1 Section 3.F Exercises

Exercises with solutions from Section 3.F of [LADR].

Exercise 3.F.1. Explain why every linear functional is either surjective or the zero map.

Solution. Suppose $\varphi : V \to \mathbf{F}$ is a non-zero linear functional, so that there is a $v \in V$ such that $\varphi(v) \neq 0$. Then for any $\lambda \in \mathbf{F}$, we have

$$\varphi\left(\frac{\lambda}{\varphi(v)}v\right) = \lambda.$$

Thus φ is surjective.

Exercise 3.F.2. Give three distinct examples of linear functionals on $\mathbf{R}^{[0,1]}$.

Solution. For $i = 0, 1, 2$, define $\varphi_i : \mathbf{R}^{[0,1]} \to \mathbf{R}$ by $\varphi_i(f) = f\left(\frac{i}{2}\right)$. Then each $\varphi_i \in \left(\mathbf{R}^{[0,1]}\right)'$.

Exercise 3.F.3. Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

Solution. Set $v_1 := v$ and extend this to a basis $v_1, \ldots, v_m$ of V. Take the dual basis $\varphi_1, \ldots, \varphi_m$ of V' and note that $\varphi_1(v_1) = 1$.

Exercise 3.F.4. Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

Solution. Let $u_1, \ldots, u_m$ be a basis of U and extend this to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V. Since $U \neq V$, there must be at least one v_j, i.e. $n \geq 1$. Define $\varphi : V \to \mathbf{F}$ by

$$\varphi(u_j) = 0 \quad \text{and} \quad \varphi(v_j) = 1.$$

Then $\varphi(u) = 0$ for all $u \in U$ but $\varphi \neq 0$ since $\varphi(v_1) = 1$.

Exercise 3.F.5. Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic vector spaces.

Solution. This follows from Exercise 3.E.4.

Exercise 3.F.6. Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma : V' \to \mathbf{F}^m$ by

$$\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)).$$

(a) Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.

(b) Prove that $v_1, \ldots, v_m$ is linearly independent if and only if Γ is surjective.

Solution. Let $e_1, \ldots, e_m$ be the standard basis of $\mathbf{F}^m$ and let $\psi_1, \ldots, \psi_m$ be the dual basis of $(\mathbf{F}^m)'$, so that $\psi_j(x_1, \ldots, x_m) = x_j$. Then the map $\Phi : \mathbf{F}^m \to (\mathbf{F}^m)'$ given by $\Phi(e_j) = \psi_j$ is an isomorphism and allows us to identify $\mathbf{F}^m$ with $(\mathbf{F}^m)'$. Define $T : \mathbf{F}^m \to V$ by

$$T(x_1, \ldots, x_m) = x_1 v_1 + \cdots + x_m v_m.$$

For any $\varphi \in V'$ and $(x_1, \ldots, x_m) \in \mathbf{F}^m$, observe that

$$[T'(\varphi)](x_1, \ldots, x_m) = \varphi(T(x_1, \ldots, x_m)) = \varphi(x_1 v_1 + \cdots + x_m v_m) = x_1\varphi(v_1) + \cdots + x_m\varphi(v_m).$$

Furthermore,

$$(\Phi \circ \Gamma)(\varphi) = \Phi(\varphi(v_1)e_1 + \cdots + \varphi(v_m)e_m) = \varphi(v_1)\psi_1 + \cdots + \varphi(v_m)\psi_m.$$

This implies that

$$[(\Phi \circ \Gamma)(\varphi)](x_1, \ldots, x_m) = x_1\varphi(v_1) + \cdots + x_m\varphi(v_m).$$

Thus $T' = \Phi \circ \Gamma$. Note that since Φ is a bijection, the injectivity of Γ is equivalent to the injectivity of T' and the surjectivity of Γ is equivalent to the surjectivity of T'.

(a) By Exercise 3.B.3, the list $v_1, \ldots, v_m$ spans V if and only if T is surjective. By 3.108, T is surjective if and only if T' is injective. By the previous discussion, T' is injective if and only if Γ is injective.

(b) By Exercise 3.B.3, the list $v_1, \ldots, v_m$ is linearly independent if and only if T is injective. By 3.110, T is injective if and only if T' is surjective. By the previous discussion, T' is surjective if and only if Γ is surjective.

Exercise 3.F.7. Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \ldots, x^m$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where $\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$. Here $p^{(j)}$ denotes the j^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

Solution. In what follows, i and j range over $\{0, 1, \ldots, m\}$. The dual basis is defined by $\varphi_j(x^i) = \delta^i_j$, where δ^i_j is the Kronecker delta. Define $\psi_j : \mathcal{P}_m(\mathbf{R}) \to \mathbf{R}$ by $\psi_j(p) = \frac{p^{(j)}(0)}{j!}$; each ψ_j is a linear functional since differentiation is a linear operation. Note that since

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j} x^i = \begin{cases} 0 & \text{if } i < j, \\ \frac{i!}{(i-j)!} x^{i-j} & \text{if } i \geq j, \end{cases}$$

we have $\psi_j(x^i) = \delta^i_j$. The uniqueness part of 3.5 now implies that $\varphi_j = \psi_j$.

