1 Section 7.D Exercises

Exercises with solutions from Section 7.D of [LADR].

Exercise 7.D.1. Fix $u, x \in V$ with $u \neq 0$. Define $T \in \mathcal{L}(V)$ by

$$Tv = \langle v, u \rangle x$$

for every $v \in V$. Prove that

$$\sqrt{T^*T}v = \frac{\|x\|}{\|u\|}\langle v, u \rangle u$$

for every $v \in V$.

Solution. It follows from Example 7.4 that

$$T^*Tv = \|x\|^2 \langle v, u \rangle u$$

for $v \in V$. Let $R \in \mathcal{L}(V)$ be given by

$$Rv = \frac{\|x\|}{\|u\|}\langle v, u \rangle u.$$

Note that

$$R^2 v = \frac{\|x\|}{\|u\|}\left\langle \frac{\|x\|}{\|u\|}\langle v, u \rangle u, u \right\rangle u = \|x\|^2 \langle v, u \rangle u = T^*Tv$$

and that

$$\langle Rv, v \rangle = \left\langle \frac{\|x\|}{\|u\|}\langle v, u \rangle u, v \right\rangle = \frac{\|x\|}{\|u\|}|\langle v, u \rangle|^2 \geq 0$$

for any $v \in V$. It follows that R is the unique positive square root of T^*T, i.e. $R = \sqrt{T^*T}$.

Exercise 7.D.2. Give an example of $T \in \mathcal{L}(\mathbf{C}^2)$ such that 0 is the only eigenvalue of T and the singular values of T are 5, 0.

Solution. Let $T \in \mathcal{L}(\mathbf{C}^2)$ be the operator whose matrix with respect to the standard basis of $\mathbf{C}^2$ is

$$\begin{pmatrix} 0 & \sqrt{5} \\ 0 & 0 \end{pmatrix},$$

so that 0 is the only eigenvalue of T. The matrix of T^*T is then

$$\begin{pmatrix} 0 & 0 \\ \sqrt{5} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{5} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix},$$

from which we see that the singular values of T are 5, 0.

Exercise 7.D.3. Suppose $T \in \mathcal{L}(V)$. Prove that there exists an isometry $S \in \mathcal{L}(V)$ such that $T = \sqrt{TT^*}S$.

Solution. The Polar Decomposition (7.45) implies that there exists an isometry $R \in \mathcal{L}(V)$ such that $T^* = R\sqrt{TT^*}$. Taking adjoints of both sides of this equation gives $T = \sqrt{TT^*}R^*$; here we are using that $\sqrt{TT^*}$ is a positive operator and hence is self-adjoint. Observe that R^* is an isometry by 7.42 and thus the desired isometry is $S = R^*$.

Exercise 7.D.4. Suppose $T \in \mathcal{L}(V)$ and s is a singular value of T. Prove that there exists a vector $v \in V$ such that $\|v\| = 1$ and $\|Tv\| = s$.

Solution. Suppose T has singular values $s_1, \ldots, s_n$, with $s = s_j$ for some $1 \leq j \leq n$. The Singular Value Decomposition (7.51) implies that there are orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$. In particular, we have

$$Te_j = s_1\langle e_j, e_1\rangle f_1 + \cdots + s_j\langle e_j, e_j\rangle f_j + \cdots + s_n\langle e_j, e_n\rangle f_n = sf_j.$$

Thus $\|e_j\| = 1$ and $\|Te_j\| = \|sf_j\| = |s|\|f_j\| = s$, where we have used that singular values are non-negative and that $\|f_j\| = 1$.

Exercise 7.D.5. Suppose $T \in \mathcal{L}(\mathbf{C}^2)$ is defined by $T(x, y) = (-4y, x)$. Find the singular values of T.

Solution. The matrix of T with respect to the standard basis of $\mathbf{C}^2$ is

$$A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix},$$

from which we find

$$A^*A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix}.$$

It then follows from 7.52 that the singular values of T are 1 and 4.

Exercise 7.D.6. Find the singular values of the differentiation operator $D \in \mathcal{L}(\mathcal{P}_2(\mathbf{R}))$ defined by $Dp = p'$, where the inner product on $\mathcal{P}_2(\mathbf{R})$ is as in Example 6.33. (See errata.)

