Linear Algebra Done Right Solutions

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Contents

Chapter 1. Vector Spaces

1.A. $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 1.A.1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi$ and $\beta = u + vi$, then

$$\alpha + \beta = (x + u) + (y + v)i = (u + x) + (v + y)i = \beta + \alpha,$$

where we have used the commutativity of addition in $\mathbf{R}$.

Exercise 1.A.2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}(\alpha + \beta) + \lambda &= ((x + u) + (y + v))i + \lambda = ((x + u) + s) + ((y + v) + t)i \\ &= (x + (u + s)) + (y + (v + t))i = \alpha + ((u + s) + (v + t)i) = \alpha + (\beta + \lambda),\end{aligned}$$

where we have used the associativity of addition in $\mathbf{R}$.

Exercise 1.A.3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}(\alpha\beta)\lambda &= [(xu - yv) + (xv + yu)i]\lambda \\ &= [(xu - yv)s - (xv + yu)t] + [(xu - yv)t + (xv + yu)s]i \\ &= [(xu)s - (yv)s - (xv)t - (yu)t] + [(xu)t - (yv)t + (xv)s + (yu)s]i \\ &= [x(us) - x(vt) - y(ut) - y(vs)] + [x(ut) + x(vs) + y(us) - y(vt)]i \\ &= [x(us - vt) - y(ut + vs)] + [x(ut + vs) + y(us - vt)]i \\ &= \alpha[(us - vt) + (ut + vs)i] \\ &= \alpha(\beta\lambda),\end{aligned}$$

where we have used several algebraic properties of $\mathbf{R}$.

Exercise 1.A.4. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Solution. If $\alpha = x + yi, \beta = u + vi$, and $\lambda = s + ti$, then

$$\begin{aligned}
\lambda(\alpha + \beta) &= [s(x+u) - t(y+v)] + [s(y+v) + t(x+u)i] \\
&= (sx + su - ty - tv) + (sy + sv + tx + tu)i \\
&= [(sx - ty) + (sy + tx)i] + [(su - tv) + (sv + tu)i] \\
&= \lambda\alpha + \lambda\beta,
\end{aligned}$$

where we have used distributivity in $\mathbf{R}$.

Exercise 1.A.5. Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Solution. Suppose that $\alpha = x + yi$. Let $\beta = -x - yi$ and observe that

$$\alpha + \beta = (x - x) + (y - y)i = 0 + 0i = 0.$$

To see that β is unique, suppose that β' also satisfies $\alpha + \beta' = 0$ and notice that

$$\beta = \beta = 0 = \beta + (\alpha + \beta') = (\alpha + \beta) + \beta' = 0 + \beta' = \beta',$$

where we have used the associativity of addition in $\mathbf{C}$ (Exercise 1.A.2) and the commutativity of addition in $\mathbf{C}$ (Exercise 1.A.1).

Exercise 1.A.6. Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Solution. Suppose that $\alpha = x + yi$. Since $\alpha \neq 0$, it must be the case that x and y are not both zero, so that $x^2 + y^2 \neq 0$. Let $\beta = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$ and observe that

$$\alpha\beta = (x + yi)\left(\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i\right) = \frac{x^2+y^2}{x^2+y^2} + \frac{xy - xy}{x^2+y^2}i = 1 + 0i = 1.$$

To see that β is unique, suppose β' also satisfies $\alpha\beta' = 1$ and notice that

$$\beta = \beta 1 = \beta(\alpha\beta') = (\alpha\beta)\beta' = 1\beta' = \beta',$$

where we have used the associativity of multiplication in $\mathbf{C}$ (Exercise 1.A.3) and the commutativity of multiplication in $\mathbf{C}$ (1.4).

Exercise 1.A.7. Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution. Let $z = \frac{-1+\sqrt{3}i}{2}$, so that $2z = -1+\sqrt{3}i$. Observe that

$$\begin{aligned}(2z)^2 = 4z^2 &= \left(-1+\sqrt{3}i\right)^2 = 1 - 2\sqrt{3}i - 3 = -2 - 2\sqrt{3}i \\ \Rightarrow (2z)^3 = (4z^2)(2z) &= \left(-2-2\sqrt{3}i\right)\left(-1+\sqrt{3}i\right) = 2 - 2\sqrt{3}i + 2\sqrt{3}i + 6 = 8,\end{aligned}$$

i.e. $8z^3 = 8$. It follows that $z^3 = 1$.

Exercise 1.A.8. Find two distinct square roots of i.

Solution. Let $z_1 = \frac{1+i}{\sqrt{2}}$ and $z_2 = -z_1$ (z_1 and z_2 are distinct since $z_1 \neq 0$) and observe that

$$2z_1^2 = (1+i)^2 = 2i \quad \Rightarrow \quad z_1^2 = i,$$

i.e. z_1 is a square root of i. Furthermore, $z_2^2 = (-z_1)^2 = z_1^2 = i$, so that z_2 is a square root of i also.

Exercise 1.A.9. Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution. The unique solution is $x = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right)$.