Exercise 3.F.8. Suppose m is a positive integer.

(a) Show that $1, x-5, \ldots, (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) What is the dual basis of the basis in part (a)?

Solution. (a) If there are scalars $a_0, \ldots, a_m$ such that

$$a_0 + a_1(x-5) + \cdots + a_m(x-5)^m = 0,$$

then by considering the degree of each side of this equation we can see that $a_0 = \cdots = a_m = 0$. Thus $1, x-5, \ldots, (x-5)^m$ is a linearly independent list of $m+1$ vectors in an $(m+1)$-dimensional vector space and hence must be a basis.

(b) An analogous argument to the one given in Exercise 3.F.7 shows that the dual basis $\varphi_0, \ldots, \varphi_m$ to the basis in part (a) is given by

$$\varphi_j(p) = \frac{p^{(j)}(5)}{j!}.$$

Exercise 3.F.9. Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

Solution. Let $v \in V$ be given. There are scalars $a_1, \ldots, a_n$ such that $v = \sum_{j=1}^n a_j v_j$. Observe that

$$\begin{aligned}
\left(\sum_{i=1}^n \psi(v_i)\varphi_i\right)(v) &= \sum_{i=1}^n \psi(v_i)\varphi_i(v) \\
&= \sum_{i=1}^n \psi(v_i)\left[\varphi_i\left(\sum_{j=1}^n a_j v_j\right)\right] \\
&= \sum_{i=1}^n \psi(v_i)\sum_{j=1}^n a_j\varphi_i(v_j) \\
&= \sum_{i=1}^n a_i\psi(v_i) \\
&= \psi\left(\sum_{i=1}^n a_i v_i\right) \\
&= \psi(v).
\end{aligned}$$

Thus $\psi = \sum_{i=1}^n \psi(v_i)\varphi_i$.

Exercise 3.F.10. Prove the first two bullet points in 3.101.

Solution. The first bullet point says that $(S+T)' = S' + T'$ for any $S, T \in \mathcal{L}(V, W)$. Indeed, for any $\psi \in W'$ and $v \in V$ we have

$$\begin{aligned}[(S+T)'(\psi)](v) &= \psi((S+T)(v)) = \psi(Sv + Tv) = \psi(Sv) + \psi(Tv) \\ &= [S'(\psi)](v) + [T'(\psi)](v) = [S'(\psi) + T'(\psi)](v) = [(S' + T')(\psi)](v).\end{aligned}$$

The second bullet point says that $(\lambda T)' = \lambda T'$ for any $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Indeed, for any $\psi \in W'$ and $v \in V$ we have

$$[(\lambda T)'(\psi)](v) = \psi((\lambda T)(v)) = \psi(\lambda Tv) = \lambda\psi(Tv) = \lambda[T'(\psi)](v) = [\lambda T'(\psi)](v) = [(\lambda T')(\psi)](v).$$

Exercise 3.F.11. Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \dots, c_m) \in \mathbf{F}^m$ and $(d_1, \dots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$.

Solution. Suppose there exist $(c_1, \dots, c_m) \in \mathbf{F}^m$ and $(d_1, \dots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$. If we define

$$C := \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbf{F}^{m,1} \quad \text{and} \quad D := \begin{pmatrix} d_1 & \cdots & d_n \end{pmatrix} \in \mathbf{F}^{1,n},$$

then by assumption we have $A = CD$. Note that since $A \neq 0$, we have $\operatorname{rank} A \geq 1$. Furthermore, we must have $C, D \neq 0$, which implies that $\operatorname{rank} C = \operatorname{rank} D = 1$. Exercise 3.B.23 and 3.117 then give us

$$\operatorname{rank} A = \operatorname{rank} CD \leq \min\{\operatorname{rank} C, \operatorname{rank} D\} = 1.$$

We may conclude that $\operatorname{rank} A = 1$.

Now suppose that $\operatorname{rank} A = 1$, i.e. the span of the columns of A has dimension 1, so that there is a column

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbf{F}^{m,1}$$

of A such that each column of A is a scalar multiple of c. In other words, there are scalars $d_1, \dots, d_n$ such that

$$A_{\cdot,k} = d_k c$$

for each $1 \leq k \leq n$. It follows that $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$.

Exercise 3.F.12. Show that the dual map of the identity map on V is the identity map on V'.

Solution. Let $I : V \to V$ be the identity map. Then $I' : V' \to V'$ is defined by

$$I'(\psi) = \psi \circ I = \psi.$$

Thus I' is the identity on V'.

