Complete Solutions Manual

Linear Algebra
A Modern Introduction

FOURTH EDITION

David Poole
Trent University

Prepared by

Roger Lipsett

Australia • Brazil • Japan • Korea • Mexico • Singapore • Spain • United Kingdom • United States

For product information and technology assistance, contact us at **Cengage Learning Customer & Sales Support, 1-800-354-9706**.

For permission to use material from this text or product, submit all requests online at **www.cengage.com/permissions**
Further permissions questions can be emailed to **permissionrequest@cengage.com**.

ISBN-13: 978-128586960-5
ISBN-10: 1-28586960-5

Cengage Learning
200 First Stamford Place, 4th Floor
Stamford, CT 06902
USA

Cengage Learning is a leading provider of customized learning solutions with office locations around the globe, including Singapore, the United Kingdom, Australia, Mexico, Brazil, and Japan. Locate your local office at: **www.cengage.com/global**.

Cengage Learning products are represented in Canada by Nelson Education, Ltd.

To learn more about Cengage Learning Solutions, visit **www.cengage.com**.

Purchase any of our products at your local college store or at our preferred online store **www.cengagebrain.com**.

Printed in the United States of America
1 2 3 4 5 6 7 17 16 15 14 13

Contents

1 Vectors **3**
1.1 The Geometry and Algebra of Vectors . . . 3
1.2 Length and Angle: The Dot Product . . . 10
Exploration: Vectors and Geometry . . . 25
1.3 Lines and Planes . . . 27
Exploration: The Cross Product . . . 41
1.4 Applications . . . 44
Chapter Review . . . 48

2 Systems of Linear Equations **53**
2.1 Introduction to Systems of Linear Equations . . . 53
2.2 Direct Methods for Solving Linear Systems . . . 58
Exploration: Lies My Computer Told Me . . . 75
Exploration: Partial Pivoting . . . 75
Exploration: An Introduction to the Analysis of Algorithms . . . 77
2.3 Spanning Sets and Linear Independence . . . 79
2.4 Applications . . . 93
2.5 Iterative Methods for Solving Linear Systems . . . 112
Chapter Review . . . 123

3 Matrices **129**
3.1 Matrix Operations . . . 129
3.2 Matrix Algebra . . . 138
3.3 The Inverse of a Matrix . . . 150
3.4 The LU Factorization . . . 164
3.5 Subspaces, Basis, Dimension, and Rank . . . 176
3.6 Introduction to Linear Transformations . . . 192
3.7 Applications . . . 209
Chapter Review . . . 230

4 Eigenvalues and Eigenvectors **235**
4.1 Introduction to Eigenvalues and Eigenvectors . . . 235
4.2 Determinants . . . 250
Exploration: Geometric Applications of Determinants . . . 263
4.3 Eigenvalues and Eigenvectors of $n \times n$ Matrices . . . 270
4.4 Similarity and Diagonalization . . . 291
4.5 Iterative Methods for Computing Eigenvalues . . . 308
4.6 Applications and the Perron-Frobenius Theorem . . . 326
Chapter Review . . . 365

5 Orthogonality **371**
5.1 Orthogonality in $\mathbb{R}^n$ . . . 371
5.2 Orthogonal Complements and Orthogonal Projections . . . 379
5.3 The Gram-Schmidt Process and the QR Factorization . . . 388
Exploration: The Modified QR Process . . . 398
Exploration: Approximating Eigenvalues with the QR Algorithm . . . 402
5.4 Orthogonal Diagonalization of Symmetric Matrices . . . 405
5.5 Applications . . . 417
Chapter Review . . . 442

6 Vector Spaces **451**
6.1 Vector Spaces and Subspaces . . . 451
6.2 Linear Independence, Basis, and Dimension . . . 463
Exploration: Magic Squares . . . 477
6.3 Change of Basis . . . 480
6.4 Linear Transformations . . . 491
6.5 The Kernel and Range of a Linear Transformation . . . 498
6.6 The Matrix of a Linear Transformation . . . 507
Exploration: Tiles, Lattices, and the Crystallographic Restriction . . . 525
6.7 Applications . . . 527
Chapter Review . . . 531

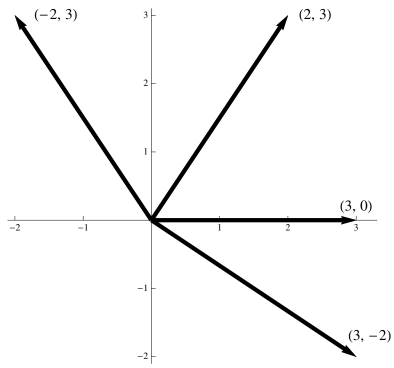
7 Distance and Approximation **537**
7.1 Inner Product Spaces . . . 537
Exploration: Vectors and Matrices with Complex Entries . . . 546
Exploration: Geometric Inequalities and Optimization Problems . . . 553
7.2 Norms and Distance Functions . . . 556
7.3 Least Squares Approximation . . . 568
7.4 The Singular Value Decomposition . . . 590
7.5 Applications . . . 614
Chapter Review . . . 625

8 Codes **633**
8.1 Code Vectors . . . 633
8.2 Error-Correcting Codes . . . 637
8.3 Dual Codes . . . 641
8.4 Linear Codes . . . 647
8.5 The Minimum Distance of a Code . . . 650

Chapter 1

Vectors

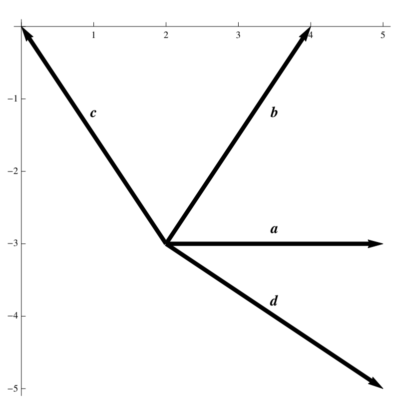
1.1 The Geometry and Algebra of Vectors

1. 

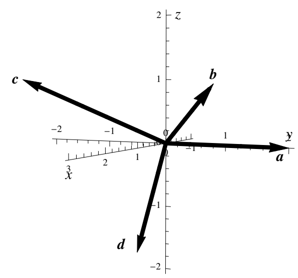
2. Since

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix},$$

plotting those vectors gives



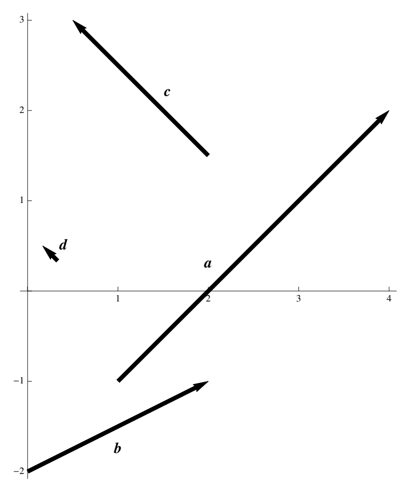
3.



4. Since the heads are all at $(3, 2, 1)$, the tails are at

$$\begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}0\\2\\0\end{bmatrix}=\begin{bmatrix}3\\0\\1\end{bmatrix},\qquad \begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\qquad \begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}2\\4\\0\end{bmatrix},\qquad \begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}-1\\-1\\-2\end{bmatrix}=\begin{bmatrix}4\\3\\3\end{bmatrix}.$$

5. The four vectors $\overrightarrow{AB}$ are



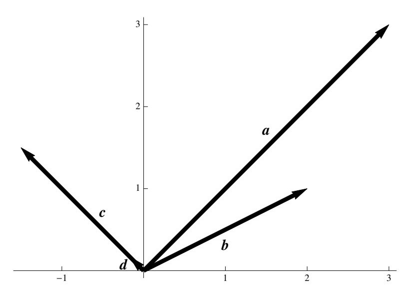
In standard position, the vectors are

(a) $\overrightarrow{AB} = [4-1, 2-(-1)] = [3, 3]$.

(b) $\overrightarrow{AB} = [2-0, -1-(-2)] = [2, 1]$

(c) $\overrightarrow{AB} = \left[\frac{1}{2}-2, 3-\frac{3}{2}\right] = \left[-\frac{3}{2}, \frac{3}{2}\right]$

(d) $\overrightarrow{AB} = \left[\frac{1}{6}-\frac{1}{3}, \frac{1}{2}-\frac{1}{3}\right] = \left[-\frac{1}{6}, \frac{1}{6}\right]$.



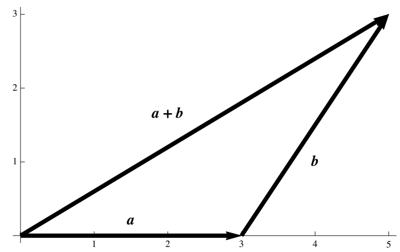
6. Recall the notation that $[a, b]$ denotes a move of a units horizontally and b units vertically. Then during the first part of the walk, the hiker walks 4 km north, so $\mathbf{a} = [0, 4]$. During the second part of the walk, the hiker walks a distance of 5 km northeast. From the components, we get

$$\mathbf{b} = [5\cos 45^\circ,\ 5\sin 45^\circ] = \left[\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right].$$

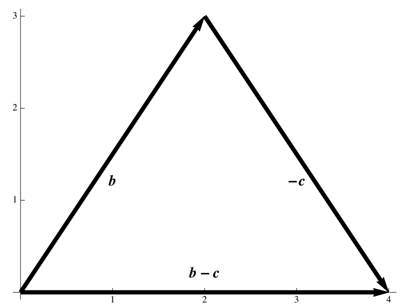
Thus the net displacement vector is

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = \left[\frac{5\sqrt{2}}{2}, 4 + \frac{5\sqrt{2}}{2}\right].$$

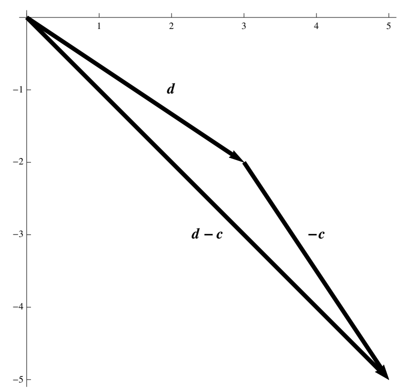
7. $\mathbf{a} + \mathbf{b} = \begin{bmatrix}3\\0\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3+2\\0+3\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix}.$



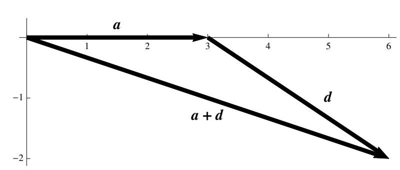
8. $\mathbf{b} - \mathbf{c} = \begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}2-(-2)\\3-3\end{bmatrix} = \begin{bmatrix}4\\0\end{bmatrix}.$



9. $\mathbf{d} - \mathbf{c} = \begin{bmatrix}3\\-2\end{bmatrix} - \begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}5\\-5\end{bmatrix}.$



10. $\mathbf{a} + \mathbf{d} = \begin{bmatrix}3\\0\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}3+3\\0+(-2)\end{bmatrix} = \begin{bmatrix}6\\-2\end{bmatrix}.$



11. $2\mathbf{a} + 3\mathbf{c} = 2[0, 2, 0] + 3[1, -2, 1] = [2 \cdot 0, 2 \cdot 2, 2 \cdot 0] + [3 \cdot 1, 3 \cdot (-2), 3 \cdot 1] = [3, -2, 3].$

12.

$$\begin{aligned} 3\mathbf{b} - 2\mathbf{c} + \mathbf{d} &= 3[3, 2, 1] - 2[1, -2, 1] + [-1, -1, -2] \\ &= [3 \cdot 3, 3 \cdot 2, 3 \cdot 1] + [-2 \cdot 1, -2 \cdot (-2), -2 \cdot 1] + [-1, -1, -2] \\ &= [6, 9, -1]. \end{aligned}$$

13. $\mathbf{u} = [\cos 60°, \sin 60°] = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$, and $\mathbf{v} = [\cos 210°, \sin 210°] = \left[-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right]$, so that

$$\mathbf{u}+\mathbf{v} = \left[\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} - \frac{1}{2}\right], \quad \mathbf{u}-\mathbf{v} = \left[\frac{1}{2} + \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} + \frac{1}{2}\right].$$

14. **(a)** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

(b) Since $\overrightarrow{OC} = \overrightarrow{AB}$, we have $\overrightarrow{BC} = \overrightarrow{OC} - \mathbf{b} = (\mathbf{b} - \mathbf{a}) - \mathbf{b} = -\mathbf{a}$.

(c) $\overrightarrow{AD} = -2\mathbf{a}$.

(d) $\overrightarrow{CF} = -2\overrightarrow{OC} = -2\overrightarrow{AB} = -2(\mathbf{b} - \mathbf{a}) = 2(\mathbf{a} - \mathbf{b})$.

(e) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = (\mathbf{b} - \mathbf{a}) + (-\mathbf{a}) = \mathbf{b} - 2\mathbf{a}$.

(f) Note that $\overrightarrow{FA}$ and $\overrightarrow{OB}$ are equal, and that $\overrightarrow{DE} = -\overrightarrow{AB}$. Then

$$\overrightarrow{BC} + \overrightarrow{DE} + \overrightarrow{FA} = -\mathbf{a} - \overrightarrow{AB} + \overrightarrow{OB} = -\mathbf{a} - (\mathbf{b} - \mathbf{a}) + \mathbf{b} = \mathbf{0}.$$

15. $2(\mathbf{a} - 3\mathbf{b}) + 3(2\mathbf{b} + \mathbf{a}) \overset{\substack{\text{property e.}\\ \text{distributivity}}}{=} (2\mathbf{a} - 6\mathbf{b}) + (6\mathbf{b} + 3\mathbf{a}) \overset{\substack{\text{property b.}\\ \text{associativity}}}{=} (2\mathbf{a} + 3\mathbf{a}) + (-6\mathbf{b} + 6\mathbf{b}) = 5\mathbf{a}$.

16.

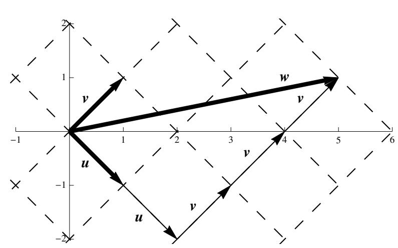
$$\begin{aligned} -3(\mathbf{a} - \mathbf{c}) + 2(\mathbf{a} + 2\mathbf{b}) + 3(\mathbf{c} - \mathbf{b}) &\overset{\substack{\text{property e.}\\ \text{distributivity}}}{=} (-3\mathbf{a} + 3\mathbf{c}) + (2\mathbf{a} + 4\mathbf{b}) + (3\mathbf{c} - 3\mathbf{b}) \\ &\overset{\substack{\text{property b.}\\ \text{associativity}}}{=} (-3\mathbf{a} + 2\mathbf{a}) + (4\mathbf{b} - 3\mathbf{b}) + (3\mathbf{c} + 3\mathbf{c}) \\ &= -\mathbf{a} + \mathbf{b} + 6\mathbf{c}. \end{aligned}$$

17. $\mathbf{x} - \mathbf{a} = 2(\mathbf{x} - 2\mathbf{a}) = 2\mathbf{x} - 4\mathbf{a} \Rightarrow \mathbf{x} - 2\mathbf{x} = \mathbf{a} - 4\mathbf{a} \Rightarrow -\mathbf{x} = -3\mathbf{a} \Rightarrow \mathbf{x} = 3\mathbf{a}$.

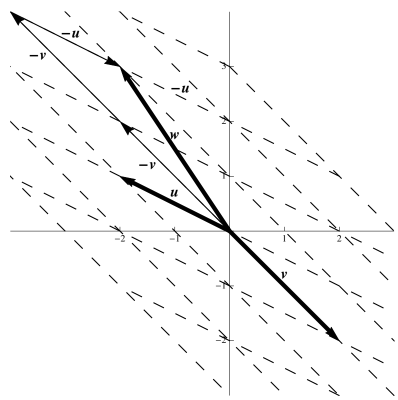
18.

$$\begin{aligned} \mathbf{x} + 2\mathbf{a} - \mathbf{b} = 3(\mathbf{x} + \mathbf{a}) - 2(2\mathbf{a} - \mathbf{b}) = 3\mathbf{x} + 3\mathbf{a} - 4\mathbf{a} + 2\mathbf{b} \quad &\Rightarrow \\ \mathbf{x} - 3\mathbf{x} = -\mathbf{a} - 2\mathbf{a} + 2\mathbf{b} + \mathbf{b} \quad &\Rightarrow \\ -2\mathbf{x} = -3\mathbf{a} + 3\mathbf{b} \quad &\Rightarrow \\ \mathbf{x} = \frac{3}{2}\mathbf{a} - \frac{3}{2}\mathbf{b}. \end{aligned}$$

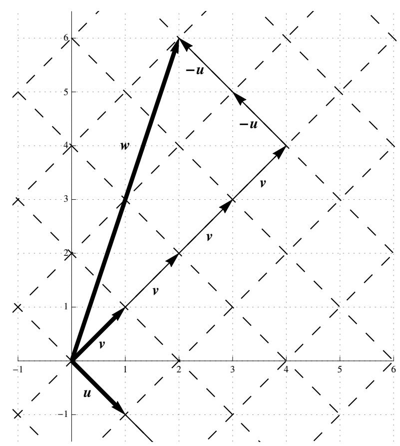
19. We have $2\mathbf{u} + 3\mathbf{v} = 2[1, -1] + 3[1, 1] = [2 \cdot 1 + 3 \cdot 1, 2 \cdot (-1) + 3 \cdot 1] = [5, 1]$. Plots of all three vectors are



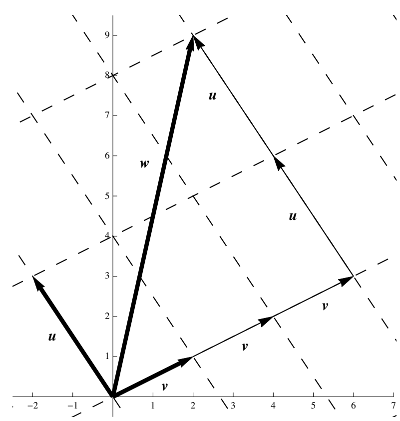
20. We have $-\mathbf{u}-2\mathbf{v}=-[-2,1]-2[2,-2]=[-(-2)-2\cdot 2,-1-2\cdot(-2)]=[-2,3]$. Plots of all three vectors are



21. From the diagram, we see that $\mathbf{w}=-2\mathbf{u}+4\mathbf{v}$.

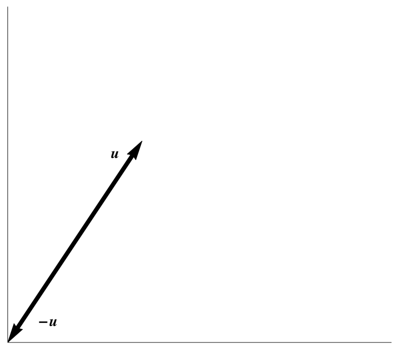


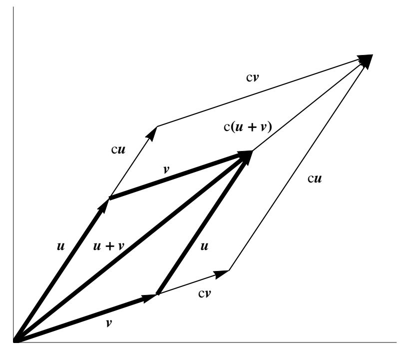
22. From the diagram, we see that $\mathbf{w}=2\mathbf{u}+3\mathbf{v}$.



23. Property (d) states that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$. The first diagram below shows $\mathbf{u}$ along with $-\mathbf{u}$. Then, as the diagonal of the parallelogram, the resultant vector is $\mathbf{0}$.

Property (e) states that $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$. The second figure illustrates this.





24. Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, and let c and d be scalars in $\mathbb{R}$.

Property (d):

$$\begin{aligned}\mathbf{u} + (-\mathbf{u}) &= [u_1, u_2, \ldots, u_n] + (-1[u_1, u_2, \ldots, u_n]) \\ &= [u_1, u_2, \ldots, u_n] + [-u_1, -u_2, \ldots, -u_n] \\ &= [u_1 - u_1, u_2 - u_2, \ldots, u_n - u_n] \\ &= [0, 0, \ldots, 0] = \mathbf{0}.\end{aligned}$$

Property (e):

$$\begin{aligned}c(\mathbf{u} + \mathbf{v}) &= c\left([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]\right) \\ &= c\left([u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]\right) \\ &= [c(u_1 + v_1), c(u_2 + v_2), \ldots, c(u_n + v_n)] \\ &= [cu_1 + cv_1, cu_2 + cv_2, \ldots, cu_n + cv_n] \\ &= [cu_1, cu_2, \ldots, cu_n] + [cv_1, cv_2, \ldots, cv_n] \\ &= c[u_1, u_2, \ldots, u_n] + c[v_1, v_2, \ldots, v_n] \\ &= c\mathbf{u} + c\mathbf{v}.\end{aligned}$$

Property (f):

$$\begin{aligned}(c + d)\mathbf{u} &= (c + d)[u_1, u_2, \ldots, u_n] \\ &= [(c + d)u_1, (c + d)u_2, \ldots, (c + d)u_n] \\ &= [cu_1 + du_1, cu_2 + du_2, \ldots, cu_n + du_n] \\ &= [cu_1, cu_2, \ldots, cu_n] + [du_1, du_2, \ldots, du_n] \\ &= c[u_1, u_2, \ldots, u_n] + d[u_1, u_2, \ldots, u_n] \\ &= c\mathbf{u} + d\mathbf{u}.\end{aligned}$$

Property (g):

$$\begin{aligned}c(d\mathbf{u}) &= c(d[u_1, u_2, \ldots, u_n]) \\ &= c[du_1, du_2, \ldots, du_n] \\ &= [cdu_1, cdu_2, \ldots, cdu_n] \\ &= [(cd)u_1, (cd)u_2, \ldots, (cd)u_n] \\ &= (cd)[u_1, u_2, \ldots, u_n] \\ &= (cd)\mathbf{u}.\end{aligned}$$

25. $\mathbf{u} + \mathbf{v} = [0, 1] + [1, 1] = [1, 0]$.

26. $\mathbf{u} + \mathbf{v} = [1, 1, 0] + [1, 1, 1] = [0, 0, 1]$.

27. $\mathbf{u}+\mathbf{v}=[1,0,1,1]+[1,1,1,1]=[0,1,0,0]$.

28. $\mathbf{u}+\mathbf{v}=[1,1,0,1,0]+[0,1,1,1,0]=[1,0,1,0,0]$.

29.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

30.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

31. $2+2+2=6=0$ in $\mathbb{Z}_3$.

32. $2\cdot 2\cdot 2=3\cdot 2=0$ in $\mathbb{Z}_3$.

33. $2(2+1+2)=2\cdot 2=3\cdot 1+1=1$ in $\mathbb{Z}_3$.

34. $3+1+2+3=4\cdot 2+1=1$ in $\mathbb{Z}_4$.

35. $2\cdot 3\cdot 2=4\cdot 3+0=0$ in $\mathbb{Z}_4$.

36. $3(3+3+2)=4\cdot 6+0=0$ in $\mathbb{Z}_4$.

37. $2+1+2+2+1=2$ in $\mathbb{Z}_3$, $2+1+2+2+1=0$ in $\mathbb{Z}_4$, $2+1+2+2+1=3$ in $\mathbb{Z}_5$.

38. $(3+4)(3+2+4+2)=2\cdot 1=2$ in $\mathbb{Z}_5$.

39. $8(6+4+3)=8\cdot 4=5$ in $\mathbb{Z}_9$.

40. $2^{100}=\left(2^{10}\right)^{10}=(1024)^{10}=1^{10}=1$ in $\mathbb{Z}_{11}$.

41. $[2,1,2]+[2,0,1]=[1,1,0]$ in $\mathbb{Z}_3^3$.

42. $2[2,\ 2,\ 1]=[2\cdot 2,\ 2\cdot 2,\ 2\cdot 1]=[1,\ 1,\ 2]$ in $\mathbb{Z}_3^3$.

43. $2([3,1,1,2]+[3,3,2,1])=2[2,0,3,3]=[2\cdot 2,\ 2\cdot 0,\ 2\cdot 3,\ 2\cdot 3]=[0,0,2,2]$ in $\mathbb{Z}_4^4$.
$2([3,1,1,2]+[3,3,2,1])=2[1,4,3,3]=[2\cdot 1,\ 2\cdot 4,\ 2\cdot 3,\ 2\cdot 3]=[2,3,1,1]$ in $\mathbb{Z}_5^4$.

44. $x=2+(-3)=2+2=4$ in $\mathbb{Z}_5$.

45. $x=1+(-5)=1+1=2$ in $\mathbb{Z}_6$

46. $x=2^{-1}=2$ in $\mathbb{Z}_3$.

47. No solution. 2 times anything is always even, so cannot leave a remainder of 1 when divided by 4.

48. $x=2^{-1}=3$ in $\mathbb{Z}_5$.

49. $x=3^{-1}4=2\cdot 4=3$ in $\mathbb{Z}_5$.

50. No solution. 3 times anything is always a multiple of 3, so it cannot leave a remainder of 4 when divided by 6 (which is also a multiple of 3).

51. No solution. 6 times anything is always even, so it cannot leave an odd number as a remainder when divided by 8.

52. $x = 8^{-1}9 = 7 \cdot 9 = 8$ in $\mathbb{Z}_{11}$

53. $x = 2^{-1}(2 + (-3)) = 3(2 + 2) = 2$ in $\mathbb{Z}_5$.

54. No solution. This equation is the same as $4x = 2 - 5 = -3 = 3$ in $\mathbb{Z}_6$. But 4 times anything is even, so it cannot leave a remainder of 3 when divided by 6 (which is also even).

55. Add 5 to both sides to get $6x = 6$, so that $x = 1$ or $x = 5$ (since $6 \cdot 1 = 6$ and $6 \cdot 5 = 30 = 6$ in $\mathbb{Z}_8$).

56. (a) All values. (b) All values. (c) All values.

57. **(a)** All $a \neq 0$ in $\mathbb{Z}_5$ have a solution because 5 is a prime number.

(b) $a = 1$ and $a = 5$ because they have no common factors with 6 other than 1.

(c) a and m can have no common factors other than 1; that is, the *greatest common divisor*, gcd, of a and m is 1.

1.2 Length and Angle: The Dot Product

1. Following Example 1.15, $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (-1) \cdot 3 + 2 \cdot 1 = -3 + 2 = -1.$

2. Following Example 1.15, $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 3 \cdot 4 + (-2) \cdot 6 = 12 - 12 = 0.$

3. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 = 2 + 6 + 3 = 11.$

4. $\mathbf{u} \cdot \mathbf{v} = 3.2 \cdot 1.5 + (-0.6) \cdot 4.1 + (-1.4) \cdot (-0.2) = 4.8 - 2.46 + 0.28 = 2.62.$

5. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -\sqrt{2} \\ 0 \\ -5 \end{bmatrix} = 1 \cdot 4 + \sqrt{2} \cdot (-\sqrt{2}) + \sqrt{3} \cdot 0 + 0 \cdot (-5) = 4 - 2 = 2.$

6. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1.12 \\ -3.25 \\ 2.07 \\ -1.83 \end{bmatrix} \cdot \begin{bmatrix} -2.29 \\ 1.72 \\ 4.33 \\ -1.54 \end{bmatrix} = -1.12 \cdot 2.29 - 3.25 \cdot 1.72 + 2.07 \cdot 4.33 - 1.83 \cdot (-1.54) = 3.6265.$

7. Finding a unit vector $\mathbf{v}$ in the same direction as a given vector $\mathbf{u}$ is called ***normalizing*** the vector $\mathbf{u}$. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5},$$

so a unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

8. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13},$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \end{bmatrix}.$$

9. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14},$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{14}}\\\frac{2}{\sqrt{14}}\\\frac{3}{\sqrt{14}}\end{bmatrix}.$$

10. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{3.2^2 + (-0.6)^2 + (-1.4)^2} = \sqrt{10.24 + 0.36 + 1.96} = \sqrt{12.56} \approx 3.544,$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{3.544}\begin{bmatrix}1.5\\0.4\\-2.1\end{bmatrix} \approx \begin{bmatrix}0.903\\-0.169\\-0.395\end{bmatrix}.$$

11. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{1^2 + \left(\sqrt{2}\right)^2 + \left(\sqrt{3}\right)^2 + 0^2} = \sqrt{6},$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\\sqrt{2}\\\sqrt{3}\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\\frac{\sqrt{2}}{\sqrt{6}}\\\frac{\sqrt{3}}{\sqrt{6}}\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{6}}{6}\\\frac{\sqrt{3}}{3}\\\frac{\sqrt{2}}{2}\\0\end{bmatrix}$$

12. Proceed as in Example 1.19:

$$\begin{aligned}\|\mathbf{u}\| = \sqrt{1.12^2 + (-3.25)^2 + 2.07^2 + (-1.83)^2} &= \sqrt{1.2544 + 10.5625 + 4.2849 + 3.3489}\\ &= \sqrt{19.4507} \approx 4.410,\end{aligned}$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v}\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{4.410}\begin{bmatrix}1.12 & -3.25 & 2.07 & -1.83\end{bmatrix} \approx \begin{bmatrix}0.254 & -0.737 & 0.469 & -0.415\end{bmatrix}.$$

13. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix}-1\\2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}-4\\1\end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-4)^2 + 1^2} = \sqrt{17}.$$

14. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix}3\\-2\end{bmatrix} - \begin{bmatrix}4\\6\end{bmatrix} = \begin{bmatrix}-1\\-8\end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1)^2 + (-8)^2} = \sqrt{65}.$$

15. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\2\end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1)^2 + (-1)^2 + 2^2} = \sqrt{6}.$$

16. Following Example 1.20, we compute: $\mathbf{u}-\mathbf{v} = \begin{bmatrix} 3.2 \\ -0.6 \\ -1.4 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 4.1 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 1.7 \\ -4.7 \\ -1.2 \end{bmatrix}$, so

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{1.7^2+(-4.7)^2+(-1.2)^2} = \sqrt{26.42} \approx 5.14.$$

17. **(a)** $\mathbf{u}\cdot\mathbf{v}$ is a real number, so $\|\mathbf{u}\cdot\mathbf{v}\|$ is the norm of a number, which is not defined.

(b) $\mathbf{u}\cdot\mathbf{v}$ is a scalar, while $\mathbf{w}$ is a vector. Thus $\mathbf{u}\cdot\mathbf{v}+\mathbf{w}$ adds a scalar to a vector, which is not a defined operation.

(c) $\mathbf{u}$ is a vector, while $\mathbf{v}\cdot\mathbf{w}$ is a scalar. Thus $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$ is the dot product of a vector and a scalar, which is not defined.

(d) $c\cdot(\mathbf{u}+\mathbf{v})$ is the dot product of a scalar and a vector, which is not defined.

18. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{3\cdot(-1)+0\cdot 1}{\sqrt{3^2+0^2}\sqrt{(-1)^2+1^2}} = -\frac{3}{3\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Thus $\cos\theta < 0$ (in fact, $\theta = \frac{3\pi}{4}$), so the angle between $\mathbf{u}$ and $\mathbf{v}$ is obtuse.

19. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{2\cdot 1+(-1)\cdot(-2)+1\cdot(-1)}{\sqrt{2^2+(-1)^2+1^2}\sqrt{1^2+(-2)^2+1^2}} = \frac{1}{2}.$$

Thus $\cos\theta > 0$ (in fact, $\theta = \frac{\pi}{3}$), so the angle between $\mathbf{u}$ and $\mathbf{v}$ is acute.

20. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{4\cdot 1+3\cdot(-1)+(-1)\cdot 1}{\sqrt{4^2+3^2+(-1)^2}\sqrt{1^2+(-1)^2+1^2}} = \frac{0}{\sqrt{26}\sqrt{3}} = 0.$$

Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is a right angle.

21. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Note that we can determine whether θ is acute, right, or obtuse by examining the sign of $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$, which is determined by the sign of $\mathbf{u}\cdot\mathbf{v}$. Since

$$\mathbf{u}\cdot\mathbf{v} = 0.9\cdot(-4.5)+2.1\cdot 2.6+1.2\cdot(-0.8) = 0.45 > 0,$$

we have $\cos\theta > 0$ so that θ is acute.

22. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Note that we can determine whether θ is acute, right, or obtuse by examining the sign of $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$, which is determined by the sign of $\mathbf{u}\cdot\mathbf{v}$. Since

$$\mathbf{u}\cdot\mathbf{v} = 1\cdot(-3)+2\cdot 1+3\cdot 2+4\cdot(-2) = -3,$$

we have $\cos\theta < 0$ so that θ is obtuse.

23. Since the components of both $\mathbf{u}$ and $\mathbf{v}$ are positive, it is clear that $\mathbf{u}\cdot\mathbf{v} > 0$, so the angle between them is acute since it has a positive cosine.

24. From Exercise 18, $\cos\theta = -\frac{\sqrt{2}}{2}$, so that $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4} = 135^\circ$.

25. From Exercise 19, $\cos\theta = \frac{1}{2}$, so that $\theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} = 60^\circ$.

26. From Exercise 20, $\cos\theta = 0$, so that $\theta = \frac{\pi}{2} = 90^\circ$ is a right angle.

27. As in Example 1.21, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ and the norms of the two vectors:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 0.9\cdot(-4.5)+2.1\cdot 2.6+1.2\cdot(-0.8)=0.45,\\ \|\mathbf{u}\| &= \sqrt{0.9^2+2.1^2+1.2^2}=\sqrt{6.66},\\ \|\mathbf{v}\| &= \sqrt{(-4.5)^2+2.6^2+(-0.8)^2}=\sqrt{27.65}.\end{aligned}$$

So if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{0.45}{\sqrt{6.66}\sqrt{27.65}}\approx\frac{0.45}{\sqrt{182.817}},$$

so that

$$\theta=\cos^{-1}\left(\frac{0.45}{\sqrt{182.817}}\right)\approx 1.5375\approx 88.09^\circ.$$

Note that it is important to maintain as much precision as possible until the last step, or roundoff errors may build up.

28. As in Example 1.21, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ and the norms of the two vectors:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 1\cdot(-3)+2\cdot 1+3\cdot 2+4\cdot(-2)=-3,\\ \|\mathbf{u}\| &= \sqrt{1^2+2^2+3^2+4^2}=\sqrt{30},\\ \|\mathbf{v}\| &= \sqrt{(-3)^2+1^2+2^2+(-2)^2}=\sqrt{18}.\end{aligned}$$

So if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=-\frac{3}{\sqrt{30}\sqrt{18}}=-\frac{1}{2\sqrt{15}}\quad\text{so that}\quad\theta=\cos^{-1}\left(-\frac{1}{2\sqrt{15}}\right)\approx 1.7\approx 97.42^\circ.$$

29. As in Example 1.21, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ and the norms of the two vectors:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 1\cdot 5+2\cdot 6+3\cdot 7+4\cdot 8=70,\\ \|\mathbf{u}\| &= \sqrt{1^2+2^2+3^2+4^2}=\sqrt{30},\\ \|\mathbf{v}\| &= \sqrt{5^2+6^2+7^2+8^2}=\sqrt{174}.\end{aligned}$$

So if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{70}{\sqrt{30}\sqrt{174}}=\frac{35}{3\sqrt{145}}\quad\text{so that}\quad\theta=\cos^{-1}\left(\frac{35}{3\sqrt{145}}\right)\approx 0.2502\approx 14.34^\circ.$$

30. To show that $\triangle ABC$ is right, we need only show that one pair of its sides meets at a right angle. So let $\mathbf{u}=\overrightarrow{AB}$, $\mathbf{v}=\overrightarrow{BC}$, and $\mathbf{w}=\overrightarrow{AC}$. Then we must show that one of $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{w}$ or $\mathbf{v}\cdot\mathbf{w}$ is zero in order to show that one of these pairs is orthogonal. Then

$$\begin{gathered}\mathbf{u}=\overrightarrow{AB}=[1-(-3),\,0-2]=[4,-2]\,,\quad \mathbf{v}=\overrightarrow{BC}=[4-1,\,6-0]=[3,6]\,,\\ \mathbf{w}=\overrightarrow{AC}=[4-(-3),\,6-2]=[7,4]\,,\end{gathered}$$

and

$$\mathbf{u}\cdot\mathbf{v}=4\cdot 3+(-2)\cdot 6=12-12=0.$$

Since this dot product is zero, these two vectors are orthogonal, so that $\overrightarrow{AB}\perp\overrightarrow{BC}$ and thus $\triangle ABC$ is a right triangle. It is unnecessary to test the remaining pairs of sides.

31. To show that $\triangle ABC$ is right, we need only show that one pair of its sides meets at a right angle. So let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{AC}$. Then we must show that one of $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{w}$ or $\mathbf{v}\cdot\mathbf{w}$ is zero in order to show that one of these pairs is orthogonal. Then

$$\begin{aligned}\mathbf{u} &= \overrightarrow{AB} = [-3-1,\ 2-1,\ (-2)-(-1)] = [-4,1,-1]\,,\\ \mathbf{v} &= \overrightarrow{BC} = [2-(-3),\ 2-2,\ -4-(-2)] = [5,0,-2],\\ \mathbf{w} &= \overrightarrow{AC} = [2-1,\ 2-1,\ -4-(-1)] = [1,1,-3],\end{aligned}$$

and

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= -4\cdot 5 + 1\cdot 0 - 1\cdot(-2) = -18\\ \mathbf{u}\cdot\mathbf{w} &= -4\cdot 1 + 1\cdot 1 - 1\cdot(-3) = 0.\end{aligned}$$

Since this dot product is zero, these two vectors are orthogonal, so that $\overrightarrow{AB} \perp \overrightarrow{AC}$ and thus $\triangle ABC$ is a right triangle. It is unnecessary to test the remaining pair of sides.

32. As in Example 1.22, the dimensions of the cube do not matter, so we work with a cube with side length 1. Since the cube is symmetric, we need only consider one diagonal and adjacent edge. Orient the cube as shown in Figure 1.34; take the diagonal to be $[1,1,1]$ and the adjacent edge to be $[1,0,0]$. Then the angle θ between these two vectors satisfies

$$\cos\theta = \frac{1\cdot 1 + 1\cdot 0 + 1\cdot 0}{\sqrt{3}\sqrt{1}} = \frac{1}{\sqrt{3}}, \qquad \text{so} \qquad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74^\circ.$$

Thus the diagonal and an adjacent edge meet at an angle of 54.74°.

33. As in Example 1.22, the dimensions of the cube do not matter, so we work with a cube with side length 1. Since the cube is symmetric, we need only consider one pair of diagonals. Orient the cube as shown in Figure 1.34; take the diagonals to be $\mathbf{u} = [1,1,1]$ and $\mathbf{v} = [1,1,0] - [0,0,1] = [1,1,-1]$. Then the dot product is

$$\mathbf{u}\cdot\mathbf{v} = 1\cdot 1 + 1\cdot 1 + 1\cdot(-1) = 1 + 1 - 1 = 1 \neq 0.$$

Since the dot product is nonzero, the diagonals are not orthogonal.

34. To show a parallelogram is a rhombus, it suffices to show that its diagonals are perpendicular (Euclid). But

$$\mathbf{d}_1\cdot\mathbf{d}_2 = \begin{bmatrix}2\\2\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\3\end{bmatrix} = 2\cdot 1 + 2\cdot(-1) + 0\cdot 3 = 0.$$

To determine its side length, note that since the diagonals are perpendicular, one half of each diagonal are the legs of a right triangle whose hypotenuse is one side of the rhombus. So we can use the Pythagorean Theorem. Since

$$\|\mathbf{d}_1\|^2 = 2^2 + 2^2 + 0^2 = 8, \qquad \|\mathbf{d}_2\|^2 = 1^2 + (-1)^2 + 3^2 = 11,$$

we have for the side length

$$s^2 = \left(\frac{\|\mathbf{d}_1\|}{2}\right)^2 + \left(\frac{\|\mathbf{d}_2\|}{2}\right)^2 = \frac{8}{4} + \frac{11}{4} = \frac{19}{4},$$

so that $s = \frac{\sqrt{19}}{2} \approx 2.18$.

35. Since $ABCD$ is a rectangle, opposite sides BA and CD are parallel and congruent. So we can use the method of Example 1.1 in Section 1.1 to find the coordinates of vertex D: we compute $\overrightarrow{BA} = [1-3, 2-6, 3-(-2)] = [-2,-4,5]$. If $\overrightarrow{BA}$ is then translated to $\overrightarrow{CD}$, where $C = (0,5,-4)$, then

$$D = (0+(-2),\ 5+(-4),\ -4+5) = (-2,1,1).$$

36. The resultant velocity of the airplane is the sum of the velocity of the airplane and the velocity of the wind:

$$\mathbf{r} = \mathbf{p} + \mathbf{w} = \begin{bmatrix} 200 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -40 \end{bmatrix} = \begin{bmatrix} 200 \\ -40 \end{bmatrix}.$$

37. Let the x direction be east, in the direction of the current, and the y direction be north, across the river. The speed of the boat is 4 mph north, and the current is 3 mph east, so the velocity of the boat is

$$\mathbf{v} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

38. Let the x direction be the direction across the river, and the y direction be downstream. Since $\mathbf{v}t = \mathbf{d}$, use the given information to find $\mathbf{v}$, then solve for t and compute $\mathbf{d}$. Since the speed of the boat is 20 km/h and the speed of the current is 5 km/h, we have $\mathbf{v} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}$. The width of the river is 2 km, and the distance downstream is unknown; call it y. Then $\mathbf{d} = \begin{bmatrix} 2 \\ y \end{bmatrix}$. Thus

$$\mathbf{v}t = \begin{bmatrix} 20 \\ 5 \end{bmatrix} t = \begin{bmatrix} 2 \\ y \end{bmatrix}.$$

Thus $20t = 2$ so that $t = 0.1$, and then $y = 5 \cdot 0.1 = 0.5$. Therefore

(a) Ann lands 0.5 km, or half a kilometer, downstream;

(b) It takes Ann 0.1 hours, or six minutes, to cross the river.

Note that the river flow does not increase the time required to cross the river, since its velocity is perpendicular to the direction of travel.

39. We want to find the angle between Bert's resultant vector, $\mathbf{r}$, and his velocity vector upstream, $\mathbf{v}$. Let the first coordinate of the vector be the direction across the river, and the second be the direction upstream. Bert's velocity vector directly across the river is unknown, say $\mathbf{u} = \begin{bmatrix} x \\ 0 \end{bmatrix}$. His velocity vector upstream compensates for the downstream flow, so $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So the resultant vector is $\mathbf{r} = \mathbf{u} + \mathbf{v} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 1 \end{bmatrix}$. Since Bert's speed is 2 mph, we have $\|\mathbf{r}\| = 2$. Thus

$$x^2 + 1 = \|\mathbf{r}\|^2 = 4, \quad \text{so that} \quad x = \sqrt{3}.$$

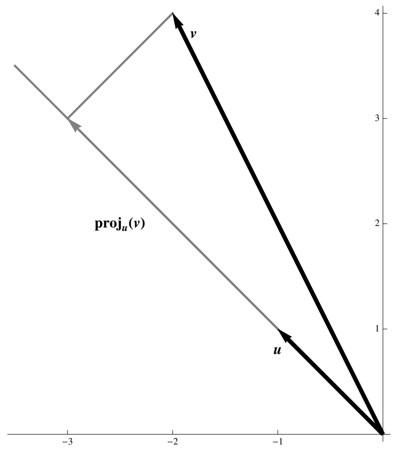
If θ is the angle between $\mathbf{r}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{r} \cdot \mathbf{v}}{\|\mathbf{r}\| \, \|\mathbf{v}\|} = \frac{\sqrt{3}}{2}, \quad \text{so that} \quad \theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ.$$

40. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{(-1) \cdot (-2) + 1 \cdot 4}{(-1) \cdot (-1) + 1 \cdot 1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

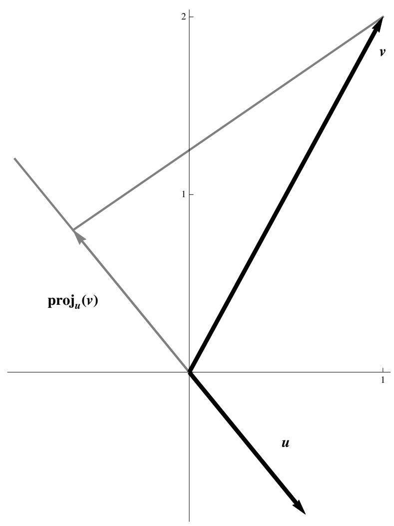
A graph of the situation is (with $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ in gray, and the perpendicular from $\mathbf{v}$ to the projection also drawn)



41. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{\frac{3}{5} \cdot 1 + \left(-\frac{4}{5} \cdot 2\right)}{\frac{3}{5} \cdot \frac{3}{5} + \left(-\frac{4}{5}\right) \cdot \left(-\frac{4}{5}\right)} \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix} = -\frac{1}{1}\mathbf{u} = -\mathbf{u}.$$

A graph of the situation is (with $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ in gray, and the perpendicular from $\mathbf{v}$ to the projection also drawn)



42. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{\frac{1}{2} \cdot 2 - \frac{1}{4} \cdot 2 - \frac{1}{2} \cdot (-2)}{\frac{1}{2} \cdot \frac{1}{2} + \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix} = \frac{8}{3} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ -\frac{4}{3} \end{bmatrix}.$$

43. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{1 \cdot 2 + (-1) \cdot (-3) + 1 \cdot (-1) + (-1) \cdot (-2)}{1 \cdot 1 + (-1) \cdot (-1) + 1 \cdot 1 + (-1) \cdot (-1)} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \frac{6}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix} = \frac{3}{2}\mathbf{u}.$$

44. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{0.5 \cdot 2.1 + 1.5 \cdot 1.2}{0.5 \cdot 0.5 + 1.5 \cdot 1.5} \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} = \frac{2.85}{2.5} \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 0.57 \\ 1.71 \end{bmatrix} = 1.14\mathbf{u}.$$

45. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{3.01\cdot 1.34 - 0.33\cdot 4.25 + 2.52\cdot(-1.66)}{3.01\cdot 3.01 - 0.33\cdot(-0.33) + 2.52\cdot 2.52}\begin{bmatrix}3.01\\-0.33\\2.52\end{bmatrix}$$

$$= -\frac{1.5523}{15.5194}\begin{bmatrix}3.01\\-0.33\\2.52\end{bmatrix} \approx \begin{bmatrix}-0.301\\0.033\\-0.252\end{bmatrix} \approx -\frac{1}{10}\mathbf{u}.$$

46. Let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix}2-1\\2-(-1)\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix}4-1\\0-(-1)\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$.

(a) To compute the projection, we need

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix} = 6, \qquad \mathbf{u}\cdot\mathbf{u} = \begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix} = 10.$$

Thus

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{3}{5}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}\frac{3}{5}\\\frac{9}{5}\end{bmatrix},$$

so that

$$\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v} = \begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}\frac{3}{5}\\\frac{9}{5}\end{bmatrix} = \begin{bmatrix}\frac{12}{5}\\-\frac{4}{5}\end{bmatrix}.$$

Then

$$\|\mathbf{u}\| = \sqrt{1^2+3^2} = \sqrt{10}, \quad \|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{4\sqrt{10}}{5},$$

so that finally

$$\mathcal{A} = \frac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \frac{1}{2}\sqrt{10}\cdot\frac{4\sqrt{10}}{5} = 4.$$

(b) We already know $\mathbf{u}\cdot\mathbf{v} = 6$ and $\|\mathbf{u}\| = \sqrt{10}$ from part (a). Also, $\|\mathbf{v}\| = \sqrt{3^2+1^2} = \sqrt{10}$. So

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{6}{\sqrt{10}\sqrt{10}} = \frac{3}{5},$$

so that

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}.$$

Thus

$$\mathcal{A} = \frac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta = \frac{1}{2}\sqrt{10}\sqrt{10}\cdot\frac{4}{5} = 4.$$

47. Let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix}4-3\\-2-(-1)\\6-4\end{bmatrix} = \begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix}5-3\\0-(-1)\\2-4\end{bmatrix} = \begin{bmatrix}2\\1\\-2\end{bmatrix}$.

(a) To compute the projection, we need

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}1\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix} = -3, \qquad \mathbf{u}\cdot\mathbf{u} = \begin{bmatrix}1\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix} = 6.$$

Thus

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = -\frac{3}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\-1\end{bmatrix},$$

so that

$$\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

Then

$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}, \quad \|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2} = \frac{\sqrt{30}}{2},$$

so that finally

$$\mathcal{A} = \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \frac{1}{2}\sqrt{6} \cdot \frac{\sqrt{30}}{2} = \frac{3\sqrt{5}}{2}.$$

(b) We already know $\mathbf{u} \cdot \mathbf{v} = -3$ and $\|\mathbf{u}\| = \sqrt{6}$ from part (a). Also, $\|\mathbf{v}\| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$. So

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-3}{3\sqrt{6}} = -\frac{\sqrt{6}}{6},$$

so that

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{\sqrt{6}}{6}\right)^2} = \frac{\sqrt{30}}{6}.$$

Thus

$$\mathcal{A} = \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta = \frac{1}{2}\sqrt{6} \cdot 3 \cdot \frac{\sqrt{30}}{6} = \frac{3\sqrt{5}}{2}.$$

48. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their dot product $\mathbf{u} \cdot \mathbf{v} = 0$. So we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for k:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} k+1 \\ k-1 \end{bmatrix} = 0 \;\Rightarrow\; 2(k+1) + 3(k-1) = 0 \;\Rightarrow\; 5k - 1 = 0 \;\Rightarrow k = \frac{1}{5}.$$

Substituting into the formula for $\mathbf{v}$ gives

$$\mathbf{v} = \begin{bmatrix} \frac{1}{5} + 1 \\ \frac{1}{5} - 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \end{bmatrix}.$$

As a check, we compute

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \end{bmatrix} = \frac{12}{5} - \frac{12}{5} = 0,$$

and the vectors are indeed orthogonal.

49. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their dot product $\mathbf{u} \cdot \mathbf{v} = 0$. So we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for k:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix} = 0 \;\Rightarrow\; k^2 - k - 6 = 0 \;\Rightarrow\; (k+2)(k-3) = 0 \;\Rightarrow k = 2, -3.$$

Substituting into the formula for $\mathbf{v}$ gives

$$k = 2: \;\mathbf{v}_1 = \begin{bmatrix} (-2)^2 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}, \qquad k = -3: \;\mathbf{v}_2 = \begin{bmatrix} 3^2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix}.$$

As a check, we compute

$$\mathbf{u} \cdot \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix} = 1 \cdot 4 - 1 \cdot (-2) + 2 \cdot (-3) = 0, \; \mathbf{u} \cdot \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix} = 1 \cdot 9 - 1 \cdot 3 + 2 \cdot (-3) = 0$$

and the vectors are indeed orthogonal.

50. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their dot product $\mathbf{u}\cdot\mathbf{v}=0$. So we set $\mathbf{u}\cdot\mathbf{v}=0$ and solve for y in terms of x:

$$\mathbf{u}\cdot\mathbf{v}=\begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=0 \ \Rightarrow\ 3x+y=0 \ \Rightarrow\ y=-3x.$$

Substituting $y=-3x$ back into the formula for $\mathbf{v}$ gives

$$\mathbf{v}=\begin{bmatrix}x\\-3x\end{bmatrix}=x\begin{bmatrix}1\\-3\end{bmatrix}.$$

Thus any vector orthogonal to $\begin{bmatrix}3\\1\end{bmatrix}$ is a multiple of $\begin{bmatrix}1\\-3\end{bmatrix}$. As a check,

$$\mathbf{u}\cdot\mathbf{v}=\begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}x\\-3x\end{bmatrix}=3x-3x=0 \text{ for any value of } x,$$

so that the vectors are indeed orthogonal.

51. As noted in the remarks just prior to Example 1.16, the zero vector $\mathbf{0}$ is orthogonal to all vectors in $\mathbb{R}^2$. So if $\begin{bmatrix}a\\b\end{bmatrix}=\mathbf{0}$, any vector $\begin{bmatrix}x\\y\end{bmatrix}$ will do. Now assume that $\begin{bmatrix}a\\b\end{bmatrix}\neq\mathbf{0}$; that is, that either a or b is nonzero. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their dot product $\mathbf{u}\cdot\mathbf{v}=0$. So we set $\mathbf{u}\cdot\mathbf{v}=0$ and solve for y in terms of x:

$$\mathbf{u}\cdot\mathbf{v}=\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=0 \ \Rightarrow\ ax+by=0.$$

First assume $b\neq 0$. Then $y=-\frac{a}{b}x$, so substituting back into the expression for $\mathbf{v}$ we get

$$\mathbf{v}=\begin{bmatrix}x\\-\frac{a}{b}x\end{bmatrix}=x\begin{bmatrix}1\\-\frac{a}{b}\end{bmatrix}=\frac{x}{b}\begin{bmatrix}b\\-a\end{bmatrix}.$$

Next, if $b=0$, then $a\neq 0$, so that $x=-\frac{b}{a}y$, and substituting back into the expression for $\mathbf{v}$ gives

$$\mathbf{v}=\begin{bmatrix}-\frac{b}{a}y\\y\end{bmatrix}=y\begin{bmatrix}-\frac{b}{a}\\1\end{bmatrix}=-\frac{y}{a}\begin{bmatrix}b\\-a\end{bmatrix}.$$

So in either case, a vector orthogonal to $\begin{bmatrix}a\\b\end{bmatrix}$, if it is not the zero vector, is a multiple of $\begin{bmatrix}b\\-a\end{bmatrix}$. As a check, note that

$$\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}rb\\-ra\end{bmatrix}=rab-rab=0 \text{ for all values of } r.$$

52. **(a)** The geometry of the vectors in Figure 1.26 suggests that if $\|\mathbf{u}+\mathbf{v}\|=\|\mathbf{u}\|+\|\mathbf{v}\|$, then $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. This means that the angle between them must be 0. So we first prove

Lemma 1. *For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\,\|\mathbf{v}\|$ if and only if the vectors point in the same direction.*

Proof. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},$$

so that $\cos\theta=1$ if and only if $\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\,\|\mathbf{v}\|$. But $\cos\theta=1$ if and only if $\theta=0$, which means that $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. □

We can now show

Theorem 2. *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$, $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *point in the same direction.*

Proof. First assume that $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. Then $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|$, and thus

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} && \text{By Example 1.9}\\
&= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 && \text{Since } \mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2 \text{ for any vector } \mathbf{w}\\
&= \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \text{By the lemma}\\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 .
\end{aligned}$$

Since $\|\mathbf{u}+\mathbf{v}\|$ and $\|\mathbf{u}\|+\|\mathbf{v}\|$ are both nonnegative, taking square roots gives $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\|+\|\mathbf{v}\|$. For the other direction, if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$, then their squares are equal, so that

$$\begin{gathered}
(\|\mathbf{u}\| + \|\mathbf{v}\|)^2 = \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \text{ and}\\
\|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}
\end{gathered}$$

are equal. But $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u}$ and similarly for $\mathbf{v}$, so that canceling those terms gives $2\mathbf{u}\cdot\mathbf{v} = 2\,\|\mathbf{u}\|\,\|\mathbf{v}\|$ and thus $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|$. Using the lemma again shows that $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. ☐

(b) The geometry of the vectors in Figure 1.26 suggests that if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$, then $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions. In addition, since $\|\mathbf{u}+\mathbf{v}\| \geq 0$, we must also have $\|\mathbf{u}\| \geq \|\mathbf{v}\|$. If they point in opposite directions, the angle between them must be π. This entire proof is exactly analogous to the proof in part (a). We first prove

Lemma 3. *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$, $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\,\|\mathbf{v}\|$ *if and only if the vectors point in opposite directions.*

Proof. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},$$

so that $\cos\theta = -1$ if and only if $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\,\|\mathbf{v}\|$. But $\cos\theta = -1$ if and only if $\theta = \pi$, which means that $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions. ☐

We can now show

Theorem 4. *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$, $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *point in opposite directions and* $\|\mathbf{u}\| \geq \|\mathbf{v}\|$.

Proof. First assume that $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions and $\|\mathbf{u}\| \geq \|\mathbf{v}\|$. Then $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\,\|\mathbf{v}\|$, and thus

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} && \text{By Example 1.9}\\
&= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 && \text{Since } \mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2 \text{ for any vector } \mathbf{w}\\
&= \|\mathbf{u}\|^2 - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \text{By the lemma}\\
&= (\|\mathbf{u}\| - \|\mathbf{v}\|)^2 .
\end{aligned}$$

Now, since $\|\mathbf{u}\| \geq \|\mathbf{v}\|$ by assumption, we see that both $\|\mathbf{u}+\mathbf{v}\|$ and $\|\mathbf{u}\| - \|\mathbf{v}\|$ are nonnegative, so that taking square roots gives $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$. For the other direction, if $\|\mathbf{u}+\mathbf{v}\| =$

$\|\mathbf{u}\| - \|\mathbf{v}\|$, then first of all, since the left-hand side is nonnegative, the right-hand side must be as well, so that $\|\mathbf{u}\| \geq \|\mathbf{v}\|$. Next, we can square both sides of the equality, so that

$$(\|\mathbf{u}\| - \|\mathbf{v}\|)^2 = \|\mathbf{u}\|^2 - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \text{ and}$$
$$\|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}$$

are equal. But $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u}$ and similarly for $\mathbf{v}$, so that canceling those terms gives $2\mathbf{u}\cdot\mathbf{v} = -2\,\|\mathbf{u}\|\,\|\mathbf{v}\|$ and thus $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\,\|\mathbf{v}\|$. Using the lemma again shows that $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions. □

53. Prove Theorem 1.2(b) by applying the definition of the dot product:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_1(v_1+w_1) + u_2(v_2+w_2) + \cdots + u_n(v_n+w_n)\\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \cdots + u_nv_n + u_nw_n\\ &= (u_1v_1 + u_2v_2 + \cdots + u_nv_n) + (u_1w_1 + u_2w_2 + \cdots + u_nw_n)\\ &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}.\end{aligned}$$

54. Prove the three parts of Theorem 1.2(d) by applying the definition of the dot product and various properties of real numbers:

***Part 1*:** For any vector $\mathbf{u}$, we must show $\mathbf{u}\cdot\mathbf{u} \geq 0$. But

$$\mathbf{u}\cdot\mathbf{u} = u_1u_1 + u_2u_2 + \cdots + u_nu_n = u_1^2 + u_2^2 + \cdots + u_n^2.$$

Since for any real number x we know that $x^2 \geq 0$, it follows that this sum is also nonnegative, so that $\mathbf{u}\cdot\mathbf{u} \geq 0$.

***Part 2*:** We must show that if $\mathbf{u} = \mathbf{0}$ then $\mathbf{u}\cdot\mathbf{u} = 0$. But $\mathbf{u} = 0$ means that $u_i = 0$ for all i, so that

$$\mathbf{u}\cdot\mathbf{u} = 0\cdot 0 + 0\cdot 0 + \cdots + 0\cdot 0 = 0.$$

***Part 3*:** We must show that if $\mathbf{u}\cdot\mathbf{u} = 0$, then $\mathbf{u} = \mathbf{0}$. From part 1, we know that

$$\mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2 + \cdots + u_n^2,$$

and that $u_i^2 \geq 0$ for all i. So if the dot product is to be zero, each u_i^2 must be zero, which means that $u_i = 0$ for all i and thus $\mathbf{u} = \mathbf{0}$.

55. We must show $\mathrm{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|\mathbf{v}-\mathbf{u}\| = \mathrm{d}(\mathbf{v},\mathbf{u})$. By definition, $\mathrm{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$. Then by Theorem 1.3(b) with $c = -1$, we have $\|-\mathbf{w}\| = \|\mathbf{w}\|$ for any vector $\mathbf{w}$; applying this to the vector $\mathbf{u}-\mathbf{v}$ gives

$$\|\mathbf{u}-\mathbf{v}\| = \|-(\mathbf{u}-\mathbf{v})\| = \|\mathbf{v}-\mathbf{u}\|,$$

which is by definition equal to $\mathrm{d}(\mathbf{v},\mathbf{u})$.

56. We must show that for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ that $\mathrm{d}(\mathbf{u},\mathbf{w}) \leq \mathrm{d}(\mathbf{u},\mathbf{v}) + \mathrm{d}(\mathbf{v},\mathbf{w})$. This is equivalent to showing that $\|\mathbf{u}-\mathbf{w}\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\|$. Now substitute $\mathbf{u}-\mathbf{v}$ for x and $\mathbf{v}-\mathbf{w}$ for y in Theorem 1.5, giving

$$\|\mathbf{u}-\mathbf{w}\| = \|(\mathbf{u}-\mathbf{v}) + (\mathbf{v}-\mathbf{w})\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\|.$$

57. We must show that $\mathrm{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = 0$ if and only if $\mathbf{u} = \mathbf{v}$. This follows immediately from Theorem 1.3(a), $\|\mathbf{w}\| = 0$ if and only if $\mathbf{w} = \mathbf{0}$, upon setting $\mathbf{w} = \mathbf{u}-\mathbf{v}$.

58. Apply the definitions:

$$\begin{aligned}\mathbf{u}\cdot c\mathbf{v} &= [u_1, u_2, \ldots, u_n]\cdot[cv_1, cv_2, \ldots, cv_n]\\ &= u_1cv_1 + u_2cv_2 + \cdots + u_ncv_n\\ &= cu_1v_1 + cu_2v_2 + \cdots + cu_nv_n\\ &= c(u_1v_1 + u_2v_2 + \cdots + u_nv_n)\\ &= c(\mathbf{u}\cdot\mathbf{v}).\end{aligned}$$

59. We want to show that $\|\mathbf{u}-\mathbf{v}\| \geq \|\mathbf{u}\| - \|\mathbf{v}\|$. This is equivalent to showing that $\|\mathbf{u}\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}\|$. This follows immediately upon setting $\mathbf{x} = \mathbf{u}-\mathbf{v}$ and $\mathbf{y} = \mathbf{v}$ in Theorem 1.5.

60. If $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$, it does *not* follow that $\mathbf{v} = \mathbf{w}$. For example, since $\mathbf{0}\cdot\mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathbb{R}^n$, the zero vector is orthogonal to every vector $\mathbf{v}$. So if $\mathbf{u} = \mathbf{0}$ in the above equality, we know nothing about $\mathbf{v}$ and $\mathbf{w}$. (as an example, $\mathbf{0}\cdot[1,2] = \mathbf{0}\cdot[-17,12]$). Note, however, that $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$ implies that $\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{w} = \mathbf{u}(\mathbf{v}-\mathbf{w}) = \mathbf{0}$, so that $\mathbf{u}$ is orthogonal to $\mathbf{v}-\mathbf{w}$.

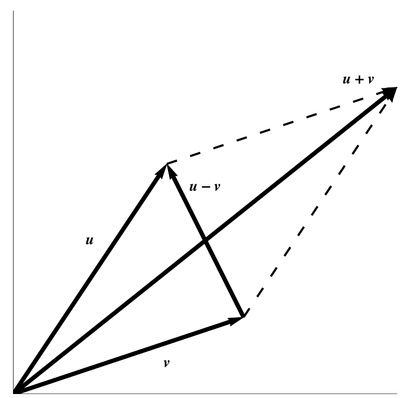
61. We must show that $(\mathbf{u}+\mathbf{v})(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for all vectors in $\mathbb{R}^n$. Recall that for any $\mathbf{w}$ in $\mathbb{R}^n$ that $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2$, and also that $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$. Then

$$(\mathbf{u}+\mathbf{v})(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 + \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2.$$

62. **(a)** Let $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. Then

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\
&= (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v}) \\
&= \left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) + 2\mathbf{u}\cdot\mathbf{v} + \left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) - 2\mathbf{u}\cdot\mathbf{v} \\
&= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.
\end{aligned}$$

(b) Part (a) tells us that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its four sides.



