{\epsilon_n}(r_n),$$

where the last equality follows because $V_{\epsilon_{N+1}}(r_{N+1}) = \emptyset$.

Combining (IH2) with $\epsilon_{N+1} = 0$, we see that for all $1 \leq n \leq N+1$, either $\epsilon_n = 0$ or ϵ_n is positive, irrational, and satisfies $\epsilon_n \leq \frac{\sqrt{2}}{2^n}$.

Similarly, combining (IH3) with $V_{\epsilon_{N+1}}(r_{N+1}) = \emptyset$, we have $\overline{V_{\epsilon_m}(r_m)} \cap \overline{V_{\epsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $1 \leq m < n \leq N+1$.

Case 2. This is the harder case. If $r_{N+1} \notin U$ then let $\epsilon_{n_1}, \ldots, \epsilon_{n_J}$ be those ϵ's from $\epsilon_1, \ldots, \epsilon_N$ which are non-zero; there must be at least one such ϵ_{n_j} by (IH1) and each ϵ_{n_j} must be positive and irrational by (IH2). Observe that

$$U = \bigcup_{n=1}^{N} V_{\epsilon_n}(r_n) = \bigcup_{j=1}^{J} V_{\epsilon_{n_j}}(r_{n_j}),$$

where each $V_{\epsilon_{n_j}}(r_{n_j})$ is a proper open interval. For each $1 \leq j \leq J$, note that since $r_{N+1} \notin U$, we must have $r_{N+1} \notin V_{\epsilon_{n_j}}(r_{n_j})$. Both of the endpoints of $V_{\epsilon_{n_j}}(r_{n_j})$ are the sum of a rational number and an irrational number and hence are irrational; since r_{N+1} is rational, we see that $r_{N+1} \notin [r_{n_j} - \epsilon_{n_j}, r_{n_j} + \epsilon_{n_j}]$. Given this, if we let d be the minimum of the distances from r_{N+1} to the endpoints of each $V_{\epsilon_{n_j}}$, i.e.,

$$d = \min\Big\{\big|r_{n_j} - \epsilon_{n_j} - r_{N+1}\big|, \big|r_{n_j} + \epsilon_{n_j} - r_{N+1}\big| : 1 \le j \le J\Big\},$$

then d must be positive. Furthermore, d must be irrational since it is the sum of a rational number and an irrational number, and for each $1 \le j \le J$, we have

$$\big[r_{N+1} - \tfrac{d}{2}, r_{N+1} + \tfrac{d}{2}\big] \cap [r_{n_j} - \epsilon_{n_j}, r_{n_j} + \epsilon_{n_j}] = \emptyset. \tag{†}$$

Let $\epsilon_{N+1} = \min\Big\{\frac{\sqrt{2}}{2^{N+1}}, \frac{d}{2}\Big\}$ and note that ϵ_{N+1} is positive, so that $r_{N+1} \in V_{\epsilon_{N+1}}(r_{N+1})$. Combining this with (IH1) gives us

$$\{r_1, \ldots, r_N, r_{N+1}\} \subseteq \bigcup_{n=1}^{N+1} V_{\epsilon_n}(r_n).$$

As noted before, d is positive and irrational, so ϵ_{N+1} is positive, irrational, and satisfies $\epsilon_{N+1} \le \frac{\sqrt{2}}{2^{N+1}}$; combining this with (IH2) shows that for all $1 \le n \le N+1$, either $\epsilon_n = 0$ or ϵ_n is positive, irrational, and satisfies $\epsilon_n \le \frac{\sqrt{2}}{2^n}$.

Let $1 \le n \le N$ be given. If $\epsilon_n = 0$, then the identity $\overline{V_{\epsilon_n}(r_n)} \cap \overline{V_{\epsilon_{N+1}}(r_{N+1})} = \emptyset$ is clear, since $V_{\epsilon_n}(r_n) = \emptyset$. If $\epsilon_n \ne 0$, then $n = n_j$ for some $1 \le j \le J$. In this case, we have

$$\begin{aligned} &\overline{V_{\epsilon_n}(r_n)} = \overline{V_{\epsilon_{n_j}}(r_{n_j})} = [r_{n_j} - \epsilon_{n_j}, r_{n_j} + \epsilon_{n_j}] \quad \text{and} \\ &\qquad\qquad \overline{V_{\epsilon_{N+1}}(r_{N+1})} = [r_{N+1} - \epsilon_{N+1}, r_{N+1} + \epsilon_{N+1}] \subseteq \big[r_{N+1} - \tfrac{d}{2}, r_{N+1} + \tfrac{d}{2}\big]. \end{aligned}$$

By equation (†), we see that $\overline{V_{\epsilon_n}(r_n)} \cap \overline{V_{\epsilon_{N+1}}(r_{N+1})} = \emptyset$. Combining this with (IH3), we see that $\overline{V_{\epsilon_m}(r_m)} \cap \overline{V_{\epsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $1 \le m < n \le N+1$.

This completes the recursive step; for the base case, simply let $\epsilon_1 = \frac{\sqrt{2}}{2}$. Thus we obtain a sequence (ϵ_n) which satisfies (IH1), (IH2), and (IH3) for all $N \in \mathbf{N}$. In other words, the sequence (ϵ_n) has the following properties:

(A1) $\mathbf{Q} \subseteq \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$;

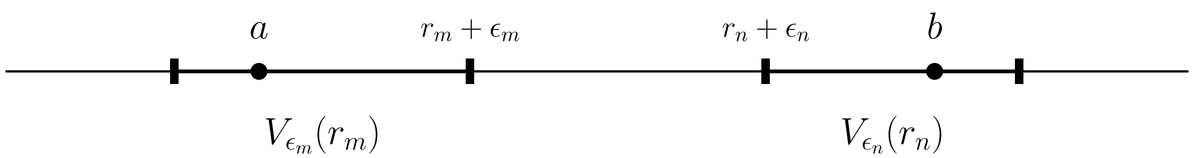
(A2) for all $n \in \mathbf{N}$, either $\epsilon_n = 0$ or ϵ_n is positive, irrational, and satisfies $\epsilon_n \leq \frac{\sqrt{2}}{2^n}$;

(A3) $\overline{V_{\epsilon_m}(r_m)} \cap \overline{V_{\epsilon_n}(r_n)} = \emptyset$ for all $m, n \in \mathbf{N}$ with $m < n$.

Let $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$ and $F = O^{\mathsf{c}}$. As in part (a), F is closed and, by (A1), consists solely of irrational numbers. By (A2), we have $\epsilon_n \leq \frac{\sqrt{2}}{2^n}$ for each $n \in \mathbf{N}$; a similar argument as in part (a) shows that O cannot be the entire real line and thus F is non-empty.

To see that F is perfect, suppose by way of contradiction that $x \in F$ is isolated, i.e., there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \cap F = \{x\}$. This implies that the intervals $(x - \delta, x)$ and $(x, x + \delta)$ are contained in O. We claim that if an interval such as $(x - \delta, x)$ is to be contained in O, then it must be entirely contained inside a single $V_{\epsilon_n}(r_n)$. To see this, suppose by way of contradiction that $a, b \in (x - \delta, x)$ are such that $a < b, a \in V_{\epsilon_m}(r_m)$, and $b \in V_{\epsilon_n}(r_n)$, with $m \neq n$. By (A3), it must then be the case that

$$a < r_m + \epsilon_m < r_n - \epsilon_n < b.$$



So $r_m + \epsilon_m \in (a, b) \subseteq (x - \delta, x) \subseteq O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$; it follows that there exists a $k \in \mathbf{N}$ such that $r_m + \epsilon_m$ belongs to $V_{\epsilon_k}(r_k)$. If $k = m$, this says that an open interval contains one of its endpoints, and if $k \neq m$ then this violates (A3). In either case, we have a contradiction.

Thus if an interval such as $(x - \delta, x)$ is to be contained in O, it must be entirely contained inside a single $V_{\epsilon_n}(r_n)$. Since $(x - \delta, x)$ and $(x, x + \delta)$ are disjoint, there exist positive integers $m \neq n$ such that

$$(x - \delta, x) \subseteq V_{\epsilon_m}(r_m) \qquad \text{and} \qquad (x, x + \delta) \subseteq V_{\epsilon_n}(r_n).$$

This implies that

$$[x - \delta, x] \subseteq \overline{V_{\epsilon_m}(r_m)} \qquad \text{and} \qquad [x, x + \delta] \subseteq \overline{V_{\epsilon_n}(r_n)},$$

which in turn gives $x \in \overline{V_{\epsilon_m}(r_m)} \cap \overline{V_{\epsilon_n}(r_n)}$, contradicting (A3). We may conclude that F is a perfect set.

3.5 Baire's Theorem

Exercise 3.5.1. Argue that a set A is a G_δ set if and only if its complement is an F_σ set.

Solution. This is immediate from De Morgan's Laws (see Exercise 3.2.9).

Exercise 3.5.2. Replace each _____ with the word *finite* or *countable* depending on which is more appropriate.

(a) The _____ union of F_σ sets is an F_σ set.

(b) The _____ intersection of F_σ sets is an F_σ set.

(c) The _____ union of G_δ sets is a G_δ set.

(d) The _____ intersection of G_δ sets is a G_δ set.

Solution. (a) The countable union of F_σ sets is an F_σ set. Suppose we have a countable collection $\{A_m : m \in \mathbf{N}\}$ of F_σ sets, i.e., for each $m \in \mathbf{N}$ there is a countable collection $\{B_{m,n} : n \in \mathbf{N}\}$ of closed sets such that $A_m = \bigcup_{n=1}^\infty B_{m,n}$. Notice that

$$\bigcup_{m=1}^\infty A_m = \bigcup_{m=1}^\infty \bigcup_{n=1}^\infty B_{m,n} = \bigcup_{(m,n)\in\mathbf{N}^2} B_{m,n}.$$

Since $\mathbf{N}^2$ is countable (Lemma L.5), we have expressed $\bigcup_{m=1}^\infty A_m$ as a countable union of closed sets; it follows that $\bigcup_{m=1}^\infty A_m$ is an F_σ set.

(b) The finite intersection of F_σ sets is an F_σ set. To see this, it will suffice to show that if A and B are F_σ sets, then $A\cap B$ is an F_σ set; the general case will then follow from a straightforward induction argument. Suppose therefore that $A = \bigcup_{m=1}^\infty A_m$ and $B = \bigcup_{n=1}^\infty B_n$, where $\{A_m : m \in \mathbf{N}\}$ and $\{B_n : n \in \mathbf{N}\}$ are countable collections of closed sets, and observe that

$$A \cap B = \left(\bigcup_{m=1}^\infty A_m\right) \cap \left(\bigcup_{n=1}^\infty B_n\right) = \bigcup_{(m,n)\in\mathbf{N}^2} (A_m \cap B_n).$$

Since each $A_m \cap B_n$ is closed (being an intersection of closed sets) and $\mathbf{N}^2$ is countable (Lemma L.5), we have expressed $A \cap B$ as a countable union of closed sets; it follows that $A \cap B$ is an F_σ set.

The countable intersection of F_σ sets need not be an F_σ set. For a counterexample, let $\{r_1, r_2, \ldots\}$ be an enumeration of $\mathbf{Q}$ and for positive integers m, n, let

$$B_{m,n} := \left(-\infty, r_m - \frac{1}{n}\right] \cup \left[r_m + \frac{1}{n}, \infty\right).$$

Each $B_{m,n}$ is a closed set, so if we let $A_m := \bigcup_{n=1}^{\infty} B_{m,n}$ for each $m \in \mathbf{N}$, then each A_m is an F_σ set. We claim that $\bigcap_{m=1}^{\infty} A_m = \mathbf{I}$, the set of irrational numbers. To see this, we will show that $(\bigcap_{m=1}^{\infty} A_m)^{\mathsf{c}} = \mathbf{Q}$. By De Morgan's Laws (Exercise 3.2.9), we have

$$\begin{aligned}\left(\bigcap_{m=1}^{\infty} A_m\right)^{\mathsf{c}} &= \bigcup_{m=1}^{\infty} A_m^{\mathsf{c}} = \bigcup_{m=1}^{\infty}\left(\bigcup_{n=1}^{\infty} B_{m,n}\right)^{\mathsf{c}} \\ &= \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty} B_{m,n}^{\mathsf{c}} = \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\left(r_m - \frac{1}{n}, r_m + \frac{1}{n}\right) = \bigcup_{m=1}^{\infty}\{r_m\} = \mathbf{Q}.\end{aligned}$$

Thus $\bigcap_{m=1}^{\infty} A_m = \mathbf{I}$. As we will show in Exercise 3.5.6, $\mathbf{I}$ is not an F_σ set.

(c) The finite union of G_δ sets is a G_δ set, but the countable union of G_δ sets need not be a G_δ set; these statements follow from part (b) of this exercise, Exercise 3.5.1, and De Morgan's Laws (Exercise 3.2.9).

(d) The countable intersection of G_δ sets is a G_δ set. Again, this follows from part (a) of this exercise, Exercise 3.5.1, and De Morgan's Laws (Exercise 3.2.9).

Exercise 3.5.3. (This exercise has already appeared as Exercise 3.2.15.)

(a) Show that a closed interval $[a, b]$ is a G_δ set.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.

(c) Show that $\mathbf{Q}$ is an F_σ set, and the set of irrationals $\mathbf{I}$ forms a G_δ set.

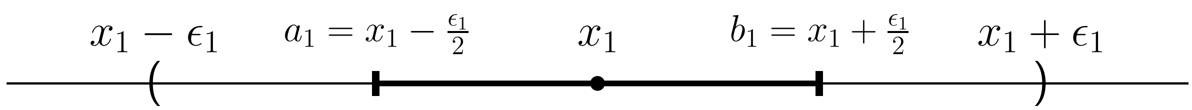
Solution. See Exercise 3.2.15.

Exercise 3.5.4. Starting with $n = 1$, inductively construct a nested sequence of *closed* intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ satisfying $I_n \subseteq G_n$. Give special attention to the issue of the endpoints of each I_n. Show how this leads to a proof of the theorem.

Solution. Since G_1 is dense, it must be non-empty, i.e., there exists some $x_1 \in G_1$, and then since G_1 is open there exists an $\epsilon_1 > 0$ such that $(x_1 - \epsilon_1, x_1 + \epsilon_1) \subseteq G_1$. Let

$$a_1 := x_1 - \frac{\epsilon_1}{2}, \qquad b_1 := x_1 + \frac{\epsilon_1}{2}, \qquad \text{and} \qquad I_1 := [a_1, b_1],$$

and note that $I_1 \subseteq (x_1 - \epsilon_1, x_1 + \epsilon_1) \subseteq G_1$. This handles the base case.



Suppose that after n steps we have chosen nested, closed intervals $I_1 = [a_1, b_1] \supseteq \cdots \supseteq I_n = [a_n, b_n]$ such that $I_1 \subseteq G_1, \ldots, I_n \subseteq G_n$ and $a_1 < b_1, \ldots, a_n < b_n$. Since G_{n+1} is dense, there exists some $x_{n+1} \in G_{n+1}$ such that $a_n < x_{n+1} < b_n$, and since G_{n+1} is open there exists some $\epsilon_{n+1} > 0$ such that $(x_{n+1} - \epsilon_{n+1}, x_{n+1} + \epsilon_{n+1}) \subseteq G_{n+1}$. Let $\delta = \min\left\{\frac{\epsilon_{n+1}}{2}, x_{n+1} - a_n, b_n - x_{n+1}\right\}$, and define

$$a_{n+1} := x_{n+1} - \delta, \qquad b_{n+1} := x_{n+1} + \delta, \qquad \text{and} \qquad I_{n+1} := [a_{n+1}, b_{n+1}].$$

Note that $a_{n+1} < b_{n+1}$, and since $\delta \leq x_{n+1} - a_n$ and $\delta \leq b_n - x_{n+1}$, we have $I_{n+1} \subseteq I_n$. Moreover, because $\delta \leq \frac{\epsilon_{n+1}}{2}$, we also have $I_{n+1} \subseteq (x_{n+1} - \epsilon_{n+1}, x_{n+1} + \epsilon_{n+1}) \subseteq G_{n+1}$. This completes the induction step.

We then obtain a nested sequence of closed intervals $(I_n)_{n=1}^\infty$ such that $I_n \subseteq G_n$ for each $n \in \mathbf{N}$. We may now appeal to the Nested Interval Property (Theorem 1.4.1) to obtain some $x \in \bigcap_{n=1}^\infty I_n$, which must also belong to $\bigcap_{n=1}^\infty G_n$.

Exercise 3.5.5. Show that it is impossible to write

$$\mathbf{R} = \bigcup_{n=1}^{\infty} F_n,$$

where for each $n \in \mathbf{N}, F_n$ is a closed set containing no nonempty open intervals.

Solution. Suppose that $\{F_n : n \in \mathbf{N}\}$ is a collection of closed sets, each of which contains no non-empty open intervals. Let $n \in \mathbf{N}$ be given and let $x < z$ be arbitrary real numbers. By assumption $(x, z) \not\subseteq F_n$, so there must exist some $y \in (x, z) \cap F_n^\mathsf{c}$; it follows that F_n^c is dense.

Thus $\{F_n^{\mathsf{c}} : n \in \mathbf{N}\}$ is a collection of open, dense sets. Theorem 3.5.2 (Exercise 3.5.4) and De Morgan's Laws (Exercise 3.2.9) now imply that

$$\bigcap_{n=1}^{\infty} F_n^{\mathsf{c}} \neq \emptyset \iff \bigcup_{n=1}^{\infty} F_n \neq \mathbf{R}.$$

Exercise 3.5.6. Show how the previous exercise implies that the set $\mathbf{I}$ of irrationals cannot be an F_σ set, and $\mathbf{Q}$ cannot be a G_δ set.

Solution. We will argue by contradiction. Suppose that $\mathbf{I}$ is an F_σ set, so that $\mathbf{I} = \bigcup_{m=1}^{\infty} F_m$, where each F_m is closed. Note that for any $m \in \mathbf{N}$, it must be the case that F_m contains no non-empty open interval; otherwise, F_m would contain infinitely many rational numbers. Let $\{r_1, r_2, \ldots\}$ be an enumeration of $\mathbf{Q}$, so that $\mathbf{Q} = \bigcup_{n=1}^{\infty}\{r_n\}$, and note that each singleton $\{r_n\}$ is closed and contains no non-empty interval. Observe that

$$\mathbf{R} = \mathbf{I} \cup \mathbf{Q} = \left(\bigcup_{m=1}^{\infty} F_m\right) \cup \left(\bigcup_{n=1}^{\infty}\{r_n\}\right) = \bigcup_{(m,n)\in\mathbf{N}^2} (F_m \cup \{r_n\}).$$

For any $(m,n) \in \mathbf{N}^2$, the union $F_m \cup \{r_n\}$ is closed and contains no non-empty intervals. However, since $\mathbf{N}^2$ is countable (Lemma L.5), this expression for $\mathbf{R}$ contradicts Exercise 3.5.5. Thus it must be that $\mathbf{I}$ is not an F_σ set, which by Exercise 3.5.1 implies that $\mathbf{Q}$ cannot be a G_δ set.

Exercise 3.5.7. Using Exercise 3.5.6 and versions of the statements in Exercise 3.5.2, construct a set that is neither in F_σ nor in G_δ.

Solution. Define $E := (\mathbf{I} \cap (-\infty, 0]) \cup (\mathbf{Q} \cap [0, \infty))$; we claim that E is neither an F_σ nor a G_δ set. Seeking a contradiction, suppose that E is an F_σ set. Any interval is an F_σ set (see Exercise 3.5.3), so by Exercise 3.5.2 (b) we have that

$$E \cap (-\infty, 0) = \mathbf{I} \cap (-\infty, 0)$$

is an F_σ set, i.e., there is a countable collection $\{F_m : m \in \mathbf{N}\}$ of closed sets such that

$$\mathbf{I} \cap (-\infty, 0) = \bigcup_{m=1}^{\infty} F_m.$$

For $m \in \mathbf{N}$, let $-F_m = \{-x : x \in F_m\}$. Since $(x_n) \to x$ implies $(-x_n) \to -x$, each $-F_m$ is also closed. Furthermore, we have

$$\mathbf{I} \cap (0, \infty) = \bigcup_{m=1}^{\infty} -F_m.$$

It follows that $\mathbf{I} \cap (0, \infty)$ is an F_σ set. However, Exercise 3.5.2 (a) now implies that

$$\mathbf{I} = (\mathbf{I} \cap (-\infty, 0)) \cup (\mathbf{I} \cap (0, \infty))$$

is an F_σ set, contradicting Exercise 3.5.6. Thus E cannot be an F_σ set and a similar argument with $\mathbf{Q}$ shows that E cannot be a G_δ set either.

Exercise 3.5.8. Show that a set E is nowhere-dense in $\mathbf{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbf{R}$.

Solution. We will show that $A \subseteq \mathbf{R}$ contains no non-empty open intervals if and only if A^{c} is dense in $\mathbf{R}$. By A containing no non-empty open intervals, we mean that for all $x, y \in \mathbf{R}$ such that $x < y$, we have $(x, y) \not\subseteq A$. This is equivalent to saying that for all $x, y \in \mathbf{R}$ such that $x < y$, there exists some $t \in \mathbf{R}$ such that $x < t < y$ and $t \notin A$. In other words, A^{c} is dense in $\mathbf{R}$.

Exercise 3.5.9. Decide whether the following sets are dense in $\mathbf{R}$, nowhere-dense in $\mathbf{R}$, or somewhere in between.

(a) $A = \mathbf{Q} \cap [0, 5]$.

(b) $B = \{1/n : n \in \mathbf{N}\}$.

(c) the set of irrationals.

(d) the Cantor set.

Solution. (a) We have $\overline{A} = [0, 5]$, which is not the entire real line and also contains non-empty open intervals. Thus A is neither dense nor nowhere-dense.

(b) We have $\overline{B} = \{0\} \cup B \neq \mathbf{R}$, so that B is not dense. Note that if $\overline{B}$ contained a non-empty open interval then $\overline{B}$ would contain at least one irrational number, but $\overline{B} \subseteq \mathbf{Q}$. Thus $\overline{B}$ contains no non-empty open intervals and it follows that B is nowhere-dense.

(c) $\mathbf{I}$ is dense in $\mathbf{R}$ (see Exercise 1.4.5) and hence cannot be nowhere-dense (a dense subset $E \subseteq \mathbf{R}$ certainly cannot be nowhere-dense; $\overline{E} = \mathbf{R}$ contains every non-empty open interval).

(d) The Cantor set is closed, so $\overline{C} = C \neq \mathbf{R}$; it follows that C is not dense in $\mathbf{R}$. C also does not contain any non-empty open intervals; given any $x < y$ in C, it is always possible to find some $t \notin C$ such that $x < t < y$ (see Exercise 3.4.8). Thus C is nowhere-dense in $\mathbf{R}$.

Exercise 3.5.10. Finish the proof by finding a contradiction to the results in this section.

Solution. Since $E_n \subseteq \overline{E_n}$ for each $n \in \mathbf{N}$, we have $\mathbf{R} = \bigcup_{n=1}^{\infty} \overline{E_n}$. However, each $\overline{E_n}$ is closed and by assumption contains no non-empty open intervals, so this contradicts Exercise 3.5.5.