Exercise 3.F.13. Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Suppose φ_1, φ_2 denotes the dual basis of the standard basis of $\mathbf{R}^2$ and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of $\mathbf{R}^3$.

(a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.

(b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.

Solution. (a) By the definition of the dual map, we have

$$[T'(\varphi_1)](x, y, z) = \varphi_1(T(x, y, z)) = \varphi_1(4x + 5y + 6z, 7x + 8y + 9z) = 4x + 5y + 6z,$$
$$[T'(\varphi_2)](x, y, z) = \varphi_2(T(x, y, z)) = \varphi_2(4x + 5y + 6z, 7x + 8y + 9z) = 7x + 8y + 9z.$$

(b) Note that

$$\psi_1(x, y, z) = x, \quad \psi_2(x, y, z) = y, \quad \text{and} \quad \psi_3(x, y, z) = z.$$

It follows that

$$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3 \quad \text{and} \quad T'(\varphi_2) = 6\psi_1 + 7\psi_2 + 8\psi_3.$$

Exercise 3.F.14. Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for $x \in \mathbf{R}$.

(a) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbf{R})$.

(b) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\int_0^1 p(x)dx$. Evaluate $(T'(\varphi))(x^3)$.

Solution. (a) We have

$$\begin{aligned}
[T'(\varphi)](p) &= \varphi(Tp) \\
&= \varphi(x^2 p + p'') \\
&= (x^2 p(x) + p''(x))'|_{x=4} \\
&= (2xp(x) + x^2 p'(x) + p'''(x))|_{x=4} \\
&= 8p(4) + 16p'(4) + p'''(4).
\end{aligned}$$

(b) We have

$$[T'(\varphi)](x^3) = \varphi(Tx^3) = \varphi(x^5 + 6x) = \int_0^1 x^5 + 6x \, \mathrm{d}x = \tfrac{19}{6}.$$

Exercise 3.F.15. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

Solution. Suppose $T = 0$ and $\varphi \in W'$. Then

$$T'(\varphi) = \varphi \circ T = \varphi \circ 0 = 0.$$

Thus $T' = 0$.

Now suppose that $T' = 0$. Let $w_1, \dots, w_n$ be a basis of W and let $\psi_1, \dots, \psi_n$ be the corresponding dual basis of W'. For any $v \in V$, there are scalars $a_1, \dots, a_n$ such that $Tv = a_1 w_1 + \cdots + a_n w_n$. For each $1 \leq j \leq n$, note that

$$0 = [T'(\psi_j)](v) = \psi_j(Tv) = \psi_j(a_1 w_1 + \cdots + a_n w_n) = a_j.$$

Thus $Tv = 0$ and we see that $T = 0$.

Exercise 3.F.16. Suppose V and W are finite-dimensional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

Solution. Let Φ be the map in question, i.e. $\Phi(T) = T'$. 3.101 shows that Φ is linear and Exercise 3.F.15 shows that Φ is injective. 3.61 and 3.95 give us $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$ and so 3.69 allows us to conclude that Φ is an isomorphism.

Exercise 3.F.17. Suppose $U \subset V$. Explain why $U^0 = \{\varphi \in V' : U \subset \operatorname{null} \varphi\}$.

Solution. This follows since $\varphi(u) = 0 \iff u \in \operatorname{null} \varphi$.

Exercise 3.F.18. Suppose V is finite-dimensional and $U \subset V$. Show that $U = \{0\}$ if and only if $U^0 = V'$.

Solution. Suppose that $U = \{0\}$. Then because each $\varphi \in V'$ is a linear map, we have $\varphi(0) = 0$ and thus $\varphi \in U^0$. It follows that $U^0 = V'$.

Now suppose that $U \neq \{0\}$, i.e. there exists $u \in U$ with $u \neq 0$. By Exercise 3.F.3, there exists a linear functional $\varphi \in V'$ such that $\varphi(u) = 1$. It follows that $\varphi \notin U^0$, so that $U^0 \neq V'$.

Exercise 3.F.19. Suppose V is finite-dimensional and U is a subspace of V. Show that $U = V$ if and only if $U^0 = \{0\}$.

Solution. Suppose $U = V$. Then if $\varphi \in U^0$, we have $\varphi(v) = 0$ for all $v \in V$, i.e. $\varphi = 0$, or $U^0 = \{0\}$.

Now suppose that $U \neq V$. By Exercise 3.F.4, there is a linear functional $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$, i.e. $\varphi \in U^0$, but $\varphi \neq 0$. Thus $U^0 \neq \{0\}$.

Exercise 3.F.20. Suppose U and W are subsets of V with $U \subset W$. Prove that $W^0 \subset U^0$.

Solution. If $\varphi \in W^0$, then in particular $\varphi(u) = 0$ for each $u \in U$, since $U \subseteq W$. Thus $\varphi \in U^0$.

Exercise 3.F.21. Suppose V is finite-dimensional and U and W are subspaces of V with $W^0 \subset U^0$. Prove that $U \subset W$.