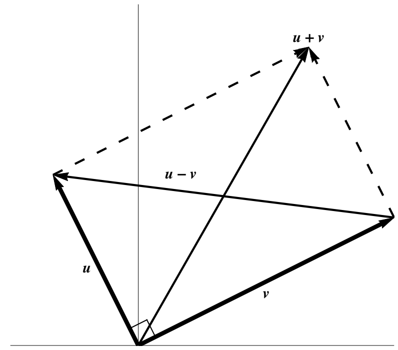
63. Let $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. Then

$$\begin{aligned}
\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 &= \frac{1}{4}\left[(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) - ((\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\right] \\
&= \frac{1}{4}\left[(\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v})\right] \\
&= \frac{1}{4}\left[\left(\|\mathbf{u}\|^2 - \|\mathbf{u}\|^2\right) + \left(\|\mathbf{v}\|^2 - \|\mathbf{v}\|^2\right) + 4\mathbf{u}\cdot\mathbf{v}\right] \\
&= \mathbf{u}\cdot\mathbf{v}.
\end{aligned}$$

64. **(a)** Let $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. Then using the previous exercise,

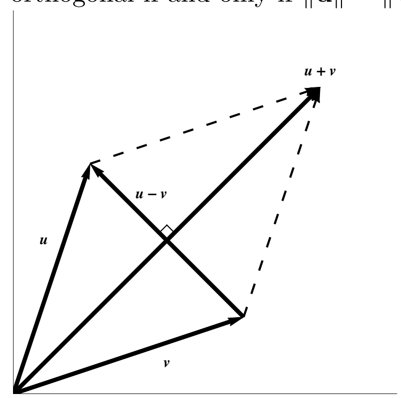
$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\| &\Leftrightarrow \|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2 \\
&\Leftrightarrow \|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = 0 \\
&\Leftrightarrow \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 = 0 \\
&\Leftrightarrow \mathbf{u}\cdot\mathbf{v} = 0 \\
&\Leftrightarrow \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}
\end{aligned}$$

(b) Part (a) tells us that a parallelogram is a rectangle if and only if the lengths of its diagonals are equal.



65. **(a)** By Exercise 55, $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2-\|\mathbf{v}\|^2$. Thus $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = 0$ if and only if $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$. It follows immediately that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

(b) Part (a) tells us that the diagonals of a parallelogram are perpendicular if and only if the lengths of its sides are equal, i.e., if and only if it is a rhombus.



66. From Example 1.9 and the fact that $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2$, we have $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$. Taking the square root of both sides yields $\|\mathbf{u}+\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2}$. Now substitute in the given values $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = \sqrt{3}$, and $\mathbf{u}\cdot\mathbf{v} = 1$, giving

$$\|\mathbf{u}+\mathbf{v}\| = \sqrt{2^2 + 2\cdot 1 + \left(\sqrt{3}\right)^2} = \sqrt{4+2+3} = \sqrt{9} = 3.$$

67. From Theorem 1.4 (the Cauchy-Schwarcz inequality), we have $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$. If $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 2$, then $|\mathbf{u}\cdot\mathbf{v}| \le 2$, so we cannot have $\mathbf{u}\cdot\mathbf{v} = 3$.

68. **(a)** If $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$. Then

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} = 0+0 = 0,$$

so that $\mathbf{u}$ is orthogonal to $\mathbf{v}+\mathbf{w}$.

(b) If $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$. Then

$$\mathbf{u}\cdot(s\mathbf{v}+t\mathbf{w}) = \mathbf{u}\cdot(s\mathbf{v}) + \mathbf{u}\cdot(t\mathbf{w}) = s(\mathbf{u}\cdot\mathbf{v}) + t(\mathbf{u}\cdot\mathbf{w}) = s\cdot 0 + t\cdot 0 = 0,$$

so that $\mathbf{u}$ is orthogonal to $s\mathbf{v}+t\mathbf{w}$.

69. We have

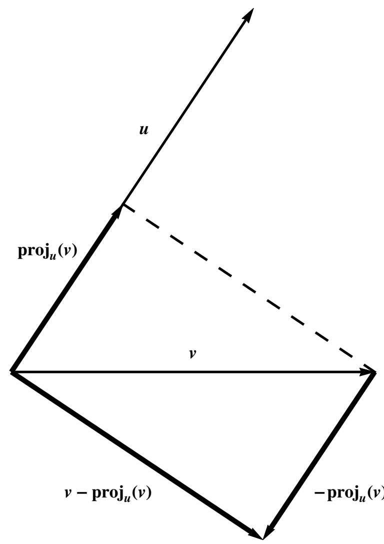
$$\begin{aligned}
\mathbf{u}\cdot(\mathbf{v}-\operatorname{proj}_{\mathbf{u}}\mathbf{v}) &= \mathbf{u}\cdot\left(\mathbf{v} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right)\\
&= \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\\
&= \mathbf{u}\cdot\mathbf{v} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)(\mathbf{u}\cdot\mathbf{u})\\
&= \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v}\\
&= 0.
\end{aligned}$$

70. **(a)** $\text{proj}_{\mathbf{u}}(\text{proj}_{\mathbf{u}}\,\mathbf{v}) = \text{proj}_{\mathbf{u}}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\,\text{proj}_{\mathbf{u}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \text{proj}_{\mathbf{u}}\,\mathbf{v}$.

(b) Using part (a),

$$\begin{aligned}\text{proj}_{\mathbf{u}}\,(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}) = \text{proj}_{\mathbf{u}}\left(\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\text{proj}_{\mathbf{u}}\,\mathbf{u} \\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \mathbf{0}.\end{aligned}$$

(c) From the diagram, we see that $\text{proj}_{\mathbf{u}}\,\mathbf{v} \| \mathbf{u}$, so that $\text{proj}_{\mathbf{u}}\,(\text{proj}_{\mathbf{u}}\,\mathbf{v}) = \text{proj}_{\mathbf{u}}\,\mathbf{v}$. Also, $(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}) \perp \mathbf{u}$, so that $\text{proj}_{\mathbf{u}}\,(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}) = \mathbf{0}$.



71. **(a)** We have

$$\begin{aligned}(u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1v_1 + u_2v_2)^2 &= u_1^2v_1^2 + u_1^2v_2^2 + u_2^2v_1^2 + u_2^2v_2^2 - u_1^2v_1^2 - 2u_1v_1u_2v_2 - u_2^2v_2^2 \\ &= u_1^2v_2^2 + u_2^2v_1^2 - 2u_1u_2v_1v_2 \\ &= (u_1v_2 - u_2v_1)^2.\end{aligned}$$

But the final expression is nonnegative since it is a square. Thus the original expression is as well, showing that $(u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1v_1 + u_2v_2)^2 \ge 0$.

(b) We have

$$\begin{aligned}&(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\ &\quad= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \\ &\qquad\quad - u_1^2v_1^2 - 2u_1v_1u_2v_2 - u_2^2v_2^2 - 2u_1v_1u_3v_3 - u_3^2v_3^2 - 2u_2v_2u_3v_3 \\ &\quad= u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 \\ &\qquad\quad - 2u_1u_2v_1v_2 - 2u_1v_1u_3v_3 - 2u_2v_2u_3v_3 \\ &\quad= (u_1v_2 - u_2v_1)^2 + (u_1v_3 - u_3v_1)^2 + (u_3v_2 - u_2v_3)^2.\end{aligned}$$

But the final expression is nonnegative since it is the sum of three squares. Thus the original expression is as well, showing that $(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \ge 0$.

72. **(a)** Since $\text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$, we have

$$\begin{aligned}\text{proj}_{\mathbf{u}}\,\mathbf{v}\cdot(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}) &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\cdot\left(\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) \\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\left(\mathbf{u}\cdot\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\cdot\mathbf{u}\right) \\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}(\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v}) \\ &= 0,\end{aligned}$$

so that $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ is orthogonal to $\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}$. Since their vector sum is $\mathbf{v}$, those three vectors form a right triangle with hypotenuse $\mathbf{v}$, so by Pythagoras' Theorem,

$$\left\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\right\|^2 \le \left\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\right\|^2 + \left\|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\right\|^2 = \|\mathbf{v}\|^2 .$$

Since norms are always nonnegative, taking square roots gives $\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\| \le \|\mathbf{v}\|$.

(b)

$$\begin{aligned} \|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\| \le \|\mathbf{v}\| &\iff \left\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right\| \le \|\mathbf{v}\| \\ &\iff \left\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\right)\mathbf{u}\right\| \le \|\mathbf{v}\| \\ &\iff \left|\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\right| \|\mathbf{u}\| \le \|\mathbf{v}\| \\ &\iff \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|} \le \|\mathbf{v}\| \\ &\iff |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\| \, \|\mathbf{v}\| , \end{aligned}$$

which is the Cauchy-Schwarcz inequality.

73. Suppose $\operatorname{proj}_{\mathbf{u}} \mathbf{v} = c\mathbf{u}$. From the figure, we see that $\cos\theta = \frac{c\|\mathbf{u}\|}{\|\mathbf{v}\|}$. But also $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. Thus these two expressions are equal, i.e.,

$$\frac{c\,\|\mathbf{u}\|}{\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \Rightarrow c\,\|\mathbf{u}\| = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|} \Rightarrow c = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{u}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}.$$

74. The basis for induction is the cases $n = 1$ and $n = 2$. The $n = 1$ case is the assertion that $\|\mathbf{v}_1\| \le \|\mathbf{v}_2\|$, which is obviously true. The $n = 2$ case is the Triangle Inequality, which is also true.

Now assume the statement holds for $n = k \ge 2$; that is, for any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$,

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k\| \le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k\| .$$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}$ be any vectors. Then

$$\begin{aligned} \|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k + \mathbf{v}_{k+1}\| &= \|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + (\mathbf{v}_k + \mathbf{v}_{k+1})\| \\ &\le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k + \mathbf{v}_{k+1}\| \end{aligned}$$

using the inductive hypothesis. But then using the Triangle Inequality (or the case $n = 2$ in this theorem), $\|\mathbf{v}_k + \mathbf{v}_{k+1}\| \le \|\mathbf{v}_k\| + \|\mathbf{v}_{k+1}\|$. Substituting into the above gives

$$\begin{aligned} \|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k + \mathbf{v}_{k+1}\| &\le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k + \mathbf{v}_{k+1}\| \\ &\le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k\| + \|\mathbf{v}_{k+1}\| , \end{aligned}$$

which is what we were trying to prove.

Exploration: Vectors and Geometry

1. As in Example 1.25, let $\mathbf{p} = \overrightarrow{OP}$. Then $\mathbf{p} - \mathbf{a} = \overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$, so that $\mathbf{p} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{1}{3}(2\mathbf{a} + \mathbf{b})$. More generally, if P is the point $\frac{1}{n}$ of the way from A to B along $\overrightarrow{AB}$, then $\mathbf{p} - \mathbf{a} = \overrightarrow{AP} = \frac{1}{n}\overrightarrow{AB} = \frac{1}{n}(\mathbf{b} - \mathbf{a})$, so that $\mathbf{p} = \mathbf{a} + \frac{1}{n}(\mathbf{b} - \mathbf{a}) = \frac{1}{n}((n-1)\mathbf{a} + \mathbf{b})$.

2. Use the notation that the vector $\overrightarrow{OX}$ is written $\mathbf{x}$. Then from exercise 1, we have $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$ and $\mathbf{q} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, so that

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\overrightarrow{AB}.$$

3. Draw $\overrightarrow{AC}$. Then from exercise 2, we have $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB} = \overrightarrow{SR}$. Also draw $\overrightarrow{BD}$. Again from exercise 2, we have $\overrightarrow{PS} = \frac{1}{2}\overrightarrow{BD} = \overrightarrow{QR}$. Thus opposite sides of the quadrilateral $PQRS$ are equal. They are also parallel: indeed, $\triangle BPQ$ and $\triangle BAC$ are similar, since they share an angle and $BP : BA = BQ : BC$. Thus $\angle BPQ = \angle BAC$; since these angles are equal, $PQ \| AC$. Similarly, $SR \| AC$ so that $PQ \| SR$. In a like manner, we see that $PS \| RQ$. Thus $PQRS$ is a parallelogram.

4. Following the hint, we find $\mathbf{m}$, the point that is two-thirds of the distance from A to P. From exercise 1, we have

$$\mathbf{p} = \frac{1}{2}(\mathbf{b}+\mathbf{c}), \text{ so that } \mathbf{m} = \frac{1}{3}(2\mathbf{p}+\mathbf{a}) = \frac{1}{3}\left(2\cdot\frac{1}{2}(\mathbf{b}+\mathbf{c})+\mathbf{a}\right) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Next we find $\mathbf{m}'$, the point that is two-thirds of the distance from B to Q. Again from exercise 1, we have

$$\mathbf{q} = \frac{1}{2}(\mathbf{a}+\mathbf{c}), \text{ so that } \mathbf{m}' = \frac{1}{3}(2\mathbf{q}+\mathbf{b}) = \frac{1}{3}\left(2\cdot\frac{1}{2}(\mathbf{a}+\mathbf{c})+\mathbf{b}\right) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Finally we find $\mathbf{m}''$, the point that is two-thirds of the distance from C to R. Again from exercise 1, we have

$$\mathbf{r} = \frac{1}{2}(\mathbf{a}+\mathbf{b}), \text{ so that } \mathbf{m}'' = \frac{1}{3}(2\mathbf{r}+\mathbf{c}) = \frac{1}{3}\left(2\cdot\frac{1}{2}(\mathbf{a}+\mathbf{b})+\mathbf{c}\right) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Since $\mathbf{m} = \mathbf{m}' = \mathbf{m}''$, all three medians intersect at the centroid, G.

5. With notation as in the figure, we know that $\overrightarrow{AH}$ is orthogonal to $\overrightarrow{BC}$; that is, $\overrightarrow{AH}\cdot\overrightarrow{BC} = 0$. Also $\overrightarrow{BH}$ is orthogonal to $\overrightarrow{AC}$; that is, $\overrightarrow{BH}\cdot\overrightarrow{AC} = 0$. We must show that $\overrightarrow{CH}\cdot\overrightarrow{AB} = 0$. But

$$\begin{aligned}\overrightarrow{AH}\cdot\overrightarrow{BC} = 0 &\Rightarrow (\mathbf{h}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c}) = 0 \Rightarrow \mathbf{h}\cdot\mathbf{b}-\mathbf{h}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} = 0\\ \overrightarrow{BH}\cdot\overrightarrow{AC} = 0 &\Rightarrow (\mathbf{h}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a}) = 0 \Rightarrow \mathbf{h}\cdot\mathbf{c}-\mathbf{h}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a} = 0.\end{aligned}$$

Adding these two equations together and canceling like terms gives

$$0 = \mathbf{h}\cdot\mathbf{b}-\mathbf{h}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} = (\mathbf{h}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{a}) = \overrightarrow{CH}\cdot\overrightarrow{AB},$$

so that these two are orthogonal. Thus all the altitudes intersect at the orthocenter H.

6. We are given that $\overrightarrow{QK}$ is orthogonal to $\overrightarrow{AC}$ and that $\overrightarrow{PK}$ is orthogonal to $\overrightarrow{CB}$, and must show that $\overrightarrow{RK}$ is orthogonal to $\overrightarrow{AB}$. By exercise 1, we have $\mathbf{q} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$, $\mathbf{p} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$, and $\mathbf{r} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$. Thus

$$\begin{aligned}\overrightarrow{QK}\cdot\overrightarrow{AC} = 0 &\Rightarrow (\mathbf{k}-\mathbf{q})\cdot(\mathbf{c}-\mathbf{a}) = 0 \Rightarrow \left(\mathbf{k}-\frac{1}{2}(\mathbf{a}+\mathbf{c})\right)\cdot(\mathbf{c}-\mathbf{a}) = 0\\ \overrightarrow{PK}\cdot\overrightarrow{CB} = 0 &\Rightarrow (\mathbf{k}-\mathbf{p})\cdot(\mathbf{b}-\mathbf{c}) = 0 \Rightarrow \left(\mathbf{k}-\frac{1}{2}(\mathbf{b}+\mathbf{c})\right)\cdot(\mathbf{b}-\mathbf{c}) = 0.\end{aligned}$$

Expanding the two dot products gives

$$\begin{aligned}\mathbf{k}\cdot\mathbf{c}-\mathbf{k}\cdot\mathbf{a}-\frac{1}{2}\mathbf{a}\cdot\mathbf{c}+\frac{1}{2}\mathbf{a}\cdot\mathbf{a}-\frac{1}{2}\mathbf{c}\cdot\mathbf{c}+\frac{1}{2}\mathbf{a}\cdot\mathbf{c} = 0\\ \mathbf{k}\cdot\mathbf{b}-\mathbf{k}\cdot\mathbf{c}-\frac{1}{2}\mathbf{b}\cdot\mathbf{b}+\frac{1}{2}\mathbf{b}\cdot\mathbf{c}-\frac{1}{2}\mathbf{c}\cdot\mathbf{b}+\frac{1}{2}\mathbf{c}\cdot\mathbf{c} = 0.\end{aligned}$$

Add these two together and cancel like terms to get

$$0 = \mathbf{k}\cdot\mathbf{b}-\mathbf{k}\cdot\mathbf{a}-\frac{1}{2}\mathbf{b}\cdot\mathbf{b}+\frac{1}{2}\mathbf{a}\cdot\mathbf{a} = \left(\mathbf{k}-\frac{1}{2}(\mathbf{b}+\mathbf{a})\right)\cdot(\mathbf{b}-\mathbf{a}) = (\mathbf{k}-\mathbf{r})\cdot(\mathbf{b}-\mathbf{a}) = \overrightarrow{RK}\cdot\overrightarrow{AB}.$$

Thus $\overrightarrow{RK}$ and $\overrightarrow{AB}$ are indeed orthogonal, so all the perpendicular bisectors intersect at the circumcenter.

7. Let O, the center of the circle, be the origin. Then $\mathbf{b} = -\mathbf{a}$ and $\|\mathbf{a}\|^2 = \|\mathbf{c}\|^2 = r^2$ where r is the radius of the circle. We want to show that $\overrightarrow{AC}$ is orthogonal to $\overrightarrow{BC}$. But

$$\begin{aligned}\overrightarrow{AC} \cdot \overrightarrow{BC} &= (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{b}) \\ &= (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} + \mathbf{a}) \\ &= \|\mathbf{c}\|^2 + \mathbf{c} \cdot \mathbf{a} - \|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{c} \\ &= (\mathbf{a} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{a}) + (r^2 - r^2) = 0.\end{aligned}$$

Thus the two are orthogonal, so that $\angle ACB$ is a right angle.

8. As in exercise 5, we first find $\mathbf{m}$, the point that is halfway from P to R. We have $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\mathbf{r} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$, so that

$$\mathbf{m} = \frac{1}{2}(\mathbf{p} + \mathbf{r}) = \frac{1}{2}\left(\frac{1}{2}(\mathbf{a} + \mathbf{b}) + \frac{1}{2}(\mathbf{c} + \mathbf{d})\right) = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}).$$

Similarly, we find $\mathbf{m}'$, the point that is halfway from Q to S. We have $\mathbf{q} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$ and $\mathbf{s} = \frac{1}{2}(\mathbf{a} + \mathbf{d})$, so that

$$\mathbf{m}' = \frac{1}{2}(\mathbf{q} + \mathbf{s}) = \frac{1}{2}\left(\frac{1}{2}(\mathbf{b} + \mathbf{c}) + \frac{1}{2}(\mathbf{a} + \mathbf{d})\right) = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}).$$

Thus $\mathbf{m} = \mathbf{m}'$, so that $\overrightarrow{PR}$ and $\overrightarrow{QS}$ intersect at their mutual midpoints; thus, they bisect each other.

1.3 Lines and Planes

1. (a) The normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, or $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \left(\mathbf{x} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = 0.$

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, we get

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 3x + 2y = 0.$$

The general form is $3x + 2y = 0$.

2. (a) The normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, or $\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \left(\mathbf{x} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 0.$

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, we get

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix} = 3(x - 1) - 4(y - 2) = 0.$$

Expanding and simplifying gives the general form $3x - 4y = -5$.

3. (a) In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and expanding the vector form from part (a) gives $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 - t \\ 3t \end{bmatrix}$, which yields the parametric form $x = 1 - t$, $y = 3t$.

4. (a) In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and expanding the vector form from part (a) gives the parametric form $x = -4 + t$, $y = 4 + t$.

5. (a) In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and expanding the vector form from part (a) gives the parametric form $x = t$, $y = -t$, $z = 4t$.

6. (a) In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and expanding the vector form from part (a) gives $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3+2t \\ 5t \\ -2 \end{bmatrix}$, which yields the parametric form $x = 3 + 2t$, $y = 5t$, $z = -2$.

7. (a) The normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, or $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \left(\mathbf{x} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = 0$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we get

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y-1 \\ z \end{bmatrix} = 3x + 2(y-1) + z = 0.$$

Expanding and simplifying gives the general form $3x + 2y + z = 2$.

8. (a) The normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, or $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \cdot \left(\mathbf{x} - \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \right) = 0$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we get

$$\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x-3 \\ y \\ z+2 \end{bmatrix} = 2(x-3) + 5y = 0.$$

Expanding and simplifying gives the general form $2x + 5y = 6$.

9. (a) In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, or

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and expanding the vector form from part (a) gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2s - 3t \\ s + 2t \\ 2s + t \end{bmatrix}$$

which yields the parametric form the parametric form $x = 2s - 3t$, $y = s + 2t$, $z = 2s + t$.

10. (a) In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, or

$$\mathbf{x} = \begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and expanding the vector form from part (a) gives the parametric form $x = 6-t$, $y = -4 + s + t$, $z = -3 + s + t$.

11. Any pair of points on ℓ determine a direction vector, so we use P and Q. We choose P to represent the point on the line. Then a direction vector for the line is $\mathbf{d} = \overrightarrow{PQ} = (3, 0) - (1, -2) = (2, 2)$. The vector equation for the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

12. Any pair of points on ℓ determine a direction vector, so we use P and Q. We choose P to represent the point on the line. Then a direction vector for the line is $\mathbf{d} = \overrightarrow{PQ} = (-2, 1, 3) - (0, 1, -1) = (-2, 0, 4)$. The vector equation for the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$.

13. We must find two direction vectors, $\mathbf{u}$ and $\mathbf{v}$. Since P, Q, and R lie in a plane, we compute We get two direction vectors

$$\begin{aligned} \mathbf{u} &= \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (4, 0, 2) - (1, 1, 1) = (3, -1, 1) \\ \mathbf{v} &= \overrightarrow{PR} = \mathbf{r} - \mathbf{p} = (0, 1, -1) - (1, 1, 1) = (-1, 0, -2). \end{aligned}$$

Since $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of each other, they will serve as direction vectors (if they were parallel to each other, we would have not a plane but a line). So the vector equation for the plane is $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, or $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$.

14. We must find two direction vectors, $\mathbf{u}$ and $\mathbf{v}$. Since P, Q, and R lie in a plane, we compute We get two direction vectors

$$\begin{aligned} \mathbf{u} &= \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (1, 0, 1) - (1, 1, 0) = (0, -1, 1) \\ \mathbf{v} &= \overrightarrow{PR} = \mathbf{r} - \mathbf{p} = (0, 1, 1) - (1, 1, 0) = (-1, 0, 1). \end{aligned}$$

Since $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of each other, they will serve as direction vectors (if they were parallel to each other, we would have not a plane but a line). So the vector equation for the plane is $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, or $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

15. The parametric and associated vector forms $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ found below are not unique.

(a) As in the remarks prior to Example 1.20, we start by letting $x = t$. Substituting $x = t$ into $y = 3x - 1$ gives $y = 3t - 1$. So we get parametric equations $x = t$, $y = 3t - 1$, and corresponding vector form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(b) In this case since the coefficient of y is 2, we start by letting $x = 2t$. Substituting $x = 2t$ into $3x + 2y = 5$ gives $3 \cdot 2t + 2y = 5$, which gives $y = -3t + \frac{5}{2}$. So we get parametric equations $x = 2t$, $y = \frac{5}{2} - 3t$, with corresponding vector equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{2} \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

Note that the equation was of the form $ax + by = c$ with $a = 3$, $b = 2$, and that a direction vector was given by $\begin{bmatrix} b \\ -a \end{bmatrix}$. This is true in general.

16. Note that $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$ is the line that passes through $\mathbf{p}$ (when $t = 0$) and $\mathbf{q}$ (when $t = 1$). We write $\mathbf{d} = \mathbf{q} - \mathbf{p}$; this is a direction vector for the line through $\mathbf{p}$ and $\mathbf{q}$.

(a) As noted above, the line $\mathbf{p} + t\mathbf{d}$ passes through P at $t = 0$ and through Q at $t = 1$. So as t varies from 0 to 1, the line describes the line segment $\overline{PQ}$.

(b) As shown in **Exploration: Vectors and Geometry**, to find the midpoint of $\overline{PQ}$, we start at P and travel half the length of $\overline{PQ}$ in the direction of the vector $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$. That is, the midpoint of $\overline{PQ}$ is the head of the vector $\mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p})$. Since $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$, we see that this line passes through the midpoint at $t = \frac{1}{2}$, and that the midpoint is in fact $\mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p}) = \frac{1}{2}(\mathbf{p} + \mathbf{q})$.

(c) From part (b), the midpoint is $\frac{1}{2}\left([2, -3] + [0, 1]\right) = \frac{1}{2}[2, -2] = [1, -1]$.

(d) From part (b), the midpoint is $\frac{1}{2}\left([1, 0, 1] + [4, 1, -2]\right) = \frac{1}{2}[5, 1, -1] = \left[\frac{5}{2}, \frac{1}{2}, -\frac{1}{2}\right]$.

(e) Again from **Exploration: Vectors and Geometry**, the vector whose head is $\frac{1}{3}$ of the way from P to Q along $\overline{PQ}$ is $\mathbf{x}_1 = \frac{1}{3}(2\mathbf{p} + \mathbf{q})$. Similarly, the vector whose head is $\frac{2}{3}$ of the way from P to Q along $\overline{PQ}$ is also the vector one third of the way from Q to P along $\overline{QP}$; applying the same formula gives for this point $\mathbf{x}_2 = \frac{1}{3}(2\mathbf{q} + \mathbf{p})$. When $\mathbf{p} = [2, -3]$ and $\mathbf{q} = [0, 1]$, we get

$$\begin{aligned} \mathbf{x}_1 &= \frac{1}{3}(2[2, -3] + [0, 1]) = \frac{1}{3}[4, -5] = \left[\frac{4}{3}, -\frac{5}{3}\right] \\ \mathbf{x}_2 &= \frac{1}{3}(2[0, 1] + [2, -3]) = \frac{1}{3}[2, -1] = \left[\frac{2}{3}, -\frac{1}{3}\right]. \end{aligned}$$

(f) Using the formulas from part (e) with $\mathbf{p} = [1, 0, -1]$ and $\mathbf{q} = [4, 1, -2]$ gives

$$\begin{aligned} \mathbf{x}_1 &= \frac{1}{3}(2[1, 0, -1] + [4, 1, -2]) = \frac{1}{3}[6, 1, -4] = \left[2, \frac{1}{3}, -\frac{4}{3}\right] \\ \mathbf{x}_2 &= \frac{1}{3}(2[4, 1, -2] + [1, 0, -1]) = \frac{1}{3}[9, 2, -5] = \left[3, \frac{2}{3}, -\frac{5}{3}\right]. \end{aligned}$$

17. A line ℓ_1 with slope m_1 has equation $y = m_1x + b_1$, or $-m_1x + y = b_1$. Similarly, a line ℓ_2 with slope m_2 has equation $y = m_2x + b_2$, or $-m_2x + y = b_2$. Thus the normal vector for ℓ_1 is $\mathbf{n}_1 = \begin{bmatrix} -m_1 \\ 1 \end{bmatrix}$, and the normal vector for ℓ_2 is $\mathbf{n}_2 = \begin{bmatrix} -m_2 \\ 1 \end{bmatrix}$. Now, ℓ_1 and ℓ_2 are perpendicular if and only if their normal vectors are perpendicular, i.e., if and only if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. But

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \begin{bmatrix} -m_1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -m_2 \\ 1 \end{bmatrix} = m_1m_2 + 1,$$

so that the normal vectors are perpendicular if and only if $m_1m_2 + 1 = 0$, i.e., if and only if $m_1m_2 = -1$.

18. Suppose the line ℓ has direction vector $\mathbf{d}$, and the plane $\mathscr{P}$ has normal vector $\mathbf{n}$. Then if $\mathbf{d} \cdot \mathbf{n} = 0$ ($\mathbf{d}$ and $\mathbf{n}$ are orthogonal), then the line ℓ is parallel to the plane $\mathscr{P}$. If on the other hand $\mathbf{d}$ and $\mathbf{n}$ are parallel, so that $\mathbf{d} = \mathbf{n}$, then ℓ is perpendicular to $\mathscr{P}$.

(a) Since the general form of $\mathscr{P}$ is $2x + 3y - z = 1$, its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Since $\mathbf{d} = 1\mathbf{n}$, we see that ℓ is perpendicular to $\mathscr{P}$.

(b) Since the general form of $\mathscr{P}$ is $4x - y + 5z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$. Since

$$\mathbf{d} \cdot \mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-1) - 1 \cdot 5 = 0,$$

ℓ is parallel to $\mathscr{P}$.

(c) Since the general form of $\mathscr{P}$ is $x - y - z = 3$, its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Since

$$\mathbf{d} \cdot \mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 2 \cdot 1 + 3 \cdot (-1) - 1 \cdot (-1) = 0,$$

ℓ is parallel to $\mathscr{P}$.

(d) Since the general form of $\mathscr{P}$ is $4x + 6y - 2z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$. Since

$$\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = \frac{1}{2}\mathbf{n},$$

ℓ is perpendicular to $\mathscr{P}$.

19. Suppose the plane $\mathscr{P}_1$ has normal vector $\mathbf{n}_1$, and the plane $\mathscr{P}$ has normal vector $\mathbf{n}$. Then if $\mathbf{n}_1 \cdot \mathbf{n} = 0$ ($\mathbf{n}_1$ and $\mathbf{n}$ are orthogonal), then $\mathscr{P}_1$ is perpendicular to $\mathscr{P}$. If on the other hand $\mathbf{n}_1$ and $\mathbf{n}$ are parallel, so that $\mathbf{n}_1 = c\mathbf{n}$, then $\mathscr{P}_1$ is parallel to $\mathscr{P}$. Note that in this exercise, $\mathscr{P}_1$ has the equation $4x - y + 5z = 2$, so that $\mathbf{n}_1 = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$.

(a) Since the general form of $\mathscr{P}$ is $2x + 3y - z = 1$, its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Since

$$\mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = 4 \cdot 2 - 1 \cdot 3 + 5 \cdot (-1) = 0,$$

the normal vectors are perpendicular, and thus $\mathscr{P}_1$ is perpendicular to $\mathscr{P}$.

(b) Since the general form of $\mathscr{P}$ is $4x - y + 5z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$. Since $\mathbf{n}_1 = \mathbf{n}$, $\mathscr{P}_1$ is parallel to $\mathscr{P}$.

(c) Since the general form of $\mathscr{P}$ is $x - y - z = 3$, its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Since

$$\mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 4 \cdot 1 - 1 \cdot (-1) + 5 \cdot (-1) = 0,$$

the normal vectors are perpendicular, and thus $\mathscr{P}_1$ is perpendicular to $\mathscr{P}$.

(d) Since the general form of $\mathscr{P}$ is $4x + 6y - 2z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$. Since

$$\mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = 4 \cdot 4 - 1 \cdot 6 + 5 \cdot (-2) = 0,$$

the normal vectors are perpendicular, and thus $\mathscr{P}_1$ is perpendicular to $\mathscr{P}$.

20. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine $\mathbf{p}$ and $\mathbf{d}$. The general equation of the given line is $2x - 3y = 1$, so its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Our line is perpendicular to that line, so it has direction vector $\mathbf{d} = \mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Furthermore, since our line passes through the point $P = (2, -1)$, we have $\mathbf{p} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Thus the vector form of the line perpendicular to $2x - 3y = 1$ through the point $P = (2, -1)$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

21. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine $\mathbf{p}$ and $\mathbf{d}$. The general equation of the given line is $2x - 3y = 1$, so its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Our line is parallel to that line, so it has direction vector $\mathbf{d} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (note that $\mathbf{d} \cdot \mathbf{n} = 0$). Since our line passes through the point $P = (2, -1)$, we have $\mathbf{p} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, so that the vector equation of the line parallel to $2x - 3y = 1$ through the point $P = (2, -1)$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

22. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine $\mathbf{p}$ and $\mathbf{d}$. A line is perpendicular to a plane if its direction vector $\mathbf{d}$ is the normal vector $\mathbf{n}$ of the plane. The general equation of the given plane is $x - 3y + 2z = 5$, so its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$. Thus the direction vector of our line is $\mathbf{d} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$. Furthermore, since our line passes through the point $P = (-1, 0, 3)$, we have $\mathbf{p} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$. So the vector form of the line perpendicular to $x - 3y + 2z = 5$ through $P = (-1, 0, 3)$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}.$$

23. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine $\mathbf{p}$ and $\mathbf{d}$. Since the given line has parametric equations

$$x = 1 - t, \quad y = 2 + 3t, \quad z = -2 - t, \quad \text{it has vector form} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

So its direction vector is $\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$, and this must be the direction vector $\mathbf{d}$ of the line we want, which is parallel to the given line. Since our line passes through the point $P = (-1, 0, 3)$, we have $\mathbf{p} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$. So the vector form of the line parallel to the given line through $P = (-1, 0, 3)$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

24. Since the normal form is $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$, we use the given information to determine $\mathbf{n}$ and $\mathbf{p}$. Note that a plane is parallel to a given plane if their normal vectors are equal. Since the general form of the given plane is $6x - y + 2z = 3$, its normal vector is $\mathbf{n} = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$, so this must be a normal vector of the desired plane as well. Furthermore, since our plane passes through the point $P = (0, -2, 5)$, we have $\mathbf{p} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$. So the normal form of the plane parallel to $6x - y + 2z = 3$ through $(0, -2, 5)$ is

$$\begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 12.$$

25. Using Figure 1.34 in Section 1.2 for reference, we will find a normal vector $\mathbf{n}$ and a point vector $\mathbf{p}$ for each of the sides, then substitute into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ to get an equation for each plane.

(a) Start with $\mathscr{P}_1$ determined by the face of the cube in the xy-plane. Clearly a normal vector for $\mathscr{P}_1$ is $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, or any vector parallel to the x-axis. Also, the plane passes through $P = (0, 0, 0)$, so we set $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then substituting gives

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad x = 0.$$

So the general equation for $\mathscr{P}_1$ is $x = 0$. Applying the same argument above to the plane $\mathscr{P}_2$ determined by the face in the xz-plane gives a general equation of $y = 0$, and similarly the plane $\mathscr{P}_3$ determined by the face in the xy-plane gives a general equation of $z = 0$.
Now consider $\mathscr{P}_4$, the plane containing the face parallel to the face in the yz-plane but passing through $(1, 1, 1)$. Since $\mathscr{P}_4$ is parallel to $\mathscr{P}_1$, its normal vector is also $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; since it passes through $(1, 1, 1)$, we set $\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then substituting gives

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{or} \quad x = 1.$$

So the general equation for $\mathscr{P}_4$ is $x = 1$. Similarly, the general equations for $\mathscr{P}_5$ and $\mathscr{P}_6$ are $y = 1$ and $z = 1$.

(b) Let $\mathbf{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a normal vector for the desired plane $\mathscr{P}$. Since $\mathscr{P}$ is perpendicular to the xy-plane, their normal vectors must be orthogonal. Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x \cdot 0 + y \cdot 0 + z \cdot 1 = z = 0.$$

Thus $z = 0$, so the normal vector is of the form $\mathbf{n} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. But the normal vector is also perpendicular to the plane in question, by definition. Since that plane contains both the origin and $(1, 1, 1)$, the normal vector is orthogonal to $(1, 1, 1) - (0, 0, 0)$:

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = x \cdot 1 + y \cdot 1 + z \cdot 0 = x + y = 0.$$

Thus $x + y = 0$, so that $y = -x$. So finally, a normal vector to $\mathscr{P}$ is given by $\mathbf{n} = \begin{bmatrix} x \\ -x \\ 0 \end{bmatrix}$ for any nonzero x. We may as well choose $x = 1$, giving $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Since the plane passes through $(0, 0, 0)$, we let $\mathbf{p} = \mathbf{0}$. Then substituting in $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ gives

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad x - y = 0.$$

Thus the general equation for the plane perpendicular to the xy-plane and containing the diagonal from the origin to $(1, 1, 1)$ is $x - y = 0$.

(c) As in Example 1.22 (Figure 1.34) in Section 1.2, use $\mathbf{u} = [0, 1, 1]$ and $\mathbf{v} = [1, 0, 1]$ as two vectors in the required plane. If $\mathbf{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a normal vector to the plane, then $\mathbf{n} \cdot \mathbf{u} = 0 = \mathbf{n} \cdot \mathbf{v}$:

$$\mathbf{n} \cdot \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = y + z = 0 \Rightarrow y = -z, \qquad \mathbf{n} \cdot \mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = x + z = 0 \Rightarrow x = -z.$$

Thus the normal vector is of the form $\mathbf{n} = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix}$ for any z. Taking $z = -1$ gives $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Now, the side diagonals pass through $(0, 0, 0)$, so set $\mathbf{p} = \mathbf{0}$. Then $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ yields

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad x + y - z = 0.$$

The general equation for the plane containing the side diagonals is $x + y - z = 0$.

26. Finding the distance between points A and B is equivalent to finding $\mathrm{d}(\mathbf{a}, \mathbf{b})$, where $\mathbf{a}$ is the vector from the origin to A, and similarly for $\mathbf{b}$. Given $\mathbf{x} = [x, y, z]$, $\mathbf{p} = [1, 0, -2]$, and $\mathbf{q} = [5, 2, 4]$, we want to solve $\mathrm{d}(\mathbf{x}, \mathbf{p}) = \mathrm{d}(\mathbf{x}, \mathbf{q})$; that is,

$$\mathrm{d}(\mathbf{x}, \mathbf{p}) = \sqrt{(x-1)^2 + (y-0)^2 + (z+2)^2} = \sqrt{(x-5)^2 + (y-2)^2 + (z-4)^2} = \mathrm{d}(\mathbf{x}, \mathbf{q}).$$

Squaring both sides gives

$$\begin{aligned}(x-1)^2+(y-0)^2+(z+2)^2 &= (x-5)^2+(y-2)^2+(z-4)^2 \quad \Rightarrow\\ x^2-2x+1+y^2+z^2+4z+4 &= x^2-10x+25+y^2-4y+4+z^2-8z+16 \quad \Rightarrow\\ 8x+4y+12z &= 40 \quad \Rightarrow\\ 2x+y+3z &= 10.\end{aligned}$$

Thus all such points (x,y,z) lie on the plane $2x+y+3z=10$.

27. To calculate $d(Q,\ell)=\frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$, we first put ℓ into general form. With $\mathbf{d}=\begin{bmatrix}1\\-1\end{bmatrix}$, we get $\mathbf{n}=\begin{bmatrix}1\\1\end{bmatrix}$ since then $\mathbf{n}\cdot\mathbf{d}=0$. Then we have

$$\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p} \;\Rightarrow\; \begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=1.$$

Thus $x+y=1$ and thus $a=b=c=1$. Since $Q=(2,2)=(x_0,y_0)$, we have

$$d(Q,\ell)=\frac{|1\cdot 2+1\cdot 2-1|}{\sqrt{1^2+1^2}}=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}.$$

28. Comparing the given equation to $\mathbf{x}=\mathbf{p}+t\mathbf{d}$, we get $P=(1,1,1)$ and $\mathbf{d}=\begin{bmatrix}-2\\0\\3\end{bmatrix}$. As suggested by Figure 1.63, we need to calculate the length of $\overrightarrow{RQ}$, where R is the point on the line at the foot of the perpendicular from Q. So if $\mathbf{v}=\overrightarrow{PQ}$, then

$$\overrightarrow{PR}=\operatorname{proj}_{\mathbf{d}}\mathbf{v}, \qquad \overrightarrow{RQ}=\mathbf{v}-\operatorname{proj}_{\mathbf{d}}\mathbf{v}.$$

Now, $\mathbf{v}=\overrightarrow{PQ}=\mathbf{q}-\mathbf{p}=\begin{bmatrix}0\\1\\0\end{bmatrix}-\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}-1\\0\\-1\end{bmatrix}$, so that

$$\operatorname{proj}_{\mathbf{d}}\mathbf{v}=\left(\frac{\mathbf{d}\cdot\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d}=\left(\frac{-2\cdot(-1)+3\cdot(-1)}{-2\cdot(-2)+3\cdot 3}\right)\begin{bmatrix}-2\\0\\3\end{bmatrix}=\begin{bmatrix}\frac{2}{13}\\0\\-\frac{3}{13}\end{bmatrix}.$$

Thus

$$\mathbf{v}-\operatorname{proj}_{\mathbf{d}}\mathbf{v}=\begin{bmatrix}-1\\0\\-1\end{bmatrix}-\begin{bmatrix}\frac{2}{13}\\0\\-\frac{3}{13}\end{bmatrix}=\begin{bmatrix}-\frac{15}{13}\\0\\-\frac{10}{13}\end{bmatrix}.$$

Then the distance $d(Q,\ell)$ from Q to ℓ is

$$\|\mathbf{v}-\operatorname{proj}_{\mathbf{d}}\mathbf{v}\|=\frac{5}{13}\left\|\begin{bmatrix}3\\0\\2\end{bmatrix}\right\|=\frac{5}{13}\sqrt{3^2+2^2}=\frac{5\sqrt{13}}{13}.$$

29. To calculate $d(Q,\mathscr{P})=\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$, we first note that the plane has equation $x+y-z=0$, so that $a=b=1$, $c=-1$, and $d=0$. Also, $Q=(2,2,2)$, so that $x_0=y_0=z_0=2$. Hence

$$d(Q,\mathscr{P})=\frac{|1\cdot 2+1\cdot 2-1\cdot 2-0|}{\sqrt{1^2+1^2+(-1)^2}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}.$$

30. To calculate $\mathrm{d}(Q,\mathscr{P}) = \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$, we first note that the plane has equation $x - 2y + 2z = 1$, so that $a = 1$, $b = -2$, $c = 2$, and $d = 1$. Also, $Q = (0,0,0)$, so that $x_0 = y_0 = z_0 = 0$. Hence

$$\mathrm{d}(Q,\mathscr{P}) = \frac{|1\cdot 0 - 2\cdot 0 + 2\cdot 0 - 1|}{\sqrt{1^2+(-2)^2+2^2}} = \frac{1}{3}.$$

31. Figure 1.66 suggests that we let $\mathbf{v} = \overrightarrow{PQ}$; then $\mathbf{w} = \overrightarrow{PR} = \operatorname{proj}_{\mathbf{d}} \mathbf{v}$. Comparing the given line ℓ to $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we get $P = (-1,2)$ and $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Next,

$$\mathbf{w} = \operatorname{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{d}\cdot\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = \left(\frac{1\cdot 3 + (-1)\cdot 0}{1\cdot 1 + (-1)\cdot(-1)}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So

$$\mathbf{r} = \mathbf{p} + \overrightarrow{PR} = \mathbf{p} + \operatorname{proj}_{\mathbf{d}} \mathbf{v} = \mathbf{p} + \mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

So the point R on ℓ that is closest to Q is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

32. Figure 1.66 suggests that we let $\mathbf{v} = \overrightarrow{PQ}$; then $\overrightarrow{PR} = \operatorname{proj}_{\mathbf{d}} \mathbf{v}$. Comparing the given line ℓ to $\mathbf{x} = \mathbf{p}+t\mathbf{d}$, we get $P = (1,1,1)$ and $\mathbf{d} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$. Then $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$. Next,

$$\operatorname{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{d}\cdot\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = \left(\frac{-2\cdot(-1) + 3\cdot(-1)}{(-2)^2+3^2}\right)\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{13} \\ 0 \\ -\frac{3}{13} \end{bmatrix}.$$

So

$$\mathbf{r} = \mathbf{p} + \overrightarrow{PR} = \mathbf{p} + \operatorname{proj}_{\mathbf{d}} \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{2}{13} \\ 0 \\ -\frac{3}{13} \end{bmatrix} = \begin{bmatrix} \frac{15}{13} \\ 1 \\ \frac{10}{13} \end{bmatrix}.$$

So the point R on ℓ that is closest to Q is $\left(\frac{15}{13}, 1, \frac{10}{13}\right)$.

33. Figure 1.67 suggests we let $\mathbf{v} = \overrightarrow{PQ}$, where P is some point on the plane; then $\overrightarrow{QR} = \operatorname{proj}_{\mathbf{n}} \mathbf{v}$. The equation of the plane is $x + y - z = 0$, so $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Setting $y = 0$ shows that $P = (1,0,1)$ is a point on the plane. Then

$$\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

so that

$$\operatorname{proj}_{\mathbf{n}} \mathbf{v} = \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\frac{1\cdot 1 + 1\cdot 1 - 1\cdot 1}{1^2+1^2+(-1)^2}\right)\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}.$$

Finally,

$$\mathbf{r} = \mathbf{p} + \overrightarrow{PQ} + \overrightarrow{PR} = \mathbf{p} + \mathbf{v} - \operatorname{proj}_{\mathbf{n}} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{4}{3} \\ \frac{8}{3} \end{bmatrix}.$$

Therefore, the point R in $\mathscr{P}$ that is closest to Q is $\left(\frac{4}{3}, \frac{4}{3}, \frac{8}{3}\right)$.

34. Figure 1.67 suggests we let $\mathbf{v} = \overrightarrow{PQ}$, where P is some point on the plane; then $\overrightarrow{QR} = \operatorname{proj}_{\mathbf{n}} \mathbf{v}$. The equation of the plane is $x - 2y + 2z = 1$, so $\mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$. Setting $y = z = 0$ shows that $P = (1, 0, 0)$ is a point on the plane. Then

$$\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

so that

$$\operatorname{proj}_{\mathbf{n}} \mathbf{v} = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n} = \left(\frac{1 \cdot (-1)}{1^2 + (-2)^2 + 2^2}\right) \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{bmatrix}.$$

Finally,

$$\mathbf{r} = \mathbf{p} + \overrightarrow{PQ} + \overrightarrow{PR} = \mathbf{p} + \mathbf{v} - \operatorname{proj}_{\mathbf{n}} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{bmatrix}.$$

Therefore, the point R in $\mathscr{P}$ that is closest to Q is $\left(-\frac{1}{9}, \frac{2}{9}, -\frac{2}{9}\right)$.

35. Since the given lines ℓ_1 and ℓ_2 are parallel, choose arbitrary points Q on ℓ_1 and P on ℓ_2, say $Q = (1, 1)$ and $P = (5, 4)$. The direction vector of ℓ_2 is $\mathbf{d} = [-2, 3]$. Then

$$\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix},$$

so that

$$\operatorname{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}}\right) \mathbf{d} = \left(\frac{-2 \cdot (-4) + 3 \cdot (-3)}{(-2)^2 + 3^2}\right) \begin{bmatrix} -2 \\ 3 \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Then the distance between the lines is given by

$$\|\mathbf{v} - \operatorname{proj}_{\mathbf{d}} \mathbf{v}\| = \left\| \begin{bmatrix} -4 \\ -3 \end{bmatrix} + \frac{1}{13} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\| = \left\| \frac{1}{13} \begin{bmatrix} -54 \\ -36 \end{bmatrix} \right\| = \frac{18}{13} \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\| = \frac{18}{13}\sqrt{13}.$$

36. Since the given lines ℓ_1 and ℓ_2 are parallel, choose arbitrary points Q on ℓ_1 and P on ℓ_2, say $Q = (1, 0, -1)$ and $P = (0, 1, 1)$. The direction vector of ℓ_2 is $\mathbf{d} = [1, 1, 1]$. Then

$$\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix},$$

so that

$$\operatorname{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}}\right) \mathbf{d} = \left(\frac{1 \cdot 1 + 1 \cdot (-1) + 1 \cdot (-2)}{1^2 + 1^2 + 1^2}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -\frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then the distance between the lines is given by

$$\|\mathbf{v} - \operatorname{proj}_{\mathbf{d}} \mathbf{v}\| = \left\| \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{5}{3} \\ -\frac{1}{3} \\ -\frac{4}{3} \end{bmatrix} \right\| = \frac{1}{3}\sqrt{5^2 + (-1)^2 + (-4)^2} = \frac{\sqrt{42}}{3}.$$

37. Since $\mathscr{P}_1$ and $\mathscr{P}_2$ are parallel, we choose an arbitrary point on $\mathscr{P}_1$, say $Q = (0, 0, 0)$, and compute $\mathrm{d}(Q, \mathscr{P}_2)$. Since the equation of $\mathscr{P}_2$ is $2x + y - 2z = 5$, we have $a = 2$, $b = 1$, $c = -2$, and $d = 5$; since $Q = (0, 0, 0)$, we have $x_0 = y_0 = z_0 = 0$. Thus the distance is

$$\mathrm{d}(\mathscr{P}_1, \mathscr{P}_2) = \mathrm{d}(Q, \mathscr{P}_2) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|2 \cdot 0 + 1 \cdot 0 - 2 \cdot 0 - 5|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{5}{3}.$$

38. Since $\mathscr{P}_1$ and $\mathscr{P}_2$ are parallel, we choose an arbitrary point on $\mathscr{P}_1$, say $Q = (1, 0, 0)$, and compute $\mathrm{d}(Q, \mathscr{P}_2)$. Since the equation of $\mathscr{P}_2$ is $x + y + z = 3$, we have $a = b = c = 1$ and $d = 3$; since $Q = (1, 0, 0)$, we have $x_0 = 1$, $y_0 = 0$, and $z_0 = 0$. Thus the distance is

$$\mathrm{d}(\mathscr{P}_1, \mathscr{P}_2) = \mathrm{d}(Q, \mathscr{P}_2) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 - 3|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

39. We wish to show that $\mathrm{d}(B, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2+b^2}}$, where $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\mathbf{n} \cdot \mathbf{a} = c$, and $B = (x_0, y_0)$. If $\mathbf{v} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, then

$$\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - c = ax_0 + by_0 - c.$$

Then from Figure 1.65, we see that

$$\mathrm{d}(B, \ell) = \|\mathrm{proj}_{\mathbf{n}}\, \mathbf{v}\| = \left\| \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} \right\| = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}.$$

40. We wish to show that $\mathrm{d}(B, \ell) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$, where $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\mathbf{n} \cdot \mathbf{a} = d$, and $B = (x_0, y_0, z_0)$. If $\mathbf{v} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, then

$$\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - d = ax_0 + by_0 + cz_0 - d.$$

Then from Figure 1.65, we see that

$$\mathrm{d}(B, \ell) = \|\mathrm{proj}_{\mathbf{n}}\, \mathbf{v}\| = \left\| \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} \right\| = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

41. Choose $B = (x_0, y_0)$ on ℓ_1; since ℓ_1 and ℓ_2 are parallel, the distance between them is $\mathrm{d}(B, \ell_2)$. Then since B lies on ℓ_1, we have $\mathbf{n} \cdot \mathbf{b} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = ax_0 + by_0 = c_1$. Choose A on ℓ_2, so that $\mathbf{n} \cdot \mathbf{a} = c_2$. Set $\mathbf{v} = \mathbf{b} - \mathbf{a}$. Then using the formula in Exercise 39, the distance is

$$\mathrm{d}(\ell_1, \ell_2) = \mathrm{d}(B, \ell_2) = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a}|}{\|\mathbf{n}\|} = \frac{|c_1 - c_2|}{\|\mathbf{n}\|}.$$

42. Choose $B = (x_0, y_0, z_0)$ on $\mathscr{P}_1$; since $\mathscr{P}_1$ and $\mathscr{P}_2$ are parallel, the distance between them is $\mathrm{d}(B, \mathscr{P}_2)$. Then since B lies on $\mathscr{P}_1$, we have $\mathbf{n} \cdot \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = ax_0 + by_0 + cz_0 = d_1$. Choose A on $\mathscr{P}_2$, so that $\mathbf{n} \cdot \mathbf{a} = d_2$. Set $\mathbf{v} = \mathbf{b} - \mathbf{a}$. Then using the formula in Exercise 40, the distance is

$$\mathrm{d}(\mathscr{P}_1, \mathscr{P}_2) = \mathrm{d}(B, \mathscr{P}_2) = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a}|}{\|\mathbf{n}\|} = \frac{|d_1 - d_2|}{\|\mathbf{n}\|}.$$

43. Since $\mathscr{P}_1$ has normal vector $\mathbf{n}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathscr{P}_2$ has normal vector $\mathbf{n}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, the angle θ between the normal vectors satisfies

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{1 \cdot 2 + 1 \cdot 1 + 1 \cdot (-2)}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{1}{3\sqrt{3}}.$$

Thus

$$\theta = \cos^{-1}\left(\frac{1}{3\sqrt{3}}\right) \approx 78.9^\circ.$$

44. Since $\mathscr{P}_1$ has normal vector $\mathbf{n}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $\mathscr{P}_2$ has normal vector $\mathbf{n}_2 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, the angle θ between the normal vectors satisfies

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{3\cdot 1 - 1\cdot 4 + 2\cdot(-1)}{\sqrt{3^2+(-1)^2+2^2}\sqrt{1^2+4^2+(-1)^2}} = -\frac{3}{\sqrt{14}\sqrt{18}} = -\frac{1}{\sqrt{28}}.$$

This is an obtuse angle, so the acute angle is

$$\pi - \theta = \pi - \cos^{-1}\left(-\frac{1}{\sqrt{28}}\right) \approx 79.1^\circ.$$

45. First, to see that $\mathscr{P}$ and ℓ intersect, substitute the parametric equations for ℓ into the equation for $\mathscr{P}$, giving

$$x + y + 2z = (2+t) + (1-2t) + 2(3+t) = 9 + t = 0,$$

so that $t = -9$ represents the point of intersection, which is thus $(2+(-9), 1-2(-9), 3+(-9)) = (-7, 19, -6)$. Now, the normal to $\mathscr{P}$ is $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and a direction vector for ℓ is $\mathbf{d} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. So if θ is the angle between $\mathbf{n}$ and $\mathbf{d}$, then θ satisfies

$$\cos\theta = \frac{\mathbf{n}\cdot\mathbf{d}}{\|\mathbf{n}\|\,\|\mathbf{d}\|} = \frac{1\cdot 1 + 1\cdot(-2) + 2\cdot 1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+(-2)^2+1^2}} = \frac{1}{6},$$

so that

$$\theta = \cos^{-1}\left(\frac{1}{6}\right) \approx 80.4^\circ.$$

Thus the angle between the line and the plane is $90^\circ - 80.4^\circ \approx 9.6^\circ$.

46. First, to see that $\mathscr{P}$ and ℓ intersect, substitute the parametric equations for ℓ into the equation for $\mathscr{P}$, giving

$$4x - y - z = 4\cdot t - (1+2t) - (2+3t) = -t - 3 = 6,$$

so that $t = -9$ represents the point of intersection, which is thus $(-9, 1+2\cdot(-9), 2+3\cdot(-9)) = (-9, -17, -25)$. Now, the normal to $\mathscr{P}$ is $\mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$, and a direction vector for ℓ is $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. So if θ is the angle between $\mathbf{n}$ and $\mathbf{d}$, then θ satisfies

$$\cos\theta = \frac{\mathbf{n}\cdot\mathbf{d}}{\|\mathbf{n}\|\,\|\mathbf{d}\|} = \frac{4\cdot 1 - 1\cdot 2 - 1\cdot 3}{\sqrt{4^2+1^2+1^2}\sqrt{1^2+2^2+3^2}} = -\frac{1}{\sqrt{18}\sqrt{14}}.$$

This corresponds to an obtuse angle, so the acute angle between the two is

$$\theta = \pi - \cos^{-1}\left(-\frac{1}{\sqrt{18}\sqrt{14}}\right) \approx 86.4^\circ.$$

Thus the angle between the line and the plane is $90^\circ - 86.4^\circ \approx 3.6^\circ$.

47. We have $\mathbf{p} = \mathbf{v} - c\,\mathbf{n}$, so that $c\,\mathbf{n} = \mathbf{v} - \mathbf{p}$. Take the dot product of both sides with $\mathbf{n}$, giving

$$\begin{aligned}
(c\,\mathbf{n})\cdot\mathbf{n} &= (\mathbf{v}-\mathbf{p})\cdot\mathbf{n} \quad \Rightarrow \\
c(\mathbf{n}\cdot\mathbf{n}) &= \mathbf{v}\cdot\mathbf{n} - \mathbf{p}\cdot\mathbf{n} \quad \Rightarrow \\
c(\mathbf{n}\cdot\mathbf{n}) &= \mathbf{v}\cdot\mathbf{n} \ \ \text{(since } \mathbf{p} \text{ and } \mathbf{n} \text{ are orthogonal)} \quad \Rightarrow \\
c &= \frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}.
\end{aligned}$$

Note that another interpretation of the figure is that $c\mathbf{n} = \operatorname{proj}_{\mathbf{n}} \mathbf{v} = \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}$, which also implies that $c = \frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}$.

Now substitute this value of c into the original equation, giving

$$\mathbf{p} = \mathbf{v} - c\mathbf{n} = \mathbf{v} - \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}.$$

48. (a) A normal vector to the plane $x+y+z=0$ is $\mathbf{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Then

$$\mathbf{n}\cdot\mathbf{v} = \begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-2\end{bmatrix} = 1\cdot 1 + 1\cdot 0 + 1\cdot(-2) = -1$$
$$\mathbf{n}\cdot\mathbf{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = 1\cdot 1 + 1\cdot 1 + 1\cdot 1 = 3,$$

so that $c = -\frac{1}{3}$. Then

$$\mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \begin{bmatrix}1\\0\\-2\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{4}{3}\\\frac{1}{3}\\-\frac{5}{3}\end{bmatrix}.$$

(b) A normal vector to the plane $3x-y+z=0$ is $\mathbf{n} = \begin{bmatrix}3\\-1\\1\end{bmatrix}$. Then

$$\mathbf{n}\cdot\mathbf{v} = \begin{bmatrix}3\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-2\end{bmatrix} = 3\cdot 1 - 1\cdot 0 + 1\cdot(-2) = 1$$
$$\mathbf{n}\cdot\mathbf{n} = \begin{bmatrix}3\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}3\\-1\\1\end{bmatrix} = 3\cdot 3 - 1\cdot(-1) + 1\cdot 1 = 11,$$

so that $c = \frac{1}{11}$. Then

$$\mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \begin{bmatrix}1\\0\\-2\end{bmatrix} - \frac{1}{11}\begin{bmatrix}3\\-1\\1\end{bmatrix} = \begin{bmatrix}\frac{8}{11}\\\frac{1}{11}\\-\frac{23}{11}\end{bmatrix}.$$

(c) A normal vector to the plane $x-2z=0$ is $\mathbf{n} = \begin{bmatrix}1\\0\\-2\end{bmatrix}$. Then

$$\mathbf{n}\cdot\mathbf{v} = \begin{bmatrix}1\\0\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-2\end{bmatrix} = 1\cdot 1 + 0\cdot 0 - 2\cdot(-2) = 5$$
$$\mathbf{n}\cdot\mathbf{n} = \begin{bmatrix}1\\0\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-2\end{bmatrix} = 1\cdot 1 + 0\cdot 0 - 2\cdot(-2) = 5,$$

so that $c = 1$. Then

$$\mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \mathbf{0}.$$

Note that the projection is $\mathbf{0}$ because the vector is normal to the plane, so its projection onto the plane is a single point.

(d) A normal vector to the plane $2x - 3y + z = 0$ is $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. Then

$$\begin{aligned} \mathbf{n}\cdot\mathbf{v} &= \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 2\cdot 1 - 3\cdot 0 + 1\cdot(-2) = 0 \\ \mathbf{n}\cdot\mathbf{n} &= \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 2\cdot 2 - 3\cdot(-3) + 1\cdot 1 = 14, \end{aligned}$$

so that $c = 0$. Thus $\mathbf{p} = \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$. Note that the projection is the vector itself because the vector is parallel to the plane, so it is orthogonal to the normal vector.

Exploration: The Cross Product

1. (a) $\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 - 0\cdot(-1) \\ 1\cdot 3 - 0\cdot 2 \\ 0\cdot(-1) - 1\cdot 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}.$

(b) $\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} -1\cdot 1 - 2\cdot 1 \\ 2\cdot 0 - 3\cdot 1 \\ 3\cdot 1 - (-1)\cdot 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}.$

(c) $\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} 2\cdot(-6) - 3\cdot(-4) \\ 3\cdot 2 - (-1)\cdot(-6) \\ -1\cdot(-4) - 2\cdot 2 \end{bmatrix} = \mathbf{0}.$

(d) $\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 - 1\cdot 2 \\ 1\cdot 1 - 1\cdot 3 \\ 1\cdot 2 - 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$

2. We have

$$\begin{aligned} \mathbf{e}_1\times\mathbf{e}_2 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0\cdot 0 - 0\cdot 1 \\ 0\cdot 0 - 1\cdot 0 \\ 1\cdot 1 - 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{e}_3 \\ \mathbf{e}_2\times\mathbf{e}_3 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 - 0\cdot 0 \\ 0\cdot 0 - 0\cdot 1 \\ 0\cdot 0 - 1\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{e}_1 \\ \mathbf{e}_3\times\mathbf{e}_1 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0\cdot 0 - 1\cdot 0 \\ 1\cdot 1 - 0\cdot 0 \\ 0\cdot 0 - 0\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{e}_2. \end{aligned}$$

3. Two vectors are orthogonal if their dot product equals zero. But

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} &= \begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\\ &= (u_2v_3-u_3v_2)u_1+(u_3v_1-u_1v_3)u_2+(u_1v_2-u_2v_1)u_3\\ &= (u_2v_3u_1-u_1v_3u_2)+(u_3v_1u_2-u_2v_1u_3)+(u_1v_2u_3-u_3v_2u_1)=0\\ (\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} &= \begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\\ &= (u_2v_3-u_3v_2)v_1+(u_3v_1-u_1v_3)v_2+(u_1v_2-u_2v_1)v_3\\ &= (u_2v_3v_1-u_2v_1v_3)+(u_3v_1v_2-u_3v_2v_1)+(u_1v_2v_3-u_1v_3v_2)=0.\end{aligned}$$

4. **(a)** By Exercise 1, a vector normal to the plane is

$$\mathbf{n}=\mathbf{u}\times\mathbf{v}=\begin{bmatrix}0\\1\\1\end{bmatrix}\times\begin{bmatrix}3\\-1\\2\end{bmatrix}=\begin{bmatrix}1\cdot 2-1\cdot(-1)\\1\cdot 3-0\cdot 2\\0\cdot(-1)-1\cdot 3\end{bmatrix}=\begin{bmatrix}3\\3\\-3\end{bmatrix}.$$

So the normal form for the equation of this plane is $\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p}$, or

$$\begin{bmatrix}3\\3\\-3\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\3\\-3\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-2\end{bmatrix}=9.$$

This simplifies to $3x+3y-3z=9$, or $x+y-z=3$.