Solution. As shown in Example 6.33, the list

$$e_1 = \frac{1}{\sqrt{2}}, \quad e_2 = \sqrt{\frac{3}{2}}x, \quad e_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

is an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$ with respect to the inner product given in Example 6.33. Note that

$$De_1 = 0, \quad De_2 = \sqrt{\frac{3}{2}} = \sqrt{3}e_1, \quad \text{and} \quad De_3 = \sqrt{\frac{45}{8}}(2x) = \sqrt{15}e_2.$$

It follows that the matrix of D with respect to e_1, e_2, e_3 is

$$A = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix},$$

from which we find that

$$A^*A = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}.$$

We can now use 7.52 to see that the singular values of D are $\sqrt{3}$ and $\sqrt{15}$.

Exercise 7.D.7. Define $T \in \mathcal{L}(\mathbf{F}^3)$ by

$$T(z_1, z_2, z_3) = (z_3, 2z_1, 3z_2).$$

Find (explicitly) an isometry $S \in \mathcal{L}(\mathbf{F}^3)$ such that $T = S\sqrt{T^*T}$.

Solution. A straightforward calculation shows that $\sqrt{T^*T}$ is given by

$$\sqrt{T^*T}(z_1, z_2, z_3) = (2z_1, 3z_2, z_3).$$

Thus we should take S to be the map $S(z_1, z_2, z_3) = (z_3, z_1, z_2)$, which is evidently an isometry.

Exercise 7.D.8. Suppose $T \in \mathcal{L}(V), S \in \mathcal{L}(V)$ is an isometry, and $R \in \mathcal{L}(V)$ is a positive operator such that $T = SR$. Prove that $R = \sqrt{T^*T}$.
[*The exercise above shows that if we write T as the product of an isometry and a positive operator (as in the Polar Decomposition 7.45), then the positive operator equals $\sqrt{T^*T}$.*]

Solution. Since R is given as a positive operator, it will suffice to show that R is a square root of T^*T; it will then follow that R is the unique positive square root of T^*T, i.e. $R = \sqrt{T^*T}$. Taking the adjoint of both sides of the equation $T = SR$ gives us $T^* = RS^*$ (we have used that R is self-adjoint), from which we see that

$$T^*T = RS^*SR = R^2;$$

here we have used that $S^* = S^{-1}$ since S is an isometry (7.42).

Exercise 7.D.9. Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if there exists a unique isometry S in $\mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$.

Solution. Suppose that T is invertible. The Polar Decomposition (7.45) implies that there always exists some isometry S such that $T = S\sqrt{T^*T}$; suppose that there are two isometries S and U such that

$$T = S\sqrt{T^*T} = U\sqrt{T^*T}.$$

Since T and S are invertible, Exercise 3.D.9 implies that $\sqrt{T^*T}$ is invertible and thus

$$S = U = T\left(\sqrt{T^*T}\right)^{-1}.$$

Now suppose that T is not invertible and consider the proof of the Polar Decomposition (7.45). Since T is not invertible, it must be the case that $\dim \operatorname{range} T < \dim V$ and hence

$$\dim\left(\operatorname{range} \sqrt{T^*T}\right)^{\perp} = \dim(\operatorname{range} T)^{\perp} = m \geq 1.$$

The next step of the proof involves choosing an orthonormal basis $e_1, \ldots, e_m$ of $\left(\operatorname{range} \sqrt{T^*T}\right)^{\perp}$, an orthonormal basis $f_1, \ldots, f_m$ of $(\operatorname{range} T)^{\perp}$, and then obtaining an isometry $S \in \mathcal{L}(V)$ such that

$$S(a_1e_1 + \cdots + a_me_m) = a_1f_1 + \cdots + a_mf_m$$

and such that $T = S\sqrt{T^*T}$. Observe that the list $-f_1, f_2, \ldots, f_m$ is also an orthonormal basis of $(\operatorname{range} T)^{\perp}$. Define an isometry $S' \in \mathcal{L}(V)$ as in the proof and note that $T = S'\sqrt{T^*T}$, however

$$S'e_1 = -f_1 \neq f_1 = Se_1;$$

here we are crucially using that $m \geq 1$. Thus the isometry S is not unique.

Exercise 7.D.10. Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.