Solution. We will prove the contrapositive statement. Suppose that $U \not\subseteq W$, i.e. there exists $u \in U$ such that $u \notin W$. Let $w_1, \ldots, w_m$ be a basis of W. Since $u \notin W$, the list $w_1, \ldots, w_m, u$ must be linearly independent and thus we can extend this list to a basis $w_1, \ldots, w_m, u, v_1, \ldots, v_n$ for V. Define $\varphi \in V'$ by

$$\varphi(w_j) = \varphi(v_j) = 0 \quad \text{and} \quad \varphi(u) = 1.$$

Then $\varphi \in W^0$ but $\varphi \notin U^0$, i.e. $W^0 \not\subseteq U^0$.

Exercise 3.F.22. Suppose U, W are subspaces of V. Show that $(U + W)^0 = U^0 \cap W^0$.

Solution. Suppose that $\varphi \in (U + W)^0$. Since $U \subseteq U + W$ and $W \subseteq U + W$, we have in particular that $\varphi(u) = 0$ and $\varphi(w) = 0$ for all $u \in U$ and $w \in W$, i.e. $\varphi \in U^0 \cap W^0$. Thus $(U+W)^0 \subseteq U^0 \cap W^0$.

Now suppose that $\varphi \in U^0 \cap W^0$. For any $u + w \in U + W$, we have

$$\varphi(u + w) = \varphi(u) + \varphi(w) = 0 + 0 = 0.$$

It follows that $\varphi \in (U + W)^0$ and hence that $U^0 \cap W^0 \subseteq (U + W)^0$. We may conclude that $(U + W)^0 = U^0 \cap W^0$.

Exercise 3.F.23. Suppose V is finite-dimensional and U and W are subspaces of V. Prove that $(U \cap W)^0 = U^0 + W^0$.

Solution. Suppose that $\varphi \in U^0 + W^0$, so that $\varphi = \psi_1 + \psi_2$ for some $\psi_1 \in U^0$ and some $\psi_2 \in W^0$. If $v \in U \cap W$, then

$$\varphi(v) = \psi_1(v) + \psi_2(v) = 0 + 0 = 0.$$

Thus $\varphi \in (U \cap W)^0$ and we see that $U^0 + W^0 \subseteq (U \cap W)^0$.

For the reverse inclusion, let $t_1, \ldots, t_k$ be a basis of $U \cap W$. We extend this list twice: first to a basis $t_1, \ldots, t_k, u_1, \ldots, u_l$ of U and also to a basis $t_1, \ldots, t_k, w_1, \ldots, w_m$ of W. As the proof of 2.43 shows, the list $t_1, \ldots, t_k, u_1, \ldots, u_l, w_1, \ldots, w_m$ is a basis of $U + W$. Finally, extend this to a basis

$$t_1, \ldots, t_k, u_1, \ldots, u_l, w_1, \ldots, w_m, x_1, \ldots, x_n$$

of V. Let $\varphi \in (U \cap W)^0$ be given and define $\psi_1, \psi_2 \in V'$ by

$$\begin{aligned} \psi_1(t_j) = \psi_1(u_j) = 0, \quad \psi_1(w_j) = \varphi(w_j) \quad \text{and} \quad \psi_1(x_j) = \tfrac{1}{2}\varphi(x_j), \\ \psi_2(t_j) = \psi_2(w_j) = 0, \quad \psi_2(u_j) = \varphi(u_j) \quad \text{and} \quad \psi_2(x_j) = \tfrac{1}{2}\varphi(x_j). \end{aligned}$$

Since ψ_1 maps the basis vectors of U to 0 and ψ_2 maps the basis vectors of W to 0, we have $\psi_1 \in U^0$ and $\psi_2 \in W^0$. We claim that $\varphi = \psi_1 + \psi_2$. Let $v \in V$ be given. Then v is of the form

$$v = \sum a_j t_j + \sum b_j u_j + \sum c_j w_j + \sum d_j x_j.$$

Observe that

$$\begin{aligned} (\psi_1 + \psi_2)(v) &= \sum a_j(\psi_1 + \psi_2)(t_j) + \sum b_j(\psi_1 + \psi_2)(u_j) \\ &\quad + \sum c_j(\psi_1 + \psi_2)(w_j) + \sum d_j(\psi_1 + \psi_2)(x_j) \\ &= \sum a_j\varphi(t_j) + \sum b_j\varphi(u_j) \\ &\quad + \sum c_j\varphi(w_j) + \sum d_j\varphi(x_j) \\ &= \varphi(v). \end{aligned}$$

Our claim follows, i.e. $\varphi \in U^0 + W^0$, so that $(U \cap W)^0 \subseteq U^0 + W^0$. We may conclude that $(U \cap W)^0 = U^0 + W^0$.