Exercise 1.A.10. Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i).$$

Solution. If there was such a λ, then

$$\lambda(2-3i) = 12-5i \quad \Rightarrow \quad \lambda = \frac{12-5i}{2-3i} = 3+2i.$$

However,

$$(3+2i)(-6+7i) = -32+9i \neq -32-9i.$$

Exercise 1.A.11. Show that $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$, and $z = (z_1, ..., z_n)$, then

$$\begin{aligned}(x+y)+z &= (x_1+y_1, ..., x_n+y_n)+z = ((x_1+y_1)+z_1, ..., (x_n+y_n)+z_n) \\ &= (x_1+(y_1+z_1), ..., x_n+(y_n+z_n)) = x+(y_1+z_1, ..., y_n+z_n) = x+(y+z),\end{aligned}$$

where we have used the associativity of addition in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.2).

Exercise 1.A.12. Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Solution. If $x = (x_1, ..., x_n)$, then

$$(ab)x = ((ab)x_1, ..., (ab)x_n) = (a(bx_1), ..., a(bx_n)) = a(bx_1, ..., bx_n) = a(bx),$$

where we have used the associativity of multiplication in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.3).

Exercise 1.A.13. Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$, then

$$1x = (1x_1, ..., 1x_n) = (x_1, ..., x_n) = x,$$

where we have used that $1x_j = x_j$ for any $x_j \in \mathbf{F}$.

Exercise 1.A.14. Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, then

$$\begin{aligned}\lambda(x+y) &= \lambda(x_1+y_1, ..., x_n+y_n) \\ &= (\lambda(x_1+y_1), ..., \lambda(x_n+y_n)) \\ &= (\lambda x_1+\lambda y_1, ..., \lambda x_n+\lambda y_n) \\ &= (\lambda x_1, ..., \lambda x_n)+(\lambda y_1, ..., \lambda y_n) \\ &= \lambda(x_1, ..., x_n)+\lambda(y_1, ..., y_n) \\ &= \lambda x+\lambda y,\end{aligned}$$

where we have used distributivity in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.4).

Exercise 1.A.15. Show that $(a+b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Solution. If $x = (x_1, ..., x_n)$, then

$$\begin{aligned}(a+b)x &= (a+b)(x_1, ..., x_n)\\ &= ((a+b)x_1, ..., (a+b)x_n)\\ &= (ax_1 + bx_1, ..., ax_n + bx_n)\\ &= (ax_1, ..., ax_n) + (bx_1, ..., bx_n)\\ &= a(x_1, ..., x_n) + b(x_1, ..., x_n)\\ &= ax + bx,\end{aligned}$$

where we have used distributivity in $\mathbf{F}$ (we proved this for $\mathbf{C}$ in Exercise 1.A.4).

1.B. Definition of Vector Space

Exercise 1.B.1. Show that $-(-v) = v$ for every $v \in V$.

Solution. Since $v + (-v) = 0$ and the additive inverse of a vector is unique (1.27), it must be the case that $-(-v) = v$.

Exercise 1.B.2. Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution. It will suffice to show that if $av = 0$ and $a \neq 0$, so that a^{-1} exists, then $v = 0$. Indeed,

$$av = 0 \quad \Rightarrow \quad a^{-1}(av) = 0 \quad \Rightarrow \quad (a^{-1}a)v = 0 \quad \Rightarrow \quad 1v = 0 \quad \Rightarrow \quad v = 0.$$

Exercise 1.B.3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution. For $v, w, x \in V$, notice that

$$v + 3x = w \quad \Leftrightarrow \quad 3x = w - v \quad \Leftrightarrow \quad x = \tfrac{1}{3}(w - v).$$

Exercise 1.B.4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Solution. The empty set does not contain an additive identity.

Exercise 1.B.5. Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

> *The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.*

Solution. If V satisfies all of the conditions in (1.20), then as shown in (1.30) we have $0v = 0$ for all $v \in V$. Suppose that V satisfies all of the conditions in (1.20), except we have replaced the additive inverse condition with the condition that $0v = 0$ for all $v \in V$. We want to show that for each $v \in V$, there exists an element $w \in V$ such that $v + w = 0$. Indeed, for $v \in V$, let $w = (-1)v$ and observe that

$$v + w = 1v + (-1)v = (1 - 1)v = 0v = 0.$$

Exercise 1.B.6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$t + \infty = \infty + t = \infty + \infty = \infty,$$
$$t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty,$$
$$\infty + (-\infty) = (-\infty) + \infty = 0.$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution. This is not a vector space over $\mathbf{R}$, since addition is not associative:

$$(1 + \infty) + (-\infty) = \infty + (-\infty) = 0 \neq 1 = 1 + 0 = 1 + (\infty + (-\infty)).$$

Exercise 1.B.7. Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Solution. We define addition and scalar multiplication on V^S as in (1.24), i.e. for $f, g \in V^S$ the sum $f + g \in V^S$ is the function

$$\begin{aligned} f + g \,:\, S &\to V \\ x &\mapsto f(x) + g(x); \end{aligned}$$

the addition $f(x) + g(x)$ is vector addition in V. Similarly, for $\lambda \in \mathbf{F}$ and $f \in V^S$, the product $\lambda f \in V^S$ is the function

$$\begin{aligned} \lambda f \,:\, S &\to V \\ x &\mapsto \lambda f(x); \end{aligned}$$

the product $\lambda f(x)$ is scalar multiplication in V. We now show that V^S with these definitions satisfies each condition in definition (1.20).