Solution. The relevant Spectral Theorem (7.24 or 7.29) implies that there is an orthonormal basis $e_1, \ldots, e_n$ of V such that $Te_j = \lambda_j e_j$ for each $1 \leq j \leq n$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of T. It follows that the adjoint of T is given by $T^*e_j = \overline{\lambda_j}e_j$, from which we see that T^*T is given by $T^*Te_j = |\lambda_j|^2 e_j$, so that the eigenvalues of T^*T are $|\lambda_1|^2, \ldots, |\lambda_n|^2$. It now follows from 7.52 that the singular values of T are $|\lambda_1|, \ldots, |\lambda_n|$.

Exercise 7.D.11. Suppose $T \in \mathcal{L}(V)$. Prove that T and T^* have the same singular values.

Solution. By 7.52, it will suffice to show that T^*T and TT^* have the same eigenvalues. Suppose that λ is an eigenvalue of T^*T, so that $T^*Tv = \lambda v$ for some $v \neq 0$. If $Tv \neq 0$, then observe that

$$TT^*(Tv) = T(\lambda v) = \lambda Tv,$$

so that λ is an eigenvalue of TT^* with a corresponding eigenvector Tv. If $Tv = 0$, then it must be the case that $\lambda = 0$. This implies that T^*T is not invertible, which in turn implies that TT^* is not invertible (Exercise 3.D.10). It follows that 0 is also an eigenvalue of TT^*. A similar argument, reversing the roles of T and T^*, shows that λ is an eigenvalue of T^*T if λ is an eigenvalue of TT^*.

Exercise 7.D.12. Prove or give a counterexample: if $T \in \mathcal{L}(V)$, then the singular values of T^2 equal the squares of the singular values of T.

Solution. This is false. For a counterexample, consider the operator $T \in \mathcal{L}(\mathbf{F}^2)$ whose matrix with respect to the standard basis of $\mathbf{F}^2$ is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

A straightforward calculation shows that the singular values of T are 0, 1. However, since $T^2 = 0$, the singular values of T^2 are 0, 0.

Exercise 7.D.13. Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if 0 is not a singular value of T.

Solution. Observe that

$$\begin{aligned}
0 \text{ is not a singular value of } T &\iff 0 \text{ is not an eigenvalue of } \sqrt{T^*T} \\
&\iff \sqrt{T^*T} \text{ is invertible} \\
&\iff T^*T \text{ is invertible} && \text{(Exercise 3.D.9)} \\
&\iff T^* \text{ and } T \text{ are invertible} && \text{(Exercise 3.D.9)} \\
&\iff T \text{ is invertible.} && \text{(Exercise 7.A.4)}
\end{aligned}$$

Exercise 7.D.14. Suppose $T \in \mathcal{L}(V)$. Prove that $\dim \operatorname{range} T$ equals the number of nonzero singular values of T.

Solution. The operator $\sqrt{T^*T}$ is self-adjoint and hence the relevant Spectral Theorem (7.24 or 7.29) implies that $\sqrt{T^*T}$ is diagonalizable. Letting $s_1, \ldots, s_n$ be the non-zero singular values of

T, it then follows from 3.22 and 5.41 that

$$\begin{aligned}\dim\operatorname{null}\sqrt{T^*T} + \dim\operatorname{range}\sqrt{T^*T} &= \dim E\Big(0, \sqrt{T^*T}\Big) + \dim\operatorname{range}\sqrt{T^*T}\\ &= \dim V = \dim E\Big(0, \sqrt{T^*T}\Big) + \dim E\Big(s_1, \sqrt{T^*T}\Big) + \cdots + \dim E\Big(s_n, \sqrt{T^*T}\Big),\end{aligned}$$

from which we see that

$$\dim\operatorname{range}\sqrt{T^*T} = \dim E\Big(s_1, \sqrt{T^*T}\Big) + \cdots + \dim E\Big(s_n, \sqrt{T^*T}\Big);$$

note that the integer on the right-hand side is the number of non-zero singular values of T. Finally, as the proof of the Polar Decomposition (7.45) shows, we have

$$\dim\operatorname{range} T = \dim\operatorname{range}\sqrt{T^*T}.$$

Exercise 7.D.15. Suppose $S \in \mathcal{L}(V)$. Prove that S is an isometry if and only if all the singular values of S equal 1.