Exercise 3.F.24. Prove 3.106 using the ideas sketched in the discussion before the statement of 3.106.

Solution. Let $u_1, \dots, u_m$ be a basis of U, which we extend to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V. Let $\varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_n$ be the corresponding dual basis of V'. We will show that $\mathscr{B} := \psi_1, \dots, \psi_n$ is a basis of U^0. Certainly, $\mathscr{B}$ is linearly independent. Furthermore, we claim that $U^0 = \operatorname{span}(\mathscr{B})$. By definition, for each $1 \leq j \leq n$ and $1 \leq i \leq m$, we have $\psi_j(u_i) = 0$, so that $\psi_j \in U^0$. Suppose that $\psi \in U^0$. There are scalars $a_1, \dots, a_m, b_1, \dots, b_n$ such that

$$\psi = a_1\varphi_1 + \cdots + a_m\varphi_m + b_1\psi_1 + \cdots + b_n\psi_n.$$

In particular, for each $1 \leq i \leq n$, we have $0 = \psi(u_i) = a_i$. Thus

$$\psi = b_1\psi_1 + \cdots + b_n\psi_n \in \operatorname{span}(\psi_1, \dots, \psi_n).$$

It follows that $U^0 = \operatorname{span}(\mathscr{B})$, as claimed. We may conclude that $\mathscr{B}$ is a basis of U^0, whence

$$\dim U + \dim U^0 = \dim V.$$

Exercise 3.F.25. Suppose V is finite-dimensional and U is a subspace of V. Show that

$$U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

Solution. Let $W = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}$. It is clear that $U \subseteq W$. For the reverse inclusion, let $u_1, \ldots, u_m$ be a basis of U, which we extend to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V. Let $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n$ be the corresponding dual basis of V'. As we showed in Exercise 3.F.24, $\psi_1, \ldots, \psi_n$ is a basis of U^0. Suppose $v \in W$. There are scalars $a_1, \ldots, a_m, b_1, \ldots, b_n$ such that

$$v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n.$$

Since $v \in W$, we have $\psi_j(v) = b_j = 0$. Thus v is of the form $v = a_1u_1 + \cdots + a_mu_m \in U$, so that $W \subseteq U$. We may conclude that $U = W$.

Exercise 3.F.26. Suppose V is finite-dimensional and Γ is a subspace of V'. Show that

$$\Gamma = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Gamma\}^0.$$

Solution. Let $W = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Gamma\}$. It is straightforward to verify that $\Gamma \subseteq W^0$, i.e. Γ is a subspace of W^0. If we let $\varphi_1, \ldots, \varphi_m$ be a basis of Γ, then $W = \operatorname{null}\varphi_1 \cap \cdots \cap \operatorname{null}\varphi_m$. Exercise 3.F.30 now implies that $\dim W = \dim V - m$, which in turn gives us $\dim W^0 = m$ by 3.106. So Γ is a subspace of W^0 such that $\dim \Gamma = \dim W^0$; it must be the case that $\Gamma = W^0$.

Exercise 3.F.27. Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}), \mathcal{P}_5(\mathbf{R}))$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that $\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}$.

Solution. By 3.107, we have $\operatorname{null} T' = (\operatorname{range} T)^0 = \operatorname{span}(\varphi)$, and by Exercise 3.F.25, we have

$$\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbf{R}) : \psi(p) = 0 \text{ for every } \psi \in (\operatorname{range} T)^0\}.$$

Since $(\operatorname{range} T)^0 = \operatorname{span}(\varphi)$, we see that

$$\begin{aligned}\operatorname{range} T &= \{p \in \mathcal{P}_5(\mathbf{R}) : \psi(p) = 0 \text{ for every } \psi \in \operatorname{span}(\varphi)\} \\ &= \{p \in \mathcal{P}_5(\mathbf{R}) : \varphi(p) = 0\} = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}.\end{aligned}$$

Exercise 3.F.28. Suppose V and W are finite-dimensional, $T \in \mathcal{L}(V, W)$, and there exists $\varphi \in W'$ such that $\operatorname{null} T' = \operatorname{span}(\varphi)$. Prove that $\operatorname{range} T = \operatorname{null} \varphi$.

Solution. By 3.107, we have $\operatorname{null} T' = (\operatorname{range} T)^0 = \operatorname{span}(\varphi)$, and by Exercise 3.F.25, we have

$$\operatorname{range} T = \{w \in W : \psi(w) = 0 \text{ for every } \psi \in (\operatorname{range} T)^0\}.$$

Since $(\operatorname{range} T)^0 = \operatorname{span}(\varphi)$, we see that

$$\operatorname{range} T = \{w \in W : \psi(w) = 0 \text{ for every } \psi \in \operatorname{span}(\varphi)\} = \{w \in W : \varphi(w) = 0\} = \operatorname{null} \varphi.$$

Exercise 3.F.29. Suppose V and W are finite-dimensional, $T \in \mathcal{L}(V,W)$, and there exists $\varphi \in V'$ such that $\operatorname{range} T' = \operatorname{span}(\varphi)$. Prove that $\operatorname{null} T = \operatorname{null} \varphi$.