Chapter 4

Functional Limits and Continuity

4.2 Functional Limits

Exercise 4.2.1. (a) Supply the details for how Corollary 4.2.4 part (ii) follows from the Sequential Criterion for Functional Limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

(b) Now, write another proof of Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the sequential criterion in Theorem 4.2.3.

(c) Repeat (a) and (b) for Corollary 4.2.4 part (iii).

Solution. (a) Suppose (x_n) is a sequence contained in A, satisfying $x_n \neq c$ and $\lim_{n\to\infty} x_n = c$. The sequential criterion (Theorem 4.2.3) implies that

$$\lim_{n\to\infty} f(x_n) = L \qquad \text{and} \qquad \lim_{n\to\infty} g(x_n) = M,$$

and thus the Algebraic Limit Theorem (for sequences, Theorem 2.3.3) gives

$$\lim_{n\to\infty} [f(x_n) + g(x_n)] = L + M.$$

The sequential criterion allows us to conclude that $\lim_{x\to c}[f(x) + g(x)] = L + M$.

(b) Let $\epsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, there exist positive real numbers δ_1 and δ_2 such that

$$0 < |x - c| < \delta_1 \text{ and } x \in A \implies |f(x) - L| < \frac{\epsilon}{2},$$

$$0 < |x - c| < \delta_2 \text{ and } x \in A \implies |g(x) - M| < \frac{\epsilon}{2}.$$

Set $\delta := \min\{\delta_1, \delta_2\}$ and suppose that $x \in A$ is such that $0 < |x - c| < \delta$. Observe that

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\lim_{x\to c}[f(x) + g(x)] = L + M$.

(c) Suppose (x_n) is a sequence contained in A, satisfying $x_n \neq c$ and $\lim_{n\to\infty} x_n = c$. The sequential criterion (Theorem 4.2.3) implies that

$$\lim_{n\to\infty} f(x_n) = L \qquad \text{and} \qquad \lim_{n\to\infty} g(x_n) = M,$$

and thus the Algebraic Limit Theorem (Theorem 2.3.3) gives

$$\lim_{n\to\infty}[f(x_n)g(x_n)] = LM.$$

The sequential criterion allows us to conclude that $\lim_{x\to c}[f(x)g(x)] = LM$.

Let $\epsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, there exist positive real numbers δ_1, δ_2, and δ_3 such that

$$0 < |x - c| < \delta_1 \text{ and } x \in A \implies |f(x) - L| < \frac{\epsilon}{2(|M| + 1)},$$

$$0 < |x - c| < \delta_2 \text{ and } x \in A \implies |g(x) - M| < \frac{\epsilon}{2(|L| + 1)},$$

$$0 < |x - c| < \delta_3 \text{ and } x \in A \implies |g(x) - M| < 1 \implies |g(x)| < |M| + 1.$$

Let $\delta := \min\{\delta_1, \delta_2, \delta_3\}$, suppose that $x \in A$ is such that $0 < |x - c| < \delta$, and observe that

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= |g(x)[f(x) - L] + L[g(x) - M]| \\
&\leq |g(x)||f(x) - L| + |L||g(x) - M| \\
&< (|M| + 1)\frac{\epsilon}{2(|M| + 1)} + |L|\frac{\epsilon}{2(|L| + 1)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus $\lim_{x\to c}[f(x)g(x)] = LM$.

Exercise 4.2.2. For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ϵ challenge.

(a) $\lim_{x\to 3}(5x-6) = 9$, where $\epsilon = 1$.

(b) $\lim_{x\to 4}\sqrt{x} = 2$, where $\epsilon = 1$.

(c) $\lim_{x\to\pi}[[x]] = 3$, where $\epsilon = 1$. (The function $[[x]]$ returns the greatest integer less than or equal to x.)

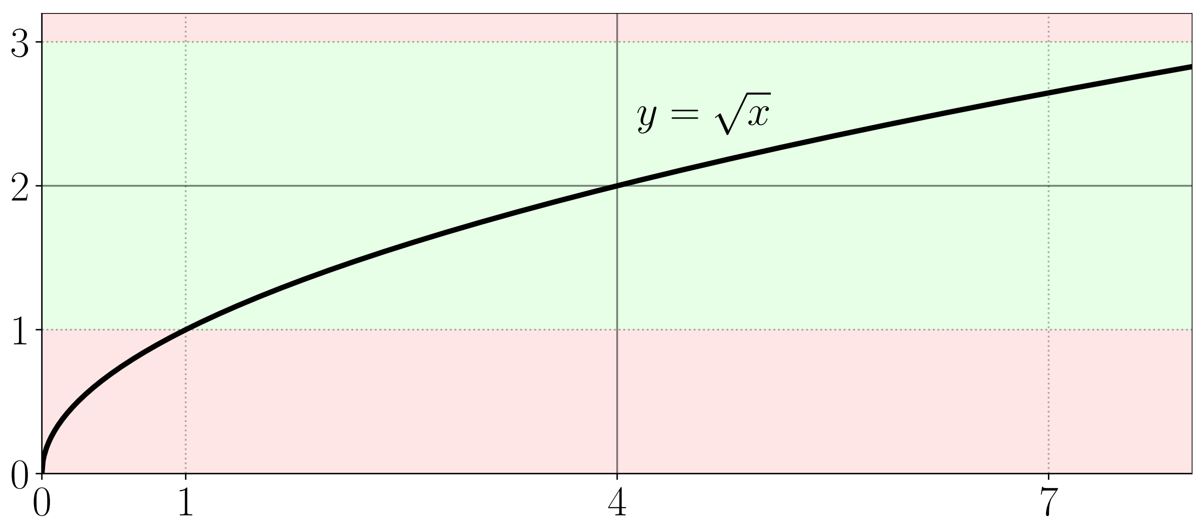
(d) $\lim_{x\to\pi}[[x]] = 3$, where $\epsilon = .01$.

Solution. (a) Observe that

$$|5x - 6 - 9| = 5|x-3| < 1 \iff |x-3| < \tfrac{1}{5}.$$

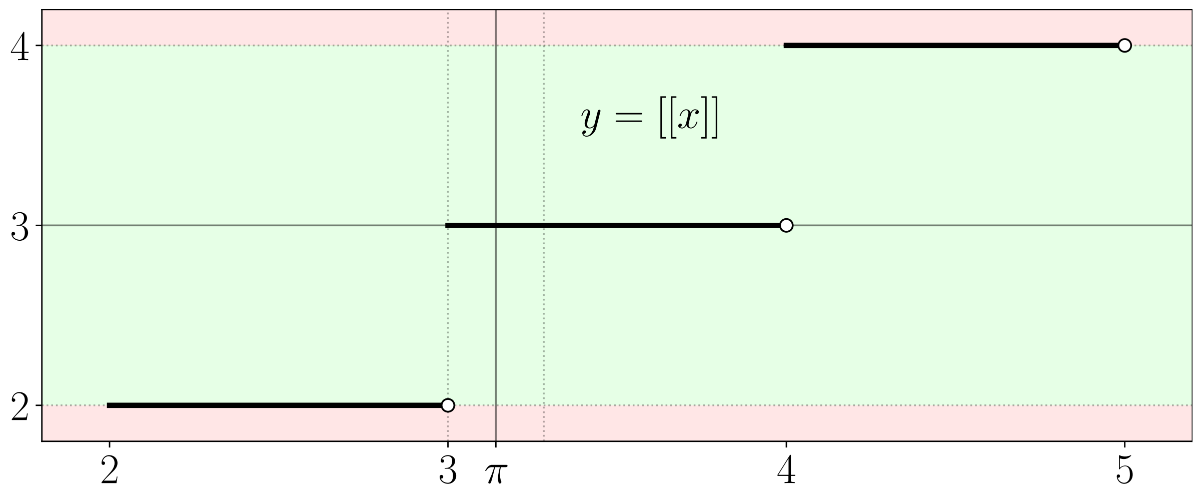
Thus $\delta = \frac{1}{5}$ is the largest possible value we can take.

(b) It is straightforward to verify that $x \in (1,7) = (4-3, 4+3)$ gives us $\sqrt{x} \in (1,3) = (2-1, 2+1)$, so that $\delta = 3$ is a valid response. No larger value of δ will work, since this would give us an $x \in [0,1)$, which implies that $\sqrt{x} \in [0,1) \not\subseteq (1,3)$.

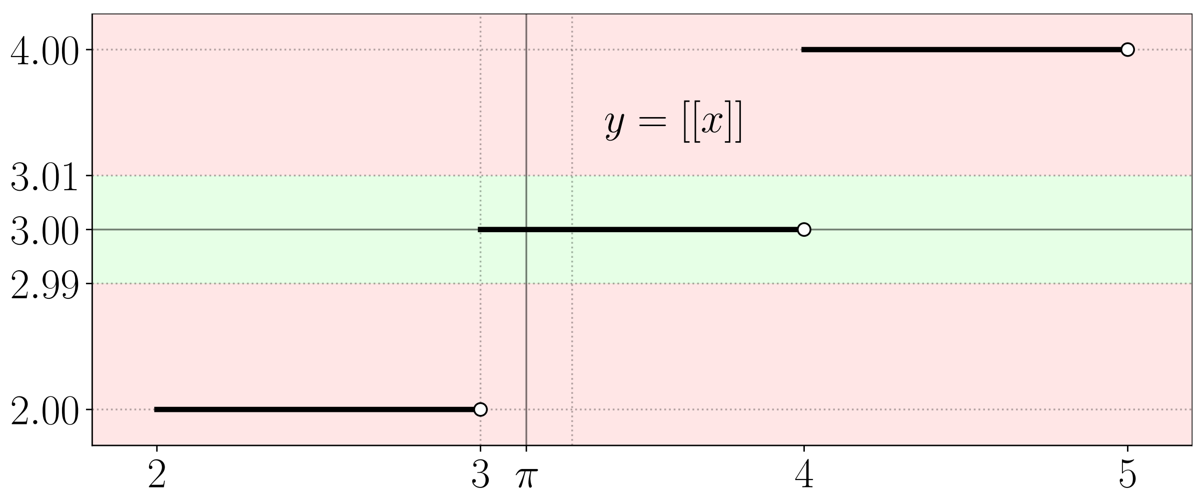


(c) Since $[[x]]$ is always an integer, we have $|[[x]] - 3| < 1$ if and only if $[[x]] = 3$, which is the case if and only if $3 \le x < 4$. So we should choose the largest possible δ such that $V_\delta(\pi) \subseteq [3,4)$, which is

$$\delta = \min\{\pi - 3, 4 - \pi\} = \pi - 3.$$



(d) As in part (c), we have $|[[x]] - 3| < 0.01$ if and only if $[[x]] = 3$, so the largest possible choice is $\delta = \pi - 3$. (The following graph is not to scale.)



Exercise 4.2.3. Review the definition of Thomae's function $t(x)$ from Section 4.1.

(a) Construct three different sequences $(x_n), (y_n)$, and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.

(b) Now, compute $\lim t(x_n), \lim t(y_n)$, and $\lim t(z_n)$.

(c) Make an educated conjecture for $\lim_{x \to 1} t(x)$, and use Definition 4.2.1B to verify the claim. (Given $\epsilon > 0$, consider the set of points $\{x \in \mathbf{R} : t(x) \geq \epsilon\}$. Argue that all the points in this set are isolated.)

Solution. (a) Take

$$x_n = 1 + \frac{1}{n}, \quad y_n = 1 - \frac{1}{n}, \qquad \text{and} \qquad z_n = 1 + \frac{\sqrt{2}}{n}.$$

(b) Since $x_n = \frac{n+1}{n}$, we have $t(x_n) = \frac{1}{n}$ and thus $\lim t(x_n) = 0$. Similarly, $y_n = \frac{n-1}{n}$, so $t(y_1) = t(0) = 1$ and $t(y_n) = \frac{1}{n}$ for $n \geq 2$, so that $\lim t(y_n) = 0$ also. Finally, since $z_n \in \mathbf{I}$ for each $n \in \mathbf{N}$, we have $\lim t(z_n) = 0$.

(c) We conjecture that $\lim_{x\to 1} t(x) = 0$. To see this, first let us prove the following lemma.

Lemma L.12. Suppose $x \in \mathbf{R}$ and $n \in \mathbf{N}$. There exists a $\delta > 0$ such that if $\frac{a}{b} \neq x$ is a rational number contained in $V_\delta(x)$ with $b > 0$, then $b > n$.

Proof. Suppose $b \in \mathbf{N}$ is such that $1 \leq b \leq n$. Since $I := [x-1, x+1]$ is an interval of length 2, there are either $2b$ or $2b+1$ rationals of the form $\frac{a}{b}$ contained in I. (To fit the most rationals inside I, we should place the first such rational $\frac{a}{b}$ on the left endpoint $x-1$; then $\frac{a+2b}{b} = \frac{a}{b} + 2 = x + 1$ is the right endpoint. Thus we have the $2b+1$ rational numbers $\frac{a}{b}, \frac{a+1}{b}, \ldots, \frac{a+2b}{b}$ contained in I. In the general case, the left endpoint will not be of the form $\frac{a}{b}$ and so there will be only $2b$ rationals of this form contained in I.) Given this, the set

$$A = \left\{ \left| x - \frac{a}{b} \right| : \frac{a}{b} \in I, \frac{a}{b} \neq x, 1 \leq b \leq n \right\}$$

is non-empty and finite, so that $\delta := \min A$ exists (Lemma L.3); notice that $\delta > 0$ since each element of A is strictly positive. It follows that $V_\delta(x)$ can contain only rationals $\frac{a}{b}$ with denominators $b > n$, other than possibly x itself. □

Now we can prove that $\lim_{x\to 1} t(x) = 0$. Let $\epsilon > 0$ be given and let $n \in \mathbf{N}$ be such that $\frac{1}{n} < \epsilon$. By Lemma L.12, there exists a $\delta > 0$ such that if $\frac{a}{b} \neq 1$ is a rational number contained in $V_\delta(1)$, then $b > n$. Suppose $x \in V_\delta(1)$. If x is irrational then $t(x) = 0 \in V_\epsilon(0)$, and if $x = \frac{a}{b} \neq 1$ is rational then

$$0 \leq t(x) = \frac{1}{b} < \frac{1}{n} < \epsilon \implies t(x) \in V_\epsilon(0).$$

In either case, $x \in V_\delta(1) \setminus \{1\}$ implies that $t(x) \in V_\epsilon(0)$ and thus $\lim_{x\to 1} t(x) = 0$.

Exercise 4.2.4. Consider the reasonable but erroneous claim that

$$\lim_{x\to 10} 1/[[x]] = 1/10.$$

(a) Find the largest δ that represents a proper response to the challenge of $\epsilon = 1/2$.

(b) Find the largest δ that represents a proper response to $\epsilon = 1/50$.

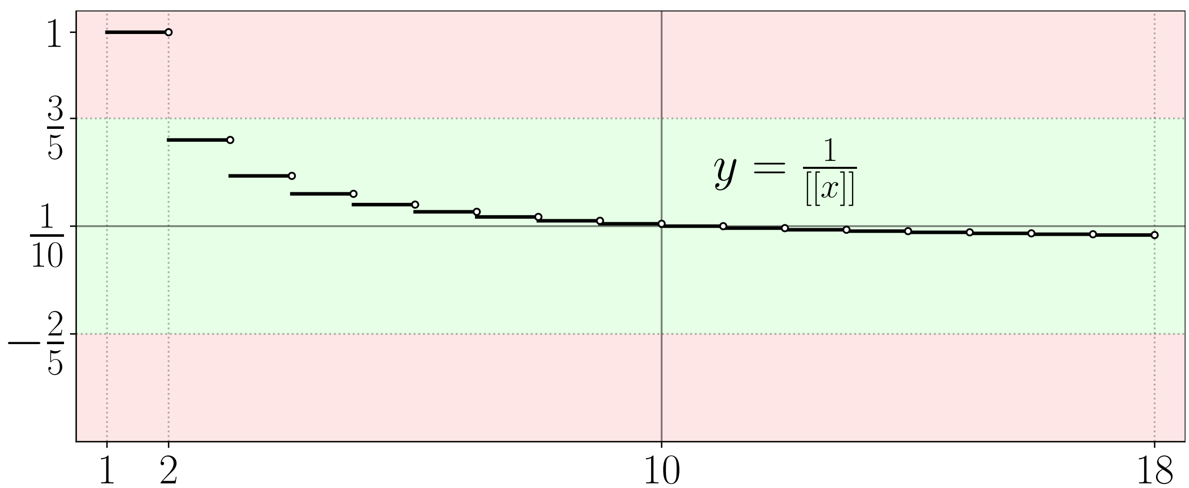
(c) Find the largest ϵ challenge for which there is no suitable δ response possible.

Solution. Let $f(x) = \frac{1}{[[x]]}$, which is defined provided $[[x]] \neq 0$, which is the case if and only if $x < 0$ or $x \geq 1$. Thus the domain of f is $A = (-\infty, 0) \cup [1, \infty)$.

(a) Let $\delta = 8$ and observe that

$$x \in V_\delta(10) = (2, 18) \implies f(x) \in \left[\frac{1}{17}, \frac{1}{2}\right] \subseteq \left(-\frac{2}{5}, \frac{3}{5}\right) = V_{1/2}\left(\frac{1}{10}\right).$$

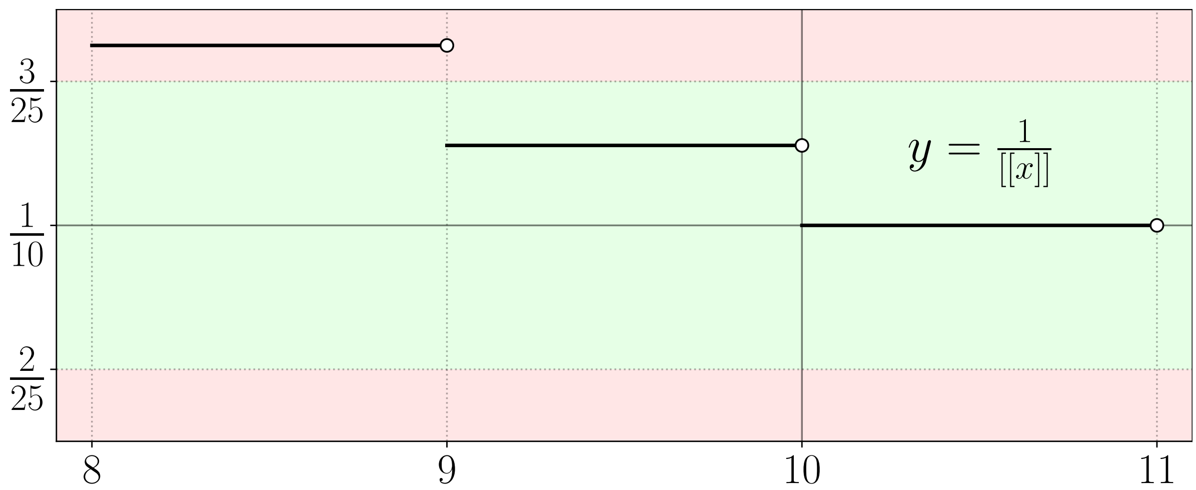
Thus $\delta = 8$ is a valid response to the challenge of $\epsilon = \frac{1}{2}$. If $\delta > 8$, then there exists an $x \in V_\delta(10)$ such that $1 \leq x < 2$, which gives $f(x) = 1 \notin \left(-\frac{2}{5}, \frac{3}{5}\right) = V_{1/2}\left(\frac{1}{10}\right)$. Hence $\delta = 8$ is the largest proper response to the challenge of $\epsilon = \frac{1}{2}$.



(b) Let $\delta = 1$ and observe that

$$x \in V_\delta(10) = (9, 11) \implies f(x) \in \left[\frac{1}{10}, \frac{1}{9}\right] \subseteq \left(\frac{2}{25}, \frac{3}{25}\right) = V_{1/50}\left(\frac{1}{10}\right).$$

Thus $\delta = 1$ is a valid response to the challenge of $\epsilon = \frac{1}{50}$. If $\delta > 1$, then there exists an $x \in V_\delta(10)$ such that $8 \leq x < 9$, which gives $f(x) = \frac{1}{8} \notin \left(\frac{2}{25}, \frac{3}{25}\right) = V_{1/50}\left(\frac{1}{10}\right)$. Hence $\delta = 1$ is the largest proper response to the challenge of $\epsilon = \frac{1}{50}$.



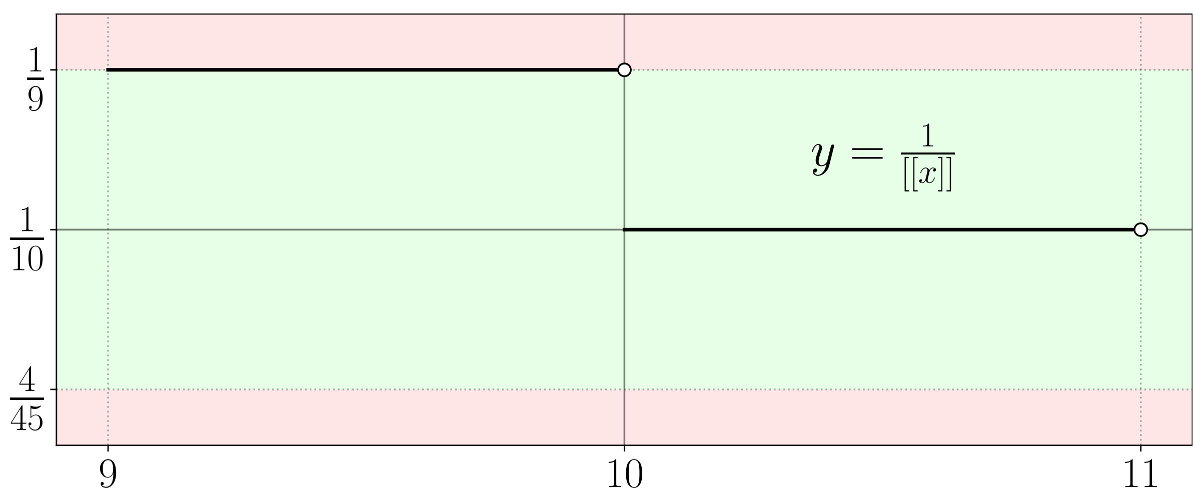
(c) Suppose that $\epsilon = \frac{1}{90}$ and $\delta > 0$. Notice that there exists an $x \in V_\delta(10)$ such that $9 \leq x < 10$, which gives $f(x) = \frac{1}{9} \notin \left(\frac{4}{45}, \frac{1}{9}\right) = V_\epsilon\left(\frac{1}{10}\right)$. Thus there is no valid δ response to the challenge of $\epsilon = \frac{1}{90}$.

Now suppose $\epsilon > \frac{1}{90}$, let $\delta = 1$, and observe that

$$x \in V_\delta(10) = (9, 11) \implies f(x) \in \left[\frac{1}{10}, \frac{1}{9}\right] \subseteq V_\epsilon\left(\frac{1}{10}\right).$$

Thus $\delta = 1$ is a valid response to this ϵ.

We may conclude that $\epsilon = \frac{1}{90}$ is the largest challenge for which there is no suitable δ response possible.



Exercise 4.2.5. Use Definition 4.2.1 to supply a proper proof for the following limit statements.