Solution. Suppose that S is an isometry. By 7.42 we then have $S^* = S^{-1}$ and thus $\sqrt{S^*S} = I$, from which it is clear that all the singular values of S equal 1. Now suppose that each singular value of S is 1. The Singular Value Decomposition (7.51) implies that there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Sv = \langle v, e_1\rangle f_1 + \cdots + \langle v, e_n\rangle f_n$$

for any $v \in V$. By 6.25 and 6.30 we then have

$$\|Sv\|^2 = \|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2,$$

from which we see that S is an isometry.

Exercise 7.D.16. Suppose $T_1, T_2 \in \mathcal{L}(V)$. Prove that T_1 and T_2 have the same singular values if and only if there exist isometries $S_1, S_2 \in \mathcal{L}(V)$ such that $T_1 = S_1T_2S_2$.

Solution. Suppose there exist such isometries S_1 and S_2. In light of 7.52, it will suffice to show that $T_1^*T_1$ and $T_2^*T_2$ have the same eigenvalues. Observe that

$$T_1^*T_1 = (S_1T_2S_2)^*(S_1T_2S_2) = S_2^*T_2^*S_1^*S_1T_2S_2 = S_2^*T_2^*T_2S_2,$$

where we have used that $S_1^* = S_1^{-1}$ (7.42). It now follows from Exercise 5.A.15 that $T_1^*T_1$ and $T_2^*T_2$ have the same eigenvalues.

Now suppose that T_1 and T_2 have the same singular values $s_1, \ldots, s_n$. The Polar Decomposition (7.45) implies that there exist orthonormal bases

$$e_1, \ldots, e_n, \quad f_1, \ldots, f_n, \quad u_1, \ldots, u_n, \quad \text{and} \quad v_1, \ldots, v_n$$

such that

$$T_1 v = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n \quad \text{and} \quad T_2 v = s_1 \langle v, u_1 \rangle v_1 + \cdots + s_n \langle v, u_n \rangle v_n$$

for any $v \in V$. Define two linear maps S_1 and S_2 by $S_1 v_j = f_j$ and $S_2 e_j = u_j$; it follows from the equivalence of (a) and (d) in 7.42 that S_1 and S_2 are isometries. Furthermore, observe that

$$S_1 T_2 S_2 e_j = S_1 T_2 u_j = s_j S_1 v_j = s_j f_j = T_1 e_j.$$

Thus $T_1 = S_1 T_2 S_2$.

Exercise 7.D.17. Suppose $T \in \mathcal{L}(V)$ has singular value decomposition given by

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$, where $s_1, \ldots, s_n$ are the singular values of T and $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are orthonormal bases of V.

(a) Prove that if $v \in V$, then

$$T^* v = s_1 \langle v, f_1 \rangle e_1 + \cdots + s_n \langle v, f_n \rangle e_n.$$

(b) Prove that if $v \in V$, then

$$T^* T v = s_1^2 \langle v, e_1 \rangle e_1 + \cdots + s_n^2 \langle v, e_n \rangle e_n.$$

(c) Prove that if $v \in V$, then

$$\sqrt{T^* T} v = s_1 \langle v, e_1 \rangle e_1 + \cdots + s_n \langle v, e_n \rangle e_n.$$

(d) Suppose T is invertible. Prove that if $v \in V$, then

$$T^{-1} v = \frac{\langle v, f_1 \rangle e_1}{s_1} + \cdots + \frac{\langle v, f_n \rangle e_n}{s_n}$$

for every $v \in V$.

Solution. (a) The given singular value decomposition implies that

$$\mathcal{M}(T, (e_1, \ldots, e_n), (f_1, \ldots, f_n)) = \begin{pmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n \end{pmatrix}.$$

It follows from 7.10 that

$$\mathcal{M}(T^*, (f_1, \ldots, f_n), (e_1, \ldots, e_n)) = \begin{pmatrix} \overline{s_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \overline{s_n} \end{pmatrix} = \begin{pmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n \end{pmatrix},$$

where we have used that singular values are necessarily non-negative real numbers, being the eigenvalues of a positive operator. This matrix implies the desired result.

(b) We have by 3.43 and part (a) that

$$\begin{aligned} &\mathcal{M}(T^*T, (e_1, \ldots, e_n), (e_1, \ldots, e_n)) \\ &\qquad = \mathcal{M}(T^*, (f_1, \ldots, f_n), (e_1, \ldots, e_n))\mathcal{M}(T, (e_1, \ldots, e_n), (f_1, \ldots, f_n)) \\ &\qquad = \begin{pmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n \end{pmatrix}^2 = \begin{pmatrix} s_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n^2 \end{pmatrix}, \end{aligned}$$

which implies the desired result.