Solution. By 3.109, we have $\operatorname{range} T' = (\operatorname{null} T)^0 = \operatorname{span}(\varphi)$, and by Exercise 3.F.25, we have

$$\operatorname{null} T = \{v \in V : \psi(v) = 0 \text{ for every } \psi \in (\operatorname{null} T)^0\}.$$

Since $(\operatorname{null} T)^0 = \operatorname{span}(\varphi)$, we see that

$$\operatorname{null} T = \{v \in V : \psi(v) = 0 \text{ for every } \psi \in \operatorname{span}(\varphi)\} = \{v \in V : \varphi(v) = 0\} = \operatorname{null} \varphi.$$

Exercise 3.F.30. Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_m$ is a linearly independent list in V'. Prove that

$$\dim((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)) = (\dim V) - m.$$

Solution. First, let us prove the following lemma.

Lemma 1. Suppose V is finite-dimensional and $\varphi \in V'$. Then $\operatorname{span}(\varphi) = (\operatorname{null} \varphi)^0$.

Proof. It is straightforward to verify that $\operatorname{span}(\varphi) \subseteq (\operatorname{null} \varphi)^0$. The Fundamental Theorem of Linear Maps (3.22) and 3.106 combine to show that $\dim \operatorname{range} \varphi = \dim(\operatorname{null} \varphi)^0$, and since $\varphi = 0 \iff \operatorname{span}(\varphi) = \{0\}$, Exercise 3.F.1 shows that $\dim \operatorname{span}(\varphi) = \dim \operatorname{range} \varphi$. Thus $\dim \operatorname{span}(\varphi) = \dim(\operatorname{null} \varphi)^0$ and we may conclude that $\operatorname{span}(\varphi) = (\operatorname{null} \varphi)^0$. □

Note that by Exercise 3.F.23 and Lemma 1, we have

$$\begin{aligned}
\dim((\operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m)^0) &= \dim((\operatorname{null} \varphi_1)^0 + \cdots + (\operatorname{null} \varphi_m)^0) \\
&= \dim(\operatorname{span}(\varphi_1) + \cdots + \operatorname{span}(\varphi_m)) \\
&= \dim \operatorname{span}(\varphi_1, \ldots, \varphi_m) \\
&= m,
\end{aligned}$$

where the last equality follows since the list $\varphi_1, \ldots, \varphi_m$ is linearly independent. 3.106 now gives us

$$\dim(\operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m) = \dim V - \dim((\operatorname{null} \varphi_1 \cap \cdots \cap \operatorname{null} \varphi_m)^0) = \dim V - m.$$

Exercise 3.F.31. Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_n$ is a basis of V'. Show that there exists a basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$.

Solution. For each $1 \leq j \leq n$, we have by Exercise 3.F.30 that $\dim\left(\bigcap_{i \neq j} \operatorname{null} \varphi_i\right) = 1$ and thus $\bigcap_{i \neq j} \operatorname{null} \varphi_i = \operatorname{span}(u_j)$ for some $u_j \neq 0$ in V. Note that Exercise 3.F.30 also implies that $\bigcap_{1 \leq i \leq n} \operatorname{null} \varphi_i = \{0\}$. Since u_j is non-zero, it must be the case that $\varphi_j(u_j) \neq 0$. Given this, we can define $v_j := \frac{u_j}{\varphi_j(u_j)}$; is straightforward to verify that $\varphi_i(v_j) = \delta^i_j$. If we have scalars $a_1, \ldots, a_n$ such that

$$a_1 v_1 + \cdots + a_n v_n = 0,$$

then applying φ_j to both sides of this equation shows that each $a_j = 0$, i.e. the list $v_1, \ldots, v_n$ is linearly independent. By 3.95, we have $\dim V = n$ and so 2.39 implies that $v_1, \ldots, v_n$ is a basis of V. Finally, the uniqueness part of 3.5 shows that $\varphi_1, \ldots, \varphi_n$ is the dual basis to $v_1, \ldots, v_n$.

Exercise 3.F.32. Suppose $T \in \mathcal{L}(V)$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Prove that the following are equivalent:

(a) T is invertible.

(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$.

(c) The columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.

(d) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$.

(e) The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$.

Solution. In what follows, let $c_1, \ldots, c_n$ be the columns of $\mathcal{M}(T)$ and let $r_1, \ldots, r_n$ be the rows of $\mathcal{M}(T)$.

Suppose (a) holds, so that T is surjective. By 3.117, we must have

$$\dim \operatorname{span}(c_1, \ldots, c_n) = \dim \operatorname{range} T = \dim V = n.$$

It follows from 2.42 that $c_1, \ldots, c_n$ is a basis of $\operatorname{span}(c_1, \ldots, c_n)$ and thus is a linearly independent list, i.e. (b) holds.