(a) $\lim_{x\to 2}(3x+4) = 10$.

(b) $\lim_{x\to 0} x^3 = 0$.

(c) $\lim_{x\to 2}(x^2+x-1) = 5$.

(d) $\lim_{x\to 3} 1/x = 1/3$.

Solution. (a) Let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{3}$. If $x \in \mathbf{R}$ is such that $0 < |x-2| < \delta$, then

$$|(3x+4)-10| = |3x-6| = 3|x-2| < 3\delta = \epsilon.$$

Thus $\lim_{x\to 2}(3x+4) = 10$.

(b) Let $\epsilon > 0$ be given and let $\delta = \epsilon^{1/3}$. If $x \in \mathbf{R}$ is such that $0 < |x| < \delta$, then

$$\left|x^3\right| = |x|^3 < \delta^3 = \epsilon.$$

Thus $\lim_{x\to 0} x^3 = 0$.

(c) Let $\epsilon > 0$ be given. Observe that if $|x-2| < 1$, i.e., $x \in (1,3)$, then $x+3 \in (4,7)$. Let $\delta = \min\left\{\frac{\epsilon}{7}, 1\right\}$. If $x \in \mathbf{R}$ is such that $0 < |x-2| < \delta$, then

$$\left|x^2+x-1-5\right| = |x+3||x-2| < 7\delta \le \epsilon.$$

Thus $\lim_{x\to 2}(x^2+x-1) = 5$.

(d) Let $\epsilon > 0$ be given. Observe that if $|x-3| < 1$, i.e., $x \in (2,4)$, then $\frac{1}{3x} \in \left(\frac{1}{12}, \frac{1}{6}\right)$. Let $\delta = \min\{6\epsilon, 1\}$ and note that if $x \in \mathbf{R}\setminus\{0\}$ is such that $0 < |x-3| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|x-3|}{|3x|} < \frac{\delta}{6} \le \epsilon.$$

Thus $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$.

Exercise 4.2.6. Decide if the following claims are true or false, and give short justifications of each conclusion.

(a) If a particular δ has been constructed as a suitable response to a particular ϵ challenge, then any smaller positive δ will also suffice.

(b) If $\lim_{x \to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.

(c) If $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} 3[f(x) - 2]^2 = 3(L - 2)^2$.

(d) If $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f.)

Solution. (a) This is true, since if $0 < \delta' < \delta$ then $V_{\delta'}(c) \subseteq V_\delta(c)$ for any $c \in \mathbf{R}$.

(b) This is false. For a counterexample, consider Thomae's function t. In Exercise 4.2.3 we showed that $\lim_{x \to 1} t(x) = 0$, but $t(1) = 1$.

(c) This is true and follows from several applications of the Algebraic Limit Theorem for Functional Limits (Corollary 4.2.4).

(d) This is false. Define $f, g : \mathbf{R} \setminus \{0\} \to \mathbf{R}$ by $f(x) = x$ and $g(x) = \frac{1}{x}$. It is straightforward to verify that $\lim_{x \to 0} f(x) = 0$, but $\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} 1 = 1$.

Exercise 4.2.7. Let $g : A \to \mathbf{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$.

Show that if $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} g(x)f(x) = 0$ as well.

Solution. Let $\epsilon > 0$ be given. Since $\lim_{x \to c} g(x) = 0$, there is a $\delta > 0$ such that $0 < |x - c| < \delta$ and $x \in A$ implies that $|g(x)| < \frac{\epsilon}{M}$. Observe that for such x we have

$$|f(x)g(x)| = |f(x)||g(x)| < M\frac{\epsilon}{M} = \epsilon.$$

Thus $\lim_{x \to c} g(x)f(x) = 0$.

Exercise 4.2.8. Compute each limit or state that it does not exist. Use the tools developed in this section to justify each conclusion.

(a) $\lim_{x \to 2} \frac{|x-2|}{x-2}$

(b) $\lim_{x \to 7/4} \frac{|x-2|}{x-2}$

(c) $\lim_{x \to 0} (-1)^{[[1/x]]}$

(d) $\lim_{x \to 0} \sqrt[3]{x}(-1)^{[[1/x]]}$

Solution. (a) Let $f : \mathbf{R} \setminus \{2\} \to \mathbf{R}$ be given by $f(x) = \frac{|x-2|}{x-2}$. Observe that

$$f(x) = \begin{cases} 1 & \text{if } x > 2, \\ -1 & \text{if } x < 2. \end{cases}$$

We claim that $\lim_{x \to 2} f(x)$ does not exist. To see this, consider the sequences (x_n) and (y_n) given by $x_n = 2 + \frac{1}{n}$ and $y_n = 2 - \frac{1}{n}$, which satisfy $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 2$. However,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1 \neq -1 = \lim_{n\to\infty} -1 = \lim_{n\to\infty} f(y_n).$$

Our claim now follows from Corollary 4.2.5.

(b) Define f as in part (a). We claim that $\lim_{x\to 7/4} f(x) = -1$. To see this, let $\epsilon > 0$ be given. If $x \in \mathbf{R} \setminus \{2\}$ is such that $0 < \left|x - \frac{7}{4}\right| < \frac{1}{4}$, i.e., $x \in \left(\frac{3}{2}, 2\right)$, then

$$|f(x) - (-1)| = |-1 + 1| = 0 < \epsilon.$$

Thus $\lim_{x\to 7/4} f(x) = -1$.

(c) We claim that $\lim_{x\to 0}(-1)^{[[1/x]]}$ does not exist. To see this, consider the sequences (x_n) and (y_n) given by $x_n = \frac{1}{2n}$ and $y_n = \frac{1}{2n+1}$, which satisfy $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. However,

$$\begin{aligned} \lim_{n\to\infty} (-1)^{[[1/x_n]]} = \lim_{n\to\infty} (-1)^{[[2n]]} &= \lim_{n\to\infty} 1 = 1 \\ &\neq -1 = \lim_{n\to\infty} -1 = \lim_{n\to\infty} (-1)^{[[2n+1]]} = \lim_{n\to\infty} (-1)^{[[1/y_n]]}. \end{aligned}$$

Our claim now follows from Corollary 4.2.5.

(d) Let $\epsilon > 0$ be given and let $\delta = \epsilon^3$. If $x \in \mathbf{R}$ is such that $0 < |x| < \delta$, then

$$\left|\sqrt[3]{x}\right| = \sqrt[3]{|x|} < \sqrt[3]{\delta} = \epsilon.$$

Thus $\lim_{x\to 0} \sqrt[3]{x} = 0$. Since the function $(-1)^{[[1/x]]}$ is evidently bounded, we may apply Exercise 4.2.7 to conclude that $\lim_{x\to 0} \sqrt[3]{x}(-1)^{[[1/x]]} = 0$.

Exercise 4.2.9 (Infinite Limits). The statement $\lim_{x\to 0} 1/x^2 = \infty$ certainly makes intuitive sense. To construct a rigorous definition in the challenge-response style of Definition 4.2.1 for an infinite limit statement of this form, we replace the (arbitrarily small) $\epsilon > 0$ challenge with an (arbitrarily large) $M > 0$ challenge:

Definition: $\lim_{x\to c} f(x) = \infty$ means that for all $M > 0$ we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.

(a) Show $\lim_{x\to 0} 1/x^2 = \infty$ in the sense described in the previous definition.

(b) Now, construct a definition for the statement $\lim_{x\to\infty} f(x) = L$. Show $\lim_{x\to\infty} 1/x = 0$.

(c) What would a rigorous definition for $\lim_{x\to\infty} f(x) = \infty$ look like? Give an example of such a limit.

Solution. (a) Let $M > 0$ be given and let $\delta = \frac{1}{\sqrt{M}} > 0$. If $x \in \mathbf{R}$ is such that $0 < |x| < \delta$, then observe that

$$\frac{1}{|x|} > \sqrt{M} > 0 \implies \frac{1}{x^2} > M.$$

It follows that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

(b) The statement $\lim_{x\to\infty} f(x) = L$ means that for all $\epsilon > 0$ we can find an $M > 0$ such that whenever $x > M$, it follows that $|f(x) - L| < \epsilon$.

To see that $\lim_{x\to\infty} \frac{1}{x} = 0$, let $\epsilon > 0$ be given, let $M = \frac{1}{\epsilon}$, and observe that

$$x > M = \frac{1}{\epsilon} \implies \frac{1}{x} < \epsilon.$$

(c) The statement $\lim_{x\to\infty} f(x) = \infty$ means that for all $M > 0$ we can find a $K > 0$ such that whenever $x > K$, it follows that $f(x) > M$; it is straightforward to verify that $\lim_{x\to\infty} x = \infty$, for example.

Exercise 4.2.10 (Right and Left Limits). Introductory calculus courses typically refer to the *right-hand limit* of a function as the limit obtained by "letting x approach a from the right-hand side."

(a) Give a proper definition in the style of Definition 4.2.1 for the right-hand and left-hand limit statements:

$$\lim_{x\to a^+} f(x) = L \qquad \text{and} \qquad \lim_{x\to a^-} f(x) = M.$$

(b) Prove that $\lim_{x\to a} f(x) = L$ if and only if both the right and left-hand limits equal L.

Solution. (a) Suppose we have a function $f : A \to \mathbf{R}$ and $a \in \mathbf{R}$ is a limit point of $A \cap (a, \infty)$. We say that $\lim_{x\to a^+} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $a < x < a + \delta$ and $x \in A$, it follows that $|f(x) - L| < \epsilon$. Similarly, if $a \in \mathbf{R}$ is a limit point of $A \cap (-\infty, a)$, we say that $\lim_{x\to a^-} f(x) = M$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $a - \delta < x < a$ and $x \in A$, it follows that $|f(x) - M| < \epsilon$.

(b) If $\lim_{x\to a} f(x) = L$, then certainly $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$, since both of the statements $a < x < a + \delta$ and $a - \delta < x < a$ imply that $0 < |x - a| < \delta$. Suppose therefore that $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ and let $\epsilon > 0$ be given. There are positive real numbers δ_1 and δ_2 such that

$$a < x < a + \delta_1 \implies |f(x) - L| < \epsilon \qquad \text{and} \qquad a - \delta_2 < x < a \implies |f(x) - L| < \epsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $x \in \mathbf{R}$ is such that $0 < |x - a| < \delta$, then either

$$a < x < a + \delta < a + \delta_1 \implies |f(x) - L| < \epsilon, \text{ or}$$

$$a - \delta_2 < a - \delta < x < a \implies |f(x) - L| < \epsilon.$$

In either case we have $|f(x) - L| < \epsilon$ and hence $\lim_{x\to a} f(x) = L$.

Exercise 4.2.11 (Squeeze Theorem). Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A, show $\lim_{x\to c} g(x) = L$.

Solution. Suppose (x_n) is a sequence contained in A satisfying $x_n \neq c$ and $\lim_{n\to\infty} x_n = c$. By assumption, we have $f(x_n) \leq g(x_n) \leq h(x_n)$ for all $n \in \mathbf{N}$, and Theorem 4.2.3 guarantees that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} h(x_n) = L$. We may now apply the Squeeze Theorem for sequences (Exercise 2.3.3) to see that $\lim_{n\to\infty} g(x_n) = L$, and Theorem 4.2.3 then allows us to conclude that $\lim_{x\to c} g(x) = L$.

4.3 Continuous Functions

Exercise 4.3.1. Let $g(x) = \sqrt[3]{x}$.

(a) Prove that g is continuous at $c = 0$.

(b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ will be helpful.)

Solution. (a) Let $\epsilon > 0$ be given and let $\delta = \epsilon^3$. If we take $x \in \mathbf{R}$ such that $|x| < \delta$, then we will have

$$\left|\sqrt[3]{x}\right| = \sqrt[3]{|x|} < \sqrt[3]{\delta} = \epsilon.$$

Thus g is continuous at $c = 0$.

(b) Taking $a = x^{1/3}$ and $b = c^{1/3}$ in the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ gives

$$x - c = \left(x^{1/3} - c^{1/3}\right)\left(x^{2/3} + (xc)^{1/3} + c^{2/3}\right)$$
$$\implies |x-c| = \left|x^{1/3} - c^{1/3}\right|\left|x^{2/3} + (xc)^{1/3} + c^{2/3}\right|.$$

If we take x close enough to c so that x and c have the same sign, i.e., take x such that $|x-c| < |c|$, then $xc > 0$ and so

$$\left|x^{2/3} + (xc)^{1/3} + c^{2/3}\right| = x^{2/3} + (xc)^{1/3} + c^{2/3} \geq c^{2/3}.$$

Let $\delta = \min\{|c|, c^{2/3}\epsilon\}$ and suppose $x \in \mathbf{R}$ is such that $|x-c| < \delta$. By the previous discussion, we then have

$$\left|x^{1/3} - c^{1/3}\right| \leq \frac{|x-c|}{c^{2/3}} < \frac{\delta}{c^{2/3}} < \epsilon.$$

Thus g is continuous at c.

Exercise 4.3.2. To gain a deeper understanding of the relationship between ϵ and δ in the definition of continuity, let's explore some modest variations of Definition 4.3.1. In all of these, let f be a function defined on all of $\mathbf{R}$.

(a) Let's say f is *onetinuous* at c if for all $\epsilon > 0$ we can choose $\delta = 1$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x-c| < \delta$. Find an example of a function that is onetinuous on all of $\mathbf{R}$.

(b) Let's say f is *equaltinuous* at c if for all $\epsilon > 0$ we can choose $\delta = \epsilon$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x-c| < \delta$. Find an example of a function that is equaltinuous on $\mathbf{R}$ that is nowhere onetinuous, or explain why there is no such function.

(c) Let's say f is *lesstinuous* at c if for all $\epsilon > 0$ we can choose $0 < \delta < \epsilon$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x-c| < \delta$. Find an example of a function that is lesstinuous on $\mathbf{R}$ that is nowhere equaltinuous, or explain why there is no such function.

(d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

Solution. (a) Let f be given by $f(x) = 0$ for all $x \in \mathbf{R}$. Fix $c \in \mathbf{R}$ and let $\epsilon > 0$ be given. If $x \in \mathbf{R}$ is such that $|x-c| < 1$, then

$$|f(x) - f(c)| = |0 - 0| = 0 < \epsilon.$$

Thus f is onetinuous on $\mathbf{R}$.

(b) Let f be given by $f(x) = x$ for all $x \in \mathbf{R}$. Fix $c \in \mathbf{R}$ and let $\epsilon > 0$ be given. If $x \in \mathbf{R}$ is such that $|x - c| < \epsilon$, then
$$|f(x) - f(c)| = |x - c| < \epsilon.$$
Thus f is equaltinuous on $\mathbf{R}$. However, f is nowhere onetinuous. Fix $c \in \mathbf{R}$ again and consider $\epsilon = \frac{1}{4}$. Note that $x = c + \frac{1}{2}$ satisfies $|x - c| = \left|c + \frac{1}{2} - c\right| = \frac{1}{2} < 1$, however
$$|f(x) - f(c)| = |x - c| = \frac{1}{2} > \frac{1}{4} = \epsilon.$$
Thus f is nowhere onetinuous.

(c) Let f be given by $f(x) = 2x$ for all $x \in \mathbf{R}$. Fix $c \in \mathbf{R}$, let $\epsilon > 0$ be given, and let $\delta = \frac{\epsilon}{2} < \epsilon$. If $x \in \mathbf{R}$ is such that $|x - c| < \delta$, then
$$|f(x) - f(c)| = 2|x - c| < 2\delta = \epsilon.$$
Thus f is lesstinuous on $\mathbf{R}$. However, f is nowhere equaltinuous. Fix $c \in \mathbf{R}$ again and let $\epsilon = 1$. Note that $x = c + \frac{3}{4}$ satisfies $|x - c| = \frac{3}{4} < \epsilon$, however
$$|f(x) - f(c)| = 2|x - c| = \frac{3}{2} > \epsilon.$$
Thus f is nowhere equaltinuous.

(d) It is clear that every lesstinuous function is continuous. We claim that every continuous function is lesstinuous. To see this, let f be a continuous function. Fix $c \in \mathbf{R}$ and $\epsilon > 0$. Since f is continuous at c, there is a $\delta' > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta'$. Let $\delta = \min\{\delta', \frac{\epsilon}{2}\}$, so that $0 < \delta < \epsilon$, and observe that if $x \in \mathbf{R}$ is such that $|x - c| < \delta$ then x also satisfies $|x - c| < \delta'$, whence $|f(x) - f(c)| < \epsilon$.

Exercise 4.3.3. (a) Supply a proof for Theorem 4.3.9 using the ϵ-δ characterization of continuity.

(b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2 (iii)).

Solution. (a) Let $a \in A$ and $\epsilon > 0$ be given. By assumption we have $f(a) \in B$, so g is continuous at $f(a)$. There then exists a $\delta_1 > 0$ such that
$$|y - f(a)| < \delta_1 \text{ and } y \in B \implies |g(y) - g(f(a))| < \epsilon. \tag{4.1}$$

Since f is continuous at a, there exists a $\delta_2 > 0$ such that

$$|x - a| < \delta_2 \text{ and } x \in A \implies |f(x) - f(a)| < \delta_1. \tag{4.2}$$

Combining (1) and (2), we have

$$\begin{aligned} |x - a| < \delta_2 \text{ and } x \in A &\implies |f(x) - f(a)| < \delta_1 \text{ and } f(x) \in B \\ &\implies |g(f(x)) - g(f(a))| < \epsilon. \end{aligned}$$

Thus $g \circ f$ is continuous at a.

(b) Let $a \in A$ be given and suppose $(a_n)_{n=1}^{\infty}$ is contained in A and satisfies $\lim_{n\to\infty} a_n = a$. Since f is continuous at a, Theorem 4.3.2 (iii) gives us $\lim_{n\to\infty} f(a_n) = f(a)$. By assumption g is continuous at $f(a) \in B$ and $(f(a_n))_{n=1}^{\infty}$ is contained in B, so Theorem 4.3.2 (iii) gives us $\lim_{n\to\infty} g(f(a_n)) = g(f(a))$. One more application of Theorem 4.3.2 (iii) allows us to conclude that $g \circ f$ is continuous at a.

Exercise 4.3.4. Assume f and g are defined on all of $\mathbf{R}$ and that $\lim_{x\to p} f(x) = q$ and $\lim_{x\to q} g(x) = r$.

(a) Give an example to show that it may not be true that

$$\lim_{x\to p} g(f(x)) = r.$$

(b) Show that the result in (a) does follow if we assume f and g are continuous.

(c) Does the result in (a) hold if we only assume f is continuous? How about if we only assume that g is continuous?

Solution. (a) Let f be given by $f(x) = 0$ for all $x \in \mathbf{R}$ and let g be given by

$$g(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

We have $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, however note that $g(f(x)) = g(0) = 1$ for all $x \in \mathbf{R}$. It follows that

$$\lim_{x\to 0} g(f(x)) = 1 \neq 0.$$

(b) By Theorem 4.3.9, the composition $g \circ f$ is continuous. Since f and g are defined on all of $\mathbf{R}$, Theorem 4.3.2 (iv) lets us write

$$\lim_{x \to p} g(f(x)) = g(f(p)) = g\left(\lim_{x \to p} f(x)\right) = g(q) = \lim_{x \to q} g(x).$$

(c) As the counterexample in part (a) shows, the result does not hold if we only assume that f is continuous. However, it does hold if we assume that g is continuous. To see this, let (x_n) be some sequence satisfying $\lim_{n\to\infty} x_n = p$ and $x_n \neq p$. Theorem 4.2.3 shows that $\lim f(x_n) = q$, and since g is continuous the sequential characterization of continuity (Theorem 4.3.2 (iii)) implies that

$$\lim g(f(x_n)) = g(q) = r,$$

where the last equality also follows from the continuity of g. Theorem 4.2.3 allows us to conclude that

$$\lim_{x \to p} g(f(x)) = r.$$

Exercise 4.3.5. Show using Definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbf{R}$, then $f : A \to \mathbf{R}$ is continuous at c.

Solution. Since c is an isolated point of A, there exists a $\delta > 0$ such that $(c-\delta, c+\delta) \cap A = \{c\}$. Let $\epsilon > 0$ be given. If $x \in A$ is such that $|x - c| < \delta$, then it must be the case that $x = c$, which gives us

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon.$$

Thus f is continuous at c.

Exercise 4.3.6. Provide an example of each or explain why the request is impossible.

(a) Two functions f and g, neither of which is continuous at 0 such that $f(x)g(x)$ and $f(x)+g(x)$ are continuous at 0.

(b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0.

(c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

(d) A function $f(x)$ not continuous at 0 such that $f(x) + \frac{1}{f(x)}$ is continuous at 0.

(e) A function $f(x)$ not continuous at 0 such that $[f(x)]^3$ is continuous at 0.

Solution. (a) Let $f, g : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Neither f nor g is continuous at 0, however note that for all $x \in \mathbf{R}$ we have

$$f(x)g(x) = 0 \qquad \text{and} \qquad f(x) + g(x) = 1.$$

Thus fg and $f + g$ are continuous at 0.

(b) This is impossible. If f and $f + g$ are continuous at 0 then Theorem 4.3.4 implies that $g = f + g - f$ is continuous at 0.

(c) If we take g as in part (a) and let $f(x) = 0$ for all $x \in \mathbf{R}$, then g is not continuous at 0 but $f = fg$ is continuous at 0.

(d) Let $f : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} \sqrt{2} - 1 & \text{if } x \neq 0, \\ \sqrt{2} + 1 & \text{if } x = 0, \end{cases}$$

and note that f is discontinuous at 0. Note further that $f(x) + \frac{1}{f(x)} = 2\sqrt{2}$ for all $x \in \mathbf{R}$; it follows that $f + \frac{1}{f}$ is continuous at 0.

(e) This is impossible. As we showed in Exercise 4.3.1, the function $g(x) = \sqrt[3]{x}$ is continuous everywhere. Thus if $[f(x)]^3$ is continuous at 0, then by Theorem 4.3.9 the composition

$$f(x) = \sqrt[3]{[f(x)]^3}$$

must also be continuous at 0.

Exercise 4.3.7. (a) Referring to the proper theorems, give a formal argument that Dirichlet's function from Section 4.1 is nowhere-continuous on $\mathbf{R}$.

(b) Review the definition of Thomae's function in Section 4.1 and demonstrate that it fails to be continuous at every rational point.

(c) Use the characterization of continuity in Theorem 4.3.2 (iii) to show that Thomae's function is continuous at every irrational point in $\mathbf{R}$. (Given $\epsilon > 0$, consider the set of points $\{x \in \mathbf{R} : t(x) \geq \epsilon\}$.)

Solution. (a) Let $g : \mathbf{R} \to \mathbf{R}$ be Dirichlet's function, i.e.,

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Suppose $c \in \mathbf{Q}$. By the density of $\mathbf{I}$ in $\mathbf{R}$, for any $\delta > 0$ there is an irrational number $x \in \mathbf{I}$ such that $x \in V_\delta(c)$; it follows that $g(x) = 0 \notin V_1(1) = V_1(g(c))$. Thus by (the negation of) Theorem 4.3.2 (ii), g is not continuous at c.

