1 Section 3.B Exercises

Exercises with solutions from Section 3.B of [LADR].

Exercise 3.B.1. Give an example of a linear map T such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Solution. Let $T : \mathbf{R}^5 \to \mathbf{R}^2$ be given by

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2).$$

Then

$$\operatorname{null} T = \{(0, 0, x_3, x_4, x_5) \in \mathbf{R}^5 : x_3, x_4, x_5 \in \mathbf{R}\} \quad \text{and} \quad \operatorname{range} T = \mathbf{R}^2.$$

Thus $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Exercise 3.B.2. Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\operatorname{range} S \subset \operatorname{null} T.$$

Prove that $(ST)^2 = 0$.

Solution. Let $v \in V$ be given. Then $S(Tv) \in \operatorname{range} S \subseteq \operatorname{null} T$, so $T(S(Tv)) = 0$. It follows that

$$(ST)^2(v) = S(T(S(Tv))) = S(0) = 0.$$

Thus $(ST)^2 = 0$.

Exercise 3.B.3. Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V?

(b) What property of T corresponds to $v_1, \ldots, v_m$ being linearly independent?

Solution. (a) The surjectivity of T corresponds to $v_1, \ldots, v_m$ spanning V, i.e. $v_1, \ldots, v_m$ spans V if and only if T is surjective. To see this, observe that T is surjective if and only if for every $v \in V$ there exists $(z_1, \ldots, z_m) \in \mathbf{F}^m$ such that $T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m = v$. This is the case if and only if $V = \operatorname{span}(v_1, \ldots, v_m)$.

(b) The injectivity of T corresponds to $v_1, \ldots, v_m$ being linearly independent, i.e. $v_1, \ldots, v_m$ is linearly independent if and only if T is injective. To see this, observe that by 3.16, T is injective if and only if $\operatorname{null} T = \{0\}$, i.e. if and only if the only choice of $(z_1, \ldots, z_m) \in \mathbf{F}^m$ which gives $z_1 v_1 + \cdots + z_m v_m = 0$ is $(0, \ldots, 0)$; this is the definition of linear independence.

Exercise 3.B.4. Show that

$$\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$$

is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

Solution. Let $W = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$. Define $S, T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$ by

$$S(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0) \quad \text{and} \quad T(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, x_4).$$

Then

$$\operatorname{null} S = \{(0, 0, x_3, x_4, x_5) \in \mathbf{R}^5 : x_3, x_4, x_5 \in \mathbf{R}\}$$
$$\text{and } \operatorname{null} T = \{(x_1, x_2, 0, 0, x_5) \in \mathbf{R}^5 : x_1, x_2, x_5 \in \mathbf{R}\},$$

so that $\dim \operatorname{null} S = \dim \operatorname{null} T = 3$ and thus $S, T \in W$. Observe that

$$(S + T)(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, x_4)$$

and so

$$\operatorname{null}(S + T) = \{(0, 0, 0, 0, x_5) \in \mathbf{R}^5 : x_5 \in \mathbf{R}\}.$$

Then $\dim \operatorname{null}(S + T) = 1$, so $S + T \notin W$. This shows that W is not closed under addition and hence is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

Exercise 3.B.5. Give an example of a linear map $T : \mathbf{R}^4 \to \mathbf{R}^4$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

Solution. Let $T \in \mathcal{L}(\mathbf{R}^4, \mathbf{R}^4)$ be given by

$$T(x_1, x_2, x_3, x_4) = (x_3, x_4, 0, 0).$$

Then

$$\operatorname{range} T = \{(x_3, x_4, 0, 0) \in \mathbf{R}^4 : x_3, x_4 \in \mathbf{R}\} \quad \text{and} \quad \operatorname{null} T = \{(x_1, x_2, 0, 0) \in \mathbf{R}^4 : x_1, x_2 \in \mathbf{R}\},$$

which are the same subspace of $\mathbf{R}^4$.

Exercise 3.B.6. Prove that there does not exist a linear map $T : \mathbf{R}^5 \to \mathbf{R}^5$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

Solution. If V is a finite-dimensional vector space and $T : V \to W$ is a linear map such that $\operatorname{range} T = \operatorname{null} T$, then by the Fundamental Theorem of Linear Maps (3.22), we have

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = 2 \dim \operatorname{null} T.$$

Thus $\dim V$ must be a non-negative even integer. Since $\dim \mathbf{R}^5 = 5$, there can be no linear map $T : \mathbf{R}^5 \to \mathbf{R}^5$ satisfying $\operatorname{range} T = \operatorname{null} T$.