Commutativity. Let $f, g \in V^S$ and $x \in S$ be given. Observe that

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x),$$

where we have used the commutativity of addition in V for the second equality. It follows that $f + g = g + f$.

Associativity. Let $f, g, h \in V^S$ and $x \in S$ be given. Observe that

$$\begin{aligned}((f+g)+h)(x) &= (f+g)(x) + h(x) = (f(x)+g(x)) + h(x) \\ &= f(x) + (g(x)+h(x)) = f(x) + (g+h)(x) = (f+(g+h))(x),\end{aligned}$$

where we have used the associativity of addition in V for the third equality. It follows that $(f+g)+h = f+(g+h)$. Similarly, let $f \in V^S$ and $a, b \in \mathbf{F}$ be given. Observe that, for any $x \in S$,

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = a((bf)(x)) = (a(bf))(x),$$

where we have used the associativity of scalar multiplication in V for the second equality. It follows that $(ab)f = a(bf)$.

Additive identity. We claim that the additive identity in V^S is the function $0 : S \to V$ given by $0(x) = 0$ for any $x \in S$; the 0 on the right-hand side is the additive identity in V. Indeed, for any $f \in V^S$ and $x \in S$ we have

$$(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x).$$

It follows that $f + 0 = f$.

Additive inverse. For $f \in V^S$, define $g : S \to V$ by $g(x) = -f(x)$ for $x \in S$, where $-f(x)$ is the additive inverse in V of $f(x)$. We claim that g is the additive inverse of f. To see this, let $x \in S$ be given and observe that

$$(f+g)(x) = f(x) + g(x) = f(x) + (-f(x)) = 0 = 0(x);$$

it follows that $f + g = 0$.

Multiplicative identity. Let $f \in V^S$ and $x \in S$ be given. Observe that

$$(1f)(x) = 1f(x) = f(x),$$

where we have used that $1v = v$ for any $v \in V$. It follows that $1f = f$.

Distributive properties. Let $a \in \mathbf{F}$ and $f, g \in V^S$ be given. Observe that, for any $x \in S$,

$$\begin{aligned}(a(f+g))(x) &= a(f+g)(x) = a((f(x)+g(x)) \\ &= af(x) + ag(x) = (af)(x) + (ag)(x) = (af+ag)(x),\end{aligned}$$

where we have used the first distributive property in V for the third equality. It follows that $a(f+g) = af + ag$. Similarly, let $a, b \in \mathbf{F}$ and $f \in V^S$ be given. For any $x \in S$, observe that

$$((a+b)f)(x) = (a+b)f(x) = af(x) + bf(x) = (af)(x) + (bf)(x) = (af+bf)(x),$$

where we have used the second distributive property in V for the second equality. It follows that $(a+b)f = af + bf$.

We may conclude that V^S is a vector space over $\mathbf{F}$.

Exercise 1.B.8. Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by
$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by
$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$
for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

> *Think of V as a subset of $V_{\mathbf{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbf{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.*

Solution. We need to verify each condition in definition (1.20). The algebraic manipulations required to show that commutativity, associativity, and the first distributive property hold for $V_{\mathbf{C}}$ are essentially the same algebraic manipulations we performed in Exercise 1.A.1, Exercise 1.A.2, Exercise 1.A.3, and Exercise 1.A.4, except instead of using the algebraic properties of $\mathbf{R}$, we use the algebraic properties of V (i.e. the properties listed in (1.20)); we will avoid repeating ourselves and instead verify the remaining conditions.

Additive identity. We claim that the additive identity in $V_{\mathbf{C}}$ is $0 + i0$, where 0 is the additive identity in V. Indeed, for any $u + iv \in V_{\mathbf{C}}$ we have
$$(u + iv) + (0 + i0) = (u + 0) + i(v + 0) = u + iv.$$

Additive inverse. We claim that the additive inverse of an element $u + iv \in V_{\mathbf{C}}$ is the element $(-u) + i(-v)$, where $-u$ is the additive inverse of u in V. Indeed,
$$(u + iv) + ((-u) + i(-v)) = (u + (-u)) + i(v + (-v)) = 0 + i0.$$

Multiplicative identity. For any $u + iv \in V_{\mathbf{C}}$, we have
$$(1 + 0i)(u + iv) = (1u - 0v) + i(1v + 0u) = u + iv.$$

Distributive properties. For the second distributive property, let $a + bi, c + di \in \mathbf{C}$ and $u + iv \in V_{\mathbf{C}}$ be given. Observe that

$$
\begin{aligned}
((a+bi)+(c+di))(u+iv) &= ((a+c)+(b+d)i)(u+iv) \\
&= ((a+c)u-(b+d)v)+i((a+c)v+(b+d)u) \\
&= (au+cu-bv-dv)+i(av+cv+bu+du) \\
&= ((au-bv)+i(av+bu))+((cu-dv)+i(cv+du)) \\
&= (a+bi)(u+iv)+(c+di)(u+iv),
\end{aligned}
$$

where we have used the second distributive property for V for the third equality.