(b) Two vectors in the plane are $\mathbf{u}=\overrightarrow{PQ}=\begin{bmatrix}2\\1\\1\end{bmatrix}$ and $\mathbf{v}=\overrightarrow{PR}=\begin{bmatrix}1\\3\\-2\end{bmatrix}$. So by Exercise 1, a vector normal to the plane is

$$\mathbf{n}=\mathbf{u}\times\mathbf{v}=\begin{bmatrix}2\\1\\1\end{bmatrix}\times\begin{bmatrix}1\\3\\-2\end{bmatrix}=\begin{bmatrix}1\cdot(-2)-1\cdot 3\\1\cdot 1-2\cdot(-2)\\2\cdot 3-1\cdot 1\end{bmatrix}=\begin{bmatrix}-5\\5\\5\end{bmatrix}.$$

So the normal form for the equation of this plane is $\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p}$, or

$$\begin{bmatrix}-5\\5\\5\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-5\\5\\5\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\1\end{bmatrix}=0.$$

This simplifies to $-5x+5y+5z=0$, or $x-y-z=0$.

5. **(a)** $\mathbf{v}\times\mathbf{u}=\begin{bmatrix}v_2u_3-v_3u_2\\v_3u_1-u_3v_1\\v_1u_2-v_2u_1\end{bmatrix}=-\begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}=-(\mathbf{u}\times\mathbf{v}).$

(b) $\mathbf{u}\times\mathbf{0}=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\times\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}u_2\cdot 0-u_3\cdot 0\\u_3\cdot 0-u_1\cdot 0\\u_1\cdot 0-u_2\cdot 0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\mathbf{0}.$

(c) $\mathbf{u}\times\mathbf{u}=\begin{bmatrix}u_2u_3-u_3u_2\\u_3u_1-u_1u_3\\u_1u_2-u_2u_1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\mathbf{0}.$

(d) $\mathbf{u}\times k\mathbf{v}=\begin{bmatrix}u_2kv_3-u_3kv_2\\u_3kv_1-u_1kv_3\\u_1kv_2-u_2kv_1\end{bmatrix}=k\begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}=k(\mathbf{u}\times\mathbf{v}).$

(e) $\mathbf{u} \times k\mathbf{u} = k(\mathbf{u} \times \mathbf{u}) = k(\mathbf{0}) = \mathbf{0}$ by parts (d) and (c).

(f) Compute the cross-product:

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \begin{bmatrix} u_2(v_3 + w_3) - u_3(v_2 + w_2) \\ u_3(v_1 + w_1) - u_1(v_3 + w_3) \\ u_1(v_2 + w_2) - u_2(v_1 + w_1) \end{bmatrix} \\
&= \begin{bmatrix} (u_2v_3 - u_3v_2) + (u_2w_3 - u_3w_2) \\ (u_3v_1 - u_1v_3) + (u_3w_1 - u_1w_3) \\ (u_1v_2 - u_2v_1) + (u_1w_2 - u_2w_1) \end{bmatrix} \\
&= \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} + \begin{bmatrix} u_2w_3 - u_3w_2 \\ u_3w_1 - u_1w_3 \\ u_1w_2 - u_2w_1 \end{bmatrix} \\
&= \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}.
\end{aligned}$$

6. In each case, simply compute:

(a)

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \\
&= u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1 \\
&= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3 \\
&= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \\
&= \begin{bmatrix} u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) \\ u_3(v_2w_3 - v_3w_2) - u_1(v_1w_2 - v_2w_1) \\ u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2) \end{bmatrix} \\
&= \begin{bmatrix} (u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1 \\ (u_1w_1 + u_2w_2 + u_3w_3)v_2 - (u_1v_1 + u_2v_2 + u_3v_3)w_2 \\ (u_1w_1 + u_2w_2 + u_3w_3)v_3 - (u_1v_1 + u_2v_2 + u_3v_3)w_3 \end{bmatrix} \\
&= (u_1w_1 + u_2w_2 + u_3w_3) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - (u_1v_1 + u_2v_2 + u_3v_3) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\
&= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.
\end{aligned}$$

(c)

$$\begin{aligned}
\|\mathbf{u} \times \mathbf{v}\|^2 &= \left\| \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \right\|^2 \\
&= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= (u_1^2 + u_2^2 + u_3^2)^2(v_1^2 + v_2^2 + v_3^2)^2 - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2.
\end{aligned}$$

7. For problem 2, use the computation in the solution above to show that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$. Then

$$\begin{aligned} \mathbf{e}_2 \times \mathbf{e}_3 &= \mathbf{e}_2 \times (\mathbf{e}_1 \times \mathbf{e}_2) \\ &= (\mathbf{e}_2 \cdot \mathbf{e}_2)\mathbf{e}_1 - (\mathbf{e}_2 \cdot \mathbf{e}_1)\mathbf{e}_2 && 6(\text{b}) \\ &= 1 \cdot \mathbf{e}_1 - 0 \cdot \mathbf{e}_2 && \mathbf{e}_i \cdot \mathbf{e}_j = 0 \text{ if } i \neq j,\ \mathbf{e}_i \cdot \mathbf{e}_i = 1 \\ &= \mathbf{e}_1. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{e}_3 \times \mathbf{e}_1 &= \mathbf{e}_3 \times (\mathbf{e}_2 \times \mathbf{e}_3) \\ &= (\mathbf{e}_3 \cdot \mathbf{e}_3)\mathbf{e}_2 - (\mathbf{e}_3 \cdot \mathbf{e}_2)\mathbf{e}_3 && 6(\text{b}) \\ &= 1 \cdot \mathbf{e}_2 - 0 \cdot \mathbf{e}_3 && \mathbf{e}_i \cdot \mathbf{e}_j = 0 \text{ if } i \neq j,\ \mathbf{e}_i \cdot \mathbf{e}_i = 1 \\ &= \mathbf{e}_2. \end{aligned}$$

For problem 3, we have $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{u}) \cdot \mathbf{v}$ by 6(a), and then since $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ by Exercise 5(c), this reduces to $\mathbf{0} \cdot \mathbf{v} = 0$. Thus $\mathbf{u}$ is orthogonal to $\mathbf{u} \times \mathbf{v}$. Similarly, $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (-\mathbf{v} \times \mathbf{u}) = -\mathbf{v} \cdot (\mathbf{v} \times \mathbf{u})$ by Exercise 5(a), and then as before, $-\mathbf{v} \cdot (\mathbf{v} \times \mathbf{u}) = -(\mathbf{v} \times \mathbf{v}) \cdot \mathbf{u} = -\mathbf{0} \cdot \mathbf{u} = 0$, so that $\mathbf{v}$ is also orthogonal to $\mathbf{u} \times \mathbf{v}$.

8. (a) We have

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2\theta \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \left(1 - \cos^2\theta\right) \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2\theta. \end{aligned}$$

Since the angle between $\mathbf{u}$ and $\mathbf{v}$ is always between 0 and π, it always has a nonnegative sine, so taking square roots gives $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta$.

(b) Recall that the area of a triangle is $A = \frac{1}{2}$(base)(height). If the angle between $\mathbf{u}$ and $\mathbf{v}$ is θ, then the length of the perpendicular from the head of $\mathbf{v}$ to the line determined by $\mathbf{u}$ is an altitude of the triangle; the corresponding base is $\mathbf{u}$. Thus the area is

$$A = \frac{1}{2}\|\mathbf{u}\| \cdot (\|\mathbf{v}\| \sin\theta) = \frac{1}{2}\|\mathbf{u}\| \|\mathbf{v}\| \sin\theta = \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|.$$

(c) Let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$. Then from part (b), we see that the area is

$$A = \frac{1}{2}\left\| \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} \right\| = \frac{1}{2}\left\| \begin{bmatrix} -5 \\ -6 \\ 1 \end{bmatrix} \right\| = \frac{1}{2}\sqrt{62}.$$

1.4 Applications

1. Use the method of example 1.34. The magnitude of the resultant force $\mathbf{r}$ is

$$\|\mathbf{r}\| = \sqrt{\|\mathbf{f}_1\|^2 + \|\mathbf{f}_2\|^2} = \sqrt{12^2 + 5^2} = 13 \text{ N},$$

while the angle between $\mathbf{r}$ and east (the direction of $\mathbf{f}_2$) is

$$\theta = \tan^{-1}\frac{12}{5} \approx 67.4^\circ.$$

Note that the resultant is closer to north than east; the larger force to the north pulls the object more strongly to the north.

2. Use the method of example 1.34. The magnitude of the resultant force $\mathbf{r}$ is

$$\|\mathbf{r}\| = \sqrt{\|\mathbf{f}_1\|^2 + \|\mathbf{f}_2\|^2} = \sqrt{15^2 + 20^2} = 25 \text{ N},$$

while the angle between $\mathbf{r}$ and west (the direction of $\mathbf{f}_1$) is

$$\theta = \tan^{-1} \frac{20}{15} \approx 53.1^\circ.$$

Note that the resultant is closer to south than west; the larger force to the south pulls the object more strongly to the south.

3. Use the method of Example 1.34. If we let $\mathbf{f}_1 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$, then $\mathbf{f}_2 = \begin{bmatrix} 8\cos 60^\circ \\ 8\sin 60^\circ \end{bmatrix} = \begin{bmatrix} 4 \\ 4\sqrt{3} \end{bmatrix}$. So the resultant force is

$$\mathbf{r} = \mathbf{f}_1 + \mathbf{f}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 4\sqrt{3} \end{bmatrix} = \begin{bmatrix} 12 \\ 4\sqrt{3} \end{bmatrix}.$$

The magnitude of $\mathbf{r}$ is $\|\mathbf{r}\| = \sqrt{12^2 + \left(4\sqrt{3}\right)^2} = \sqrt{192} = 8\sqrt{3}$ N, and the angle formed by $\mathbf{r}$ and $\mathbf{f}_1$ is

$$\theta = \tan^{-1} \frac{4\sqrt{3}}{12} = \tan^{-1} \frac{1}{\sqrt{3}} = 30^\circ.$$

Note that the resultant also forms a 30° degree angle with $\mathbf{f}_2$; since the magnitudes of the two forces are the same, the resultant points equally between them.

4. Use the method of Example 1.34. If we let $\mathbf{f}_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, then $\mathbf{f}_2 = \begin{bmatrix} 6\cos 135^\circ \\ 6\sin 135^\circ \end{bmatrix} = \begin{bmatrix} -3\sqrt{2} \\ 3\sqrt{2} \end{bmatrix}$. So the resultant force is

$$\mathbf{r} = \mathbf{f}_1 + \mathbf{f}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3\sqrt{2} \\ 3\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 - 3\sqrt{2} \\ 3\sqrt{2} \end{bmatrix}.$$

The magnitude of $\mathbf{r}$ is

$$\|\mathbf{r}\| = \sqrt{(4 - 3\sqrt{2})^2 + (3\sqrt{2})^2} = \sqrt{52 - 24\sqrt{2}} \approx 4.24 \text{ N},$$

and the angle formed by $\mathbf{r}$ and $\mathbf{f}_1$ is

$$\theta = \tan^{-1} \frac{3\sqrt{2}}{4 - 3\sqrt{2}} \approx 93.3^\circ$$

5. Use the method of Example 1.34. If we let $\mathbf{f}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, then $\mathbf{f}_2 = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$, and $\mathbf{f}_3 = \begin{bmatrix} 4\cos 60^\circ \\ 4\sin 60^\circ \end{bmatrix} = \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix}$. So the resultant force is

$$\mathbf{r} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -6 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} -2 \\ 2\sqrt{3} \end{bmatrix}.$$

The magnitude of $\mathbf{r}$ is

$$\|\mathbf{r}\| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{16} = 4 \text{ N},$$

and the angle formed by $\mathbf{r}$ and $\mathbf{f}_1$ is

$$\theta = \tan^{-1} \frac{2\sqrt{3}}{-2} = \tan^{-1}(-\sqrt{3}) = 120^\circ.$$

(Note that many CAS's will return -60° for $\tan^{-1}(-\sqrt{3})$; by convention we require an angle between 0 and 180°, so we add 180° to that answer, since $\tan\theta = \tan(180^\circ + \theta)$).

6. Use the method of Example 1.34. If we let $\mathbf{f}_1 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$, then $\mathbf{f}_2 = \begin{bmatrix} 0 \\ 13 \end{bmatrix}$, $\mathbf{f}_3 = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$, and $\mathbf{f}_4 = \begin{bmatrix} 0 \\ -8 \end{bmatrix}$. So the resultant force is

$$\mathbf{r} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \mathbf{f}_4 = \begin{bmatrix} 10 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 13 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -8 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

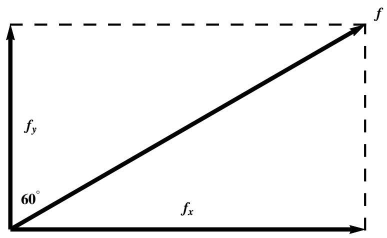
The magnitude of $\mathbf{r}$ is

$$\|\mathbf{r}\| = \sqrt{5^2 + 5^2} = \sqrt{50} = 5\sqrt{2} \text{ N},$$

and the angle formed by $\mathbf{r}$ and $\mathbf{f}_1$ is

$$\theta = \tan^{-1}\frac{5}{5} = \tan^{-1} 1 = 45^\circ.$$

7. Following Example 1.35, we have the following diagram:



Here $\mathbf{f}$ makes a 60° angle with $\mathbf{f}_y$, so that

$$\|\mathbf{f}_y\| = \|\mathbf{f}\| \cos 60^\circ, \qquad \|\mathbf{f}_x\| = \|\mathbf{f}\| \sin 60^\circ.$$

Thus $\|\mathbf{f}_x\| = 10 \cdot \frac{\sqrt{3}}{2} = 5\sqrt{3} \approx 8.66$ N and $\|\mathbf{f}_y\| = 10 \cdot \frac{1}{2} = 5$ N. Finally, this gives

$$\mathbf{f}_x = \begin{bmatrix} 5\sqrt{3} \\ 0 \end{bmatrix}, \qquad \mathbf{f}_y = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

8. The force that must be applied parallel to the ramp is the force needed to counteract the component of the force due to gravity that acts parallel to the ramp. Let $\mathbf{f}$ be the force due to gravity; this acts downwards. We can decompose it into components $\mathbf{f}_p$, acting parallel to the ramp, and $\mathbf{f}_r$, acting orthogonally to the ramp. Since the ramp makes an angle of 30° with the horizontal, it makes an angle of 60° with $\mathbf{f}$. Thus

$$\|\mathbf{f}_p\| = \|\mathbf{f}\| \cos 60^\circ = \frac{1}{2}\|\mathbf{f}\| = 5 \text{ N}.$$

9. The vertical force is the vertical component of the force vector $\mathbf{f}$. Since this acts at an angle of 45° to the horizontal, the magnitude of the component of this force in the vertical direction is

$$\|\mathbf{f}_y\| = \|\mathbf{f}\| \cos 45^\circ = 1500 \cdot \frac{\sqrt{2}}{2} = 750\sqrt{2} \approx 1060.66 \text{ N}.$$

10. The vertical force is the vertical component of the force vector $\mathbf{f}$, which has magnitude 100 N and acts at an angle of 45° to the horizontal. Since it acts at an angle of 45° to the horizontal, the magnitude of the component of this force in the vertical direction is

$$\|\mathbf{f}_y\| = \|\mathbf{f}\| \cos 45^\circ = 100 \cdot \frac{\sqrt{2}}{2} = 50\sqrt{2} \approx 70.7 \text{ N}.$$

Note that the mass of the lawnmower itself is irrelevant; we are not considering the gravitational force in this exercise, only the force imparted by the person mowing the lawn.

11. Use the method of Example 1.36. Let $\mathbf{t}$ be the force vector on the cable; then the tension on the cable is $\|\mathbf{t}\|$. The force imparted by the hanging sign is the y component of $\mathbf{t}$; call it $\mathbf{t}_y$. Since $\mathbf{t}$ and $\mathbf{t}_y$ form an angle of 60°, we have

$$\|\mathbf{t}_y\| = \|\mathbf{t}\| \cos 60^\circ = \frac{1}{2}\|\mathbf{t}\|.$$

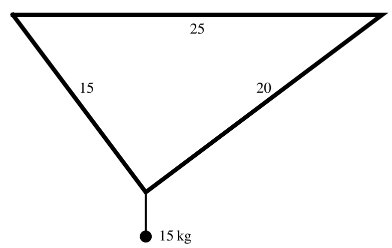
The gravitational force on the sign is its mass times the acceleration due to gravity, which is $50 \cdot 9.8 = 490$ N. Thus $\|\mathbf{t}\| = 2\|\mathbf{t}_y\| = 2 \cdot 490 = 980$ N.

12. Use the method of Example 1.36. Let $\mathbf{w}$ be the force vector created by the sign. Then $\|\mathbf{w}\| = 1 \cdot 9.8 = 9.8$ N, since the sign weighs 1 kg. By symmetry, each string carries half the weight of the sign, since the angles each string forms with the vertical are the same. Let $\mathbf{s}$ be the tension in the left-hand string. Since the angle between $\mathbf{s}$ and $\mathbf{w}$ is 45°, we have

$$\frac{1}{2}\|\mathbf{w}\| = \|\mathbf{s}\| \cos 45^\circ = \frac{\sqrt{2}}{2}\|\mathbf{s}\|.$$

Thus $\|\mathbf{s}\| = \frac{1}{\sqrt{2}}\|\mathbf{w}\| = \frac{9.8}{\sqrt{2}} = 4.9\sqrt{2} \approx 6.9$ N.

13. A diagram of the situation is



The triangle is a right triangle, since $15^2 + 20^2 = 625 = 25^2$. Thus if θ_1 is the angle that the left-hand wire makes with the ceiling, then $\sin\theta_1 = \frac{20}{25} = \frac{4}{5}$; likewise, if θ_2 is the angle that the right-hand wire makes with the ceiling, then $\sin\theta_2 = \frac{15}{25} = \frac{3}{5}$. Let $\mathbf{f}_1$ be the force on the left-hand wire and $\mathbf{f}_2$ the force on the right-hand wire. Let $\mathbf{r}$ be the force due to gravity acting on the painting. Then following Example 1.36, we have, using the Law of Sines,

$$\frac{\|\mathbf{f}_1\|}{\sin\theta_1} = \frac{\|\mathbf{f}_2\|}{\sin\theta_2} = \frac{\|\mathbf{r}\|}{\sin 90^\circ} = \frac{15 \cdot 9.8}{1} = 147 \text{ N}.$$

Then

$$\|\mathbf{f}_1\| = 147\sin\theta_1 = 147 \cdot \frac{4}{5} = 117.6 \text{ N}, \qquad \|\mathbf{f}_2\| = 147\sin\theta_2 = 147 \cdot \frac{3}{5} = 88.2 \text{ N}.$$

14. Let $\mathbf{r}$ be the force due to gravity acting on the painting, $\mathbf{f}_1$ be the tension on the wire opposite the 30° angle, and $\mathbf{f}_2$ be the tension on the wire opposite the 45° angle (don't these people know to hang paintings straight?). Then $\|\mathbf{r}\| = 20 \cdot 9.8 = 196$ N. Note that the remaining angle in the triangle is 105°. Then using the method of Example 1.36, we have, using the Law of Sines,

$$\frac{\|\mathbf{f}_1\|}{\sin 30^\circ} = \frac{\|\mathbf{f}_2\|}{\sin 45^\circ} = \frac{\|\mathbf{r}\|}{\sin 105^\circ}.$$

Thus

$$\|\mathbf{f}_1\| = \frac{\|\mathbf{r}\| \cdot \sin 30^\circ}{\sin 105^\circ} \approx \frac{196 \cdot \frac{1}{2}}{0.9659} \approx 101.46 \text{ N}$$

$$\|\mathbf{f}_2\| = \frac{\|\mathbf{r}\| \cdot \sin 45^\circ}{\sin 105^\circ} \approx \frac{196 \cdot \frac{\sqrt{2}}{2}}{0.9659} \approx 143.48 \text{ N}.$$

Chapter Review

1. **(a)** *True.* This follows from the properties of $\mathbb{R}^n$ listed in Theorem 1.1 in Section 1.1:

$$\begin{aligned}
\mathbf{u} &= \mathbf{u} + \mathbf{0} && \text{Zero Property, Property (c)}\\
&= \mathbf{u} + (\mathbf{w} + (-\mathbf{w})) && \text{Additive Inverse Property, Property (d)}\\
&= (\mathbf{u} + \mathbf{w}) + (-\mathbf{w}) && \text{Distributive Property, Property (b)}\\
&= (\mathbf{v} + \mathbf{w}) + (-\mathbf{w}) && \text{By the given condition } \mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}\\
&= \mathbf{v} + (\mathbf{w} + (-\mathbf{w})) && \text{Distributive Property, Property (b)}\\
&= \mathbf{v} + \mathbf{0} && \text{Additive Inverse Property, Property (d)}\\
&= \mathbf{v} && \text{Zero Property, Property (c).}
\end{aligned}$$

(b) *False.* See Exercise 60 in Section 1.2. For one counterexample, let $\mathbf{w} = \mathbf{0}$ and $\mathbf{u}$ and $\mathbf{v}$ be arbitrary vectors. Then $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0$ since the dot product of any vector with the zero vector is zero. But certainly $\mathbf{u}$ and $\mathbf{v}$ need not be equal. As a second counterexample, suppose that both $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to $\mathbf{w}$; clearly they need not be equal, but $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0$.

(c) *False.* For example, let $\mathbf{u}$ be any nonzero vector, and $\mathbf{v}$ any vector orthogonal to $\mathbf{u}$. Let $\mathbf{w} = \mathbf{u}$. Then $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{v}$ is orthogonal to $\mathbf{w} = \mathbf{u}$, but certainly $\mathbf{u}$ and $\mathbf{w} = \mathbf{u}$ are not orthogonal.

(d) *False.* When a line is parallel to a plane, then $\mathbf{d} \cdot \mathbf{n} = 0$; that is, $\mathbf{d}$ is *orthogonal* to the normal vector of the plane.

(e) *True.* Since a normal vector $\mathbf{n}$ for $\mathscr{P}$ and the line ℓ are both perpendicular to $\mathscr{P}$, they must be parallel. See Figure 1.62.

(f) *True.* See the remarks following Example 1.31 in Section 1.3.

(g) *False.* They can be *skew* lines, which are nonintersecting lines with nonparallel direction vectors. For example, let ℓ_1 be the line $\mathbf{x} = t\begin{bmatrix}1\\0\\0\end{bmatrix}$ (the x-axis), and ℓ_2 be the line $\mathbf{x} = \begin{bmatrix}0\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\1\\0\end{bmatrix}$ (the line through $(0,0,1)$ that is parallel to the y-axis). These two lines do not intersect, yet they are not parallel.

(h) *False.* For example, $\begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} = 1 + 0 + 1 = 0$ in $\mathbb{Z}_2$.

(i) *True.* If $ab = 0$ in $\mathbb{Z}/5$, then ab must be a multiple of 5. But 5 is prime, so either a or b must be divisible by 5, so that either $a = 0$ or $b = 0$ in $\mathbb{Z}_5$.

(j) *False.* For example, $2 \cdot 3 = 6 = 0$ in $\mathbb{Z}_6$, but neither 2 nor 3 is zero in $\mathbb{Z}_6$.

2. Let $\mathbf{w} = \begin{bmatrix}10\\-10\end{bmatrix}$. Then the head of the resulting vector is

$$4\mathbf{u} + \mathbf{v} + \mathbf{w} = 4\begin{bmatrix}-1\\5\end{bmatrix} + \begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}10\\-10\end{bmatrix} = \begin{bmatrix}9\\12\end{bmatrix}.$$

So the coordinates of the point at the head of $4\mathbf{u} + \mathbf{v}$ are $(9, 12)$.

3. Since $2\mathbf{x} + \mathbf{u} = 3(\mathbf{x} - \mathbf{v}) = 3\mathbf{x} - 3\mathbf{v}$, simplifying gives $\mathbf{u} + 3\mathbf{v} = \mathbf{x}$. Thus

$$\mathbf{x} = \mathbf{u} + 3\mathbf{v} = \begin{bmatrix}-1\\5\end{bmatrix} + 3\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}8\\11\end{bmatrix}.$$

4. Since $ABCD$ is a square, $\overrightarrow{OC} = -\overrightarrow{OA}$, so that

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -\overrightarrow{OA} - \overrightarrow{OB} = -\mathbf{a} - \mathbf{b}.$$

5. We have

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \begin{bmatrix}-1\\1\\2\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-1\end{bmatrix} = -1\cdot 2+1\cdot 1+2\cdot(-1) = -3 \\ \|\mathbf{u}\| &= \sqrt{(-1)^2+1^2+2^2} = \sqrt{6} \\ \|\mathbf{v}\| &= \sqrt{2^2+1^2+(-1)^2} = \sqrt{6}. \end{aligned}$$

Then if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, it satisfies

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = -\frac{3}{\sqrt{6}\sqrt{6}} = -\frac{1}{2}.$$

Thus

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ.$$

6. We have

$$\operatorname{proj}_{\mathbf{u}}\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \frac{1\cdot 1-2\cdot 1+2\cdot 1}{1\cdot 1-2\cdot(-2)+2\cdot 2}\begin{bmatrix}1\\-2\\2\end{bmatrix} = \frac{1}{9}\begin{bmatrix}1\\-2\\2\end{bmatrix} = \begin{bmatrix}\frac{1}{9}\\-\frac{2}{9}\\\frac{2}{9}\end{bmatrix}.$$

7. We are looking for a vector in the xy-plane; any such vector has a z-coordinate of 0. So the vector we are looking for is $\mathbf{u} = \begin{bmatrix}a\\b\\0\end{bmatrix}$ for some a, b. Then we want

$$\mathbf{u}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}a\\b\\0\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} = a+2b = 0,$$

so that $a=-2b$. So for example choose $b=1$; then $a=-2$, and the vector

$$\mathbf{u} = \begin{bmatrix}-2\\1\\0\end{bmatrix}$$

is orthogonal to $\begin{bmatrix}1\\2\\3\end{bmatrix}$. Finally, to get a unit vector, we must normalize $\mathbf{u}$. We have

$$\|\mathbf{u}\| = \sqrt{(-2)^2+1^2+0^2} = \sqrt{5}$$

to get

$$\mathbf{w} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \begin{bmatrix}-\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\\0\end{bmatrix}$$

that is orthogonal to the given vector. We could have chosen any value for b, but we would have gotten either $\mathbf{w}$ or $-\mathbf{w}$ for the normalized vector.

8. The vector form of the given line is

$$\mathbf{x} = \begin{bmatrix}2\\3\\-1\end{bmatrix} + t\begin{bmatrix}-1\\2\\1\end{bmatrix},$$

so the line has a direction vector $\mathbf{d} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. Since the plane is perpendicular to this line, $\mathbf{d}$ is a normal vector $\mathbf{n}$ to the plane. Then the plane passes through $P = (1, 1, 1)$, so equation $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ becomes

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2.$$

Expanding gives the general equation $-x + 2y + z = 2$.

9. Parallel planes have parallel normals, so the vector $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, which is a normal to the given plane, is also a normal to the desired plane. The plane we want must pass through $P = (3, 2, 5)$, so equation $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ becomes

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = 7.$$

Expanding gives the general equation $2x + 3y - z = 7$.

10. The three points give us two vectors that lie in the plane:

$$\mathbf{d}_1 = \overrightarrow{AB} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$
$$\mathbf{d}_2 = \overrightarrow{BC} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

A normal vector $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ must be orthogonal to both of these vectors, so

$$\mathbf{n} \cdot \mathbf{d}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = -b + c = 0 \quad \Rightarrow \quad b = c$$
$$\mathbf{n} \cdot \mathbf{d}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = -a + b + c = 0 \quad \Rightarrow \quad a = b + c.$$

Since $b = c$ and $a = b + c$, we get $a = 2c$, so $\mathbf{n} = \begin{bmatrix} 2c \\ c \\ c \end{bmatrix}$ for any value of c. Choosing $c = 1$ gives the vector $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Let P be the point $A = (1, 1, 0)$ (we could equally well choose B or C), and compute $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$:

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 3.$$

Expanding gives the general equation $2x + y + z = 3$.

11. Let

$$\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix} 1-1 \\ 0-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix} 0-1 \\ 1-1 \\ 2-0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Using the first method from Figure 1.39 (Exercises 46 and 47) in Section 1.2, we have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \frac{0\cdot(-1) - 1\cdot 0 + 1\cdot 2}{0^2 + (-1)^2 + 1^2}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \frac{2}{2}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Then the area of the triangle is

$$\frac{1}{2}\left\|\mathbf{u}\right\|\left\|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\right\| = \frac{1}{2}\sqrt{0^2 + (-1)^2 + 1^2}\sqrt{(-1)^2 + 1^2 + 1^2} = \frac{1}{2}\sqrt{2}\sqrt{3} = \frac{1}{2}\sqrt{6}.$$

12. From the first example in **Exploration: Vectors and Geometry**, we have a formula for the midpoint:

$$\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}\left(\begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -7 \\ 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 8 \\ -6 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}.$$

13. Suppose that $\|\mathbf{u}\| = 2$ and $\|\mathbf{v}\| = 3$. Then from the Cauchy-Schwarcz inequality, we have

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\| = 2\cdot 3 = 6.$$

Thus $-6 \le \mathbf{u}\cdot\mathbf{v} \le 6$, so the dot product cannot equal -7.

14. We will apply the formula

$$\mathrm{d}(A, \mathscr{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

where $A = (x_0, y_0, z_0)$ is a point, and a general equation for the plane is $ax + by + cz = d$. Here, we have $a = 2$, $b = 3$, $c = -1$, and $d = 0$; since the point is $(3, 2, 5)$, we have $x_0 = 3$, $y_0 = 2$, and $z_0 = 5$. So the distance from the point to the plane is

$$\mathrm{d}(A, \mathscr{P}) = \frac{|2\cdot 3 + 3\cdot 2 - 1\cdot 5 - 0|}{\sqrt{2^2 + 3^2 + (-1)^2}} = \frac{7}{\sqrt{14}} = \frac{\sqrt{14}}{2}.$$

15. As in example 1.32 in Section 1.3, we have $B = (3, 2, 5)$, and the line ℓ has vector form

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

so that $A = (0, 1, 2)$ lies on the line, and a direction vector for the line is $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\mathbf{v} = \overrightarrow{AB} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix},$$

and then

$$\operatorname{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}}\right) \mathbf{d} = \left(\frac{1 \cdot 3 + 1 \cdot 1 + 1 \cdot 3}{1^2 + 1^2 + 1^2}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{7}{3} \\ \frac{7}{3} \end{bmatrix}.$$

Then the vector from B that is perpendicular to the line is the vector

$$\mathbf{v} - \operatorname{proj}_{\mathbf{d}} \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{7}{3} \\ \frac{7}{3} \\ \frac{7}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix}.$$

So the distance from B to ℓ is

$$\|\mathbf{v} - \operatorname{proj}_{\mathbf{d}} \mathbf{v}\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{4}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \frac{1}{3}\sqrt{4 + 16 + 4} = \frac{2}{3}\sqrt{6}.$$

16. We have

$$3 - (2+4)^3(4+3)^2 = 3 - 1^3 \cdot 2^2 = 3 - 1 \cdot 4 = -1 = 4$$

in $\mathbb{Z}_5$. Note that $2 + 4 = 6 = 1$ in $\mathbb{Z}_5$, $4 + 3 = 7 = 2$ in $\mathbb{Z}_5$, and $-1 = 4$ in $\mathbb{Z}_5$.

17. $3(x + 2) = 5$ implies that $5 \cdot 3(x + 2) = 5 \cdot 5 = 25 = 4$. But $5 \cdot 3 = 15 = 1$ in $\mathbb{Z}_7$, so this is the same as $x + 2 = 4$, so that $x = 2$. To check the answer, we have

$$3(2 + 2) = 3 \cdot 4 = 12 = 5 \text{ in } \mathbb{Z}_7.$$

18. This has no solutions. For any value of x, the left-hand side is a multiple of 3, so it cannot leave a remainder of 5 when divided by 9 (which is also a multiple of 3).

19. Compute the dot product in $\mathbb{Z}_5^4$:

$$[2, 1, 3, 3] \cdot [3, 4, 4, 2] = 2 \cdot 3 + 1 \cdot 4 + 3 \cdot 4 + 3 \cdot 2 = 6 + 4 + 12 + 6 = 1 + 4 + 2 + 1 = 8 = 3.$$

20. Suppose that

$$[1, 1, 1, 0] \cdot [d_1, d_2, d_3, d_4] = d_1 + d_2 + d_3 = 0 \text{ in } \mathbb{Z}_2.$$

Then an even number (either zero or two) of d_1, d_2, and d_3 must be 1 and the others must be zero. d_4 is arbitrary (either 0 or 1). So the eight possible vectors are

$$[0, 0, 0, 0],\ [0, 0, 0, 1],\ [1, 1, 0, 0],\ [1, 1, 0, 1],\ [1, 0, 1, 0],\ [1, 0, 1, 1],\ [0, 1, 1, 0],\ [0, 1, 1, 1].$$

Chapter 2

Systems of Linear Equations

2.1 Introduction to Systems of Linear Equations

1. This is equation is linear since each of x, y, and z appear to the first power with constant coefficients. 1, π, and $\sqrt[3]{5}$ are constants.

2. This is not linear since each of x, y, and z appears to a power other than 1.

3. This is not linear since x appears to a power other than 1.

4. This is not linear since the equation contains the term xy, which is a product of two of the variables.

5. This is not linear since $\cos x$ is a function of x other than a constant times a multiple of x.

6. This is linear, since here $\cos 3$ is a constant, and x, y, and z all appear to the first power with constant coefficients $\cos 3$, -4, and 1, and the remaining term, $\sqrt{3}$, is also a constant.

7. Put the equation into the general form $ax + by + c$ by adding y to both sides to get $2x + 4y = 7$. There are no restrictions on x and y since there were none in the original equation.

8. In the given equation, the denominator must not be zero, so we must have $x \neq y$. With this restriction, we have

$$\frac{x^2 - y^2}{x - y} = 1 \iff \frac{(x+y)(x-y)}{x-y} = 1 \iff x + y = 1,\ x \neq y.$$

The corresponding linear equation is $x + y = 1$, for $x \neq y$.

9. Since x and y each appear in the denominator, neither one can be zero. With this restriction, we have

$$\frac{1}{x} + \frac{1}{y} = \frac{4}{xy} \iff \frac{y}{xy} + \frac{x}{xy} = \frac{4}{xy} \iff y + x = 4 \iff x + y = 4.$$

The corresponding linear equation is $x + y = 4$, for $x \neq 0$ and $y \neq 0$.

10. Since $\log_{10} x$ and $\log_{10} y$ are defined only for $x > 0$ and $y > 0$, these are the restrictions on the variables. With these restrictions, we have

$$\log_{10} x - \log_{10} y = 2 \iff \log_{10} \frac{x}{y} = 2 \iff \frac{x}{y} = 10^2 = 100 \iff x = 100y \iff x - 100y = 0.$$

The corresponding linear equation is $x - 100y = 0$ for $x > 0$ and $y > 0$.

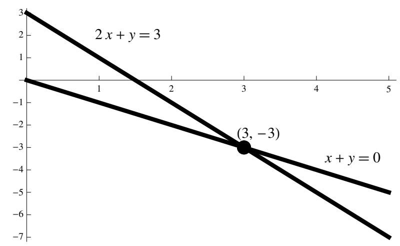
11. As in Example 2.2(a), we set $x = t$ and solve for y. Setting $x = t$ in $3x - 6y = 0$ gives $3t - 6y = 0$, so that $6y = 3t$ and thus $y = \frac{1}{2}t$. So the set of solutions has the parametric form $\left[t, \frac{1}{2}t\right]$. Note that if we had set $x = 2t$ to start with, we would have gotten the simpler (but equivalent) parametric form $[2t, t]$.

12. As in Example 2.2(a), we set $x_1 = t$ and solve for x_2. Setting $x_1 = t$ in $2x_1 + 3x_2 = 5$ gives $2t + 3x_2 = 5$, so that $3x_2 = 5 - 2t$ and thus $x_2 = \frac{5}{3} - \frac{2}{3}t$. So the set of solutions has the parametric form $\left[t, \frac{5}{3} - \frac{2}{3}t\right]$. Note that if we had set $x = 3t$ to start with, we would have gotten the simpler (but equivalent) parametric form $[3t, 5 - 2t]$.

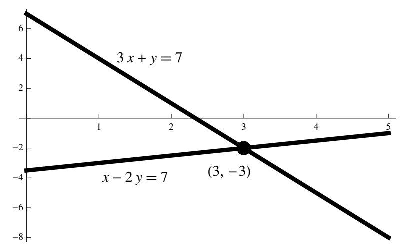
13. As in Example 2.2(b), we set $y = s$ and $z = t$ and solve for x. (We choose y and z since the coefficient of the remaining variable, x, is one, so we will not have to do any division). Then $x + 2s + 3t = 4$, so that $x = 4 - 2s - 3t$. Thus a complete set of solutions written in parametric form is $[4 - 2s - 3t, s, t]$.

14. As in Example 2.2(b), we set $x_1 = s$ and $x_2 = t$ and solve for x_3. This substitution yields $4s + 3t + 2x_3 = 1$, so that $2x_3 = 1 - 4s - 3t$ and $x_3 = \frac{1}{2} - 2s - \frac{3}{2}t$. Thus a complete set of solutions written in parametric form is $\left[s, t, \frac{1}{2} - 2s - \frac{3}{2}t\right]$.

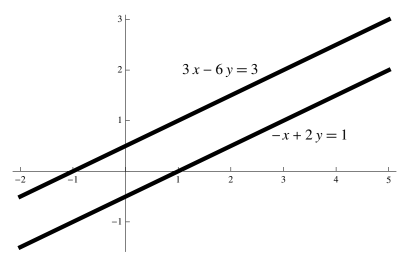
15. Subtract the first equation from the second to get $x = 3$; substituting $x = 3$ back into the first equation gives $y = -3$. Thus the unique intersection point is $(3, -3)$.



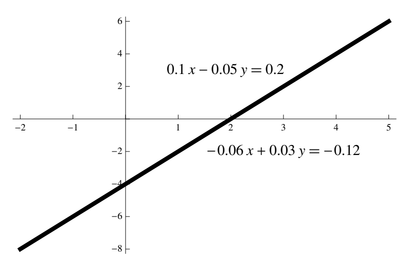
16. Add twice the second equation to the first equation to get $7x = 21$ and thus $x = 3$. Substitute $x = 3$ back into the first equation to get $y = -2$. So the unique intersection point is $(3, -2)$.



17. Add three times the second equation to the first equation. This gives $0 = 6$, which is impossible. So this system has no solutions — the two lines are parallel.



18. Add $\frac{3}{5}$ times the first equation to the second, giving $0 = 0$. Thus the system has an infinite number of solutions — the two lines are the same line (they are coincident).



19. Starting from the second equation, we find $y = 3$. Substitute $y = 3$ into $x - 2y = 1$ to get $x - 2 \cdot 3 = 1$, so that $x = 7$. The unique solution is $[x, y] = [7, 3]$.

20. Starting from the second equation, we find $2v = 6$ so that $v = 3$. Substitute $v = 3$ into $2u - 3v = 5$ to get $2u - 3 \cdot 3 = 5$, or $2u = 14$, so that $u = 7$. The unique solution is $[u, v] = [7, 3]$.

21. Starting from the third equation, we have $3z = -1$, so that $z = -\frac{1}{3}$. Now substitute this value into the second equation:

$$2y - z = 1 \Rightarrow 2y - \left(-\frac{1}{3}\right) = 1 \Rightarrow 2y + \frac{1}{3} = 1 \Rightarrow 2y = \frac{2}{3} \Rightarrow y = \frac{1}{3}.$$

Finally, substitute these values for y and z into the first equation:

$$x - y + z = 0 \Rightarrow x - \frac{1}{3} + \left(-\frac{1}{3}\right) = 0 \Rightarrow x - \frac{2}{3} = 0 \Rightarrow x = \frac{2}{3}.$$

So the unique solution is $[x, y, z] = \left[\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}\right]$.

22. The third equation gives $x_3 = 0$; substituting into the second equation gives $-5x_2 = 0$, so that $x_2 = 0$. Substituting these values into the first equation gives $x_1 = 0$. So the unique solution is $[x_1, x_2, x_3] = [0, 0, 0]$.

23. Substitute $x_4 = 1$ from the fourth equation into the third equation, giving $x_3 - 1 = 0$, so that $x_3 = 1$. Substitute these values into the second equation, giving $x_2 + 1 + 1 = 0$ and thus $x_2 = -2$. Finally, substitute into the first equation:

$$x_1 + x_2 - x_3 - x_4 = 1 \Rightarrow x_1 + (-2) - 1 - 1 = 1 \Rightarrow x_1 - 4 = 1 \Rightarrow x_1 = 5.$$

The unique solution is $[x_1, x_2, x_3, x_4] = [5, -2, 1, 1]$.

24. Let $z = t$, so that $y = 2t - 1$. Substitute for y and z in the first equation, giving

$$x - 3y + z = 5 \Rightarrow x - 3(2t - 1) + t = 5 \Rightarrow x - 5t + 3 = 5 \Rightarrow x = 5t + 2.$$

This system has an infinite number of solutions: $[x, y, z] = [5t + 2, 2t - 1, t]$.

25. Working forward, we start with $x = 2$. Substitute that into the second equation to get $2 \cdot 2 + y = -3$, or $4 + y = -3$, so that $y = -7$. Finally, substitute those two values into the third equation:

$$-3x - 4y + z = -10 \Rightarrow -3 \cdot 2 - 4 \cdot (-7) + z = -10 \Rightarrow 22 + z = -10 \Rightarrow z = -32.$$

The unique solution is $[x, y, z] = [2, -7, -32]$.

26. Working forward, we start with $x_1 = -1$. Substituting into the second equation gives $-\frac{1}{2} \cdot (-1) + x_2 = 5$, or $\frac{1}{2} + x_2 = 5$. Thus $x_2 = \frac{9}{2}$. Finally, substitute those two values into the third equation:

$$\frac{3}{2}x_1 + 2x_2 + x_3 = 7 \Rightarrow \frac{3}{2} \cdot (-1) + 2 \cdot \frac{9}{2} + x_3 = 7 \Rightarrow \frac{15}{2} + x_3 = 7 \Rightarrow x_3 = -\frac{1}{2}.$$

The unique solution is $[x_1, x_2, x_3] = \left[-1, \frac{9}{2}, -\frac{1}{2}\right]$.

27. As in Example 2.6, we create the augmented matrix from the coefficients:

$$\left[\begin{array}{rr|r} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array}\right]$$

28. As in Example 2.6, we create the augmented matrix from the coefficients:

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 \end{array}\right]$$

29. As in Example 2.6, we create the augmented matrix from the coefficients:

$$\left[\begin{array}{rr|r} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array}\right]$$

30. As in Example 2.6, we create the augmented matrix from the coefficients:

$$\left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ -1 & 1 & -1 & -3 & 1 \end{array}\right]$$

31. There are three variables, since there are three columns not counting the last one. Use x, y, and z. Then the system becomes

$$\begin{aligned} y + z &= 1 \\ x - y \qquad &= 1 \\ 2x - y + z &= 1\,. \end{aligned}$$

32. There are five variables, since there are five columns not counting the last one. Use x_1, x_2, x_3, x_4, and x_5. Then the system becomes

$$\begin{aligned} x_1 - x_2 \qquad\quad + 3x_4 + \ x_5 &= 2 \\ x_1 + x_2 + 2x_3 + \ x_4 - \ x_5 &= 4 \\ x_2 \qquad\quad + 2x_4 - 3x_5 &= 0\,. \end{aligned}$$

33. Add the first equation to the second, giving $3x = 3$, so that $x = 1$. Substituting into the first equation gives $1 - y = 0$, so that $y = 1$. The unique solution is $[x, y] = [1, 1]$.

34. As in Example 2.6, we perform row operations on the augmented matrix of this system (from Exercise 28) to get it into a form where we can read off the solution.

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & -2 & 0 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 3 & -3 & 1 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \cdots$$

$$\cdots \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \end{array}\right] \begin{array}{c} \xrightarrow{R_1 - R_3} \\ \xrightarrow{R_2 + R_3} \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \end{array}\right].$$

From this we get the unique solution $[x_1, x_2, x_3] = \left[\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right]$.

35. Add the first two equations to give $6y = -6$, so that $y = -1$. Substituting back into the second equation gives $-x - 1 = -5$, so that $x = 4$. Further, $2 \cdot 4 + 4 \cdot (-1) = 4$, so this solution satisfies the third equation as well. Thus the unique solution is $[x, y] = [4, -1]$.

36. As in Example 2.6, we perform row operations on the augmented matrix of this system (from Exercise 30) to get it into a simpler form.

$$\left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ -1 & 1 & -1 & -3 & 1 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 & 3 \end{array}\right] \xrightarrow{-R_2} \cdots$$

$$\cdots \left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & -3 \end{array}\right] \xrightarrow{R_1 + 2R_2} \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 5 & -4 \\ 0 & 1 & 1 & 2 & -3 \end{array}\right].$$

This augmented matrix corresponds to the linear system

$$\begin{aligned} a \qquad + 2c + 5d &= -4 \\ b + \ c + 2d &= -3 \end{aligned}$$

so that, letting $c = s$ and $d = t$, we get $a = -4 - 2s - 5t$ and $b = -3 - s - 2t$. So a complete set of solutions is parametrized by $[a, b, c, d] = [-4 - 2s - 5t, -3 - s - 2t, s, t]$.

37. Starting with the linear system given in Exercise 31, add the first equation to the second and to the third, giving

$$\begin{aligned} x + \ z &= 2 \\ 2x + 2z &= 2\,. \end{aligned}$$

Subtracting twice the first of these from the second gives $0 = -2$, which is impossible. So this system is inconsistent and has no solution.

38. As in Example 2.6, we perform row operations on the augmented matrix of this system (from Exercise 32) to get it into a simpler form.

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 2 & 2 & -2 & -2 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \cdots$$

$$\cdots \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 2 & 2 & -2 & -2 & 2 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & -6 & -8 & 2 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \cdots$$

$$\cdots \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & -3 & -4 & 1 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 4 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & -3 & -4 & 1 \end{array}\right]$$

Then setting $x_4 = s$ and $x_5 = t$, we have

$$\begin{aligned} x_1 \qquad\quad + 5s + 4t &= 2 \\ x_2 \quad + 2s + 3t &= 0 \\ x_3 - 3s - 4t &= 1\,, \end{aligned} \qquad \text{so that} \qquad \begin{aligned} x_1 &= 2 - 5s - 4t \\ x_2 &= -2s - 3t \\ x_3 &= 1 + 3s + 4t. \end{aligned}$$

So the general solution to this system has parametric form

$$[x_1, x_2, x_3, x_4, x_5] = [2 - 5s - 4t, -2s - 3t, 1 + 3s + 4t, s, t]\,.$$

39. **(a)** Since $t = x$ and $y = 3 - 2t$, we can substitute x for t in the equation for y to get $y = 3 - 2x$, or $2x + y = 3$.

(b) If $y = s$, then substituting s for y in the general equation above gives $2x + s = 3$, or $2x = 3 - s$, so that $x = \frac{3}{2} - \frac{1}{2}s$. So the parametric solution is $[x, y] = \left[\frac{3}{2} - \frac{1}{2}s, s\right]$.

40. **(a)** Since $x_1 = t$, substitute x_1 for t in the second and third equations, giving

$$x_2 = 1 + x_1 \text{ and } x_3 = 2 - x_1.$$

This gives the linear system

$$\begin{aligned} -x_1 + x_2 \qquad &= 1 \\ x_1 \qquad + x_3 &= 2\,. \end{aligned}$$

(b) Substitute s for x_3 into the second equation, giving $x_1 + s = 2$, so that $x_1 = 2 - s$. Substitute that value for x_1 into the first equation, giving $-(2 - s) + x_2 = 1$, or $x_2 = 3 - s$. So the parametric form given in this way is

$$[x_1, x_2, x_3] = [2 - s, 3 - s, s].$$

41. Let $u = \frac{1}{x}$ and $v = \frac{1}{y}$. Then the system of equations becomes $2u + 3v = 0$ and $3u + 4v = 1$. Solving the second equation for v gives $v = \frac{1}{4} - \frac{3}{4}u$. Substitute that value into the first equation, giving

$$2u + 3\left(\frac{1}{4} - \frac{3}{4}u\right) = 0, \quad \text{or} \quad -\frac{1}{4}u = -\frac{3}{4}, \quad \text{so} \quad u = 3.$$

Then $v = \frac{1}{4} - \frac{3}{4} \cdot 3 = -2$, so the solution is $[u, v] = [3, -2]$, or $[x, y] = \left[\frac{1}{u}, \frac{1}{v}\right] = \left[\frac{1}{3}, -\frac{1}{2}\right]$.

42. Let $u = x^2$ and $v = y^2$. Then the system of equations becomes $u + 2v = 6$ and $u - v = 3$. Subtracting the second of these from the first gives $3v = 3$, so that $v = 1$. Substituting that into the second equation gives $u - 1 = 3$, so that $u = 4$. The solution in terms of u and v is thus $[u, v] = [x^2, y^2] = [4, 1]$. Since 1 and 4 have both a positive and a negative square root, the original system has the four solutions

$$[x, y] = [2, 1], \quad [-2, 1], \quad [2, -1], \quad \text{or} \quad [-2, -1].$$

43. Let $u = \tan x$, $v = \sin y$, and $w = \cos z$. Then the system becomes

$$\begin{aligned} u - 2v \phantom{{}+w} &= 2 \\ u - v + w &= 2 \\ v - w &= -1\,. \end{aligned}$$

Subtract the first equation from the second, giving $v + w = 0$. Add that to the third equation, so that $2v = -1$ and $v = -\frac{1}{2}$. Thus $w = \frac{1}{2}$. Substituting these into the second equation gives $u - \left(-\frac{1}{2}\right) + \frac{1}{2} = 2$, so that $u = 1$. The solution is $[u, v, w] = \left[1, -\frac{1}{2}, \frac{1}{2}\right]$, so that

$$\begin{aligned} x &= \tan^{-1} u = \tan^{-1} 1 = \frac{\pi}{4} + k\pi, \\ y &= \sin^{-1} v = \sin^{-1}\left(-\frac{1}{2}\right) = \frac{7\pi}{6} + 2\ell\pi \text{ or } -\frac{\pi}{6} + 2\ell\pi, \\ z &= \cos^{-1} w = \cos^{-1}\frac{1}{2} = \pm\frac{\pi}{3} + 2n\pi, \end{aligned}$$

where k, ℓ, and n are integers.

44. Let $r = 2^a$ and $s = 3^b$. Then the system becomes $-r + 2s = 1$ and $3r - 4s = 1$. Adding three times the first equation to the second gives $2s = 4$, so that $s = 2$. Substituting that back into the first equation gives $-r + 4 = 1$, so that $r = 3$. So the solution is $[r, s] = [2^a, 3^b] = [3, 2]$. In terms of a and b, this gives $[a, b] = [\log_2 3, \log_3 2]$.

2.2 Direct Methods for Solving Linear Systems

1. This matrix is not in row echelon form, since the leading entry in row 2 appears to the right of the leading entry in row 3, violating condition 2 of the definition.

2. This matrix is in row echelon form, since the row of zeros is at the bottom and the leading entry in each nonzero row is to the left of the ones in all succeeding rows. It is not in reduced row echelon form since the leading entry in the first row is not 1.

3. This matrix is in reduced row echelon form: the leading entry in each nonzero row is 1 and is to the left of the ones in all succeeding rows, and any column containing a leading 1 contains zeros everywhere else.

4. This matrix is in reduced row echelon form: all zero rows are at the bottom, and all other conditions are vacuously satisfied (that is, they are true since no rows or columns satisfy the conditions).

5. This matrix is not in row echelon form since it has a zero row, but that row is not at the bottom.

6. This matrix is not in row echelon form since the leading entries in rows 1 and 2 appear to the right of leading entries in succeeding rows.

7. This matrix is not in row echelon form since the leading entry in row 1 does not appear to the left of the leading entry in row 2. It appears in the same column, but that is not sufficient for the definition.

8. This matrix is in row echelon form, since the zero row is at the bottom and the leading entry in each row appears in a column to the left of the leading entries in succeeding rows. However, it is not in reduced row echelon form since the leading entries in rows 1 and 3 are not 1. (Also, it is not in reduced row echelon form since the third column contains a leading 1 but also contains other nonzero entries).

9. **(a)**

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdots$$

(b)

$$\cdots \xrightarrow[R_2-R_3]{R_1-R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10. (a)

$$\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{1}{2} \\ 4 & 3 \end{bmatrix} \xrightarrow{R_2-4R_1} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \cdots$$

(b)

$$\cdots \xrightarrow{R_1-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

11. (a)

$$\begin{bmatrix} 3 & 5 \\ 5 & -2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 2 & 4 \\ 5 & -2 \\ 3 & 5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 \\ 5 & -2 \\ 3 & 5 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-5R_1} \begin{bmatrix} 1 & 2 \\ 0 & -12 \\ 0 & -1 \end{bmatrix} \cdots$$

(b)

$$\cdots \xrightarrow{-\frac{1}{12}R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow[R_3+R_2]{R_1-2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

12. (a)

$$\begin{bmatrix} 2 & -4 & -2 & 6 \\ 3 & 1 & 6 & 6 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & 1 & 6 & 6 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 7 & 9 & -3 \end{bmatrix} \cdots$$

(b)

$$\cdots \xrightarrow{\frac{1}{7}R_2} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & \frac{9}{7} & -\frac{3}{7} \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 0 & \frac{11}{7} & \frac{15}{7} \\ 0 & 1 & \frac{9}{7} & -\frac{3}{7} \end{bmatrix}$$

13. (a)

$$\begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix} \xrightarrow[-3R_3]{3R_2} \begin{bmatrix} 3 & -2 & -1 \\ 6 & -3 & -3 \\ -12 & 9 & 3 \end{bmatrix} \xrightarrow[R_3+4R_1]{R_2-2R_1} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \cdots$$

(b)

$$\cdots \xrightarrow{R_1+2R_2} \begin{bmatrix} 3 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

14. (a)

$$\begin{bmatrix} -2 & -4 & 7 \\ -3 & -6 & 10 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -3 \\ -3 & -6 & 10 \\ -2 & -4 & 7 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2+3R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdots$$

(b)

$$\cdots \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

15.

$$\begin{bmatrix}1&2&-4&-4&5\\0&-1&10&9&-5\\0&0&1&1&-1\\0&0&0&0&24\end{bmatrix}\xrightarrow{R_4+29R_3}\begin{bmatrix}1&2&-4&-4&5\\0&-1&10&9&-5\\0&0&1&1&-1\\0&0&29&29&-5\end{bmatrix}\xrightarrow{8R_3}\begin{bmatrix}1&2&-4&-4&5\\0&-1&10&9&-5\\0&0&8&8&-8\\0&0&29&29&-5\end{bmatrix}$$

$$\xrightarrow{R_4-3R_2}\begin{bmatrix}1&2&-4&-4&5\\0&-1&10&9&-5\\0&0&8&8&-8\\0&3&-1&2&10\end{bmatrix}\xrightarrow{R_3\leftrightarrow R_2}\begin{bmatrix}1&2&-4&-4&5\\0&0&8&8&-8\\0&-1&10&9&-5\\0&3&-1&2&10\end{bmatrix}$$

$$\begin{matrix}\xrightarrow{R_2+2R_1}\\\xrightarrow{R_3+2R_1}\\\xrightarrow{R_4-R_1}\end{matrix}\begin{bmatrix}1&2&-4&-4&5\\2&4&0&0&2\\2&3&2&1&5\\-1&1&3&6&5\end{bmatrix}$$

16. $R_i \leftrightarrow R_j$ undoes itself; $\frac{1}{k}R_i$ undoes kR_1; and $R_i - kR_j$ undoes $R_i + kR_J$.

17. $A = \begin{bmatrix}1&2\\3&4\end{bmatrix}\xrightarrow{R_2-2R_1}\begin{bmatrix}1&2\\1&0\end{bmatrix}\xrightarrow{-\frac{1}{2}R_1}\begin{bmatrix}-\frac{1}{2}&-1\\1&0\end{bmatrix}\xrightarrow{R_1+\frac{7}{2}R_2}\begin{bmatrix}3&-1\\1&0\end{bmatrix} = B$. So A and B are row equivalent, and we can convert A to B by the sequence $R_2 - 2R_1$, $-\frac{1}{2}R_1$, and $R_1 + \frac{7}{2}R_2$.

18.

$$A = \begin{bmatrix}2&0&-1\\1&1&0\\-1&1&1\end{bmatrix}\xrightarrow{R_1+R_2}\begin{bmatrix}3&1&-1\\1&1&0\\-1&1&1\end{bmatrix}\xrightarrow{R_2+2R_3}\begin{bmatrix}3&1&-1\\-1&3&2\\-1&1&1\end{bmatrix}$$

$$\xrightarrow{R_3+R_1}\begin{bmatrix}3&1&-1\\-1&3&2\\2&2&0\end{bmatrix}\xrightarrow{R_2+R_1}\begin{bmatrix}3&1&-1\\2&4&1\\2&2&0\end{bmatrix}\xrightarrow{R_2+\frac{1}{2}R_3}\begin{bmatrix}3&1&-1\\3&5&1\\2&2&0\end{bmatrix} = B.$$

Therefore the matrices A and B are row equivalent.

19. Performing $R_2 + R_1$ gives a new second row $R_2' = R_2 + R_1$. Then performing $R_1 + R_2$ gives a new first row with value $R_1 + R_2' = 2R_1 + R_2$. Only one row operation can be performed at a time (although in a sequence of row operations, we may perform more than one in one step as long as any rows affected do not appear in other transformations in that step, just for efficiency).

20. $\begin{bmatrix}x_1\\x_2\end{bmatrix}\xrightarrow{R_2+R_1}\begin{bmatrix}x_1\\x_1+x_2\end{bmatrix}\xrightarrow{R_1-R_2}\begin{bmatrix}-x_2\\x_1+x_2\end{bmatrix}\xrightarrow{R_2+R_1}\begin{bmatrix}-x_2\\x_1\end{bmatrix}\xrightarrow{-R_1}\begin{bmatrix}x_2\\x_1\end{bmatrix}$. The net effect is to interchange the first and second rows. So this sequence of elementary row operations is the same as $R_1 \leftrightarrow R_2$.

21. Note that $3R_2 - 2R_1$ is not an elementary row operation since it is not of the form $R_i \leftrightarrow R_j$, kR_i, or $R_i + kR_j$ (it is not of the third form since the coefficient of R_2 is 3, not 1). However,

$$\begin{bmatrix}3&1\\2&4\end{bmatrix}\xrightarrow{R_2-\frac{2}{3}R_1}\begin{bmatrix}3&1\\0&\frac{10}{3}\end{bmatrix}\xrightarrow{3R_2}\begin{bmatrix}3&1\\0&10\end{bmatrix},$$

so this is in fact a composition of elementary row operations.

22. We can create a 1 in the top left corner as follows:

$$\begin{bmatrix}3&2\\1&4\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}1&4\\3&2\end{bmatrix},\quad\begin{bmatrix}3&2\\1&4\end{bmatrix}\xrightarrow{\frac{1}{3}R_1}\begin{bmatrix}1&\frac{2}{3}\\1&4\end{bmatrix},\quad\begin{bmatrix}3&2\\1&4\end{bmatrix}\xrightarrow{R_1-2R_2}\begin{bmatrix}1&-6\\1&4\end{bmatrix}.$$

The first method, simply interchanging the two rows, requires the least computation and seems simplest. Our next choice would probably be the third method, since at least it produces integer results.

23. The rank of a matrix is the number of nonzero rows in its row echelon form.

(1) Since A is not in row echelon form, we first put it in row echelon form:

$$\begin{bmatrix}1 & 0 & 1\\0 & 0 & 3\\0 & 1 & 0\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 3\end{bmatrix}.$$

Since the row echelon form has three nonzero rows, $\operatorname{rank} A = 3$.

(2) A is already in row echelon form, and it has two nonzero rows, so $\operatorname{rank} A = 2$.

(3) A is already in row echelon form, and it has two nonzero rows, so $\operatorname{rank} A = 2$.

(4) A is already in row echelon form, and it has no nonzero rows, so $\operatorname{rank} A = 0$.

(5) Since A is not in row echelon form, we first put it in row echelon form:

$$\begin{bmatrix}1 & 0 & 3 & -4 & 0\\0 & 0 & 0 & 0 & 0\\0 & 1 & 5 & 0 & 1\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}1 & 0 & 3 & -4 & 0\\0 & 1 & 5 & 0 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Since the row echelon form has two nonzero rows, $\operatorname{rank} A = 2$.

(6) Since A is not in row echelon form, we first put it in row echelon form:

$$\begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Since the row echelon form has three nonzero rows, $\operatorname{rank} A = 3$.

(7) Since A is not in row echelon form, we first put it in row echelon form:

$$\begin{bmatrix}1 & 2 & 3\\1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix}1 & 2 & 3\\0 & -2 & -3\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \xrightarrow{R_3 + \frac{1}{2}R_2} \begin{bmatrix}1 & 2 & 3\\0 & -2 & -3\\0 & 0 & -\frac{1}{2}\\0 & 0 & 1\end{bmatrix} \xrightarrow{R_4 + 2R_3} \begin{bmatrix}1 & 2 & 3\\0 & -2 & -3\\0 & 0 & -\frac{1}{2}\\0 & 0 & 0\end{bmatrix}.$$

Since the row echelon form has three nonzero rows, $\operatorname{rank} A = 3$.

(8) A is already in row echelon form, and it has three nonzero rows, so $\operatorname{rank} A = 3$.

24. Such a matrix has 0, 1, 2, or 3 nonzero rows.

- If it has no nonzero rows, then all rows are zero, so the only such matrix is $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$.
- If it has one nonzero row, that must be the first row, and any entries in that row following the 1 are arbitrary, so the possibilities are

$$\begin{bmatrix}1 & * & *\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & *\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix},$$

  where $*$ denotes that any entry is allowed.
- If it has two nonzero rows, then the third row is zero. There are several possibilities for the arrangements of 1s in the first two rows: they could be in columns 1 and 2, 1 and 3, or 2 and 3. In each case, the entries following each 1 are arbitrary, except for the column in the first row above the 1 in the second row. This gives the possibilities

$$\begin{bmatrix}1 & 0 & *\\0 & 1 & *\\0 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}1 & * & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}.$$

- If the matrix has all nonzero rows, then the only possibility is $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. Note that all other entries must be zero since each column has a one in it.

25. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c}1&2&-3&9\\2&-1&1&0\\4&-1&1&4\end{array}\right]\xrightarrow[R_3-4R_1]{R_2-2R_1}\left[\begin{array}{ccc|c}1&2&-3&9\\0&-5&7&-18\\0&-9&13&-32\end{array}\right]\xrightarrow{-\frac{1}{5}R_2}\left[\begin{array}{ccc|c}1&2&-3&9\\0&1&-\frac{7}{5}&\frac{18}{5}\\0&-9&13&-32\end{array}\right]$$

$$\xrightarrow{R_3+9R_2}\left[\begin{array}{ccc|c}1&2&-3&9\\0&1&-\frac{7}{5}&\frac{18}{5}\\0&0&\frac{2}{5}&\frac{2}{5}\end{array}\right]\xrightarrow{\frac{5}{2}R_3}\left[\begin{array}{ccc|c}1&2&-3&9\\0&1&-\frac{7}{5}&\frac{18}{5}\\0&0&1&1\end{array}\right]$$

$$\xrightarrow[R_2+\frac{7}{5}R_3]{R_1+3R_3}\left[\begin{array}{ccc|c}1&2&0&12\\0&1&0&5\\0&0&1&1\end{array}\right]\xrightarrow{R_1-2R_2}\left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&5\\0&0&0&1\end{array}\right]$$

Thus the solution is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\5\\1\end{bmatrix}$.

26. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c}1&-1&1&0\\-1&3&1&5\\3&1&7&2\end{array}\right]\xrightarrow[R_3-3R_1]{R_2+R_1}\left[\begin{array}{ccc|c}1&-1&1&0\\0&2&2&5\\0&4&4&2\end{array}\right]\xrightarrow{\frac{1}{2}R_2}\left[\begin{array}{ccc|c}1&-1&1&0\\0&1&1&\frac{5}{2}\\0&4&4&2\end{array}\right]\xrightarrow{R_3-4R_2}\left[\begin{array}{ccc|c}1&-1&1&0\\0&1&1&\frac{5}{2}\\0&0&0&-8\end{array}\right]$$

This is an inconsistent system since the bottom row of the matrix is zero but the constant is nonzero. Thus the system has no solution.

27. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c}1&-3&-2&0\\-1&2&1&0\\2&4&6&0\end{array}\right]\xrightarrow[R_3-2R_1]{R_2+R_1}\left[\begin{array}{ccc|c}1&-3&-2&0\\0&-1&-1&0\\0&10&10&0\end{array}\right]\xrightarrow{-R_2}$$

$$\left[\begin{array}{ccc|c}1&-3&-2&0\\0&1&1&0\\0&10&10&0\end{array}\right]\xrightarrow{R_3-10R_2}\left[\begin{array}{ccc|c}1&-3&-2&0\\0&1&1&0\\0&0&0&0\end{array}\right]\xrightarrow{R_1+3R_2}\left[\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&0&0&0\end{array}\right]$$

Since there is a row of zeros, and no leading 1 in the third column, we know that x_3 is a free variable, say $x_3=t$. Back substituting gives $x_2+t=0$ and $x_1+t=0$, so that $x_1=x_2=-t$. So the solution is

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-t\\-t\\t\end{bmatrix}=t\begin{bmatrix}-1\\-1\\1\end{bmatrix}.$$

28. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & 4 & 1 \\ 3 & -1 & 0 & 1 & 1 \\ 3 & -4 & 1 & -1 & 2 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 2 & \frac{1}{2} \\ 3 & -1 & 0 & 1 & 1 \\ 3 & -4 & 1 & -1 & 2 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-3R_1} \left[\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 2 & \frac{1}{2} \\ 0 & -\frac{11}{2} & \frac{3}{2} & -5 & -\frac{1}{2} \\ 0 & -\frac{17}{2} & \frac{5}{2} & -7 & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{-\frac{2}{11}R_2} \left[\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 1 & -\frac{3}{11} & \frac{10}{11} & \frac{1}{11} \\ 0 & -\frac{17}{2} & \frac{5}{2} & -7 & \frac{1}{2} \end{array}\right] \xrightarrow{R_3+\frac{17}{2}R_2} \left[\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 1 & -\frac{3}{11} & \frac{10}{11} & \frac{1}{11} \\ 0 & 0 & \frac{2}{11} & \frac{8}{11} & \frac{14}{11} \end{array}\right]$$

This matrix is in row echelon form, but we can continue, putting it in reduced row echelon form:

$$\cdots \xrightarrow{\frac{11}{2}R_3} \left[\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 1 & -\frac{3}{11} & \frac{10}{11} & \frac{1}{11} \\ 0 & 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow{R_1-\frac{3}{2}R_2} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{11} & \frac{7}{11} & \frac{4}{11} \\ 0 & 1 & -\frac{3}{11} & \frac{10}{11} & \frac{1}{11} \\ 0 & 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow[R_2+\frac{3}{11}R_3]{R_1+\frac{1}{11}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 4 & 7 \end{array}\right]$$

Using $z = t$ as the parameter gives $w = 1 - t$, $x = 2 - 2t$, $y = 7 - 4t$, $z = t$ for the solution, or

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-t \\ 2-2t \\ 7-4t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ -4 \\ 1 \end{bmatrix}.$$

29. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-4R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -1 & 1 \\ 0 & 4 & -4 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -1 \\ 0 & 4 & -4 \end{array}\right] \xrightarrow[R_3-4R_2]{R_1-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

30. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right] \xrightarrow[\substack{R_2+2R_1 \\ R_3+R_1}]{-R_1} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system has been reduced to

$$\begin{aligned} x_1 - 3x_2 &= -2 \\ x_3 - 2x_4 &= 1. \end{aligned}$$

x_2 and x_4 are free variables; setting $x_2 = s$ and $x_4 = t$ and back-substituting gives for the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s-2 \\ s \\ 2t+1 \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

31. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{ccccc|c} \frac{1}{2} & 1 & -1 & -6 & 0 & 2 \\ \frac{1}{6} & \frac{1}{2} & 0 & -3 & 1 & -1 \\ \frac{1}{3} & 0 & -2 & 0 & -4 & 8 \end{array}\right] \begin{array}{c} \xrightarrow{2R_1} \\ \xrightarrow{6R_2} \\ \xrightarrow{3R_3} \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -12 & 0 & 4 \\ 1 & 3 & 0 & -18 & 6 & -6 \\ 1 & 0 & -6 & 0 & -12 & 24 \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{R_2-R_1} \\ \xrightarrow{R_3-R_1} \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -12 & 0 & 4 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & -2 & -4 & 12 & -12 & 20 \end{array}\right]$$

$$\xrightarrow{R_3+2R_2} \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -12 & 0 & 4 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2} \left[\begin{array}{ccccc|c} 1 & 0 & -6 & 0 & -12 & 24 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the matrix has rank 2 but there are 5 variables, there are three free variables, say $x_3 = r$, $x_4 = s$, and $x_5 = t$. Back-substituting gives $x_1 - 6r - 12t = 24$ and $x_2 + 2r - 6s + 6t = -10$, so that $x_1 = 24 + 6r + 12t$ and $x_2 = -10 - 2r + 6s - 6t$. Thus the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 24 \\ -10 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 12 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

32. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 1 \end{array}\right] \begin{array}{c} \xrightarrow{\frac{1}{\sqrt{2}}R_1} \\ \xrightarrow{\frac{1}{\sqrt{2}}R_2} \end{array} \left[\begin{array}{ccc|c} 1 & \frac{\sqrt{2}}{2} & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & 1 & -\frac{3\sqrt{2}}{2} & -1 \\ 0 & -1 & \sqrt{2} & 1 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-\frac{\sqrt{2}}{2}R_2} \\ \xrightarrow{R_3+R_2} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & \sqrt{2}+\frac{3}{2} & \sqrt{2} \\ 0 & 1 & -\frac{3\sqrt{2}}{2} & -1 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 \end{array}\right]$$

$$\xrightarrow{\sqrt{2}\,R_3} \left[\begin{array}{ccc|c} 1 & 0 & \sqrt{2}+\frac{3}{2} & \sqrt{2} \\ 0 & 1 & -\frac{3\sqrt{2}}{2} & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-(\sqrt{2}+\frac{3}{2})R_3} \\ \xrightarrow{R_2+\frac{3\sqrt{2}}{2}R_3} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

33. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 2 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-R_1} \\ \xrightarrow{R_4-R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3+2R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_4-2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right]$$

We need not proceed any further — since the last row says $0 = 7$, we know the system is inconsistent and has no solution.

34. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 4 & 10 \\ 1 & 3 & 6 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-R_1} \\ \xrightarrow{R_3-R_1} \\ \xrightarrow{R_4-R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 2 & 5 & 9 & 16 \\ 0 & 3 & 9 & 19 & 31 \end{array}\right] \begin{array}{c} \xrightarrow{R_3-2R_2} \\ \xrightarrow{R_4-3R_2} \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 3 & 10 & 13 \end{array}\right]$$

$$\xrightarrow{R_4-3R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-R_4} \\ \xrightarrow{R_2-3R_4} \\ \xrightarrow{R_3-3R_4} \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-R_3} \\ \xrightarrow{R_2-2R_3} \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

so that $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is the unique solution.

35. If we reverse the first and third rows, we get a system in row echelon form with a leading 1 in every row, so $\operatorname{rank} A = 3$. Thus this is a system of three equations in three variables with $3 - 3 = 0$ free variables, so it has a unique solution.

36. Note that $R_3 - 2R_2$ results in the row $0\ 0\ 0\ 0 \mid 2$, so that the system is inconsistent and has no solutions.

37. This system is a homogeneous system with four variables and only three equations, so the rank of the matrix is at most 3 and thus there is at least one free variable. So this system has infinitely many solutions.

38. Note that $R_3 \leftarrow R_3 - R_1 - R_2$ gives a zero row, and then $R_2 \leftarrow R_2 - 6R_1$ gives a matrix in row echelon form. So there are three free variables, and this system has infinitely many solutions.

39. It suffices to show that the row-reduced matrix has no zero rows, since it then has rank 2, so there are no free variables and no possibility of an inconsistent system.

First, if $a = 0$, then since $ad - bc = -bc \neq 0$, we see that both b and c are nonzero. Then row-reduce the augmented matrix:

$$\left[\begin{array}{cc|c} 0 & b & r \\ c & d & s \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} c & d & s \\ 0 & b & r \end{array}\right].$$

Since b and c are both nonzero, this is a consistent system with a unique solution.

Second, if $c = 0$, then since $ad - bc = ad \neq 0$, we see that both a and d are nonzero. Then the augmented matrix is

$$\left[\begin{array}{cc|c} a & b & r \\ 0 & d & s \end{array}\right].$$

This is row-reduced since both a and d are nonzero, and it clearly has rank 2, so this is a consistent system with a unique solution.

Finally, if $a \neq 0$ and $c \neq 0$, then row-reducing gives

$$\left[\begin{array}{cc|c} a & b & r \\ c & d & s \end{array}\right] \begin{array}{c} \xrightarrow{cR_1} \\ \xrightarrow{aR_2} \end{array} \left[\begin{array}{cc|c} ac & bc & rc \\ ac & ad & as \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{cc|c} ac & bc & rc \\ 0 & ad-bc & as \end{array}\right].$$

Since $ac \neq 0$ and also $ad - bc \neq 0$, this is a rank 2 matrix, so it has a unique solution.

40. First reduce the augmented matrix of the given system to row echelon form:

$$\left[\begin{array}{cc|c} k & 2 & 3 \\ 2 & -4 & -6 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 2 & -4 & -6 \\ k & 2 & 3 \end{array}\right] \xrightarrow{R_2 - \frac{1}{2}kR_1} \left[\begin{array}{cc|c} 2 & -4 & -6 \\ 0 & 2+2k & 3+3k \end{array}\right]$$

(a) The system has no solution when this matrix has a zero row with a corresponding nonzero constant. Now, $2 + 2k = 0$ for $k = -1$, in which case $3 + 3k = 0$ as well. Thus there is no value of k for which the system has no solutions.

(b) The system has a unique solution when rank $A = 2$, which happens when the bottom row is nonzero, i.e., for $k \neq -1$.

(c) The system has an infinite number of solutions when A has a zero row with a zero constant. From part (a), this happens when $k = -1$.

41. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right] \xrightarrow{R_2 - kR_1} \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{array}\right]$$

(a) If $1 - k^2 = 0$ but $1 - k \neq 0$, this system will be inconsistent and have no solution. $1 - k^2 = 0$ for $k = 1$ and $k = -1$; of these, only $k = -1$ produces $1 - k \neq 0$. Thus the system has no solution for $k = -1$.

(b) The system has a unique solution when $1 - k^2 \neq 0$, so for $k \neq \pm 1$.

(c) The system has infinitely many solutions when the last row is zero, so when $1 - k^2 = 1 - k = 0$. This happens for $k = 1$.

42. First reduce the augmented matrix of the given system to row echelon form:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 1 & 1 & 1 & k \\ 2 & -1 & 4 & k^2 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 3 & -2 & k^2-4 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 0 & 0 & k^2-k-2 \end{array}\right]$$

(a) The system has no solution when the matrix has a zero row with a corresponding nonzero constant. This occurs for $k^2 - k - 2 \neq 0$, i.e. for $k \neq 2, -1$.

(b) Since the augmented matrix has a zero row, the system does not have a unique solution for any value of k; z is a free variable.

(c) The system has an infinite number of solutions when the augmented matrix has a zero row with a zero constant. This occurs for $k^2 - k - 2 = 0$, i.e. for $k = 2, -1$.

43. First reduce the augmented matrix of the given system to row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{array}\right] \xrightarrow[R_3 - kR_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & -2-k \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & 2-k-k^2 & -2-k \end{array}\right]$$

(a) The system has no solution when the matrix has a zero row with a corresponding nonzero constant. This occurs when $2 - k - k^2 = (1 - k)(2 + k) = 0$ but $-2 - k \neq 0$. For $k = 1$, $-2 - k = -3 \neq 0$, but for $k = -2$, $-2 - k = 0$. Thus the system has no solution only when $k = 1$.

(b) The system has a unique solution when $2 - k - k^2 \neq 0$; that is, when $k \neq 1$ and $k \neq -2$.

(c) The system has an infinite number of solutions when the augmented matrix has a zero row with a zero constant. This occurs when $2 - k - k^2 = 0$ and $-2 - k = 0$. From part (a), we see that this is exactly when $k = -2$.

44. There are many examples. For instance,

(a) The following system of n equations in n variables has infinitely many solutions (for example, choose k and set $x_1 = k$ and $x_2 = -k$, with the other $x_i = 0$):

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 0 \\ 2x_1 + 2x_2 + \cdots + 2x_n &= 0 \\ &\vdots \\ nx_1 + nx_2 + \cdots + nx_n &= 0\,. \end{aligned}$$

Let m be any integer greater than n. Then the following system of m equations in n variables has infinitely many solutions (for example, choose k and set $x_1 = k$ and $x_2 = -k$, with the other $x_i = 0$):

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 0 \\ 2x_1 + 2x_2 + \cdots + 2x_n &= 0 \\ &\vdots \\ mx_1 + mx_2 + \cdots + mx_n &= 0\,. \end{aligned}$$

(b) The following system of n equations in n variables has a unique solution:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ &\vdots \\ x_n &= 0\,. \end{aligned}$$

Let m be any integer greater than n. Then a system whose first n equations are as above, and which continues with equations $x_1+x_2 = 0$, $x_1+2x_2 = 0$, and so forth until there are m equations clearly also has a unique solution.

45. Observe that the normal vectors to the planes, which are $[3, 2, 1]$ and $[2, -1, 4]$, are not parallel, so the two planes will indeed intersect in a line. The line of intersection is the points in the solution set of the system

$$\begin{aligned} 3x + 2y + z &= -1 \\ 2x - y + 4z &= 5 \end{aligned}$$

Reduce the corresponding augmented matrix to row echelon form:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 2 & -1 & 4 & 5 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 2 & -1 & 4 & 5 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{7}{3} & \frac{10}{3} & \frac{17}{3} \end{array}\right]$$

$$\xrightarrow{-\frac{3}{7}R_2} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{10}{7} & -\frac{17}{7} \end{array}\right] \xrightarrow{R_1-\frac{2}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{9}{7} & \frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{17}{7} \end{array}\right]$$

Thus z is a free variable; we set $z = 7t$ to eliminate some fractions. Then

$$x = -9t + \frac{9}{7}, \qquad y = 10t - \frac{17}{7},$$

so that the line of intersection is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{7} \\ -\frac{17}{7} \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ 10 \\ 7 \end{bmatrix}.$$

46. Observe that the normal vectors to the planes, which are $[4, 1, 1]$ and $[2, -1, 3]$, are not parallel, so the two planes will indeed intersect in a line. The line of intersection is the points in the solution set of the system

$$\begin{aligned} 4x + y + z &= 0 \\ 2x - y + 3z &= 2 \end{aligned}$$

Reduce the corresponding augmented matrix to row echelon form:

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 0 \\ 2 & -1 & 3 & 2 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 2 & -1 & 3 & 2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{3}{2} & \frac{5}{2} & 2 \end{array}\right]$$

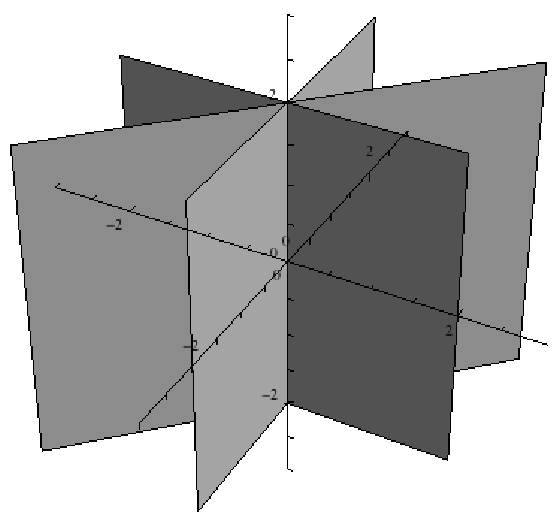
$$\xrightarrow{-\frac{2}{3}R_2} \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{5}{3} & -\frac{4}{3} \end{array}\right] \xrightarrow{R_1 - \frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{5}{3} & -\frac{4}{3} \end{array}\right]$$

Letting the parameter be $z = t$, we get the parametric form of the solution:

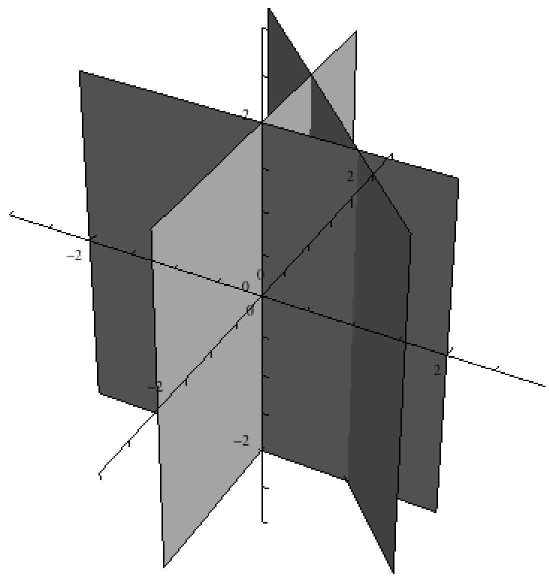
$$x = \frac{1}{3} - \frac{2}{3}t, \qquad y = -\frac{4}{3} + \frac{5}{3}t, \qquad z = t.$$

47. We are looking for examples, so we should keep in mind simple planes that we know, such as $x = 0$, $y = 0$, and $z = 0$.

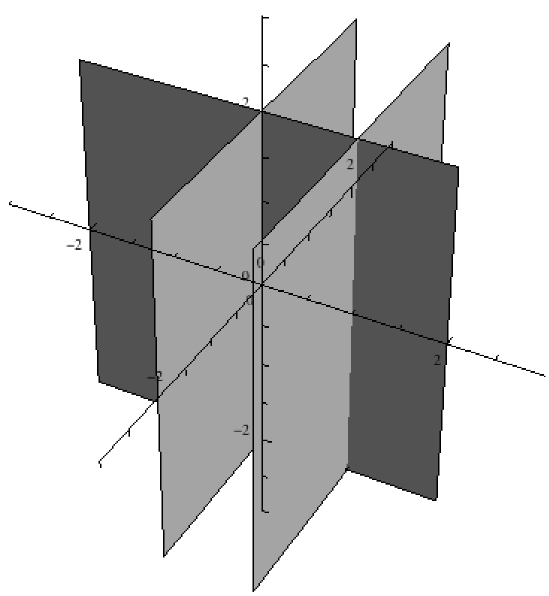
(a) Starting with $x = 0$ and $y = 0$, these planes intersect when $x = y = 0$, so they intersect on the z-axis. So we must find another plane that contains the z-axis. Since the line $y = x$ in $\mathbb{R}^2$ passes through the origin, the plane $y = x$ in $\mathbb{R}^3$ will have z as a free variable, and it contains $(0, 0, 0)$, so it contains the z-axis. Thus the three planes $x = 0$, $y = 0$, and $x = y$ intersect in a line, the z-axis. A sketch of these three planes is:



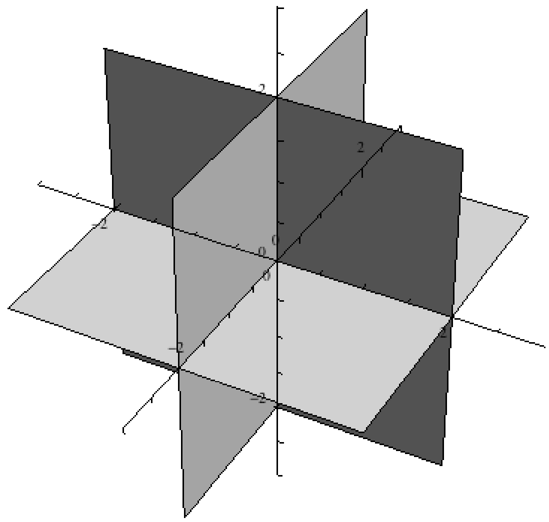
(b) Again start with $x = 0$ and $y = 0$; we are looking for a plane that intersects both of these, but not on the z-axis. Looking down from the top, it seems that any plane whose projection onto the xy plane is a line in the first quadrant will suffice. Since $x + y = 1$ is such a line, the plane $x + y = 0$ should intersect each of the other planes, but there will be no common points of intersection. To find the line of intersection of $x = 0$ and $x + y = 1$, solve that pair of equations. Using forward substitution we get $y = 1$, so the line of intersection is the line $x = 0$, $y = 1$, which is the line $[0, 1, t]$. To find the line of intersection of $y = 0$ and $x + y = 1$, again solve using forward substitution, giving the line $x = 1$, $y = 0$, which is the line $[1, 0, t]$. Clearly these two lines have no common points. A sketch of these three planes is:



(c) Clearly $x = 0$ and $x = 1$ are parallel, while $y = 0$ intersects each of them in a line. A sketch of these three planes is:



(d) The three planes $x = 0$, $y = 0$, and $z = 0$ intersect only at the origin:



48. The two lines have parametric forms $\mathbf{p} + s\mathbf{u}$ and $\mathbf{q} + t\mathbf{v}$, where s and t are the parameters. These two are equal when

$$\mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}, \text{ or } s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}.$$

Substitute the given values of the four vectors to get

$$s\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} s + t &= 3 \\ 2s - t &= 0 \\ -s &= -1\,. \end{aligned}$$

From the third equation, $s = 1$; substituting into the first equation gives $t = 2$. Since $s = 1$, $t = 2$ satisfies the second equation, this is the solution. Thus the point of intersection is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{p} + 1\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}.$$

49. The two lines have parametric forms $\mathbf{p} + s\mathbf{u}$ and $\mathbf{q} + t\mathbf{v}$, where s and t are the parameters. These two are equal when

$$\mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}, \text{ or } s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}.$$

Substitute the given values of the four vectors to get

$$s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - t\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} s - 2t &= -4 \\ -3t &= 0 \\ s - t &= -1\,. \end{aligned}$$

From the second equation, $t = 0$; setting $t = 0$ in the other two equations gives $s = -4$ and $s = -1$. This is impossible, so the system has no solution and the lines do not intersect.

50. Let $Q = (a, b, c)$; then we want to find values of Q such that the line $\mathbf{q} + s\mathbf{v}$ intersects the line $\mathbf{p} + t\mathbf{u}$. The two lines intersect when

$$\mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}, \text{ or } s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}.$$

Substitute the values of the vectors to get

$$s\begin{bmatrix}1\\1\\-1\end{bmatrix}-t\begin{bmatrix}2\\1\\0\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}-\begin{bmatrix}1\\2\\-3\end{bmatrix}=\begin{bmatrix}a-1\\b-2\\c+3\end{bmatrix}\Rightarrow\begin{aligned}s-2t&=a-1\\s-t&=b-2\\-s&=c+3\,.\end{aligned}$$

Solving those equations for a, b, and c gives

$$Q=(a,b,c)=(s-2t+1,s-t+2,-s-3).$$

Thus any pair of numbers s and t determine a point Q at which the two given lines intersect.

51. If $\mathbf{x}=[x_1,x_2,x_3]$ satisfies $\mathbf{u}\cdot\mathbf{x}=\mathbf{v}\cdot\mathbf{x}=0$, then the components of $\mathbf{x}$ satisfy the system

$$\begin{aligned}u_1x_1+u_2x_2+u_3x_3&=0\\v_1x_1+v_2x_2+v_3x_3&=0.\end{aligned}$$

There are several cases. First, if either vector is zero, then the cross product is zero, and the statement does not hold: any vector that is orthogonal to the other vector is orthogonal to both, but is not a multiple of the zero vector. Otherwise, both vectors are nonzero. Assume that $\mathbf{u}$ is the vector with a lowest-numbered nonzero component (that is, if $u_1=0$ but $u_2\neq 0$, while $v_1\neq 0$, then reverse $\mathbf{u}$ and $\mathbf{v}$). The augmented matrix of the system is

$$\left[\begin{array}{ccc|c}u_1&u_2&u_3&0\\v_1&v_2&v_3&0\end{array}\right].$$

If $u_1=u_2=0$ but $u_3\neq 0$, then $v_1=v_2=0$ and $v_3\neq 0$ as well, so that $\mathbf{u}$ and $\mathbf{v}$ are parallel and thus $\mathbf{u}\times\mathbf{v}=\mathbf{0}$. However, any vector of the form

$$\begin{bmatrix}s\\t\\0\end{bmatrix}$$

is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, so the result does not hold in this case either.

Next, if $u_1=0$ but $u_2\neq 0$, then $v_1=0$ as well, so the augmented matrix has the form

$$\left[\begin{array}{ccc|c}0&u_2&u_3&0\\0&v_2&v_3&0\end{array}\right]\xrightarrow{u_2R_2}\left[\begin{array}{ccc|c}0&u_2&u_3&0\\0&u_2v_2&u_2v_3&0\end{array}\right]\xrightarrow{R_2-v_2R_1}\left[\begin{array}{ccc|c}0&u_2&u_3&0\\0&0&u_2v_3-u_3v_2&0\end{array}\right].$$

Then x_1 is a free variable. If $u_2v_3-u_3v_2=0$, then $\mathbf{u}=\begin{bmatrix}0\\u_2\\u_3\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}0\\v_2\\v_3\end{bmatrix}$ are again parallel, so their dot product is zero and again the result does not hold. But if $u_2v_3-u_3v_2\neq 0$, then $x_3=0$, which implies that $x_2=0$ as well. So any vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ in this case is of the form

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}s\\0\\0\end{bmatrix},$$

and

$$\begin{bmatrix}0\\u_2\\u_3\end{bmatrix}\times\begin{bmatrix}0\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}u_2v_3-u_3v_2\\u_3\cdot 0-0\cdot v_3\\0\cdot v_2-0\cdot u_2\end{bmatrix}=\begin{bmatrix}u_2v_3-u_3v_2\\0\\0\end{bmatrix},$$

and since $u_2v_3-u_3v_2\neq 0$, the orthogonal vectors are multiples of the cross product.

Finally, assume that $u_1\neq 0$. Then the augmented matrix reduces as follows:

$$\left[\begin{array}{ccc|c}u_1&u_2&u_3&0\\v_1&v_2&v_3&0\end{array}\right]\xrightarrow{u_1R_2}\left[\begin{array}{ccc|c}u_1&u_2&u_3&0\\u_1v_1&u_1v_2&u_1v_3&0\end{array}\right]\xrightarrow{R_2-v_1R_1}\left[\begin{array}{ccc|c}u_1&u_2&u_3&0\\0&u_1v_2-u_2v_1&u_1v_3-u_3v_1&0\end{array}\right].$$

Now consider the last row of this matrix. If it consists of all zeros, then $u_1v_2 = u_2v_1$ and $u_1v_3 = u_3v_1$. If $v_1 \neq 0$, then $\frac{u_2}{u_1} = \frac{v_2}{v_1}$ and also $\frac{u_3}{u_1} = \frac{v_3}{v_1}$, so that again $\mathbf{u}$ and $\mathbf{v}$ are parallel and the result does not hold. Otherwise, if $v_1 = 0$, then $u_1v_2 = u_1v_3 = 0$, and u_1 was assumed nonzero, so that $v_2 = v_3 = 0$ and thus $\mathbf{v}$ is the zero vector, which it was assumed not to be. So at least one of $u_1v_2 - u_2v_1$ and $u_1v_3 - u_3v_1$ is nonzero.

If they are both nonzero, then $x_3 = t$ is free, and

$$(u_1v_2 - u_2v_1)x_2 = (u_3v_1 - u_1v_3)t, \quad \text{so that} x_2 = \frac{u_3v_1 - u_1v_3}{u_1v_2 - u_2v_1}t.$$

But then $u_1x_1 + u_2x_2 + u_3x_3 = 0$, so that

$$u_1x_1 + u_2\left(\frac{u_3v_1 - u_1v_3}{u_1v_2 - u_2v_1}\right)t + u_3t = u_1x_1 + \frac{u_1(u_3v_2 - u_2v_3)}{u_1v_2 - u_2v_1}t = 0,$$

and thus

$$x_1 = \frac{u_2v_3 - u_3v_2}{u_1v_2 - u_2v_1}t.$$

Thus the orthogonal vectors are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} \frac{u_2v_3 - u_3v_2}{u_1v_2 - u_2v_1} \\ \frac{u_3v_1 - u_1v_3}{u_1v_2 - u_2v_1} \\ 1 \end{bmatrix},$$

or, clearing fractions,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix},$$

so that orthogonal vectors are multiples of the cross product.

If $u_1v_2 - u_2v_1 \neq 0$ but $u_1v_3 - u_3v_1 = 0$, then again $v_1 = 0$ forces $\mathbf{v} = 0$, which is impossible. So $v_1 \neq 0$ and thus $\frac{u_3}{u_1} = \frac{v_3}{v_1}$. Now, $x_3 = t$ is free and $x_2 = 0$, so that $u_1x_1 + u_3t = 0$ and then $x_1 = -\frac{u_3}{u_1}t$. Let $t = (u_1v_2 - u_2v_1)s$; then

$$x_1 = -\frac{u_3(u_1v_2 - u_2v_1)}{u_1}s = -\frac{u_3u_1v_2}{u_1}s + \frac{u_3}{u_1}(u_2v_1)s = -u_3v_2s + \frac{v_3}{v_1}(u_2v_1)s = (u_2v_3 - u_3v_2)s.$$

Thus orthogonal vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} u_2v_3 - u_3v_2 \\ 0 \\ u_1v_2 - u_2v_1 \end{bmatrix},$$

and since $u_1v_3 - u_3v_1 = 0$, this column vector is $\mathbf{u} \times \mathbf{v}$, and again the result holds.

The final case is that $u_1v_3 - u_3v_1 \neq 0$ but $u_1v_2 - u_2v_1 = 0$. Again $v_1 = 0$ forces $\mathbf{v} = 0$, which is impossible, so that $v_1 \neq 0$ and thus $\frac{u_2}{u_1} = \frac{v_2}{v_1}$. Now, $x_2 = t$ is free and $x_3 = 0$, so that $u_1x_1 + u_2t = 0$ and then $x_1 = -\frac{u_2}{u_1}t$. Let $t = (u_1v_3 - u_3v_1)s$; then

$$x_1 = -\frac{u_2(u_1v_3 - u_3v_1)}{u_1}s = -\frac{u_2u_1v_3}{u_1}s + \frac{u_2}{u_1}u_3v_1s = -u_2v_3s + \frac{v_2}{v_1}u_3v_1s = (u_3v_2 - u_2v_3)s.$$

Thus orthogonal vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} u_2v_3 - u_3v_2 \\ u_1v_3 - u_3v_1 \\ 0 \end{bmatrix},$$

and since $u_1v_2 - u_2v_1 = 0$, this column vector is $\mathbf{u} \times \mathbf{v}$, and again the result holds.

Whew.

52. To show that the lines are skew, note first that they have nonparallel direction vectors, so that they are not parallel. Suppose there is some $\mathbf{x}$ such that

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}.$$

Then $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$. Substituting the given values into this equation gives

$$s\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} - t\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} 2s \quad\quad &= -1 \\ -3s - 6t &= 0 \\ s + t &= -1\,. \end{aligned}$$

But the first equation gives $s = -\frac{1}{2}$, so that from the second equation $t = \frac{1}{4}$. But this pair of values does not satisfy the third equation. So the system has no solution and thus the lines do not intersect. Since they are not parallel, they are skew.

Next, we wish to find a pair of parallel planes, one containing each line. Since $\mathbf{u}$ and $\mathbf{v}$ are direction vectors for the lines, those direction vectors will lie in *both* planes since the planes are required to be parallel. So a normal vector is given by

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} -3 \cdot (-1) - 1 \cdot 6 \\ 1 \cdot 0 - 2 \cdot (-1) \\ 2 \cdot 6 - (-3) \cdot 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 12 \end{bmatrix}.$$

So the planes we are looking for will have equations of the form $-3x + 2y + 12z = d$. The plane that passes through $P = (1, 1, 0)$ must satisfy $-3 \cdot 1 + 2 \cdot 1 + 12 \cdot 0 = d$, so that $d = -1$. The plane that passes through $Q = (0, 1, -1)$ must satisfy $-3 \cdot 0 + 2 \cdot 1 + 12 \cdot (-1) = d$, so that $d = -10$. Hence the two planes are

$$-3x + 2y + 12z = -1 \text{ containing } \mathbf{p} + s\mathbf{u}, \qquad -3x + 2y + 12z = -10 \text{ containing } \mathbf{q} + t\mathbf{v}.$$

53. Over $\mathbb{Z}_3$,

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 2 & 1 \end{array}\right] \xrightarrow{2R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right].$$

Thus the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

54. Over $\mathbb{Z}_2$,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $z = t$ is a free variable, and we have $x + t = 1$, so that $x = 1 - t = 1 + t$ and also $y + t = 0$, so that $y - t = t$. Thus the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 + t \\ t \\ t \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\},$$

since the only possible values for t in $\mathbb{Z}_2$ are 0 and 1.

55. Over $\mathbb{Z}_3$,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3+2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_3+R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{2R_3} \cdots$$

$$\cdots \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2+2R_3]{R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus the unique solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

56. Over $\mathbb{Z}_5$,

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 4 & 1 \end{array}\right] \xrightarrow{2R_1} \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 1 & 4 & 1 \end{array}\right] \xrightarrow{R_2+4R_1} \left[\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & 0 & 4 \end{array}\right].$$

Since the bottom row says $0 = 4$, this system is inconsistent and has no solutions.

57. Over $\mathbb{Z}_7$,

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 4 & 1 \end{array}\right] \xrightarrow{5R_1} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 1 & 4 & 1 \end{array}\right] \xrightarrow{R_2+6R_1} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1+4R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right].$$

Thus the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

58. Over $\mathbb{Z}_5$,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 1 & 2 & 4 & 0 & 3 \\ 2 & 2 & 0 & 1 & 1 \\ 1 & 0 & 3 & 0 & 2 \end{array}\right] \begin{array}{c} \xrightarrow{R_2+4R_1} \\ \xrightarrow{R_3+3R_1} \\ \xrightarrow{R_4+4R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 0 & 2 & 4 & 1 & 2 \\ 0 & 2 & 0 & 3 & 4 \\ 0 & 0 & 3 & 1 & 1 \end{array}\right] \xrightarrow{R_3+4R_2} \cdots$$

$$\cdots \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 0 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 1 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{3R_2} \\ \xrightarrow{R_4+2R_3} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+3R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $x_4 = t$ is a free variable, and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-4t \\ 2-4t \\ 2-2t \\ t \end{bmatrix} = \begin{bmatrix} 1+t \\ 2+t \\ 2+3t \\ t \end{bmatrix}.$$

Since $t = 0$, 1, 2, 3, or 4, there are five solutions:

$$\begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 4 \\ 4 \end{bmatrix}.$$

59. The Rank Theorem, which holds over $\mathbb{Z}_p$ as well as over $\mathbb{R}^n$, says that if A is the matrix of a consistent system, then the number of free variables is $n - \operatorname{rank}(A)$, where n is the number of variables. If there is one free variable, then it can take on any of the p values in $\mathbb{Z}_p$; since $1 = n - \operatorname{rank}(A)$, the system has exactly $p = p^1 = p^{n-\operatorname{rank}(A)}$ solutions. More generally, suppose there are k free variables. Then $k = n - \operatorname{rank}(A)$. Each of the free variables can take any of p possible values, so altogether there are p^k choices for the values of the free variables. Each of these yields a different solution, so the total number of solutions is $p^k = p^{n-\operatorname{rank}(A)}$.

60. The first step in row-reducing the augmented matrix of this system would be to multiply by the appropriate number to get a 1 in the first column. However, neither 2 nor 3 has a multiplicative inverse in $\mathbb{Z}_6$, so this is impossible. So the best we can do in terms of row-reduction is

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & 3 & 2 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

Thus y is a free variable, and $2x + 3y = 4$. To solve this equation, choose each possible value for y in turn:

- When $y = 0$, $y = 2$, or $y = 4$ we get $2x = 4$, so that $x = 2$ or $x = 5$.
- When $y = 1$ or $y = 3$, we get $2x + 3 = 4$, or $2x = 1$, which has no solution.

So altogether there are six solutions:

$$\begin{bmatrix}2\\0\end{bmatrix}, \quad \begin{bmatrix}5\\0\end{bmatrix}, \quad \begin{bmatrix}2\\2\end{bmatrix}, \quad \begin{bmatrix}5\\2\end{bmatrix}, \quad \begin{bmatrix}2\\4\end{bmatrix}, \quad \begin{bmatrix}5\\4\end{bmatrix}.$$

Exploration: Lies My Computer Told Me

1.

$$\begin{aligned} x + \phantom{\tfrac{801}{800}}y &= 0 \\ x + \tfrac{801}{800}y &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -800x - 800y &= 0 \\ 800x + 801y &= 800 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= -800 \\ y &= 800\,. \end{aligned}$$

2.

$$\begin{aligned} x + y &= 0 \\ x + 1.0012y &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -x - y &= 0 \\ x + 1.0012y &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= -833.33 \\ y &= 833.33\,. \end{aligned}$$

3. At four significant digits, we get

$$\begin{aligned} x + y &= 0 \\ x + 1.001y &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -x - y &= 0 \\ x + 1.001y &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= -1000 \\ y &= 1000\,, \end{aligned}$$

while at three significant digits, we get

$$\begin{aligned} x + y &= 0 \\ x + 1.00y &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -x - y &= 0 \\ x + 1.00y &= 1 \end{aligned} \quad \Rightarrow \quad \text{Inconsistent system; no solution.}$$

4. For the equations we just looked at, the slopes of the two lines were very close (the angle between the lines was very small). That means that changing one of the slopes even a little bit will result in a large change in their intersection point. That is the cause of the wild changes in the computed output values above — rounding to five significant digits is indeed the same as a small change in the slope of the second line, and it caused a large change in the computed solution.

For the second example, note that the two slopes are

$$\frac{7.083}{4.552} \approx 1.55602, \qquad \frac{2.693}{1.731} \approx 1.55575,$$

so that again the slopes are close, so we would expect the same kind of behavior.

Exploration: Partial Pivoting

1. (a) Solving $0.00021x = 1$ to five significant digits gives $x = \frac{1}{0.00021} \approx 4761.9$.

(b) If only four significant digits are kept, we have $0.0002x = 1$, so that $x = \frac{1}{0.0002} = 5000$. Thus the effect of an error of 0.00001 in the input is an error of 238.1 in the result.

2. (a) Start by pivoting on 0.4. Multiply the first row by $\frac{1}{0.4} = 250$ to get $[1, 249, 250]$. Now we want to subtract 75.3 times that row from the second row. Multiplying by 75.3 and keeping three significant digits gives

$$75.3[1, 249, 250] = [75.3, 18749.7, 18825.0] \approx [75.3, 18700, 18800].$$

Then the second row becomes (again keeping only three significant digits)

$$[75.3, 45.3, 30] - [75.3, 18700, 18800] = [0, 18654.7, 18770] \approx [0, 18700, 18800].$$

This gives $y = \frac{18800}{18700} \approx 1.00535 \approx 1.01$. Substituting back into the modified first row, which says $x + 249y = 250$ gives $x = 250 - 251.49 \approx 250 - 251 = -1$.

Substituting $x = y = 1$ into the original system, however, yields two true equations, so this is in fact the solution. The problem was that the large numbers in the second row, combined with the fact that we were keeping only three significant digits, overwhelmed the information from adding in the first row, so all that information was lost.

(b) First interchanging the rows gives the matrix

$$\left[\begin{array}{cc|c} 75.3 & -45.3 & 30 \\ 0.4 & 99.6 & 100 \end{array}\right]$$

Multiply the first row by $\frac{1}{75.3}$ to get (after reducing to three significant digits) $[1, -0.602, 0.398]$. Then subtract 0.4 times that row from the second row:

$$[0.4, 99.6, 100] - 0.4[1, -0.602, 0.398] \approx [0, 99.8, 99.8].$$

This gives $99.8y = 99.8$, so that $y = 1$; from the first equation, we have $x - 0.602y = 0.398$, so that $x - 0.602 \cdot 1 = 0.398$ and thus $x = 1$. We get the correct answer.

3. (a) Without partial pivoting, we pivot on 0.001 to three significant digits, so we divide the first row by 0.001, giving $[1, 995, 1000]$, then multiply it by 10.2 and add the result to row 2, giving the matrix

$$\left[\begin{array}{cc|c} 1 & 995 & 1000 \\ 0 & -10100 & -10200 \end{array}\right] \qquad \Rightarrow \qquad y \approx 1.01.$$

Back-substituting gives $x = \frac{1.00 - 1.00}{2} = 0$ (again remembering to keep only three significant digits). This error was introduced because 1.00 and -50.0 were ignored when reducing to three significant digits in the original row-reduction, being overwhelmed by numbers in the range of 10000.

Using partial pivoting, we pivot on -10.2 to three significant digits: the second row becomes $[1, 0.098, 4.9]$. Then multiply it by 0.001 and subtract it from the first row, giving $[0, 0.995, 1.00]$. Thus $0.995y = 1.00$, so that to three significant digits $y = 1.00$. Back-substituting gives $x = \frac{-50.0-1.00}{-10.2} = \frac{51.0}{10.2} = 5$. Pivoting on the largest absolute value reduced the error in the solution.

(b) Start by pivoting on the 10.0 at upper left:

$$\left[\begin{array}{ccc|c} 10.0 & -7.00 & 0.00 & 7.00 \\ -3.0 & 2.09 & 6.00 & 3.91 \\ 5.0 & 1.00 & 5.00 & 6.00 \end{array}\right] \to \left[\begin{array}{ccc|c} 10.0 & -7.00 & 0.00 & 7.00 \\ 0.0 & -0.01 & 6.00 & 6.01 \\ 0.0 & 2.50 & 5.00 & 2.50 \end{array}\right].$$

If we continue, pivoting on -0.01 to three significant digits, the second row then becomes $[0, 1, -600, -601]$. Multiply it by 2.5 and add it to the third row:

$$\left[\begin{array}{ccc|c} 10.0 & -7.00 & 0.00 & 7.00 \\ 0.0 & -0.01 & 6.00 & 6.01 \\ 0.0 & 2.50 & 5.00 & 2.50 \end{array}\right] \to \left[\begin{array}{ccc|c} 10.0 & -7.00 & 0.00 & 7.00 \\ 0.0 & -0.01 & 6.00 & 6.01 \\ 0.0 & 0 & -1500 & -1500 \end{array}\right].$$

This gives $z = 1.00$, $y = -1.00$, and $x = 0.00$.

If instead we reversed the second and third rows and pivoted on the 2.5, we would first divide that row by 2.5, giving $[0, 1, 2, 1]$, then multiply it by 0.01 and add it to the third row:

$$\left[\begin{array}{ccc|c} 10.0 & -7.00 & 0.00 & 7.00 \\ 0.0 & 2.50 & 5.00 & 2.50 \\ 0.0 & -0.01 & 6.00 & 6.01 \end{array}\right] \to \left[\begin{array}{ccc|c} 10.0 & -7.00 & 0.00 & 7.00 \\ 0.0 & 1 & 2 & 1 \\ 0.0 & -0.01 & 6.00 & 6.01 \end{array}\right] \to \left[\begin{array}{ccc|c} 10.0 & -7.00 & 0.00 & 7.00 \\ 0.0 & 1 & 2 & 1 \\ 0.0 & 0 & 6.02 & 6.02 \end{array}\right].$$

This also gives $z = 1.00$, $y = -1.00$, and $x = 0.00$.

Exploration: An Introduction to the Analysis of Algorithms

1. We count the number of operations, one step at a time.

$$\left[\begin{array}{rrr|r} 2 & 4 & 6 & 8 \\ 3 & 9 & 6 & 12 \\ -1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 3 & 9 & 6 & 12 \\ -1 & 1 & -1 & 1 \end{array}\right]$$

There were three operations here: $\frac{1}{2} \cdot 4 = 2$, $\frac{1}{2} \cdot 6 = 3$, and $\frac{1}{2} \cdot 8 = 4$. We don't count $\frac{1}{2} \cdot 2 = 1$ since we don't actually perform that operation — once we chose 2, we knew we were going to turn it into 1.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 3 & 9 & 6 & 12 \\ -1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 0 \\ -1 & 1 & -1 & 1 \end{array}\right]$$

There were three operations here: $3 \cdot 2 = 6$, $3 \cdot 3 = 9$, and $3 \cdot 4 = 12$, fr a total of six operations so far. Note that we are counting only multiplications and divisions, and that again we don't count $3 \cdot 1 = 3$ since we didn't actually have to perform it — we just replace the 3 in the second row by a 0.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 0 \\ -1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_3 + R_1} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & 2 & 5 \end{array}\right]$$

There were 3 operations here: $1 \cdot 2 = 2$, $1 \cdot 3 = 3$, and $1 \cdot 4 = 4$, for a total of nine operations so far.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & 2 & 5 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 5 \end{array}\right]$$

In this step, there were 2 operations: $\frac{1}{3} \cdot (-3)$ and $\frac{1}{3} \cdot 0 = 0$, for a total of eleven so far.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 5 \end{array}\right] \xrightarrow{R_3 - 3R_2} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{array}\right]$$

In this step, there were 2 operations: $-3 \cdot (-1) = 3$ and $-3 \cdot 0 = 0$, for a total of 13 so far.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{array}\right] \xrightarrow{\frac{1}{5}R_3} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

In this step, there was one operation, $\frac{1}{5} \cdot 5 = 1$, for a total of 14.

Finally, to complete the back-substitution, we need three more operations: $-1 \cdot x_3$ to find x_2, and $2 \cdot x_2$ and $3 \cdot x_3$ to find x_1. So in total 17 operations were required.

2. To compute the reduced row echelon form, we first compute the row echelon form above, which took 14 operations. Then:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{rrr|r} 1 & 0 & 5 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This required two operations, $-2 \cdot (-1)$ and $-2 \cdot 0$, for a total of 16. Finally,

$$\left[\begin{array}{rrr|r} 1 & 0 & 5 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{R_1 - 5R_3} \\ \xrightarrow{R_2 + R_3} \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This required two operations, both in the constant column: $1 \cdot 1$ and $-5 \cdot 1$, for a total of 18. So for this example, Gauss-Jordan required one more operation than Gaussian elimination. We would probably need more data to conclude that it is generally less efficient.

3. **(a)** There are n operations required to create the first leading 1 because we have to divide every entry in the first row, except a_{11}, by a_{11} (we don't have to divide a_{11} by itself because the 1 results from the choice of pivot. Next, there are n operations required to create the first zero in column 1, because we have to multiply every entry in the first row, except the 1 in column one, by a_{21}. Again, we don't have to multiply the 1 by itself, since we know we are simply replacing a_{21} by zero. Similarly, there are n operations required to create each zero in column 1. Since there are $n-1$ rows excluding the first row, introducing the 1 in row one and the zeros below it requires $n + (n-1)n = n^2$ operations.

(b) We can think of the resulting matrix, below and to the right of a_{22}, as an $(n-1) \times n$ augmented matrix. Applying the process in part (a) to this matrix, we see that turning a_{22} into a 1 and creating zeros below it requires $(n-1) + (n-2)(n-1) = (n-1)^2$ operations. Continuing to the matrix starting at row 3, column 3; this requires $(n-2)^2$ operations. So altogether, to reduce the matrix to row echelon form requires $n^2 + (n-1)^2 + \cdots + 2^2 + 1^2$ operations.

(c) To find x_{n-1} requires one operation: we must multiply x_n by the entry $a_{n-1,n}$. Then there are two operations required to find x_{n-2}, and so forth, until we require $n-1$ operations to find x_1. So the total number required for the back-substitution is $1 + 2 + \cdots + (n-1)$.

(d) From Appendix B, we see that

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 &= \frac{k(k+1)(2k+1)}{6} \\ 1 + 2 + \cdots + k &= \frac{k(k+1)}{2}. \end{aligned}$$

Thus the total number of operations required is

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + 1 + 2 + \cdots + (n-1) &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n-1)}{2} \\ &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + \frac{1}{2}n^2 - \frac{1}{2}n \\ &= \frac{1}{3}n^3 + n^2 - \frac{1}{3}n. \end{aligned}$$

For large values of n, the cubic term dominates, so the total number of operations required is $\approx \frac{1}{3}n^3$.

4. As we saw in Exercise 2 in this section, we get the same $n^2 + (n-1)^2 + \cdots + 2^2 + 1^2$ operations to get the matrix into row echelon form. Then to create the 1 zero above row 2 requires $1(n-1)$ operations, since each of the remaining $n-1$ entries in row 2 must be multiplied. To create the 2 zeros above row 3 requires $2(n-2)$ operations, since each of the remaining entries in row 3 must be multiplied twice. Continuing, we see that the total number of operations is

$$\begin{aligned} &(n-1) + 2(n-2) + \cdots + (n-1)(n-(n-1)) \\ &\qquad = (1 \cdot n + 2 \cdot n + \cdots + (n-1) \cdot n) - (1^2 + 2^2 + \cdots + (n-1)^2) \\ &\qquad = n(1 + 2 + \cdots + (n-1)) - (1^2 + 2^2 + \cdots + (n-1)^2). \end{aligned}$$

Adding that to the operations required to get the matrix into row echelon form, and using the formulas from Appendix B, we get for the total number of operations

$$\begin{aligned} n^2 + (n-1)^2 + \cdots + 2^2 + 1^2 + n(1 + 2 + \cdots + (n-1)) - (1^2 + 2^2 + \cdots + (n-1))^2 \\ = n^2 + n \cdot \frac{n(n-1)}{2} \\ = n^2 + \frac{1}{2}n^3 - \frac{1}{2}n^2 \\ = \frac{1}{2}n^3 + \frac{1}{2}n^2. \end{aligned}$$

Again, for large values of n, the cubic term dominates, so the total number of operations required is $\approx \frac{1}{2}n^3$.

2.3 Spanning Sets and Linear Independence

1. Yes, it is. We want to write

$$x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

so we want to solve the system of linear equations

$$\begin{aligned} x + 2y &= 1 \\ -x - y &= 2. \end{aligned}$$

Row reducing the augmented matrix for the system gives

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & -1 & 2 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 3 \end{array}\right].$$

So the solution is $x = -5$ and $y = 3$, and the linear combination is $\mathbf{v} = -5\mathbf{u}_1 + 3\mathbf{u}_2$.

2. No, it is not. We want to write

$$x\begin{bmatrix} 4 \\ -2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

so we want to solve the system of linear equations

$$\begin{aligned} 4x - 2y &= 2 \\ -2x + y &= 1 \end{aligned}$$

Row-reducing the augmented matrix for the system gives

$$\left[\begin{array}{cc|c} 4 & -2 & 2 \\ -2 & 1 & 1 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 & 1 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 2 \end{array}\right]$$

So the system is inconsistent, and the answer follows from Theorem 2.4.

3. No, it is not. We want to write

$$x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

so we want to solve the system of linear equations

$$\begin{aligned} x &= 1 \\ x + y &= 2 \\ y &= 3. \end{aligned}$$

Rather than constructing the matrix and reducing it, note that the first and third equations force $x = 1$ and $y = 3$, and that these values of x and y do not satisfy the second equation. Thus the system is inconsistent, so that $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

4. Yes, it is. We want to write

$$x\begin{bmatrix}1\\1\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}3\\2\\-1\end{bmatrix},$$

so we want to solve the system of linear equations

$$\begin{aligned} x \qquad &= 3\\ x+y &= 2\\ y &= -1 \end{aligned}$$

Rather than constructing the matrix and reducing it, note that the first and third equations force $x=3$ and $y=-1$, and that these values of x and y also satisfy the second equation. Thus $\mathbf{v}=3\mathbf{u}_1-\mathbf{u}_2$.

5. Yes, it is. We want to write

$$x\begin{bmatrix}1\\1\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\1\end{bmatrix}+z\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix},$$

so we want to solve the system of linear equations

$$\begin{aligned} x \qquad +z &= 1\\ x+y \qquad &= 2\\ y+z &= 3. \end{aligned}$$

Row-reducing the associated augmented matrix gives

$$\left[\begin{array}{ccc|c}1&0&1&1\\1&1&0&2\\0&1&1&3\end{array}\right]\xrightarrow{R_2-R_1}\left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&1\\0&1&1&3\end{array}\right]\xrightarrow{R_3-R_2}\left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&1\\0&0&2&2\end{array}\right]\xrightarrow{\frac{1}{2}R_3}\cdots$$

$$\cdots\left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&1\\0&0&1&1\end{array}\right]\xrightarrow[R_2+R_3]{R_1-R_3}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&2\\0&0&1&1\end{array}\right]$$

This gives the solution $x=0$, $y=2$, and $z=1$, so that $\mathbf{v}=2\mathbf{u}_2+\mathbf{u}_3$.

6. Yes, it is. The associated linear system is

$$\begin{aligned} 1.0x+3.4y-1.2z &= 3.2\\ 0.4x+1.4y+0.2z &= 2.0\\ 4.8x-6.4y-1.0z &= -2.6 \end{aligned}$$

Solving this system with a CAS gives $x=1.0$, $y=1.0$, $z=1.0$, so that $\mathbf{v}=\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3$.

7. Yes, it is. $\mathbf{b}$ is in the span of the columns of A if and only if the system $A\mathbf{x}=\mathbf{b}$ has a solution, which is to say if and only if the linear system

$$\begin{aligned} x+2y &= 5\\ 3x+4y &= 6 \end{aligned}$$

has a solution. Row-reducing the augmented matrix for the system gives

$$\left[\begin{array}{cc|c}1&2&5\\3&4&6\end{array}\right]\xrightarrow{R_2-3R_1}\left[\begin{array}{cc|c}1&2&5\\0&-2&-9\end{array}\right]\xrightarrow{-\frac{1}{2}R_2}\left[\begin{array}{cc|c}1&2&5\\0&1&\frac{9}{2}\end{array}\right]\xrightarrow{R_1-2R_2}\left[\begin{array}{cc|c}1&0&-4\\0&1&\frac{9}{2}\end{array}\right]$$

Thus $x=-4$ and $y=\frac{9}{2}$, so that

$$A\begin{bmatrix}-4\\\frac{9}{2}\end{bmatrix}=\begin{bmatrix}5\\6\end{bmatrix}.$$

8. Yes, it is. $\mathbf{b}$ is in the span of the columns of A if and only if the system $A\mathbf{x} = \mathbf{b}$ has a solution, which is to say if and only if the linear system

$$\begin{aligned} x + \ 2y + \ 3z &= \ 4 \\ 5x + \ 6y + \ 7z &= \ 8 \\ 9x + 10y + 11z &= 12 \end{aligned}$$

Row-reducing the augmented matrix for the system gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \xrightarrow[R_3-9R_1]{R_2-5R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -24 \end{array}\right]$$
$$\xrightarrow{R_3+8R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system has an infinite number of solutions, so that $\mathbf{b}$ is indeed in the span of the columns of A.

9. We must show that for any vector $\begin{bmatrix} a \\ b \end{bmatrix}$, we can write

$$x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

for some x, y. Row-reduce the associated augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b-a \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & \frac{a-b}{2} \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{a+b}{2} \\ 0 & 1 & \frac{a-b}{2} \end{array}\right]$$

So given a and b, we have

$$\frac{a+b}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{a-b}{2} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

10. We want to show that for any vector $\begin{bmatrix} a \\ b \end{bmatrix}$, we can write

$$x \begin{bmatrix} 3 \\ -2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

for some x, y. Row-reduce the associated augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 0 & a \\ -2 & 1 & b \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{a}{3} \\ -2 & 1 & b \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{a}{3} \\ 0 & 1 & b+\frac{2a}{3} \end{array}\right]$$

which shows that the vector can be written

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{a}{3} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \left(b + \frac{2a}{3}\right) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

11. We want to show that any vector can be written as a linear combination of the three given vectors, i.e. that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

for some x, y, z. Row-reduce the associated augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & -1 & 1 & c-a \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 2 & b+c-a \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & \frac{b+c-a}{2} \end{array}\right] \xrightarrow{R_2-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 0 & \frac{a+b-c}{2} \\ 0 & 0 & 1 & \frac{b+c-a}{2} \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{a-b+c}{2} \\ 0 & 1 & 0 & \frac{a+b-c}{2} \\ 0 & 0 & 1 & \frac{b+c-a}{2} \end{array}\right]$$

Thus

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2}(a-b+c) \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}(a+b-c) \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}(-a+b+c) \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

12. We want to show that any vector can be written as a linear combination of the three given vectors, i.e. that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

for some x, y, z. Row-reduce the associated augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 2 & 1 & b \\ 0 & 3 & -1 & c \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & -1 & b-a \\ 0 & 3 & -1 & c \end{array}\right] \xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 2 & c+3a-3b \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 1 & \frac{1}{2}(c+3a-3b) \end{array}\right] \xrightarrow[R_2+R_3]{R_1-2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3b-2a-c \\ 0 & 1 & 0 & \frac{1}{2}(c+a-b) \\ 0 & 0 & 1 & \frac{1}{2}(c+3a-3b) \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(7b-5a-3c) \\ 0 & 1 & 0 & \frac{1}{2}(c+a-b) \\ 0 & 0 & 1 & \frac{1}{2}(c+3a-3b) \end{array}\right]$$

Thus

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2}(7b-5a-3c) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}(c+a-b) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{1}{2}(c+3a-3b) \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

13. **(a)** Since $\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, these vectors are parallel, so their span is the line through the origin with direction vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(b) The vector equation of this line is $\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Note that this is just another way of saying that the line is the span of the given vector. To derive the general equation of the line, expand the above, giving the parametric equations $x = -t$ and $y = 2t$. Substitute $-x$ for t in the second equation, giving $y = -2x$, or $2x + y = 0$.

14. **(a)** Since the span of $\begin{bmatrix}0\\0\end{bmatrix}$ is just the origin, and the origin is in the span of $\begin{bmatrix}3\\4\end{bmatrix}$, the span of these two vectors is just the span of $\begin{bmatrix}3\\4\end{bmatrix}$, which is the line through the origin with direction vector $\begin{bmatrix}3\\4\end{bmatrix}$.

(b) The vector equation of this line is $\begin{bmatrix}x\\y\end{bmatrix} = t\begin{bmatrix}3\\4\end{bmatrix}$. Note that this is just another way of saying that the line is the span of the given vector. To derive the general equation of the line, expand the above, giving the parametric equations $x = 3t$ and $y = 4t$. Substitute $\frac{1}{3}x$ for t in the second equation, giving $y = \frac{4}{3}x$. Clearing fractions gives $3y = 4x$, or $4x - 3y = 0$.

15. **(a)** Since the given vectors are not parallel, their span is a plane through the origin containing these two vectors as direction vectors.

(b) The vector equation of the plane is

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = s\begin{bmatrix}1\\2\\0\end{bmatrix} + t\begin{bmatrix}3\\2\\-1\end{bmatrix}.$$

One way of deriving the general equation of the plane is to solve the system of equations arising from the above vector equation. Row-reducing the corresponding augmented system gives

$$\left[\begin{array}{cc|c}1 & 3 & x\\2 & 2 & y\\0 & -1 & z\end{array}\right] \xrightarrow{R_2-2R_2} \left[\begin{array}{cc|c}1 & 3 & x\\0 & -4 & y-2x\\0 & -1 & z\end{array}\right] \xrightarrow{R_3-\frac{1}{4}R_2} \left[\begin{array}{cc|c}1 & 3 & x\\0 & -4 & y-4x\\0 & 0 & z+\frac{1}{4}(2x-y)\end{array}\right]$$

Since we know the system is consistent, the bottom row must consist entirely of zeros. Thus the points on the plane must satisfy $z + \frac{1}{4}(2x - y) = 0$, or $4z + 2x - y = 0$. This is the plane $2x - y + 4z = 0$.

An alternative method is to find a normal vector to the plane by taking the cross-product of the two given vectors:

$$\begin{bmatrix}1\\2\\0\end{bmatrix} \times \begin{bmatrix}3\\2\\-1\end{bmatrix} = \begin{bmatrix}0\cdot 2 - 2\cdot(-1)\\1\cdot(-1) - 0\cdot 3\\2\cdot 3 - 1\cdot 2\end{bmatrix} = \begin{bmatrix}2\\-1\\4\end{bmatrix}.$$

So the equation of the plane is $2x - y + 4z = d$; since it passes through the origin, we again get $2x - y + 4z = 0$.

16. **(a)** First, note that

$$\begin{bmatrix}0\\-1\\1\end{bmatrix} = -\begin{bmatrix}1\\0\\-1\end{bmatrix} - \begin{bmatrix}-1\\1\\0\end{bmatrix},$$

so that the third vector is in the span of the first two. So we need only consider the first two vectors. Their span is the plane through the origin with direction vectors $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\0\end{bmatrix}$.

(b) The vector equation of this plane is

$$s\begin{bmatrix}1\\0\\-1\end{bmatrix} + t\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}.$$

To find the general equation of the plane, we can solve the corresponding system of equations for s and t. Row-reducing the augmented matrix gives

$$\left[\begin{array}{cc|c}1 & -1 & x\\0 & 1 & y\\-1 & 0 & z\end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{cc|c}1 & -1 & x\\0 & 1 & y\\0 & -1 & x+z\end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{cc|c}1 & -1 & x\\0 & 1 & y\\0 & 0 & x+y+z\end{array}\right]$$

Since we know the system is consistent, the bottom row must consist of all zeros, so that points on the plane satisfy the equation $x + y + z = 0$.

Alternatively, we could find a normal vector to the plane by taking the cross product of the direction vectors:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot 0 - (-1) \cdot 1 \\ -1 \cdot -1 - 1 \cdot 0 \\ 1 \cdot 1 - 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus the equation of the plane is $x + y + z = d$; since it passes through the origin, we again get $x + y + z = 0$.

17. Substituting the two points $(1, 0, 3)$ and $(-1, 1, -3)$ into the equation for the plane gives a linear system in a, b, and c:

$$\begin{aligned} a \qquad + 3c &= 0 \\ -a + b - 3c &= 0. \end{aligned}$$

Row-reducing the associated augmented matrix gives

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right],$$

so that $b = 0$ and $a = -3c$. Thus the equation of the plane is $-3cx + cz = 0$. Clearly we must have $c \neq 0$, since otherwise we do not get a plane. So dividing through by c gives $-3x + z = 0$, or $3x - z = 0$, for the general equation of the plane.