Suppose (b) holds. Then since $\mathbf{F}^{n,1}$ is n-dimensional, 2.39 implies that $c_1, \ldots, c_n$ is a basis of $\mathbf{F}^{n,1}$ and thus (c) holds.

Suppose (c) holds, so that $\dim \operatorname{span}(c_1, \ldots, c_n) = \dim \mathbf{F}^{n,1} = n$. By 3.118, we must also have $\dim \operatorname{span}(r_1, \ldots, r_n) = n$. It follows from 2.42 that $r_1, \ldots, r_n$ is a basis of $\operatorname{span}(r_1, \ldots, r_n)$ and thus is a linearly independent list, i.e. (d) holds.

Suppose (d) holds. Then since $\mathbf{F}^{1,n}$ is n-dimensional, 2.39 implies that $r_1, \ldots, r_n$ is a basis of $\mathbf{F}^{1,n}$ and thus (e) holds.

Suppose (e) holds, so that $\dim \operatorname{span}(r_1, \ldots, r_n) = n$. 3.118 and 3.117 then imply that $\dim \operatorname{range} T = n$ and we see that T is surjective. It follows from 3.69 that T is invertible, i.e. (a) holds.

Exercise 3.F.33. Suppose m and n are positive integers. Prove that the function that takes A to A^{t} is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$. Furthermore, prove that this linear map is invertible.

Solution. Let $\Psi : \mathbf{F}^{m,n} \to \mathbf{F}^{n,m}$ be the map $\Psi(A) = A^{\mathrm{t}}$. If A, B are m-by-n matrices and $\lambda \in \mathbf{F}$, then:

$$(A + \lambda B)^{\mathrm{t}}_{j,k} = (A + \lambda B)_{k,j} = A_{k,j} + \lambda B_{k,j} = A^{\mathrm{t}}_{j,k} + \lambda B^{\mathrm{t}}_{j,k}.$$

It follows that Ψ is a linear map. To see that Ψ is invertible, define $\Phi : \mathbf{F}^{n,m} \to \mathbf{F}^{m,n}$ by $\Phi(A) = A^{\mathrm{t}}$; it is clear that Ψ and Φ are mutual inverses.

Exercise 3.F.34. The ***double dual space*** of V, denoted V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda : V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for $v \in V$ and $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

(c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

[*Suppose V is finite-dimensional. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is considered more natural.*]

Solution. (a) Suppose $u, v \in V$ and $\lambda \in \mathbf{F}$. Then for any $\varphi \in V'$, we have

$$(\Lambda(u + \lambda v))(\varphi) = \varphi(u + \lambda v) = \varphi(u) + \lambda\varphi(v) = (\Lambda u)(\varphi) + \lambda(\Lambda v)(\varphi) = (\Lambda u + \lambda \Lambda v)(\varphi).$$

It follows that Λ is a linear map.

(b) $T'' \circ \Lambda$ and $\Lambda \circ T$ are both maps $V \to V''$. Let $v \in V$ be given. Then $\Lambda(Tv) \in V''$ is given by

$$(\Lambda(Tv))(\varphi) = \varphi(Tv).$$

The dual map T'' sends $\Lambda v \in V''$ to $(\Lambda v) \circ T' \in V''$ and hence

$$(T''(\Lambda v))(\varphi) = (\Lambda v)(T'(\varphi)) = (\Lambda v)(\varphi \circ T) = \varphi(Tv).$$

Thus $\Lambda \circ T = T'' \circ \Lambda$.

(c) Let $v_1, \ldots, v_n$ be a basis of V and $\varphi_1, \ldots, \varphi_n$ the corresponding dual basis of V'. Suppose $v = a_1 v_1 + \cdots + a_n v_n$ is such that $\Lambda v = 0$, i.e. $\varphi(v) = 0$ for every $\varphi \in V'$. In particular, we have $\varphi_j(v) = a_j = 0$ for each $1 \leq j \leq n$, so that $v = 0$. Hence $\operatorname{null} \Lambda = \{0\}$ and we see that Λ is injective. By 3.95 we have $\dim V = \dim V' = \dim V''$ and so 3.69 allows us to conclude that Λ is an isomorphism.

Exercise 3.F.35. Show that $(\mathcal{P}(\mathbf{R}))'$ and $\mathbf{R}^\infty$ are isomorphic.