Similarly, suppose $c \in \mathbf{I}$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, for any $\delta > 0$ there is a rational number $x \in \mathbf{Q}$ such that $x \in V_\delta(c)$; it follows that $g(x) = 1 \notin V_1(0) = V_1(g(c))$. Thus by (the negation of) Theorem 4.3.2 (ii), g is not continuous at c.

We have now shown that g fails to be continuous at each $c \in \mathbf{R}$, i.e., g is nowhere-continuous on $\mathbf{R}$.

(b) Let $t : \mathbf{R} \to \mathbf{R}$ be Thomae's function, i.e.,

$$t(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbf{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Suppose $c \in \mathbf{Q}$. The density of $\mathbf{I}$ in $\mathbf{R}$ allows us to pick a sequence of irrational numbers (x_n) such that $\lim_{n\to\infty} x_n = c$. We then have $t(x_n) = 0$ for each $n \in \mathbf{N}$ and so $\lim_{n\to\infty} t(x_n) = 0$. However, $t(c)$ is strictly positive; it follows that $\lim_{n\to\infty} t(x_n) \neq t(c)$ and so Corollary 4.3.3 allows us to conclude that t is not continuous at $c \in \mathbf{Q}$. Thus t fails to be continuous on $\mathbf{Q}$.

(c) Suppose $c \in \mathbf{I}$ and suppose we have a sequence (x_n) such that $\lim_{n\to\infty} x_n = c$. Our aim is to show that $\lim_{n\to\infty} t(x_n) = t(c) = 0$. Let $\epsilon > 0$ be given and choose $K \in \mathbf{N}$ such that $\frac{1}{K} < \epsilon$. By Lemma L.12, there exists a $\delta > 0$ such that if $y = \frac{a}{b}$ is a rational number contained in $V_\delta(c)$ with $b > 0$, then $b > K$. For such a y, we then have $t(y) = \frac{1}{b} < \frac{1}{K} < \epsilon$. Since $\lim_{n\to\infty} x_n = c$, there is an $N \in \mathbf{N}$ such that $x_n \in V_\delta(c)$ for all $n \geq N$. Suppose $n \in \mathbf{N}$ satisfies $n \geq N$. There are two cases.

Case 1. If $x_n \in \mathbf{I}$, then $|t(x_n)| = 0 < \epsilon$.

Case 2. If $x_n \in \mathbf{Q}$, then since $x_n \in V_\delta(c)$ we have $|t(x_n)| < \frac{1}{K} < \epsilon$.

In either case we have $|t(x_n)| < \epsilon$ and thus $\lim_{n\to\infty} t(x_n) = t(c) = 0$ as desired. Theorem 4.3.2 (iii) allows us to conclude that t is continuous at $c \in \mathbf{I}$ and hence that t is continuous on $\mathbf{I}$.

Exercise 4.3.8. Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbf{R}$.

(a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.

(b) If $g(r) = 0$ for all $r \in \mathbf{Q}$, then $g(x) = 0$ for all $x \in \mathbf{R}$.

(c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbf{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

Solution. (a) This is true. Let (x_n) be the sequence given by $x_n = 1 - \frac{1}{n}$. Since g is continuous at 1 and $\lim_{n\to\infty} x_n = 1$, Theorem 4.3.2 (iii) implies that $\lim_{n\to\infty} g(x_n) = g(1)$. Note that $x_n < 1$ for each $n \in \mathbf{N}$, so that $g(x_n) \geq 0$ for each $n \in \mathbf{N}$. The Order Limit Theorem (Theorem 2.3.4) allows us to conclude that $\lim_{n\to\infty} g(x_n) = g(1) \geq 0$ also.

(b) This is true. Let $x \in \mathbf{R}$ be given. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there is a sequence (r_n) of rational numbers such that $\lim_{n\to\infty} r_n = x$. On the one hand, by the continuity of g at x, we must have $\lim_{n\to\infty} g(r_n) = g(x)$ (Theorem 4.3.2 (iii)). On the other hand, $g(r_n) = 0$ for all $n \in \mathbf{N}$ and thus $\lim_{n\to\infty} g(r_n) = 0$. Since the limit of a sequence is unique (Theorem 2.2.7), we see that $g(x) = 0$.

(c) This is true. Since g is continuous at x_0, for $\epsilon = g(x_0) > 0$ there is a $\delta > 0$ such that $g(x) \in V_\epsilon(g(x_0)) = (0, 2g(x_0))$ whenever $x \in V_\delta(x_0)$. In other words, for each of the uncountably many $x \in (x_0 - \delta, x_0 + \delta)$, we have $g(x) > 0$.

Exercise 4.3.9. Assume $h : \mathbf{R} \to \mathbf{R}$ is continuous on $\mathbf{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Solution. Suppose that (x_n) is a convergent sequence contained in K with $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbf{R}$; the continuity of h implies that $\lim_{n\to\infty} h(x_n) = h(x)$. Since each $x_n \in K$, we have $h(x_n) = 0$ for each $n \in \mathbf{N}$ and thus $\lim_{n\to\infty} h(x_n) = 0$. The uniqueness of the limit of a sequence (Theorem 2.2.7) now implies that $h(x) = 0$, i.e., $x \in K$. Theorem 3.2.8 allows us to conclude that K is closed.

Exercise 4.3.10. Observe that if a and b are real numbers, then

$$\max\{a, b\} = \frac{1}{2}[(a + b) + |a - b|].$$

(a) Show that if $f_1, f_2, \ldots, f_n$ are continuous functions, then

$$g(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is a continuous function.

(b) Let's explore whether the result in (a) extends to the infinite case. For each $n \in \mathbf{N}$, define f_n on $\mathbf{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \geq 1/n \\ n|x| & \text{if } |x| < 1/n. \end{cases}$$

Now explicitly compute $h(x) = \sup\{f_1(x), f_2(x), f_3(x), \ldots\}$.

Solution. (a) First, let us show that the function $x \mapsto |x|$ is continuous. If $y \in \mathbf{R}$ and $\epsilon > 0$, let $\delta = \epsilon$ and suppose that $|x - y| < \delta$. By the reverse triangle inequality (Exercise 1.2.6 (d)), we have

$$||x| - |y|| \leq |x - y| < \delta = \epsilon.$$

It follows that $x \mapsto |x|$ is continuous on $\mathbf{R}$.

Now suppose that $f_1, f_2 : A \to \mathbf{R}$ are two continuous functions defined on some domain $A \subseteq \mathbf{R}$. For any $x \in A$, note that

$$g(x) = \max\{f_1(x), f_2(x)\} = \frac{1}{2}[(f_1(x) + f_2(x)) + |f_1(x) - f_2(x)|].$$

Since f_1 and f_2 are continuous on A, and we showed that $x \mapsto |x|$ is continuous everywhere, Theorem 4.3.9 and several applications of Theorem 4.3.4 show that g is also continuous on A.

Using the observation that

$$\max\{f_1(x), f_2(x), \ldots, f_n(x)\} = \max\{\max\{f_1(x), f_2(x), \ldots, f_{n-1}(x)\}, f_n(x)\},$$

a straightforward induction argument on n (the base case was handled in the previous paragraph) shows that the maximum of n continuous functions is a continuous function.

(b) If $x = 0$, then for each $n \in \mathbf{N}$ we have $f_n(0) = 0$ and thus $h(0) = 0$. If $x \neq 0$, then choose $n \in \mathbf{N}$ such that $\frac{1}{n} < |x|$. It follows that $f_n(x) = 1$ and thus $h(x) = 1$. So h is the function

$$h(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is not continuous at 0.

Exercise 4.3.11 (Contraction Mapping Theorem). Let f be a function defined on all of $\mathbf{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbf{R}$.

(a) Show that f is continuous on $\mathbf{R}$.

(b) Pick some point $y_1 \in \mathbf{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

(c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

(d) Finally, prove that if x is *any* arbitrary point in $\mathbf{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y defined by (b).

Solution. (a) Let $y \in \mathbf{R}$ and $\epsilon > 0$ be given. Let $\delta = \frac{\epsilon}{c}$, suppose that $x \in \mathbf{R}$ is such that $|x - y| < \delta$, and observe that

$$|f(x) - f(y)| \leq c|x - y| < c\delta = \epsilon.$$

Thus f is continuous at each $y \in \mathbf{R}$.

(b) Suppose $n > m$ are positive integers. Repeatedly applying the triangle inequality yields

$$|y_n - y_m| \le |y_n - y_{n-1}| + \cdots + |y_{m+1} - y_m| = \sum_{k=m}^{n-1} |y_{k+1} - y_k|.$$

Now we use the hypothesis that $|f(x) - f(y)| \le c|x - y|$ for all $x, y \in \mathbf{R}$ and the definition of the sequence $y_n = f(y_{n-1})$ to see that

$$\sum_{k=m}^{n-1} |y_{k+1} - y_k| \le \sum_{k=m}^{n-1} c|y_k - y_{k-1}| \le \cdots \le \sum_{k=m}^{n-1} c^{k-1}|y_2 - y_1| = c^{-2}|y_2 - y_1| \sum_{k=m+1}^{n} c^k.$$

If we let $s_n = \sum_{k=0}^{n} c^k$, then we have shown that for all positive integers $n > m$ we have the inequality

$$|y_n - y_m| \le c^{-2}|y_2 - y_1|(s_n - s_m). \tag{1}$$

Let $\epsilon > 0$ be given. The series $\sum_{k=0}^{\infty} c^k$ is convergent since $0 < c < 1$, so the sequence (s_n) is Cauchy. There then exists an $N \in \mathbf{N}$ such that

$$n > m \ge N \implies |s_n - s_m| = s_n - s_m < \frac{c^2}{|y_2 - y_1| + 1}\epsilon. \tag{2}$$

Suppose n, m are positive integers such that $n > m \ge N$. By (1) and (2) we then have

$$|y_n - y_m| \le c^{-2}|y_2 - y_1| \frac{c^2}{|y_2 - y_1| + 1}\epsilon < \epsilon.$$

Thus (y_n) is a Cauchy sequence.

(c) Since f is continuous at y (part (a)), we have $\lim_{n\to\infty} f(y_n) = f(y)$. It follows that

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} f(y_n) = f(y).$$

For uniqueness, observe that for any $x \in \mathbf{R}$ such that $x = f(x)$ we have

$$|x - y| = |f(x) - f(y)| \le c|x - y|.$$

If $|x - y|$ were not zero, this would imply that $c \ge 1$. Since $0 < c < 1$, it must be the case that $|x - y| = 0$, i.e., $x = y$.

(d) Let $x_1 = x$ and $x_{n+1} = f(x_n)$. As we just proved, (x_n) converges to some $y' \in \mathbf{R}$ such that $f(y') = y'$. The uniqueness part of (c) then implies that $y' = y$.

Exercise 4.3.12. Let $F \subseteq \mathbf{R}$ be a nonempty closed set and define $g(x) = \inf\{|x - a| : a \in F\}$. Show that g is continuous on all of $\mathbf{R}$ and $g(x) \neq 0$ for all $x \notin F$.

Solution. If A and B are non-empty and bounded below subsets of $\mathbf{R}$ such that $a \leq b$ for all $a \in A$ and $b \in B$, then it is straightforward to verify that $\inf A \leq \inf B$.

Fix $c \in \mathbf{R}$ and note that for any $x \in \mathbf{R}$ and $a \in F$, we have $|x - a| \leq |x - c| + |c - a|$. By the previous paragraph, this implies that

$$\inf\{|x - a| : a \in F\} \leq \inf\{|x - c| + |c - a| : a \in F\}.$$

A statement analogous to Example 1.3.7 for infima then gives us

$$\inf\{|x - a| : a \in F\} \leq |x - c| + \inf\{|c - a| : a \in F\},$$

or equivalently $g(x) - g(c) \leq |x - c|$. We can similarly derive $g(c) - g(x) \leq |x - c|$ and hence

$$|g(x) - g(c)| \leq |x - c|.$$

Thus for any $\epsilon > 0$ we can take $\delta = \epsilon$ and obtain

$$|x - c| < \delta \implies |g(x) - g(c)| < \epsilon.$$

It follows that g is continuous at each $c \in \mathbf{R}$.

Suppose that $g(x) = 0$. Using Exercise 1.3.1 (b), we can choose a sequence (a_n) contained in F and satisfying $\lim_{n\to\infty}|x - a_n| = g(x) = 0$, which is equivalent to $\lim_{n\to\infty} a_n = x$. Since F is closed, Theorem 3.2.8 then implies that $x \in F$. Thus if $x \notin F$, it must be the case that $g(x) \neq 0$.

Exercise 4.3.13. Let f be a function defined on all of $\mathbf{R}$ that satisfies the additive condition $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.

(b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbf{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbf{Z}$. Now, prove that $f(r) = kr$ for any rational number r.

(c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbf{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbf{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Solution. (a) We have $f(0) = f(0+0) = f(0) + f(0)$ and so $f(0) = 0$. Furthermore, for any $x \in \mathbf{R}$,

$$0 = f(0) = f(x - x) = f(x) + f(-x) \implies f(-x) = -f(x).$$

(b) We will show that $f(n) = kn$ for all $n \in \mathbf{N}$ by induction on n. The base case is clear, so suppose that $f(n) = kn$ for some $n \in \mathbf{N}$ and observe that

$$f(n+1) = f(n) + f(1) = kn + k = k(n+1).$$

This completes the induction step and the proof.

Combining the identity $f(n) = kn$ with $f(-x) = -f(x)$ from part (a), we see that $f(z) = kz$ for all $z \in \mathbf{Z}$.

Now suppose that $r = \frac{m}{n}$ is a rational number. On one hand, using what we just proved,

$$f\left(n \cdot \frac{m}{n}\right) = f(m) = km.$$

On the other hand, using the additivity of f,

$$f\left(n \cdot \frac{m}{n}\right) = f\left(\sum_{j=1}^{n} \frac{m}{n}\right) = \sum_{j=1}^{n} f\left(\frac{m}{n}\right) = nf\left(\frac{m}{n}\right).$$

Thus $nf\left(\frac{m}{n}\right) = km$, i.e., $f(r) = kr$.

(c) Let $c \in \mathbf{R}$ be given and suppose (x_n) is a convergent sequence satisfying $\lim_{n\to\infty} x_n = c$. Since $\lim_{n\to\infty}(x_n - c) = 0$ and f is continuous at 0, we must have $\lim_{n\to\infty} f(x_n - c) = f(0) = 0$. By the additivity of f, for each $n \in \mathbf{N}$ we have $f(x_n - c) = f(x_n) - f(c)$. It follows that

$$0 = \lim_{n\to\infty} f(x_n - c) = \lim_{n\to\infty} (f(x_n) - f(c)) = \left(\lim_{n\to\infty} f(x_n)\right) - f(c),$$

which implies that $\lim_{n\to\infty} f(x_n) = f(c)$. Thus f is continuous at each $c \in \mathbf{R}$.

By Theorem 4.3.4, the function $f(x) - kx$ is continuous on all of $\mathbf{R}$ and, by part (b), satisfies $f(r) - kr = 0$ for each $r \in \mathbf{Q}$. Exercise 4.3.8 (b) allows us to conclude that $f(x) - kx = 0$, i.e., $f(x) = kx$, for all $x \in \mathbf{R}$.

Exercise 4.3.14. (a) Let F be a closed set. Construct a function $f : \mathbf{R} \to \mathbf{R}$ such that the set of points where f fails to be continuous is precisely F. (The concept of the interior of a set, discussed in Exercise 3.2.14, may be useful.)

(b) Now consider an open set O. Construct a function $g : \mathbf{R} \to \mathbf{R}$ whose set of discontinuous points is precisely O. (For this problem, the function in Exercise 4.3.12 may be useful.)

Solution. (a) Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \cap F, \\ -1 & \text{if } x \in \mathbf{I} \cap F, \\ 0 & \text{if } x \notin F. \end{cases}$$

If $x \notin F$, then x belongs to the open set F^{c} and so there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq F^{\mathsf{c}}$. Since f vanishes on this proper interval, we see that f is continuous at x.

Suppose $x \in \mathbf{Q} \cap F$ and let $\delta > 0$ be given. We consider two cases.

Case 1. If $(x - \delta, x + \delta) \subseteq F$, then we can find an irrational $y \in (x - \delta, x + \delta)$. It follows that

$$f(y) = -1 \notin (0, 2) = (f(x) - 1, f(x) + 1).$$

Case 2. If $(x - \delta, x + \delta) \not\subseteq F$, then we can find some $y \in (x - \delta, x + \delta)$ such that $y \notin F$. It follows that

$$f(y) = 0 \notin (0, 2) = (f(x) - 1, f(x) + 1).$$

Thus f is not continuous at x and a similar argument shows that f is not continuous at any $x \in \mathbf{I} \cap F$ either. We may conclude that the set of points where f fails to be continuous is precisely F.

(b) Let $d : \mathbf{R} \to \mathbf{R}$ be Dirichlet's function and let $h : \mathbf{R} \to \mathbf{R}$ be the function given by

$$h(x) = \inf\{|x - a| : a \in O^{\mathsf{c}}\}.$$

In Exercise 4.3.12 we showed that h is continuous everywhere. Furthermore, since O^{c} is closed, Exercise 4.3.12 also shows that h satisfies $h(x) > 0$ for all $x \in O$ and $h(x) = 0$ for all $x \notin O$. Define $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = d(x)h(x)$ and suppose that $x \in O$. Since $h(x) > 0$ and h is continuous at x, by Exercise 4.3.8 (c) there is some $\delta > 0$ such that h is strictly positive on the interval $I = (x - \delta, x + \delta)$. It follows that for all $t \in I$ we have $d(t) = \frac{g(t)}{h(t)}$. If g were continuous at x then Theorem 4.3.4 would imply that d is continuous at x; since Dirichlet's function is nowhere-continuous, it must then be the case that g fails to be continuous at x. Thus g is discontinuous on O.

Now suppose that $x \notin O$, so that $h(x) = 0$ and thus $g(x) = 0$. For any $y \in \mathbf{R}$, we then have

$$|g(y) - g(x)| = |g(y)| = |d(y)h(y)| = |d(y)||h(y)| \leq |h(y)|.$$

Since h is continuous at x and $h(x) = 0$, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|y - x| < \delta \implies |h(y)| < \epsilon.$$

It follows that $|g(y)| < \epsilon$ for such y and thus g is continuous at x. We may conclude that the set of points where g fails to be continuous is precisely O.

4.4 Continuous Functions on Compact Sets

Exercise 4.4.1. (a) Show that $f(x) = x^3$ is continuous on all of $\mathbf{R}$.

(b) Argue, using Theorem 4.4.5, that f is not uniformly continuous on $\mathbf{R}$.

(c) Show that f is uniformly continuous on any bounded subset of $\mathbf{R}$.

Solution. (a) As Example 4.3.5 shows, any polynomial is continuous on all of $\mathbf{R}$.

(b) Define sequences (x_n) and (y_n) by $x_n = n + \frac{1}{n}$ and $y_n = n$, and observe that

$$|x_n - y_n| = \frac{1}{n} \to 0 \qquad \text{and} \qquad |f(x_n) - f(y_n)| = 3n + \frac{3}{n} + \frac{1}{n^3} > 3.$$

Theorem 4.4.5 allows us to conclude that f is not uniformly continuous on $\mathbf{R}$.

(c) Suppose that $A \subseteq \mathbf{R}$ is a bounded subset of $\mathbf{R}$, so that there is an $M > 0$ such that $A \subseteq [-M, M]$. For any $x, y \in A$, it follows that

$$\left|x^2 + xy + y^2\right| \leq |x|^2 + |x||y| + |y|^2 \leq 3M^2.$$

Let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{3M^2}$. For any $x, y \in A$ we then have

$$\left|x^3 - y^3\right| = |x - y|\left|x^2 + xy + y^2\right| \leq 3M^2\delta = \epsilon.$$

Thus f is uniformly continuous on A.

Exercise 4.4.2. (a) Is $f(x) = 1/x$ uniformly continuous on $(0,1)$?

(b) Is $g(x) = \sqrt{x^2+1}$ uniformly continuous on $(0,1)$?

(c) Is $h(x) = x\sin(1/x)$ uniformly continuous on $(0,1)$?

Solution. (a) Define sequences (x_n) and (y_n) by $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. Observe that

$$|x_n - y_n| = \frac{1}{n} - \frac{1}{n+1} \to 0 \qquad \text{and} \qquad |f(x_n) - f(y_n)| = 1.$$

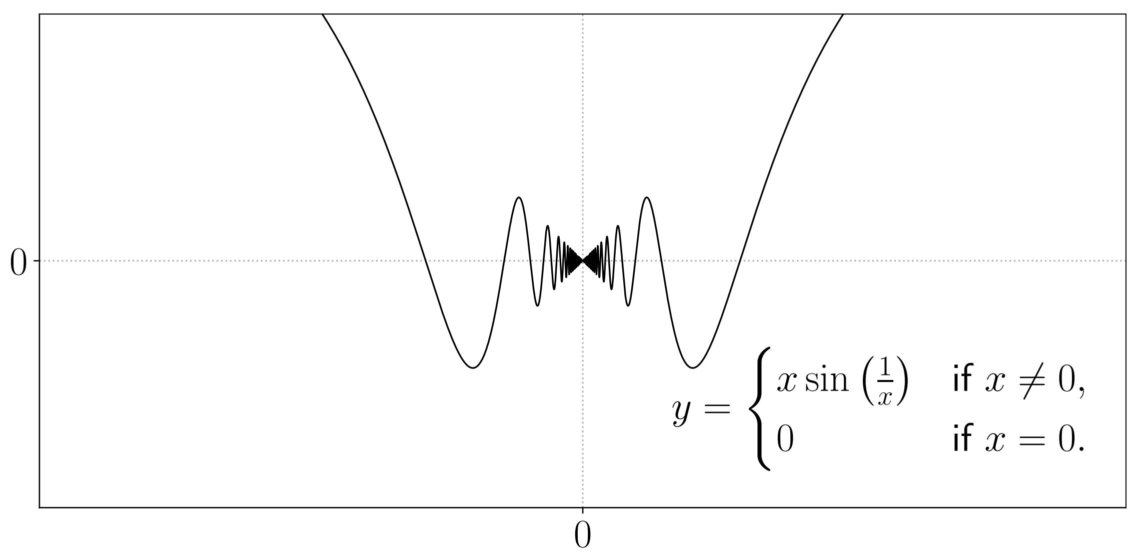
Theorem 4.4.5 allows us to conclude that f is not uniformly continuous on $\mathbf{R}$.

(b) If a function is uniformly continuous on some $B \subseteq \mathbf{R}$, then it is also uniformly continuous on any subset $A \subseteq B$. The function $g(x) = \sqrt{x^2+1}$ is continuous on all of $\mathbf{R}$, hence uniformly continuous on the compact set $[0,1]$ (Theorem 4.4.7), and hence uniformly continuous on the subset $(0,1)$.

(c) Define $h : \mathbf{R} \to \mathbf{R}$ by

$$h(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The continuity of h away from the origin is clear. As shown in Example 4.3.6, h is also continuous at the origin and thus continuous on all of $\mathbf{R}$. It follows that h is uniformly continuous on the compact set $[0,1]$ (Theorem 4.4.7) and hence uniformly continuous on the subset $(0,1)$.