18. Note that

$$\mathbf{u} = 1\mathbf{u} + 0\mathbf{v} + 0\mathbf{w}, \quad \mathbf{v} = 0\mathbf{u} + 1\mathbf{v} + 0\mathbf{w}, \quad \mathbf{w} = 0\mathbf{u} + 0\mathbf{v} + 1\mathbf{w}.$$

Or, if one adopts the usual convention of not writing terms with coefficients of zero, $\mathbf{u}$ is in the span because $\mathbf{u} = \mathbf{u}$.

19. Note that

$$\begin{aligned} \mathbf{u} &= \mathbf{u}, \\ \mathbf{v} &= (\mathbf{u} + \mathbf{v}) - \mathbf{u}, \\ \mathbf{w} &= (\mathbf{u} + \mathbf{v} + \mathbf{w}) - (\mathbf{u} + \mathbf{v}), \end{aligned}$$

and the terms in parentheses on the right, together with $\mathbf{u}$, are the elements of the given set.

20. **(a)** To prove the statement, we must show that anything that can be written as a linear combination of the elements of S can also be written as a linear combination of the elements of T. But if $\mathbf{x}$ is a linear combination of elements of S, then

$$\begin{aligned} \mathbf{x} &= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k \\ &= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k + 0\mathbf{u}_{k+1} + 0\mathbf{u}_{k+1} + \cdots + 0\mathbf{u}_m. \end{aligned}$$

This shows that $\mathbf{x}$ is also a linear combination of elements of T, proving the result.

(b) We know that $\text{span}(S) \subseteq \text{span}(T)$ from part (a). Since T is a set of vectors in $\mathbb{R}^n$, we know that $\text{span}(T) \subseteq \mathbb{R}^n$. But then

$$\mathbb{R}^n = \text{span}(S) \subseteq \text{span}(T) \subseteq \mathbb{R}^n,$$

so that $\text{span}(T) = \mathbb{R}^n$.

21. **(a)** Suppose that

$$\mathbf{u}_i = u_{i1}\mathbf{v}_1 + u_{i2}\mathbf{v}_2 + \cdots + u_{im}\mathbf{v}_m, \quad i = 1,\ 2,\ \ldots,\ k.$$

Suppose that $\mathbf{w}$ is a linear combination of the $\mathbf{u}_i$. Then

$$\begin{aligned}
\mathbf{w} &= w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + \cdots + w_k\mathbf{u}_k \\
&= w_1(u_{11}\mathbf{v}_1 + u_{12}\mathbf{v}_2 + \cdots + u_{1m}\mathbf{v}_m) + w_2(u_{21}\mathbf{v}_1 + u_{22}\mathbf{v}_2 + \cdots + u_{2m}\mathbf{v}_m) + \cdots \\
&\qquad w_k(u_{k1}\mathbf{v}_1 + u_{k2}\mathbf{v}_2 + \cdots + u_{km}\mathbf{v}_m) \\
&= (w_1u_{11} + w_2u_{21} + \cdots + w_ku_{k1})\mathbf{v}_1 + (w_1u_{12} + w_2u_{22} + \cdots + w_ku_{k2})\mathbf{v}_2 + \cdots \\
&\qquad (w_1u_{1m} + w_2u_{2m} + \cdots + w_ku_{km})\mathbf{v}_m \\
&= w_1'\mathbf{v}_1 + w_2'\mathbf{v}_2 + \cdots + w_m'\mathbf{v}_m.
\end{aligned}$$

This shows that $\mathbf{w}$ is a linear combination of the $\mathbf{v}_j$. Thus $\operatorname{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k) \subseteq \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

(b) Reversing the roles of the $\mathbf{u}$'s and the $\mathbf{v}$'s in part (a) gives $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) \subseteq \operatorname{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$. Combining that with part (a) shows that each of the spans is contained in the other, so they are equal.

(c) Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We want to show that each $\mathbf{u}_i$ is a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, and the reverse. The first of these is obvious: since $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ span $\mathbb{R}^3$, any vector is $\mathbb{R}^3$ is a linear combination of those, so in particular $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are. Next, note that

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{u}_1 \\
\mathbf{e}_2 &= \mathbf{u}_2 - \mathbf{u}_1 \\
\mathbf{e}_3 &= \mathbf{u}_3 - \mathbf{u}_2,
\end{aligned}$$

so that the $\mathbf{e}_j$ are linear combinations of the $\mathbf{u}_i$. Part (b) then implies that $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3$.

22. The vectors $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ are not scalar multiples of one another, so are linearly independent.

23. There is no obvious dependence relation, so follow Example 2.23. We want scalars c_1, c_2, c_3 so that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row-reduce the associated augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \xrightarrow{\frac{1}{5}R_3} \cdots$$

$$\cdots \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2+2R_3]{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus the system has only the solution $c_1 = c_2 = c_3 = 0$, so the vectors are linearly independent.

24. There is no obvious dependence relation, so follow Example 2.23. We want scalars c_1, c_2, c_3 so that

$$c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row-reduce the associated augmented matrix:

$$\left[\begin{array}{rrr|r} 2 & 3 & 1 & 0 \\ 2 & 1 & -5 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_3} \\ \xrightarrow{R_1} \\ \xrightarrow{R_2} \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 2 & 0 \\ 2 & 3 & 1 & 0 \\ 2 & 1 & -5 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-2R_1} \\ \xrightarrow{R_3-2R_1} \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 2 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -3 & -9 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{rrr|r} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -3 & -9 & 0 \end{array}\right] \xrightarrow{R_3+3R_2} \left[\begin{array}{rrr|r} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{rrr|r} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the system has a nontrivial solution, and the vectors are linearly dependent. From the reduced matrix, the general solution is $c_1 = 4t$, $c_2 = -3t$, $c_3 = t$. Setting $t = 1$, for example, we get the relation

$$4 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

25. Since by inspection

$$\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0},$$

the vectors are linearly dependent.

26. Since we are given 4 vectors in $\mathbb{R}^3$, we know they are linearly dependent vectors by Theorem 2.8. To find a dependence relation, we want to find scalars c_1, c_2, c_3, c_4 such that

$$c_1 \begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + c_4 \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row-reduce the associated augmented matrix:

$$\left[\begin{array}{rrrr|r} -2 & 4 & 3 & 5 & 0 \\ 3 & -1 & 1 & 0 & 0 \\ 7 & 5 & 3 & 2 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{rrrr|r} 1 & -2 & -\frac{3}{2} & -\frac{5}{2} & 0 \\ 3 & -1 & 1 & 0 & 0 \\ 7 & 5 & 3 & 2 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-3R_1} \\ \xrightarrow{R_3-7R_1} \end{array} \left[\begin{array}{rrrr|r} 1 & -2 & -\frac{3}{2} & -\frac{5}{2} & 0 \\ 0 & 5 & \frac{11}{2} & \frac{15}{2} & 0 \\ 0 & 19 & \frac{27}{2} & \frac{39}{2} & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{rrrr|r} 1 & -2 & -\frac{3}{2} & -\frac{5}{2} & 0 \\ 0 & 1 & \frac{11}{10} & \frac{3}{2} & 0 \\ 0 & 19 & \frac{27}{2} & \frac{39}{2} & 0 \end{array}\right] \xrightarrow{R_3-19R_2} \left[\begin{array}{rrrr|r} 1 & -2 & -\frac{3}{2} & -\frac{5}{2} & 0 \\ 0 & 1 & \frac{11}{10} & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{37}{5} & -9 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{5}{37}R_3} \left[\begin{array}{rrrr|r} 1 & -2 & -\frac{3}{2} & -\frac{5}{2} & 0 \\ 0 & 1 & \frac{11}{10} & \frac{3}{2} & 0 \\ 0 & 0 & 1 & \frac{45}{37} & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_1+\frac{3}{2}R_3} \\ \xrightarrow{R_2-\frac{11}{10}R_3} \end{array} \left[\begin{array}{rrrr|r} 1 & -2 & 0 & -\frac{25}{37} & 0 \\ 0 & 1 & 0 & \frac{6}{37} & 0 \\ 0 & 0 & 1 & \frac{45}{37} & 0 \end{array}\right]$$

$$\xrightarrow{R_1+2R_2} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{13}{37} & 0 \\ 0 & 1 & 0 & \frac{6}{37} & 0 \\ 0 & 0 & 1 & \frac{45}{37} & 0 \end{array}\right]$$

Thus the system has a nontrivial solution, and the vectors are linearly dependent. From the reduced matrix, the general solution is $c_1 = \frac{13}{37}t$, $c_2 = -\frac{6}{37}t$, $c_3 = -\frac{45}{37}t$, $c_4 = t$. Setting $t = 37$, for example, we get the relation

$$13\begin{bmatrix}-2\\3\\7\end{bmatrix} - 6\begin{bmatrix}4\\-1\\5\end{bmatrix} - 45\begin{bmatrix}3\\1\\3\end{bmatrix} + 37\begin{bmatrix}5\\0\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

27. Since $\mathbf{v}_3 = \begin{bmatrix}0\\0\\0\end{bmatrix}$ is the zero vector, these vectors are linearly dependent. For example,

$$0\cdot\begin{bmatrix}3\\4\\5\end{bmatrix} + 0\cdot\begin{bmatrix}6\\7\\8\end{bmatrix} + 137\begin{bmatrix}0\\0\\0\end{bmatrix} = \mathbf{0}.$$

Any set of vectors containing the zero vector is linearly dependent.

28. There is no obvious dependence relation, so follow Example 2.23. We want scalars c_1, c_2, c_3 so that

$$c_1\begin{bmatrix}-1\\1\\2\\1\end{bmatrix} + c_2\begin{bmatrix}3\\2\\2\\4\end{bmatrix} + c_3\begin{bmatrix}2\\3\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Row-reduce the associated augmented matrix:

$$\left[\begin{array}{rrr|r}-1&3&2&0\\1&2&3&0\\2&2&1&0\\1&4&-1&0\end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{rrr|r}1&-3&-2&0\\1&2&3&0\\2&2&1&0\\1&4&-1&0\end{array}\right] \begin{array}{c}\xrightarrow{R_2-R_1}\\ \xrightarrow{R_3-2R_2}\\ \xrightarrow{R_4-R_1}\end{array} \left[\begin{array}{rrr|r}1&-3&-2&0\\0&5&5&0\\0&6&5&0\\0&7&1&0\end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{rrr|r}1&-3&-2&0\\0&1&1&0\\0&6&5&0\\0&7&1&0\end{array}\right]$$

$$\begin{array}{c}\xrightarrow{R_3-6R_2}\\ \xrightarrow{R_4-7R_2}\end{array} \left[\begin{array}{rrr|r}1&-3&-2&0\\0&1&1&0\\0&0&-1&0\\0&0&-6&0\end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{rrr|r}1&-3&-2&0\\0&1&1&0\\0&0&1&0\\0&0&-6&0\end{array}\right] \xrightarrow{R_4+6R_3} \left[\begin{array}{rrr|r}1&-3&-2&0\\0&1&1&0\\0&0&1&0\\0&0&0&0\end{array}\right]$$

$$\begin{array}{c}\xrightarrow{R_1+2R_3}\\ \xrightarrow{R_2-R_3}\end{array} \left[\begin{array}{rrr|r}1&-3&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right] \xrightarrow{R_1+3R_2} \left[\begin{array}{rrr|r}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]$$

Since the system has only the trivial solution, the given vectors are linearly independent.

29. There is no obvious dependence relation, so follow Example 2.23. We want scalars c_1, c_2, c_3, c_4 so that

$$c_1\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}-1\\1\\0\\1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\\-1\end{bmatrix} + c_4\begin{bmatrix}0\\1\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Row-reduce the associated augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_3-R_1]{R_2+R_1} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4-R_2} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{R_4+R_3} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_4} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3-R_4]{R_2+R_4} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1+R_2-R_3} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Since the system has only the trivial solution, the given vectors are linearly independent.

30. The vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

are linearly dependent. For suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = 0$. Then c_4 must be zero in order for the first component of the sum to be zero. But then the only vector containing a nonzero entry in the second component is $\mathbf{v}_3$, so that $c_3 = 0$ as well. Now for the same reason $c_2 = 0$, looking at the third component. Finally, that implies that $c_1 = 0$. So the vectors are linearly independent.

31. By inspection

$$\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0},$$

so the vectors are linearly dependent.

32. Construct a matrix A with these vectors as its rows and try to produce a zero row:

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1'=R_1+R_2} \begin{bmatrix} 1 & 1 & 6 \\ -1 & 2 & 3 \end{bmatrix} \xrightarrow{R_2'=R_2+2R_1'} \begin{bmatrix} 1 & 1 & 6 \\ 0 & 3 & 9 \end{bmatrix}$$

This matrix is in row echelon form and has no zero rows, so the matrix has rank 2 and the vectors are linearly independent by Theorem 2.7.

33. Construct a matrix A with these vectors as its rows and try to produce a zero row:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow[R_3'=R_3-R_1]{R_2'=R_2-R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_3''=R_3'+2R_2'} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

This matrix is in row echelon form and has no zero rows, so the matrix has rank 3 and the vectors are linearly independent by Theorem 2.7.

34. Construct a matrix A with these vectors as its rows and try to produce a zero row:

$$\begin{bmatrix}2&2&1\\3&1&2\\1&-5&2\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_3}\begin{bmatrix}1&-5&2\\3&1&2\\2&2&1\end{bmatrix}\xrightarrow[R_3'=R_3-2R_1]{R_2'=R_2-3R_1}\begin{bmatrix}1&-5&2\\0&16&-4\\0&12&-3\end{bmatrix}\xrightarrow{R_3''=R_3'-\frac{3}{4}R_2'}\begin{bmatrix}1&-5&2\\0&12&-3\\0&0&0\end{bmatrix}$$

Since there is a zero row, the matrix has rank less than 3, so the vectors are linearly dependent by Theorem 2.7. Working backwards, we have

$$\begin{aligned}[0,0,0]=R_3''=R_3'-\frac{3}{4}R_2'=R_3-2R_1-\frac{3}{4}(R_2-3R_1)=\frac{1}{4}R_1-\frac{3}{4}R_2+R_3\\ =\frac{1}{4}\begin{bmatrix}1\\-5\\2\end{bmatrix}-\frac{3}{4}\begin{bmatrix}3\\1\\2\end{bmatrix}+\begin{bmatrix}2\\2\\1\end{bmatrix}.\end{aligned}$$

35. Construct a matrix A with these vectors as its rows and try to produce a zero row:

$$\begin{bmatrix}0&1&2\\2&1&3\\2&0&1\end{bmatrix}\xrightarrow[R_2'=R_1]{R_1'=R_2}\begin{bmatrix}2&1&3\\0&1&2\\2&0&1\end{bmatrix}\xrightarrow{R_3'=R_3-R_1'}\begin{bmatrix}2&1&3\\0&1&2\\0&-1&-2\end{bmatrix}\xrightarrow{R_3''=R_3'+R_2'}\begin{bmatrix}2&1&3\\0&1&1\\0&0&0\end{bmatrix}$$

This matrix is in row echelon form and has a zero row, so the vectors are linearly dependent by Theorem 2.7. Working backwards, we have

$$[0,0,0]=R_3''=R_3'+R_2'=R_3-R_1'+R_2'=R_3-R_2+R_1=R_1-R_2+R_3=\begin{bmatrix}0\\1\\2\end{bmatrix}-\begin{bmatrix}2\\1\\3\end{bmatrix}+\begin{bmatrix}2\\0\\1\end{bmatrix}.$$

36. Construct a matrix A with these vectors and try to produce a zero row:

$$\begin{bmatrix}-2&3&7\\4&-1&5\\3&1&3\\5&0&2\end{bmatrix}\xrightarrow[\substack{R_3'=R_3+\frac{3}{2}R_1\\ R_4'=R_4+\frac{5}{2}R_1}]{R_2'=R_2+2R_1}\begin{bmatrix}-2&3&7\\0&5&19\\0&\frac{11}{2}&\frac{27}{2}\\0&\frac{15}{2}&\frac{39}{2}\end{bmatrix}\xrightarrow[R_4''=2R_4']{R_3''=2R_3'}\begin{bmatrix}-2&3&7\\0&5&19\\0&11&27\\0&15&39\end{bmatrix}\cdots$$

$$\cdots\xrightarrow[R_4'''=R_4''-3R_2']{R_3'''=R_3''-\frac{11}{5}R_2'}\begin{bmatrix}-2&3&7\\0&5&19\\0&0&-\frac{74}{5}\\0&0&-18\end{bmatrix}\xrightarrow{R_4''''=R_4'''-\frac{5\cdot 18}{74}R_3'''}\begin{bmatrix}-2&3&7\\0&5&19\\0&0&-\frac{74}{5}\\0&0&0\end{bmatrix}$$

Since this matrix has a zero row, it has rank less than 2, so the vectors are linearly dependent. Working backwards, we have

$$\begin{aligned}[0,0,0]=R_4''''&=R_4'''-\frac{90}{74}R_3'''\\&=R_4''-3R_2'-\frac{45}{37}\left(R_3''-\frac{11}{5}R_2'\right)\\&=2R_4'-3R_2'-\frac{90}{37}R_3'+\frac{99}{37}R_2'\\&=2\left(R_4+\frac{5}{2}R_1\right)-3(R_2+2R_1)-\frac{90}{37}\left(R_3+\frac{3}{2}R_1\right)+\frac{99}{37}(R_2+2R_1)\\&=\frac{26}{37}R_1-\frac{12}{37}R_2-\frac{90}{37}R_3+2R_4.\end{aligned}$$

Clearing fractions gives

$$[0,0,0] = 26R_1 - 12R_2 - 90R_3 + 74R_4.$$

Thus

$$26\begin{bmatrix}-2\\3\\7\end{bmatrix} - 12\begin{bmatrix}4\\-1\\5\end{bmatrix} - 90\begin{bmatrix}3\\1\\3\end{bmatrix} + 74\begin{bmatrix}5\\0\\2\end{bmatrix} = \mathbf{0}.$$

Note that instead of doing all this work, we could just have applied Theorem 2.8.

37. Construct a matrix A with these vectors and try to produce a zero row:

$$\begin{bmatrix}3&4&5\\6&7&8\\0&0&0\end{bmatrix}$$

This matrix already has a zero row, so we know by Theorem 2.7 that the vectors are linearly dependent, and we have an obvious dependence relation: let the coefficients of each of the first two vectors be zero, and the coefficient of the zero vector be any nonzero number.

38. Construct a matrix A with these vectors and try to produce a zero row:

$$\begin{bmatrix}-1&1&2&1\\3&2&2&4\\2&3&1&-1\end{bmatrix} \xrightarrow[R_3'=R_3+2R_1]{R_2'=R_2+3R_1} \begin{bmatrix}-1&1&2&1\\0&5&8&7\\0&5&5&1\end{bmatrix} \xrightarrow{R_3''=R_3-R_2} \begin{bmatrix}-1&1&2&1\\0&5&8&7\\0&0&-3&-6\end{bmatrix}.$$

This matrix is in row echelon form, and it has no zero row, so it has rank 4 and thus the vectors are linearly independent by Theorem 2.7.

39. Construct a matrix A with these vectors and try to produce a zero row:

$$\begin{bmatrix}1&-1&1&0\\-1&1&0&1\\1&0&1&-1\\0&1&-1&1\end{bmatrix} \xrightarrow[R_3'=R_3-R_1]{R_2'=R_2+R_1} \begin{bmatrix}1&-1&1&0\\0&0&1&1\\0&1&0&-1\\0&1&-1&1\end{bmatrix} \xrightarrow{R_4'=R_4-R_3'}$$

$$\begin{bmatrix}1&-1&1&0\\0&0&1&1\\0&1&0&-1\\0&0&-1&2\end{bmatrix} \xrightarrow[R_3''=R_2']{R_2''=R_3'} \begin{bmatrix}1&-1&1&0\\0&1&0&-1\\0&0&1&1\\0&0&-1&2\end{bmatrix} \xrightarrow{R_4''=R_4'+R_3'} \begin{bmatrix}1&-1&1&0\\0&1&0&-1\\0&0&1&1\\0&0&0&3\end{bmatrix}.$$

This matrix is in row echelon form, and it has no zero row, so it has rank 4 and thus the vectors are linearly independent by Theorem 2.7.

40. Construct a matrix A with these vectors and try to produce a zero row:

$$\begin{bmatrix}0&0&0&1\\0&0&2&1\\0&3&2&1\\4&3&2&1\end{bmatrix} \xrightarrow{\substack{R_2'=R_2-R_1\\R_3'=R_3-R_1\\R_4'=R_4-R_1}} \begin{bmatrix}0&0&0&1\\0&0&2&0\\0&3&2&0\\4&3&2&0\end{bmatrix} \xrightarrow[R_4''=R_4'-R_2']{R_3''=R_3'-R_2'} \begin{bmatrix}0&0&0&1\\0&0&2&0\\0&3&0&0\\4&3&0&0\end{bmatrix} \xrightarrow{R_4'''=R_4''-R_3''}$$

$$\begin{bmatrix}0&0&0&1\\0&0&2&0\\0&3&0&0\\4&0&0&0\end{bmatrix} \xrightarrow{\substack{R_1'=R_4'''\\R_2''=R_3''\\R_3'''=R_2'\\R_4''''=R_1}} \begin{bmatrix}4&0&0&0\\0&3&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix}.$$

This matrix is in row echelon form, and it has no zero row, so it has rank 4 and thus the vectors are linearly independent by Theorem 2.7.

41. Construct a matrix A with these vectors and try to produce a zero row:

$$\begin{bmatrix} 3 & -1 & 1 & -1 \\ -1 & 3 & 1 & -1 \\ 1 & 1 & 3 & 1 \\ -1 & -1 & 1 & 3 \end{bmatrix} \xrightarrow[R_3'=R_1]{R_1'=R_3} \begin{bmatrix} 1 & 1 & 3 & 1 \\ -1 & 3 & 1 & -1 \\ 3 & -1 & 1 & -1 \\ -1 & -1 & 1 & 3 \end{bmatrix} \begin{array}{c} \xrightarrow{R_2'=R_2+R_1'} \\ \xrightarrow{R_3''=R_3'-3R_1'} \\ \xrightarrow{R_4'=R_4+R_1'} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 4 & 4 & 0 \\ 0 & -4 & -8 & -4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \xrightarrow{R_3'''=R_3''+R_2'+R_4'} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}.$$

Since this matrix has a zero row, it has rank less than 4, so the vectors are linearly dependent. Working backwards, we have

$$\begin{aligned} [0,0,0,0] = R_3''' &= R_3'' + R_2' + R_4' \\ &= R_3' - 3R_1' + R_2 + R_1' + R_4 + R_1' \\ &= R_3' - R_1' + R_2 + R_4 \\ &= R_1 - R_3 + R_2 + R_4. \end{aligned}$$

Thus

$$\begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0}.$$

42. **(a)** In this case, $\operatorname{rank}(A) = n$. To see this, recall that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if the only solution of $[A \mid 0] = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n \mid 0]$ is the trivial solution. Since the trivial solution is always a solution, it is the only one if and only if the number of free variables in the associated system is zero. Then the Rank Theorem (Theorem 2.2 in Section 2.2) says that the number of free variables is $n - \operatorname{rank}(A)$, so that $n = \operatorname{rank}(A)$.

(b) Let

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{bmatrix}.$$

Then by Theorem 2.7, the vectors, which are the rows of A, are linearly independent if and only if the rank of A is greater than or equal to n. But A has n rows, so its rank is at most n. Thus, the rows of A are linearly independent if and only if the rank of A is exactly n.

43. **(a)** Yes, they are. Suppose that $c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{v}+\mathbf{w}) + c_3(\mathbf{u}+\mathbf{w}) = 0$. Expanding gives

$$c_1\mathbf{u} + c_1\mathbf{v} + c_2\mathbf{v} + c_2\mathbf{w} + c_3\mathbf{u} + c_3\mathbf{w} = (c_1+c_3)\mathbf{u} + (c_1+c_2)\mathbf{v} + (c_2+c_3)\mathbf{w} = 0.$$

Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, all three of these coefficients are zero. So to see what values of c_1, c_2, and c_3 satisfy this equation, we must solve the linear system

$$\begin{aligned} c_1 \phantom{{}+c_2} + c_3 &= 0 \\ c_1 + c_2 \phantom{{}+c_3} &= 0 \\ c_2 + c_3 &= 0. \end{aligned}$$

Row-reducing the coefficient matrix gives

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Since the rank of the reduced matrix is 3, the only solution is the trivial solution, so $c_1 = c_2 = c_3 = 0$. Thus the original vectors $\mathbf{u}+\mathbf{v}$, $\mathbf{v}+\mathbf{w}$, and $\mathbf{u}+\mathbf{w}$ are linearly independent.

(b) No, they are not. Suppose that $c_1(\mathbf{u}-\mathbf{v}) + c_2(\mathbf{v}-\mathbf{w}) + c_3(\mathbf{u}-\mathbf{w}) = 0$. Expanding gives

$$c_1\mathbf{u} - c_1\mathbf{v} + c_2\mathbf{v} - c_2\mathbf{w} + c_3\mathbf{u} - c_3\mathbf{w} = (c_1+c_3)\mathbf{u} + (-c_1+c_2)\mathbf{v} + (-c_2-c_3)\mathbf{w} = 0.$$

Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, all three of these coefficients are zero. So to see what values of c_1, c_2, and c_3 satisfy this equation, we must solve the linear system

$$\begin{aligned} c_1 \qquad + c_3 &= 0 \\ -c_1 + c_2 \qquad &= 0 \\ - c_2 - c_3 &= 0. \end{aligned}$$

Row-reducing the coefficient matrix gives

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So c_3 is a free variable, and a solution to the original system is $c_1 = c_2 = -c_3$. Taking $c_3 = -1$, for example, we have

$$(\mathbf{u}-\mathbf{v}) + (\mathbf{v}-\mathbf{w}) - (\mathbf{u}-\mathbf{w}) = \mathbf{0}.$$

44. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the two vectors. Suppose first that at least one of them, say $\mathbf{v}_1$, is zero. Then they are linearly dependent since $\mathbf{v}_1 + 0\mathbf{v}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$; further, $\mathbf{v}_1 = 0\mathbf{v}_2$, so one is a scalar multiple of the other.

Now suppose that neither vector is the zero vector. We want to show that one is a multiple of the other if and only if they are linearly dependent. If one is a multiple of the other, say $\mathbf{v}_1 = k\mathbf{v}_2$, then clearly $1\mathbf{v}_1 - k\mathbf{v}_2 = \mathbf{0}$, so that they are linearly dependent. To go the other way, suppose they are linearly dependent. Then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$, where c_1 and c_2 are not both zero. Suppose that $c_1 \neq 0$ (if instead $c_2 \neq 0$, reverse the roles of the first and second subscripts). Then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0 \quad \Rightarrow \quad c_1\mathbf{v}_1 = -c_2\mathbf{v}_2 \quad \Rightarrow \quad \mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2,$$

so that $\mathbf{v}_1$ is a multiple of $\mathbf{v}_2$, as was to be shown.

45. Suppose we have m row vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in $\mathbb{R}^n$ with $m > n$. Let

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}.$$

By Theorem 2.7, the rows (the $\mathbf{v}_i$) are linearly dependent if and only if $\operatorname{rank}(A) < m$. By definition, $\operatorname{rank}(A)$ is the number of nonzero rows in its row echelon form. But since there can be at most one leading 1 in each column, the number of nonzero rows is at most $n < m$. Thus $\operatorname{rank}(A) \le n < m$, so that by Theorem 2.7, the vectors are linearly dependent.

46. Suppose that $A = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a set of linearly independent vectors, and that B is a subset of A, say $B = \{\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \dots, \mathbf{v}_{i_k}\}$ where $1 \le i_1 < i_1 < \cdots < i_k \le n$ for $j = 1, 2, \dots, k$. We want to show that B is a linearly independent set. Suppose that

$$c_{i_1}\mathbf{v}_{i_1} + c_{i_2}\mathbf{v}_{i_2} + \cdots + c_{i_k}\mathbf{v}_{i_k} = 0.$$

Set $c_r = 0$ if r is not in $\{i_1, i_2, \dots, i_k\}$, and consider

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

This sum is zero since the sum of the $i_1, i_2, \dots, i_k$ elements is zero, and the other coefficients are zero. But A forms a linearly independent set, so all the coefficients must in fact be zero. Thus the linear combination of the $\mathbf{v}_{i_j}$ above must be the trivial linear combination, so B is a linearly independent set.

47. Since $S' = \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset S = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}\}$, Exercise 20(a) proves that $\operatorname{span}(S') \subseteq \operatorname{span}(S)$. Using Exercise 21(a), to show that $\operatorname{span}(S) \subseteq \operatorname{span}(S')$, we must show that each element of S is a linear combination of elements of S'. But clearly each $\mathbf{v}_i$ is a linear combination of elements of S', since $\mathbf{v}_i \in S'$. Also, we are given that $\mathbf{v}$ is linear combination of the $\mathbf{v}_i$, so it too is a linear combination of elements of S'. Thus $\operatorname{span}(S) \subseteq \operatorname{span}(S')$. Since we have proven both inclusions, the two sets are equal.

48. Suppose $b\mathbf{v} + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k = 0$. Substitute for $\mathbf{v}$ from the given equation, to get

$$b(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k = bc_1\mathbf{v}_1 + (bc_2 + b_2)\mathbf{v}_2 + \cdots + (bc_k + b_k)\mathbf{v}_k = 0.$$

Since the $\mathbf{v}_i$ are a linearly independent set, all of the coefficients must be zero:

$$bc_1 = 0, \qquad bc_2 + b_2 = 0, \qquad bc_3 + b_3 = 0, \qquad \dots \qquad bc_k + b_k = 0.$$

In particular, $bc_1 = 0$; since we are given that $c_1 \neq 0$, it follows that $b = 0$. Then substituting 0 for b in the remaining equations above

$$b_2 = 0, \qquad b_3 = 0, \qquad \dots \qquad b_k = 0.$$

Thus b and all of the b_k are zero, so the original linear combination was the trivial one. It follows that $\{\mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly independent set.

2.4 Applications

1. Let x_1, x_2, and x_3 be the number of bacteria of strains I, II, and III, respectively. Then from the consumption of A, B, and C, we get the following system:

$$\begin{aligned} x_1 + 2x_2 \phantom{{}+ 2x_3} &= 400 \\ 2x_1 + x_2 + x_3 &= 600 \\ x_1 + x_2 + 2x_3 &= 600 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 400 \\ 2 & 1 & 1 & 600 \\ 1 & 1 & 2 & 600 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 160 \\ 0 & 1 & 0 & 120 \\ 0 & 0 & 1 & 160 \end{array}\right].$$

So 160 bacteria of strain I, 120 bacteria of strain II, and 160 bacteria of strain III can coexist.

2. Let x_1, x_2, and x_3 be the number of bacteria of strains I, II, and III, respectively. Then from the consumption of A, B, and C, we get the following system:

$$\begin{aligned} x_1 + 2x_2 \phantom{{}+ 3x_3} &= 400 \\ 2x_1 + x_2 + 3x_3 &= 500 \\ x_1 + x_2 + x_3 &= 600 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 400 \\ 2 & 1 & 3 & 500 \\ 1 & 1 & 1 & 600 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

This is an inconsistent system, so the bacteria cannot coexist.

3. Let x_1, x_2, and x_3 be the number of small, medium, and large arrangements. Then from the consumption of flowers in each arrangement we get:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 24 \\ 3x_1 + 4x_2 + 8x_3 &= 50 \\ 3x_1 + 6x_2 + 6x_3 &= 48 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 24 \\ 3 & 4 & 8 & 50 \\ 3 & 6 & 6 & 48 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

So 2 small, 3 medium, and 4 large arrangements were sold that day.

4. **(a)** Let x_1, x_2, and x_3 be the number of nickels, dimes, and quarters. Then we get

$$\begin{aligned} x_1 + \quad x_2 + \quad x_3 &= 20 \\ 2x_1 - \quad x_2 \qquad\quad &= 0 \\ 5x_1 + 10x_2 + 25x_3 &= 300 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 2 & -1 & 0 & 0 \\ 5 & 10 & 25 & 300 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 8 \end{array}\right].$$

So there are 4 nickels, 8 dimes, and 8 quarters.

(b) This is similar to part (a), except that the constraint $2x_1 - x_2$ no longer applies, so we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 5 & 10 & 25 & 300 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & -20 \\ 0 & 1 & 4 & 40 \end{array}\right].$$

The general solution is $x_1 = -20 + 3t$, $x_2 = 40 - 4t$, and $x_3 = t$. From the first equation, we must have $3t - 20 \geq 0$, so that $t \geq 7$. From the second, we have $t \leq 10$. So the possible values for t are 7, 8, 9, and 10, which give the following combinations of nickels, dimes, and quarters:

$$[x_1, x_2, x_3] = [1, 12, 7] \text{ or } [4, 8, 8] \text{ or } [7, 4, 9] \text{ or } [10, 0, 10].$$

5. Let x_1, x_2, and x_3 be the number of house, special, and gourmet blends. Then from the consumption of beans in each blend we get

$$\begin{aligned} 300x_1 + 200x_2 + 100x_3 &= 30{,}000 \\ 200x_2 + 200x_3 &= 15{,}000 \\ 200x_1 + 100x_2 + 200x_3 &= 25{,}000\,. \end{aligned}$$

Forming the augmented matrix and reducing it gives

$$\left[\begin{array}{ccc|c} 300 & 200 & 100 & 30000 \\ 0 & 200 & 200 & 15000 \\ 200 & 100 & 200 & 25000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 65 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 45 \end{array}\right].$$

The merchant should make 65 house blend, 30 special blend, and 45 gourmet blend.

6. Let x_1, x_2, and x_3 be the number of house, special, and gourmet blends. Then from the consumption of beans in each blend we get

$$\begin{aligned} 300x_1 + 200x_2 + 100x_3 &= 30{,}000 \\ 50x_1 \quad 200x_2 + 350x_3 &= 15{,}000 \\ 150x_1 + 100x_2 + \ 50x_3 &= 15{,}000\,. \end{aligned}$$

Forming the augmented matrix and reducing it gives

$$\left[\begin{array}{ccc|c} 300 & 200 & 100 & 30000 \\ 50 & 200 & 350 & 15000 \\ 150 & 100 & 50 & 15000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 60 \\ 0 & 1 & 2 & 60 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $x_1 = 60 + t$, $x_2 = 60 - 2t$, and $x_3 = t$. From the first equation, we get $t \geq -60$; from the second, we get $t \leq 30$, and the third gives us $t \geq 0$. So the range of possible values is $0 \leq t \leq 30$. We wish to maximize the profit

$$P = \frac{1}{2}x_1 + \frac{3}{2}x_2 + 2x_3 = \frac{1}{2}(60 + t) + \frac{3}{2}(60 - t) + 2t = 120 - \frac{1}{2}t.$$

Since $0 \leq t \leq 30$, the profit is maximized if $t = 0$, in which case $x_1 = x + 2 = 60$ and $x_3 = 0$. Therefore the merchant should make 60 house and special blends, and no gourmet blends.

7. Let x, y, z, and w be the number of FeS_2, O_2, Fe_2O_3, and SO_2 molecules respectively. Then compare the number of iron, sulfur, and oxygen atoms in reactants and products:

$$\begin{aligned} &\text{Iron:} && x = 2z \\ &\text{Sulfur:} && 2x = w \\ &\text{Oxygen:} && 2y = 3z + 2w, \end{aligned}$$

so we get the matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -3 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{11}{8} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & 0 \end{array}\right].$$

Thus $z = \frac{1}{4}w$, $y = \frac{11}{8}w$, and $x = \frac{1}{2}w$. The smallest positive value of w that will produce integer values for all four variables is 8, so letting $w = 8$ gives $x = 4$, $y = 11$, $z = 2$, and $w = 8$. The balanced equation is $4FeS_2 + 11O_2 \longrightarrow 2Fe_2O_3 + 8SO_2$.

8. Let x, y, z, and w be the number of CO_2, H_2O, $C_6H_{12}O_6$, and O_2 molecules respectively. The compare the number of carbon, oxygen, and hydrogen atoms in reactants and products:

$$\begin{aligned} &\text{Carbon:} && x = 6z \\ &\text{Oxygen:} && 2x + y = 6z + 2w \\ &\text{Hydrogen:} && 2y = 12z. \end{aligned}$$

This system is easy to solve using back-substitution: the first equation gives $x = 6z$ and the third gives $y = 6z$. Substituting $6z$ for both x and y in the second equation gives $2 \cdot 6z + 6z = 6z + 2w$, so that $w = 6z$ as well. The smallest positive value of z giving integers is $z = 1$, so that $x = y = w = 6$ and $z = 1$. The balanced equation is $6CO_2 + 6H_2O \longrightarrow C_6H_{12}O_6 + 6O_2$.

9. Let x, y, z, and w be the number of C_4H_{10}, O_2, CO_2, and H_2O molecules respectively. The compare the number of carbon, oxygen, and hydrogen atoms in reactants and products:

$$\begin{aligned} &\text{Carbon:} && 4x = z \\ &\text{Oxygen:} && 2y = 2z + w \\ &\text{Hydrogen:} && 10x = 2w. \end{aligned}$$

This system is easy to solve using back-substitution: the first equation gives $z = 4x$ and the third gives $w = 5x$. Substituting these values for z and w in the second equation gives $2y = 2 \cdot 4x + 5x = 13x$, so that $y = \frac{13}{2}x$. The smallest positive value of x giving integers for all four variables is $x = 2$, resulting in $x = 2$, $y = 13$, $z = 8$, and $w = 10$. The balanced equation is $2C_4H_{10} + 13O_2 \longrightarrow 8CO_2 + 10H_2O$.

10. Let x, y, z, and w be the number of $C_7H_6O_2$, O_2, H_2O, and CO_2 molecules respectively. The compare the number of carbon, oxygen, and hydrogen atoms in reactants and products:

$$\begin{aligned} &\text{Carbon:} && 7x = w \\ &\text{Oxygen:} && 2x + 2y = z + 2w \\ &\text{Hydrogen:} && 6x = 2z. \end{aligned}$$

This system is easy to solve using back-substitution: the first equation gives $w = 7x$ and the third gives $z = 3x$. Substituting these values for w and z in the second equation gives $2x + 2y = 3x + 2 \cdot 7x$, so that $y = \frac{15}{2}x$. The smallest positive value for x giving integer values for all four variables is $x = 2$, so that the solution is $x = 2$, $y = 15$, $z = 6$, and $w = 14$. The balanced equation is $2C_7H_6O_2 + 15O_2 \longrightarrow 6H_2O + 14CO_2$.

11. Let x, y, z, and w be the number of $C_5H_{11}OH$, O_2, H_2O, and CO_2 molecules respectively. The compare the number of carbon, oxygen, and hydrogen atoms in reactants and products:

$$\begin{array}{ll} \text{Carbon:} & 5x = w \\ \text{Oxygen:} & x + 2y = z + 2w \\ \text{Hydrogen:} & 12x = 2z. \end{array}$$

This system is easy to solve using back-substitution: the first equation gives $w = 5x$ and the third gives $z = 6x$. Substituting these values for w and z in the second equation gives $x + 2y = 6x + 2 \cdot 5x$, so that $y = \frac{15}{2}x$. The smallest positive value for x giving integer values for all four variables is $x = 2$, so that the solution is $x = 2$, $y = 15$, $z = 12$, and $w = 10$. The balanced equation is $2C_5H_{11}OH+15O_2 \longrightarrow 12H_2O+10CO_2$.

12. Let x, y, z, and w be the number of $HClO_4$, P_4O_{10}, H_3PO_4, and Cl_2O_7 molecules respectively. The compare the number of hydrogen, chlorine, oxygen, and phosphorus atoms in reactants and products:

$$\begin{array}{ll} \text{Hydrogen:} & x = 3z \\ \text{Chlorine:} & x = 2w \\ \text{Oxygen:} & 4x + 10y = 4z + 7w \\ \text{Phosphorus:} & 4y = z. \end{array}$$

This gives the matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 4 & 10 & -4 & -7 & 0 \\ 0 & 4 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the general solution is $x = 2t$, $y = \frac{1}{6}t$, $z = \frac{2}{3}t$, $w = t$. The smallest positive value of t giving all integer answers is $t = 6$, so the desired solution is $x = 12$, $y = 1$, $z = 4$, and $w = 6$. The balanced equation is $12HClO_4+P_4O_{10} \longrightarrow 4H_3PO_4 + 6Cl_2O_7$.

13. Let x_1, x_2, x_3, x_4, and x_5 be the number of Na_2CO_3, C, N_2, NaCN, and CO molecules respectively. The compare the number of sodium, carbon, oxygen, and nitrogen atoms in reactants and products:

$$\begin{array}{ll} \text{Sodium:} & 2x_1 = x_4 \\ \text{Carbon:} & x_1 + x_2 = x_4 + x_5 \\ \text{Oxygen:} & 3x_1 = x_5 \\ \text{Nitrogen:} & 2x_3 = x_4. \end{array}$$

This gives the matrix

$$\left[\begin{array}{ccccc|c} 2 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array}\right]$$

Thus x_5 is a free variable, and

$$x_1 = \frac{1}{3}x_5, \quad x_2 = \frac{4}{3}x_5, \quad x_3 = \frac{1}{3}x_5, \quad x_4 = \frac{2}{3}x_5.$$

The smallest value of x_5 that produces integer solutions is 3, so the desired solution is $x_1 = 1$, $x_2 = 4$, $x_3 = 1$, $x_4 = 2$, and $x_5 = 3$. The balanced equation is $Na_2CO_3 + 4C+N_2 \longrightarrow 2NaCN+3CO$.

14. Let x_1, x_2, x_3, x_4, and x_5 be the number of $C_2H_2Cl_4$, $Ca(OH)_2$, C_2HCl_3, $CaCl_2$, and H_2O molecules respectively. The compare the number of carbon, hydrogen, chlorine, calcium, and oxygen atoms in reactants and products:

$$\begin{array}{lll} \text{Carbon:} & 2x_1 & = \quad 2x_3 \\ \text{Hydrogen:} & 2x_1 + 2x_2 & = \quad x_3 + 2x_5 \\ \text{Chlorine:} & 4x_1 & = 3x_3 + 2x_4 \\ \text{Calcium:} & x_2 & = \quad x_4 \\ \text{Oxygen:} & 2x_2 & = \quad x_5. \end{array}$$

This gives the matrix

$$\left[\begin{array}{ccccc|c} 2 & 0 & -2 & 0 & 0 & 0 \\ 2 & 2 & -1 & 0 & -2 & 0 \\ 4 & 0 & -3 & -2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus x_5 is a free variable, and

$$x_1 = x_5, \quad x_2 = \frac{1}{2}x_5, \quad x_3 = x_5, \quad x_4 = \frac{1}{2}x_5.$$

The smallest value of x_5 that produces integer solutions is 2, so the desired solution is $x_1 = 2$, $x_2 = 1$, $x_3 = 2$, $x_4 = 1$, and $x_5 = 2$. The balanced equation is $2C_2H_2Cl_4 + Ca(OH)_2 \longrightarrow 2C_2HCl_3 + CaCl_2 + 2H_2O$.

15. **(a)** By applying the conservation of flow rule to each node we obtain the system of equations

$$f_1 + f_2 = 20 \qquad f_2 - f_3 = -10 \qquad f_1 + f_3 = 30.$$

Form the augmented matrix and row-reduce it:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 20 \\ 0 & 1 & -1 & -10 \\ 1 & 0 & 1 & 30 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 30 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Letting $f_3 = t$, the possible flows are $f_1 = 30 - t$, $f_2 = t - 10$, $f_3 = t$.

(b) The flow through AB is f_2, so in this case $5 = t - 10$ and $t = 15$. So the other flows are $f_1 = f_3 = 15$.

(c) Since each flow must be nonnegative, the first equation gives $t \leq 30$, the second gives $t \geq 10$, and the third gives $t \geq 0$. So we have $10 \leq t \leq 30$, which means that

$$0 \leq f_1 \leq 20 \quad 0 \leq f_2 \leq 20 \quad 10 \leq f_3 \leq 30.$$

(d) If a negative flow were allowed, it would mean a flow in the opposite direction, so that a negative flow on f_2 would mean a flow from B into A.

16. **(a)** By applying the conservation of flow rule to each node we obtain the system of equations

$$f_1 + f_2 = 20 \qquad f_1 + f_3 = 25 \qquad f_2 + f_4 = 25 \qquad f_3 + f_4 = 30.$$

Form the augmented matrix and row-reduce it:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 20 \\ 1 & 0 & 1 & 0 & 25 \\ 0 & 1 & 0 & 1 & 25 \\ 0 & 0 & 1 & 1 & 30 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -5 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Letting $f_4 = t$, the possible flows are $f_1 = t - 5$, $f_2 = 25 - t$, $f_3 = 30 - t$, $f_4 = t$.

(b) If $f_4 = t = 10$, then the average flows on the other streets will be $f_1 = 10-5 = 5$, $f_2 = 25-10 = 15$, and $f_3 = 30 - 10 = 20$.

(c) Each flow must be nonnegative, so we get the constraints $t \geq 5$, $t \leq 25$, $t \leq 30$, and $t \geq 0$. Thus $5 \leq t \leq 25$. This gives the constraints

$$0 \leq f_1 \leq 20 \qquad 0 \leq f_2 \leq 20 \qquad 5 \leq f_3 \leq 25 \qquad 5 \leq f_4 \leq 25.$$

(d) Reversing all of the directions would have no effect on the solution. This would mean multiplying all rows of the augmented matrix by -1. However, multiplying a row by -1 is an elementary row operation, so the new matrix would have the same reduced row echelon form and thus the same solution.

17. **(a)** By applying the conservation of flow rule to each node we obtain the system of equations

$$f_1 + f_2 = 100 \qquad f_2 + f_3 = f_4 + 150 \qquad f_4 + f_5 = 150 \qquad f_1 + 200 = f_3 + f_5.$$

Rearrange the equations, set up the augmented matrix, and row reduce it:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 100 \\ 0 & 1 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 1 & 0 & -1 & 0 & -1 & -200 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & -200 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 0 & 0 & 0 & \end{array}\right]$$

Then both $f_3 = s$ and $f_5 = t$ are free variables, so the flows are

$$f_1 = -200 + s + t, \quad f_2 = 300 - s - t, \quad f_3 = s, \quad f_4 = 150 - t, \quad f_5 = t.$$

(b) If DC is closed, then $f_5 = t = 0$, so that $f_1 = -200 + s$ and $f_2 = 300 - s$. Thus s, which is the flow through DB, must satisfy $200 \leq s \leq 300$ liters per day.

(c) If $DB = f_3 = S$ were closed, then $f_1 = -200 + t$, so that $t = f_5 \geq 200$. But that would mean that $f_4 = 150 - t \leq -50 < 0$, which is impossible.

(d) Since $f_4 = 150 - t$, we must have $0 \leq t \leq 150$, so that $0 \leq f_4 \leq 150$. Note that all of these values for t are consistent with the first three equations as well: if $t = 0$, then $s = 250$ produces positive values for all flows, while if $t = 150$, then $s = 75$ produces all positive values. So the flow through DB is between 0 and 150 liters per day.

18. **(a)** By applying the conservation of flow rule to each node we obtain:

$$\begin{aligned} f_3 + 200 &= f_1 + 100 & f_1 + 150 &= f_2 + f_4 & f_2 + f_5 &= 300 \\ f_6 + 100 &= f_3 + 200 & f_4 + f_7 &= f_6 + 100 & f_5 + f_7 &= 250. \end{aligned}$$

Create and row-reduce the augmented matrix for the system:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 100 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -150 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \end{array}\right] \to \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 50 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $f_7 = t$ and $f_6 = s$, so the flows are

$$f_1 = s \quad f_2 = 50 + t \quad f_3 = -100 + s \quad f_4 = 100 + s - t \quad f_5 = 250 - t \quad f_6 = s \quad f_7 = t.$$

(b) From the solution in part (a), since $f_1 = s = f_6$, it is not possible for f_1 and f_6 to have different values.

(c) If $f_4 = 0$, then $s - t = 100$, so that $t = 100 + s$. Substituting this for t everywhere gives

$$f_1 = s \quad f_2 = 150 + s \quad f_3 = -100 + s \quad f_4 = 0 \quad f_5 = 150 - s \quad f_6 = s \quad f_7 = 100 + s.$$

The constraints imposed by the requirement that all the f_i be nonnegative means that $100 \le s \le 150$, so that

$$\begin{gathered} 100 \le f_1 \le 150 \quad 250 \le f_2 \le 300 \quad 0 \le f_3 \le 50 \quad f_4 = 0 \\ 50 \le f_5 \le 100 \quad 100 \le f_6 \le 150 \quad 200 \le f_7 \le 250. \end{gathered}$$

19. Applying the current law to node A gives $I_1 + I_3 = I_2$ or $I_1 - I_2 + I_3 = 0$. Applying the voltage law to the top circuit, $ABCA$, gives $-I_2 - I_1 + 8 = 0$. Similarly, for the circuit $ABDA$ we obtain $-I_2 + 13 - 4I_3 = 0$. This gives the system

$$\begin{aligned} I_1 - I_2 + \ \ I_3 &= \ \ 0 \\ I_1 + I_2 \qquad\ &= \ \ 8 \\ I_2 + 4I_3 &= 13 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 8 \\ 0 & 1 & 4 & 13 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Thus $I_1 = 3$ amps, $I_2 = 5$ amps, and $I_3 = 2$ amps.

20. Applying the current law to node B and the voltage law to the upper and lower circuits gives

$$\begin{aligned} I_1 - \ \ I_2 + \ \ I_3 &= 0 \\ I_1 + 2I_2 \qquad\ &= 5 \\ 2I_2 + 4I_3 &= 8 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 5 \\ 0 & 2 & 4 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus $I_1 = 1$ amps, $I_2 = 2$ amps, and $I_3 = 1$ amp.

21. **(a)** Applying the current and voltage laws to the circuit gives the system:

$$\begin{aligned} \text{Node } B&: & I &= I_1 + I_4 \\ \text{Node } C&: & I_1 &= I_2 + I_3 \\ \text{Node } D&: & I_2 + \ \ I_5 &= \ I \\ \text{Node } E&: & I_3 + \ \ I_4 &= I_5 \\ \text{Circuit } ABEDA&: & -2I_4 - I_5 + \ 14 &= \ 0 \\ \text{Circuit } BCEB&: & -I_1 - I_3 + 2I_4 &= \ 0 \\ \text{Circuit } CDEC&: & -2I_2 + I_3 + \ \ I_5 &= \ 0 \end{aligned}$$

Construct the augmented matrix and row-reduce it:

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the currents are $I = 10$, $I_1 = 6$, $I_2 = 4$, $I_3 = 2$, $I_4 = 4$, and $I_5 = 6$ amps.

(b) From Ohm's Law, the effective resistance is $R_{\text{eff}} = \frac{V}{I} = \frac{14}{10} = \frac{7}{5}$ ohms.

(c) To force the current in CE to be zero, we force $I_3 = 0$ and let r be the resistance in BC. This gives the following system of equations:

$$\begin{aligned}
\text{Node } B&: & I &= I_1 + I_4\\
\text{Node } C&: & I_1 &= I_2\\
\text{Node } D&: & I_2 + I_5 &= I\\
\text{Node } E&: & I_4 &= I_5\\
\text{Circuit } ABEDA&: & -2I_4 - I_5 + 14 &= 0\\
\text{Circuit } BCEB&: & -rI_1 \qquad + 2I_4 &= 0\\
\text{Circuit } CDEC&: & -2I_2 \qquad + I_5 &= 0
\end{aligned}$$

It is easier to solve this system by substitution. Substitute $I_4 = I_5$ in the circuit equation for $ABEDA$ to get $-3I_5 + 14 = 0$, so that $I_4 = I_5 = \frac{14}{3}$. Thus $-2I_2 + \frac{14}{3} = 0$ so that $I_1 = I_2 = \frac{7}{3}$. Then the circuit equation for $BCED$ gives us

$$-r \cdot \frac{7}{3} + 2 \cdot \frac{14}{3} = 0 \quad \Rightarrow r = 4.$$

So if we use a 4 ohm resistor in BC, the current in CE will be zero.

22. **(a)** Applying the voltage law to Figure 2.23(a) gives $IR_1 + IR_2 = E$. But from Ohm's Law, $E = IR_{\text{eff}}$, so that $I(R_1 + R_2) = IR_{\text{eff}}$ and thus $R_{\text{eff}} = R_1 + R_2$.

(b) Applying the current and voltage laws to Figure 2.23(b) gives the system of equations $I = I_1 + I_2$, $-I_1R_1 + I_2R_2 = 0$, $-I_1R_1 + E = 0$, and $-I_2R_2 + E = 0$. Gauss-Jordan elimination gives

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0\\ 0 & -R_1 & R_2 & 0\\ 0 & -R_1 & 0 & -E\\ 0 & 0 & -R_2 & -E \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{(R_1+R_2)E}{R_1R_2}\\ 0 & 1 & 0 & \frac{E}{R_1}\\ 0 & 0 & 1 & \frac{E}{R_2}\\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $I = \frac{(R_1+R_2)E}{R_1R_2}$, $I_1 = \frac{E}{R_1}$, $I_2 = \frac{E}{R_2}$. But $E = IR_{\text{eff}}$; substituting this value of E into the equation for I gives

$$I = \frac{(R_1 + R_2)IR_{\text{eff}}}{R_1R_2} \quad \Rightarrow \quad R_{\text{eff}} = \frac{1}{\frac{R_1+R_2}{R_1R_2}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}.$$

23. The matrix for this system, with inputs on the left, outputs along the top, is

	Farming	Manufacturing
Farming	$\frac{1}{2}$	$\frac{1}{3}$
Manufacturing	$\frac{1}{2}$	$\frac{2}{3}$

Let the output of the farm sector be f and the output of the manufacturing sector be m. Then

$$\begin{aligned}
\frac{1}{2}f + \frac{1}{3}m &= f\\
\frac{1}{2}f + \frac{2}{3}m &= m.
\end{aligned}$$

Simplifying and row-reducing the corresponding augmented matrix gives

$$\begin{aligned} -\tfrac{1}{2}f + \tfrac{1}{3}m &= 0\\ \tfrac{1}{2}f - \tfrac{1}{3}m &= 0 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{cc|c} -\frac{1}{2} & \frac{1}{3} & 0\\ \frac{1}{2} & -\frac{1}{3} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 0\\ 0 & 0 & 0 \end{array}\right]$$

Thus $f = \frac{2}{3}m$, so that farming and manufacturing must be in a $2:3$ ratio.

24. The matrix for this system, with needed resources on the left, outputs along the top, is

	Coal	Steel
Coal	0.3	0.8
Steel	0.7	0.2

Let the coal output be c and the steel output be s. Then

$$\begin{aligned} 0.3c + 0.8s &= c \\ 0.7c + 0.2s &= s. \end{aligned}$$

Simplifying and row-reducing the corresponding augmented matrix gives

$$\begin{aligned} -0.7c + 0.8s &= 0 \\ 0.7c - 0.8s &= 0 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{rr|r} -0.7 & 0.8 & 0 \\ 0.7 & -0.8 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -\frac{8}{7} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $c = \frac{8}{7}s$, so that coal and steel must be in an $8 : 7$ ratio. For a total output of \$20 million, coal output must be $\frac{8}{8+7} \cdot 20 = \frac{32}{3} \approx \$10.667$ million, while steel output is $\frac{7}{8+7} \cdot 20 = \frac{28}{3} \approx \$9.333$ million.

25. Let x be the painter's rate, y the plumber's rate, and z the electrician's rate. From the given schedule, we get the linear system

$$\begin{aligned} 2x + y + 5z &= 10x \\ 4x + 5y + z &= 10y \\ 4x + 4y + 4z &= 10z. \end{aligned}$$

Simplifying and row-reducing the corresponding augmented matrix gives

$$\begin{aligned} -8x + y + 5z &= 0 \\ 4x - 5y + z &= 0 \\ 4x + 4y - 6z &= 0 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} -8 & 1 & 5 & 0 \\ 4 & -5 & 1 & 0 \\ 4 & 4 & -6 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{13}{18} & 0 \\ 0 & 1 & -\frac{7}{9} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $x = \frac{13}{18}z$ and $y = \frac{7}{9}z$. Let $z = 54$; this is a whole number in the required range such that the others will be whole numbers as well. Then $x = 39$ and $y = 42$. The painter should charge \$39 per hour, the plumber \$42 per hour, and the electrician \$54 per hour.

26. From the given matrix, we get the linear system

$$\begin{aligned} \frac{1}{4}L + \frac{1}{8}T + \frac{1}{6}Z &= B \\ \frac{1}{2}B + \frac{1}{4}L + \frac{1}{4}T + \frac{1}{6}Z &= L \\ \frac{1}{4}B + \frac{1}{4}L + \frac{1}{2}T + \frac{1}{3}Z &= T \\ \frac{1}{4}B + \frac{1}{4}L + \frac{1}{8}T + \frac{1}{3}Z &= Z. \end{aligned}$$

Simplify and row-reduce the associated augmented matrix:

$$\begin{aligned} -B + \tfrac{1}{4}L + \tfrac{1}{8}T + \tfrac{1}{6}Z &= 0 \\ \tfrac{1}{2}B - \tfrac{3}{4}L + \tfrac{1}{4}T + \tfrac{1}{6}Z &= 0 \\ \tfrac{1}{4}B + \tfrac{1}{2}L - \tfrac{1}{2}T + \tfrac{1}{3}Z &= 0 \\ \tfrac{1}{4}B + \tfrac{1}{8}L + \tfrac{1}{8}T - \tfrac{2}{3}Z &= 0 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{rrrr|r} -1 & \frac{1}{4} & \frac{1}{8} & \frac{1}{6} & 0 \\ \frac{1}{2} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & -\frac{2}{3} & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & -\frac{6}{5} & 0 \\ 0 & 0 & 1 & -\frac{8}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $B = \frac{2}{3}Z$, $L = \frac{6}{5}Z$, and $T = \frac{8}{5}Z$. Furthermore, B is the smallest price of the four, so $B = \frac{2}{3}Z$ must be at least 50, in which case $Z = 75$. With that value of Z, we get $B = 50$, $L = 90$, and $T = 120$.

27. The matrix for this system, with needed resources on the left, outputs along the top, is

	Coal	Steel
Coal	0.15	0.2
Steel	0.1	0.1

(a) Let the coal output be c and the steel output be s. Then since 45 million of external demand is required for coal and 124 million for steel, we have

$$\begin{aligned} c &= 0.15c + 0.25s + 45 \\ s &= 0.2c + 0.1s + 124. \end{aligned}$$

Simplify and row-reduce the associated matrix:

$$\begin{array}{rcrcr} 0.85c & - & 0.25s & = & 45 \\ -0.2c & + & 0.9s & + & 124 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rr|r} 0.85 & -0.25 & 45 \\ -0.2 & 0.9 & 124 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 0 & 100 \\ 0 & 1 & 160 \end{array}\right]$$

So the coal industry should produce \$100 million and the steel industry should produce \$160 million.

(b) The new external demands for coal and steel are 40 and 130 million respectively. The only thing that changes in the equations is the constant terms, so we have

$$\begin{array}{rcrcr} 0.85c & - & 0.25s & = & 40 \\ -0.2c & + & 0.9s & + & 130 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rr|r} 0.85 & -0.25 & 40 \\ -0.2 & 0.9 & 130 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 0 & 95.8 \\ 0 & 1 & 165.73 \end{array}\right]$$

So the coal industry should reduce production by $100-95.8 = \$4.2$ million, while the steel industry should increase production by $165.73 - 160 = \$5.73$ million.

28. This is an open system with external demand given by the other city departments. From the given matrix, we get the system

$$\begin{aligned} 1 + 0.2A + 0.1H + 0.2T &= A \\ 1.2 + 0.1A + 0.1H + 0.2T &= H \\ 0.8 + 0.2A + 0.4H + 0.3T &= T. \end{aligned}$$

Simplifying and row-reducing the associated augmented matrix gives

$$\begin{array}{rcrcrcr} 0.8A & - & 0.1H & - & 0.2T & = & 1 \\ -0.1A & + & 0.9H & - & 0.2T & = & 1.2 \\ -0.2A & - & 0.4H & + & 0.7T & = & 0.8 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 0.8 & -0.1 & -0.2 & 1 \\ -0.1 & 0.9 & -0.2 & 1.2 \\ -0.2 & -0.4 & 0.7 & 0.8 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2.31 \\ 0 & 1 & 0 & 2.28 \\ 0 & 0 & 1 & 3.11 \end{array}\right].$$

So the Administrative department should produce \$2.31 million of services, the Health department \$2.28 million, and the Transportation department \$3.11 million.

29. **(a)** Over $\mathbb{Z}_2$, we need to solve $x_1\mathbf{a} + x_2\mathbf{b} + \cdots + x_5\mathbf{e} = \mathbf{t} + \mathbf{s}$:

$$\mathbf{s} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \Rightarrow \quad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence x_5 is a free variable, so there are exactly two solutions (with $x_5 = 0$ and with $x_5 = 1$). Solving for the other variables, we get $x_1 = 1 + x_5$, $x_2 = 1 + x_5$, $x_3 = 1$, and $x_4 = x_5$. So the two solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

So we can push switches 1, 2, and 3 or switches 3, 4, and 5.

(b) In this case, $\mathbf{t} = \mathbf{e}_2$ over $\mathbb{Z}_2$, so the matrix becomes

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

This is an inconsistent system, so there is no solution. Thus it is impossible to start with all the lights off and turn on only the second light.

30. (a) With $\mathbf{s} = \mathbf{e}_4$ and $\mathbf{t} = \mathbf{e}_2 + \mathbf{e}_4$ over $\mathbb{Z}_2$, so that $\mathbf{s} + \mathbf{t} = \mathbf{e}_2$ and we get the matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Since the system is inconsistent, there is no solution in this case: it is impossible to start with only the fourth light on and end up with only the second and fourth lights on.

(b) In this case $\mathbf{t} = \mathbf{e}_2$ over $\mathbb{Z}_2$, so that $\mathbf{s} + \mathbf{t} = \mathbf{e}_2 + \mathbf{e}_4$, and we get

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So there are two solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

So we can push switches 1, 2, and 3 or switches 3, 4, and 5.

31. Recall from Example 2.35 that it does not matter what order we push the switches in, and that pushing a switch twice is the same as not pushing it at all. So with

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

there are $2^5 = 32$ possible choices, since there are 5 buttons and we have the choice of pushing or not pushing each one. If we make a list of the resulting 32 resulting light states and eliminate duplicates, we see that the list of possible states is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

32. **(a)** Over $\mathbb{Z}_3$, we need to solve $x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} = \mathbf{t}$, where

$$\mathbf{t} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \text{so that} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Thus $x_1 = 1$, $x_2 = 2$, and $x_3 = 2$, so we must push switch A once and switches B and C twice each.

(b) Over $\mathbb{Z}_3$, we need to solve $x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} = \mathbf{t}$, where

$$\mathbf{t} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{so that} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

In other words, we must push each switch twice.

(c) Let x, y, and z be the final states of the three lights. Then we want to solve

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 1 & 1 & 1 & y \\ 0 & 1 & 1 & z \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & y + 2z \\ 0 & 1 & 0 & x + 2y + z \\ 0 & 0 & 1 & 2x + y \end{array}\right].$$

So we can achieve this configuration if we push switch A $y + 2z$ times, switch B $x + 2y + z$ times, and switch C $2x + y$ times (adding in $\mathbb{Z}_3$).