Exercise 3.B.7. Suppose V and W are finite-dimensional with $2 \le \dim V \le \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution. Let $X = \{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$. By 3.16 we have

$$X = \{T \in \mathcal{L}(V, W) : \operatorname{null} T \neq \{0\}\}.$$

Let $v_1, \dots, v_m$ be a basis for V and let $w_1, \dots, w_n$ be a basis for W; by assumption, we have $2 \le m \le n$. We will define two linear maps $S, T \in \mathcal{L}(V, W)$ by specifying their effect on the basis vectors $v_1, \dots, v_m$ and appealing to 3.5. Let

$$\begin{aligned} Sv_1 &= 0, \quad & Sv_2 &= w_2, \quad & Sv_j &= \tfrac{1}{2} w_j \text{ for } 3 \le j \le m \text{ if } m \ge 3, \\ Tv_1 &= w_1, \quad & Tv_2 &= 0, \quad & Tv_j &= \tfrac{1}{2} w_j \text{ for } 3 \le j \le m \text{ if } m \ge 3. \end{aligned}$$

S and T are not injective since $0 \neq v_1 \in \operatorname{null} S$ and $0 \neq v_2 \in \operatorname{null} T$, so S and T belong to X. Let L be the the map $S + T$. Then L is given by

$$Lv_j = w_j \text{ for } 1 \le j \le m.$$

We claim that L is injective. To see this, suppose that $Lv = 0$ for some $v \in V$. There are scalars $a_1, \dots, a_m$ such that $v = a_1 v_1 + \cdots + a_m v_m$. Then

$$0 = Lv = L(a_1 v_1 + \cdots + a_m v_m) = a_1 L v_1 + \cdots + a_m L v_m = a_1 w_1 + \cdots + a_m w_m.$$

Since the list $w_1, \dots, w_m$ is linearly independent, we see that $a_1 = \cdots = a_m = 0$ and thus $v = 0$. It follows that L is injective and hence that X is not closed under addition and so cannot be a subspace.

Exercise 3.B.8. Suppose V and W are finite-dimensional with $\dim V \ge \dim W \ge 2$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution. The solution is similar to Exercise 3.B.7. Let $X = \{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$. Suppose $v_1, \ldots, v_m$ is a basis for V and $w_1, \ldots, w_n$ is a basis for W; by assumption, we have $m \geq n \geq 2$. Define $S, T \in \mathcal{L}(V, W)$ by

$$Sv_j = \begin{cases} 0 & \text{if } j = 1 \text{ or } n < j \leq m \text{ if } m > n, \\ w_2 & \text{if } j = 2, \\ \frac{1}{2} w_j & \text{if } 3 \leq j \leq n \text{ if } n \geq 3. \end{cases} \qquad Tv_j = \begin{cases} w_1 & \text{if } j = 1, \\ 0 & \text{if } j = 2 \text{ or } n < j \leq m \text{ if } m > n, \\ \frac{1}{2} w_j & \text{if } 3 \leq j \leq n \text{ if } n \geq 3. \end{cases}$$

We claim that S is not surjective. To see this, we will show that $w_1 \notin \operatorname{range} S$. Suppose by way of contradiction that there exists $v \in V$ such that $Sv = w_1$. Then there are scalars $a_1, \ldots, a_m$ such that $a_1 v_1 + \cdots + a_m v_m = v$, which gives

$$w_1 = Sv = S(a_1 v_1 + \cdots + a_m v_m) = a_1 S v_1 + \cdots + a_m S v_m = a_2 w_2 + \cdots + \tfrac{1}{2} a_n w_n.$$

Thus $w_1 \in \operatorname{span}(w_2, \ldots, w_n)$, contradicting the linear independence of the basis $w_1, \ldots, w_n$. It follows that $w_1 \notin \operatorname{range} S$, so that S is not surjective. Similarly, we see that T is not surjective, since $w_2 \notin \operatorname{range} T$. Hence S and T belong to X. Let L be the map $S + T$. Then L is given by

$$Lv_j = \begin{cases} w_j & \text{if } 1 \leq j \leq n, \\ 0 & \text{if } n < j \leq m \text{ if } m > n. \end{cases}$$

We claim that L is surjective. To see this, let $w \in W$ be given. Then there are scalars $a_1, \ldots, a_n$ such that $w = a_1 w_1 + \cdots + a_n w_n$. Observe that

$$L(a_1 v_1 + \cdots + a_n v_n) = a_1 L v_1 + \cdots + a_n L v_n = a_1 w_1 + \cdots + a_n w_n = w.$$

It follows that L is surjective and hence that X is not closed under addition and so cannot be a subspace.

Exercise 3.B.9. Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

Solution. Suppose we have scalars $a_1, \ldots, a_n$ such that $a_1 T v_1 + \cdots + a_n T v_n = 0$. By linearity, this is equivalent to $T(a_1 v_1 + \cdots + a_n v_n) = 0$. Then since T is injective, we have by 3.16 that $a_1 v_1 + \cdots + a_n v_n = 0$. The linear independence of $v_1, \ldots, v_n$ then implies that $a_1 = \cdots = a_n = 0$ and hence that $Tv_1, \ldots, Tv_n$ is linearly independent.

Exercise 3.B.10. Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $Tv_1, \ldots, Tv_n$ spans $\operatorname{range} T$.