1.C. Subspaces

Exercise 1.C.1. For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

Solution. Let U denote the set in each part of this question.

(a) This is a subspace of $\mathbf{F}^3$. Certainly the zero vector belongs to U. Suppose that $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in U$ and $\alpha \in \mathbf{F}$ and observe that

$$
(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) = 0 + 0 = 0,
$$
$$
\alpha x_1 + 2(\alpha x_2) + 3(\alpha x_3) = \alpha(x_1 + 2x_2 + 3x_3) = \alpha 0 = 0.
$$

Thus $x + y$ and αx also belong to U. It follows from (1.34) that U is a subspace of V.

(b) This is not a subspace of $\mathbf{F}^3$ because it does not contain the zero vector.

(c) This is not a subspace of $\mathbf{F}^3$ because it is not closed under addition: $(1, 1, 0)$ and $(0, 0, 1)$ belong to U, but $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$ does not belong to U.

(d) This is a subspace of $\mathbf{F}^3$. Note that $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 - 5x_3 = 0\}$. Certainly the zero vector belongs to U. Suppose that $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in U$ and $\alpha \in \mathbf{F}$ and observe that

$$
(x_1 + y_1) - 5(x_3 + y_3) = (x_1 - 5x_3) + (y_1 - 5y_3) = 0 + 0 = 0,
$$
$$
\alpha x_1 - 5(\alpha x_3) = \alpha(x_1 - 5x_3) = \alpha 0 = 0.
$$

Thus $x + y$ and αx also belong to U. It follows from (1.34) that U is a subspace of V.

Exercise 1.C.2. Verify all assertions about subspaces in Example 1.35.

Solution.

(a) The assertion is that if $b \in \mathbf{F}$, then

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\} = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 - 5x_4 = b\}$$

is a subspace of $\mathbf{F}^4$ if and only if $b = 0$. Indeed, if $b \neq 0$ then U is not a subspace of $\mathbf{F}^4$ because the zero vector does not belong to U, and if $b = 0$ then we may argue as in Exercise 1.C.1 (d) to see that U is a subspace of $\mathbf{F}^4$.

(b) The assertion is that the set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$, i.e.

$$U = \{f : [0, 1] \to \mathbf{R}, f \text{ continuous}\}$$

is a subspace of $\mathbf{R}^{[0,1]}$. Certainly the zero function $x \mapsto 0$ on $[0, 1]$ is continuous and hence belongs to U, and it is well-known from elementary real analysis that sums and constant multiples of continuous functions are again continuous. It follows from (1.34) that U is a subspace of $\mathbf{R}^{[0,1]}$.

(c) The assertion is that the set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$, i.e.

$$U = \{f : \mathbf{R} \to \mathbf{R}, f \text{ differentiable}\}$$

is a subspace of $\mathbf{R}^{\mathbf{R}}$. Certainly the zero function $x \mapsto 0$ on $\mathbf{R}$ is differentiable and hence belongs to U, and it is well-known from elementary real analysis that sums and constant multiples of differentiable functions are again differentiable. It follows from (1.34) that U is a subspace of $\mathbf{R}^{\mathbf{R}}$.

(d) The assertion is that the set U of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$. If $b \neq 0$, then the zero function $x \mapsto 0$ on $(0, 3)$, which has derivative $0 \neq b$ at $x = 2$, does not belong to U and thus U is not a subspace of $\mathbf{R}^{(0,3)}$.

Suppose that $b = 0$ and note that the zero function now belongs to U. If $f, g \in U$ and $\alpha \in \mathbf{R}$, then

$$(f + g)'(2) = f'(2) + g'(2) = 0 + 0 = 0 \quad \text{and} \quad (\alpha f)'(2) = \alpha f'(2) = \alpha 0 = 0.$$

Thus $f + g$ and αf belong to U. It follows from (1.34) that U is a subspace of $\mathbf{R}^{(0,3)}$.

(e) The assertion is that the set U of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^\infty$. Certainly the zero sequence $(0, 0, 0, \ldots)$ has limit 0 and hence belongs to U. Suppose that $x = (x_n)_{n=1}^\infty$ and $y = (y_n)_{n=1}^\infty$ belong to U and $\alpha \in \mathbf{C}$. Using basic results about limits, observe that

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0 + 0 = 0$$
$$\text{and} \quad \lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n = \alpha 0 = 0.$$

Thus $x + y$ and αx belong to U. It follows from (1.34) that U is a subspace of $\mathbf{C}^{(0,3)}$.

Exercise 1.C.3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

Solution. Let U be the set in question; it is straightforward to verify that the zero function belongs to U. Suppose that $f, g \in U$ and $\alpha \in \mathbf{R}$. Observe that

$$(f+g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f(2) + g(2)) = 3(f+g)(2)$$
$$\text{and} \quad (\alpha f)'(-1) = \alpha f'(-1) = \alpha(3f(2)) = 3(\alpha f(2)) = 3(\alpha f)(2).$$

Thus $f + g$ and αf belong to U. It follows from (1.34) that U is a subspace of $\mathbf{R}^{(-4,4)}$.