Solution. Define a map $\Phi : (\mathcal{P}(\mathbf{R}))' \to \mathbf{R}^\infty$ by

$$\Phi(\varphi) = (\varphi(1), \varphi(x), \varphi(x^2), \ldots).$$

This map is linear. Indeed, if $\varphi, \psi \in (\mathcal{P}(\mathbf{R}))'$ and $\lambda \in \mathbf{F}$, then

$$\begin{aligned}\Phi(\varphi + \lambda\psi) &= ((\varphi + \lambda\psi)(1), (\varphi + \lambda\psi)(x), \ldots) = (\varphi(1) + \lambda\psi(1), \varphi(x) + \lambda\psi(x), \ldots) \\ &= (\varphi(1), \varphi(x), \ldots) + \lambda(\psi(1), \psi(x), \ldots) = \Phi(\varphi) + \lambda\Phi(\psi).\end{aligned}$$

Φ is injective: if $\varphi \in (\mathcal{P}(\mathbf{R}))'$ is such that $\Phi(\varphi) = 0$, i.e. $\varphi(x^j) = 0$ for all $j \geq 0$, then

$$\varphi(p) = \varphi\left(\sum_{j=0}^{\deg p} a_j x^j\right) = \sum_{j=0}^{\deg p} a_j \varphi(x^j) = 0.$$

It follows that $\varphi = 0$, hence that $\operatorname{null} \Phi = \{0\}$, and hence that Φ is injective.

To see that Φ is surjective, let $(y_0, y_1, y_2, \ldots) \in \mathbf{R}^\infty$ be given. Define a map $\varphi : \mathcal{P}(\mathbf{R}) \to \mathbf{R}$ by

$$\varphi(p) = \varphi\left(\sum_{j=0}^{\deg p} a_j x^j\right) = \sum_{j=0}^{\deg p} a_j y_j.$$

Let $p = \sum_{j=0}^{\deg p} a_j x^j$ and $q = \sum_{j=0}^{\deg q} b_j x^j$ be given and suppose without loss of generality that $\deg p \leq \deg q$. Then $p + q = \sum_{j=0}^{\deg p} (a_j + b_j) x^j + \sum_{j=\deg p+1}^{\deg q} b_j x^j$ (if $\deg p = \deg q$, we consider this second sum to be zero). Thus

$$\varphi(p + q) = \sum_{j=0}^{\deg p} (a_j + b_j) y_j + \sum_{j=\deg p+1}^{\deg q} b_j y_j = \sum_{j=0}^{\deg p} a_j y_j + \sum_{j=0}^{\deg q} b_j y_j = \varphi(p) + \varphi(q).$$

If $\lambda \in \mathbf{F}$, then

$$\varphi(\lambda p) = \sum_{j=0}^{\deg p} \lambda a_j y_j = \lambda \sum_{j=0}^{\deg p} a_j y_j = \lambda\varphi(p).$$

Hence φ is a linear functional on $\mathcal{P}(\mathbf{R})$. Since $\varphi(x^j) = y_j$, we see that $\Phi(\varphi) = (y_0, y_1, y_2, \ldots)$ and hence that Φ is surjective. We may conclude that Φ is an isomorphism.

Exercise 3.F.36. Suppose U is a subspace of V. Let $i : U \to V$ be the inclusion map defined by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.

(a) Show that $\operatorname{null} i' = U^0$.

(b) Prove that if V is finite-dimensional, then $\operatorname{range} i' = U'$.

(c) Prove that if V is finite-dimensional, then $\widetilde{i'}$ is an isomorphism from V'/U^0 onto U'.

[*The isomorphism in part (c) is natural in that it does not depend on a choice of basis in either vector space.*]

Solution. (a) For $\varphi \in V', i'\varphi$ is the map $\varphi \circ i : U \to \mathbf{F}$, which is simply the restriction of φ to U. Hence

$$i'\varphi = 0 \iff \varphi(u) = 0 \text{ for all } u \in U.$$

It follows that $\operatorname{null} i' = U^0$.

(b) Let $\psi \in U'$ be given. By Exercise 3.A.11, we can extend ψ to a linear functional $\varphi \in V'$ in such a way that $\varphi|_U = \psi$. It follows that $i'\varphi = \psi$ and we see that i' is surjective.

(c) 3.91 shows that $\widetilde{i'}$ is an isomorphism of $V'/(\operatorname{null} i')$ onto $\operatorname{range} i'$. Using parts (a) and (b), we see that $\widetilde{i'}$ is an isomorphism of V'/U^0 onto $\operatorname{range} i'$.

Exercise 3.F.37. Suppose U is a subspace of V. Let $\pi : V \to V/U$ be the usual quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.

(a) Show that π' is injective.

(b) Show that $\operatorname{range} \pi' = U^0$.

(c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.

[*The isomorphism in part (c) is natural in that it does not depend on a choice of basis in either vector space. In fact, there is no assumption here that any of these vector spaces are finite-dimensional.*]

Solution. Note that π' is the map Γ from Exercise 3.E.20, taking $W = \mathbf{F}$.

(a) This follows from part (b) of Exercise 3.E.20.

(b) This follows from part (c) of Exercise 3.E.20.

(c) This is immediate from parts (a) and (b) of this exercise.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edition.