Solution. Let $w \in \operatorname{range} T$ be given, so that $w = Tv$ for some $v \in V$. Since $v_1, \ldots, v_n$ spans V, there are scalars $a_1, \ldots, a_n$ such that $v = a_1 v_1 + \cdots + a_n v_n$. Then:

$$a_1 T v_1 + \cdots + a_n T v_n = T(a_1 v_1 + \cdots + a_n v_n) = Tv = w.$$

Thus $Tv_1, \ldots, Tv_n$ spans $\operatorname{range} T$.

Exercise 3.B.11. Suppose $S_1, \ldots, S_n$ are injective linear maps such that $S_1 S_2 \cdots S_n$ makes sense. Prove that $S_1 S_2 \cdots S_n$ is injective.

Solution. We will prove this by induction. Let $P(n)$ be the statement that for any collection of n injective linear maps $S_1, \ldots, S_n$ such that $S_1 S_2 \cdots S_n$ makes sense, we have that $S_1 S_2 \cdots S_n$ is injective. The base case $P(1)$ is clear. Suppose that $P(n)$ is true for some $n \in \mathbf{N}$, and suppose we have $n+1$ linear maps $S_1, \ldots, S_{n+1}$ such that $S_1 S_2 \cdots S_{n+1}$ makes sense. Let v be a vector in the domain of S_{n+1} such that

$$(S_1 S_2 \cdots S_{n+1})(v) = S_1((S_2 \cdots S_{n+1})(v)) = 0.$$

Since S_1 is injective, 3.16 implies that $(S_2 \cdots S_{n+1})(v) = 0$. Our induction hypothesis guarantees that $S_2 \cdots S_{n+1}$ is injective, so again by 3.16 we have that $v = 0$. It follows that $\operatorname{null}(S_1 S_2 \cdots S_{n+1}) = \{0\}$ and hence by 3.16 the linear map $S_1 S_2 \cdots S_{n+1}$ is injective. This completes the induction step and the proof.

Exercise 3.B.12. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{Tu : u \in U\}$.

Solution. Since $\operatorname{null} T$ is a subspace of V, 2.34 implies that there exists a subspace U of V such that $V = U \oplus \operatorname{null} T$; 1.45 then gives us $U \cap \operatorname{null} T = \{0\}$. Suppose that $w \in \operatorname{range} T$, so that $w = Tv$ for some $v \in V$. Since $V = U \oplus \operatorname{null} T$, there are unique vectors $u \in U$ and $x \in \operatorname{null} T$ such that $v = u + x$. Then

$$w = Tv = T(u + x) = Tu + Tx = Tu + 0 = Tu.$$

Thus $\operatorname{range} T \subseteq \{Tu : u \in U\}$, and since the reverse inclusion is clear, we may conclude that $\operatorname{range} T = \{Tu : u \in U\}$.

Exercise 3.B.13. Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

Solution. It is not hard to see that $(5,1,0,0),(0,0,7,1)$ is a basis for $\operatorname{null} T$, so that $\dim \operatorname{null} T = 2$. Then by the Fundamental Theorem of Linear Maps (3.22), we have

$$\dim \mathbf{F}^4 = \dim \operatorname{null} T + \dim \operatorname{range} T, \text{ i.e. } 4 = 2 + \dim \operatorname{range} T.$$

Thus $\dim \operatorname{range} T = 2 = \dim \mathbf{F}^2$. Since $\operatorname{range} T$ is a subspace of $\mathbf{F}^2$, Exercise 2.C.1 allows us to conclude that $\operatorname{range} T = \mathbf{F}^2$ and hence that T is surjective.

Exercise 3.B.14. Suppose U is a 3-dimensional subspace of $\mathbf{R}^8$ and that T is a linear map from $\mathbf{R}^8$ to $\mathbf{R}^5$ such that $\operatorname{null} T = U$. Prove that T is surjective.

Solution. Since $\dim U = 3$, we also have $\dim \operatorname{null} T = 3$. The Fundamental Theorem of Linear Maps (3.22) gives

$$\dim \mathbf{R}^8 = \dim \operatorname{null} T + \dim \operatorname{range} T, \text{ i.e. } 8 = 3 + \dim \operatorname{range} T.$$

Thus $\dim \operatorname{range} T = 5 = \dim \mathbf{R}^5$. Exercise 2.C.1 allows us to conclude that $\operatorname{range} T = \mathbf{R}^5$ and hence that T is surjective.

Exercise 3.B.15. Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals

$$\{(x_1,x_2,x_3,x_4,x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Solution. Let $U = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$. It is not hard to see that $(3,1,0,0,0),(0,0,1,1,1)$ is a basis for U, so that $\dim U = 2$. Let T be a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$. The Fundamental Theorem of Linear Maps (3.22) implies that

$$\dim \mathbf{F}^5 = \dim \operatorname{null} T + \dim \operatorname{range} T, \text{ i.e. } 5 = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Since $\operatorname{range} T$ is a subspace of $\mathbf{F}^2$, we must have $\dim \operatorname{range} T \leq \dim \mathbf{F}^2 = 2$. Combining this with the equality $5 = \dim \operatorname{null} T + \dim \operatorname{range} T$, we see that $\dim \operatorname{null} T \geq 3$. It follows that U cannot be the null space of T.