33. The switches here correspond to the same vectors as in Example 2.35, but now the numbers are interpreted in $\mathbb{Z}_3$ rather than in $\mathbb{Z}_2$. So in $\mathbb{Z}_3^5$, with $\mathbf{s} = 0$ (all lights initially off), we need to solve $x_1\mathbf{a} + x_2\mathbf{b} + \cdots + x_5\mathbf{e} = \mathbf{t}$, where

$$\mathbf{t} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \text{so that} \quad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So x_5 is a free variable, so there are three solutions, corresponding to $x_5 = 0$, 1, or 2. Solving for the other variables gives

$$x_1 = 1 - x_5, \quad x_2 = 1 - 2x_5, \quad x_3 = 2, \quad x_4 = 2 - x_5,$$

so that the solutions are (remembering that $1 - 2 = 2$ and $2 \cdot 2 = 1$ in $\mathbb{Z}_3$)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}.$$

34. The possible configurations are

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} = \begin{bmatrix} s_1 + s_2 \\ s_1 + s_2 + s_3 \\ s_2 + s_3 + s_4 \\ s_3 + s_4 + s_5 \\ s_4 + s_5 \end{bmatrix}$$

where $s_i \in \{0, 1, 2\}$ is the number of times that switch i is thrown.

35. (a) Let a value of 1 represent white, and 0 represent black. Let $\mathbf{a}_i$ correspond to touching the number i; it has a 1 in a given entry if the value of that square changes when i is pressed, and a 0 if it does not (thus the 1's correspond to the squares with the blue dots in them in the diagram). Since each square has two states, white and black, we need to solve, over $\mathbb{Z}_2$, the equation

$$\mathbf{s} + x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_9\mathbf{a}_9 = \mathbf{t}.$$

Here

$$\mathbf{s} = \begin{bmatrix} 0\\1\\1\\1\\0\\1\\1\\1\\0 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \quad \text{so that} \quad x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_9\mathbf{a}_9 = \mathbf{0} - \mathbf{s} = \mathbf{s}.$$

Row-reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccccccccc|c} 1&1&0&1&0&0&0&0&0&0\\ 1&1&1&0&1&0&0&0&0&1\\ 0&1&1&0&0&1&0&0&0&1\\ 1&0&0&1&1&0&1&0&0&1\\ 1&0&1&0&1&0&1&0&1&0\\ 0&0&1&0&1&1&0&0&1&1\\ 0&0&0&1&0&0&1&1&0&1\\ 0&0&0&0&1&0&1&1&1&1\\ 0&0&0&0&0&1&0&1&1&0 \end{array}\right] \to \left[\begin{array}{ccccccccc|c} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0 \end{array}\right].$$

So touching the third and seventh squares will turn all the squares black.

(b) In part (a), note that the row-reduced matrix consists of the identity matrix and a column vector corresponding to the constants. Changing the constants cannot produce an inconsistent system, since the matrix will still reduce to the identity matrix. So any configuration is reachable, since we can solve $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_9\mathbf{a}_9 = \mathbf{s}$ for any $\mathbf{s}$.

36. In this version of the puzzle, the squares have three states, so we work over $\mathbb{Z}_3$, with 2 corresponding to black, 1 to gray, and 0 to white. Then we need to go from

$$\mathbf{s} = \begin{bmatrix} 2\\2\\0\\1\\1\\0\\2\\0\\1 \end{bmatrix} \quad \text{to} \quad \mathbf{t} = \begin{bmatrix} 2\\2\\2\\2\\2\\2\\2\\2\\2 \end{bmatrix}.$$

Thus we want to solve

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_9\mathbf{a}_9 = \mathbf{t} - \mathbf{s} = \begin{bmatrix} 0\\0\\2\\1\\1\\2\\0\\2\\1 \end{bmatrix}.$$

Row-reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccccccccc|c} 1&1&0&1&0&0&0&0&0&0\\ 1&1&1&0&1&0&0&0&0&0\\ 0&1&1&0&0&1&0&0&0&2\\ 1&0&0&1&1&0&1&0&0&1\\ 1&0&1&0&1&0&1&0&1&1\\ 0&0&1&0&1&1&0&0&1&2\\ 0&0&0&1&0&0&1&1&0&0\\ 0&0&0&0&1&0&1&1&1&2\\ 0&0&0&0&0&1&0&1&1&1 \end{array}\right] \to \left[\begin{array}{ccccccccc|c} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&2\\ 0&0&0&0&1&0&0&0&0&2\\ 0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&1&2 \end{array}\right],$$

showing that a solution exists.

37. Let Grace and Hans' ages be g and h. Then $g = 3h$ and $g+5 = 2(h+5)$, or $g = 2(h+5)-5 = 2h+5$. It follows that $g = 3h = 2h+5$, so that $h = 5$ and $g = 15$. So Hans is 5 and Grace is 15.

38. Let the ages be a, b, and c. Then the first two conditions give us $a+b+c = 60$ and $a-b = b-c$. For the third condition, Bert will be as old as Annie is now in $a-b$ years; at that time, Annie will be $a+(a-b) = 2a-b$ years, and we are given that $2a-b = 3c$. So we get the system

$$\begin{aligned} a + b + c &= 60 \\ a - 2b + c &= 0 \\ 2a - b - 3c &= 0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 31 & 1 & 1 & 60 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 31 & 0 & 0 & 28 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 12 \end{bmatrix}.$$

Annie is 28, Bert is 20, and Chris is 12.

39. Let the areas of the fields be a and b. We have the two equations

$$\begin{aligned} a + \phantom{\frac{1}{2}}b &= 1800 \\ \frac{2}{3}a + \frac{1}{2}b &= 1100. \end{aligned}$$

Multiply the first equation by two and subtract the first equation from the result to get $\frac{1}{3}a = 400$, so that $a = 1200$ square yards and thus $b = 1800 - a = 600$ square yards.

40. Let x_1, x_2, and x_3 be the number of bundles of the first, second, and third types of corn. Then from the given conditions we get

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 39 \\ 2x_1 + 3x_2 + x_3 &= 34 \\ x_1 + 2x_2 + 3x_3 &= 26. \end{aligned}$$

Row-reduce the augmented matrix to get

$$\left[\begin{array}{ccc|c} 3&2&1&39\\ 2&3&1&34\\ 1&2&3&26 \end{array}\right] \to \left[\begin{array}{ccc|c} 1&0&0&\frac{37}{4}\\ 0&1&0&\frac{17}{4}\\ 0&0&1&\frac{11}{4} \end{array}\right].$$

Therefore there are 9.25 measures of corn in a bundle of the first type, 4.25 measures of corn in a bundle of the second type, and 2.75 measures of corn in a bundle of the third type.

41. **(a)** The addition table gives $a+c = 2$, $a+d = 4$, $b+c = 3$, $b+d = 5$. Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{cccc|c} 1&0&1&0&2\\ 1&0&0&1&4\\ 0&1&1&0&3\\ 0&1&0&1&5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1&0&0&1&4\\ 0&1&0&1&5\\ 0&0&1&-1&-2\\ 0&0&0&0&0 \end{array}\right].$$

So d is a free variable. Solving for the other variables in terms of d gives $a = 4 - d$, $b = 5 - d$, and $c = d - 2$. So there are an infinite number of solutions; for example, with $d = 1$, we get $a = 3$, $b = 4$, and $c = -1$.

(b) We have $a + c = 3$, $a + d = 4$, $b + c = 6$, $b + d = 5$. Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

So this is an inconsistent system, and there are no such values of a, b, c, and d.

42. The addition table gives $a + c = w$, $a + d = y$, $b + c = x$, $b + d = z$. Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & w \\ 1 & 0 & 0 & 1 & y \\ 0 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 1 & z \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & w \\ 0 & 1 & 0 & 1 & x \\ 0 & 0 & 1 & -1 & y - w \\ 0 & 0 & 0 & 0 & z - x - y + w \end{array}\right].$$

In order for this to be a consistent system, we must have $z - x - y + w = 0$, or $z - y = x - w$. Another way of phrasing this is $x + y = z + w$; in other words, the sum of the diagonal entries must equal the sum of the off-diagonal entries.

43. (a) The addition table gives the following system of equations:

$$\begin{array}{lll} a + d = 3 & b + d = 2 & c + d = 1 \\ a + e = 5 & b + e = 4 & c + e = 3 \\ a + f = 4 & b + f = 3 & c + f = 1. \end{array}$$

Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 1 & 0 & 5 \\ 1 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The sixth row shows that this is an inconsistent system.

(b) The addition table gives the following system of equations:

$$\begin{array}{lll} a + d = 1 & b + d = 2 & c + d = 3 \\ a + e = 3 & b + e = 4 & c + e = 5 \\ a + f = 4 & b + f = 5 & c + f = 6. \end{array}$$

Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 & 6 \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So f is a free variable, and there are an infinite number of solutions of the form $a = 4 - f$, $b = 5 - f$, $c = 6$, $d = f - 3$, $e = f - 1$.

44. Generalize the addition table to

$$\begin{array}{c|ccc} + & a & b & c \\ \hline d & m & n & o \\ e & p & q & r \\ f & s & t & u \end{array}$$

The addition table gives the following system of equations:

$$\begin{array}{ccc} a + d = m & b + d = n & c + d = o \\ a + e = p & b + e = q & c + e = r \\ a + f = s & b + f = t & c + f = u. \end{array}$$

Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & m \\ 1 & 0 & 0 & 0 & 1 & 0 & p \\ 1 & 0 & 0 & 0 & 0 & 1 & s \\ 0 & 1 & 0 & 1 & 0 & 0 & n \\ 0 & 1 & 0 & 0 & 1 & 0 & q \\ 0 & 1 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 1 & 1 & 0 & 0 & o \\ 0 & 0 & 1 & 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 & 0 & 1 & u \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 1 & m \\ 0 & 1 & 0 & 0 & 0 & 1 & n \\ 0 & 0 & 1 & 0 & 0 & 1 & o \\ 0 & 0 & 0 & 1 & 0 & -1 & m - p \\ 0 & 0 & 0 & 0 & 1 & -1 & n + p - m - t \\ 0 & 0 & 0 & 0 & 0 & 0 & m - n - p + q \\ 0 & 0 & 0 & 0 & 0 & 0 & n - o - q + r \\ 0 & 0 & 0 & 0 & 0 & 0 & p - q - s + t \\ 0 & 0 & 0 & 0 & 0 & 0 & q - r - t + u \end{array}\right].$$

For the table to be valid, the last entry in the zero rows must also be zero, so we need the conditions $m + q = n + p$, $n + r = o + q$, $p + t = q + s$, and $q + u = r + t$. In terms of the addition table, this means that for each 2×2 submatrix, the sum of the diagonal entries must equal the sum of the off-diagonal entries (compare with Exercise 42).

45. **(a)** Since the three points $(0, 1)$, $(-1, 4)$, and $(2, 1)$ must satisfy the equation $y = ax^2 + bx + c$, substitute those values to get the three equations $c = 1$, $a - b + c = 4$, and $4a + 2b + c = 1$. Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 4 \\ 4 & 2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus $a = 1$, $b = -2$, and $c = 1$, so that the equation of the parabola is $y = x^2 - 2x + 1$.

(b) Since the three points $(-3, 1)$, $(-2, 2)$, and $(-1, 5)$ must satisfy the equation $y = ax^2 + bx + c$, substitute those values to get the three equations $9a - 3b + c = 1$, $4a - 2b + c = 2$, and $a - b + c = 5$. Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 9 & -3 & 1 & 1 \\ 4 & -2 & 1 & 2 \\ 1 & 1 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 10 \end{array}\right]$$

Thus $a = 1$, $b = 6$, and $c = 10$, so that the equation of the parabola is $y = x^2 + 6x + 10$.

46. **(a)** Since the three points $(0, 1)$, $(-1, 4)$, and $(2, 1)$ must satisfy the equation $x^2 + y^2 + ax + by + c = 0$, substitute those values to get the three equations

$$\begin{array}{rcl} 1 + b + c = 0 & & b + c = -1 \\ 1 + 16 - a + 4b + c = 0 & \Rightarrow & a - 4b - c = 17 \\ 4 + 1 + 2a + b + c = 0 & & 2a + b + c = -5\,. \end{array}$$

Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & -1 \\ 1 & -4 & -1 & 17 \\ 2 & 1 & 1 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Thus $a = -2$, $b = -6$, and $c = 5$, so that the equation of the circle is $x^2 + y^2 - 2x - 6y + 5 = 0$. Completing the square gives $(x-1)^2 + (y-3)^2 + 5 - 1 - 9 = 0$, so that $(x-1)^2 + (y-3)^2 = 5$. This is a circle whose center is $(1, 3)$, with radius $\sqrt{5}$. A sketch of the circle follows part (b).

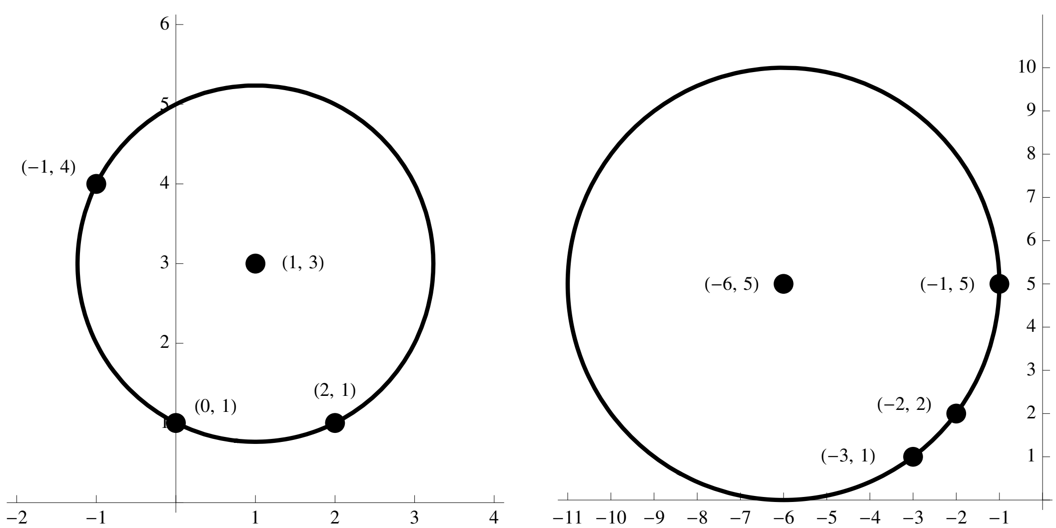
(b) Since the three points $(-3, 1)$, $(-2, 2)$, and $(-1, 5)$ must satisfy the equation $x^2+y^2+ax+by+c = 0$, substitute those values to get the three equations

$$\begin{array}{rcl} 9 + 1 - 3a + b + c = 0 & & 3a - b - c = 10 \\ 4 + 4 - 2a + 2b + c = 0 & \Rightarrow & 2a - 2b - c = 8 \\ 1 + 25 - a + 5b + c = 0 & & a - 5b - c = 26\,. \end{array}$$

Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 10 \\ 2 & -2 & -1 & 8 \\ 1 & -5 & -1 & 26 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 36 \end{array}\right]$$

Thus $a = 12$, $b = -10$, and $c = 36$, so that the equation of the circle is $x^2+y^2+12x-10y+36 = 0$. Completing the square gives $(x+6)^2+(y-5)^2+36-36-25 = 0$, so that $(x+6)^2+(y-5)^2 = 25$. This is a circle whose center is $(-6, 5)$, with radius 5. Sketches of both circles are:



47. We have $\frac{3x+1}{x^2+2x-3} = \frac{3x+1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$, so that clearing fractions, we get $(x+3)A + (x-1)B = 3x+1$. Collecting terms gives $(A+B)x + (3A-B) = 3x+1$. This gives the linear system $A+B = 3$, $3A - B = 1$. Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right].$$

Thus $A = 1$ and $B = 2$, so that $\frac{3x+1}{x^2+2x-3} = \frac{1}{x-1} + \frac{2}{x+3}$.

48. We have

$$\begin{aligned}
\frac{x^2-3x+3}{x^3+2x^2+x} = \frac{x^3-3x+3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \quad &\Rightarrow \\
x^2-3x+3 = A(x+1)^2 + Bx(x+1) + Cx \quad &\Rightarrow \\
x^2-3x+3 = (A+B)x^2 + (2A+B+C)x + A \quad &\Rightarrow \\
A+B=1, \quad 2A+B+C=-3, \quad A=3.&
\end{aligned}$$

This is most easily solved by substitution: $A = 3$, so that $A + B = 1$ gives $B = -2$. Substituting into the remaining equation gives $2 \cdot 3 - 2 + C = -3$, so that $C = -7$. Thus

$$\frac{x^2-3x+3}{x^3+2x^2+x} = \frac{3}{x} - \frac{2}{x+1} - \frac{7}{(x+1)^2}$$

49. We have

$$\begin{aligned}
\frac{x-1}{(x+1)(x^2+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4} \quad &\Rightarrow \\
x-1 = A(x^2+1)(x^2+4) + (Bx+C)(x+1)(x^2+4) + (Dx+E)(x+1)(x^2+1).&
\end{aligned}$$

Expand the right-hand side and collect terms to get

$$\begin{aligned}
x-1 = (A+B+D)x^4 + (B+C+D+E)x^3 + (5A+4B+C+D+E)x^2 \\
+ (4B+4C+D+E)x + (4A+4C+E).
\end{aligned}$$

This gives the linear system

$$\begin{array}{rrrrrrl}
A\,+ & B & & +\,D & & = & 0 \\
& B\,+ & C & +\,D & +\,E & = & 0 \\
5A\,+ & 4B\,+ & C & +\,D & +\,E & = & 0 \\
& 4B\,+ & 4C & +\,D & +\,E & = & 1 \\
4A & & +\,4C & & +\,E & = & -1.
\end{array}$$

Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{ccccc|r}
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 \\
5 & 4 & 1 & 1 & 1 & 0 \\
0 & 4 & 4 & 1 & 1 & 1 \\
4 & 0 & 4 & 0 & 1 & -1
\end{array}\right] \to \left[\begin{array}{ccccc|r}
1 & 0 & 0 & 0 & 0 & -\frac{1}{5} \\
0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -\frac{2}{15} \\
0 & 0 & 0 & 0 & 1 & -\frac{1}{5}
\end{array}\right].$$

Thus $A = -\frac{1}{5}$, $B = \frac{1}{3}$, $C = 0$, $D = -\frac{2}{15}$, and $E = -\frac{1}{5}$, giving

$$\begin{aligned}
\frac{x-1}{(x+1)(x^2+1)(x^2+4)} &= -\frac{1}{5(x+1)} + \frac{x}{3(x^2+1)} - \frac{\frac{2}{15}x + \frac{1}{5}}{x^2+4} \\
&= -\frac{1}{5(x+1)} + \frac{x}{3(x^2+1)} - \frac{2x+3}{15(x^2+4)}.
\end{aligned}$$

50. We have

$$\frac{x^3+x+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}.$$

Multiply both sides of the equation by $x(x-1)(x^2+x+1)(x^2+1)^3$, simplify, and equate coefficients of x^n to get the system

$$\begin{aligned}
A+B+C+E&=0 & -3A+3B-3C+3D-2E+F-G+H+J&=0\\
B-C+D+F&=0 & A+4B+C-3D-2F-H&=1\\
3A+4B+3C-D+2E+G&=0 & -3A+B-C+D-E-G-I&=0\\
-A+3B-3C+3D-E+2F+H&=0 & B-D-F-H-J&=1\\
3A+6B+3C-3D+E-F+G+I&=0 & -A&=1.
\end{aligned}$$

Construct and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{rrrrrrrrrr|r}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
3 & 4 & 3 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & 0\\
-1 & 3 & -3 & 3 & -1 & 2 & 0 & 1 & 0 & 0 & 0\\
3 & 6 & 3 & -3 & 1 & -1 & 1 & 0 & 1 & 0 & 0\\
-3 & 3 & -3 & 3 & -2 & 1 & -1 & 1 & 0 & 1 & 0\\
1 & 4 & 1 & -3 & 0 & -2 & 0 & -1 & 0 & 0 & 1\\
-3 & 1 & -1 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & 0\\
0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 1\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \to \left[\begin{array}{cccccccccc|r}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8}\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{15}{8}\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{9}{8}\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{7}{4}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{4}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}
\end{array}\right].$$

This gives the partial fraction decomposition

$$\begin{aligned}
\frac{x^3+x+1}{x(x-1)(x^2+x+1)(x^2+1)^3} &= -\frac{1}{x} + \frac{\frac{1}{8}}{x-1} - \frac{x}{x^2+x+1} + \frac{\frac{15}{8}x-\frac{9}{8}}{x^2+1} + \frac{\frac{7}{4}x-\frac{1}{4}}{(x^2+1)^2} + \frac{\frac{1}{2}x+\frac{1}{2}}{(x^2+1)^3}\\
&= -\frac{1}{x} + \frac{1}{8(x-1)} - \frac{x}{x^2+1} + \frac{15x-9}{8(x^2+1)} + \frac{7x-1}{4(x^2+1)^2} + \frac{x+1}{2(x^2+1)^3}.
\end{aligned}$$

51. Suppose that $1+2+\cdots+n = an^2+bn+c$, and let $n=0$, 1, and 2 to get the system

$$\begin{aligned}
c&=0\\
a+\ \ b+c&=1\\
4a+2b+c&=3.
\end{aligned}$$

Construct and row-reduce the associated augmented matrix:

$$\left[\begin{array}{ccc|c}
0 & 0 & 1 & 0\\
1 & 1 & 1 & 1\\
4 & 2 & 1 & 3
\end{array}\right] \to \left[\begin{array}{ccc|c}
1 & 0 & 0 & \frac{1}{2}\\
0 & 1 & 0 & \frac{1}{2}\\
0 & 0 & 1 & 0
\end{array}\right]$$

Thus $a=b=\frac{1}{2}$ and $c=0$, so that

$$1+2+\cdots+n = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n+1).$$

52. Suppose that $1^2+2^2+\cdots+n^2 = an^3+bn^2+cn+d$, and let $n=0$, 1, 2, and 3 to get the system

$$\begin{aligned}
d&=\ \ 0\\
a+\ \ b+\ \ c+d&=\ \ 1\\
8a+4b+2c+d&=\ \ 5\\
27a+9b+3c+d&=14.
\end{aligned}$$

Construct and row-reduce the associated augmented matrix:

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 & 5 \\ 27 & 9 & 3 & 1 & 14 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Thus $a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$, and $d = 0$, so that

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{6}(2n^3 + 3n^2 + n) = \frac{1}{6}n(n+1)(2n+1).$$

53. Suppose that $1^3 + 2^3 + \cdots + n^3 = an^4 + bn^3 + cn^2 + dn + e$, and let $n = 0$, 1, 2, 3, and 4 to get the system

$$\begin{aligned} e &= 0 \\ a + b + c + d + e &= 1 \\ 16a + 8b + 4c + 2d + e &= 9 \\ 81a + 27b + 9c + 3d + e &= 36 \\ 256a + 64b + 16c + 4d + e &= 100. \end{aligned}$$

Construct and row-reduce the associated augmented matrix:

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 & 9 \\ 81 & 27 & 9 & 3 & 1 & 36 \\ 256 & 64 & 16 & 4 & 1 & 100 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Thus $a = \frac{1}{4}$, $b = \frac{1}{2}$, $c = \frac{1}{4}$, and $d = e = 0$, so that

$$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \frac{1}{4}n^2(n^2 + 2n + 1) = \frac{1}{4}n^2(n+1)^2 = \left(\frac{n(n+1)}{2}\right)^2.$$

2.5 Iterative Methods for Solving Linear Systems

1. Start by solving the first equation for x_1 and the second for x_2:

$$\begin{aligned} x_1 &= \frac{6}{7} + \frac{1}{7}x_2 \\ x_2 &= \frac{4}{5} + \frac{1}{5}x_1. \end{aligned}$$

Using the initial vector $[x_1, x_2] = [0, 0]$, we get a sequence of approximations:

n	0	1	2	3	4	5	6
x_1	0	0.857	0.971	0.996	0.999	1.000	1.000
x_2	0	0.800	0.971	0.994	0.999	1.000	1.000

Solving the first equation for x_2 gives $x_2 = 7x_1 - 6$; substituting into the second equation gives

$$x_1 - 5(7x_1 - 6) = -4 \quad \text{so that} \quad -34x_1 = -34 \quad \Rightarrow \quad x_1 = 1.$$

Substituting this value back into either equation gives $x_2 = 1$, so the exact solution is $[x_1, x_2] = [1, 1]$.

2. Solving for x_1 and x_2 gives

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{1}{2}x_2 \\ x_2 &= -1 + \quad x_1. \end{aligned}$$

Using the initial vector $[x_1, x_2] = [0, 0]$, we get a sequence of approximations:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
x_1	0	2.5	3	1.75	1.4	2.125	2.25	1.938	1.875	2.031	2.062	1.984	1.969
x_2	0	−1.0	1.5	2.0	0.75	0.5	1.125	1.25	0.938	0.875	1.031	1.062	0.984

n	13	14	15	16	17	18	19	20	21	22	23	24	25
x_1	2.008	2.016	1.996	1.992	2.002	2.004	1.999	1.998	2.000	2.001	2.000	2.000	2.000
x_2	0.969	1.008	1.016	0.996	0.992	1.002	1.004	0.999	0.998	1.000	1.001	1.000	1.000

Using $x_2 = x_1 - 1$ in the first equation gives $x_1 = \frac{5}{2} - \frac{1}{2}(x_1 - 1)$, so that $x_1 = 2$ and $x_2 = 1$. So the exact solution is $[x_1, x_2] = [2, 1]$.

3. Solving for x_1 and x_2 gives

$$\begin{aligned} x_1 &= \frac{1}{9}x_2 + \frac{2}{9} \\ x_2 &= \frac{2}{7}x_2 + \frac{2}{7}. \end{aligned}$$

Using the initial vector $[x_1, x_2] = [0, 0]$, we get a sequence of approximations:

n	0	1	2	3	4	5	6
x_1	0	0.222	0.254	0.261	0.262	0.262	0.262
x_2	0	0.286	0.349	0.358	0.360	0.361	0.361

Solving the first equation for x_2 gives $x_2 = 9x_1 - 2$. Substitute this into the second equation to get $x_1 - 3.5(9x_2 - 2) = -1$, so that $30.5x_1 = 8$ and thus $x_1 = \frac{8}{30.5} \approx 0.262295$. This gives $x_2 = 9 \cdot \frac{8}{30.5} - 2 \approx 0.360656$.

4. Solving for x_1, x_2, and x_3 gives

$$\begin{aligned} x_1 &= -\frac{1}{20}x_2 + \frac{1}{20}x_3 + \frac{17}{20} \\ x_2 &= \frac{1}{10}x_1 + \frac{1}{10}x_3 - \frac{13}{10} \\ x_3 &= \frac{1}{10}x_1 - \frac{1}{10}x_2 + \frac{9}{5}. \end{aligned}$$

Using the initial vector $[x_1, x_2, x_3] = [0, 0, 0]$, we get a sequence of approximations:

n	0	1	2	3	4	5	6
x_1	0	0.85	1.005	1.002	1.000	1.000	1.000
x_2	0	−1.3	−1.035	−0.998	−0.999	−1.000	−1.000
x_3	0	1.8	2.015	2.004	2.000	2.000	2.000

To solve the system directly, row-reduce the augmented matrix:

$$\left[\begin{array}{rrr|r} 20 & 1 & -1 & 17 \\ 1 & -10 & 1 & 13 \\ -1 & 1 & 10 & 18 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

giving the exact solution $[x_1, x_2, x_3] = [1, -1, 2]$.

5. Solve to get

$$\begin{aligned} x_1 &= \quad \frac{1}{3}x_2 + \quad \frac{1}{3} \\ x_2 &= -\frac{1}{4}x_1 - \frac{1}{4}x_3 + \frac{1}{4} \\ x_3 &= -\frac{1}{3}x_2 + \quad \frac{1}{3}. \end{aligned}$$

Using the initial vector $[x_1, x_2, x_3] = [0, 0, 0]$, we get a sequence of approximations

n	0	1	2	3	4	5	6	7	8	9
x_1	0	0.333	0.250	0.306	0.292	0.301	0.299	0.300	0.300	0.300
x_2	0	0.250	0.083	0.125	0.097	0.104	0.100	0.101	0.100	0.100
x_3	0	0.333	0.250	0.306	0.292	0.301	0.299	0.300	0.300	0.300

To solve the system directly, row-reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & 1 \\ 1 & 4 & 1 & 1 \\ 0 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{10} \\ 0 & 1 & 0 & \frac{1}{10} \\ 0 & 0 & 1 & \frac{3}{10} \end{array}\right]$$

giving the exact solution $[x_1, x_2, x_3] = \left[\frac{3}{10}, \frac{1}{10}, \frac{3}{10}\right]$.

6. Solve to get

$$\begin{aligned} x_1 &= \frac{1}{3}x_2 + \quad \frac{1}{3} \\ x_2 &= \frac{1}{3}x_1 + \frac{1}{3}x_3 \\ x_3 &= \frac{1}{3}x_2 + \frac{1}{3}x_4 + \frac{1}{3} \\ x_4 &= \frac{1}{3}x_3 + \quad \frac{1}{3}. \end{aligned}$$

Using the initial vector $[x_1, x_2, x_3, x_4] = [0, 0, 0, 0]$, we get a sequence of approximations

n	0	1	2	3	4	5	6	7	8	9	10	11	12
x_1	0	0.333	0.333	0.407	0.420	0.440	0.444	0.450	0.452	0.453	0.454	0.454	0.454
x_2	0	0	0.222	0.259	0.321	0.333	0.351	0.355	0.360	0.361	0.363	0.363	0.363
x_3	0	0.333	0.444	0.556	0.580	0.613	0.620	0.630	0.632	0.634	0.635	0.636	0.636
x_4	0	0.333	0.444	0.481	0.519	0.527	0.538	0.540	0.543	0.544	0.545	0.545	0.545

To solve the system directly, row-reduce the augmented matrix:

$$\left[\begin{array}{cccc|c} 3 & -1 & 0 & 0 & 1 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 1 \\ 0 & 0 & -1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{5}{11} \\ 0 & 1 & 0 & 0 & \frac{4}{11} \\ 0 & 0 & 1 & 0 & \frac{7}{11} \\ 0 & 0 & 0 & 1 & \frac{6}{11} \end{array}\right]$$

giving the exact solution $[x_1, x_2, x_3, x_4] = \left[\frac{5}{11}, \frac{4}{11}, \frac{7}{11}, \frac{6}{11}\right]$.

7. Using the equations from Exercise 1, we get (to three decimal places, but keeping four during the calculations)

n	0	1	2	3	4
x_1	0	0.857	0.996	1.000	1.000
x_2	0	0.971	0.999	1.000	1.000

The Gauss-Seidel method takes 3 iterations, while the Jacobi method takes 5.

8. Using the equations from Exercise 2, we get (to three decimal places, but keeping four during the calculations)

n	0	1	2	3	4	5	6	7	8	9	10	11	12
x_1	0	2.5	1.75	2.125	1.938	2.031	1.984	2.008	1.996	2.002	1.999	2.000	2.000
x_2	0	1.5	0.75	1.125	0.938	1.031	0.984	1.008	0.996	1.002	0.999	1.000	1.000

The Gauss-Seidel method takes 11 iterations, while the Jacobi method takes 24.

9. Using the equations from Exercise 3, we get (to three decimal places, but keeping four during the calculations)

n	0	1	2	3	4
x_1	0	0.222	0.261	0.262	0.262
x_2	0	0.349	0.360	0.361	0.361

The Gauss-Seidel method takes 3 iterations, while the Jacobi method takes 5.

10. Using the equations from Exercise 4, we get (to three decimal places, but keeping four during the calculations)

n	0	1	2	3	4
x_1	0	0.850	1.011	1.000	1.000
x_2	0	-1.215	-0.998	-1.000	-1.000
x_3	0	2.006	2.001	2.000	2.000

The Gauss-Seidel method takes 3 iterations, while the Jacobi method takes 5.

11. Using the equations from Exercise 5, we get (to three decimal places, but keeping four during the calculations)

n	0	1	2	3	4	5	6
x_1	0	0.333	0.278	0.296	0.299	0.300	0.300
x_2	0	0.167	0.111	0.102	0.100	0.100	0.100
x_3	0	0.278	0.296	0.299	0.300	0.300	0.300

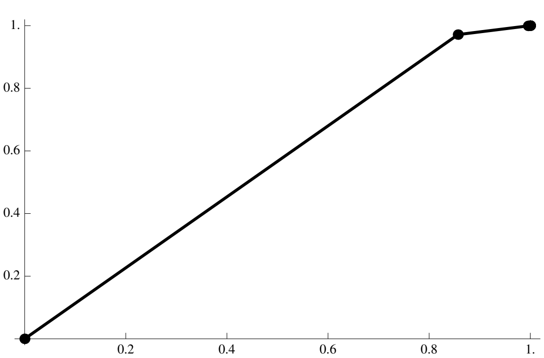
The Gauss-Seidel method takes 5 iterations, while the Jacobi method takes 8.

12. Using the equations from Exercise 6, we get (to three decimal places, but keeping four during the calculations)

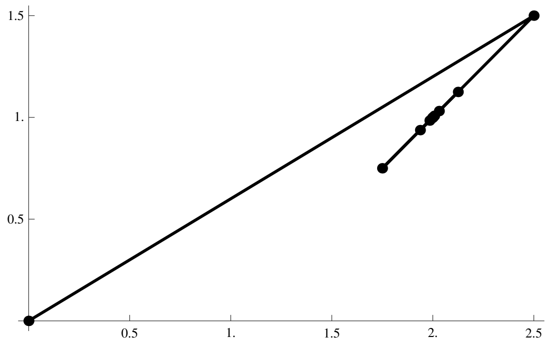
n	0	1	2	3	4	5	6	7
x_1	0	0.333	0.370	0.416	0.433	0.451	0.454	0.454
x_2	0	0.111	0.247	0.328	0.353	0.361	0.363	0.363
x_3	0	0.370	0.568	0.617	0.631	0.635	0.636	0.636
x_4	0	0.457	0.523	0.539	0.544	0.545	0.545	0.545

The Gauss-Seidel method takes 6 iterations, while the Jacobi method takes 11.

13.



14.



15. Applying the Gauss-Seidel method to $x_1 - 2x_2 = 3$, $3x_1 + 2x_2 = 1$ with an initial estimate of $[0, 0]$ gives

n	0	1	2	3	4
x_1	0	3	-5	19	-53
x_2	0	-4	8	-28	80

which evidently diverges. If, however, we reverse the two equations to get $3x_1 + 2x_2 = 1$, $x_1 - 2x_2 = 3$, this is strictly diagonally dominant since $|3| > |2|$ and $|-2| > |1|$. Applying the Gauss-Seidel method gives

n	0	1	2	3	4	5	6	7	8	9
x_1	0	0.333	1.222	0.926	1.025	0.992	1.003	0.999	1.000	1.000
x_2	0	-1.333	-0.889	-1.037	-0.988	-1.004	-0.999	-1.000	-1.000	-1.000

The exact solution is $[x_1, x_2] = [1, -1]$.

16. Applying the Gauss-Seidel method to $x_1 - 4x_2 + 2x_3 = 2$, $2x_2 + 4x_3 = 1$, $6x_1 - x_2 - 2x_3 = 1$ with an initial estimate of $[0, 0, 0]$ gives

n	0	1	2	3	4
x_1	0	2	-6.5	-8	203.5
x_2	0	0.5	-10	30.5	80
x_3	0	5.25	-15	-39.75	570

which evidently diverges. If, however, we arrange the equations as

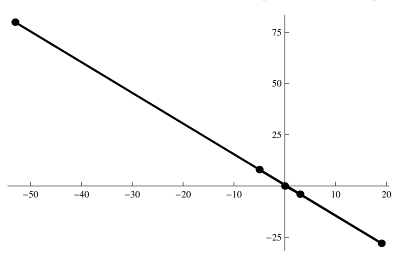
$$\begin{aligned} 6x_1 - x_2 - 2x_3 &= 1 \\ x_1 - 4x_2 + 2x_3 &= 2 \\ 2x_2 + 4x_3 &= 1, \end{aligned}$$

we get a strictly diagonally dominant system. Applying the Gauss-Seidel method gives

n	0	1	2	3	4	5	6	7
x_1	0	0.167	0.250	0.250	0.250	0.250	0.250	0.250
x_2	0	-0.458	-0.198	-0.263	-0.247	-0.251	-0.250	-0.250
x_3	0	0.479	0.349	0.382	0.373	0.375	0.375	0.375

The exact solution is $[x_1, x_2, x_3] = \left[\frac{1}{4}, -\frac{1}{4}, \frac{3}{8}\right]$.

17.



18. Applying the Gauss-Seidel method to the equations

$$\begin{aligned} -4x_1 + 5x_2 &= 14 \\ x_1 - 3x_2 &= -7 \end{aligned}$$

gives

n	0	1	2	3	4	5	6	7	8
x_1	0	-3.5	-2.04	-1.43	-1.18	-1.08	-1.03	-1.01	-1.01
x_2	0	1.17	1.65	1.86	1.94	1.97	1.99	2.00	2.00

This gives an approximate solution (to 0.01) of $x_1 = -1.01$, $x_2 = 2.00$. The exact solution is $[x_1, x_2] = [-1, 2]$.

19. Applying the Gauss-Seidel method to the equations

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -8 \\ x_1 + 4x_2 - 4x_3 &= 102 \\ -2x_1 - 2x_2 + 4x_3 &= 90 \end{aligned}$$

gives

n	0	1	2	3	4	5	6	7	8
x_1	0	-1.6	14.97	8.55	10.74	9.84	10.12	9.99	10.02
x_2	0	25.9	11.41	14.05	11.62	11.72	11.25	11.19	11.08
x_3	0	-10.35	-9.31	-11.2	-11.32	-11.72	-11.82	-11.90	-11.95

n	9	10	11	12	13	14
x_1	10.00	10.00	10.00	10.00	10.00	10.00
x_2	11.05	11.03	11.01	11.01	11.00	11.00
x_3	-11.97	-11.98	-11.99	-12.00	-12.00	-12.00

This gives an approximate solution (to 0.01) of $x_1 = 10.00$, $x_2 = 11.00$, $x_3 = -12.00$. The exact solution is $[x_1, x_2, x_3] = [10, 11, -12]$.

20. Redoing the computation from Exercise 18, showing three decimal places, and computing to an accuracy of 0.001, gives

n	0	1	2	3	4	5	6	7	8
x_1	0	-3.5	-2.042	-1.434	-1.181	-1.075	-1.031	-1.013	-1.005
x_2	0	1.167	1.653	1.855	1.940	1.975	1.990	1.996	1.998

n	9	10	11	12
x_1	-1.002	-1.001	-1.000	-1.000
x_2	1.999	2.000	2.000	2.000

This gives an approximate solution (to 0.001) of $x_1 = -1.000$, $x_2 = 2.000$. The exact solution is $[x_1, x_2] = [-1, 2]$.

21. Redoing the computation from Exercise 19, showing three decimal places, and computing to an accuracy of 0.001, gives

n	0	1	2	3	4	5	6	7	8
x_1	0	−1.6	14.97	8.55	10.74	9.839	10.120	9.989	10.022
x_2	0	25.9	11.408	14.051	11.615	11.718	11.249	11.187	11.082
x_3	0	−10.35	−9.311	−11.199	−11.322	−11.721	−11.816	−11.912	−11.948

n	9	10	11	12	13
x_1	10.002	10.005	10.001	10.001	10.000
x_2	11.052	11.026	11.015	11.008	11.005
x_3	−11.973	−11.985	−11.992	−11.996	−11.998

n	14	15	16	17	18
x_1	10.000	10.000	10.000	10.000	10.000
x_2	11.002	11.001	11.001	11.000	11.000
x_3	−11.999	−11.999	−12.000	−12.000	−12.000

This gives an approximate solution (to 0.001) of $x_1 = 10.000$, $x_2 = 11.000$, $x_3 = -12.000$. The exact solution is $[x_1, x_2, x_3] = [10, 11, -12]$.

22. With the interior points labeled as shown, we use the temperature-averaging property to get the following system:

$$\begin{aligned}
t_1 &= \frac{1}{4}(0 + 40 + 40 + t_2) = 20 + \frac{1}{4}t_2 \\
t_2 &= \frac{1}{4}(0 + t_1 + t_3 + 5) = \frac{5}{4} + \frac{1}{4}t_1 + \frac{1}{4}t_3 \\
t_3 &= \frac{1}{4}(t_2 + 40 + 40 + 5) = \frac{85}{4} + \frac{1}{4}t_2.
\end{aligned}$$

The Gauss-Seidel method gives the following with initial approximation $[0, 0, 0]$:

n	0	1	2	3	4	5	6	7	8
t_1	0	20.000	21.562	23.086	23.276	23.300	23.303	23.304	23.304
t_2	0	6.250	12.344	13.105	13.201	13.213	13.214	13.214	13.214
t_3	0	22.812	24.336	24.526	24.550	24.553	24.554	24.554	24.554

This gives equilibrium temperatures (to 0.001) of $t_1 = 23.304$, $t_2 = 13.214$, and $t_3 = 24.554$.

23. With the interior points labeled as shown, we use the temperature-averaging property to get the following system:

$$\begin{aligned}
t_1 &= \frac{1}{4}(t_2 + t_3) \\
t_2 &= \frac{1}{4}(t_1 + t_4) \\
t_3 &= \frac{1}{4}(t_1 + t_4 + 200) \\
t_4 &= \frac{1}{4}(t_2 + t_3 + 200)
\end{aligned}$$

The Gauss-Seidel method gives the following with initial approximation $[0, 0, 0]$:

n	0	1	2	3	4	5	6	7	8	9	10
t_1	0	0	12.5	21.875	24.219	24.805	24.951	24.988	24.997	24.999	25.000
t_2	0	0	18.75	23.438	24.609	24.902	24.976	24.994	24.998	25.000	25.000
t_3	0	50	68.75	73.438	74.609	74.902	74.976	74.994	74.998	75.000	75.000
t_4	0	62.5	71.875	74.219	74.805	74.951	74.988	74.997	74.998	75.000	75.000

This gives equilibrium temperatures (to 0.001) of $t_1 = t_2 = 25$ and $t_3 = t_4 = 75$.

24. With the interior points labeled as shown, we use the temperature-averaging property to get the following system:

$$\begin{aligned} t_1 &= \frac{1}{4}(t_2 + t_3) \\ t_2 &= \frac{1}{4}(t_1 + t_4 + 40) \\ t_3 &= \frac{1}{4}(t_1 + t_4 + 80) \\ t_4 &= \frac{1}{4}(t_2 + t_3 + 200) \end{aligned}$$

The Gauss-Seidel method gives the following with initial approximation $[0, 0, 0]$:

n	0	1	2	3	4	5	6	7	8	9	10
t_1	0	0	7.5	15.625	17.656	18.164	18.291	18.323	18.331	18.333	18.333
t_2	0	10	26.25	30.312	31.328	31.582	31.646	31.661	31.665	31.666	31.667
t_3	0	20	36.25	40.312	41.328	41.582	41.646	41.661	41.665	41.666	41.667
t_4	0	57.5	65.625	67.656	68.164	68.291	68.323	68.331	68.333	68.333	68.333

This gives equilibrium temperatures (to 0.001) of $t_1 = 18.333$, $t_2 = 31.667$, $t_3 = 41.667$, and $t_4 = 68.333$.

25. With the interior points labeled as shown, we use the temperature-averaging property to get the following system:

$$\begin{aligned} t_1 &= \frac{1}{4}(t_2 + 80) \\ t_2 &= \frac{1}{4}(t_1 + t_3 + t_4) \\ t_3 &= \frac{1}{4}(t_2 + t_5 + 80) \\ t_4 &= \frac{1}{4}(t_2 + t_5 + 5) \\ t_5 &= \frac{1}{4}(t_3 + t_4 + t_6 + 5) \\ t_6 &= \frac{1}{4}(t_5 + 85). \end{aligned}$$

The Gauss-Seidel method gives the following with initial approximation $[0, 0, 0]$:

n	0	1	2	3	4	5	6
t_1	0	20	21.25	22.812	23.33	23.66	23.773
t_2	0	5	11.25	13.32	14.639	15.093	15.237
t_3	0	21.25	24.609	26.987	27.730	27.963	28.035
t_4	0	2.5	5.859	8.237	8.980	9.213	9.285
t_5	0	7.188	14.629	16.283	16.758	16.904	16.949
t_6	0	23.047	24.097	25.321	25.439	25.476	25.487

n	7	8	9	10	11	12
t_1	23.809	23.821	23.824	23.825	23.826	23.826
t_2	15.282	15.297	15.301	15.302	15.303	15.303
t_3	28.058	28.065	28.067	28.068	28.068	28.068
t_4	9.308	9.315	9.317	9.318	9.318	9.318
t_5	16.963	16.968	16.969	16.970	16.970	16.970
t_6	25.491	25.492	25.492	25.492	25.492	25.492

This gives equilibrium temperatures (to 0.001) of $t_1 = 23.826$, $t_2 = 15.303$, $t_3 = 28.068$, $t_4 = 9.318$, $t_5 = 16.970$, and $t_6 = 25.492$.

26. With the interior points labeled as shown, we use the temperature-averaging property to get the following system:

$$
\begin{aligned}
t_1 &= \frac{1}{4}(t_2 + t_5) & t_9 &= \frac{1}{4}(t_5 + t_{10} + t_{13} + 40)\\
t_2 &= \frac{1}{4}(t_1 + t_3 + t_6) & t_{10} &= \frac{1}{4}(t_6 + t_9 + t_{11} + t_{14})\\
t_3 &= \frac{1}{4}(t_2 + t_4 + t_7 + 20) & t_{11} &= \frac{1}{4}(t_7 + t_{10} + t_{12} + t_{15})\\
t_4 &= \frac{1}{4}(t_3 + t_8 + 40) & t_{12} &= \frac{1}{4}(t_8 + t_{11} + t_{16} + 100)\\
t_5 &= \frac{1}{4}(t_1 + t_6 + t_9) & t_{13} &= \frac{1}{4}(t_9 + t_{14} + 80)\\
t_6 &= \frac{1}{4}(t_2 + t_5 + t_7 + t_{10}) & t_{14} &= \frac{1}{4}(t_{10} + t_{13} + t_{15} + 40)\\
t_7 &= \frac{1}{4}(t_3 + t_6 + t_8 + t_{11}) & t_{15} &= \frac{1}{4}(t_{11} + t_{14} + t_{16} + 100)\\
t_8 &= \frac{1}{4}(t_4 + t_7 + t_{12} + 20) & t_{16} &= \frac{1}{4}(t_{12} + t_{15} + 200).
\end{aligned}
$$

n	0	1	2	3	4	5	6	7	8	9
t_1	0	0	0	0.938	1.934	2.853	3.996	5.194	6.142	6.816
t_2	0	0	1.25	2.812	4.443	6.66	9.035	10.924	12.27	13.185
t_3	0	5.	8.438	10.449	13.424	16.637	19.081	20.781	21.923	22.68
t_4	0	11.25	14.141	16.753	19.533	21.544	22.892	23.781	24.365	24.748
t_5	0	0	2.5	4.922	6.968	9.323	11.742	13.645	14.995	15.911
t_6	0	0	1.875	5.391	10.364	15.505	19.421	22.156	24.	25.223
t_7	0	1.25	4.844	12.5	20.356	25.747	29.306	31.642	33.172	34.174
t_8	0	8.125	16.562	24.707	29.538	32.488	34.345	35.537	36.31	36.813
t_9	0	10.	16.875	20.547	24.077	27.466	29.965	31.682	32.83	33.589
t_{10}	0	2.5	8.984	17.544	25.682	31.161	34.748	37.092	38.625	39.628
t_{11}	0	0.938	17.598	32.928	41.307	46.234	49.285	51.227	52.482	53.298
t_{12}	0	27.266	49.575	58.263	62.661	65.182	66.725	67.702	68.331	68.74
t_{13}	0	22.5	28.281	31.797	34.859	36.958	38.334	39.232	39.818	40.202
t_{14}	0	16.25	26.641	35.359	40.367	43.372	45.246	46.444	47.218	47.722
t_{15}	0	29.297	52.095	60.926	65.368	67.903	69.451	70.429	71.058	71.467
t_{16}	0	64.141	75.417	79.797	82.007	83.271	84.044	84.533	84.847	85.052

n	10	11	12	13	14	15	16	17	18	19
t_1	7.274	7.579	7.78	7.912	7.999	8.056	8.093	8.118	8.133	8.144
t_2	13.794	14.197	14.461	14.635	14.748	14.823	14.871	14.903	14.924	14.938
t_3	23.179	23.506	23.721	23.861	23.953	24.013	24.053	24.078	24.095	24.106
t_4	24.998	25.162	25.269	25.34	25.386	25.416	25.435	25.448	25.457	25.462
t_5	16.522	16.924	17.189	17.362	17.476	17.55	17.599	17.631	17.651	17.665
t_6	26.029	26.559	26.906	27.133	27.282	27.379	27.443	27.485	27.512	27.53
t_7	34.83	35.259	35.54	35.724	35.845	35.923	35.975	36.009	36.031	36.045
t_8	37.142	37.357	37.497	37.589	37.65	37.689	37.715	37.732	37.743	37.75
t_9	34.088	34.415	34.63	34.77	34.862	34.922	34.962	34.988	35.004	35.015
t_{10}	40.284	40.714	40.995	41.179	41.299	41.378	41.43	41.463	41.485	41.5
t_{11}	53.83	54.178	54.406	54.554	54.652	54.716	54.757	54.785	54.803	54.814
t_{12}	69.006	69.18	69.294	69.368	69.417	69.449	69.47	69.483	69.492	69.498
t_{13}	40.452	40.617	40.724	40.794	40.84	40.87	40.89	40.903	40.911	40.917
t_{14}	48.051	48.266	48.406	48.498	48.559	48.598	48.624	48.641	48.652	48.659
t_{15}	71.733	71.907	72.021	72.095	72.144	72.176	72.197	72.21	72.219	72.225
t_{16}	85.185	85.272	85.329	85.366	85.39	85.406	85.417	85.423	85.428	85.431

n	20	21	22	23	24	25	26	27	28	29
t_1	8.151	8.155	8.158	8.16	8.161	8.162	8.163	8.163	8.163	8.163
t_2	14.947	14.953	14.956	14.959	14.961	14.962	14.962	14.963	14.963	14.963
t_3	24.114	24.118	24.121	24.123	24.125	24.126	24.126	24.127	24.127	24.127
t_4	25.466	25.468	25.47	25.471	25.471	25.472	25.472	25.472	25.472	25.473
t_5	17.674	17.68	17.684	17.686	17.688	17.689	17.69	17.69	17.69	17.691
t_6	27.541	27.549	27.554	27.557	27.56	27.561	27.562	27.562	27.563	27.563
t_7	36.055	36.061	36.065	36.068	36.069	36.071	36.071	36.072	36.072	36.072
t_8	37.755	37.758	37.76	37.761	37.762	37.763	37.763	37.763	37.763	37.763
t_9	35.023	35.027	35.03	35.033	35.034	35.035	35.035	35.036	35.036	35.036
t_{10}	41.509	41.516	41.52	41.522	41.524	41.525	41.526	41.526	41.527	41.527
t_{11}	54.822	54.827	54.83	54.832	54.834	54.835	54.835	54.836	54.836	54.836
t_{12}	69.502	69.504	69.506	69.507	69.508	69.508	69.509	69.509	69.509	69.509
t_{13}	40.92	40.923	40.924	40.925	40.926	40.926	40.927	40.927	40.927	40.927
t_{14}	48.664	48.667	48.669	48.67	48.671	48.672	48.672	48.672	48.672	48.673
t_{15}	72.229	72.232	72.233	72.234	72.235	72.235	72.236	72.236	72.236	72.236
t_{16}	85.433	85.434	85.435	85.435	85.436	85.436	85.436	85.436	85.436	85.436

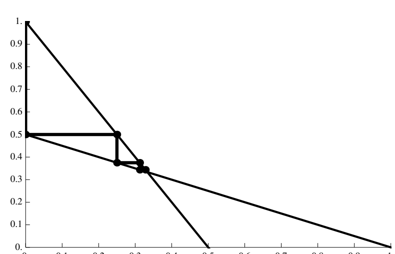
n	30	31	32	33	34
t_1	8.163	8.164	8.164	8.164	8.164
t_2	14.963	14.963	14.964	14.964	14.964
t_3	24.127	24.127	24.127	24.127	24.127
t_4	25.473	25.473	25.473	25.473	25.473
t_5	17.691	17.691	17.691	17.691	17.691
t_6	27.563	27.563	27.563	27.564	27.564
t_7	36.072	36.073	36.073	36.073	36.073
t_8	37.764	37.764	37.764	37.764	37.764
t_9	35.036	35.036	35.036	35.036	35.036
t_{10}	41.527	41.527	41.527	41.527	41.527
t_{11}	54.836	54.836	54.836	54.836	54.836
t_{12}	69.509	69.509	69.509	69.509	69.509
t_{13}	40.927	40.927	40.927	40.927	40.927
t_{14}	48.673	48.673	48.673	48.673	48.673
t_{15}	72.236	72.236	72.236	72.236	72.236
t_{16}	85.436	85.436	85.436	85.436	85.436

Column 34 gives the equilibrium temperature to an accuracy of 0.001.

27. (a) Let x_1 correspond to the left end of the paper and x_2 to the right end, and let n be the number of folds. Then the first six values of $[x_1, x_2]$ are

n	0	1	2	3	4	5	6
x_1	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$
x_2	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{11}{32}$	$\frac{11}{32}$

The graph below shows these points, together with the two lines determined in part (b):



(b) From the table, we see that $x_1 = 0$ gives $x_2 = \frac{1}{2}$ in the next iteration, and $x_1 = \frac{1}{4}$ gives $x_2 = \frac{3}{8}$ in the next iteration. Similarly, $x_2 = 1$ gives $x_1 = 0$ in the next iteration, and $x_2 = \frac{1}{2}$ gives $x_1 = \frac{1}{4}$ in the next iteration. Thus

$$x_2 - \frac{1}{2} = \frac{\frac{3}{8} - \frac{1}{2}}{\frac{1}{4} - 0}(x_1 - 0), \text{ or } x_2 = -\frac{1}{2}x_1 + \frac{1}{2}$$

$$x_1 - 0 = \frac{\frac{1}{4} - 0}{\frac{1}{2} - 1}(x_2 - 1), \text{ or } x_1 = -\frac{1}{2}x_2 + \frac{1}{2}$$

(c) Switching to decimal representations, we apply Gauss-Seidel to these equations after $n = 6$, starting with the estimates $x_1 = \frac{21}{64}$ and $x_2 = \frac{11}{32}$, to obtain an approximate point of convergence:

n	0	1	2	3	4	5	6	7	8	9	10
x_1	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$	0.328	0.332	0.333	0.333
x_2	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{11}{32}$	$\frac{11}{32}$	0.336	0.334	0.333	0.333

(d) Row-reduce the augmented matrix for this pair of equations, giving

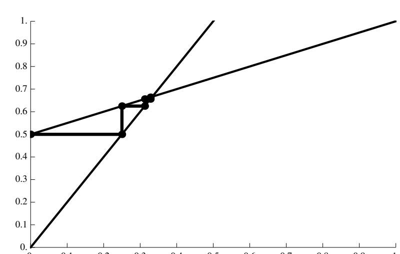
$$\left[\begin{array}{cc|c} \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{array}\right].$$

The exact solution to the system of equations is $[x_1, x_2] = \left[\frac{1}{3}, \frac{1}{3}\right]$. The ends of the piece of paper converge to $\frac{1}{3}$.

28. **(a)** Let x_1 record the positions of the left endpoints and x_2 the positions of the right endpoints at the end of each walks. Then the first six values of $[x_1, x_2]$ are

n	0	1	2	3	4	5	6
x_1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$	$\frac{21}{64}$
x_2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{21}{32}$	$\frac{21}{32}$	$\frac{85}{128}$

The graph below shows these points, together with the two lines determined in part (b):



(b) From the table, we see that $x_1 = 0$ gives $x_2 = \frac{1}{2}$ in the next iteration, and $x_1 = \frac{1}{4}$ gives $x_2 = \frac{5}{8}$ in the next iteration. Similarly, $x_2 = \frac{1}{2}$ gives $x_1 = \frac{1}{4}$ in the next iteration, and $x_2 = \frac{5}{8}$ gives $x_1 = \frac{5}{16}$ in the next iteration. Thus

$$x_2 - \frac{1}{2} = \frac{\frac{5}{8} - \frac{1}{2}}{\frac{1}{4} - 0}(x_1 - 0), \text{ or } x_2 = \frac{1}{2}x_1 + \frac{1}{2}$$

$$x_1 - \frac{1}{4} = \frac{\frac{5}{8} - \frac{1}{2}}{\frac{5}{16} - \frac{1}{4}}\left(x_2 - \frac{1}{2}\right), \text{ or } x_1 = \frac{1}{2}x_2.$$

(c) Switching to decimal representations, we apply Gauss-Seidel to these equations after $n = 6$, starting with the estimates $x_1 = \frac{21}{64}$ and $x_2 = \frac{85}{128}$, to obtain an approximate point of convergence:

n	0	1	2	3	4	5	6	7	8	9
x_1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$	$\frac{21}{64}$	0.332	0.333	0.333
x_2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{21}{32}$	$\frac{21}{32}$	$\frac{85}{128}$	0.666	0.667	0.667

(d) Row-reduce the augmented matrix for this pair of equations, giving

$$\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \end{array}\right].$$

The exact solution to the system of equations is $[x_1, x_2] = \left[\frac{1}{3}, \frac{2}{3}\right]$. So the ant eventually oscillates between $\frac{1}{3}$ and $\frac{2}{3}$.

Chapter Review

1. (a) False. Some linear systems are inconsistent, and have no solutions. See Section 2.1. One example is $x + y = 0$ and $x + y = 1$.

(b) True. See Section 2.2. Since in a homogeneous system, the lines defined by the equations all pass through the origin, the system has at least one solution; namely, the solution in which all variables are zero.

(c) False. See Theorem 2.6. This is guaranteed only if the system is homogeneous, but not if it is not (for example, $x + y = 0$, $x + y = 1$ is inconsistent in $\mathbb{R}^3$).

(d) False. See the discussion of homogeneous systems in Section 2.2. It can have either a unique solution, infinitely many solutions, or no solutions.

(e) True. See Theorem 2.4 in Section 2.3.

(f) False. If $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent (i.e., one is a multiple of the other), we get a line, not a plane. And if $\mathbf{u}$ and $\mathbf{v}$ are both zero, we get only the origin. If the two vectors are linearly independent, then their span is indeed a plane through the origin.

(g) True. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\mathbf{v} = c\mathbf{u}$, so that $-c\mathbf{u} + \mathbf{v} = 0$, so that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

(h) True. If they can be drawn that way, then the sum of the vectors is $\mathbf{0}$, since tracing the vectors in turn returns you to the starting point.

(i) False. For example, $\mathbf{u} = [1, 0, 0]$, $\mathbf{v} = [0, 1, 0]$ and $\mathbf{w} = [1, 1, 0]$ are linearly dependent since $\mathbf{u} + \mathbf{v} - \mathbf{w} = \mathbf{0}$, yet none of these is a multiple of another.

(j) True. See Theorem 2.8 in Section 2.3. m vectors in $\mathbb{R}^n$ are linearly dependent if $m > n$.

2. The rank is the number of nonzero rows in row echelon form:

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 2 \\ 3 & -1 & 1 & 3 & 4 \\ 3 & 4 & 2 & -3 & 2 \\ 0 & -5 & -1 & 6 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & -2 & 0 & 3 & 2 \\ 0 & -5 & -1 & 6 & 2 \\ 3 & 4 & 2 & -3 & 2 \\ 3 & -1 & 1 & 3 & 4 \end{bmatrix} \xrightarrow[R_4 - 3R_1 + R_2]{R_3 - 3R_1 + 2R_2} \begin{bmatrix} 1 & -2 & 0 & 3 & 2 \\ 0 & -5 & -1 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This row echelon form of A has two nonzero rows, so that $\operatorname{rank}(A) = 2$.

3. Construct the augmented matrix of the system and row-reduce it:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 1 & 3 & -1 & 7 \\ 2 & 1 & -5 & 7 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow{-2R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 2 & 2 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3]{R_1 + 2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

Thus the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

4. Construct the augmented matrix and row-reduce it:

$$\left[\begin{array}{cccc|c} 3 & 8 & -18 & 1 & 35 \\ 1 & 2 & -4 & 0 & 11 \\ 1 & 3 & -7 & 1 & 10 \end{array}\right] \begin{array}{c} \xrightarrow{R_2} \\ \xrightarrow{R_3} \\ \xrightarrow{R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 1 & 3 & -7 & 1 & 10 \\ 3 & 8 & -18 & 1 & 35 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-R_1} \\ \xrightarrow{R_3-3R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 0 & 1 & -3 & 1 & -1 \\ 0 & 2 & -6 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 0 & 1 & -3 & 1 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 0 & 1 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

$$\xrightarrow{R_2-R_3} \left[\begin{array}{cccc|c} 1 & 2 & -4 & 0 & 11 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

Then $y = t$ is a free variable, and the general solution is $w = 5 - 2t$, $x = 3 + 3t$, $y = t$, $z = -4$:

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5-2t \\ 3+3t \\ t \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ -4 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

5. Construct the augmented matrix and row-reduce it over $\mathbb{Z}_7$:

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 1 & 2 & 3 \end{array}\right] \xrightarrow{4R_1} \left[\begin{array}{cc|c} 1 & 5 & 2 \\ 1 & 2 & 3 \end{array}\right] \xrightarrow{R_2+6R_1} \left[\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & 4 & 1 \end{array}\right] \xrightarrow{R_1+4R_2} \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 4 & 2 \end{array}\right].$$

Thus the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$.

6. Construct the augmented matrix and row-reduce over $\mathbb{Z}_5$:

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 4 & 2 \end{array}\right] \xrightarrow{R_2+3R_1} \left[\begin{array}{cc|c} 3 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

So the solutions are all of the form $3x + 2y = 1$. Setting $y = 0, 1, 2, 3, 4$ in turn, we get

$$\begin{aligned} y = 0 &\Rightarrow 3x = 1 \Rightarrow x = 2 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ y = 1 &\Rightarrow 3x + 2 = 1 \Rightarrow 3x = 4 \Rightarrow x = 3 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ y = 2 &\Rightarrow 3x + 4 = 1 \Rightarrow 3x = 2 \Rightarrow x = 4 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\ y = 3 &\Rightarrow 3x + 1 = 1 \Rightarrow x = 0 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ y = 4 &\Rightarrow 3x + 3 = 1 \Rightarrow 3x = 3 \Rightarrow x = 1 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}. \end{aligned}$$

7. Construct the augmented matrix and row-reduce it:

$$\left[\begin{array}{cc|c} k & 2 & 1 \\ 1 & 2k & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & 2k & 1 \\ k & 2 & 1 \end{array}\right] \xrightarrow{R_2 - kR_1} \left[\begin{array}{cc|c} 1 & 2k & 1 \\ 0 & 2-2k^2 & 1-k \end{array}\right]$$

The system is inconsistent when $2 - 2k^2 = -2(k+1)(k-1) = 0$ but $1 - k \neq 0$. The first of these is zero for $k = -1$ and for $k = 1$; when $k = -1$ we have $1 - k = 2 \neq 0$. So the system is inconsistent when $k = -1$.