Exercise 4.4.3. Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Solution. For any $x, y \in [1, \infty)$, we have

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2 y^2}\right| = \frac{x + y}{x^2 y^2}|x - y| = \left(\frac{1}{xy^2} + \frac{1}{x^2 y}\right)|x - y| \leq 2|x - y|.$$

Let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{2}$. For any $x, y \in [1, \infty)$ such that $|x - y| < \delta$, we then have

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \leq 2|x - y| < 2\delta = \epsilon.$$

Thus f is uniformly continuous on $[1, \infty)$.

Define the sequences (x_n) and (y_n) in $(0, 1]$ by $x_n = \frac{1}{\sqrt{n}}$ and $y_n = \frac{1}{\sqrt{n+1}}$. Observe that

$$|x_n - y_n| = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \to 0 \qquad \text{and} \qquad |f(x_n) - f(y_n)| = 1.$$

It follows from Theorem 4.4.5 that f is not uniformly continuous on $(0, 1]$.

Exercise 4.4.4. Decide whether each of the following statements is true or false, justifying each conclusion.

(a) If f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$ (meaning $1/f$ has bounded range).

(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

(c) If f is defined on $\mathbf{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbf{R}$.

Solution. (a) This is true. Since f is continuous on the compact set $[a, b]$, Theorem 4.4.2 implies that there exist $x_0, x_1 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in [a, b]$. By assumption we have $f(x_0) > 0$ and so

$$0 < f(x_0) \leq f(x) \leq f(x_1) \iff 0 < \frac{1}{f(x_1)} \leq \frac{1}{f(x)} \leq \frac{1}{f(x_0)}$$

for all $x \in [a, b]$, i.e., $1/f$ is bounded on $[a, b]$.

(b) This is true. Since A is bounded, there is a $K > 0$ such that $A \subseteq [-K, K]$, and since f is uniformly continuous on A, there is a $\delta > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta \implies |f(x) - f(y)| < 1.$$

Let $N \in \mathbf{N}$ be such that $\frac{2K}{N} < \delta$ and for each $j \in \{1, 2, \dots, N\}$ define

$$I_j = \left[-K + (j-1)\frac{2K}{N}, -K + j\frac{2K}{N}\right],$$

so that $I_1 \cup \cdots \cup I_N = [-K, K]$. For $j \in \{1, 2, \dots, N\}$, if $I_j \cap A \neq \emptyset$, then there exists some $a_j \in I_j \cap A$. Let

$$M = \max\{1 + |f(a_j)| : j \in \{1, 2, \dots, N\} \text{ and } I_j \cap A \neq \emptyset\};$$

we are justified by Lemma L.3 in taking the maximum of this set as it is finite and must be non-empty, since if A is non-empty (which we may as well assume) there must be some j such that $I_j \cap A \neq \emptyset$.

Suppose $x \in A$. Since $I_1 \cup \cdots \cup I_N = [-K, K]$ and $A \subseteq [-K, K]$, there must be some $j \in \{1, 2, \dots, N\}$ such that $x \in I_j \cap A$. Since $x, a_j \in I_j \cap A$, we then have $|x - a_j| \leq |I_j| = \frac{2K}{N} < \delta$ and thus

$$|f(x) - f(a_j)| < 1 \implies |f(x)| < 1 + |f(a_j)| \leq M.$$

It follows that $f(A) \subseteq [-M, M]$, i.e., $f(A)$ is bounded.

(c) This is false. Let $f : \mathbf{R} \to \mathbf{R}$ be Dirichlet's function, i.e.,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

For any subset $A \subseteq \mathbf{R}$, the only possibilities for $f(A)$ are $\emptyset, \{0\}, \{1\}$, and $\{0, 1\}$; each of these is compact. However, f is nowhere-continuous.

Exercise 4.4.5. Assume that g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that g is uniformly continuous on (a, c).

Solution. Let $\epsilon > 0$ be given. There exist positive reals δ_1 and δ_2 such that

$$x, y \in (a, b] \text{ and } |x - y| < \delta_1 \implies |g(x) - g(y)| < \frac{\epsilon}{2},$$
$$x, y \in [b, c) \text{ and } |x - y| < \delta_2 \implies |g(x) - g(y)| < \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and suppose that $x, y \in (a, c)$ are such that $|x - y| < \delta$. There are four cases.

Case 1. If $x, y \in (a, b]$, then since $|x - y| < \delta \leq \delta_1$, we have $|g(x) - g(y)| < \frac{\epsilon}{2} < \epsilon$.

Case 2. If $x, y \in [b, c)$, then since $|x - y| < \delta \leq \delta_2$, we have $|g(x) - g(y)| < \frac{\epsilon}{2} < \epsilon$.

Case 3. If $x \in (a, b]$ and $y \in [b, c)$, then note that $|x - b| \leq |x - y| < \delta \leq \delta_1$ and $|b - y| \leq |x - y| < \delta \leq \delta_2$. It follows that $|g(x) - g(b)| < \frac{\epsilon}{2}$ and that $|g(b) - g(y)| < \frac{\epsilon}{2}$, which gives us

$$|g(x) - g(y)| \leq |g(x) - g(b)| + |g(b) - g(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Case 4. The case where $x \in [b, c)$ and $y \in (a, b]$ is handled similarly to Case 3.

In any case, we have $|g(x) - g(y)| < \epsilon$. It follows that g is uniformly continuous on (a, c).

Exercise 4.4.6. Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case.

(a) A continuous function $f : (0, 1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

(b) A uniformly continuous function $f : (0, 1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

(c) A continuous function $f : [0, \infty) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Solution. (a) Let $f : (0, 1) \to \mathbf{R}$ be given by $f(x) = \frac{1}{x}$ and consider the Cauchy sequence (x_n) given by $x_n = \frac{1}{n+1}$. Notice that $f(x_n) = n + 1$, which is not convergent and hence not Cauchy.

(b) This is impossible, as we will show in Exercise 4.4.13 (a).

(c) This is impossible. Let $f : [0, \infty) \to \mathbf{R}$ be continuous and let (x_n) be a Cauchy sequence contained in $[0, \infty)$; by Theorem 2.6.4 we must have $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbf{R}$. Since $[0, \infty)$ is a closed set, we have $x \in [0, \infty)$, and because f is continuous at x it follows that $\lim_{n\to\infty} f(x_n) = f(x)$. Thus $(f(x_n))$ is a Cauchy sequence (Theorem 2.6.4).

Exercise 4.4.7. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Solution. f is continuous on the compact set $[0, 1]$ and hence is uniformly continuous on $[0, 1]$ (Theorem 4.4.7). Note that for any $x, y \in [1, \infty)$ we have

$$\left|\sqrt{x} - \sqrt{y}\right| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{2}|x - y|.$$

It is now straightforward to show that f is uniformly continuous on $[1, \infty)$ (see Exercise 4.4.9). By an argument analogous to the one given in Exercise 4.4.5, we may now conclude that f is uniformly continuous on $[0, \infty)$.

Exercise 4.4.8. Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0, 1]$ with range $(0, 1)$.

(b) A continuous function defined on $(0, 1)$ with range $[0, 1]$.

(c) A continuous function defined on $(0, 1]$ with range $(0, 1)$.

Solution. (a) This is impossible. If $f : [0, 1] \to \mathbf{R}$ is continuous, then since $[0, 1]$ is compact, the image of f must be compact (Theorem 4.4.1). However, $(0, 1)$ is not a compact set.

(b) Consider $f : (0, 1) \to \mathbf{R}$ given by $f(x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{2}$; the image of f is then $[0, 1]$.

(c) Consider $f : (0, 1] \to \mathbf{R}$ given by $f(x) = \frac{1}{2}(1 - x)\sin\left(\frac{1}{x}\right) + \frac{1}{2}$; the image of f is then $(0, 1)$.

Exercise 4.4.9 (Lipschitz Functions). A function $f : A \to \mathbf{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

for all $x \neq y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbf{R}$ is Lipschitz, then it is uniformly continuous on A.

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Solution. (a) Since f is Lipschitz, there is an $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in A$. Let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{M}$. For any $x, y \in A$ satisfying $|x - y| < \delta$, we then have

$$|f(x) - f(y)| \leq M|x - y| < M\delta = \epsilon.$$

It follows that f is uniformly continuous on A.

(b) The converse statement is not true. Consider $f : [0, \infty) \to \mathbf{R}$ given by $f(x) = \sqrt{x}$. As we showed in Exercise 4.4.7, this function is uniformly continuous on $[0, \infty)$. However, we claim that f is not Lipschitz on $[0, \infty)$. To show this, for each $M > 0$ we need to find some $x \neq y \in [0, \infty)$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| > M.$$

So, for $M > 0$, let $x = \frac{1}{4M^2}$ and $y = 0$, and observe that

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \left| \frac{\frac{1}{2M}}{\frac{1}{4M^2}} \right| = 2M > M.$$

Exercise 4.4.10. Assume that f and g are uniformly continuous functions defined on a common domain A. Which of the following combinations are necessarily uniformly continuous on A:

$$f(x) + g(x), \quad f(x)g(x), \quad \frac{f(x)}{g(x)}, \quad f(g(x))\,?$$

(Assume that the quotient and the composition are properly defined and thus at least continuous.)

Solution. We claim that $f + g$ is uniformly continuous on A. To see this, let $\epsilon > 0$ be given. There exist $\delta_1, \delta_2 > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta_1 \implies |f(x) - f(y)| < \frac{\epsilon}{2},$$

$$x, y \in A \text{ and } |x - y| < \delta_2 \implies |g(x) - g(y)| < \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and observe that for any $x, y \in A$ satisfying $|x - y| < \delta$, we then have

$$|f(x) + g(x) - f(y) - g(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $f + g$ is uniformly continuous on A.

The product fg need not be uniformly continuous. For a counterexample, consider $f, g : \mathbf{R} \to \mathbf{R}$ given by $f(x) = g(x) = x$. This function is clearly Lipschitz and hence uniformly continuous on all of $\mathbf{R}$ (Exercise 4.4.9). However, the product $f(x)g(x) = x^2$ is not uniformly continuous on $\mathbf{R}$; this can be seen using the same sequences as in Exercise 4.4.1 (b) and appealing to Theorem 4.4.5.

The quotient $\frac{f}{g}$ need not be uniformly continuous. For a counterexample, consider $f, g : (0, 1] \to \mathbf{R}$ given by $f(x) = 1$ and $g(x) = x$. Both are uniformly continuous, but the quotient $\frac{f(x)}{g(x)} = \frac{1}{x}$ is not (Exercise 4.4.2 (a)).

Suppose that $g(A) \subseteq A$, so that the composition $f \circ g : A \to \mathbf{R}$ is well-defined. We claim that this composition is also uniformly continuous. To see this, let $\epsilon > 0$ be given. There exists a $\delta_2 > 0$ such that

$$s, t \in A \text{ and } |s - t| < \delta_2 \implies |f(s) - f(t)| < \epsilon.$$

There then exists a $\delta_1 > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta_1 \implies |g(x) - g(y)| < \delta_2.$$

By assumption, if $x, y \in A$ then $g(x), g(y) \in A$. Thus

$$\begin{aligned} x, y \in A \text{ and } |x - y| < \delta_1 &\implies g(x), g(y) \in A \text{ and } |g(x) - g(y)| < \delta_2 \\ &\implies |f(g(x)) - f(g(y))| < \epsilon. \end{aligned}$$

Thus $f \circ g$ is uniformly continuous on A.

Exercise 4.4.11 (Topological Characterization of Continuity). Let g be defined on all of $\mathbf{R}$. If B is a subset of $\mathbf{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbf{R} : g(x) \in B\}.$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbf{R}$ is an open set.

Solution. Suppose g is continuous and $O \subseteq \mathbf{R}$ is an open set. Fix $c \in g^{-1}(O)$, so that $g(c) \in O$. Since O is open, there exists an $\epsilon > 0$ such that $V_\epsilon(g(c)) \subseteq O$, and since g is continuous at c, there is a $\delta > 0$ such that $x \in V_\delta(c)$ implies that $g(x) \in V_\epsilon(g(c)) \subseteq O$ (Theorem 4.3.2 (ii)). In other words, any $x \in V_\delta(c)$ also belongs to $g^{-1}(O)$, so that $V_\delta(c) \subseteq g^{-1}(O)$. It follows that $g^{-1}(O)$ is an open set.

Now suppose that $g^{-1}(O)$ is open whenever $O \subseteq \mathbf{R}$ is an open set. Fix $c \in \mathbf{R}$ and let $\epsilon > 0$ be given. The set $V_\epsilon(g(c))$ is open, so by assumption the set $g^{-1}[V_\epsilon(g(c))]$ is also open. Certainly we have $c \in g^{-1}[V_\epsilon(g(c))]$, so there exists a $\delta > 0$ such that $V_\delta(c) \subseteq g^{-1}[V_\epsilon(g(c))]$. It follows that if $x \in V_\delta(c)$, then $g(x) \in V_\epsilon(g(c))$ and so Theorem 4.3.2 (ii) allows us to conclude that g is continuous at each $c \in \mathbf{R}$.

Exercise 4.4.12. Review Exercise 4.4.11, and then determine which of the following statements is true about a continuous function defined on $\mathbf{R}$:

(a) $f^{-1}(B)$ is finite whenever B is finite.

(b) $f^{-1}(K)$ is compact whenever K is compact.

(c) $f^{-1}(A)$ is bounded whenever A is bounded.

(d) $f^{-1}(F)$ is closed whenever F is closed.

Solution. (a) This is false. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 0$, which satisfies $f^{-1}(\{0\}) = \mathbf{R}$.

(b) This is false; see part (a) for a counterexample.

(c) This is false; see part (a) for a counterexample.

(d) This is true. If F is closed, then F^{c} is open. Since f is continuous, we have that $f^{-1}(F^{\mathsf{c}})$ is also open (Exercise 4.4.11) and it follows that $(f^{-1}(F^{\mathsf{c}}))^{\mathsf{c}}$ is closed. This set is nothing but $f^{-1}(F)$:

$$x \in (f^{-1}(F^{\mathsf{c}}))^{\mathsf{c}} \iff x \notin f^{-1}(F^{\mathsf{c}}) \iff f(x) \notin F^{\mathsf{c}} \iff f(x) \in F \iff x \in f^{-1}(F).$$

Exercise 4.4.13 (Continuous Extension Theorem). (a) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbf{R}$ is uniformly continuous and $(x_n) \subseteq A$ is a Cauchy sequence, then show $f(x_n)$ is a Cauchy sequence.

(b) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended g is continuous.)

Solution. (a) Let $\epsilon > 0$ be given. Since f is uniformly continuous, there is a $\delta > 0$ such that for any $x, y \in A$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Since $(x_n) \subseteq A$ is a Cauchy sequence, there is an $N \in \mathbf{N}$ such that for all $n > m \geq N$, we have $|x_n - x_m| < \delta$, which implies that $|f(x_n) - f(x_m)| < \epsilon$. Thus $(f(x_n))$ is also a Cauchy sequence.

(b) Suppose that g is uniformly continuous on (a, b). Define a sequence $a_n = a + \frac{b-a}{2n}$, so that (a_n) is contained in (a, b) and satisfies $\lim_{n\to\infty} a_n = a$. Because (a_n) is Cauchy, part (a) implies that the sequence $(g(a_n))$ is also Cauchy and hence convergent, say $\lim_{n\to\infty} g(a_n) = y \in \mathbf{R}$. Define $g(a) := y$.

We claim that this extended g is continuous at a. Let (x_n) be a sequence contained in (a, b) such that $\lim_{n\to\infty} x_n = a$ and let $\epsilon > 0$ be given. Since g is uniformly continuous on (a, b), there is a $\delta > 0$ such that for any $x, y \in (a, b)$ satisfying $|x - y| < \delta$, we have $|g(x) - g(y)| < \epsilon$. Note that $\lim_{n\to\infty} |x_n - a_n| = 0$ since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n = a$, so there is an $N \in \mathbf{N}$ such that $n \geq N$ implies that $|x_n - a_n| < \delta$, which gives us $|g(x_n) - g(a_n)| < \epsilon$. Thus $\lim_{n\to\infty} |g(x_n) - g(a_n)| = 0$. Combining this with $\lim_{n\to\infty} g(a_n) = g(a)$, we see that $\lim_{n\to\infty} g(x_n) = g(a)$ also and hence g is continuous at a.

An analogous argument shows that we can also continuously extend g to be defined at b by considering the sequence $b_n = b - \frac{b-a}{2n}$.

For the converse implication, we apply Theorem 4.4.7 to see that g is uniformly continuous on the compact set $[a, b]$ and hence uniformly continuous on the subset (a, b).

Exercise 4.4.14. Construct an alternate proof of Theorem 4.4.7 using the open cover characterization of compactness from the Heine-Borel Theorem (Theorem 3.3.8 (iii)).

Solution. Suppose $f : K \to \mathbf{R}$ is continuous, where K is compact. Let $\epsilon > 0$ be given. Since f is continuous on K, for each $t \in K$ there exists a $\delta_t > 0$ such that

$$x \in K \text{ and } |x - t| < \delta_t \implies |f(x) - f(t)| < \frac{\epsilon}{2}.$$

Observe that the collection $\{V_{\delta_t/2}(t) : t \in K\}$ forms an open cover of K. Since K is compact, there exists a finite subcover $\{V_{\delta_{t_1}/2}(t_1), \ldots, V_{\delta_{t_n}/2}(t_n)\}$. Let $\delta = \min\{\delta_{t_1}, \ldots, \delta_{t_n}\}$ and suppose that $x, y \in K$ are such that $|x - y| < \frac{\delta}{2}$. There is a $j \in \{1, \dots, n\}$ such that $x \in V_{\delta_{t_j}/2}(t_j)$, so that $|x - t_j| < \frac{\delta_{t_j}}{2} < \delta_{t_j}$ and thus $|f(x) - f(t_j)| < \frac{\epsilon}{2}$. Note that

$$|y - t_j| \leq |x - y| + |x - t_j| < \frac{\delta}{2} + \frac{\delta_{t_j}}{2} \leq \frac{\delta_{t_j}}{2} + \frac{\delta_{t_j}}{2} = \delta_{t_j}.$$

It follows that $|f(y) - f(t_j)| < \frac{\epsilon}{2}$ and hence that

$$|f(x) - f(y)| \leq |f(x) - f(t_j)| + |f(y) - f(t_j)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus f is uniformly continuous on K.

4.5 The Intermediate Value Theorem

Exercise 4.5.1. Show how the Intermediate Value Theorem follows as a corollary to Theorem 4.5.2.

Solution. Let $f : [a, b] \to \mathbf{R}$ be continuous and let $L \in \mathbf{R}$ be such that either $f(a) < L < f(b)$ or $f(b) < L < f(a)$; our aim is to show that there exists $c \in (a, b)$ such that $f(c) = L$. Theorem 3.4.7 shows that $[a, b]$ is connected and hence Theorem 4.5.2 implies that the image $f([a, b])$ is also connected. Clearly $f(a), f(b) \in f([a, b])$, so Theorem 3.4.7 implies that $L \in f([a, b])$, i.e., there exists $c \in [a, b]$ such that $f(c) = L$. In fact, since $f(a) \neq L$ and $f(b) \neq L$ we have $c \in (a, b)$.

Exercise 4.5.2. Provide an example of each of the following, or explain why the request is impossible.

(a) A continuous function defined on an open interval with range equal to a closed interval.

(b) A continuous function defined on a closed interval with range equal to an open interval.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbf{R}$.

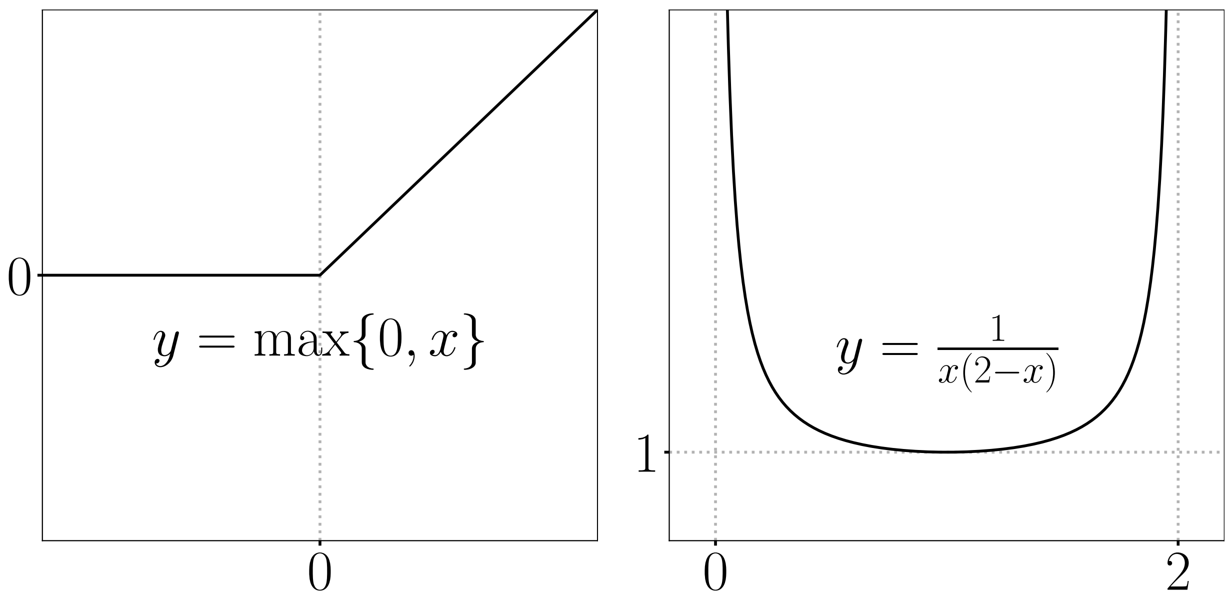
(d) A continuous function defined on all of $\mathbf{R}$ with range equal to $\mathbf{Q}$.

Solution. (I am not sure if Abbott allows unbounded intervals here.)

(a) If we allow unbounded intervals, then $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x$ is an example of such a function. For bounded intervals, see Exercise 4.4.8 (b) for an example of such a function.

(b) If we allow unbounded intervals, then $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x$ is an example of such a function. If we do not allow unbounded intervals, then such a function cannot exist by Theorem 4.4.1 (Preservation of Compact Sets).

(c) If we allow unbounded intervals, then $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = \max\{0, x\}$ is an example of such a function; the image of f is $[0, \infty)$. For bounded intervals, consider the function $f : (0, 2) \to \mathbf{R}$ given by $f(x) = \frac{1}{x(2-x)}$; the image of f is $[1, \infty)$.



(d) This is impossible. $\mathbf{R}$ is connected (Theorem 3.4.7) and so its image under a continuous function must also be connected (Theorem 4.5.2), but $\mathbf{Q}$ is not connected (Theorem 3.4.7).

Exercise 4.5.3. A function f is *increasing* on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a, b]$.

Solution. First, let us prove the following lemma.

Lemma L.13. Suppose $a < b$ and $f : [a, b] \to \mathbf{R}$ is increasing.