Exercise 3.B.16. Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Solution. Let $T : V \to V$ be the linear map in question. There is a basis $u_1, \ldots, u_m$ for $\operatorname{range} T$ and a basis $w_1, \ldots, w_n$ for $\operatorname{null} T$. Since each $u_i \in \operatorname{range} T$, there exists a $v_i \in V$ such that $Tv_i = u_i$. We claim that the list $v_1, \ldots, v_m, w_1, \ldots, w_n$ spans V. To see this, let $v \in V$ be given. Then $Tv \in \operatorname{range} T$, so there are scalars $a_1, \ldots, a_m$ such that

$$\begin{aligned}
&Tv = a_1u_1 + \cdots + a_mu_m = a_1Tv_1 + \cdots + a_mTv_m = T(a_1v_1 + \cdots + a_mv_m) \\
\implies &T(v - (a_1v_1 + \cdots + a_mv_m)) = 0 \\
\implies &v - (a_1v_1 + \cdots + a_mv_m) \in \operatorname{null} T.
\end{aligned}$$

Hence there are scalars $b_1, \ldots, b_n$ such that

$$\begin{aligned} v - (a_1v_1 + \cdots + a_mv_m) &= b_1w_1 + \cdots b_nw_n \\ \implies v &= a_1v_1 + \cdots + a_mv_m + b_1w_1 + \cdots b_nw_n. \end{aligned}$$

Thus the list $v_1, \ldots, v_m, w_1, \ldots, w_n$ spans V. We may conclude that V is finite-dimensional.

Exercise 3.B.17. Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Solution. If $\dim V > \dim W$, then 3.23 guarantees that no linear map from V to W is injective. Suppose therefore that $\dim V \leq \dim W$. Then there is a basis $v_1, \ldots, v_m$ for V and a basis $w_1, \ldots, w_n$ for W, where $m \leq n$. Define a linear map $T : V \to W$ by $Tv_j = w_j$. As shown in Exercise 3.B.7 (with the map L), such a map is injective.

Exercise 3.B.18. Suppose V and W are both finite-dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim V \geq \dim W$.

Solution. If $\dim V < \dim W$, then 3.24 guarantees that no linear map from V to W is surjective. Suppose therefore that $\dim V \geq \dim W$. Then there is a basis $v_1, \ldots, v_m$ for V and a basis $w_1, \ldots, w_n$ for W, where $m \geq n$. Define a linear map $T : V \to W$ by $Tv_j = w_j$ for $1 \leq j \leq n$, and $Tv_j = 0$ for $n < j \leq m$, if $m > n$. As shown in Exercise 3.B.8 (with the map L), such a map is surjective.

Exercise 3.B.19. Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Solution. Suppose that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$. The Fundamental Theorem of Linear Maps (3.22) implies that

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim U + \dim \operatorname{range} T.$$

Since $\operatorname{range} T$ is a subspace of W, we have $\dim \operatorname{range} T \leq \dim W$. Combining this with the equality $\dim V - \dim U = \dim \operatorname{range} T$, we see that $\dim U \geq \dim V - \dim W$.

Now suppose that $\dim U \geq \dim V - \dim W$. Let $u_1, \ldots, u_m$ be a basis of U, which we extend to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V, and let $w_1, \ldots, w_k$ be a basis of W. By assumption, we have $m \geq m + n - k$, or $k \geq n$. Define a map $T : V \to W$ by $Tu_i = 0$ for $1 \leq i \leq m$, and $Tv_i = w_i$ for $1 \leq i \leq n$; this is possible precisely because $k \geq n$, i.e. there are enough w_i's to define this map. It is not hard to see that $U \subseteq \operatorname{null} T$. Suppose that $v \in \operatorname{null} T$. There are scalars $a_1, \ldots, a_m, b_1, \ldots, b_n$ such that $v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n$. Then

$$\begin{aligned} 0 = Tv = T(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) \\ = a_1Tu_1 + \cdots + a_mTu_m + b_1Tv_1 + \cdots + b_nTv_n = b_1w_1 + \cdots b_nw_n. \end{aligned}$$

The linear independence of $w_1, \ldots, w_n$ then implies that $b_1 = \cdots = b_n = 0$, so that $v = a_1u_1 + \cdots + a_mu_m$. Hence $v \in U$ and we may conclude that $\operatorname{null} T = U$.

Exercise 3.B.20. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Solution. Suppose there exists such a map S and suppose that $v \in V$ is such that $Tv = 0$. Then

$$(ST)(v) = S(0) = 0 \implies v = 0,$$

since ST is the identity map on V. Thus $\operatorname{null} T = \{0\}$, which is the case if and only if T is injective.