Exercise 1.C.4. Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

Solution. Let U be the set in question. If $b \neq 0$ then the zero function $x \mapsto 0$ on $[0, 1]$, which has integral $0 \neq b$ over $[0, 1]$, does not belong to U and thus U is not a subspace of $\mathbf{R}^{[0,1]}$. Suppose that $b = 0$ and note that the zero function now belongs to U. If $f, g \in U$ and $\alpha \in \mathbf{R}$, then using basic properties of integration we have

$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0 + 0 = 0 \quad \text{and} \quad \int_0^1 (\alpha f) = \alpha \int_0^1 f = \alpha 0 = 0.$$

Thus $f + g$ and αf belong to U. It follows from (1.34) that U is a subspace of $\mathbf{R}^{[0,1]}$.

Exercise 1.C.5. Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

Solution. The question is whether the subset

$$\mathbf{R}^2 = \{(x, y) : x, y \in \mathbf{R}\} \subseteq \{(z, w) : z, w \in \mathbf{C}\} = \mathbf{C}^2$$

is a subspace, where we are taking complex scalars in $\mathbf{C}^2$. This is not a subspace because it is not closed under scalar multiplication: $(1, 0) \in \mathbf{R}^2$ but $i(1, 0) = (i, 0) \notin \mathbf{R}^2$.

Exercise 1.C.6.

(a) Is $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?

(b) Is $\{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of $\mathbf{C}^3$?

Solution.

(a) Let U be the set in question. For $a, b \in \mathbf{R}$ we have $a^3 = b^3$ if and only if $a = b$ and thus the set U can be expressed as

$$U = \{(a, a, c) \in \mathbf{R}^3 : a, c \in \mathbf{R}\}.$$

Certainly $(0, 0, 0) \in U$. If $(a, a, c), (x, x, y) \in U$ and $\lambda \in \mathbf{R}$, then observe that

$$(a, a, c) + (x, x, y) = (a + x, a + x, c + y) \in U \quad \text{and} \quad \lambda(a, a, c) = (\lambda a, \lambda a, \lambda c) \in U.$$

It follows from (1.34) that U is a subspace of $\mathbf{R}^3$.

(b) Let U be the set in question. Observe that

$$\left(\frac{-1 + \sqrt{3}i}{2}\right)^3 = \left(\frac{-1 - \sqrt{3}i}{2}\right)^3 = 1.$$

It follows that $u := \left(\frac{-1+\sqrt{3}i}{2}, 1, 0\right)$ and $v := \left(\frac{-1-\sqrt{3}i}{2}, 1, 0\right)$ belong to U. However,

$$u + v = (-1, 2, 0) \notin U.$$

Thus U is not a subspace of $\mathbf{C}^3$ because it is not closed under addition.

Exercise 1.C.7. Prove or give a counterexample: If U is a nonempty subset of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbf{R}^2$.

Solution. For a counterexample, consider $U = \{(a, b) : a, b \in \mathbf{Q}\} \subseteq \mathbf{R}^2$, which satisfies the required conditions since the sum of two rational numbers is a rational number and the additive inverse of a rational number is a rational number. However, U is not a subspace of $\mathbf{R}^2$ because it is not closed under scalar multiplication: $(1, 0) \in U$ but $\sqrt{2}(1, 0) = \left(\sqrt{2}, 0\right) \notin U$.

Exercise 1.C.8. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

Solution. Let U be the union of the x- and y-axes, i.e.

$$U = \{(x, 0) : x \in \mathbf{R}\} \cup \{(0, y) : y \in \mathbf{R}\}.$$

It is straightforward to verify that U is closed under scalar multiplication. However, U is not a subspace of $\mathbf{R}^2$ because it is not closed under addition: $(1,0)$ and $(0,1)$ belong to U, but $(1,0)+(0,1)=(1,1)$ does not.

Exercise 1.C.9. A function $f:\mathbf{R}\to\mathbf{R}$ is called *periodic* if there exists a positive number p such that $f(x)=f(x+p)$ for all $x\in\mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution. Consider the periodic functions $\sin(x)$ and $\sin\left(\sqrt{2}x\right)$ and let $f(x)=\sin(x)+\sin\left(\sqrt{2}x\right)$. We will show that f is not periodic.