(i) If $c \in (a, b]$, then

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : a < x < c\}.$$

(ii) If $c \in [a, b)$, then
$$\lim_{x \to c^+} f(x) = \inf\{f(x) : c < x < b\}.$$

Proof. (i) Fix $c \in (a, b]$. Note that since f is increasing, we have $f([a, b]) \subseteq [f(a), f(b)]$; it follows that $\{f(x) : a < x < c\}$ is bounded and non-empty, so $S := \sup\{f(x) : a < x < c\}$ exists. Let $\epsilon > 0$ be given. By Lemma 1.3.8, there exists a $y \in (a, c)$ such that $S - \epsilon < f(y) \leq S$. Since f is increasing, we then have
$$x \in (y, c) \implies S - \epsilon < f(y) \leq f(x) \leq S.$$
In other words, letting $\delta = c - y$, for any x satisfying $c - \delta < x < c$ we have $|f(x) - S| < \epsilon$. It follows that $\lim_{x \to c^-} f(x) = S$.

(ii) The proof is similar to part (i). □

Returning to the exercise, we will now prove the contrapositive statement: if f is increasing and not continuous on $[a, b]$, then f does not satisfy the intermediate value property. Suppose therefore that f is not continuous at some $c \in [a, b]$ i.e., suppose that $\lim_{x \to c} f(x) \neq f(c)$ (Theorem 4.3.2 (iv)).

Case 1. Suppose $c \in (a, b)$. Since f is increasing on $[a, b]$, Lemma L.13 implies that both of the one-sided limits exist:
$$\alpha := \lim_{x \to c^-} f(x) = \sup\{f(x) : a < x < c\},$$
$$\beta := \lim_{x \to c^+} f(x) = \inf\{f(x) : c < x < b\}.$$
By Exercise 4.2.10 (b), it must be the case that at least one of these limits is not equal to $f(c)$. Since f is increasing, we must then have $\alpha < \beta$; it follows that the infinite set $(\alpha, \beta) \setminus \{f(c)\}$, which is contained in $[f(a), f(b)]$, does not intersect the image of f. Thus f does not satisfy the intermediate value property on $[a, b]$.

Case 2. Suppose $c = a$, i.e., f is not continuous at a. Since f is increasing on $[a, b]$, Lemma L.13 implies that the limit from the right exists:
$$\beta := \lim_{x \to a^+} f(x) = \inf\{f(x) : a < x < b\}.$$

Since a is the minimum element of the domain of f, we have $\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \beta$, and since f is not continuous at a and increasing on $[a,b]$, it must then be the case that $f(a) < \beta$. It follows that the infinite set $(f(a), \beta)$, which is contained in $[f(a), f(b)]$, does not intersect the image of f. Thus f does not satisfy the intermediate value property on $[a,b]$.

Case 3. If f fails to be continuous at b, then an argument similar to the one given in Case 2, this time using the limit from the left, shows that f does not satisfy the intermediate value property on $[a,b]$.

Exercise 4.5.4. Let g be continuous on an interval A and let F be the set of points where g fails to be one-to-one; that is,

$$F = \{x \in A : f(x) = f(y) \text{ for some } y \neq x \text{ and } y \in A\}.$$

Show F is either empty or uncountable.

Solution. It will suffice to show that if F is not empty then F is uncountable. Suppose therefore that there exist $x < y$ in A such that $g(x) = g(y)$. If g is constant on $[x,y]$, then F contains the uncountable subset $[x,y]$ and so must itself be uncountable. Otherwise, there exists some $a \in (x,y)$ such that $g(a) \neq g(x)$. Let

$$I := \big(\min\{g(x), g(a)\}, \max\{g(x), g(a)\}\big)$$

and note that I is non-empty since $g(a) \neq g(x)$. Since g is continuous on A, the Intermediate Value Theorem (Theorem 4.5.1) implies that for each $t \in I$ there exist $x_t \in (x,a)$ and $y_t \in (a,y)$ such that $g(x_t) = g(y_t) = t$, so that $x_t \in F$. Because g is a function, each $t \in I$ gives rise to a distinct $x_t \in F$, i.e., the map $I \to F$ given by $t \mapsto x_t$ is injective; since I is uncountable, it then follows that F is uncountable.

Exercise 4.5.5. (a) Finish the proof of the Intermediate Value Theorem using the Axiom of Completeness started previously.

(b) Finish the proof of the Intermediate Value Theorem using the Nested Interval Property started previously.

Solution. (a) (Here is the start of the proof from the textbook.) To simplify matters a bit, let's consider the special case where f is a continuous function satisfying $f(a) < 0 < f(b)$ and show that $f(c) = 0$ for some $c \in (a, b)$. First let

$$K = \{x \in [a, b] : f(x) \leq 0\}.$$

Notice that K is bounded above by b, and $a \in K$ so K is not empty. Thus we may appeal to the Axiom of Completeness to assert that $c = \sup K$ exists.

There are three cases to consider:

$$f(c) > 0, \;\; f(c) < 0, \;\text{ and } f(c) = 0.$$

Case 1. Suppose that $f(c) > 0$. Since f is continuous at c, there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$ (see Exercise 4.3.8 (c)). This implies the existence of a $t \in (c-\delta, c) \cap [a, b]$ such that t is an upper bound of K, which contradicts that c is the supremum of K.

Case 2. Suppose that $f(c) < 0$. Since f is continuous at c, there is a $\delta > 0$ such that $f(x) < 0$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$ (see Exercise 4.3.8 (c)). This implies the existence of a $t \in (c, c + \delta) \cap [a, b]$ such that t belongs to K, which contradicts that c is the supremum of K.

So the only possibility is that $f(c) = 0$; note that c lies strictly between a and b since $f(a) < 0 < f(b)$.

The more general statement of the Intermediate Value Theorem can be obtained from this special case by considering either the function $g(x) = f(x) - L$ if $f(a) < f(b)$ or the function $g(x) = L - f(x)$ if $f(a) > f(b)$.

(b) (Here is the start of the proof from the textbook.) Again, consider the special case where $L = 0$ and $f(a) < 0 < f(b)$. Let $I_0 = [a, b]$, and consider the midpoint

$$z = (a + b)/2.$$

If $f(z) \geq 0$, then set $a_1 = a$ and $b_1 = z$. If $f(z) < 0$, then set $a_1 = z$ and $b_1 = b$. In either case, the interval $I_1 = [a_1, b_1]$ has the property that f is negative at the left endpoint and nonnegative at the right.

We repeat this procedure inductively, obtaining a sequence $(I_n = [a_n, b_n])_{n=1}^{\infty}$ of nested intervals such that $f(a_n) < 0, f(b_n) \geq 0$, and $|I_n| = 2^{-n}(b-a)$ for all $n \in \mathbf{N}$. We can now appeal to the Nested Interval Property (Theorem 1.4.1) to assert that $\bigcap_{n=1}^{\infty} I_n = \{c\}$ for some $c \in [a,b]$ (the intersection is non-empty as the intervals are closed and nested, and the intersection is a singleton since $\lim_{n\to\infty}|I_n| = 0$); furthermore, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$. Since f is continuous at c, it follows that $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(c)$. The Order Limit Theorem (Theorem 2.3.4) implies that $f(c) \leq 0$, since $f(a_n) < 0$ for all $n \in \mathbf{N}$, and that $f(c) \geq 0$, since $f(b_n) \geq 0$ for all $n \in \mathbf{N}$. Thus $f(c) = 0$.

Again, c lies strictly between a and b since $f(a) < 0 < f(b)$, and the more general statement of the Intermediate Value Theorem can be obtained from this special case by considering either the function $g(x) = f(x) - L$ if $f(a) < f(b)$ or the function $g(x) = L - f(x)$ if $f(a) > f(b)$.

Exercise 4.5.6. Let $f : [0,1] \to \mathbf{R}$ be continuous with $f(0) = f(1)$.

(a) Show that there must exist $x, y \in [0,1]$ satisfying $|x-y| = 1/2$ and $f(x) = f(y)$.

(b) Show that for each $n \in \mathbf{N}$ there exist $x_n, y_n \in [0,1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

(c) If $h \in (0, 1/2)$ is not of the form $1/n$, there does not necessarily exist $|x-y| = h$ satisfying $f(x) = f(y)$. Provide an example that illustrates this using $h = 2/5$.

Solution. (a) Define $g : \left[0, \frac{1}{2}\right] \to \mathbf{R}$ by $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$ and note that g is continuous by Theorems 4.3.4 and 4.3.9. If $g(0) = 0$ then $f(0) = f\left(\frac{1}{2}\right)$ and we are done. Otherwise, note that

$$g(0) = f(0) - f\left(\tfrac{1}{2}\right) = f(1) - f\left(\tfrac{1}{2}\right) = -\left(f\left(\tfrac{1}{2}\right) - f(1)\right) = -g\left(\tfrac{1}{2}\right).$$

It follows that $g(0)$ and $g\left(\frac{1}{2}\right)$ have opposite signs. The Intermediate Value Theorem (Theorem 4.5.1) now implies that there exists a $c \in \left(0, \frac{1}{2}\right)$ such that $g(c) = 0$, i.e., $f(c) = f\left(c + \frac{1}{2}\right)$.

(b) For $n = 1$, we can take $x_1 = 0$ and $y_1 = 1$. For $n \geq 2$, define $g : \left[0, \frac{n-1}{n}\right] \to \mathbf{R}$ by $g(x) = f(x) - f\left(x + \frac{1}{n}\right)$ and note that g is continuous by Theorems 4.3.4 and 4.3.9. If $g(0) = 0$ then $f(0) = f\left(\frac{1}{n}\right)$ and we are done. Otherwise, note that

$$g(0) = f(0) - f\left(\tfrac{1}{n}\right),$$

$$g\left(\tfrac{1}{n}\right) = f\left(\tfrac{1}{n}\right) - f\left(\tfrac{2}{n}\right),$$

$$g\big(\tfrac{2}{n}\big) = f\big(\tfrac{2}{n}\big) - f\big(\tfrac{3}{n}\big),$$

$$\vdots$$

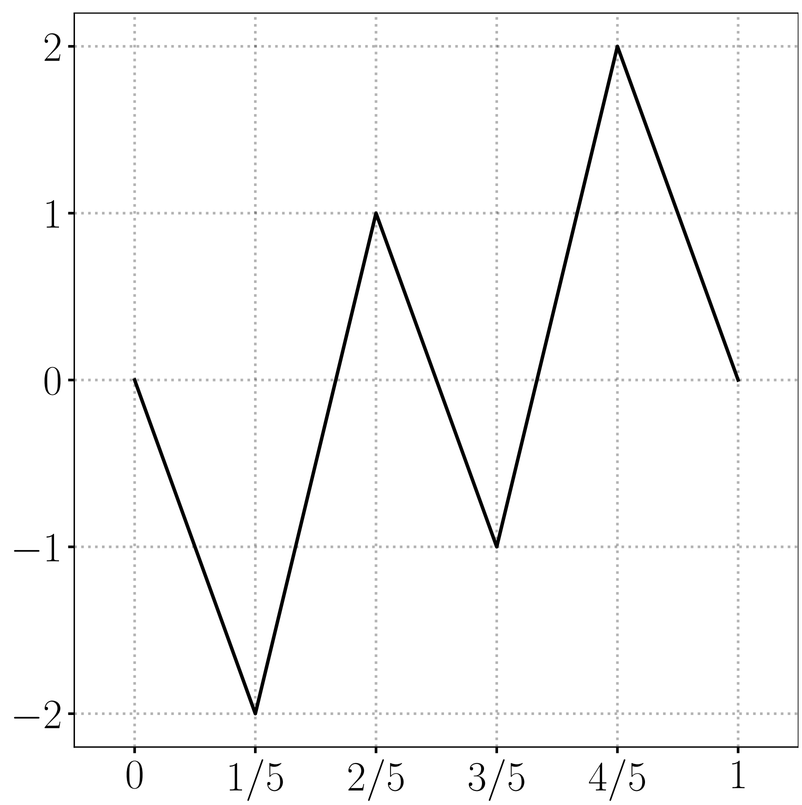
$$g\big(\tfrac{n-1}{n}\big) = f\big(\tfrac{n-1}{n}\big) - f(1).$$

Since $f(0) = f(1)$, this implies that

$$g(0) + g\big(\tfrac{1}{n}\big) + g\big(\tfrac{2}{n}\big) + \cdots + g\big(\tfrac{n-1}{n}\big) = 0.$$

Because $g(0) \neq 0$, there must exist some $k \in \{1, \ldots, n-1\}$ such that $g\big(\tfrac{k}{n}\big)$ has the opposite sign to $g(0)$. The Intermediate Value Theorem (Theorem 4.5.1) now implies that there exists a $c \in \big(0, \tfrac{k}{n}\big)$ such that $g(c) = 0$, i.e., $f(c) = f\big(c + \tfrac{1}{n}\big)$. Thus we can take $x_n = c$ and $y_n = c + \tfrac{1}{n}$.

(c) Consider the following piecewise linear function $f : [0, 1] \to \mathbf{R}$:



This function has the property that $f\big(x + \tfrac{2}{5}\big) - f(x) = 1$ for every $x \in \big[0, \tfrac{3}{5}\big]$, so that there cannot possibly exist $x, y \in [0, 1]$ satisfying $|x - y| = \tfrac{2}{5}$ and $f(x) = f(y)$.

Exercise 4.5.7. Let f be a continuous function on the closed interval $[0,1]$ with range also contained in $[0,1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0,1]$.

Solution. Define $g : [0,1] \to \mathbf{R}$ by $g(x) = f(x) - x$ and note that g is continuous by Theorem 4.3.4. Furthermore, fixed points of f correspond precisely to zeros of g. If $g(0) = 0$ or $g(1) = 0$, then we are done. Suppose therefore that $g(0) \neq 0$ and $g(1) \neq 0$. Since $0 \leq f(x) \leq 1$ for all $x \in [0,1]$, it must then be the case that $0 < f(0) \leq 1$ and $0 \leq f(1) < 1$, which implies that $g(0)$ is positive and $g(1)$ is negative. The Intermediate Value Theorem (Theorem 4.5.1) can now be applied to obtain some $x \in (0,1)$ such that $g(x) = 0$.

Exercise 4.5.8 (Inverse functions). If a function $f : A \to \mathbf{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range of f in the natural way: $f^{-1}(y) = x$ where $y = f(x)$.

Show that if f is continuous on an interval $[a,b]$ and one-to-one, then f^{-1} is also continuous.

Solution. Here are a couple of useful lemmas.

Lemma L.14. If $f : A \to \mathbf{R}$ is continuous and injective, then f is strictly monotone, i.e., f is either strictly increasing or strictly decreasing.

Proof. We will prove the contrapositive result: if f is continuous and neither strictly increasing nor strictly decreasing, then f is not injective.

Since f is not strictly increasing, there exist $x < y$ in A such that $f(y) \leq f(x)$, and since f is not strictly decreasing, there exist $s < t$ in A such that $f(s) \leq f(t)$. There are a number of cases to check. We will check one case only; the others are handled similarly.

Suppose that $x < s < t$ and $f(s) < f(x) < f(t)$. Since f is continuous, the Intermediate Value Theorem (Theorem 4.5.1) implies the existence of some $c \in (s,t)$, so that $c \neq x$, such that $f(c) = f(x)$. Thus f is not injective. $\square$

Lemma L.15. If $g : A \to I$ is a strictly monotone surjection, where I is an interval, then g is continuous.

Proof. The cases where I is empty or a singleton (which are precisely the cases where A is empty or a singleton) are easily handled, so we may assume that I is a proper interval. We may also assume that g is strictly increasing (if g is strictly decreasing, consider the function $-g$ instead).

Fix $b \in A$ and $\epsilon > 0$. We consider four cases.

Case 1. Suppose $g(b) - \epsilon$ and $g(b) + \epsilon$ both belong to I. Since g is a surjection, there exist $a, c \in A$ such that $g(a) = g(b) - \epsilon$ and $g(c) = g(b) + \epsilon$, and since g is strictly increasing it must be the case that $a < b < c$. Let $\delta = \min\{b - a, c - b\}$ and observe that, because g is strictly increasing,

$$x \in (b - \delta, b + \delta) \cap A \implies g(x) \in (g(b) - \epsilon, g(b) + \epsilon).$$

Case 2. Suppose $g(b) - \epsilon \in I$ and $g(b) + \epsilon \notin I$. Since g is a surjection, there is an $a \in A$ such that $g(a) = g(b) - \epsilon$, and since I is an interval it must be the case that I is bounded above by $g(b) + \epsilon$, so that $\sup I$ exists and is less than or equal to $g(b) + \epsilon$.

If $g(b) = \sup I$, then let $\delta = b - a$ and note that δ is positive since $g(a) = g(b) - \epsilon < g(b)$ implies $a < b$ by the monotonicity of g. Since $g(b)$ is the supremum of the image of g and g is strictly increasing, we then have

$$x \in (b - \delta, b + \delta) \cap A \implies g(x) \in (g(b) - \epsilon, g(b)] \subseteq (g(b) - \epsilon, g(b) + \epsilon).$$

If $g(b) < \sup I$ then since I is an interval we must have $s := \frac{g(b)+\sup I}{2} \in I$. The surjectivity of g then implies that there exists a $c \in A$ such that $g(c) = s$. Since g is strictly increasing and $g(b) - \epsilon < g(b) < s$, we must have $a < b < c$. Let $\delta = \min\{b - a, c - b\}$ and observe that, because g is strictly increasing,

$$x \in (b - \delta, b + \delta) \cap A \implies g(x) \in (g(b) - \epsilon, g(b) + \epsilon).$$

Case 3. The case where $g(b) - \epsilon \notin I$ and $g(b) + \epsilon \in I$ is handled similarly to Case 2, this time by considering the infimum of I.

Case 4. The case where neither one of $g(b) - \epsilon$ and $g(b) + \epsilon$ belongs to I is handled similarly to Cases 2 and 3, by considering both the infimum and supremum of I. Note that since I is a proper interval, we must have $\inf I < \sup I$, so that $g(b)$ could never be equal to both $\inf I$ and $\sup I$.

In any case, we obtain a $\delta > 0$ such that

$$x \in (b - \delta, b + \delta) \cap A \implies g(x) \in (g(b) - \epsilon, g(b) + \epsilon),$$

so that g is continuous at b. $\square$

Returning to the exercise, we have a continuous and bijective function $f : [a, b] \to f([a, b])$ defined on the compact and connected set $[a, b]$ (we may as well assume $a < b$); the image of f must be compact and connected (Theorems 4.4.1 and 4.5.2). The only possibility is $f([a, b]) = [c, d]$ for some $c < d$ (Theorems 3.3.8 and 3.4.7; it must be the case that c is strictly less than d since f is injective).

Now let $g : [c, d] \to [a, b]$ be the inverse of f. By Lemma L.14, f must be strictly monotone and thus its inverse g must also be strictly monotone. Since the image of g is the interval $[a, b]$, we may now apply Lemma L.15 to conclude that g is continuous.

4.6 Sets of Discontinuity

Exercise 4.6.1. Using modifications of these functions, construct a function $f : \mathbf{R} \to \mathbf{R}$ so that

(a) $D_f = \mathbf{Z}^c$.

(b) $D_f = \{x : 0 < x \leq 1\}$.

Solution. (a) Since $\mathbf{Z}^c$ is an open set, the construction given in Exercise 4.3.14 (b) will result in an f such that $D_f = \mathbf{Z}^c$.

(b) By Exercise 4.3.14, there exist functions $g, h : \mathbf{R} \to \mathbf{R}$ such that $D_g = \left(0, \frac{1}{2}\right)$ and $D_h = \left[\frac{1}{2}, 1\right]$. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = g(x) + h(x)$; it follows from Theorem 4.3.4 that $D_f = (0, 1]$.

Exercise 4.6.2. Given a countable set $A = \{a_1, a_2, a_3, \ldots\}$, define $f(a_n) = 1/n$ and $f(x) = 0$ for all $x \notin A$. Find D_f.

Solution. Our claim is that $D_f = A$. First, fix $c \notin A$; we will show that f is continuous at c. Let $\epsilon > 0$ be given and let $N \in \mathbf{N}$ be such that $\frac{1}{N} < \epsilon$. Consider the set

$$E = \{|c - a_n| : 1 \leq n \leq N\}.$$

This set is non-empty and finite, so we are justified (by Lemma L.3) in letting $\delta = \min E$. Each element of E must be strictly positive as $c \notin A$ and hence δ is also strictly positive. Furthermore, the interval $(c - \delta, c + \delta)$ has the property that if $a_n \in (c - \delta, c + \delta)$, then $n > N$. It follows that

$$|f(a_n) - f(c)| = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Additionally, if $x \in (c-\delta, c+\delta)$ and $x \notin A$, then $|f(x) - f(c)| = 0 < \epsilon$.

We have now shown that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$x \in (c-\delta, c+\delta) \implies f(x) \in (f(c)-\epsilon, f(c)+\epsilon).$$

Thus f is continuous at each $c \notin A$.

Now fix $a_n \in A$; we will show that f is not continuous at a_n. Let $\epsilon = \frac{1}{n} > 0$ and let $\delta > 0$ be given. Since the interval $(a_n - \delta, a_n + \delta)$ is uncountable and A is countable, it must be the case that there exists an $x \in (a_n - \delta, a_n + \delta)$ such that $x \notin A$. It follows that

$$|f(x) - f(a_n)| = \frac{1}{n} = \epsilon.$$

Thus f is not continuous at a_n and our claim follows.

Exercise 4.6.3. State a similar definition for the left-hand limit

$$\lim_{x \to c^-} f(x) = L.$$

Solution. See Exercise 4.2.10 (a).

Exercise 4.6.4. Supply a proof for this proposition.

Solution. See Exercise 4.2.10 (b).

Exercise 4.6.5. Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

Solution. Suppose $f : \mathbf{R} \to \mathbf{R}$ is monotone. (For simplicity, we will assume that the domain of f is all of $\mathbf{R}$. A more general statement can certainly be made for monotone functions $A \to \mathbf{R}$ defined on any domain $A \subseteq \mathbf{R}$, but Abbott's definitions of left- and right-hand limits are slightly awkward here. For example, if $f : [0,1] \to \mathbf{R}$ is a function, then Abbott's definition of the left-hand limit of f at 0 implies that $\lim_{x \to 0^-} f(x) = L$ for *any* $L \in \mathbf{R}$; we may choose any $\delta > 0$ we like and obtain a statement beginning with $(\forall x \in \emptyset)$, which is always true. It would be better not to talk about $\lim_{x \to 0^-} f(x)$ at all in such a case.)

First, note that a small modification of Lemma L.13 shows that if f is increasing then for each $c \in \mathbf{R}$,

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : x < c\} \qquad \text{and} \qquad \lim_{x \to c^+} f(x) = \inf\{f(x) : c < x\}.$$

(If f is decreasing, then the supremum and the infimum should be swapped.) So for a monotone function $f : \mathbf{R} \to \mathbf{R}$, the left- and right-hand limits at some point $c \in \mathbf{R}$ always exist. It follows that if f is discontinuous at c, it must be the case that these left- and right-hand limits are not equal (Theorem 4.6.3/Exercise 4.6.4), i.e., f has a jump discontinuity at c.

Exercise 4.6.6. Construct a bijection between the set of jump discontinuities of a monotone function f and a subset of $\mathbf{Q}$. Conclude that D_f for a monotone function f must either be finite or countable, but not uncountable.