Now suppose that T is injective. Let $u_1, \ldots, u_m$ be a basis for $\operatorname{range} T$, which we extend to a basis $u_1, \ldots, u_m, w_1, \ldots, w_n$ for W. Since each $u_i \in \operatorname{range} T$, there is a $v_i \in V$ such that $u_i = Tv_i$. Define a linear map $S : W \to V$ by $Su_i = v_i$ and $Sw_i = 0$. We claim that ST is the identity map on V. To see this, let $v \in V$ be given. Then $Tv \in \operatorname{range} T$, so there are scalars $a_1, \ldots, a_m$ such that $Tv = a_1u_1 + \cdots + a_mu_m$, which gives

$$Tv = a_1Tv_1 + \cdots + a_mTv_m = T(a_1v_1 + \cdots + a_mv_m).$$

Since T is injective, we must then have $v = a_1v_1 + \cdots + a_mv_m$. Applying S to both sides of $Tv = a_1u_1 + \cdots + a_mu_m$ gives

$$(ST)(v) = a_1Su_1 + \cdots + a_mSu_m = a_1v_1 + \cdots + a_mv_m = v.$$

Thus ST is the identity map on V.

Exercise 3.B.21. Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

Solution. Suppose there exists such a map S and let $w \in W$ be given. Then $T(Sw) = w$ and thus $w \in \operatorname{range} T$. It follows that T is surjective.

Now suppose that T is surjective, i.e. that $\operatorname{range} T = W$. Then by the Fundamental Theorem of Linear Maps (3.22), W is finite-dimensional, so let $w_1, \ldots, w_n$ be a basis of W. Since $\operatorname{range} T = W$, there are vectors $v_1, \ldots, v_n$ such that $Tv_i = w_i$. Define a linear map $S : W \to V$ by $Sw_i = v_i$. We claim that TS is the identity map on W. To see this, let w be given. There are scalars $a_1, \ldots, a_n$ such that $w = a_1w_1 + \cdots + a_nw_n$. Then

$$\begin{aligned}(TS)(w) = (TS)(a_1w_1 + \cdots + a_nw_n) &= a_1(TS)(w_1) + \cdots + a_n(TS)(w_n)\\ &= a_1Tv_1 + \cdots + a_nTv_n = a_1w_1 + \cdots + a_nw_n = w.\end{aligned}$$

Thus TS is the identity map on W.

Exercise 3.B.22. Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

Solution. If $u \in U$ is such that $Tu = 0$, then $(ST)(u) = S(0) = 0$. It follows that $\operatorname{null} T$ is a subspace of $\operatorname{null} ST$. Thus if we let $u_1, \ldots, u_m$ be a basis of $\operatorname{null} T$, then we can extend this to a basis $u_1, \ldots, u_m, x_1, \ldots, x_n$ of $\operatorname{null} ST$. Letting $X = \operatorname{span}(x_1, \ldots, x_n)$, we then have $\operatorname{null} ST = \operatorname{null} T \oplus X$. Let $v_1, \ldots, v_k$ be a basis for $\operatorname{null} S$. Proving that $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$ is then equivalent to showing that $m + n \leq m + k$, i.e. $n \leq k$.

First, we claim that the list $Tx_1, \ldots, Tx_n$ is linearly independent. To see this, suppose we have scalars $a_1, \ldots, a_n$ such that

$$a_1 T x_1 + \cdots + a_n T x_n = T(a_1 x_1 + \cdots + a_n x_n) = 0.$$

Then $a_1 x_1 + \cdots + a_n x_n \in \operatorname{null} T$. Evidently, we have $a_1 x_1 + \cdots + a_n x_n \in X$. Since the sum $\operatorname{null} T \oplus X$ is direct, we have $\operatorname{null} T \cap X = \{0\}$; it follows that $a_1 x_1 + \cdots + a_n x_n = 0$. The linear independence of the list $x_1, \ldots, x_n$ then implies that $a_1 = \cdots = a_n = 0$ and our claim follows.

Now, since each $x_i \in \operatorname{null} ST$, we have $S(Tx_i) = 0$, so that each Tx_i belongs to $\operatorname{null} S$. Since $\operatorname{null} S$ has a basis of length k and we showed that the list $Tx_1, \ldots, Tx_n$ is linearly independent, 2.23 implies that $n \leq k$, as desired.

Exercise 3.B.23. Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}.$$

Solution. If $w \in \operatorname{range} ST$, then $w = S(Tu)$ for some $u \in U$, so that $w \in \operatorname{range} S$ also. It follows that $\operatorname{range} ST$ is a subspace of $\operatorname{range} S$, which implies that $\dim \operatorname{range} ST \leq \dim \operatorname{range} S$.

Let $w_1, \ldots, w_m$ be a basis for $\operatorname{range} ST$ and let $v_1, \ldots, v_n$ be a basis for $\operatorname{range} T$. We claim that the list $Sv_1, \ldots, Sv_n$ spans $\operatorname{range} ST$. To see this, let $w \in \operatorname{range} ST$ be given, so that $w = S(Tu)$ for some $u \in U$. Since $Tu \in \operatorname{range} T$, there are scalars $a_1, \ldots, a_n$ such that $Tu = a_1 v_1 + \cdots + a_n v_n$. Then

$$w = S(Tu) = S(a_1 v_1 + \cdots + a_n v_n) = a_1 S v_1 + \cdots + a_n S v_n.$$

Thus $w \in \operatorname{span}(Sv_1, \ldots, Sv_n)$ and our claim follows. 2.23 now implies that $m \leq n$, i.e. $\dim \operatorname{range} ST \leq \dim \operatorname{range} T$.