Suppose there was a positive real number p such that $f(x)=f(x+p)$ for all $x\in\mathbf{R}$, i.e.

$$\sin(x)+\sin\left(\sqrt{2}x\right)=\sin(x+p)+\sin\left(\sqrt{2}x+\sqrt{2}p\right)\text{ for all }x\in\mathbf{R}. \tag{1}$$

By differentiating this equation twice, we see that

$$\sin(x)+2\sin\left(\sqrt{2}x\right)=\sin(x+p)+2\sin\left(\sqrt{2}x+\sqrt{2}p\right)\text{ for all }x\in\mathbf{R}. \tag{2}$$

Subtracting equation (1) from equation (2) gives us

$$\sin\left(\sqrt{2}x\right)=\sin\left(\sqrt{2}x+\sqrt{2}p\right)\text{ for all }x\in\mathbf{R}, \tag{3}$$

which together with equation (1) implies that

$$\sin(x)=\sin(x+p)\text{ for all }x\in\mathbf{R}. \tag{4}$$

By taking $x=0$ in equation (4) we see that $0=\sin(p)$, which is the case if and only if $p=n\pi$ for some positive integer n (p was assumed to be positive). Substituting this value of p and $x=0$ into equation (3) gives $0=\sin\left(n\sqrt{2}\pi\right)$, which is the case if and only if $n\sqrt{2}\pi=m\pi$ for some integer m, which must be positive since n is positive. It follows that $\sqrt{2}=\frac{m}{n}$, contradicting the irrationality of $\sqrt{2}$.

Thus f is not periodic and we may conclude that the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ is not a subspace of $\mathbf{R}^{\mathbf{R}}$ because it is not closed under addition.

Exercise 1.C.10. Suppose V_1 and V_2 are subspaces of V. Prove that the intersection $V_1\cap V_2$ is a subspace of V.

Solution. Because V_1 and V_2 are subspaces of V, we have $0\in V_1$ and $0\in V_2$ and thus $0\in V_1\cap V_2$. Suppose $u,v\in V_1\cap V_2$ and $\lambda\in\mathbf{F}$. Since $u,v\in V_1$ and V_1 is a subspace of V, we have $u+v\in V_1$ and $\lambda u\in V_1$. Similarly, $u+v\in V_2$ and $\lambda u\in V_2$. Thus $u+v\in V_1\cap V_2$ and $\lambda u\in V_1\cap V_2$. We may use (1.34) to conclude that $V_1\cap V_2$ is a subspace of V.

Exercise 1.C.11. Prove that the intersection of every collection of subspaces of V is a subspace of V.

Solution. Let $\mathcal{U}$ be an arbitrary collection of subspaces of V. We will show that $\bigcap \mathcal{U}$ is a subspace of V. The zero vector belongs to $\bigcap \mathcal{U}$ because each $U \in \mathcal{U}$ is a subspace of V and hence contains the zero vector. If $u, v \in \bigcap \mathcal{U}$, $\lambda \in \mathbf{F}$, and $U \in \mathcal{U}$, then $u, v \in U$ and thus $u + v \in U$ and $\lambda u \in U$ since U is a subspace of V. Because $U \in \mathcal{U}$ was arbitrary, it follows that $u + v \in \bigcap \mathcal{U}$ and $\lambda u \in \bigcap \mathcal{U}$. We may use (1.34) to conclude that $\bigcap \mathcal{U}$ is a subspace of V.

Exercise 1.C.12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution. Suppose that U and W are subspaces of V. We want to show that $U \cup W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$. If one of U or W is contained in the other then either $U \cup W = U$ or $U \cup W = W$; in either case, $U \cup W$ is then a subspace of V by assumption.

For the converse, it will suffice to show that if $U \cup W$ is a subspace of V and $U \not\subseteq W$, then $W \subseteq U$. Since $U \not\subseteq W$, there is some $u \in U$ such that $u \notin W$. Let $w \in W$ be given and note that, because $U \cup W$ is a subspace of V and $u, w \in U \cup W$, we must have $u + w \in U \cup W$. It cannot be the case that $u + w \in W$, since then $u + w - w = u \in W$, so it must be the case that $u + w \in U$. It follows that $u + w - u = w \in U$ and hence that $W \subseteq U$, as desired.

Exercise 1.C.13. Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

This exercise is surprisingly harder than Exercise 1.C.12, possibly because this exercise is not true if we replace $\mathbf{F}$ *with a field containing only two elements.*

Solution. Let U_1, U_2, and U_3 be subspaces of V. We want to show that $U = U_1 \cup U_2 \cup U_3$ is a subspace of V if and only if some U_j contains the other two. If some U_j contains the other two, then $U = U_j$ is a subspace of V by assumption.

Suppose that U is a subspace of V. If any U_j is contained in the union of the other two, say $U_1 \subseteq U_2 \cup U_3$, then $U = U_2 \cup U_3$ and we may apply Exercise 1.C.12 to see that either $U_2 \subseteq U_3$ or $U_3 \subseteq U_2$; in either case, one U_j contains the other two. Suppose therefore that no U_j is contained in the union of the other two. Seeking a contradiction, suppose further that no U_j contains the other two, so that

$$U_1 \not\subseteq (U_2 \cup U_3) \quad \text{and} \quad (U_2 \cup U_3) \not\subseteq U_1.$$

It follows that there exists some $u \in U_1$ such that $u \notin U_2 \cup U_3$ and some $v \in U_2 \cup U_3$ such that $v \notin U_1$. Let $W = \{v + \lambda u : \lambda \in \mathbf{F}\} \subseteq U$ and observe that no element of W belongs to U_1, for if $v + \lambda u \in U_1$ then $v + \lambda u - \lambda u = v \in U_1$—but $v \notin U_1$. Thus

$$W \cap U_1 = \emptyset \quad \text{and} \quad W \subseteq (U_1 \cup U_2 \cup U_3) \quad \Rightarrow \quad W \subseteq (U_2 \cup U_3).$$