Solution. Suppose $f : \mathbf{R} \to \mathbf{R}$ is monotone increasing (the case where f is decreasing is handled similarly) and let D_f be the set of jump discontinuities of f (by Exercise 4.6.5, D_f is the set of all discontinuities of f). Fix $c \in D_f$. As we showed in Exercise 4.6.5, we have

$$\ell_c := \lim_{x \to c^-} f(x) = \sup\{f(x) : x < c\} \qquad \text{and} \qquad u_c := \lim_{x \to c^+} f(x) = \inf\{f(x) : c < x\}.$$

Since f is discontinuous at c and increasing, we must have $\ell_c < u_c$ and hence (ℓ_c, u_c) is a proper open interval. If $d \in D_f$ is such that $c < d$, then $u_c \leq f\left(\frac{c+d}{2}\right) \leq \ell_d$, so that the open intervals (ℓ_c, u_c) and (ℓ_d, u_d) are disjoint. It follows that the set

$$\{(\ell_c, u_c) : c \in D_f\}$$

consists of pairwise disjoint open intervals. Given this, for each $c \in D_f$ we can choose a rational number $r_c \in (\ell_c, u_c)$ and be sure that the function $g : D_f \to \mathbf{Q}$ mapping $c \mapsto r_c$ is injective. This sets up a bijection between D_f and $g(D_f) \subseteq \mathbf{Q}$. It follows from Theorem 1.5.6 and Theorem 1.5.7 (i) that D_f is finite or countable, but not uncountable.

Exercise 4.6.7. (a) Show that in each of the above cases we get an F_σ set as the set where the function is discontinuous.

(b) Show that the two sets of discontinuity in Exercise 4.6.1 are F_σ sets.

Solution. (a) For Dirichlet's function, $\mathbf{R}$ is a closed set. For the modified Dirichlet function, we have

$$\mathbf{R} \setminus \{0\} = \bigcup_{n=1}^{\infty} \left(-\infty, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, \infty\right).$$

For Thomae's function, we have

$$\mathbf{Q} = \bigcup_{q \in \mathbf{Q}} \{q\}.$$

(b) Observe that

$$\mathbf{Z}^{c} = \bigcup_{(m,n)\in\mathbf{Z}\times\mathbf{N}} \left[m + \frac{1}{n+1}, m + 1 - \frac{1}{n+1}\right] \qquad \text{and} \qquad (0,1] = \bigcup_{n=1}^{\infty}\left[\frac{1}{n}, 1\right].$$

Exercise 4.6.8. Prove that, for a fixed $\alpha > 0$, the set D_f^α is closed.

Solution. First, let us write down the negation of α-continuity. A function f is not α-continuous at a point $x \in \mathbf{R}$ if for all $\delta > 0$ there exist $y, z \in (x-\delta, x+\delta)$ such that $|f(y) - f(z)| \geq \alpha$.

Now, to show that D_f^α is closed, let (x_n) be a sequence contained in D_f^α such that $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbf{R}$. Our aim is to show that f is not α-continuous at x. Let $\delta > 0$ be given. Since $\lim_{n\to\infty} x_n = x$, there is an $N \in \mathbf{N}$ such that $x_N \in \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)$, and since f is not α-continuous at x_N, there exist $y, z \in \left(x_N - \frac{\delta}{2}, x_N + \frac{\delta}{2}\right)$ such that $|f(y) - f(z)| \geq \alpha$. In fact, the triangle inequality implies that $y, z \in (x-\delta, x+\delta)$ and thus f is not α-continuous at x.

It follows that D_f^α contains its limit points and hence that D_f^α is a closed set.

Exercise 4.6.9. If $\alpha < \alpha'$, show that $D_f^{\alpha'} \subseteq D_f^\alpha$.

Solution. A function f is not α'-continuous at a point $x \in \mathbf{R}$ if for all $\delta > 0$ there exist $y, z \in (x-\delta, x+\delta)$ such that $|f(y) - f(z)| \geq \alpha' > \alpha$; it follows that f is also not α-continuous at x.

Exercise 4.6.10. Let $\alpha > 0$ be given. Show that if f is continuous at x, then it is α-continuous at x as well. Explain how it follows that $D_f^\alpha \subseteq D_f$.

Solution. Since f is continuous at x, there is a $\delta > 0$ such that

$$y \in (x-\delta, x+\delta) \implies |f(y) - f(x)| < \frac{\alpha}{2}.$$

If $y, z \in (x-\delta, x+\delta)$, then

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(z) - f(x)| < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Thus f is α-continuous at x. The contrapositive of this result states that if f is not α-continuous at x, then f is not continuous at x. It follows that $D_f^\alpha \subseteq D_f$.

Exercise 4.6.11. Show that if f is not continuous at x, then f is not α-continuous for some $\alpha > 0$. Now explain why this guarantees that

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n},$$

where $\alpha_n = 1/n$.

Solution. If f is not continuous at x, then there exists an $\epsilon > 0$ such that for all $\delta > 0$, there is a $y \in (x - \delta, x + \delta)$ such that $|f(y) - f(x)| \geq \epsilon$. It follows that f is not α-continuous at x, where we take $\alpha = \epsilon$.

Suppose $x \in D_f$. As we just showed, there exists an $\alpha > 0$ such that $x \in D_f^{\alpha}$. Let $n \in \mathbf{N}$ be such that $\frac{1}{n} < \alpha$. We then have $D_f^{\alpha} \subseteq D_f^{\alpha_n}$ (Exercise 4.6.9) and so $x \in D_f^{\alpha_n}$. It follows that

$$D_f \subseteq \bigcup_{n=1}^{\infty} D_f^{\alpha_n}.$$

For the reverse inclusion, note that for each $n \in \mathbf{N}$, we have $D_f^{\alpha_n} \subseteq D_f$ by Exercise 4.6.10. We may conclude that

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n}.$$

Chapter 5

The Derivative

5.2 Derivatives and the Intermediate Value Property

Exercise 5.2.1. Supply proofs for parts (i) and (ii) of Theorem 5.2.4.

Solution. (i) Observe that

$$\lim_{x\to c}\frac{(f+g)(x)-(f+g)(c)}{x-c}=\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}+\frac{g(x)-g(c)}{x-c}\right)=f'(c)+g'(c),$$

where we have used Corollary 4.2.4 (ii).

(ii) Observe that

$$\lim_{x\to c}\frac{(kf)(x)-(kf)(c)}{x-c}=\lim_{x\to c}k\left(\frac{f(x)-f(c)}{x-c}\right)=kf'(c),$$

where we have used Corollary 4.2.4 (i).

Exercise 5.2.2. Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbf{R}$.

(a) Functions f and g not differentiable at zero but where fg is differentiable at zero.

(b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.

(c) A function f not differentiable at zero and a function g differentiable at zero where $f+g$ is differentiable at zero.

(d) A function f differentiable at zero but not differentiable at any other point.

Solution. (a) Let $f, g : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = g(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Notice that f and g are not continuous at zero and hence not differentiable at zero (Theorem 5.2.3), however the product fg is given by $(fg)(x) = 1$ for all $x \in \mathbf{R}$, which is differentiable everywhere.

(b) Let $f, g : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases}$$

and $g(x) = 0$ for all $x \in \mathbf{R}$. Notice that f is not continuous at zero and hence not differentiable at zero (Theorem 5.2.3), however we have $(fg)(x) = g(x) = 0$ for all $x \in \mathbf{R}$, which is differentiable everywhere.

(c) This is impossible. If g and $f+g$ are differentiable at zero, then $f = f+g-g$ must be differentiable at zero by Theorem 5.2.4.

(d) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \in \mathbf{I}. \end{cases}$$

This function is only continuous at zero and hence fails to be differentiable at each non-zero point. We claim that $f'(0) = 0$, i.e., that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = 0.$$

For any given $\epsilon > 0$, let $\delta = \epsilon$ and suppose that $x \in \mathbf{R}$ satisfies $0 < |x| < \delta$. If $x \in \mathbf{I}$, then $\left|\frac{f(x)}{x}\right| = 0 < \epsilon$, and if $x \in \mathbf{Q}$, then $\left|\frac{f(x)}{x}\right| = |x| < \delta = \epsilon$. Thus $f'(0) = 0$.

Exercise 5.2.3. (a) Use Definition 5.2.1 to produce the proper formula for the derivative of $h(x) = 1/x$.

(b) Combine the result of part (a) with the Chain Rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.4 (iii).

Solution. (a) Suppose $x \neq 0$ and observe that

$$h'(x) = \lim_{t \to x} \frac{h(t) - h(x)}{t - x} = \lim_{t \to x} \left[\left(\frac{1}{t} - \frac{1}{x} \right) \frac{1}{t - x} \right] = \lim_{t \to x} \left[\left(\frac{x - t}{tx} \right) \frac{1}{t - x} \right] = \lim_{t \to x} \frac{-1}{tx} = \frac{-1}{x^2},$$

where we have used Corollary 4.2.4 (iv).

(b) Keeping the definition of h from part (a), note that $\frac{f(x)}{g(x)} = f(x)h(g(x))$ for any x such that $g(x) \neq 0$. It follows from Theorem 5.2.4 (iii) and the Chain Rule (Theorem 5.2.5) that

$$(f/g)'(x) = f'(x)h(g(x)) + f(x)h'(g(x))g'(x).$$

We can use the result from part (a) to rewrite this as

$$(f/g)'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

(c) Suppose $x \in \mathbf{R}$ is such that $g(x) \neq 0$. For any $t \neq x$ (and such that $g(t) \neq 0$; since $g(x) \neq 0$, the continuity of g at x (Theorem 5.2.3) implies that there is some neighbourhood of x where g is non-zero), we have

$$\begin{aligned}
\frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t - x} &= \frac{f(t)g(x) - f(x)g(t)}{(t - x)[g(t)g(x)]} \\
&= \frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(t)}{(t - x)[g(t)g(x)]} \\
&= \frac{f(t) - f(x)}{t - x} \frac{g(x)}{g(t)g(x)} - \frac{g(t) - g(x)}{t - x} \frac{f(x)}{g(t)g(x)}.
\end{aligned}$$

It follows that

$$(f/g)'(x) = \lim_{t\to x} \frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t-x} = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)\lim_{t\to x}\left(\frac{g(x)}{g(t)g(x)}\right)$$
$$- \lim_{t\to x}\left(\frac{g(t)-g(x)}{t-x}\right)\lim_{t\to x}\left(\frac{f(x)}{g(t)g(x)}\right) = \frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2},$$

where we have used that f and g are differentiable at x, the continuity of g at x (Theorem 5.2.3), and several algebraic properties of functional limits (Corollary 4.2.4).

Exercise 5.2.4. Follow these steps to provide a slightly modified proof of the Chain Rule.

(a) Show that a function $h : A \to \mathbf{R}$ is differentiable at $a \in A$ if and only if there exists a function $l : A \to \mathbf{R}$ which is continuous at a and satisfies

$$h(x) - h(a) = l(x)(x-a) \qquad \text{for all } x \in A.$$

(b) Use this criterion for differentiablilty (in both directions) to prove Theorem 5.2.5.

Solution. (a) Suppose there exists such a function $l : A \to \mathbf{R}$, so that for all $x \in A$ such that $x \neq a$ we have

$$\frac{h(x)-h(a)}{x-a} = l(x).$$

It follows that $h'(a) = l(a)$ since l is continuous at a.

Now suppose that $h : A \to \mathbf{R}$ is differentiable at a. Define $l : A \to \mathbf{R}$ by

$$l(x) = \begin{cases} \frac{h(x)-h(a)}{x-a} & \text{if } x \neq a, \\ h'(a) & \text{if } x = a. \end{cases}$$

Notice that l satisfies $h(x)-h(a) = l(x)(x-a)$ for all $x \in A$, and furthermore l is continuous at a:

$$\lim_{x\to a} l(x) = \lim_{x\to a} \frac{h(x)-h(a)}{x-a} = h'(a) = l(a).$$

(b) Suppose $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ are functions such that $f(A) \subseteq B$, so that the composition $g \circ f : A \to \mathbf{R}$ is defined. Suppose f is differentiable at $c \in A$ and g is differentiable at $f(c) \in B$. By part (a), there exist functions $l : A \to \mathbf{R}$ and $L : B \to \mathbf{R}$ such that l is continuous at c, L is continuous at $f(c)$, and

$$f(x) - f(c) = l(x)(x-a) \qquad \text{for all } x \in A,$$

$$g(y) - g(f(c)) = L(y)(y - f(c)) \qquad \text{for all } y \in B.$$

In particular, we have for all $x \in A$:

$$g(f(x)) - g(f(c)) = L(f(x))(f(x) - f(c)) = L(f(x))l(x)(x - a).$$

Since f is differentiable at c, it is also continuous at c (Theorem 5.2.3), and since L is continuous at $f(c)$, the composition $L \circ f$ is continuous at c (Theorem 4.3.9). Thus the product $(L \circ f)l$ is continuous at c (Theorem 4.3.4 (iii)). It follows from part (a) that $g \circ f$ is differentiable at c and furthermore that

$$(g \circ f)'(c) = L(f(c))l(c) = g'(f(c))f'(c).$$

Exercise 5.2.5. Let $f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$

(a) For which values of a is f continuous at zero?

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

(c) For which values of a is f twice-differentiable?

Solution. (a) For $a > 0$, we have $\lim_{x \to 0} f_a(x) = 0 = f_a(0)$ and thus f_a is continuous at zero. For $a = 0$, we have

$$\lim_{x \to 0^+} f_a(x) = 1 \neq 0 = \lim_{x \to 0^-} f_a(x)$$

and thus f_a is not continuous at zero. For $a < 0$, we have

$$\lim_{x \to 0^+} f_a(x) = +\infty \neq 0 = \lim_{x \to 0^-} f_a(x)$$

and thus f_a is not continuous at zero. We may conclude that f_a is continuous at zero if and only if $a > 0$.

(b) As we showed in part (a), f_a is not continuous, and hence not differentiable, at zero for $a \leq 0$. For $0 < a < 1$, observe that

$$\lim_{x \to 0^+} \frac{f_a(x) - f_a(0)}{x - 0} = \lim_{x \to 0^+} x^{a-1} = +\infty \neq 0 = \lim_{x \to 0^-} \frac{f_a(x) - f_a(0)}{x - 0}.$$

Thus f_a is not differentiable at zero. For $a = 1$, we have

$$\lim_{x\to 0^+} \frac{f_a(x) - f_a(0)}{x - 0} = 1 \neq 0 = \lim_{x\to 0^-} \frac{f_a(x) - f_a(0)}{x - 0}$$

and thus f_a is not differentiable at zero. For $a > 1$, we have

$$\lim_{x\to 0^+} \frac{f_a(x) - f_a(0)}{x - 0} = \lim_{x\to 0^+} x^{a-1} = 0 = \lim_{x\to 0^-} \frac{f_a(x) - f_a(0)}{x - 0}$$

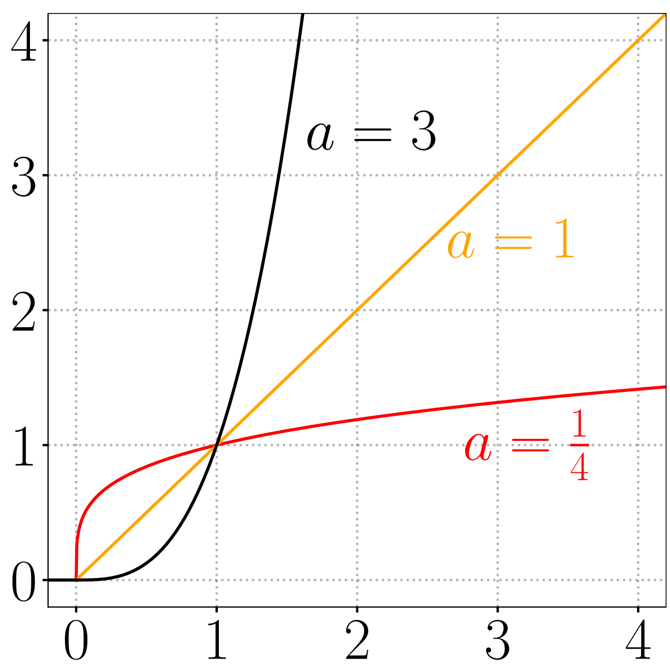
and so $f_a'(0) = 0$. The derivative function $f_a' : \mathbf{R} \to \mathbf{R}$ is given by

$$f_a'(x) = \begin{cases} ax^{a-1} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

which is continuous since $a > 1$.

(c) Similarly to part (b), f_a is twice-differentiable if and only if $a > 2$, and the second derivative function $f_a'' : \mathbf{R} \to \mathbf{R}$ is given by

$$f_a''(x) = \begin{cases} a(a-1)x^{a-2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$



Exercise 5.2.6. Let g be defined on an interval A, and let $c \in A$.

(a) Explain why $g'(c)$ in Definition 5.2.1 could have been given by

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}.$$

(b) Assume A is open. If g is differentiable at $c \in A$, show

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h}.$$

Solution. (a) Taking $x = c + h$ gives

$$\lim_{x \to c} \frac{g(x) - g(c)}{x - c} = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}.$$

(b) Let ϵ be given. By part (a), there is a $\delta > 0$ such that

$$0 < |h| < \delta \implies \left| \frac{g(c+h) - g(c)}{h} - g'(c) \right| < \epsilon.$$

Note that since $|-h| = |h|$, we also have

$$0 < |h| < \delta \implies \left| \frac{g(c-h) - g(c)}{-h} - g'(c) \right| < \epsilon.$$

For any h such that $0 < |h| < \delta$, it follows that

$$\begin{aligned}
\left| \frac{g(c+h) - g(c-h)}{2h} - g'(c) \right| &= \left| \frac{g(c+h) - g(c) + g(c) - g(c-h)}{2h} - \frac{2g'(c)}{2} \right| \\
&\leq \frac{1}{2} \left| \frac{g(c+h) - g(c)}{h} - g'(c) \right| + \frac{1}{2} \left| \frac{g(c-h) - g(c)}{-h} - g'(c) \right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus

$$\lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h} = g'(c).$$

Exercise 5.2.7. Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbf{R}$ but such that g_a' is unbounded on $[0,1]$.

(b) g_a is differentiable on $\mathbf{R}$ with g_a' continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbf{R}$ and g_a' is differentiable on $\mathbf{R}$, but such that g_a'' is not continuous at zero.

Solution. (a) Take $a = \frac{5}{3}$. For $x \neq 0$, we have by the usual rules of differentiation that

$$g_a'(x) = \frac{5x\sin\left(\frac{1}{x}\right) - 3\cos\left(\frac{1}{x}\right)}{3\sqrt[3]{x}}.$$

For $x = 0$, we have

$$g_a'(0) = \lim_{t\to 0}\frac{g_a(t)}{t} = \lim_{t\to 0} t^{2/3}\sin\left(\tfrac{1}{t}\right).$$

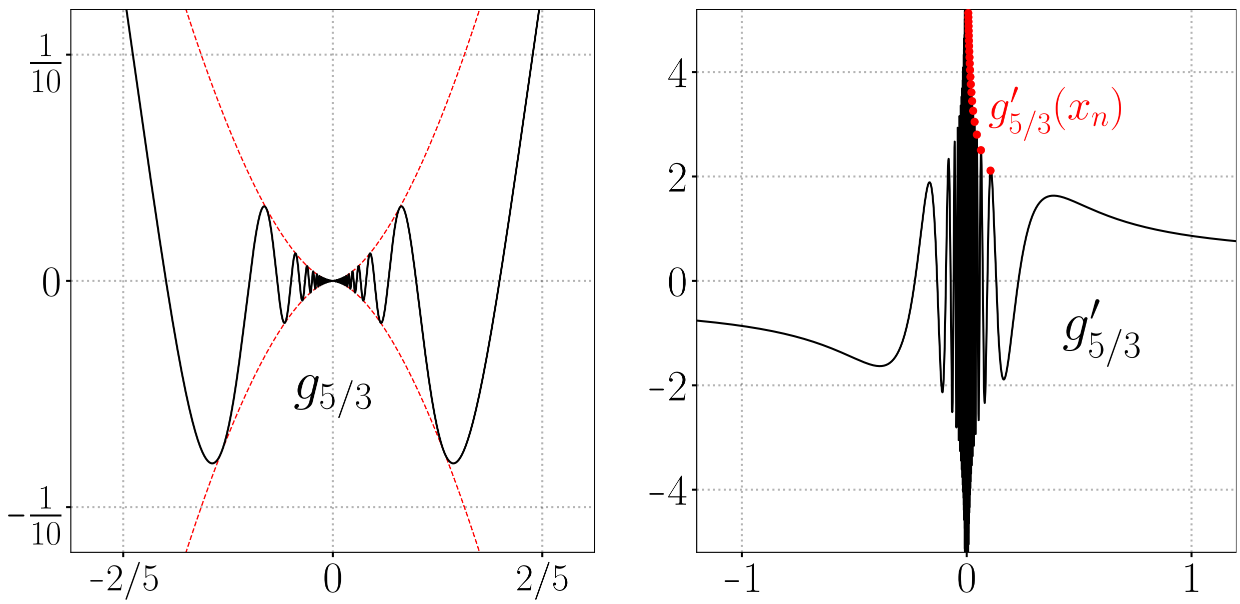
Since $-t^{2/3} \leq t^{2/3}\sin\left(\frac{1}{t}\right) \leq t^{2/3}$ for every $t \in \mathbf{R}$, the Squeeze Theorem implies that $g_a'(0) = 0$. Thus the derivative function $g_a' : \mathbf{R} \to \mathbf{R}$ is given by

$$g_a'(x) = \begin{cases} \frac{5x\sin\left(\frac{1}{x}\right) - 3\cos\left(\frac{1}{x}\right)}{3\sqrt[3]{x}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Consider the sequence (x_n) contained in $[0,1]$ given by $x_n = \frac{1}{\pi(1+2n)}$. Observe that

$$g_a'(x_n) = \frac{5x_n\sin(\pi(1+2n)) - 3\cos(\pi(1+2n))}{3\sqrt[3]{x_n}} = \frac{1}{\sqrt[3]{x_n}}.$$

It follows that $\lim_{n\to\infty} g_a'(x_n) = +\infty$ since the sequence x_n is positive and satisfies $\lim_{n\to\infty} x_n = 0$. Thus g_a' is unbounded on $[0,1]$.



(b) Take $a = 3$. For $x \neq 0$, we have by the usual rules of differentiation that

$$g_a'(x) = 3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right).$$

For $x = 0$, we have

$$g_a'(0) = \lim_{t \to 0} \frac{g_a(t)}{t} = \lim_{t \to 0} t^2 \sin\left(\frac{1}{t}\right) = 0,$$

where we have used the Squeeze Theorem as in part (a). Thus the derivative function $g_a' : \mathbf{R} \to \mathbf{R}$ is given by

$$g_a'(x) = \begin{cases} 3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Note that for $x \neq 0$, the function g_a' is given by various sums, products, and compositions of continuous functions and hence is itself continuous. For $x = 0$, the Squeeze Theorem shows that $\lim_{x \to 0} g_a'(x) = 0 = g_a'(0)$ and thus g_a' is continuous everywhere.