We now have both the inequalities $\dim \operatorname{range} ST \leq \dim \operatorname{range} S$ and $\dim \operatorname{range} ST \leq \dim \operatorname{range} T$. It follows that

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}.$$

Exercise 3.B.24. Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 \subset \operatorname{null} T_2$ if and only if there exists $S \in \mathcal{L}(W, W)$ such that $T_2 = ST_1$.

Solution. Suppose that there exists such a map S and let $v \in \operatorname{null} T_1$. Then $T_2 v = S(T_1 v) = S(0) = 0$, so that $v \in \operatorname{null} T_2$ also.

Now suppose that $\operatorname{null} T_1 \subseteq \operatorname{null} T_2$. Let $x_1, \dots, x_m$ be a basis of $\operatorname{range} T_1$, which we extend to a basis $x_1, \dots, x_m, y_1, \dots, y_n$ of W. Since each $x_j \in \operatorname{range} T_1$, we have $x_j = T_1 v_j$ for some $v_j \in V$. Define a linear map $S : W \to W$ by $Sx_j = T_2 v_j$ for $1 \leq j \leq m$ and $Sy_j = 0$ for $1 \leq j \leq n$. We claim that $T_2 = ST_1$. To see this, let $v \in V$ be given. Then $T_1 v \in \operatorname{range} T_1$, so there are scalars $a_1, \dots, a_m$ such that

$$\begin{aligned} &T_1 v = a_1 x_1 + \cdots + a_m x_m = a_1 T_1 v_1 + \cdots + a_m T_1 v_m = T_1(a_1 v_1 + \cdots + a_m v_m) \\ \Longrightarrow\ &T_1(v - (a_1 v_1 + \cdots + a_m v_m)) = 0 \\ \Longrightarrow\ &v - (a_1 v_1 + \cdots + a_m v_m) \in \operatorname{null} T_1. \end{aligned}$$

Then since $\operatorname{null} T_1 \subseteq \operatorname{null} T_2$, we have $v - (a_1 v_1 + \cdots + a_m v_m) \in \operatorname{null} T_2$. Following the algebra above in reverse with T_2 in place of T_1 shows that $T_2 v = a_1 T_2 v_1 + \cdots + a_m T_2 v_m$, and applying S to both sides of the equality $T_1 v = a_1 x_1 + \cdots + a_m x_m$ gives us

$$S(T_1 v) = a_1 Sx_1 + \cdots + a_m Sx_m = a_1 T_2 v_1 + \cdots + a_m T_2 v_m = T_2 v.$$

Thus $T_2 = ST_1$.

Exercise 3.B.25. Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} T_1 \subset \operatorname{range} T_2$ if and only if there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2 S$.

Solution. Suppose that there exists such a map S and let $w \in \operatorname{range} T_1$ be given, so that $w = T_1 v$ for some $v \in V$. Then

$$w = T_1 v = T_2(Sv) \in \operatorname{range} T_2.$$

Thus $\operatorname{range} T_1 \subseteq \operatorname{range} T_2$.

Now suppose that $\operatorname{range} T_1 \subseteq \operatorname{range} T_2$. Let $v_1, \dots, v_m$ be a basis for V. By assumption, each $T_1 v_j$ belongs to $\operatorname{range} T_2$, so we have $T_1 v_j = T_2 u_j$ for some $u_j \in V$. Define a linear map $S : V \to V$ by $Sv_j = u_j$ for $1 \leq j \leq m$. We claim that $T_1 = T_2 S$. To see this, let $v \in V$ be given. Then there are scalars $a_1, \dots, a_m$ such that $v = a_1 v_1 + \cdots + a_m v_m$. Observe that

$$\begin{aligned} T_2(Sv) &= T_2(S(a_1 v_1 + \cdots + a_m v_m)) = T_2(a_1 Sv_1 + \cdots + a_m Sv_m) = T_2(a_1 u_1 + \cdots + a_m u_m) \\ &= a_1 T_2 u_1 + \cdots + a_m T_2 u_m = a_1 T_1 v_1 + \cdots + a_m T_1 v_m = T_1(a_1 v_1 + \cdots + a_m v_m) = T_1 v. \end{aligned}$$

Thus $T_1 = T_2 S$.

Exercise 3.B.26. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ is such that $\deg Dp = (\deg p) - 1$ for every nonconstant polynomial $p \in \mathcal{P}(\mathbf{R})$. Prove that D is surjective.
[*The notation D is used above to remind you of the differentiation map that sends a polynomial p to p'. Without knowing the formula for the derivative of a polynomial (except that it reduces the degree by 1), you can use the exercise above to show that for every polynomial $q \in \mathcal{P}(\mathbf{R})$, there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that $p' = q$.*]

Solution. For a non-negative integer n, let $A(n)$ be the statement that there exists a polynomial $p_n \in \mathcal{P}(\mathbf{R})$ such that $D(p_n) = x^n$. We will use strong induction to show that $A(n)$ holds for all non-negative integers n. For the base case $n = 0$, note that by assumption the polynomial $D(x)$ must have degree 0, i.e. must be a non-zero constant, say $D(x) = b \neq 0$. Define $p_0 := b^{-1}x$. Then by linearity,

$$D(p_0) = D(b^{-1}x) = b^{-1}D(x) = b^{-1}b = 1.$$

Thus the base case $A(0)$ holds.