Because W contains infinitely many elements, there must be some $i \in \{2,3\}$ such that U_i contains infinitely many elements of W. There then exist $\lambda, \mu \in \mathbf{F}$ such that $\lambda \neq \mu$ and such that $v + \lambda u$ and $v + \mu u$ both belong to U_i, which implies that $(\lambda - \mu)u \in U_i$ since U_i is a subspace of V. This gives $u \in U_i$ since $\lambda - \mu \neq 0$, contradicting that $u \notin U_2 \cup U_3$. We may conclude that some U_j contains the other two.

Exercise 1.C.14. Suppose

$$U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}.$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

Solution. We claim that $U + W$ is the subspace

$$E = \{(x, y, 2x) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}.$$

To see this, let $(x, -x, 2x) \in U$ and $(y, y, 2y) \in W$ be given and notice that

$$(x, -x, 2x) + (y, y, 2y) = (x + y, -x + y, 2(x + y)) \in E.$$

Thus $U + W \subseteq E$. For the reverse inclusion, let $(x, y, 2x) \in E$ be given and observe that

$$(x, y, 2x) = \left(\frac{x-y}{2}, \frac{y-x}{2}, x - y\right) + \left(\frac{x+y}{2}, \frac{x+y}{2}, x + y\right) \in U + W.$$

Thus $U + W = E$, as claimed. In words, $U + W$ is the subspace of $\mathbf{F}^3$ consisting of those vectors whose third coordinate is twice their first coordinate.

Exercise 1.C.15. Suppose U is a subspace of V. What is $U + U$?

Solution. For $u + v \in U + U$ we have $u + v \in U$ since U is a subspace of V, and for $u \in U$ we have $u = u + 0 \in U + U$. Thus $U + U = U$.

Exercise 1.C.16. Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Solution. The operation is commutative, since addition of vectors in V is commutative. If $u + w \in U + W$, then $u + w = w + u \in W + U$, so that $U + W \subseteq W + U$. Similarly, $W + U \subseteq U + W$.

Exercise 1.C.17. Is the operation of addition on the subspaces of V associative? In other words, if V_1, V_2, V_3 are subspaces of V, is

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)?$$

Solution. The operation is associative, since addition of vectors in V is associative. If $(u_1 + u_2) + u_3 \in (U_1 + U_2) + U_3$, then

$$(u_1 + u_2) + u_3 = u_1 + (u_2 + u_3) \in U_1 + (U_2 + U_3),$$

so that $(U_1 + U_2) + U_3 \subseteq U_1 + (U_2 + U_3)$. Similarly, $U_1 + (U_2 + U_3) \subseteq (U_1 + U_2) + U_3$.

Exercise 1.C.18. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution. The subspace $\{0\}$ is the additive identity for the operation. If U is a subspace of V then $u + 0 = u$ for any $u \in U$; it follows that $U + \{0\} = U$.

Since $\{0\} + \{0\} = \{0\}$, the subspace $\{0\}$ is its own additive inverse. We claim that no other subspace of V has an additive inverse, i.e. if U is a subspace of V with $U \neq \{0\}$, then there does not exist a subspace W satisfying $U + W = \{0\}$. Indeed, simply observe that $U \subseteq U + W$ for any subspace W, so that $U + W \neq \{0\}$.

Exercise 1.C.19. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

Solution. This is false. For a counterexample, consider the real vector space $\mathbf{R}$ and observe that

$$\{0\} + \mathbf{R} = \mathbf{R} + \mathbf{R} = \mathbf{R},$$

but $\{0\} \neq \mathbf{R}$.

Exercise 1.C.20. Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

Solution. Let

$$W = \{(0, a, 0, b) \in \mathbf{F}^4 : a, b \in \mathbf{F}\};$$

it is straightforward to verify that W is a subspace of $\mathbf{F}^4$. If $v \in U \cap W$, then

$$v \in W \quad \Rightarrow \quad v = (0, a, 0, b) \text{ for some } a, b \in \mathbf{F},$$

$$v \in U \quad \Rightarrow \quad a = b = 0 \quad \Rightarrow \quad v = 0.$$

Thus $U \cap W = \{0\}$ and it follows from (1.46) that the sum $U + W$ is direct.
Let $(v_1, v_2, v_3, v_4) \in \mathbf{F}^4$ be given and observe that

$$(v_1, v_2, v_3, v_4) = (v_1, v_1, v_3, v_3) + (0, v_2 - v_1, 0, v_4 - v_3) \in U \oplus W.$$

Thus $\mathbf{F}^4 = U \oplus W$.