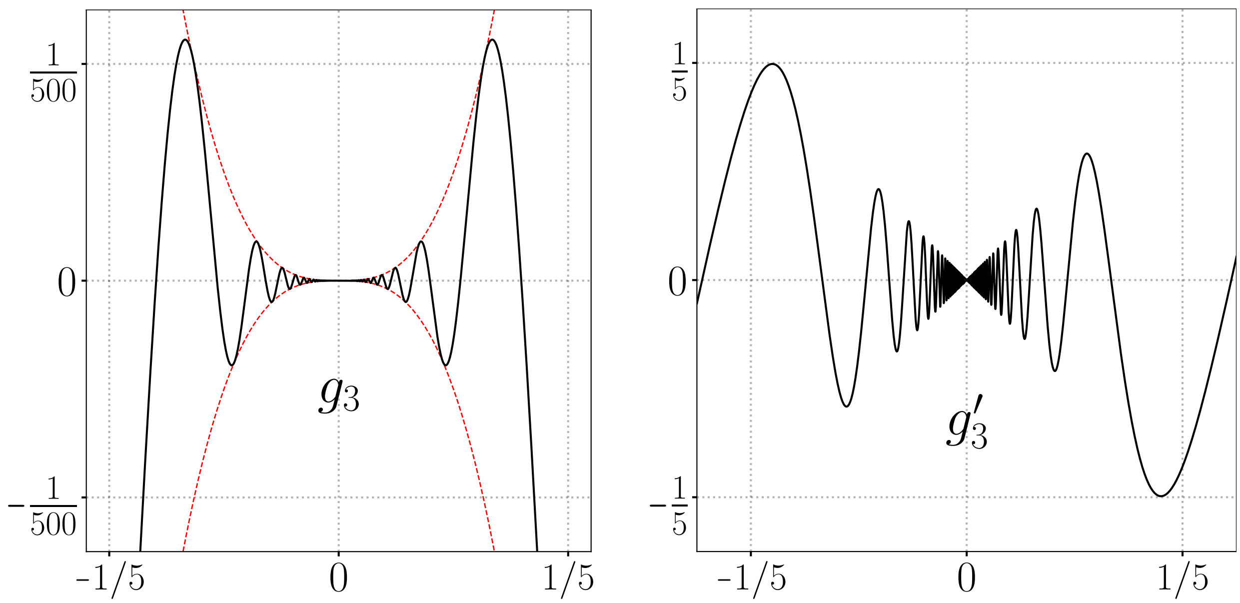
To see that g_a' is not differentiable at zero, observe that

$$\lim_{t \to 0} 3t \sin\left(\frac{1}{t}\right) = 0 \qquad \text{and} \qquad \lim_{t \to 0} \cos\left(\frac{1}{t}\right) \text{ does not exist.}$$

It follows from Corollary 4.2.4 that

$$\lim_{t\to 0}\frac{3t^2\sin\left(\frac{1}{t}\right)-t\cos\left(\frac{1}{t}\right)}{t}=\lim_{t\to 0}\left(3t\sin\left(\tfrac{1}{t}\right)-\cos\left(\tfrac{1}{t}\right)\right)$$

does not exist, i.e., g_a' is not differentiable at zero.



(c) Take $a = 4$. For $x \neq 0$, we have by the usual rules of differentiation that

$$g_a'(x) = 4x^3\sin\left(\frac{1}{x}\right) - x^2\cos\left(\frac{1}{x}\right).$$

For $x = 0$, we have

$$g_a'(0) = \lim_{t\to 0} t^3\sin\left(\frac{1}{t}\right) = 0,$$

where we have used the Squeeze Theorem as in parts (a) and (b). The derivative function is given by

$$g_a'(x) = \begin{cases} 4x^3\sin\left(\frac{1}{x}\right) - x^2\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For $x \neq 0$, we have by the usual rules of differentiation that

$$g_a''(x) = (12x^2-1)\sin\left(\frac{1}{x}\right) - 6x\cos\left(\frac{1}{x}\right).$$

For $x = 0$, we have

$$g_a''(0) = \lim_{t\to 0}\left(4t^2 \sin\left(\frac{1}{t}\right) - t\cos\left(\frac{1}{t}\right)\right) = 0,$$

where we have again used the Squeeze Theorem. Thus the second derivative function is given by

$$g_a''(x) = \begin{cases} (12x^2 - 1)\sin\left(\frac{1}{x}\right) - 6x\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

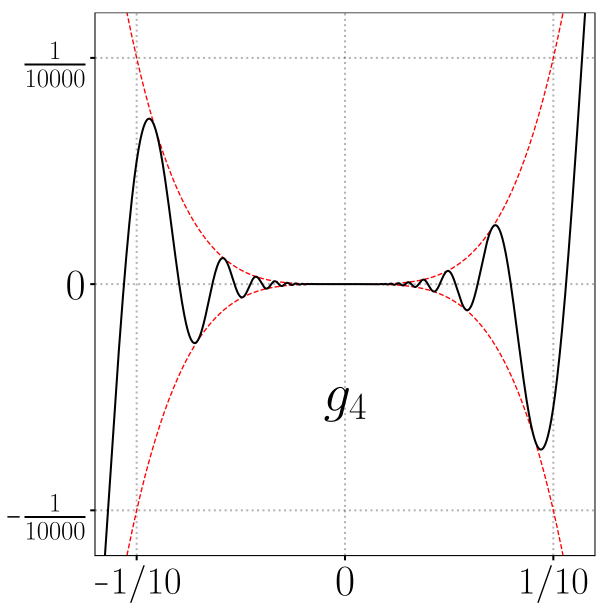
To see that g_a'' is not continuous at zero, note that

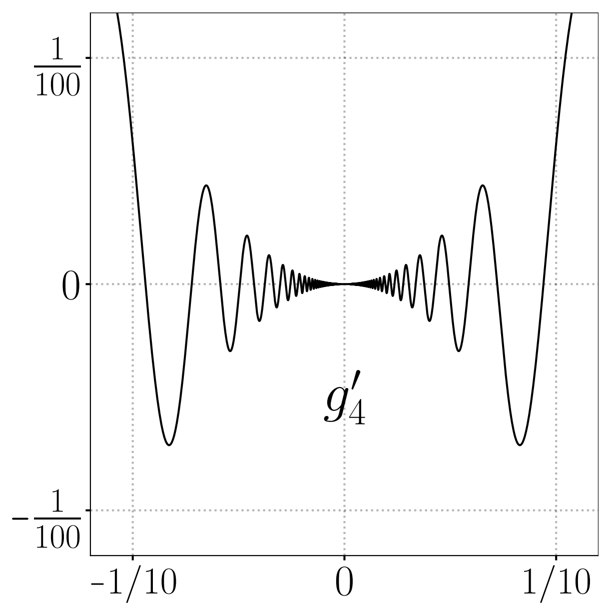
$$\lim_{x\to 0} 12x^2 \sin\left(\frac{1}{x}\right) = 0, \quad \lim_{x\to 0} 6x\cos\left(\frac{1}{x}\right) \qquad \text{and} \qquad \lim_{x\to 0}\sin\left(\frac{1}{x}\right) \text{ does not exist.}$$

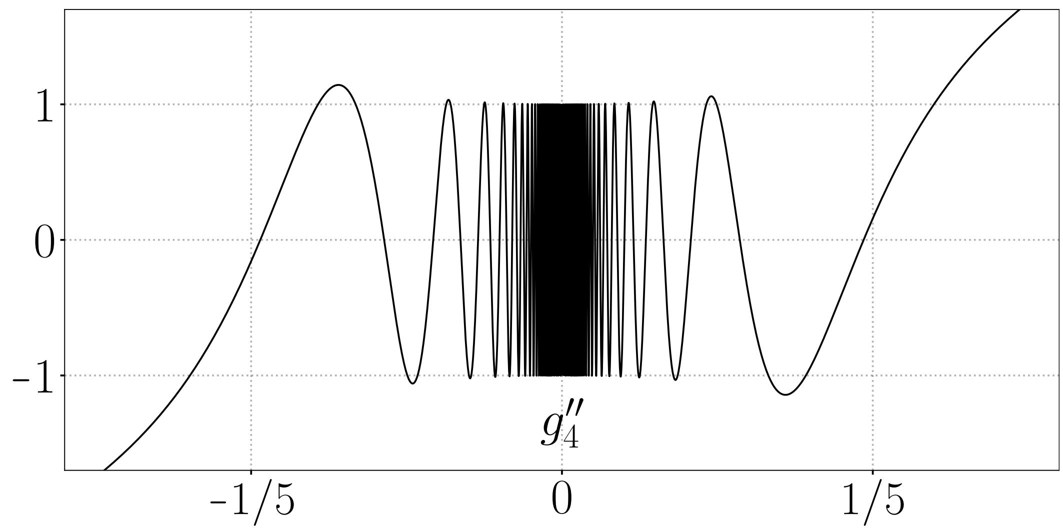
It follows from Corollary 4.2.4 that

$$\lim_{x\to 0} g_a''(x) = \lim_{x\to 0}\left(12x^2\sin\left(\frac{1}{x}\right) - 6x\cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right)\right)$$

does not exist.







Exercise 5.2.8. Review the definition of uniform continuity (Definition 4.4.4). Given a differentiable function $f : A \to \mathbf{R}$, let's say that f is *uniformly differentiable* on A if, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| < \epsilon \quad \text{whenever } 0 < |x-y| < \delta.$$

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbf{R}$? How about $g(x) = x^3$?

(b) Show that if a function is uniformly differentiable on an interval A, then the derivative must be continuous on A.

(c) Is there a theorem analogous to Theorem 4.4.7 for differentiation? Are functions that are differentiable on a closed interval $[a, b]$ necessarily uniformly differentiable?

Solution. (a) f is uniformly differentiable on $\mathbf{R}$. Let $\epsilon > 0$ be given, let $\delta = \epsilon$, and suppose $x, y \in \mathbf{R}$ are such that $0 < |x-y| < \delta$. It follows that

$$\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| = \left|\frac{x^2-y^2}{x-y} - 2y\right| = |x-y| < \delta = \epsilon.$$

However, g is not uniformly differentiable on $\mathbf{R}$. To see this, we need to show that there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exist real numbers x, y such that $0 < |x-y| < \delta$ and

$$\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| \geq \epsilon.$$

We claim that $\epsilon = 1$ satisfies the above. Indeed, let $\delta > 0$ be given. Let $x = \frac{2}{\delta}$ and $y = x + \frac{\delta}{2}$, so that $0 < |x - y| < \delta$, and observe that

$$\begin{aligned}
\left|\frac{g(x) - g(y)}{x - y} - g'(y)\right| &= \left|\frac{x^3 - y^3}{x - y} - 3y^2\right| \\
&= \left|\frac{(x - y)(x^2 + xy + y^2)}{x - y} - 3y^2\right| \\
&= \left|x^2 + xy - 2y^2\right| \\
&= |x - y||x + 2y| \\
&= \frac{\delta}{2}|3x + \delta| \\
&= \frac{\delta}{2}(3x + \delta) \\
&= \frac{3x\delta}{2} + \frac{\delta^2}{2} \\
&> \frac{x\delta}{2} \\
&= 1.
\end{aligned}$$

(b) Suppose $f : A \to \mathbf{R}$ is uniformly differentiable. Fix $\epsilon > 0$. Since f is uniformly differentiable, there exists a $\delta > 0$ such that

$$\left|\frac{f(s) - f(t)}{s - t} - f'(t)\right| < \frac{\epsilon}{2} \quad \text{whenever } 0 < |s - t| < \delta.$$

Fix $y \in A$ and suppose $x \in A$ is such that $0 < |x - y| < \delta$. Observe that

$$|f'(x) - f'(y)| \leq \left|\frac{f(y) - f(x)}{y - x} - f'(x)\right| + \left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus f' is continuous.

(c) There is no analogous theorem. Consider the function

$$f(x) = \begin{cases} x^{5/3} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

As we showed in Exercise 5.2.7 (a), f is differentiable on $\mathbf{R}$, and hence on $[0,1]$, but f' is unbounded on $[0,1]$. It follows that f' is not continuous on $[0,1]$ (since continuous functions preserve compactness) and hence by part (b) of this exercise, f cannot be uniformly differentiable on $[0,1]$.

Exercise 5.2.9. Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If f' exists on an interval and is not constant, then f' must take on some irrational values.

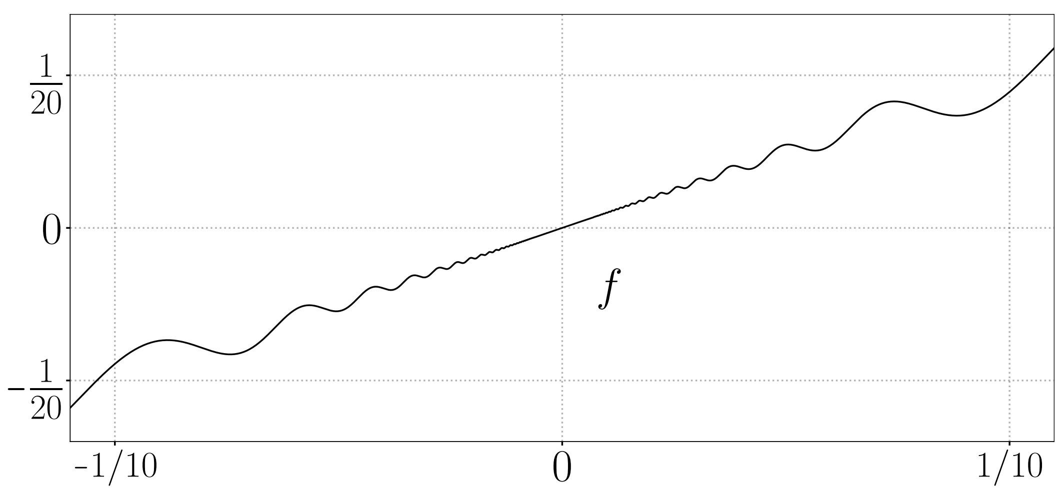
(b) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.

Solution. (a) This is true. If $f : I \to \mathbf{R}$ is differentiable and not constant, where I is an interval, then there exist distinct $x, y \in I$ such that $f'(x) \neq f'(y)$; we may assume that $f'(x) < f'(y)$. Darboux's Theorem (Theorem 5.2.7) implies that $[f'(x), f'(y)] \subseteq f'(I)$, from which it follows that f' takes on at least one (indeed, infinitely many) irrational values in the proper interval $[f'(x), f'(y)]$.

(b) This is false. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



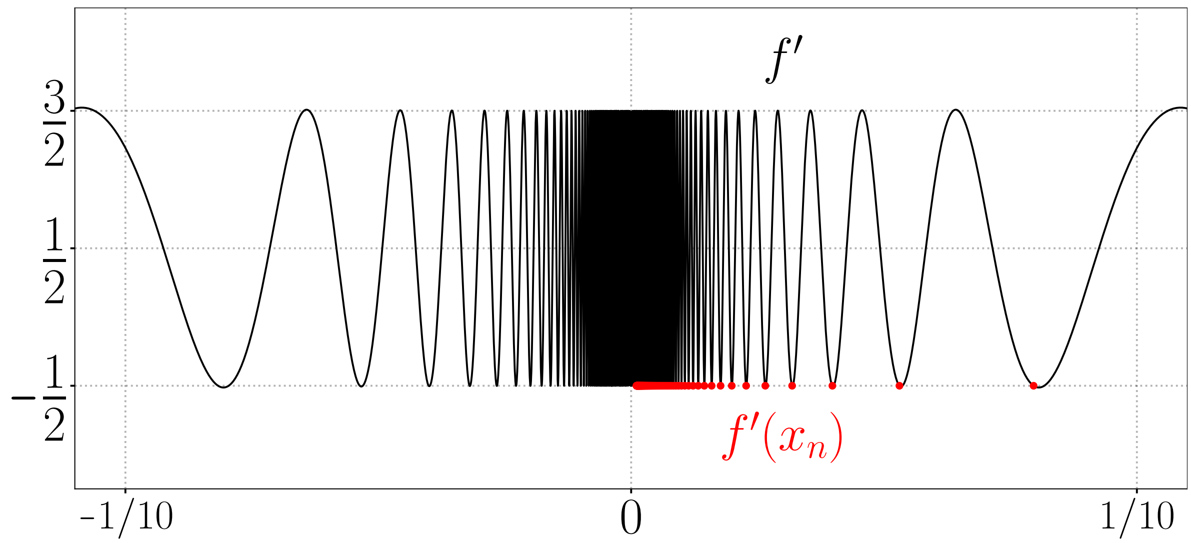
By the usual rules of differentiation and the Squeeze Theorem, the derivative $f' : \mathbf{R} \to \mathbf{R}$ is given by

$$f'(x) = \begin{cases} \frac{1}{2} + 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ \frac{1}{2} & \text{if } x = 0; \end{cases}$$

note that $f'(0) > 0$. Let (x_n) be the sequence given by $x_n = \frac{1}{2\pi n}$. Observe that $\lim_{n\to\infty} x_n = 0$ and

$$f'(x_n) = -\frac{1}{2} < 0.$$

It follows that for every δ-neighborhood $V_\delta(0)$ we can find some $x_n \in V_\delta(0)$ such that $f(x_n) < 0$.



(c) This is true and we will argue it by contradiction. Suppose that $L > f'(0)$; the case where $L < f'(0)$ is handled similarly. Let $\epsilon = L - f'(0) > 0$. Since $\lim_{x\to 0} f'(x) = L$, there is a $\delta > 0$ such that

$$x \in (-\delta, \delta) \text{ and } x \neq 0 \implies f'(x) \in \left(L - \frac{\epsilon}{2}, L + \frac{\epsilon}{2}\right). \tag{1}$$

In particular, we have $f'\left(\frac{\delta}{2}\right) \in \left(L - \frac{\epsilon}{2}, L + \frac{\epsilon}{2}\right)$. Since

$$f'(0) < L - \frac{3\epsilon}{4} < L - \frac{\epsilon}{2} < f'\left(\frac{\delta}{2}\right),$$

Darboux's Theorem (Theorem 5.2.7) implies that there is a $y \in \left(0, \frac{\delta}{2}\right)$ such that $f'(y) = L - \frac{3\epsilon}{4}$; this contradicts (1).

Exercise 5.2.10. Recall that a function $f : (a, b) \to \mathbf{R}$ is *increasing* on (a, b) if $f(x) \leq f(y)$ whenever $x < y$ in (a, b). A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate.

Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbf{R}$ and satisfies $g'(0) > 0$. Now, prove that g is *not* increasing over any open interval containing 0.

In the next section we will see that f is indeed increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

Solution. As we showed in Exercise 5.2.9 (b), the function g is differentiable on $\mathbf{R}$ and satisfies $g'(0) = \frac{1}{2} > 0$. For $n \in \mathbf{N}$ let

$$x_n := \frac{1}{\frac{\pi}{2} + 2\pi n} \qquad \text{and} \qquad y_n := \frac{1}{-\frac{\pi}{2} + 2\pi n}.$$

Suppose (a, b) is some open interval containing 0 and let N be such that $y_N < b$, so that $0 < x_N < y_N < b$. Observe that

$$g(x_N) = \frac{1}{\pi + 4\pi N} + \frac{1}{\left(\frac{\pi}{2} + 2\pi N\right)^2} \qquad \text{and} \qquad g(y_N) = \frac{1}{-\pi + 4\pi N} + \frac{1}{\left(-\frac{\pi}{2} + 2\pi N\right)^2}.$$

Some algebraic manipulation reveals that

$$g(x_N) - g(y_N) = \frac{2(16(4-\pi)N^2 + \pi + 4)}{\pi^2(4N-1)^2(4N+1)^2}.$$

The numerator and denominator of this fraction are both positive and so $g(x_N) - g(y_N)$ is also positive. Thus $x_N < y_N$ but $g(x_N) > g(y_N)$; it follows that g is not increasing on (a, b).

Exercise 5.2.11. Assume that g is differentiable on $[a, b]$ and satisfies $g'(a) < 0 < g'(b)$.

(a) Show that there exists a point $x \in (a,b)$ where $g(a) > g(x)$, and a point $y \in (a,b)$ where $g(y) < g(b)$.

(b) Now complete the proof of Darboux's Theorem started earlier.

Solution. (a) We will prove the contrapositive statement:

$$\text{if } g(a) \leq g(x) \text{ for all } x \in (a,b), \text{ then } g'(a) \geq 0.$$

Suppose therefore that for all $x \in (a,b)$ we have $g(a) \leq g(x)$. Let (x_n) be the sequence given by $x_n = a + \frac{b-a}{2n}$, so that $x_n \in (a,b)$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} x_n = a$. It follows from Theorem 4.2.3 that

$$\lim_{n\to\infty} \frac{g(x_n) - g(a)}{x_n - a} = g'(a).$$

The denominator of $\frac{g(x_n)-g(a)}{x_n-a}$ is positive for each $n \in \mathbf{N}$ and our assumption that $g(a) \leq g(x)$ for all $x \in (a,b)$ implies that the numerator is non-negative for each $n \in \mathbf{N}$. The Order Limit Theorem (Theorem 2.3.4) allows us to conclude that $g'(a) \geq 0$.

The existence of a point $y \in (a,b)$ where $g(y) < g(b)$ can be proved analogously.

(b) The function g is differentiable and hence continuous on $[a,b]$ (Theorem 5.2.3) and thus achieves a minimum value at some $c \in [a,b]$ by the Extreme Value Theorem (Theorem 4.4.2). By part (a), we actually have $c \in (a,b)$ and thus $g'(c) = 0$ by the Interior Extremum Theorem (Theorem 5.2.6).

Exercise 5.2.12 (Inverse functions). If $f : [a,b] \to \mathbf{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. In Exercise 4.5.8 we saw that if f is continuous on $[a,b]$, then f^{-1} is continuous on its domain. Let's add the assumption that f is differentiable on $[a,b]$ with $f'(x) \neq 0$ for all $x \in [a,b]$. Show f^{-1} is differentiable with

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} \quad \text{where } y = f(x).$$

Solution. Since $f'(x) \neq 0$, we have

$$\lim_{s\to x} \frac{s-x}{f(s)-f(x)} = \frac{1}{f'(x)}$$

by Corollary 4.2.4 (iv). Let $\epsilon > 0$ be given. There is a $\delta_1 > 0$ such that

$$0 < |s-x| < \delta_1 \implies \left| \frac{s-x}{f(s)-f(x)} - \frac{1}{f'(x)} \right| < \epsilon. \tag{1}$$

Since f^{-1} is continuous on its domain, we have $\lim_{t\to y} f^{-1}(t) = f^{-1}(y) = x$. Thus there exists a $\delta_2 > 0$ such that

$$0 < |t-y| < \delta_2 \implies 0 < \left|f^{-1}(t) - f^{-1}(y)\right| = \left|f^{-1}(t) - x\right| < \delta_1.$$

(The fact that $f^{-1}(t) \neq x$ follows since $t \neq y$ and f^{-1} is injective.) We may now take $s = f^{-1}(t)$ in (1) to see that

$$0 < |t-y| < \delta_2 \implies \left|\frac{f^{-1}(t) - x}{f(f^{-1}(t)) - f(x)} - \frac{1}{f'(x)}\right| = \left|\frac{f^{-1}(t) - f^{-1}(y)}{t-y} - \frac{1}{f'(x)}\right| < \epsilon.$$

Thus

$$\left(f^{-1}\right)'(y) = \lim_{t\to y} \frac{f^{-1}(t) - f^{-1}(y)}{t-y} = \frac{1}{f'(x)}.$$