Now suppose that $A(0), A(1), \dots, A(n)$ all hold for some non-negative integer n. By assumption, the polynomial $D(x^{n+2})$ must have degree $n+1$, i.e. must be of the form

$$D(x^{n+2}) = b_{n+1}x^{n+1} + b_n x^n + \cdots + b_1 x + b_0,$$

where $b_{n+1} \neq 0$. Our induction hypothesis guarantees the existence of polynomials $p_0, p_1, \dots, p_n$ such that $D(p_j) = x^j$ for $1 \leq j \leq n$. Thus we can write

$$b_{n+1}^{-1} D(x^{n+2}) = x^{n+1} + b_{n+1}^{-1}(b_n D(p_n) + \cdots + b_1 D(p_1) + b_0 D(p_0)),$$

which by the linearity of D implies

$$x^{n+1} = D(b_{n+1}^{-1}(x^{n+2} - (b_n p_n + \cdots + b_1 p_1 + b_0 p_0))).$$

So if we define $p_{n+1} := b_{n+1}^{-1}(x^{n+2} - (b_n p_n + \cdots + b_1 p_1 + b_0 p_0))$, then we have $D(p_{n+1}) = x^{n+1}$. Thus $A(n+1)$ holds. This completes the induction step.

So $A(n)$ holds for all non-negative integers n. We can now show that D is surjective. Let p be an arbitrary polynomial in $\mathcal{P}(\mathbf{R})$ and let $n = \deg p$. Then $p = \sum_{j=0}^{n} a_j x^j$ for some coefficients $a_0, \dots, a_n$ (with $a_n \neq 0$). Set $q = \sum_{j=0}^{n} a_j p_j$. Then we have

$$D(q) = \sum_{j=0}^{n} a_j D(p_j) = \sum_{j=0}^{n} a_j x^j = p.$$

Thus D is surjective.

Exercise 3.B.27. Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.
[*This exercise can be done without linear algebra, but it's more fun to do it using linear algebra.*]

Solution. Define a map $D : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Dq = 5q'' + 3q'$. It is not hard to see that this map is linear, since differentiation is a linear operation. Suppose $q \in \mathcal{P}(\mathbf{R})$ is a non-constant polynomial of degree $n \geq 1$, so that $q = \sum_{j=0}^{n} a_j x^j$ where $a_n \neq 0$. Some algebra gives

$$Dq = \begin{cases} 3a_n & \text{if } n = 1, \\ 3na_n x^{n-1} + \sum_{j=0}^{n-2} (j+1)[3a_{j+1} + 5(j+2)a_{j+2}]x^j & \text{if } n \geq 2. \end{cases}$$

In either case, since $a_n \neq 0$, Dq is a polynomial of degree $n - 1$. Thus the linear map D satisfies the hypotheses of Exercise 3.B.26 and hence must be surjective.

Exercise 3.B.28. Suppose $T \in \mathcal{L}(V, W)$, and $w_1, \ldots, w_m$ is a basis of $\operatorname{range} T$. Prove that there exist $\varphi_1, \ldots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$$

for every $v \in V$.

Solution. Let $v \in V$ be given. Since $w_1, \ldots, w_m$ is a basis of $\operatorname{range} T$, there are unique scalars $b_1, \ldots, b_m$ such that $Tv = b_1 w_1 + \cdots + b_m w_m$. For $1 \leq j \leq m$, define $\varphi_j : V \to \mathbf{F}$ by $\varphi_j(v) := b_j$; since the scalars $b_1, \ldots, b_m$ are unique, each φ_j is well-defined. We claim that each φ_j is linear. To see this, let $u, v \in V$ be given. Then

$$Tu = a_1 w_1 + \cdots + a_m w_m \quad \text{and} \quad Tv = b_1 w_1 + \cdots + b_m w_m$$

for unique scalars $a_1, \ldots, a_m, b_1, \ldots, b_m$. Since

$$T(u + v) = Tu + Tv = (a_1 + b_1)w_1 + \cdots + (a_m + b_m)w_m,$$

the scalars $a_1 + b_1, \ldots, a_m + b_m$ must be the unique coefficients for $T(u+v)$ as a linear combination of the basis vectors $w_1, \ldots, w_m$. Given this, for any $1 \leq j \leq m$ we have

$$\varphi_j(u) = a_j, \quad \varphi_j(v) = b_j, \quad \text{and} \quad \varphi_j(u + v) = a_j + b_j.$$

Thus $\varphi_j(u + v) = \varphi_j(u) + \varphi_j(v)$. Similarly, let $\lambda \in \mathbf{F}$ be a scalar. Then since

$$T(\lambda u) = \lambda Tu = \lambda a_1 w_1 + \cdots + \lambda a_m w_m,$$

the scalars $\lambda a_1, \ldots, \lambda a_m$ must be the unique coefficients for $T(\lambda u)$ as a linear combination of the basis vectors $w_1, \ldots, w_m$. Given this, for any $1 \leq j \leq m$ we have

$$\varphi_j(u) = a_j \quad \text{and} \quad \varphi_j(\lambda u) = \lambda a_j.$$

Thus $\varphi_j(\lambda u) = \lambda \varphi_j(u)$ and we see that each φ_j is linear, as claimed. Furthermore, given the definition of each φ_j, it is clear that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$$

for every $v \in V$.