Exercise 1.C.21. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Solution. Let

$$W = \{(0, 0, a, b, c) \in \mathbf{F}^5 : a, b, c \in \mathbf{F}\};$$

it is straightforward to verify that W is a subspace of $\mathbf{F}^5$. If $v \in U \cap W$, then

$$v \in U \quad \Rightarrow \quad v = (x, y, x + y, x - y, 2x) \text{ for some } x, y \in \mathbf{F},$$

$$v \in W \quad \Rightarrow \quad x = y = 0 \quad \Rightarrow \quad v = 0.$$

Thus $U \cap W = \{0\}$ and it follows from (1.46) that the sum $U + W$ is direct.
Let $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbf{F}^5$ be given and observe that

$$\begin{aligned}(v_1, v_2, v_3, v_4, v_5) &= (v_1, v_2, v_1 + v_2, v_1 - v_2, 2v_1) \\ &\qquad + (0, 0, v_3 - (v_1 + v_2), v_4 - (v_1 - v_2), v_5 - 2v_1) \in U \oplus W.\end{aligned}$$

Thus $\mathbf{F}^5 = U \oplus W$.

Exercise 1.C.22. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution. Let

$$W_1 = \{(0, 0, a, 0, 0) \in \mathbf{F}^5 : a \in \mathbf{F}\}, \quad W_2 = \{(0, 0, 0, b, 0) \in \mathbf{F}^5 : b \in \mathbf{F}\},$$
$$W_3 = \{(0, 0, 0, 0, c) \in \mathbf{F}^5 : c \in \mathbf{F}\};$$

it is straightforward to verify that W_1, W_2, and W_3 are subspaces of $\mathbf{F}^5$. Suppose that

$$u = (x, y, x + y, x - y, 2x) \in U, \qquad w_1 = (0, 0, a, 0, 0) \in W_1,$$
$$w_2 = (0, 0, 0, b, 0) \in W_2, \quad \text{and} \quad w_3 = (0, 0, 0, 0, c) \in W_3$$

are such that $u + w_1 + w_2 + w_3 = 0$. That is,

$$(x, y, x + y + a, x - y + b, 2x + c) = (0, 0, 0, 0, 0),$$

from which it follows that $x = y = a = b = c = 0$. Thus the only way to express the zero vector as a sum $u + w_1 + w_2 + w_3 \in U + W_1 + W_2 + W_3$ is to take $u = w_1 = w_2 = w_3 = 0$ and so it follows from (1.45) that the sum $U + W_1 + W_2 + W_3$ is direct.

Let $(v_1, v_2, v_3, v_4, v_5) \in \mathbf{F}^5$ be given and observe that

$$\begin{aligned}(v_1, v_2, v_3, v_4, v_5) &= (v_1, v_2, v_1 + v_2, v_1 - v_2, 2v_1) + (0, 0, v_3 - (v_1 + v_2), 0, 0) \\ &\quad + (0, 0, 0, v_4 - (v_1 - v_2), 0) + (0, 0, 0, 0, v_5 - 2v_1) \in U \oplus W_1 \oplus W_2 \oplus W_3.\end{aligned}$$

Thus $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Exercise 1.C.23. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$

then $V_1 = V_2$.

Hint: When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in $\mathbf{F}^2$.

Solution. This is false. For a counterexample, consider $V = \mathbf{R}^2$,

$$U = \{(x, 0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, \quad V_1 = \{(0, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, \quad V_2 = \{(y, y) \in \mathbf{R}^2 : y \in \mathbf{R}\}.$$

It is straightforward to verify that $U \cap V_1 = U \cap V_2 = \{0\}$, so that $U + V_1$ and $U + V_2$ are both direct sums (1.46), and that $U \oplus V_1 = U \oplus V_2 = \mathbf{R}^2$. However, $V_1 \neq V_2$ since $(1,1) \in V_2$ but $(1,1) \notin V_1$.

Exercise 1.C.24. A function $f : \mathbf{R} \to \mathbf{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f : \mathbf{R} \to \mathbf{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$. Let V_{e} denote the set of real-valued even functions on $\mathbf{R}$ and let V_{o} denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^{\mathbf{R}} = V_{\mathrm{e}} \oplus V_{\mathrm{o}}$.

Solution. Suppose that $f \in V_{\mathrm{e}} \cap V_{\mathrm{o}}$, so that $f(x) = -f(x)$ for all $x \in \mathbf{R}$. This implies that $f(x) = 0$ for all $x \in \mathbf{R}$, i.e. $f = 0$. Thus $V_{\mathrm{e}} \cap V_{\mathrm{o}} = \{0\}$ and it follows from (1.46) that the sum $V_{\mathrm{e}} + V_{\mathrm{o}}$ is direct. For $f : \mathbf{R} \to \mathbf{R}$, define $f_{\mathrm{e}} : \mathbf{R} \to \mathbf{R}$ and $f_{\mathrm{o}} : \mathbf{R} \to \mathbf{R}$ by

$$f_{\mathrm{e}}(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_{\mathrm{o}}(x) = \frac{f(x) - f(-x)}{2}.$$

It is straightforward to verify that f_{e} is an even function, f_{o} is an odd function, and $f = f_{\mathrm{e}} + f_{\mathrm{o}}$. We may conclude that $\mathbf{R}^{\mathbf{R}} = V_{\mathrm{e}} \oplus V_{\mathrm{o}}$.