(c) Define an operator $R \in \mathcal{L}(V)$ by

$$Rv = s_1\langle v, e_1\rangle e_1 + \cdots + s_n\langle v, e_n\rangle e_n,$$

so that

$$\mathcal{M}(R, (e_1, \ldots, e_n)) = \begin{pmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n \end{pmatrix}.$$

It is then clear, given part (b), that $R^2 = T^*T$. Furthermore, since each s_j is a non-negative real number, we see that R is self-adjoint and that each eigenvalue of R is a non-negative real number. It follows that R is a positive operator (7.35) and hence that $R = \sqrt{T^*T}$ (7.36).

(d) Define an operator $S \in \mathcal{L}(V)$ by

$$Sv = \frac{\langle v, f_1 \rangle e_1}{s_1} + \cdots + \frac{\langle v, f_n \rangle e_n}{s_n}$$

and observe that

$$STe_j = s_j S f_j = e_j.$$

It follows that $ST = I$, which is the case if and only if $TS = I$ (Exercise 3.D.10), and hence by the uniqueness of the inverse (3.54) we have $S = T^{-1}$.

Exercise 7.D.18. Suppose $T \in \mathcal{L}(V)$. Let $\hat{s}$ denote the smallest singular value of T, and let s denote the largest singular value of T.

(a) Prove that $\hat{s}\|v\| \le \|Tv\| \le s\|v\|$ for every $v \in V$.

(b) Suppose λ is an eigenvalue of T. Prove that $\hat{s} \le |\lambda| \le s$.

Solution. (a) Let $s_1, \ldots, s_n$ be the singular values of T. The Singular Value Decomposition (7.51) implies that there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$, from which we see that

$$\|Tv\|^2 = s_1^2 |\langle v, e_1 \rangle|^2 + \cdots + s_n^2 |\langle v, e_n \rangle|^2.$$

It is then clear that

$$\hat{s}^2 |\langle v, e_1 \rangle|^2 + \cdots + \hat{s}^2 |\langle v, e_n \rangle|^2 = \hat{s}^2 \|v\|^2 \le \|Tv\|^2 \le s^2 \|v\|^2 = s^2 |\langle v, e_1 \rangle|^2 + \cdots + s^2 |\langle v, e_n \rangle|^2,$$

which implies the desired result upon taking square roots.

(b) Let v be an eigenvector associated with λ; by replacing v with $\frac{v}{\|v\|}$ if necessary, we may assume that $\|v\| = 1$. Part (a) then gives

$$\hat{s} \le \|Tv\| = \|\lambda v\| = |\lambda| \le s.$$

Exercise 7.D.19. Suppose $T \in \mathcal{L}(V)$. Show that T is uniformly continuous with respect to the metric d on V defined by $d(u, v) = \|u - v\|$.

Solution. As we showed in the solution to Exercise 6.B.16, there exists a non-negative constant $C \in \mathbf{R}$ such that $\|Tv\| \leq C\|v\|$ for all $v \in V$. Let $\epsilon > 0$ be given and suppose $u, v \in V$ are such that $\|u - v\| < \frac{\epsilon}{1+C}$. Then

$$\|Tu - Tv\| = \|T(u - v)\| \leq C\|u - v\| < \frac{C\epsilon}{1 + C} < \epsilon$$

and thus T is uniformly continuous.

Exercise 7.D.20. Suppose $S, T \in \mathcal{L}(V)$. Let s denote the largest singular value of S, let t denote the largest singular value of T, and let r denote the largest singular value of $S + T$. Prove that $r \leq s + t$.

Solution. Let $R = S + T$. Since r is a singular value of R, there exists an eigenvector $v \in V$ such that $\|v\| = 1$ and $\sqrt{R^*R}v = rv$. It follows that

$$r = \|rv\| = \left\|\sqrt{R^*R}v\right\| = \|Rv\| = \|Sv + Tv\| \leq \|Sv\| + \|Tv\| \leq s + t;$$

the third equality follows from 7.46 and the last inequality follows from Exercise 7.D.18.

[LADR] Axler, S. (2015) *Linear Algebra Done Right.* 3rd edition.