Exercise 3.B.29. Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$. Suppose $u \in V$ is not in $\operatorname{null} \varphi$. Prove that

$$V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}.$$

Solution. Let $v \in V$ be given. If $\varphi(v) = 0$, then certainly $v \in \operatorname{null} \varphi + \{au : a \in \mathbf{F}\}$. If $\varphi(v) \neq 0$, then observe that

$$\begin{aligned} \varphi\left(\frac{v}{\varphi(v)}\right) &= 1 = \varphi\left(\frac{u}{\varphi(u)}\right) \\ \implies \varphi\left(\frac{v}{\varphi(v)} - \frac{u}{\varphi(u)}\right) &= 0 \\ \implies \frac{v}{\varphi(v)} - \frac{u}{\varphi(u)} &\in \operatorname{null} \varphi. \end{aligned}$$

Thus $\frac{v}{\varphi(v)} - \frac{u}{\varphi(u)} = w$ for some $w \in \operatorname{null} \varphi$, which gives

$$v = \varphi(v)w + \frac{\varphi(v)}{\varphi(u)}u \in \operatorname{null} \varphi + \{au : a \in \mathbf{F}\}.$$

Hence V is the sum $\operatorname{null} \varphi + \{au : a \in \mathbf{F}\}$. To see that this sum is direct, suppose that $v \in \operatorname{null} \varphi$ and $v \in \{au : a \in \mathbf{F}\}$, so that $v = au$ for some $a \in \mathbf{F}$. Then

$$0 = \varphi(v) = \varphi(au) = a\varphi(u).$$

Since $\varphi(u) \neq 0$, it must be the case that $a = 0$ and hence that $v = 0$. Thus

$$\operatorname{null} \varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$$

and we see that the sum $\operatorname{null} \varphi + \{au : a \in \mathbf{F}\}$ is direct.

Exercise 3.B.30. Suppose φ_1 and φ_2 are linear maps from V to $\mathbf{F}$ that have the same null space. Show that there exists a constant $c \in \mathbf{F}$ such that $\varphi_1 = c\varphi_2$.

Solution. If $\operatorname{null}\varphi_1 = \operatorname{null}\varphi_2 = V$, then φ_1 and φ_2 are both the map $v \mapsto 0$, and so any $c \in \mathbf{F}$ will do. Suppose therefore that $\operatorname{null}\varphi_1 \neq V$, so that there is a $u \in V$ with $u \notin \operatorname{null}\varphi_1$ and $u \notin \operatorname{null}\varphi_2$. Define $c := \frac{\varphi_1 u}{\varphi_2 u}$. We claim that $\varphi_1 = c\varphi_2$. To see this, first observe that by Exercise 3.B.29, we have

$$V = \operatorname{null}\varphi_1 \oplus \{au : a \in \mathbf{F}\}.$$

Let $v \in V$ be given, so that $v = x + au$ for some $a \in \mathbf{F}$, where $x \in \operatorname{null}\varphi_1 = \operatorname{null}\varphi_2$. Then

$$c\varphi_2 v = \frac{\varphi_1 u}{\varphi_2 u}\varphi_2(x + au) = \frac{\varphi_1 u}{\varphi_2 u} a\varphi_2 u = a\varphi_1 u = \varphi_1(x + au) = \varphi_1 v.$$

Thus $\varphi_1 = c\varphi_2$.

Exercise 3.B.31. Give an example of two linear maps T_1 and T_2 from $\mathbf{R}^5$ to $\mathbf{R}^2$ that have the same null space but are such that T_1 is not a scalar multiple of T_2.

Solution. Let T_1 and T_2 be the linear maps given by

$$T_1(x_1, x_2, x_3, x_4, x_5) = (x_4, x_5) \quad \text{and} \quad T_2(x_1, x_2, x_3, x_4, x_5) = (x_5, x_4).$$

Then

$$\operatorname{null} T_1 = \operatorname{null} T_2 = \{(x_1, x_2, x_3, 0, 0) \in \mathbf{R}^5 : x_1, x_2, x_3 \in \mathbf{R}\}.$$

However, T_1 is not a scalar multiple of T_2. To see this, note that

$$T_1(0, 0, 0, 1, 0) = (1, 0) \quad \text{and} \quad T_2(0, 0, 0, 1, 0) = (0, 1),$$

which are two linearly independent vectors in $\mathbf{R}^2$ and thus not scalar multiples of one another.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edition.