8. Form the augmented matrix of the system and row reduce it (see Example 2.14 in Section 2.2):

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array}\right] \xrightarrow{R_2-5R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

Thus $z = t$ is a free variable, and $x = -2 + t$, $y = 3 - 2t$, so the parametric equation of the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2+t \\ 3-2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

9. As in Example 2.15 in Section 2.2, we wish to find $\mathbf{x} = [x, y, z]$ that satisfies both equations simultaneously; that is, $\mathbf{x} = \mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}$. This means $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$. Substituting the given vectors into this equation gives

$$s \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -7 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} s + t &= 4 \\ -s - t &= -4 \\ 2s - t &= -7 \end{aligned}$$

Form the augmented matrix of this system and row-reduce it to determine s and t:

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ -1 & -1 & -4 \\ 2 & -1 & -7 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & -1 & 7 \\ -1 & -1 & -4 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-2R_1} \\ \xrightarrow{R_3+R_1} \end{array} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -3 & -15 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $s = -1$ and $t = 5$, so the point of intersection is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

(We could equally well have substituted $t = 5$ into the equation for the other line; check that this gives the same answer.)

10. As in Example 2.18 in Section 2.3, we want to find scalars x and y such that

$$x \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x + y &= 3 \\ x + 2y &= 5 \\ 3x - 2y &= -1 \end{aligned}$$

Form the augmented matrix of this system and row-reduce it to determine x and y:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 3 & -2 & -1 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-R_1} \\ \xrightarrow{R_3-3R_1} \end{array} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -10 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-R_2} \\ \xrightarrow{R_3+5R_3} \end{array} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

So this system has a solution, and thus so the vector is in the span of the other two vectors:

$$\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

11. As in Example 2.21 in Section 2.3, the equation of the plane we are looking for is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} s + 3t &= x \\ s + 2t &= y \\ s + t &= z \end{aligned}$$

Form the augmented matrix and row-reduce it to find conditions for x, y, and z:

$$\left[\begin{array}{cc|c} 1 & 3 & x \\ 1 & 2 & y \\ 1 & 1 & z \end{array}\right] \xrightarrow[R_3 - R_2]{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & -1 & y - x \\ 0 & -2 & z - x \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & 1 & x - y \\ 0 & -2 & z - x \end{array}\right] \xrightarrow{R_3 + 2R_2} \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & 1 & x - y \\ 0 & 0 & x - 2y + z \end{array}\right]$$

We know the system is consistent, since the two given vectors span some plane. So we must have $x - 2y + z = 0$, and this is the general equation of the plane we were seeking. (An alternative method would be to take the cross product of the two given vectors, giving $[-1, 2, -1]$, so that the plane is of the form $-x + 2y - z = d$; plugging in $x = y = z = 0$ gives $-x + 2y - z = 0$, or $x - 2y + z = 0$.)

12. As in Example 2.23 in Section 2.3, we want to find scalars c_1, c_2, and c_3 so that

$$c_1 \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 9 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Form the augmented matrix of the resulting system and row-reduce it:

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 9 & 0 \\ -3 & -2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since $c_1 = -4c_3$, $c_2 = 5c_3$ is a solution, the vectors are linearly dependent. For example, setting $c_3 = -1$ gives

$$4 \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 9 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

13. By part (a) of Exercise 21 in Section 2.3, it is sufficient to prove that $\mathbf{e}_i$ is in $\operatorname{span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \subset \mathbb{R}^3$, since then we have $\mathbb{R}_3 = \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \subset \operatorname{span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \subset \mathbb{R}^3$.

(a) Begin with $\mathbf{e}_1$. We want to find scalars x, y, and z such that

$$x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Form the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \quad \Rightarrow \quad \mathbf{e}_1 = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{w}.$$

Similarly, for $\mathbf{e}_2$,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \quad \Rightarrow \quad \mathbf{e}_1 = \frac{1}{2}\mathbf{u} - \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}.$$

Similarly, for $\mathbf{e}_3$,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \quad \Rightarrow \quad \mathbf{e}_1 = -\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}.$$

Thus $\mathbb{R}^3 = \operatorname{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

(b) Since $\mathbf{w} = \mathbf{u} + \mathbf{v}$, these vectors are not linearly independent. But any set of vectors that spans $\mathbb{R}^3$ must contain 3 linearly independent vectors, so $\operatorname{span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \neq \mathbb{R}^3$.

14. All three statements are true, so (d) is the correct answer. If the vectors are linearly independent, and $\mathbf{b}$ is any vector in $\mathbb{R}^3$, then the augmented matrix $[A \mid \mathbf{b}]$, when reduced to row echelon form, cannot have a zero row (since otherwise, the portion of the matrix corresponding to A would have a zero row, so the three vectors would be linearly dependent). Thus $[A \mid \mathbf{b}]$ has a unique solution, which can be read off from the row-reduced matrix. This proves statement (c). Next, since there is not a zero row, continuing and putting the system in reduced row-echelon form results in a matrix with 3 leading ones, so it is the identity matrix. Hence the reduced row echelon form of A is I_3, and thus A has rank 3.

15. Since A is a 3×3 matrix, $\operatorname{rank}(A) \leq 3$. Since the vectors are linearly dependent, $\operatorname{rank}(A)$ cannot be 3; since they are not all zero, $\operatorname{rank}(A)$ cannot be zero. Thus $\operatorname{rank}(A) = 1$ or $\operatorname{rank}(A) = 2$.

16. By Theorem 2.8 in Section 5.3, the maximum rank of a 5×3 matrix is 3: since the rows are vectors in $\mathbb{R}^3$, any set of four or more of them are linearly dependent. Thus when we row reduce the matrix, we must introduce at least two zero rows. The minimum rank is zero, achieved when all entries are zero.

17. Suppose that $c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{u}-\mathbf{v}) = \mathbf{0}$. Rearranging gives $(c_1+c_2)\mathbf{u} + (c_1-c_2)\mathbf{v} = \mathbf{0}$. Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, it follows that $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. Adding the two equations gives $2c_1 = 0$, so that $c_1 = 0$; then $c_2 = 0$ as well. It follows that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are linearly independent.

18. Using Exercise 21 in Section 2.3, we show first that $\mathbf{u}$ and $\mathbf{v}$ are linear combinations of $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$. But this is clear: $\mathbf{u} = \mathbf{u}$, and $\mathbf{v} = (\mathbf{u}+\mathbf{v}) - \mathbf{u}$. This shows that $\operatorname{span}(\mathbf{u}, \mathbf{v}) \subseteq \operatorname{span}(\mathbf{u}, \mathbf{u}+\mathbf{v})$. Next, we show that $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ are linear combinations of $\mathbf{u}$ and $\mathbf{v}$. But this is also clear. Thus $\operatorname{span}(\mathbf{u}, \mathbf{u}+\mathbf{v}) \subseteq \operatorname{span}(\mathbf{u}, \mathbf{v})$. Hence the two spans are equal.

19. Note that $\operatorname{rank}(A) \leq \operatorname{rank}([A \mid \mathbf{b}])$, since any zero rows in an echelon form of $[A \mid \mathbf{b}]$ are also zero rows in a row echelon form of A, so that a row echelon form of A has at least as many zero rows as a row echelon form of $[A \mid \mathbf{b}]$. If $\operatorname{rank}(A) < \operatorname{rank}([A \mid \mathbf{b}])$, then a row echelon form of $[A \mid \mathbf{b}]$ will contain at least one row that is zero in the columns corresponding to A but nonzero in the last column on that row, so that the system will be inconsistent. Hence the system is consistent only if $\operatorname{rank}(A) = \operatorname{rank}([A \mid \mathbf{b}])$.

20. Row-reduce both A and B:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 4 & 1 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 5 & 2 \end{bmatrix} \xrightarrow{R_3-5R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since in row echelon form each of these matrices has no zero rows, it follows that both matrices have rank 3, so that their reduced row echelon forms are both I_3. Thus the matrices are row-equivalent. The sequence of row operations required to derive B from A is the sequence of row operations required to reduce A to I_3, followed by the sequence of row operations, in reverse, required to reduce B to I_3.

Chapter 3

Matrices

3.1 Matrix Operations

1. Since A and D have the same shape, this operation makes sense.

$$A + 2D = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} + 2\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 3+0 & 0-6 \\ -1-4 & 5+2 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}$$

2. Since D and A have the same shape, this operation makes sense.

$$3D - 2A = 3\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} - 2\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ -6 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ -2 & 10 \end{bmatrix} = \begin{bmatrix} 0-6 & -9-0 \\ -6-(-2) & 3-10 \end{bmatrix} = \begin{bmatrix} -6 & -9 \\ -4 & -7 \end{bmatrix}$$

3. Since B is a 2×3 matrix and C is a 3×2 matrix, they do not have the same shape, so they cannot be subtracted.

4. Since both C and B^T are 3×2 matrices, this operation makes sense.

$$C - B^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ -2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1-4 & 2-0 \\ 3-(-2) & 4-2 \\ 5-1 & 6-3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5 & 2 \\ 4 & 3 \end{bmatrix}$$

5. Since A has two columns and B has two rows, the product makes sense, and

$$AB = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 4 + 0 \cdot 0 & 3 \cdot (-2) + 0 \cdot 2 & 3 \cdot 1 + 0 \cdot 3 \\ -1 \cdot 4 + 5 \cdot 0 & -1 \cdot (-2) + 5 \cdot 2 & -1 \cdot 1 + 5 \cdot 3 \end{bmatrix} = \begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}.$$

6. Since the number of columns of B (3) does not equal the number of rows of D (2), this matrix product does not make sense.

7. Since B has three columns and C has three rows, the multiplication BC makes sense, and results in a 2×2 matrix (since B has 2 rows and C has 2 columns). Since D is also a 2×2 matrix, they can be subtracted, so this formula makes sense. First,

$$BC = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 - 2 \cdot 3 + 1 \cdot 5 & 4 \cdot 2 - 2 \cdot 4 + 1 \cdot 6 \\ 0 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 & 0 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 21 & 26 \end{bmatrix}.$$

Then

$$D + BC = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 21 & 26 \end{bmatrix} = \begin{bmatrix} 0+3 & -3+6 \\ -2+21 & 1+26 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 19 & 27 \end{bmatrix}.$$

8. Since B has three columns and B^T has three rows, the product makes sense, and

$$\begin{aligned}
BB^T &= \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}^T \\
&= \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -2 & 2 \\ 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 4 \cdot 4 - 2 \cdot (-2) + 1 \cdot 1 & 4 \cdot 0 - 2 \cdot 2 + 1 \cdot 3 \\ 0 \cdot 4 + 2 \cdot (-2) + 3 \cdot 1 & 0 \cdot 0 + 2 \cdot 2 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 21 & -1 \\ -1 & 13 \end{bmatrix}
\end{aligned}$$

9. Since A has two columns and F has two rows, AF makes sense; since A has two rows and F has one column, the result is a 2×1 matrix. So this matrix has two rows, and E has two columns, so we can multiply by E on the left. Thus this product makes sense. Then

$$\begin{aligned}
AF &= \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \cdot (-1) + 0 \cdot 2 \\ -1 \cdot (-1) + 5 \cdot 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \end{bmatrix} \text{ and then} \\
E(AF) &= \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \cdot (-3) + 2 \cdot 11 \end{bmatrix} = \begin{bmatrix} 10 \end{bmatrix}.
\end{aligned}$$

10. F has shape 2×1. D has shape 2×2, so DF has shape 2×1. Since the number of columns (1) of F does not equal the number of rows (2) of DF, the product does not make sense.

11. Since F has one column and E has one row, the product makes sense, and

$$FE = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 \cdot 4 & -1 \cdot 2 \\ 2 \cdot 4 & 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}.$$

12. Since E has two columns and F has one row, the product makes sense, and

$$EF = \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot (-1) + 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}.$$

(Note that FE from Exercise 11 and EF are not the same matrix. In fact, they don't even have the same shape! In general, matrix multiplication is not commutative.)

13. Since B has two rows, B^T has two columns; since C has two columns, C^T has two rows. Thus B^TC^T makes sense, and the result is a 3×3 matrix (since B^T has three rows and C^T has three columns). Next, CB is defined (3×2 multiplied by 2×3) and also yields a 3×3 matrix, so its transpose is also a 3×3 matrix. Thus $B^TC^T - (BC)^T$ is defined.

$$\begin{aligned}
B^TC^T &= \begin{bmatrix} 4 & 0 \\ -2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 + 0 \cdot 2 & 4 \cdot 3 + 0 \cdot 4 & 4 \cdot 5 + 0 \cdot 6 \\ -2 \cdot 1 + 2 \cdot 2 & -2 \cdot 3 + 2 \cdot 4 & -2 \cdot 5 + 2 \cdot 6 \\ 1 \cdot 1 + 3 \cdot 2 & 1 \cdot 3 + 3 \cdot 4 & 1 \cdot 5 + 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 4 & 12 & 20 \\ 2 & 2 & 2 \\ 7 & 15 & 23 \end{bmatrix} \\
CB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 0 & 1 \cdot (-2) + 2 \cdot 2 & 1 \cdot 1 + 2 \cdot 3 \\ 3 \cdot 4 + 4 \cdot 0 & 3 \cdot (-2) + 4 \cdot 2 & 3 \cdot 1 + 4 \cdot 3 \\ 5 \cdot 4 + 6 \cdot 0 & 5 \cdot (-2) + 6 \cdot 2 & 5 \cdot 1 + 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 7 \\ 12 & 2 & 15 \\ 20 & 2 & 23 \end{bmatrix} \\
(CB)^T &= \begin{bmatrix} 4 & 2 & 7 \\ 12 & 2 & 15 \\ 20 & 2 & 23 \end{bmatrix}^T = \begin{bmatrix} 4 & 12 & 20 \\ 2 & 2 & 2 \\ 7 & 15 & 23 \end{bmatrix}.
\end{aligned}$$

Thus $B^TC^T = (CB)^T$, so their difference is the zero matrix.

14. Since A and D are both 2×2 matrices, the products in both orders are defined, and each results in a 2×2 matrix, so their difference is defined as well.

$$\begin{aligned}
DA &= \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 0\cdot 3 - 3\cdot(-1) & 0\cdot 0 - 3\cdot 5 \\ -2\cdot 3 + 1\cdot(-1) & -2\cdot 0 + 1\cdot 5 \end{bmatrix} = \begin{bmatrix} 3 & -15 \\ -7 & 5 \end{bmatrix} \\
AD &= \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 0 + 0\cdot(-2) & 3\cdot(-3) + 0\cdot 1 \\ -1\cdot 0 + 5\cdot(-2) & -1\cdot(-3) + 5\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ -10 & 8 \end{bmatrix} \\
DA - AD &= \begin{bmatrix} 3 & -15 \\ -7 & 5 \end{bmatrix} - \begin{bmatrix} 0 & -9 \\ -10 & 8 \end{bmatrix} = \begin{bmatrix} 3-0 & -15-(-9) \\ -7-(-10) & 5-8 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 3 & -3 \end{bmatrix}.
\end{aligned}$$

15. Since A is 2×2, A^2 is defined and is also a 2×2 matrix, so we can multiply by A again to get A^3.

$$\begin{aligned}
A^2 &= \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3\cdot 3 + 0\cdot(-1) & 3\cdot 0 + 0\cdot 5 \\ -1\cdot 3 + 5\cdot(-1) & -1\cdot 0 + 5\cdot 5 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ -8 & 25 \end{bmatrix} \\
A^3 = AA^2 &= \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 9 & 0 \\ -8 & 25 \end{bmatrix} = \begin{bmatrix} 3\cdot 9 + 0\cdot(-8) & 3\cdot 0 + 0\cdot 25 \\ -1\cdot 9 + 5\cdot(-8) & -1\cdot 0 + 5\cdot 25 \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ -49 & 125 \end{bmatrix}.
\end{aligned}$$

16. Since D is a 2×2 matrix, $I_2 - A$ makes sense and is also a 2×2 matrix. Since it has the same number of rows as columns, squaring it (which is multiplying it by itself) also makes sense.

$$I_2 - D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}.$$

Thus

$$(I_2 - D)^2 = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1\cdot 1 + 3\cdot 2 & 1\cdot 3 + 3\cdot 0 \\ 2\cdot 1 + 0\cdot 2 & 2\cdot 3 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$$

17. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then if $A^2 = 0$,

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If $c = 0$, so that the upper right-hand entry is zero, then we must also have $a = 0$ and $d = 0$ so that the diagonal entries are zero. So $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is one solution. Alternatively, if $c \neq 0$, then $ac = -cd$ and thus $a = -d$. This will make the lower left entry zero as well, and the two diagonal entries will both be equal to $a^2 + bc$. So any values of a, b, and $c \neq 0$ with $a^2 = -bc$ will work. For example, $a = b = 1$, $c = -1$, $d = -1$:

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

18. Note that the first column of A is twice its second column. Thus

$$A\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

19. The cost of shipping one unit of each product is given by $B = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix}$, where the first row gives the costs of shipping each product by truck, and the second row the costs when shipping by train. Then BA gives the costs of shipping all the products to each of the warehouses by truck or train:

$$BA = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix}\begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} = \begin{bmatrix} 650.00 & 462.50 \\ 675.00 & 406.25 \end{bmatrix}.$$

For example, the upper left entry in the product is the sum of $1.50 \cdot 200$, the cost of shipping doohickies to warehouse 1 by truck, $1.00 \cdot 150$, the cost of shipping gizmos to warehouse 1 by truck, and $2.00 \cdot 100$, the cost of shipping widgets to warehouse 1 by truck. So 650.00 is the cost of shipping all of the items to warehouse 1 by truck. Comparing, we see that it is cheaper to ship items to warehouse 1 by truck (\$650.00 vs \$675.00), but to warehouse 2 by train (\$406.25 vs \$462.50).

20. The cost of distributing one unit of the product is given by $C = \begin{bmatrix} 0.75 & 0.75 & 0.75 \\ 1.00 & 1.00 & 1.00 \end{bmatrix}$, where the entry in row i, column j is the cost of shipping one unit of product j from warehouse i. Then the cost of distribution is

$$CA = \begin{bmatrix} 0.75 & 0.75 & 0.75 \\ 1.00 & 1.00 & 1.00 \end{bmatrix} \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} = \begin{bmatrix} 337.50 & 225.00 \\ 450.00 & 300.00 \end{bmatrix}$$

In this product, the entry in row i, column i represents the total cost of shipping all the products from warehouse i. The off-diagonal entries do not represent anything in particular, since they are a result of multiplying shipping costs from one warehouse with products from the other warehouse.

21. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, so that $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

22. $\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, so that $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}.$$

23. The column vectors of B are

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix},$$

so the matrix-column representation is

$$\begin{aligned} A\mathbf{b}_1 &= 2 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 5 \end{bmatrix} \\ A\mathbf{b}_2 &= 3 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ -4 \\ 0 \end{bmatrix} \\ A\mathbf{b}_3 &= 0 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \\ -4 \end{bmatrix}. \end{aligned}$$

Thus

$$AB = \begin{bmatrix} 4 & -9 & -8 \\ -6 & -4 & 5 \\ 5 & 0 & -4 \end{bmatrix}.$$

24. The row vectors of A are

$$\mathbf{A}_1 = \begin{bmatrix}1 & 0 & -2\end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix}-3 & 1 & 1\end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix}2 & 0 & -1\end{bmatrix},$$

so the row-matrix representation is

$$\begin{aligned}
\mathbf{A}_1 B &= 1\begin{bmatrix}2 & 3 & 0\end{bmatrix} + 0\begin{bmatrix}1 & -1 & 1\end{bmatrix} - 2\begin{bmatrix}-1 & 6 & 4\end{bmatrix} = \begin{bmatrix}4 & -9 & -8\end{bmatrix}\\
\mathbf{A}_2 B &= -3\begin{bmatrix}2 & 3 & 0\end{bmatrix} + 1\begin{bmatrix}1 & -1 & 1\end{bmatrix} + 1\begin{bmatrix}-1 & 6 & 4\end{bmatrix} = \begin{bmatrix}-6 & -4 & 5\end{bmatrix}\\
\mathbf{A}_3 B &= 2\begin{bmatrix}2 & 3 & 0\end{bmatrix} + 0\begin{bmatrix}1 & -1 & 1\end{bmatrix} - 1\begin{bmatrix}-1 & 6 & 4\end{bmatrix} = \begin{bmatrix}5 & 0 & -4\end{bmatrix}.
\end{aligned}$$

Thus

$$AB = \begin{bmatrix}4 & -9 & -8\\ -6 & -4 & 5\\ 5 & 0 & -4\end{bmatrix}.$$

25. The outer product expansion is

$$\begin{aligned}
\mathbf{a}_1\mathbf{B}_1 + \mathbf{a}_2\mathbf{B}_2 + \mathbf{a}_3\mathbf{B}_3 &= \begin{bmatrix}1\\ -3\\ 2\end{bmatrix}\begin{bmatrix}2 & 3 & 0\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\end{bmatrix} + \begin{bmatrix}-2\\ 1\\ -1\end{bmatrix}\begin{bmatrix}-1 & 6 & 4\end{bmatrix}\\
&= \begin{bmatrix}2 & 3 & 0\\ -6 & -9 & 0\\ 4 & 6 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 1 & -1 & 1\\ 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}2 & -12 & -8\\ -1 & 6 & 4\\ 1 & -6 & -4\end{bmatrix}\\
&= \begin{bmatrix}4 & -9 & -8\\ -6 & -4 & 5\\ 5 & 0 & -4\end{bmatrix}.
\end{aligned}$$

26. We have

$$\begin{aligned}
B\mathbf{a}_1 &= 1\begin{bmatrix}2\\ 1\\ -1\end{bmatrix} - 3\begin{bmatrix}3\\ -1\\ 6\end{bmatrix} + 2\begin{bmatrix}0\\ 1\\ 4\end{bmatrix} = \begin{bmatrix}-7\\ 6\\ -11\end{bmatrix}\\
B\mathbf{a}_2 &= 0\begin{bmatrix}2\\ 1\\ -1\end{bmatrix} + 1\begin{bmatrix}3\\ -1\\ 6\end{bmatrix} + 0\begin{bmatrix}0\\ 1\\ 4\end{bmatrix} = \begin{bmatrix}3\\ -1\\ 6\end{bmatrix}\\
B\mathbf{a}_3 &= -2\begin{bmatrix}2\\ 1\\ -1\end{bmatrix} + 1\begin{bmatrix}3\\ -1\\ 6\end{bmatrix} - 1\begin{bmatrix}0\\ 1\\ 4\end{bmatrix} = \begin{bmatrix}-1\\ -4\\ 4\end{bmatrix}.
\end{aligned}$$

Thus

$$BA = \begin{bmatrix}-7 & 3 & -1\\ 6 & -1 & -4\\ -11 & 6 & 4\end{bmatrix}.$$

27. We have

$$\begin{aligned}
\mathbf{B}_1 A &= 2\begin{bmatrix}1 & 0 & -2\end{bmatrix} + 3\begin{bmatrix}-3 & 1 & 1\end{bmatrix} + 0\begin{bmatrix}2 & 0 & -1\end{bmatrix} = \begin{bmatrix}-7 & 3 & -1\end{bmatrix}\\
\mathbf{B}_2 A &= 1\begin{bmatrix}1 & 0 & -2\end{bmatrix} - 1\begin{bmatrix}-3 & 1 & 1\end{bmatrix} + 1\begin{bmatrix}2 & 0 & -1\end{bmatrix} = \begin{bmatrix}6 & -1 & -4\end{bmatrix}\\
\mathbf{B}_3 A &= -1\begin{bmatrix}1 & 0 & -2\end{bmatrix} + 6\begin{bmatrix}-3 & 1 & 1\end{bmatrix} + 4\begin{bmatrix}2 & 0 & -1\end{bmatrix} = \begin{bmatrix}-11 & 6 & 4\end{bmatrix}.
\end{aligned}$$

Thus

$$BA = \begin{bmatrix}-7 & 3 & -1\\ 6 & -1 & -4\\ -11 & 6 & 4\end{bmatrix}.$$

28. The outer product expansion is

$$\begin{aligned}
\mathbf{b}_1\mathbf{A}_1 + \mathbf{b}_2\mathbf{A}_2 + \mathbf{b}_3\mathbf{A}_3 &= \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \begin{bmatrix} -3 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & -4 \\ 1 & 0 & -2 \\ -1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} -9 & 3 & 3 \\ 3 & -1 & -1 \\ -18 & 6 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -1 \\ 8 & 0 & -4 \end{bmatrix} \\
&= \begin{bmatrix} -7 & 3 & -1 \\ 6 & -1 & -4 \\ -11 & 6 & 4 \end{bmatrix}.
\end{aligned}$$

29. Assume that the columns of $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{bmatrix}$ are linearly dependent. Then there exists a solution to $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n = \mathbf{0}$ with at least one $x_i \neq 0$. Now, we have

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_n \end{bmatrix},$$

so that $A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n) = x_1(A\mathbf{b}_1) + x_2(A\mathbf{b}_2) + \cdots + x_n(A\mathbf{b}_n) = \mathbf{0}$, so that the columns of AB are linearly dependent.

30. Let

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}, \qquad \text{so that} \qquad AB = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_n B \end{bmatrix}.$$

Since the rows of A are linearly dependent, there are x_i, not all zero, such that $x_1\mathbf{a}_1+x_2\mathbf{a}_2+\cdots+x_n\mathbf{a}_n = \mathbf{0}$. But then $(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n)B = x_1(\mathbf{a}_1 B) + x_2(\mathbf{a}_2 B) + \cdots + x_n(\mathbf{a}_n B) = \mathbf{0}$, so that the rows of AB are linearly dependent.

31. We have the block structure

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

Now, the blocks A_{12}, A_{21}, B_{12}, and B_{21} are all zero blocks, so any products involving those block are zero. Thus AB reduces to

$$AB = \begin{bmatrix} A_{11}B_{11} & 0 \\ 0 & A_{22}B_{22} \end{bmatrix}.$$

But

$$\begin{aligned}
A_{11}B_{11} &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 - 1 \cdot (-1) & 1 \cdot 3 - 1 \cdot 1 \\ 0 \cdot 2 + 1 \cdot (-1) & 0 \cdot 3 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\
A_{22}B_{22} &= \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}.
\end{aligned}$$

So finally,

$$AB = \begin{bmatrix} A_{11}B_{11} & 0 \\ 0 & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

32. We have

$$AB = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_1B_{11} + A_2B_{21} & A_1B_{12} + A_2B_{22} \end{bmatrix}.$$

But B_{11} is the zero matrix, and both A_2 and B_{12} are the identity matrix I_2, so this expression simplifies to

$$AB = \begin{bmatrix} I_2B_{21} & A_1I_2 + I_2B_{22} \end{bmatrix} = \begin{bmatrix} B_{21} & A_1 + B_{22} \end{bmatrix} = \begin{bmatrix} 1 & 7 & 7 \\ -2 & 7 & 7 \end{bmatrix}.$$

33. We have the block structure

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

Now, B_{21} is the zero matrix, so products involving that block are zero. Further, A_{12}, A_{21}, and B_{11} are all the identity matrix I_2. So this product simplifies to

$$AB = \begin{bmatrix} A_{11}I_2 & A_{11}B_{12} + I_2B_{22} \\ I_2I_2 & I_2B_{12} + A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{11}B_{12} + B_{22} \\ I_2 & B_{12} + A_{22}B_{22} \end{bmatrix}.$$

The remaining products are

$$\begin{aligned} A_{11}B_{12} + B_{22} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + B_{22} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 5 & 3 \end{bmatrix} \\ B_{12} + A_{22}B_{22} &= B_{12} + \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Thus

$$AB = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 4 & 5 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

34. We have the block structure

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

Now, A_{11} is the identity matrix I_3, and A_{21} is the zero matrix. Further, B_{22} is a 1×1 matrix, as is A_{22}, so they both behave just like scalar multiplication. Thus this product reduces to

$$AB = \begin{bmatrix} I_3B_{11} + A_{12}B_{21} & I_3B_{12} - A_{12} \\ 4B_{21} & 4 \cdot (-1) \end{bmatrix} = \begin{bmatrix} B_{11} + A_{12}B_{21} & B_{12} - A_{12} \\ 4B_{21} & -4 \end{bmatrix}.$$

Compute the matrices in the first row:

$$\begin{aligned} B_{11} + A_{12}B_{21} &= B_{11} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 6 \\ 3 & 3 & 4 \end{bmatrix} \\ B_{12} - A_{12} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}. \end{aligned}$$

So

$$AB = \begin{bmatrix} 2 & 3 & 4 & 0 \\ 2 & 3 & 6 & -1 \\ 3 & 3 & 4 & -2 \\ 4 & 4 & 4 & -4 \end{bmatrix}.$$

35. **(a)** Computing the powers of A as required, we get

$$A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},$$
$$A^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad A^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \quad A^7 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = A.$$

Note that $A^6 = I_2$ and $A_7 = A$.

(b) Since $A^6 = I_2$, we know that $A^{6k} = \left(A^6\right)^k = I_2^k = I_2$ for any positive integer value of k. Since $2015 = 2010 + 5 = 6 \cdot 370 + 5$, we have

$$A^{2015} = A^{6\cdot 370+5} = A^{6\cdot 370}A^5 = I_2 A^5 = A^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

36. We have

$$B^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B^3 = B^2 B = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad B^4 = (B^2)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Thus any power of B whose exponent is divisible by 8 is equal to I_2. Now, $2015 = 8 \cdot 251 + 7$, so that

$$B^{2015} = B^{8\cdot 251} \cdot B^7 = B^7 = B^4 \cdot B^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

37. Using induction, we show that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for $n \geq 1$. The base case, $n = 1$, is just the definition of $A^1 = A$. Now assume that this equation holds for k. Then

$$A^{k+1} = A^1 A^k \overset{\substack{\text{inductive} \\ \text{hypothesis}}}{=} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}.$$

So by induction, the formula holds for all $n \geq 1$.

38. (a)

$$A^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}.$$

But $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and $2\cos\theta\sin\theta = \sin 2\theta$, so that we get

$$A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

(b) We have already proven the cases $n = 1$ and $n = 2$, which serve as the basis for the induction. Now assume that the given formula holds for k. Then

$$\begin{aligned} A^{k+1} = A^1 A^k &\overset{\substack{\text{inductive} \\ \text{hypothesis}}}{=} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos k\theta - \sin\theta\sin k\theta & -(\cos\theta\sin k\theta + \sin\theta\cos k\theta) \\ \sin\theta\cos k\theta + \cos\theta\sin k\theta & \cos\theta\cos k\theta - \sin\theta\sin k\theta \end{bmatrix}. \end{aligned}$$

But $\cos\theta\cos k\theta - \sin\theta\sin k\theta = \cos(\theta + k\theta) = \cos(k+1)\theta$, and $\sin\theta\cos k\theta + \cos\theta\sin k\theta = \sin(\theta + k\theta) = \sin(k+1)\theta$, so this matrix becomes

$$A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}.$$

So by induction, the formula holds for all $k \geq 1$.

39. (a) $a_{ij} = (-1)^{i+j}$ means that if $i + j$ is even, then the corresponding entry is 1; otherwise, it is -1.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

(b) Each entry is its row number minus its column number:

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$

(c)

$$A = \begin{bmatrix} (1-1)^1 & (1-1)^2 & (1-1)^3 & (1-1)^4 \\ (2-1)^1 & (2-1)^2 & (2-1)^3 & (2-1)^4 \\ (3-1)^1 & (3-1)^2 & (3-1)^3 & (3-1)^4 \\ (4-1)^1 & (4-1)^2 & (4-1)^3 & (4-1)^4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \end{bmatrix}.$$

(d)

$$\begin{aligned} A &= \begin{bmatrix} \sin\frac{(1+1-1)\pi}{4} & \sin\frac{(1+2-1)\pi}{4} & \sin\frac{(1+3-1)\pi}{4} & \sin\frac{(1+4-1)\pi}{4} \\ \sin\frac{(2+1-1)\pi}{4} & \sin\frac{(2+2-1)\pi}{4} & \sin\frac{(2+3-1)\pi}{4} & \sin\frac{(2+4-1)\pi}{4} \\ \sin\frac{(3+1-1)\pi}{4} & \sin\frac{(3+2-1)\pi}{4} & \sin\frac{(3+3-1)\pi}{4} & \sin\frac{(3+4-1)\pi}{4} \\ \sin\frac{(4+1-1)\pi}{4} & \sin\frac{(4+2-1)\pi}{4} & \sin\frac{(4+3-1)\pi}{4} & \sin\frac{(4+4-1)\pi}{4} \end{bmatrix} \\ &= \begin{bmatrix} \sin\frac{\pi}{4} & \sin\frac{\pi}{2} & \sin\frac{3\pi}{4} & \sin\pi \\ \sin\frac{\pi}{2} & \sin\frac{3\pi}{4} & \sin\pi & \sin\frac{5\pi}{4} \\ \sin\frac{3\pi}{4} & \sin\pi & \sin\frac{5\pi}{4} & \sin\frac{3\pi}{2} \\ \sin\pi & \sin\frac{5\pi}{4} & \sin\frac{3\pi}{2} & \sin\frac{7\pi}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} & 0 \\ 1 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & -1 \\ 0 & -\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} \end{bmatrix}. \end{aligned}$$

40. (a) This matrix has zeros whenever the row number exceeds the column number, and the nonzero entries are the sum of the row and column numbers:

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 8 & 9 & 10 \\ 0 & 0 & 0 & 0 & 10 & 11 \\ 0 & 0 & 0 & 0 & 0 & 12 \end{bmatrix}.$$

(b) This matrix has 1's on the main diagonal and one diagonal away from the main diagonal, and zeros elsewhere:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(c) This matrix is zero except where the row number plus the column number is 6, 7, or 8:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

41. Let A be an $m \times n$ matrix with rows $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. Since $\mathbf{e}_i$ has a 1 in the i^{th} position and zeros elsewhere, we get

$$\mathbf{e}_i A = 0 \cdot \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + \cdots + 1 \cdot \mathbf{a}_i + \cdots + 0 \cdot \mathbf{a}_m = \mathbf{a}_i,$$

which is the i^{th} row of A.

3.2 Matrix Algebra

1. Since $X - 2A + 3B = O$, we have $X = 2A - 3B$, so

$$X = 2\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} - 3\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2\cdot 1 - 3\cdot(-1) & 2\cdot 2 - 3\cdot 0\\ 2\cdot 3 - 3\cdot 1 & 2\cdot 4 - 3\cdot 1\end{bmatrix} = \begin{bmatrix}5 & 4\\ 3 & 5\end{bmatrix}.$$

2. Since $2X = A - B$, we have $X = \frac{1}{2}(A - B)$, so

$$X = \frac{1}{2}(A-B) = \frac{1}{2}\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} - \begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}2 & 2\\ 2 & 3\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & \frac{3}{2}\end{bmatrix}$$

3. Solving for X gives

$$X = \frac{2}{3}(A+2B) = \frac{2}{3}\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} + 2\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}\right) = \frac{2}{3}\begin{bmatrix}-1 & 2\\ 5 & 6\end{bmatrix} = \begin{bmatrix}-\frac{2}{3} & \frac{4}{3}\\ \frac{10}{3} & 4\end{bmatrix}.$$

4. The given equation is $2A - 2B + 2X = 3X - 3A$; solving for X gives $X = 5A - 2B$.

$$X = 5A - 2B = 5\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} - 2\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}7 & 10\\ 13 & 18\end{bmatrix}$$

5. As in Example 3.16, we form a matrix whose columns consist of the entries of the matrices in question and row-reduce it:

$$\left[\begin{array}{rr|r}1 & 0 & 2\\ 2 & 1 & 5\\ -1 & 2 & 0\\ 1 & 1 & 3\end{array}\right] \to \left[\begin{array}{rr|r}1 & 0 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{array}\right]$$

Thus

$$\begin{bmatrix}2 & 5\\ 0 & 3\end{bmatrix} = 2\begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}.$$

6. As in Example 3.16, we form a matrix whose columns consist of the entries of the matrices in question and row-reduce it:

$$\left[\begin{array}{rrr|r}1 & 0 & 1 & 2\\ 0 & -1 & 1 & 3\\ 0 & 1 & 0 & -4\\ 1 & 0 & 1 & 2\end{array}\right] \to \left[\begin{array}{rrr|r}1 & 0 & 0 & 3\\ 0 & 1 & 0 & -4\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{array}\right]$$

so that

$$\begin{bmatrix}2 & 3\\ -4 & 2\end{bmatrix} = 3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - 4\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} - \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

7. As in Example 3.16, we form a matrix whose columns consist of the entries of the matrices in question and row-reduce it:

$$\left[\begin{array}{rrr|r}1 & -1 & 1 & 3\\ 0 & 2 & 1 & 1\\ -1 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\end{array}\right] \to \left[\begin{array}{rrr|r}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

So the system is inconsistent, and B is not a linear combination of A_1, A_2, and A_3.

8. As in Example 3.16, we form a matrix whose columns consist of the entries of the matrices in question and row-reduce it:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the linear system is consistent, we conclude that $B = A_1 + 2A_2 + 3A_3 + 4A_4$.

9. As in Example 3.17, let $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ be an element in the span of the matrices. Then form a matrix whose columns consist of the entries of the matrices in question and row-reduce it:

$$\left[\begin{array}{cc|c} 1 & 0 & w \\ 2 & 1 & x \\ -1 & 2 & y \\ 1 & 1 & z \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & w \\ 0 & 1 & x-2w \\ 0 & 0 & y+5w-2x \\ 0 & 0 & z+w-x \end{array}\right].$$

This gives the restrictions $y + 5w - 2x = 0$, so that $y = 2x - 5w$, and $z + w - x = 0$, so that $z = x - w$. So the span consists of matrices of the form

$$\begin{bmatrix} w & x \\ 2x-5w & x-w \end{bmatrix}.$$

10. As in Example 3.17, let $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ be an element in the span of the matrices. Then form a matrix whose columns consist of the entries of the matrices in question and row-reduce it:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & w \\ 0 & -1 & 1 & x \\ 0 & 1 & 0 & y \\ 1 & 0 & 1 & z \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & w+\frac{x-y}{2} \\ 0 & 1 & 0 & \frac{x+y}{2} \\ 0 & 0 & 1 & \frac{y-x}{2} \\ 0 & 0 & 0 & z-w \end{array}\right]$$

This gives the single restriction $z - w = 0$, so that $z = w$. So the span consists of matrices of the form

$$\begin{bmatrix} w & x \\ y & w \end{bmatrix}.$$

11. As in Example 3.17, let $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ be an element in the span of the matrices. Then form a matrix whose columns consist of the entries of the matrices in question and row-reduce it:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 2 & 1 & b \\ -1 & 0 & 1 & c \\ 0 & 0 & 0 & d \\ 1 & 1 & 0 & e \\ 0 & 0 & 0 & f \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & e \\ 0 & 1 & 0 & c+e \\ 0 & 0 & 1 & b-2c-2e \\ 0 & 0 & 0 & a+3b-4c-5e \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & f \end{array}\right].$$

We get the restrictions $a + 3b - 4c - 5e = 0$, so that $a = -3b + 4c + 5e$, as well as $d = f = 0$. Thus the span consists of matrices of the form

$$\begin{bmatrix} -3b+4c+5e & b & c \\ 0 & e & 0 \end{bmatrix}.$$

12. As in Example 3.17, let $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$ be an element in the span of the matrices. Then form a matrix whose columns consist of the entries of the matrices in question and row-reduce it:

$$\left[\begin{array}{rrrr|c} 1 & 0 & -1 & 1 & x_1 \\ 0 & 1 & 0 & -1 & x_2 \\ 0 & 1 & -1 & 1 & x_3 \\ 0 & 0 & 0 & 0 & x_4 \\ 1 & 0 & 1 & -1 & x_5 \\ 0 & 1 & 0 & -1 & x_6 \\ 0 & 0 & 0 & 0 & x_7 \\ 0 & 0 & 0 & 0 & x_8 \\ 1 & 0 & -1 & 1 & x_9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{x_1+x_5}{2} \\ 0 & 1 & 0 & 0 & x_3 + \frac{x_5-x_1}{2} \\ 0 & 0 & 1 & 0 & x_3 + x_5 - x_1 - x_2 \\ 0 & 0 & 0 & 1 & x_3 - x_2 + \frac{x_5-x_1}{2} \\ 0 & 0 & 0 & 0 & x_6 - x_2 \\ 0 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & x_7 \\ 0 & 0 & 0 & 0 & x_8 \\ 0 & 0 & 0 & 0 & x_9 - x_1 \end{array}\right]$$

We get the restrictions

$$x_6 = x_2, \quad x_9 = x_1, \quad x_4 = x_7 = x_8 = 0.$$

Thus the span consists of matrices of the form

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ 0 & x_5 & x_6 \\ 0 & 0 & x_1 \end{bmatrix}.$$

13. Following Example 3.18, we construct an augmented matrix whose columns are the entries of the given matrices and with zeros in the constant column, and row-reduce it:

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 0 \\ 4 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Since the only solution is the trivial solution, we conclude that these matrices are linearly independent.

14. Following Example 3.18, we construct an augmented matrix whose columns are the entries of the given matrices and with zeros in the constant column, and row-reduce it:

$$\left[\begin{array}{crc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 4 & -1 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the system has a nontrivial solution, the original matrices are not linearly independent; in fact, from the solutions we get from the row-reduced matrix,

$$\frac{1}{3}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

15. Following Example 3.18, we construct an augmented matrix whose columns are the entries of the given matrices and with zeros in the constant column, and row-reduce it:

$$\left[\begin{array}{rrrr|c} 0 & 1 & -2 & -1 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 9 & 0 \\ -1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the only solution is the trivial solution, we conclude that these matrices are linearly independent.

16. Following Example 3.18, we construct an augmented matrix whose columns are the entries of the given matrices and with zeros in the constant column, and row-reduce it:

$$\left[\begin{array}{rrrr|r} 1 & 2 & 1 & -1 & 0 \\ -1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 3 & 0 & 0 \\ 6 & 9 & 5 & -4 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The only solution is the trivial solution, so the matrices are linearly independent.

17. Let A, B, and C be $m \times n$ matrices.

(a)

$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} + a_{11} & \dots & b_{1n} + a_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} + a_{m1} & \dots & b_{mn} + a_{mn} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} + \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \\ &= B + A. \end{aligned}$$

(b)

$$\begin{aligned}
(A+B)+C &= \left(\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix}\right) + \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \dots & c_{mn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \dots & a_{mn}+b_{mn} \end{bmatrix} + \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \dots & c_{mn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}+b_{11}+c_{11} & \dots & a_{1n}+b_{1n}+c_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1}+c_{m1} & \dots & a_{mn}+b_{mn}+c_{mn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & \dots & b_{1n}+c_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1}+c_{m1} & \dots & b_{mn}+c_{mn} \end{bmatrix} \\
&= A + \left(\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} + \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \dots & c_{mn} \end{bmatrix}\right) \\
&= A + (B+C).
\end{aligned}$$

(c)

$$\begin{aligned}
A+O = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} &= \begin{bmatrix} a_{11}+0 & \dots & a_{1n}+0 \\ \vdots & \ddots & \vdots \\ a_{m1}+0 & \dots & a_{mn}+0 \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = A.
\end{aligned}$$

(d)

$$\begin{aligned}
A+(-A) &= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \left(-\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}\right) \\
&= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} -a_{11} & \dots & -a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{m1} & \dots & -a_{mn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}-a_{11} & \dots & a_{1n}-a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}-a_{m1} & \dots & a_{mn}-a_{mn} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \\
&= O.
\end{aligned}$$

18. Let A and B be $m \times n$ matrices, and c and d be scalars.

(a)

$$
\begin{aligned}
c(A+B) &= c\left(\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix}\right) \\
&= c\left(\begin{bmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \dots & a_{mn}+b_{mn} \end{bmatrix}\right) \\
&= \begin{bmatrix} c(a_{11}+b_{11}) & \dots & c(a_{1n}+b_{1n}) \\ \vdots & \ddots & \vdots \\ c(a_{m1}+b_{m1}) & \dots & c(a_{mn}+b_{mn}) \end{bmatrix} \\
&= \begin{bmatrix} ca_{11}+cb_{11} & \dots & ca_{1n}+cb_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1}+cb_{m1} & \dots & ca_{mn}+cb_{mn} \end{bmatrix} \\
&= \begin{bmatrix} ca_{11} & \dots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \dots & ca_{mn} \end{bmatrix} + \begin{bmatrix} cb_{11} & \dots & cb_{1n} \\ \vdots & \ddots & \vdots \\ cb_{m1} & \dots & cb_{mn} \end{bmatrix} \\
&= c\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + c\begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} \\
&= cA + cB.
\end{aligned}
$$

(b)

$$
\begin{aligned}
(c+d)A &= (c+d)\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \\
&= \begin{bmatrix} (c+d)a_{11} & \dots & (c+d)a_{1n} \\ \vdots & \ddots & \vdots \\ (c+d)a_{m1} & \dots & (c+d)a_{mn} \end{bmatrix} \\
&= \begin{bmatrix} ca_{11}+da_{11} & \dots & ca_{1n}+d_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1}+da_{m1} & \dots & ca_{mn}+da_{mn} \end{bmatrix} \\
&= \begin{bmatrix} ca_{11} & \dots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \dots & ca_{mn} \end{bmatrix} + \begin{bmatrix} da_{11} & \dots & da_{1n} \\ \vdots & \ddots & \vdots \\ da_{m1} & \dots & da_{mn} \end{bmatrix} \\
&= c\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + d\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \\
&= cA + dA.
\end{aligned}
$$

(c)

$$c(dA) = c\begin{bmatrix} da_{11} & \dots & da_{1n} \\ \vdots & \ddots & \vdots \\ da_{m1} & \dots & da_{mn} \end{bmatrix} = \begin{bmatrix} cda_{11} & \dots & cda_{1n} \\ \vdots & \ddots & \vdots \\ cda_{m1} & \dots & cda_{mn} \end{bmatrix} = cd\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}.$$

(d)

$$1A = 1\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} 1a_{11} & \dots & 1a_{1n} \\ \vdots & \ddots & \vdots \\ 1a_{m1} & \dots & 1a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = A.$$

19. Let A and B be $r \times n$ matrices, and C be an $n \times s$ matrix, so that the product makes sense. Then

$$\begin{aligned}
(A+B)C &= \left(\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{r1} & \dots & b_{rn} \end{bmatrix}\right)\begin{bmatrix} c_{11} & \dots & c_{1s} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{ns} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1}+b_{r1} & \dots & a_{rn}+b_{rn} \end{bmatrix}\begin{bmatrix} c_{11} & \dots & c_{1s} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{ns} \end{bmatrix} \\
&= \begin{bmatrix} (a_{11}+b_{11})c_{11}+\cdots+(a_{1n}+b_{1n})c_{n1} & \dots & (a_{11}+b_{11})c_{1s}+\cdots+(a_{1n}+b_{1n})c_{ns} \\ \vdots & \ddots & \vdots \\ (a_{r1}+b_{r1})c_{11}+\cdots+(a_{rn}+b_{rn})c_{n1} & \dots & (a_{r1}+b_{r1})c_{1s}+\cdots+(a_{r1}+b_{r1})c_{1s} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}c_{11}+\cdots+a_{1n}c_{n1} & \dots & a_{11}c_{1s}+\cdots+a_{1n}c_{ns} \\ \vdots & \ddots & \vdots \\ a_{r1}c_{11}+\cdots+a_{rn}c_{n1} & \dots & a_{r1}c_{1s}+\cdots+a_{r1}c_{ns} \end{bmatrix} + \\
&\qquad \begin{bmatrix} b_{11}c_{11}+\cdots+b_{1n}c_{n1} & \dots & b_{11}c_{1s}+\cdots+b_{1n}c_{ns} \\ \vdots & \ddots & \vdots \\ b_{r1}c_{11}+\cdots+b_{rn}c_{n1} & \dots & b_{r1}c_{1s}+\cdots+b_{r1}c_{ns} \end{bmatrix} \\
&= AC + BC.
\end{aligned}$$

20. Let A be an $r \times n$ matrix and C be an $n \times s$ matrix, so that the product makes sense. Let k be a scalar. Then

$$\begin{aligned}
k(AB) &= k\left(\begin{bmatrix} a_{11}c_{11}+\cdots+a_{1n}c_{n1} & \cdots & a_{11}c_{1s}+\cdots+a_{1n}c_{ns} \\ \vdots & \ddots & \vdots \\ a_{r1}c_{11}+\cdots+a_{rn}c_{n1} & \cdots & a_{r1}c_{1s}+\cdots+a_{rn}c_{ns} \end{bmatrix}\right) \\
&= \begin{bmatrix} k(a_{11}c_{11}+\cdots+a_{1n}c_{n1}) & \cdots & k(a_{11}c_{1s}+\cdots+a_{1n}c_{ns}) \\ \vdots & \ddots & \vdots \\ k(a_{r1}c_{11}+\cdots+a_{rn}c_{n1}) & \cdots & k(a_{r1}c_{1s}+\cdots+a_{rn}c_{ns}) \end{bmatrix} \\
&= \begin{bmatrix} (ka_{11})c_{11}+\cdots+(ka_{1n})c_{n1} & \cdots & (ka_{11})c_{1s}+\cdots+(ka_{1n})c_{ns} \\ \vdots & \ddots & \vdots \\ (ka_{r1})c_{11}+\cdots+(ka_{rn})c_{n1} & \cdots & (ka_{r1})c_{1s}+\cdots+(ka_{rn})c_{ns} \end{bmatrix} \\
&= \begin{bmatrix} ka_{11} & \dots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{r1} & \dots & ka_{rn} \end{bmatrix}\begin{bmatrix} c_{11} & \dots & c_{1s} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{ns} \end{bmatrix} \\
&= (kA)B.
\end{aligned}$$

That proves the first equality. Regrouping the expression on the second line above differently, we get

$$\begin{aligned}
k(AB) &= \begin{bmatrix} a_{11}(kc_{11}) + \cdots + a_{1n}(kc_{n1}) & \cdots & a_{11}(kc_{1s}) + \cdots + a_{1n}(kc_{ns}) \\ \vdots & \ddots & \vdots \\ a_{r1}(kc_{11}) + \cdots + a_{rn}(kc_{n1}) & \cdots & a_{r1}(kc_{1s}) + \cdots + a_{rn}(kc_{ns}) \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rn} \end{bmatrix} \begin{bmatrix} kc_{11} & \dots & kc_{1s} \\ \vdots & \ddots & \vdots \\ kc_{n1} & \dots & kc_{ns} \end{bmatrix} \\
&= A(kB).
\end{aligned}$$

21. If A is $m \times n$, then to prove $I_m A = A$, note that

$$I_m = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_m \end{bmatrix},$$

where $\mathbf{e}_i$ is a standard (row) unit vector; then

$$I_m A = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_m \end{bmatrix} A = \begin{bmatrix} \mathbf{e}_1 A \\ \mathbf{e}_2 A \\ \vdots \\ \mathbf{e}_m A \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{bmatrix} = A.$$

22. Note that

$$(A - B)(A + B) = A(A + B) - B(A + B) = A^2 + AB - BA + B^2 = (A^2 + B^2) + (AB - BA),$$

so that $(A - B)(A + B) = A^2 + B^2$ if and only if $AB - BA = O$; that is, if and only if $AB = BA$.

23. Compute both AB and BA:

$$\begin{aligned}
AB &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + c & b + d \\ c & d \end{bmatrix} \\
BA &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a + b \\ c & c + d \end{bmatrix}.
\end{aligned}$$

So if $AB = BA$, then $a + c = a$, $b + d = a + b$, $c = c$, and $d = c + d$. The first equation gives $c = 0$ and the second gives $d = a$. So the matrices that commute with A are matrices of the form

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

24. Compute both AB and BA:

$$\begin{aligned}
AB &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - c & b - d \\ -a + c & -b + d \end{bmatrix} \\
BA &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a - b & -a + b \\ c - d & -c + d \end{bmatrix}.
\end{aligned}$$

So if $AB = BA$, then $a - c = a - b$, $b - d = -a + b$, $-a + c = c - d$, and $-b + d = -c + d$. The first (or fourth) equation gives $b = c$ and the second (or third) gives $d = a$. So the matrices that commute with A are matrices of the form

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

25. Compute both AB and BA:

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} \\ BA &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}. \end{aligned}$$

So if $AB = BA$, then $a + 2c = a + 3b$, $b + 2d = 2a + 4b$, $3a + 4c = c + 3d$, and $3b + 4d = 2c + 4d$. The first (or fourth) equation gives $3b = 2c$ and the third gives $a = d - c$. Substituting $a = d - c$ into the second equation gives $3b = 2c$ again. Thus those are the only two conditions, and the matrices that commute with A are those of the form

$$\begin{bmatrix} d-c & \frac{2}{3}c \\ c & d \end{bmatrix}.$$

26. Let A_1 be the first matrix and A_4 be the second. Then

$$\begin{aligned} A_1B &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \\ BA_1 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \\ A_4B &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \\ BA_4 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}. \end{aligned}$$

So the conditions imposed by requiring that $A_1B = BA_1$ and $A_4B = BA_4$ are $b = c = 0$. Thus the matrices that commute with both A_1 and A_4 are those of the form

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

27. Since we want B to commute with every 2×2 matrix, in particular it must commute with A_1 and A_4 from the previous exercise, so that $B = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. In addition, it must commute with

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Computing the products, we get

$$\begin{aligned} A_2B &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} \\ BA_2 &= \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \\ A_3B &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \\ BA_3 &= \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix}. \end{aligned}$$

The condition that arises from these equalities is $a = d$. Thus

$$B = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

Finally, note that for any 2×2 matrix M,

$$M = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = xA_1 + yA_2 + zA_3 + wA_4,$$

so that if B commutes with A_1, A_2, A_3, and A_4, then it will commute with M. So the matrices B that satisfy this condition are the matrices above: those where the off-diagonal entries are zero and the diagonal entries are the same.

28. Suppose A is an $m\times n$ matrix and B is an $a\times b$ matrix. Since AB is defined, the number of columns (n) of A must equal the number of rows (a) of B; since BA is defined, the number of columns (b) of B must equal the number of rows (m) of A. Since $m=b$ and $n=a$, we see that A is $m\times n$ and B is $n\times m$. But then AB is $m\times m$ and BA is $n\times n$, so both are square.

29. Suppose $A=(a_{ij})$ and $B=(b_{ij})$ are both upper triangular $n\times n$ matrices. This means that $a_{ij} = b_{ij} = 0$ if $i>j$. But if $C=(c_{ij})=AB$, then

$$c_{kl} = a_{k1}b_{1l} + a_{k2}b_{2l} + \cdots + a_{kl}b_{ll} + a_{k(l+1)}b_{(l+1)l} + \cdots + a_{kn}b_{nl}.$$

If $k>l$, then $a_{k1}b_{1l} + a_{k2}b_{2l} + \cdots + a_{kl}b_{ll} = 0$ since each of the a_{kj} is zero. But also $a_{k(l+1)}b_{(l+1)l} + \cdots + a_{kn}b_{nl} = 0$ since each of the b_{il} is zero. Thus $c_{kl}=0$ for $k>l$, so that C is upper triangular.

30. Let A and B whose sizes permit the indicated operations, and let k be a scalar. Denote the i^{th} row of a matrix X by $\text{row}_i(X)$ and its j^{th} column by $\text{col}_j(X)$. Then

Theorem 3.4(a): For any i and j,

$$\left[\left(A^T\right)^T\right]_{ij} = \left[A^T\right]_{ji} = \left[A\right]_{ij}, \text{ so that } (A^T)^T = A.$$

Theorem 3.4(b): For any i and j,

$$\left[(A+B)^T\right]_{ij} = \left[A+B\right]_{ji} = \left[A\right]_{ji} + \left[B\right]_{ji} = \left[A^T\right]_{ij} + \left[B\right]_{ij}, \text{ so that } (A+B)^T = A^T + B^T.$$

Theorem 3.4(c): For any i and j,

$$\left[(kA)^T\right]_{ij} = \left[kA\right]_{ji} = k\left[A\right]_{ji} = k\left[A\right]_{ij}, \text{ so that } (kA)^T = k(A^T).$$

31. We prove that $(A^r)^T = (A^T)^r$ by induction on r. For $r=1$, this says that $(A^1)^T = (A^T)^1$, which is clear. Now assume that $(A^k)^T = (A^T)^k$. Then

$$(A^{k+1})^T = (A\cdot A^k)^T = (A^k)^T A^T = (A^T)^k A^T = (A^T)^{k+1}.$$

(The second equality is from Theorem 3.4(d), and the third is the inductive assumption).

32. We prove that $(A_1 + A_2 + \cdots + A_n)^T = A_1^T + A_2^T + \cdots + A_n^T$ by induction on n. For $n=1$, this says that $A_1^T = A_1^T$, which is clear. Now assume that $(A_1 + A_2 + \cdots + A_k)^T = A_1^T + A_2^T + \cdots + A_k^T$. Then

$$\begin{aligned}(A_1 + A_2 + \cdots + A_k + A_{k+1})^T &= ((A_1 + A_2 + \cdots + A_k) + A_{k+1})^T \\ &= (A_1 + A_2 + \cdots + A_k)^T + A_{k+1}^T \qquad \textit{(Thm. 3.4(b))} \\ &= A_1^T + A_2^T + \cdots + A_k^T + A_{k+1}^T \qquad \textit{(Inductive assumption).}\end{aligned}$$

33. We prove that $(A_1A_2\cdots A_n)^T = A_n^T A_{n-1}^T \cdots A_2^T A_1^T$ by induction on n. For $n=1$, this says that $A_1^T = A_1^T$, which is clear. Now assume that $(A_1A_2\cdots A_k)^T = A_k^T A_{k-1}^T \cdots A_2^T A_1^T$. Then

$$\begin{aligned}(A_1A_2\cdots A_kA_{k+1})^T &= ((A_1A_2\cdots A_k)A_{k+1})^T \\ &= A_{k+1}^T(A_1A_2\cdots A_k)^T \qquad \textit{(Thm. 3.4(d))} \\ &= A_{k+1}^T A_k^T \cdots A_2^T A_1^T \qquad \textit{(Inductive assumption).}\end{aligned}$$

34. Using Theorem 3.4, we have $(AA^T)^T = (A^T)^T A^T = AA^T$ and $(A^T A)^T = A^T (A^T)^T = A^T A$. So each of AA^T and $A^T A$ is equal to its own transpose, so both matrices are symmetric.

35. Let A and B be symmetric $n \times n$ matrices, and let k be a scalar.

(a) By Theorem 3.4(b), $(A+B)^T = A^T + B^T$; since both A and B are symmetric, this is equal to $A+B$. Thus $A+B$ is equal to its transpose, so is symmetric.

(b) By Theorem 3.4(c), $(kA)^T = kA^T$; since A is symmetric, this is equal to kA. Thus kA is equal to its own transpose, so is symmetric

36. **(a)** For example, let $n = 2$, with $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then both A and B are symmetric, but

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which is not symmetric.

(b) Suppose A and B are symmetric. Then if AB is symmetric, we have

$$AB = (AB)^T = B^T A^T = BA$$

using Theorem 3.4(d) and the fact that A and B are symmetric. In the other direction, if A and B are symmetric and $AB = BA$, then

$$(AB)^T = B^T A^T = BA = AB,$$

so that AB is symmetric.

37. **(a)** $A^T = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$ while $-A = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix}$, so $A^T \neq -A$ and A is not skew-symmetric.

(b) $A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A$, so A is skew-symmetric.

(c) $A^T = \begin{bmatrix} 0 & -3 & 1 \\ 3 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix} = -A$, so A is skew-symmetric.

(d) $A^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 5 \\ 2 & 5 & 0 \end{bmatrix} \neq -A$, so A is not skew-symmetric.

38. A matrix A is skew-symmetric if $A^T = -A$. But this means that for every i and j, $(A^T)_{ij} = A_{ji} = -A_{ij}$. Thus the components must satisfy $a_{ij} = -a_{ji}$ for every i and j.

39. By the previous exercise, we must have $a_{ij} = -a_{ji}$ for every i and j. So when $i = j$, we get $a_{ii} = -a_{ii}$, which implies that $a_{ii} = 0$ for all i.

40. If A and B are skew-symmetric, so that $A^T = -A$ and $B^T = -B$, then

$$(A+B)^T = A^T + B^T = -A - B = -(A+B),$$

so that $A+B$ is also skew-symmetric.

41. Suppose that $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ are skew-symmetric. Then

$$AB = \begin{bmatrix} -ab & 0 \\ 0 & -ab \end{bmatrix}, \text{ and thus } (AB)^T = \begin{bmatrix} -ab & 0 \\ 0 & -ab \end{bmatrix}.$$

So $AB = -(AB)^T$ only if $ab = -ab$, which happens if either a or b is zero. Thus either A or B must be the zero matrix.

42. We must show that $A-A^T$ is skew-symmetric, so we must show that $(A-A^T)^T = -(A-A^T) = A^T - A$. But

$$(A-A^T)^T = A^T - (A^T)^T = A^T - A$$

as desired, proving the result.

43. **(a)** If A is a square matrix, let $S = A + A^T$ and $S' = A - A^T$. Then clearly $A = \frac{1}{2}S + \frac{1}{2}S'$. S is symmetric by Theorem 3.5, so $\frac{1}{2}S$ is as well. S' is symmetric by Exercise 42, so $\frac{1}{2}S'$ is as well. Thus A is a sum of a symmetric and a skew-symmetric matrix.

(b) We have

$$\begin{aligned} S = A + A^T &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} \\ S = A - A^T &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}. \end{aligned}$$

Thus

$$A = \frac{1}{2}S + \frac{1}{2}S' = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

44. Let A and B be $n \times n$ matrices, and k be a scalar. Then

(a)

$$\begin{aligned} \operatorname{tr}(A+B) &= (a_{11}+b_{11}) + (a_{22}+b_{22}) + \cdots + (a_{nn}+b_{nn}) \\ &= (a_{11}+a_{22}+\cdots+a_{nn}) + (b_{11}+b_{22}+\cdots+b_{nn}) = \operatorname{tr}(A) + \operatorname{tr}(B). \end{aligned}$$

(b) $\operatorname{tr}(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k(a_{11}+a_{22}+\cdots+a_{nn}) = k\operatorname{tr}(A)$.

45. Let A and B be $n \times n$ matrices. Then

$$\begin{aligned} \operatorname{tr}(AB) &= (AB)_{11} + (AB)_{22} + \cdots + (AB)_{nn} \\ &= (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) + (a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2}) + \cdots \\ &\qquad (a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn}) \\ &= (b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1}) + (b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2n}a_{n2}) + \cdots \\ &\qquad (b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nn}a_{nn}) \\ &= \operatorname{tr}(BA). \end{aligned}$$

46. Note that the ii entry of AA^T is equal to

$$(AA^T)_{ii} = (A)_{i1}(A^T)_{1i} + (A)_{i2}(A^T)_{2i} + \cdots (A)_{in}(A^T)_{ni} = a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2,$$

so that

$$\operatorname{tr}(AA^T) = (a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2) + (a_{21}^2 + a_{22}^2 + \cdots + a_{2n}^2) + (a_{n1}^2 + a_{n2}^2 + \cdots + a_{nn}^2).$$

That is, it is the sum of the squares of the entries of A.

47. Note that by Exercise 45, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. By Exercise 44, then,

$$\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = \operatorname{tr}(AB) - \operatorname{tr}(AB) = 0.$$

Since $\operatorname{tr}(I_2) = 2$, it cannot be the case that $AB - BA = I_2$.

3.3 The Inverse of a Matrix

1. Using Theorem 3.8, note that

$$\det\begin{bmatrix}4 & 7\\ 1 & 2\end{bmatrix} = ad - bc = 4\cdot 2 - 1\cdot 7 = 1,$$

so that

$$\begin{bmatrix}4 & 7\\ 1 & 2\end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix}2 & -7\\ -1 & 4\end{bmatrix} = \begin{bmatrix}2 & -7\\ -1 & 4\end{bmatrix}.$$

As a check,

$$\begin{bmatrix}4 & 7\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & -7\\ -1 & 4\end{bmatrix} = \begin{bmatrix}4\cdot 2 + 7\cdot(-1) & 4\cdot(-7) + 7\cdot 4\\ 1\cdot 2 + 2\cdot(-1) & 1\cdot(-7) + 2\cdot 4\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

2. Using Theorem 3.8, note that

$$\det\begin{bmatrix}4 & -2\\ 2 & 0\end{bmatrix} = ad - bc = 4\cdot 0 - (-2)\cdot 2 = 4,$$

so that

$$\begin{bmatrix}4 & -2\\ 2 & 0\end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix}0 & 2\\ -2 & 4\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{2}\\ -\frac{1}{2} & 1\end{bmatrix}$$

As a check,

$$\begin{bmatrix}4 & -2\\ 2 & 0\end{bmatrix}\cdot\begin{bmatrix}0 & \frac{1}{2}\\ -\frac{1}{2} & 1\end{bmatrix} = \begin{bmatrix}4\cdot 0 - 2\cdot\left(-\frac{1}{2}\right) & 4\cdot\frac{1}{2} - 2\cdot 1\\ 2\cdot 0 + 0\cdot\left(-\frac{1}{2}\right) & 2\cdot\frac{1}{2} + 0\cdot 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

3. Using Theorem 3.8, note that

$$\det\begin{bmatrix}3 & 4\\ 6 & 8\end{bmatrix} = ad - bc = 3\cdot 8 - 4\cdot 6 = 0.$$

Since the determinant is zero, the matrix is not invertible.

4. Using Theorem 3.8, note that

$$\det\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = ad - bc = 0\cdot 0 - 1\cdot(-1) = 1,$$

so that

$$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}^{-1} = 1\cdot\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

As a check,

$$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\cdot\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0\cdot 0 + 1\cdot 1 & 0\cdot(-1) + 1\cdot 0\\ -1\cdot 0 + 0\cdot 1 & -1\cdot(-1) + 0\cdot 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

5. Using Theorem 3.8, note that

$$\det\begin{bmatrix}\frac{3}{4} & \frac{3}{5}\\ \frac{5}{6} & \frac{2}{3}\end{bmatrix} = ad - bc = \frac{3}{4}\cdot\frac{2}{3} - \frac{3}{5}\cdot\frac{5}{6} = \frac{1}{2} - \frac{1}{2} = 0.$$

Since the determinant is zero, the matrix is not invertible.

6. Using Theorem 3.8, note that

$$\det\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} = ad - bc = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2}+\frac{1}{2} = 1$$

so that

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}^{-1} = 1\cdot\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

As a check,

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}}\cdot-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

7. Using Theorem 3.8, note that

$$\det\begin{bmatrix}-1.5 & -4.2\\ 0.5 & 2.4\end{bmatrix} = ad - bc = -1.5\cdot 2.4 - (-4.2)\cdot 0.5 = -3.6 + 2.1 = -1.5.$$

Then

$$\begin{bmatrix}-1.5 & -4.2\\ 0.5 & 2.4\end{bmatrix}^{-1} = -\frac{1}{1.5}\cdot\begin{bmatrix}2.4 & 4.2\\ -0.5 & -1.5\end{bmatrix} = \begin{bmatrix}-1.6 & -2.8\\ 0.333 & 1\end{bmatrix}$$

As a check,

$$\begin{bmatrix}-1.5 & -4.2\\ 0.5 & 2.4\end{bmatrix}\begin{bmatrix}-1.6 & -2.8\\ 0.333 & 1\end{bmatrix} = \begin{bmatrix}-1.5\cdot(-1.6) - 4.2\cdot 0.333 & -1.5\cdot(-2.8) - 4.2\cdot 1\\ 0.5\cdot(-1.6) + 2.4\cdot 0.333 & 0.5\cdot(-2.8) + 2.4\cdot 1\end{bmatrix} \approx \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

8. Using Theorem 3.8, note that

$$\det\begin{bmatrix}3.55 & 0.25\\ 8.52 & 0.50\end{bmatrix} = ad - bc = 3.55\cdot 0.5 - 0.25\cdot 8.52 = 1.775 - 2.13 = -0.335.$$

Then

$$\begin{bmatrix}3.55 & 0.25\\ 8.52 & 0.50\end{bmatrix}^{-1} = -\frac{1}{0.335}\cdot\begin{bmatrix}0.5 & -0.25\\ -8.52 & 3.55\end{bmatrix} \approx \begin{bmatrix}-1.408 & 0.704\\ 24 & -10\end{bmatrix}$$

As a check,

$$\begin{bmatrix}3.55 & 0.25\\ 8.52 & 0.50\end{bmatrix}\begin{bmatrix}-1.408 & 0.704\\ 24 & -10\end{bmatrix} = \begin{bmatrix}3.55\cdot(-1.408) + 0.25\cdot 24 & 3.55\cdot 0.704 + 0.25\cdot(-10)\\ 8.52\cdot(-1.408) + 0.50\cdot 24 & 8.52\cdot 0.704 + 0.50\cdot(-10)\end{bmatrix} \approx \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

9. Using Theorem 3.8, note that

$$\det\begin{bmatrix}a & -b\\ b & a\end{bmatrix} = a\cdot a - (-b)\cdot b = a^2 + b^2.$$

Then

$$\begin{bmatrix}a & -b\\ b & a\end{bmatrix}^{-1} = \frac{1}{a^2+b^2}\cdot\begin{bmatrix}a & b\\ -b & a\end{bmatrix} = \begin{bmatrix}\frac{a}{a^2+b^2} & \frac{b}{a^2+b^2}\\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2}\end{bmatrix}$$

As a check,

$$\begin{aligned}\begin{bmatrix}a & -b\\ b & a\end{bmatrix}\begin{bmatrix}\frac{a}{a^2+b^2} & \frac{b}{a^2+b^2}\\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2}\end{bmatrix} &= \begin{bmatrix}a\cdot\frac{a}{a^2+b^2} - b\cdot\left(-\frac{b}{a^2+b^2}\right) & a\cdot\frac{b}{a^2+b^2} - b\cdot\frac{a}{a^2+b^2}\\ b\cdot\frac{a}{a^2+b^2} + a\cdot\left(-\frac{b}{a^2+b^2}\right) & b\cdot\frac{b}{a^2+b^2} + a\cdot\frac{a}{a^2+b^2}\end{bmatrix}\\ &= \begin{bmatrix}\frac{a^2+b^2}{a^2+b^2} & \frac{ab-ab}{a^2+b^2}\\ \frac{ab-ab}{a^2+b^2} & \frac{a^2+b^2}{a^2+b^2}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.\end{aligned}$$

10. Using Theorem 3.8, note that

$$\det\begin{bmatrix}\frac{1}{a} & \frac{1}{b}\\ \frac{1}{c} & \frac{1}{d}\end{bmatrix} = \frac{1}{a}\cdot\frac{1}{d} - \frac{1}{b}\cdot\frac{1}{c} = \frac{1}{ad} - \frac{1}{bc} = \frac{bc-ad}{abcd}$$

Then

$$\begin{bmatrix}\frac{1}{a} & \frac{1}{b}\\ \frac{1}{c} & \frac{1}{d}\end{bmatrix}^{-1} = \frac{abcd}{bc-ad}\cdot\begin{bmatrix}\frac{1}{d} & -\frac{1}{b}\\ -\frac{1}{c} & \frac{1}{a}\end{bmatrix} = \begin{bmatrix}\frac{abc}{bc-ad} & -\frac{acd}{bc-ad}\\ -\frac{abd}{bc-ad} & \frac{bcd}{bc-ad}\end{bmatrix}.$$

As a check,

$$\begin{bmatrix}\frac{1}{a} & \frac{1}{b}\\ \frac{1}{c} & \frac{1}{d}\end{bmatrix}\begin{bmatrix}\frac{abc}{bc-ad} & -\frac{acd}{bc-ad}\\ -\frac{abd}{bc-ad} & \frac{bcd}{bc-ad}\end{bmatrix} = \begin{bmatrix}\frac{bc}{bc-ad} - \frac{ad}{bc-ad} & -\frac{cd}{bc-ad} + \frac{cd}{bc-ad}\\ \frac{ab}{bc-ad} - \frac{ab}{bc-ad} & -\frac{ad}{bc-ad} + \frac{bc}{bc-ad}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

11. Following Example 3.25, we first invert the coefficient matrix

$$\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}.$$

The determinant of this matrix is $2\cdot 3 - 1\cdot 5 = 1$, so its inverse is

$$\frac{1}{1}\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix} = \begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}.$$

Thus the solution to the system is given by

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}^{-1}\cdot\begin{bmatrix}-1\\ 2\end{bmatrix} = \begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}\cdot\begin{bmatrix}-1\\ 2\end{bmatrix} = \begin{bmatrix}3\cdot(-1) - 1\cdot 2\\ -5\cdot(-1) + 2\cdot 2\end{bmatrix} = \begin{bmatrix}-5\\ 9\end{bmatrix},$$

so the solution is $x = -5$, $y = 9$.

12. Following Example 3.25, we first invert the coefficient matrix

$$\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}.$$

The determinant of this matrix is $1\cdot 1 - (-1)\cdot 2 = 3$, so its inverse is

$$\frac{1}{3}\cdot\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{1}{3}\end{bmatrix}$$

and thus the solution to the system is given by

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}^{-1}\cdot\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{1}{3}\end{bmatrix}\cdot\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 2\\ -\frac{2}{3}\cdot 1 + \frac{1}{3}\cdot 2\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}$$

and the solution is $x = 1$, $y = 0$.

13. **(a)** Following Example 3.25, we first invert the coefficient matrix

$$\begin{bmatrix}1 & 2\\ 2 & 6\end{bmatrix}.$$

The determinant of this matrix is $1\cdot 6 - 2\cdot 2 = 2$, so its inverse is

$$\frac{1}{2}\cdot\begin{bmatrix}6 & -2\\ -2 & 1\end{bmatrix} = \begin{bmatrix}3 & -1\\ -1 & \frac{1}{2}\end{bmatrix}.$$

Then the solutions for the three vectors given are

$$\begin{aligned}
\mathbf{x}_1 = A^{-1}\mathbf{b}_1 &= \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -\frac{1}{2} \end{bmatrix} \\
\mathbf{x}_2 = A^{-1}\mathbf{b}_2 &= \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \\
\mathbf{x}_3 = A^{-1}\mathbf{b}_3 &= \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.
\end{aligned}$$

(b) To solve all three systems at once, form the augmented matrix $\begin{bmatrix} A & | & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$ and row reduce to solve.

$$\left[\begin{array}{cc|ccc} 1 & 2 & 3 & -1 & 2 \\ 2 & 6 & 5 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|ccc} 1 & 0 & 4 & -5 & 6 \\ 0 & 1 & -\frac{1}{2} & 2 & -2 \end{array}\right]$$

(c) The method of constructing A^{-1} requires 7 multiplications, while row reductions requires only 6.

14. To show the result, we must show that $(cA) \cdot \frac{1}{c}A^{-1} = I$. But

$$(cA) \cdot \frac{1}{c}A^{-1} = \left(\frac{1}{c} \cdot c\right) AA^{-1} = AA^{-1} = I.$$

15. To show the result, we must show that $A^T(A^{-1})^T = I$. But

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I,$$

where the first equality follows from Theorem 3.4(d).

16. First we show that $I_n^{-1} = I_n$. To see that, we need to prove that $I_n \cdot I_n = I_n$, but that is true. Thus $I_n^{-1} = I_n$, and it also follows that I_n is invertible, since we found its inverse.

17. (a) There are many possible counterexamples. For instance, let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then $\det(AB) = 1 \cdot (-1) - 0 \cdot 3 = -1$, $\det A = 1 \cdot 1 - 0 \cdot 2 = 1$, and $\det B = 1 \cdot (-1) - 0 \cdot 1 = -1$, so that

$$\begin{aligned}
(AB)^{-1} &= \frac{1}{-1} \begin{bmatrix} -1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \\
A^{-1} &= \frac{1}{1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \\
B^{-1} &= \frac{1}{-1} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.
\end{aligned}$$

Finally,

$$A^{-1}B^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix},$$

and $(AB)^{-1} \neq A^{-1}B^{-1}$.

(b) Claim that $(AB)^{-1} = A^{-1}B^{-1}$ if and only if $AB = BA$. First, if $(AB)^{-1} = A^{-1}B^{-1}$, then

$$AB = ((AB)^{-1})^{-1} = (A^{-1}B^{-1})^{-1} = (B^{-1})^{-1}(A^{-1})^{-1} = BA.$$

To prove the reverse, suppose that $AB = BA$. Then

$$(AB)^{-1} = (BA)^{-1} = A^{-1}B^{-1}.$$

This proves the claim.

18. Use induction on n. For $n = 1$, the statement says that $(A_1)^{-1} = A_1^{-1}$, which is clear. Now assume that the statement holds for $n = k$. Then

$$(A_1A_2\cdots A_kA_{k+1})^{-1} = ((A_1A_2\cdots A_k)A_{k+1})^{-1} = A_{k+1}^{-1}(A_1A_2\cdots A_k)^{-1}$$

by Theorem 3.9(c). But then by the inductive hypothesis, this becomes

$$A_{k+1}^{-1}(A_1A_2\cdots A_k)^{-1} = A_{k+1}^{-1}(A_k^{-1}\cdots A_2^{-1}A_1^{-1}) = A_{k+1}^{-1}A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$

So by induction, the statement holds for all $k \geq 1$.

19. There are many examples. For instance, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = -A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $A^{-1} = A$ and $B^{-1} = B = -A$, so that $A^{-1} + B^{-1} = O$. But $A + B = O$, so $(A+B)^{-1}$ does not exist.

20. $XA^2 = A^{-1} \Rightarrow (XA^2)A^{-2} = A^{-1}A^{-2} \Rightarrow X(A^2A^{-2}) = A^{-1+(-2)} \Rightarrow XI = A^{-3} \Rightarrow X = A^{-3}$.

21. $AXB = (BA)^2 \Rightarrow A^{-1}(AXB)B^{-1} = A^{-1}(BA)^2B^{-1} \Rightarrow (A^{-1}A)X(BB^{-1}) = A^{-1}(BA)^2B^{-1}$. Thus $X = A^{-1}(BA)^2B^{-1}$.

22. By Theorem 3.9, $(A^{-1}X)^{-1} = X^{-1}(A^{-1})^{-1} = X^{-1}A$ and $(B^{-2}A)^{-1} = A^{-1}(B^{-2})^{-1} = A^{-1}B^2$, so the original equation becomes $X^{-1}A = AA^{-1}B^2 = B^2$. Then

$$X^{-1}A = B^2 \Rightarrow XX^{-1}A = XB^2 \Rightarrow A = XB^2 \Rightarrow AB^{-2} = XB^2B^{-2} = X, \quad \text{so that} \quad X = AB^{-2}.$$

23. $ABXA^{-1}B^{-1} = I + A$, so that

$$B^{-1}A^{-1}(ABXA^{-1}B^{-1})BA = B^{-1}A^{-1}(I + A)BA = B^{-1}A^{-1}BA + B^{-1}A^{-1}ABA$$

But

$$B^{-1}A^{-1}(ABXA^{-1}B^{-1})BA = (B^{-1}A^{-1}AB)X(A^{-1}B^{-1}BA) = X,$$

and

$$B^{-1}A^{-1}BA + B^{-1}A^{-1}ABA = B^{-1}A^{-1}BA + A.$$

Thus $X = B^{-1}A^{-1}BA + A = (AB)^{-1}BA + A$.

24. To get from A to B, reverse the first and third rows. Thus $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

25. To get from B to A, reverse the first and third rows. Thus $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

26. To get from A to C, add the first row to the third row. Thus $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

27. To get from C to A, subtract the first row from the third row. Thus $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

28. To get from C to D, subtract twice the third row from the second row. Thus $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

29. To get from D to C, add twice the third row to the second row. Thus $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

30. The row operations $R_2 - 2R_1 - 2R_3 \to R_2$, $R_3 + R_1 \to R_3$ will turn matrix A into matrix D. Thus

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \text{ satisfies } EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -1 & 3 \\ 2 & 1 & -1 \end{bmatrix} = D.$$

However, E is not an elementary matrix since it includes several row operations, not just one.

31. This matrix multiplies the first row by 3, so its inverse multiplies the first row by $\frac{1}{3}$:

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}.$$

32. This matrix adds twice the second row to the first row, so its inverse subtracts twice the second row from the first row:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

33. This matrix reverses rows 1 and 2, so its inverse does the same thing. Hence this matrix is its own inverse.

34. This matrix adds $-\frac{1}{2}$ times the first row to the second row, so its inverse adds $\frac{1}{2}$ times the first row to the second row:

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

35. This matrix subtracts twice the third row from the second row, so its inverse adds twice the third row to the second row:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

36. This matrix reverses rows 1 and 3, so its inverse does the same thing. Hence this matrix is its own inverse.

37. This matrix multiplies row 2 by c. Since $c \neq 0$, its inverse multiplies the second row by $\frac{1}{c}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

38. This matrix adds c times the third row to the second row, so its inverse subtracts c times the third row from the second row:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

39. As in Example 3.29, we first row-reduce A:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The first row operation adds the first row to the second row, and the second row operation multiplies the second row by $-\frac{1}{2}$. These correspond to the elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

So $E_2E_1A = I$ and thus $A = E_1^{-1}E_2^{-1}$. But E_1^{-1} is $R_2 - R_1 \to R_2$, and E_2^{-1} is $-2R_2 \to R_2$, so

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$

so that

$$A = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Taking the inverse of both sides of the equation $A = E_1^{-1}E_2^{-1}$ gives

$$A^{-1} = (E_1^{-1}E_2^{-1})^{-1} = (E_2^{-1})^{-1}(E_1^{-1})^{-1} = E_2E_1,$$

so that

$$A^{-1} = E_2E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

40. As in Example 3.29, we first row-reduce A:

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Converting each of these steps to an elementary matrix gives

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

So $E_4E_3E_2E_1A = I$ and thus $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$. We can invert each of the elementary matrices above by considering the row operation which reverses its action, giving

$$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad E_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

so that

$$A = E_1^{-1}E_2^{-1}E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Taking the inverse of both sides of the equation $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$ gives $A^{-1} = E_4E_3E_2E_1$, so that

$$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 2 \\ \frac{1}{2} & -1 \end{bmatrix}.$$

41. Suppose $AB = I$ and consider the equation $B\mathbf{x} = \mathbf{0}$. Left-multiplying by A gives $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. But $AB = I$, so $AB\mathbf{x} = I\mathbf{x} = \mathbf{x}$. Thus $\mathbf{x} = \mathbf{0}$. So the system represented by $B\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$. From the equivalence of (c) and (a) in the Fundamental Theorem, we know that B is invertible (i.e., B^{-1} exists and satisfies $BB^{-1} = I = B^{-1}B$.) So multiply both sides of the equation $AB = I$ on the right by B^{-1}, giving

$$ABB^{-1} = IB^{-1} \Rightarrow AI = IB^{-1} \Rightarrow A = B^{-1}.$$

This proves the theorem.

42. **(a)** Let A be invertible and suppose that $AB = O$. Left-multiply both sides of this equation by A^{-1}, giving $A^{-1}AB = A^{-1}O$. But $A^{-1}A = I$ and $A^{-1}O = O$, so we get $IB = B = O$. Thus B is the zero matrix.

(b) For example, let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ but } B \neq O.$$

43. **(a)** Since A is invertible, we can multiply both sides of $BA = CA$ on the right by A^{-1}, giving $BAA^{-1} = CAA^{-1}$, so that $BI = CI$ and thus $B = C$.

(b) For example, let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = CA, \text{ but } B \neq C.$$

44. **(a)** There are many possibilities. For example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} A^2 &= I^2 = I = A \\ B^2 &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = B \\ C^2 &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = C, \end{aligned}$$

so that A, B, and C are idempotent.

(b) Suppose that A and $n \times n$ matrix that is both invertible and idempotent. Then $A^2 = A$ since A is idempotent. Multiply both sides of this equation on the right by A^{-1}, giving $AAA^{-1} = AA^{-1} = I$. Then $AAA^{-1} = A$, so we get $A = I$. Thus A is the identity matrix.

45. To show that $A^{-1} = 2I - A$, it suffices to show that $A(2I - A) = I$. But $A^2 - 2A + I = O$ means that $I = 2A - A^2 = A(2I - A)$. This equation says that A multiplied by $2I - A$ is the identity, which is what we wanted to prove.

46. Let A be an invertible symmetric matrix. Then $AA^{-1} = I$. Take the transpose of both sides of this equation:

$$(AA^{-1})^T = I^T \Rightarrow (A^{-1})^T A^T = I \Rightarrow (A^{-1})^T A = I \Rightarrow (A^{-1})^T AA^{-1} = IA^{-1} = A^{-1}.$$

Thus $(A^{-1})^T AA^{-1} = A^{-1}$. But also clearly $(A^{-1})^T AA^{-1} = (A^{-1})^T I = (A^{-1})^T$, so that $(A^{-1})^T = A^{-1}$. This says that A^{-1} is its own transpose, so that A^{-1} is symmetric.

47. Let A and B be square matrices and assume that AB is invertible. Then

$$AB(AB)^{-1} = A(B(AB)^{-1}) = I,$$

so that A is invertible with $A^{-1} = B(AB)^{-1}$. But this means that

$$B\left((AB)^{-1}A\right) = \left(B(AB)^{-1}\right)A = I,$$

so that also B is invertible, with inverse $(AB)^{-1}A$.

48. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow{R_1 - 5R_2} \left[\begin{array}{cc|cc} 1 & 0 & -4 & 5 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

Thus the inverse of the matrix is

$$\begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix}.$$

49. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} -2 & 4 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & -2 & -\frac{1}{2} & 0 \\ 3 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & -2 & -\frac{1}{2} & 0 \\ 0 & 5 & \frac{3}{2} & 1 \end{array}\right] \xrightarrow{\frac{1}{5}R_2} \cdots$$

$$\left[\begin{array}{cc|cc} 1 & -2 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{10} & \frac{1}{5} \end{array}\right] \xrightarrow{R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{10} & \frac{2}{5} \\ 0 & 1 & \frac{3}{10} & \frac{1}{5} \end{array}\right]$$

Thus the inverse of the matrix is

$$\begin{bmatrix} \frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}.$$

50. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 4 & -2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right] \xrightarrow{R_1 + \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right]$$

Thus the inverse of the matrix is

$$\begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}.$$

51. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & a & 1 & 0 \\ -a & 1 & 0 & 1 \end{array}\right] \xrightarrow{aR_1 + R_2} \left[\begin{array}{cc|cc} 1 & a & 1 & 0 \\ 0 & a^2+1 & a & 1 \end{array}\right] \xrightarrow{\frac{1}{a^2+1}R_2} \cdots$$

$$\left[\begin{array}{cc|cc} 1 & a & 1 & 0 \\ 0 & 1 & \frac{a}{a^2+1} & \frac{1}{a^2+1} \end{array}\right] \xrightarrow{R_1 - aR_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a^2+1} & -\frac{a}{a^2+1} \\ 0 & 1 & \frac{a}{a^2+1} & \frac{1}{a^2+1} \end{array}\right].$$

Thus the inverse of the matrix is

$$\begin{bmatrix} \frac{1}{a^2+1} & -\frac{a}{a^2+1} \\ \frac{a}{a^2+1} & \frac{1}{a^2+1} \end{bmatrix}.$$

52. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{rrr|rrr} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{R_2 - 2R_1} \\ \xrightarrow{R_3 - 2R_1} \end{array} \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 7 & 2 & 1 & -2 & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right] \xrightarrow{R_3 - 4R_2} \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} & -\frac{6}{7} & 1 \end{array}\right]$$

$$\xrightarrow{-7R_3} \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right] \begin{array}{c} \xrightarrow{R_1 + R_3} \\ \xrightarrow{R_2 - \frac{2}{7}R_3} \end{array} \left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 4 & 7 & -7 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right]$$

The inverse of the matrix is

$$\begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}.$$

53. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{R_2 - 3R_1} \\ \xrightarrow{R_3 - 2R_1} \end{array} \left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -3 & 1 & 0 \\ 0 & 5 & -5 & -2 & 0 & 1 \end{array}\right] \xrightarrow{4R_3 - 5R_2} \left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -3 & 1 & 0 \\ 0 & 0 & 0 & 7 & -5 & 4 \end{array}\right].$$

Since the matrix on the left has a zero row, it cannot be reduced to the identity matrix, so that the given matrix is not invertible.

54. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_2 + R_3} \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

The inverse of the matrix is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

55. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{\frac{1}{a}R_1} \\ \xrightarrow{\frac{1}{a}R_2} \\ \xrightarrow{\frac{1}{a}R_3} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ \frac{1}{a} & 1 & 0 & 0 & \frac{1}{a} & 0 \\ 0 & \frac{1}{a} & 1 & 0 & 0 & \frac{1}{a} \end{array}\right] \xrightarrow{R_2 - \frac{1}{a}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{a^2} & \frac{1}{a} & 0 \\ 0 & \frac{1}{a} & 1 & 0 & 0 & \frac{1}{a} \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{a}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{a^2} & \frac{1}{a} & 0 \\ 0 & 0 & 1 & \frac{1}{a^3} & -\frac{1}{a^2} & \frac{1}{a} \end{array}\right].$$

Thus the inverse of the matrix is

$$\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{1}{a^2} & \frac{1}{a} & 0 \\ \frac{1}{a^3} & -\frac{1}{a^2} & \frac{1}{a} \end{bmatrix}.$$

Note that these calculations assume $a \neq 0$. But if $a = 0$, then the original matrix is not invertible since the first row is zero.

56. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Note that if $a = 0$, then the first row is zero, so the matrix is clearly not invertible. So assume $a \neq 0$. Then

$$\left[\begin{array}{ccc|ccc} 0 & a & 0 & 1 & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right]. \xrightarrow{R_3 - \frac{d}{a}R_1} \left[\begin{array}{ccc|ccc} 0 & a & 0 & 1 & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{d}{a} & 0 & 1 \end{array}\right].$$

Since the matrix on the left has a zero row, it cannot be reduced to the identity matrix, so the original matrix is not invertible.

57. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{cccc|cccc} 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -11 & -2 & 5 & -4 \\ 0 & 1 & 0 & 0 & 4 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 5 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{array}\right].$$

Thus the inverse of the matrix is

$$\begin{bmatrix} -11 & -2 & 5 & -4 \\ 4 & 1 & -2 & 2 \\ 5 & 1 & -2 & 2 \\ 9 & 2 & -4 & 3 \end{bmatrix}.$$

58. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{cccc|cccc} \sqrt{2} & 0 & 2\sqrt{2} & 0 & 1 & 0 & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 2\sqrt{2} & \frac{\sqrt{2}}{2} & -8 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -3 & 1 \end{array}\right].$$

Thus the inverse of the matrix is

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -2 & 0 \\ 2\sqrt{2} & \frac{\sqrt{2}}{2} & -8 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}.$$

59. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ a & b & c & d & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{a}{d} & -\frac{b}{d} & -\frac{c}{d} & \frac{1}{d} \end{array}\right].$$

Thus the inverse of the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{a}{d} & -\frac{b}{d} & -\frac{c}{d} & \frac{1}{d} \end{bmatrix}.$$

60. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ over $\mathbb{Z}_2$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Thus the inverse of the matrix is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

61. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ over $\mathbb{Z}_5$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 4 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{4R_1} \left[\begin{array}{cc|cc} 1 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|cc} 1 & 3 & 4 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right].$$

Since the left-hand matrix has a zero row, it cannot be reduced to the identity matrix, so the original matrix is not invertible.

62. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ over $\mathbb{Z}_3$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2+R_1]{2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 2 & 1 \end{array}\right] \xrightarrow{2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2+2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right].$$

Thus the inverse of the matrix is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

63. As in Example 3.30, we adjoin the identity matrix to A then row-reduce $\begin{bmatrix} A & I \end{bmatrix}$ over $\mathbb{Z}_7$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 3 & 6 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 6 & 4 \\ 0 & 1 & 0 & 5 & 3 & 2 \\ 0 & 0 & 1 & 0 & 6 & 5 \end{array}\right].$$

Thus the inverse of the matrix is

$$\begin{bmatrix} 4 & 6 & 4 \\ 5 & 3 & 2 \\ 0 & 6 & 5 \end{bmatrix}.$$

64. Using block multiplication gives

$$\begin{aligned}\begin{bmatrix} A & B \\ O & D \end{bmatrix}\begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix} &= \begin{bmatrix} AA^{-1}+BO & -AA^{-1}BD^{-1}+BD^{-1} \\ OA^{-1}+DO & -OA^{-1}BD^{-1}+DD^{-1} \end{bmatrix} \\ &= \begin{bmatrix} AA^{-1} & -BD^{-1}+BD^{-1} \\ O & DD^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & O \\ O & I \end{bmatrix}.\end{aligned}$$

65. Using block multiplication gives

$$\begin{aligned}&\begin{bmatrix} O & B \\ C & I \end{bmatrix}\begin{bmatrix} -(BC)^{-1} & (BC)^{-1}B \\ C(BC)^{-1} & I-C(BC)^{-1}B \end{bmatrix} \\ &\quad= \begin{bmatrix} -O(BC)^{-1}+BC(BC)^{-1} & O(BC)^{-1}B+B(I-C(BC)^{-1}B) \\ -C(BC)^{-1}+IC(BC)^{-1} & C(BC)^{-1}B+I(I-C(BC)^{-1}B) \end{bmatrix} \\ &\quad= \begin{bmatrix} BC(BC)^{-1} & BI-BC(BC)^{-1}B \\ -C(BC)^{-1}+C(BC)^{-1} & C(BC)^{-1}B+I-C(BC)^{-1}B \end{bmatrix} \\ &\quad= \begin{bmatrix} I & O \\ O & I \end{bmatrix}.\end{aligned}$$

66. Using block multiplication gives

$$\begin{aligned}&\begin{bmatrix} I & B \\ C & I \end{bmatrix}\begin{bmatrix} (I-BC)^{-1} & -(I-BC)^{-1}B \\ -C(I-BC)^{-1} & I+C(I-BC)^{-1}B \end{bmatrix} \\ &\quad= \begin{bmatrix} I(I-BC)^{-1}-BC(I-BC)^{-1} & -I(I-BC)^{-1}B+B(I+C(I-BC)^{-1}B) \\ C(I-BC)^{-1}-I(C(I-BC)^{-1}) & -C(I-BC)^{-1}B+I(I+C(I-BC)^{-1}B) \end{bmatrix} \\ &\quad= \begin{bmatrix} (I-BC)(I-BC)^{-1} & (BC-I)(I-BC)^{-1}B+B \\ C(I-BC)^{-1}-C(I-BC)^{-1} & -C(I-BC)^{-1}B+I+C(I-BC)^{-1}B \end{bmatrix} \\ &\quad= \begin{bmatrix} I & -IB+B \\ O & I \end{bmatrix} \\ &\quad= \begin{bmatrix} I & O \\ O & I \end{bmatrix}.\end{aligned}$$

67. Using block multiplication gives

$$\begin{aligned}&\begin{bmatrix} O & B \\ C & D \end{bmatrix}\begin{bmatrix} -(BD^{-1}C)^{-1} & (BD^{-1}C)^{-1}BD^{-1} \\ D^{-1}C(BD^{-1}C)^{-1} & D^{-1}-D^{-1}C(BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} \\ &\quad= \begin{bmatrix} BD^{-1}C(BD^{-1}C)^{-1} & D^{-1}-D^{-1}CBD^{-1}C)^{-1}BD^{-1} \\ -C(BD^{-1}C)^{-1}+C(BD^{-1}C)^{-1} & \begin{array}{c} C(BD^{-1}C)^{-1}BD^{-1} \\ +D(D^{-1}-D^{-1}C(BD^{-1}C)^{-1}BD^{-1}) \end{array} \end{bmatrix} \\ &\quad= \begin{bmatrix} I & D^{-1}-D^{-1}CC^{-1}DB^{-1}BD^{-1} \\ O & C(BD^{-1}C)^{-1}BD^{-1}+I-DD^{-1}C(BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} \\ &\quad= \begin{bmatrix} I & D^{-1}-D^{-1} \\ O & C(BD^{-1}C)^{-1}BD^{-1}+I-C(BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} \\ &\quad= \begin{bmatrix} I & O \\ O & I \end{bmatrix}.\end{aligned}$$

68. Using block multiplication gives

$$\begin{aligned}
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} &= \begin{bmatrix} AP+BR & AQ+BS & CP+DR & DQ+DS \end{bmatrix} \\
&= \begin{bmatrix} AP+B(-D^{-1}CP) & A(-PBD^{-1}+B(D^{-1}+D^{-1}CPBD^{-1}) \\ CP+D(-D^{-1}CP) & C(-PBD^{-1})+D(D^{-1}+D^{-1}CPBD^{-1}) \end{bmatrix} \\
&= \begin{bmatrix} (A-BD^{-1}C)P & -(A-BD^{-1}C)PBD^{-1}+BD^{-1} \\ CP-CP & -CPBD^{-1}+I+CPBD^{-1} \end{bmatrix} \\
&= \begin{bmatrix} P^{-1}P & -P^{-1}PBD^{-1}+PBD^{-1} \\ O & I \end{bmatrix} \\
&= \begin{bmatrix} I & -BD^{-1}+BD^{-1} \\ O & I \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}.
\end{aligned}$$

69. Partition the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & B \\ C & I \end{bmatrix}, \quad \text{where } B = O.$$

Then using Exercise 66, $(I - BC)^{-1} = (I - OC)^{-1} = I^{-1} = I$, so that

$$A^{-1} = \begin{bmatrix} (I-BC)^{-1} & -(I-BC)^{-1}B \\ -C(I-BC)^{-1} & I+C(I-BC)^{-1}B \end{bmatrix} = \begin{bmatrix} I & -IB \\ -CI & I+CIB \end{bmatrix} = \begin{bmatrix} I & O \\ -C & I \end{bmatrix}.$$

Substituting back in for C gives

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}.$$

70. Partition the matrix

$$M = \begin{bmatrix} \sqrt{2} & 0 & 2\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} A & B \\ O & D \end{bmatrix}$$

and use Exercise 64. Then

$$\begin{aligned}
A^{-1} &= \begin{bmatrix} \sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 2\sqrt{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \\
D^{-1} &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \\
-A^{-1}BD &= \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 2\sqrt{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -8 & 0 \end{bmatrix}.
\end{aligned}$$

Then

$$M^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -2 & 0 \\ 2\sqrt{2} & \frac{\sqrt{2}}{2} & -8 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}.$$

71. Partition the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} O & B \\ C & I \end{bmatrix}$$

and use Exercise 65. Then

$$\begin{aligned}
-(BC)^{-1} &= -\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}\right)^{-1} = -\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\\
(BC^{-1})B &= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}\\
C(BC)^{-1} &= \begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}\\
I - C(BC)^{-1}B &= I - \begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.
\end{aligned}$$

Thus

$$A^{-1} = \begin{bmatrix}-(BC)^{-1} & (BC)^{-1}B\\ C(BC)^{-1} & I - C(BC)^{-1}B\end{bmatrix} = \begin{bmatrix}-1 & 0 & 1 & 1\\ 0 & 1 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\end{bmatrix}.$$

72. Partition the matrix

$$A = \begin{bmatrix}0 & 1 & 1\\ 1 & 3 & 1\\ -1 & 5 & 2\end{bmatrix} = \begin{bmatrix}O & B\\ C & D\end{bmatrix},$$

where

$$B = \begin{bmatrix}1 & 1\end{bmatrix}, \quad C = \begin{bmatrix}1\\ -1\end{bmatrix}, \quad D = \begin{bmatrix}3 & 1\\ 5 & 2\end{bmatrix},$$

and use Exercise 67. Then

$$\begin{aligned}
-(BD^{-1}C)^{-1} &= -\left(\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ 5 & 2\end{bmatrix}^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}\right)^{-1} = -\begin{bmatrix}-5\end{bmatrix}^{-1} = \begin{bmatrix}\frac{1}{5}\end{bmatrix}\\
(BD^{-1}C)^{-1}BD^{-1} &= -\frac{1}{5}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ 5 & 2\end{bmatrix}^{-1} = \begin{bmatrix}\frac{3}{5} & -\frac{2}{5}\end{bmatrix}\\
D^{-1}C(BD^{-1}C)^{-1} &= \begin{bmatrix}3 & 1\\ 5 & 2\end{bmatrix}^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}\left(-\frac{1}{5}\right) = \begin{bmatrix}-\frac{3}{5}\\ \frac{8}{5}\end{bmatrix}\\
D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1} &= \begin{bmatrix}3 & 1\\ 5 & 2\end{bmatrix}^{-1} - \begin{bmatrix}-\frac{3}{5}\\ \frac{8}{5}\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ 5 & 2\end{bmatrix}^{-1} = \begin{bmatrix}\frac{1}{5} & \frac{1}{5}\\ -\frac{1}{5} & -\frac{1}{5}\end{bmatrix}.
\end{aligned}$$

Thus

$$A^{-1} = \begin{bmatrix}-(BD^{-1}C)^{-1} & (BD^{-1}C)^{-1}BD^{-1}\\ D^{-1}C(BD^{-1}C)^{-1} & D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1}\end{bmatrix} = \begin{bmatrix}\frac{1}{5} & \frac{3}{5} & -\frac{2}{5}\\ -\frac{3}{5} & \frac{1}{5} & \frac{1}{5}\\ \frac{8}{5} & -\frac{1}{5} & -\frac{1}{5}\end{bmatrix}.$$

3.4 The LU Factorization

1. As in the method outlined after Theorem 3.15, we want to solve $A\mathbf{x} = (LU)\mathbf{x} = \mathbf{b}$. Let $\mathbf{y} = U\mathbf{x}$ and first solve $L\mathbf{y} = \mathbf{b}$ for y by forward substitution. Then solve $U\mathbf{x} = \mathbf{y}$ by backward substitution. We have

$$A = \begin{bmatrix}-2 & 1\\ 2 & 5\end{bmatrix} = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}-2 & 1\\ 0 & 6\end{bmatrix} = LU, \text{ and } \mathbf{b} = \begin{bmatrix}5\\ 1\end{bmatrix}.$$

Then

$$L\mathbf{y} = \mathbf{b} \Rightarrow \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}5\\ 1\end{bmatrix} \Rightarrow \begin{array}{rcrcl} y_1 & & & = & 5\\ -y_1 & + & y_2 & = & 1\end{array} \Rightarrow \begin{array}{l} y_1 = 5\\ y_2 = y_1 + 1\end{array} \Rightarrow \mathbf{y} = \begin{bmatrix}5\\ 6\end{bmatrix}.$$

Likewise,

$$U\mathbf{x}=\mathbf{y}\Rightarrow\begin{bmatrix}-2&1\\0&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}5\\6\end{bmatrix}\Rightarrow\begin{array}{rcrcl}-2x_1&+&x_2&=&5\\&&6x_2&=&6\end{array}\Rightarrow\begin{array}{c}x_2=1\\-2x_1=5-x_2\end{array}\Rightarrow\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}.$$

2. As in the method outlined after Theorem 3.15, we want to solve $A\mathbf{x}=(LU)\mathbf{x}=\mathbf{b}$. Let $\mathbf{y}=U\mathbf{x}$ and first solve $L\mathbf{y}=\mathbf{b}$ for y by forward substitution. Then solve $U\mathbf{x}=\mathbf{y}$ by backward substitution. We have

$$A=\begin{bmatrix}4&-2\\2&3\end{bmatrix}=\begin{bmatrix}1&0\\\frac{1}{2}&1\end{bmatrix}\begin{bmatrix}4&-2\\0&4\end{bmatrix}=LU,\text{ and }\mathbf{b}=\begin{bmatrix}0\\8\end{bmatrix}.$$

Then

$$L\mathbf{y}=\mathbf{b}\Rightarrow\begin{bmatrix}1&0\\\frac{1}{2}&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}0\\8\end{bmatrix}\Rightarrow\begin{array}{rcrcl}y_1&&&=&0\\\frac{1}{2}y_1&+&y_2&=&8\end{array}\Rightarrow\mathbf{y}=\begin{bmatrix}0\\8\end{bmatrix}.$$

Likewise,

$$U\mathbf{x}=\mathbf{y}\Rightarrow\begin{bmatrix}4&-2\\0&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\8\end{bmatrix}\Rightarrow\begin{array}{rcrcl}4x_1&-&2x_2&=&0\\&&4x_2&=&8\end{array}\Rightarrow\begin{array}{c}x_2=2\\4x_1=2x_2\end{array}\Rightarrow\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}.$$

3. As in the method outlined after Theorem 3.15, we want to solve $A\mathbf{x}=(LU)\mathbf{x}=\mathbf{b}$. Let $\mathbf{y}=U\mathbf{x}$ and first solve $L\mathbf{y}=\mathbf{b}$ for y by forward substitution. Then solve $U\mathbf{x}=\mathbf{y}$ by backward substitution. We have

$$A=\begin{bmatrix}2&1&-2\\-2&3&-4\\4&-3&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\2&-\frac{5}{4}&1\end{bmatrix}\begin{bmatrix}2&1&-2\\0&4&-6\\0&0&-\frac{7}{2}\end{bmatrix}=LU,\text{ and }\mathbf{b}=\begin{bmatrix}-3\\1\\0\end{bmatrix}.$$

Then

$$L\mathbf{y}=\mathbf{b}\Rightarrow\begin{bmatrix}1&0&0\\-1&1&0\\2&-\frac{5}{4}&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}-3\\1\\0\end{bmatrix}\Rightarrow\begin{array}{rcrcrcl}y_1&&&&&=&-3\\-y_1&+&y_2&&&=&1\\2y_1&-&\frac{5}{4}y_2&+&y_3&=&0\end{array}$$

$$\Rightarrow\begin{array}{l}y_1=-3\\y_2=y_1+1\\y_3=-2y_1+\dfrac{5}{4}y_2\end{array}\Rightarrow\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}-3\\-2\\\frac{7}{2}\end{bmatrix}.$$

Likewise,

$$U\mathbf{x}=\mathbf{y}\Rightarrow\begin{bmatrix}2&1&-2\\0&4&-6\\0&0&-\frac{7}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-3\\-2\\\frac{7}{2}\end{bmatrix}\Rightarrow\begin{array}{rcrcrcl}2x_1&+&x_2&-&2x_3&=&-3\\&&4x_2&-&6x_3&=&-2\\&&&-&\frac{7}{2}x_3&=&\frac{7}{2}\end{array}$$

$$\Rightarrow\begin{array}{l}x_3=-1\\4x_2=6x_3-2\\2x_1=-x_2+2x_3-3\end{array}\Rightarrow\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-\frac{3}{2}\\-2\\-1\end{bmatrix}.$$

4. As in the method outlined after Theorem 3.15, we want to solve $A\mathbf{x}=(LU)\mathbf{x}=\mathbf{b}$. Let $\mathbf{y}=U\mathbf{x}$ and first solve $L\mathbf{y}=\mathbf{b}$ for y by forward substitution. Then solve $U\mathbf{x}=\mathbf{y}$ by backward substitution. We have

$$A=\begin{bmatrix}2&-4&0\\3&-1&4\\-1&2&2\end{bmatrix}=\begin{bmatrix}1&0&0\\\frac{3}{2}&1&0\\-\frac{1}{2}&0&1\end{bmatrix}\begin{bmatrix}2&-4&0\\0&5&4\\0&0&2\end{bmatrix}=LU,\text{ and }\mathbf{b}=\begin{bmatrix}2\\0\\-5\end{bmatrix}.$$

Then

$$L\mathbf{y}=\mathbf{b}\Rightarrow\begin{bmatrix}1&0&0\\ \frac{3}{2}&1&0\\ -\frac{1}{2}&0&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}2\\0\\-5\end{bmatrix}\Rightarrow\begin{array}{rcrcrcr} y_1 & & & & &=& 2\\ \frac{3}{2}y_1 &+& y_2 & & &=& 0\\ -\frac{1}{2}y_1 & & &+& y_3 &=& -5\end{array}$$

$$\Rightarrow\begin{aligned} y_1&=2\\ y_2&=-\frac{3}{2}y_1\\ y_3&=\frac{1}{2}y_1-5\end{aligned}\Rightarrow\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}2\\-3\\-4\end{bmatrix}.$$

Likewise,

$$U\mathbf{x}=\mathbf{y}\Rightarrow\begin{bmatrix}2&-4&0\\0&5&4\\0&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-3\\-4\end{bmatrix}\Rightarrow\begin{array}{rcrcrcr} 2x_1 &-& 4x_2 & & &=& 2\\ & & 5x_2 &+& 4x_3 &=& -3\\ & & & & 2x_3 &=& -4\end{array}$$

$$\Rightarrow\begin{aligned} x_3&=-2\\ 5x_2&=-4x_3-3\\ 2x_1&=4x_2+2\end{aligned}\Rightarrow\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}3\\1\\-2\end{bmatrix}.$$

5. As in the method outlined after Theorem 3.15, we want to solve $A\mathbf{x}=(LU)\mathbf{x}=\mathbf{b}$. Let $\mathbf{y}=U\mathbf{x}$ and first solve $L\mathbf{y}=\mathbf{b}$ for y by forward substitution. Then solve $U\mathbf{x}=\mathbf{y}$ by backward substitution. We have

$$A=\begin{bmatrix}2&-1&0&0\\6&-4&5&-3\\8&-4&1&0\\4&-1&0&7\end{bmatrix}=\begin{bmatrix}1&0&0&0\\3&1&0&0\\4&0&1&0\\2&-1&5&1\end{bmatrix}\begin{bmatrix}2&-1&0&0\\0&-1&5&-3\\0&0&1&0\\0&0&0&4\end{bmatrix}=LU,\text{ and }\mathbf{b}=\begin{bmatrix}1\\2\\2\\1\end{bmatrix}.$$

Then

$$L\mathbf{y}=\mathbf{b}\Rightarrow\begin{bmatrix}1&0&0&0\\3&1&0&0\\4&0&1&0\\2&-1&5&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}1\\2\\2\\1\end{bmatrix}\Rightarrow\begin{array}{rcrcrcrcr} y_1 & & & & & & &=& 1\\ 3y_1 &+& y_2 & & & & &=& 2\\ 4y_1 & & &+& y_3 & & &=& 2\\ 2y_1 &-& y_2 &+& 5y_3 &+& y_4 &=& 1\end{array}$$

$$\Rightarrow\begin{aligned} y_1&=1\\ y_2&=-3y_1+2\\ y_3&=-4y_1+2\\ y_4&=-2y_1+y_2-5y_3+1\end{aligned}\Rightarrow\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}1\\-1\\-2\\8\end{bmatrix}.$$

Likewise,

$$U\mathbf{x}=\mathbf{y}\Rightarrow\begin{bmatrix}2&-1&0&0\\0&-1&5&-3\\0&0&1&0\\0&0&0&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\-1\\-2\\8\end{bmatrix}\Rightarrow\begin{array}{rcrcrcrcr} 2x_1 &-& x_2 & & & & &=& 1\\ &-& x_2 &+& 5x_3 &-& 3x_4 &=& -1\\ & & & & x_3 & & &=& -2\\ & & & & & & 4x_4 &=& 8\end{array}$$

$$\Rightarrow\begin{aligned} x_4&=2\\ x_3&=-2\\ x_2&=5x_3-3x_4+1\\ 2x_1&=x_2+1\end{aligned}\Rightarrow\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-7\\-15\\-2\\2\end{bmatrix}.$$

6. As in the method outlined after Theorem 3.15, we want to solve $A\mathbf{x}=(LU)\mathbf{x}=\mathbf{b}$. Let $\mathbf{y}=U\mathbf{x}$ and first solve $L\mathbf{y}=\mathbf{b}$ for y by forward substitution. Then solve $U\mathbf{x}=\mathbf{y}$ by backward substitution. We have

$$A = \begin{bmatrix} 1 & 4 & 3 & 0 \\ -2 & -5 & -1 & 2 \\ 3 & 6 & -3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = LU, \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix}.$$

Then

$$L\mathbf{y} = \mathbf{b} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{rcrcrcrcr} y_1 & & & & & & & = & 1 \\ -2y_1 & + & y_2 & & & & & = & -3 \\ 3y_1 & - & 2y_2 & + & y_3 & & & = & -1 \\ -5y_1 & + & 4y_2 & - & 2y_3 & + & y_4 & = & 0 \end{array}$$

$$\Rightarrow \begin{aligned} y_1 &= 1 \\ y_2 &= 2y_1 - 3 \\ y_3 &= -3y_1 + 2y_2 - 1 \\ y_4 &= 5y_1 - 4y_2 + 2y_3 \end{aligned} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -6 \\ -3 \end{bmatrix}.$$

Likewise,

$$U\mathbf{x} = \mathbf{y} \Rightarrow \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -6 \\ -3 \end{bmatrix} \Rightarrow \begin{array}{rcrcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & & & = & 1 \\ & & 3x_2 & + & 5x_3 & + & 2x_4 & = & -1 \\ & & & - & 2x_3 & & & = & -6 \\ & & & & & & x_4 & = & -3 \end{array}$$

$$\Rightarrow \begin{aligned} x_4 &= -3 \\ x_3 &= 3 \\ 3x_2 &= -5x_3 - 2x_4 - 1 \\ x_1 &= -4x_2 - 3x_3 + 1 \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{16}{3} \\ -\frac{10}{3} \\ 3 \\ -3 \end{bmatrix}.$$

7. We use the multiplier method from Example 3.35:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} \xrightarrow{R_3-(-3R_1)} \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} = U.$$

The multiplier was -3, so

$$L = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}.$$

8. We use the multiplier method from Example 3.35:

$$A = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix} \xrightarrow{R_3-\frac{3}{2}R_1} \begin{bmatrix} 2 & -4 \\ 0 & 7 \end{bmatrix} = U.$$

The multiplier was $\frac{3}{2}$, so

$$L = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}.$$

9. We use the multiplier method from Example 3.35:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 7 & 9 \end{bmatrix} \begin{array}{c} \xrightarrow{R_2-4R_1} \\ \xrightarrow{R_3-8R_1} \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -9 & -15 \end{bmatrix} \xrightarrow{R_3-3R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 3 \end{bmatrix} = U.$$

The multipliers were $\ell_{21} = 4$, $\ell_{31} = 8$, and $\ell_{32} = 3$, so we get

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & 3 & 1 \end{bmatrix}$$

10. We use the multiplier method from Example 3.35:

$$A=\begin{bmatrix}2&2&-1\\4&0&4\\3&4&4\end{bmatrix}\xrightarrow[R_3-\frac{3}{2}R_1]{R_2-2R_1}\begin{bmatrix}2&2&-1\\0&-4&6\\0&1&\frac{11}{2}\end{bmatrix}\xrightarrow{R_3-\left(-\frac{1}{4}R_2\right)}\begin{bmatrix}2&2&-1\\0&-4&6\\0&0&7\end{bmatrix}=U.$$

The multipliers were $\ell_{21}=2$, $\ell_{31}=\frac{3}{2}$, and $\ell_{32}=-\frac{1}{4}$, so we get

$$L=\begin{bmatrix}1&0&0\\2&1&0\\\frac{3}{2}&-\frac{1}{4}&1\end{bmatrix}$$

11. We use the multiplier method from Example 3.35:

$$A=\begin{bmatrix}1&2&3&-1\\2&6&3&0\\0&6&-6&7\\-1&-2&-9&0\end{bmatrix}\xrightarrow[\substack{R_3-0R_1\\R_4-(-1R_1)}]{R_2-2R_1}\begin{bmatrix}1&2&3&-1\\0&2&-3&2\\0&6&-6&7\\0&0&-6&-1\end{bmatrix}\xrightarrow[R_4-0R_2]{R_3-3R_2}\begin{bmatrix}1&2&3&-1\\0&2&-3&2\\0&0&3&1\\0&0&-6&-1\end{bmatrix}$$

$$\xrightarrow{R_4-(-2R_3)}\begin{bmatrix}1&2&3&-1\\0&2&-3&2\\0&0&3&1\\0&0&0&1\end{bmatrix}=U.$$

The multipliers were $\ell_{21}=2$, $\ell_{31}=0$, $\ell_{41}=-1$, $\ell_{32}=3$, $\ell_{42}=0$, and $\ell_{43}=-2$, so

$$L=\begin{bmatrix}1&0&0&0\\2&1&0&0\\0&3&1&0\\-1&0&-2&1\end{bmatrix}.$$

12. We use the multiplier method from Example 3.35:

$$A=\begin{bmatrix}2&2&2&1\\-2&4&-1&2\\4&4&7&3\\6&9&5&8\end{bmatrix}\xrightarrow[\substack{R_3-2R_1\\R_4-3R_1}]{R_2-(-R_1)}\begin{bmatrix}2&2&2&1\\0&6&1&3\\0&0&3&1\\0&3&-1&5\end{bmatrix}\xrightarrow[R_4-\frac{1}{2}R_2]{R_3-0R_2}\begin{bmatrix}2&2&2&1\\0&6&1&3\\0&0&3&1\\0&0&-\frac{3}{2}&\frac{7}{2}\end{bmatrix}$$

$$\xrightarrow{R_4-\left(-\frac{1}{2}R_3\right)}\begin{bmatrix}2&2&2&1\\0&6&1&3\\0&0&3&1\\0&0&0&4\end{bmatrix}=U.$$

The multipliers were $\ell_{21}=-1$, $\ell_{31}=2$, $\ell_{41}=3$, $\ell_{32}=0$, $\ell_{42}=\frac{1}{2}$, and $\ell_{43}=-\frac{1}{2}$, so

$$L=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\2&0&1&0\\3&\frac{1}{2}&-\frac{1}{2}&1\end{bmatrix}.$$

13. We can use the multiplier method as before to make the matrix "upper triangular", i.e., in row echelon form. In this case, however, the matrix is already in row echelon form, so it is equal to U, and $L=I_3$:

$$\begin{bmatrix}1&0&1&-2\\0&3&3&1\\0&0&0&5\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&1&-2\\0&3&3&1\\0&0&0&5\end{bmatrix}.$$

14. We can use the multiplier method as before to make the matrix "upper triangular", i.e., in row echelon form.

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ -2 & -7 & 3 & 8 & -2 \\ 1 & 1 & 3 & 5 & 2 \\ 0 & 3 & -3 & -6 & 0 \end{bmatrix} \begin{array}{c} \xrightarrow{R_2-(-2R_1)} \\ \xrightarrow{R_3-1R_1} \\ \xrightarrow{R_4-0R_1} \end{array} \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & -3 & 3 & 6 & 0 \\ 0 & -1 & 3 & 6 & 1 \\ 0 & 3 & -3 & -6 & 0 \end{bmatrix} \begin{array}{c} \xrightarrow{R_3-\frac{1}{3}R_2} \\ \xrightarrow{R_4-(-1R_2)} \end{array} \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & -3 & 3 & 6 & 0 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

The multipliers were $\ell_{21} = -2$, $\ell_{31} = 1$, $\ell_{41} = 0$, $\ell_{32} = \frac{1}{3}$, and $\ell_{42} = -1$, so

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & \frac{1}{3} & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

15. In Exercise 1, we have

$$A = \begin{bmatrix} -2 & 1 \\ 2 & 5 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix}.$$

Then

$$L^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U^{-1} = -\frac{1}{12}\begin{bmatrix} 6 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{12} \\ 0 & \frac{1}{6} \end{bmatrix},$$

so that

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{12} \\ 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{12} & \frac{1}{12} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}.$$

16. In Exercise 4, we have

$$A = \begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

To compute L^{-1}, we adjoin the identity matrix to L and row-reduce:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right] \Rightarrow L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

To compute U^{-1}, we adjoin the identity matrix to U and row-reduce:

$$\left[\begin{array}{ccc|ccc} 2 & -4 & 0 & 1 & 0 & 0 \\ 0 & 5 & 4 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] \Rightarrow U^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} & -\frac{4}{5} \\ 0 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Then

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} & -\frac{4}{5} \\ 0 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{2}{5} & -\frac{4}{5} \\ -\frac{1}{2} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}.$$

17. Using the LU decomposition from Exercise 1, we compute A^{-1} one column at a time using the method outlined in the text. That is, we first solve $L\mathbf{y}_i = \mathbf{e}_i$, and then solve $U\mathbf{x}_i = \mathbf{y}_i$. Column 1 of A^{-1} is found as follows:

$$L\mathbf{y}_1 = \mathbf{e}_1 \Rightarrow \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{rcrcl} y_{11} & & & = & 1 \\ -y_{11} & + & y_{21} & = & 0 \end{array} \Rightarrow \mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$U\mathbf{x}_1 = \mathbf{y}_1 \Rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{rcrcl} -2x_{11} & + & x_{21} & = & 1 \\ & & 6x_{21} & = & 1 \end{array} \Rightarrow \begin{bmatrix} x \end{bmatrix}_1 = \begin{bmatrix} -\frac{5}{12} \\ \frac{1}{6} \end{bmatrix}.$$

Use the same procedure to find the second column:

$$L\mathbf{y}_2 = \mathbf{e}_2 \Rightarrow \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{rcrcl} y_{12} & & & = & 0 \\ -y_{12} & + & y_{22} & = & 1 \end{array} \Rightarrow \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U\mathbf{x}_2 = \mathbf{y}_2 \Rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{rcrcl} -2x_{12} & + & x_{22} & = & 0 \\ & & 6x_{21} & = & 1 \end{array} \Rightarrow \begin{bmatrix} x \end{bmatrix}_1 = \begin{bmatrix} \frac{1}{12} \\ \frac{1}{6} \end{bmatrix}.$$

So

$$A^{-1} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{12} & \frac{1}{12} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}.$$

This matches what we found in Exercise 15.

18. Using the LU decomposition from Exercise 4, we compute A^{-1} one column at a time using the method outlined in the text. That is, we first solve $L\mathbf{y}_i = \mathbf{e}_i$, and then solve $U\mathbf{x}_i = \mathbf{y}_i$. Column 1 of A^{-1} is found as follows:

$$L\mathbf{y}_1 = \mathbf{e}_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{rcrcrcl} y_{11} & & & & & = & 1 \\ \frac{3}{2}y_{11} & + & y_{21} & & & = & 0 \\ -\frac{1}{2}y_{11} & & & + & y_{31} & = & 0 \end{array} \Rightarrow \begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$U\mathbf{x}_1 = \mathbf{y}_1 = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix} \Rightarrow \begin{array}{rcrcrcl} 2x_{11} & - & 4x_{21} & & & = & 1 \\ & & 5x_{21} & + & 4x_{31} & = & -\frac{3}{2} \\ & & & & 2x_{31} & = & \frac{1}{2} \end{array} \Rightarrow \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$$

Repeat this procedure to find the second column:

$$L\mathbf{y}_2 = \mathbf{e}_2 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{rcrcrcl} y_{12} & & & & & = & 0 \\ \frac{3}{2}y_{12} & + & y_{22} & & & = & 1 \\ -\frac{1}{2}y_{12} & & & + & y_{32} & = & 0 \end{array} \Rightarrow \begin{bmatrix} y_{12} \\ y_{22} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$U\mathbf{x}_2 = \mathbf{y}_2 = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{rcrcrcl} 2x_{12} & - & 4x_{22} & & & = & 0 \\ & & 5x_{22} & + & 4x_{32} & = & 1 \\ & & & & 2x_{32} & = & 0 \end{array} \Rightarrow \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix}$$

Do this again to find the third column:

$$L\mathbf{y}_3 = \mathbf{e}_3 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{13} \\ y_{23} \\ y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{rcrcrcl} y_{13} & & & & & = & 0 \\ \frac{3}{2}y_{13} & + & y_{23} & & & = & 0 \\ -\frac{1}{2}y_{13} & & & + & y_{33} & = & 1 \end{array} \Rightarrow \begin{bmatrix} y_{13} \\ y_{23} \\ y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$U\mathbf{x}_3 = \mathbf{y}_3 = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{rcrcrcl} 2x_{13} & - & 4x_{23} & & & = & 0 \\ & & 5x_{23} & + & 4x_{33} & = & 0 \\ & & & & 2x_{33} & = & 1 \end{array} \Rightarrow \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{2}{5} \\ \frac{1}{2} \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{2}{5} & -\frac{4}{5} \\ -\frac{1}{2} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}$; this matches what we found in Exercise 16.

19. This matrix results from the identity matrix by first interchanging rows 1 and 2, and then interchanging the (new) row 1 with row 3, so it is

$$E_{13}E_{12} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Other products of elementary matrices are possible.

20. This matrix results from the identity matrix by first interchanging rows 1 and 4, and then interchanging rows 2 and 3, so it is

$$E_{14}E_{23} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Other products of elementary matrices are possible; for example, also $P = E_{13}E_{23}E_{34}$.

21. To get from the given matrix to the identity matrix, perform the following exchanges:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus we can construct the given matrix P by multiplying these elementary matrices together in the reverse order:

$$P = E_{13}E_{23}E_{34}.$$

Other combinations are possible.

22. To get from the given matrix to the identity matrix, perform the following exchanges:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_5} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_5}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_5} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus we can construct the given matrix P by multiplying these elementary matrices together in the reverse order:

$$P = E_{12}E_{25}E_{35}E_{45}.$$

Other combinations are possible.

23. To find a P^TLU factorization of A, we begin by permuting the rows of A to get it into a form that will result in a small number of operations to reduce to row echelon form. Since

$$A = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}, \text{ we have } PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ -1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 1 & 3 & 3 \end{bmatrix}.$$

Now follow the multiplier method from Example 3.35 to find the LU factorization of this matrix:

$$PA = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 1 & 3 & 3 \end{bmatrix} \xrightarrow{R_3-(-R_1)} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 5 & 4 \end{bmatrix} \xrightarrow{R_3-5R_2} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -16 \end{bmatrix} = U.$$

Then $\ell_{21} = 0$, $\ell_{31} = -1$, and $\ell_{32} = 5$, so that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 5 & 1 \end{bmatrix}$$

Thus

$$A = P^T LU = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -16 \end{bmatrix}.$$

24. To find a P^TLU factorization of A, we begin by permuting the rows of A to get it into a form that will result in a small number of operations to reduce to row echelon form. Since

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ -1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}, \text{ we have } PA = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 2 \\ -1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ -1 & 1 & 3 & 2 \end{bmatrix}.$$

Now follow the multiplier method from Example 3.35 to find the LU factorization of this matrix:

$$PA = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ -1 & 1 & 3 & 2 \end{bmatrix} \xrightarrow{R_4-(-R_1)} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix} \xrightarrow{R_4-R_2} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$\xrightarrow{R_4-R_3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U.$$

Then $\ell_{41} = -1$, $\ell_{42} = 1$, $\ell_{42} = 1$, and the other ℓ_{ij} are zero, so that

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Thus

$$A = P^T LU = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

25. To find a P^TLU factorization of A, we begin by permuting the rows of A to get it into a form that will result in a small number of operations to reduce to row echelon form. Since

$$A = \begin{bmatrix} 0 & -1 & 1 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \text{ we have } PA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

Now follow the multiplier method from Example 3.35 to find the LU factorization of this matrix:

$$PA = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_4-(-R_2)} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U.$$

Then $\ell_{42} = -1$ and the other ℓ_{ij} are zero, so that

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Thus

$$A = P^T LU = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

26. From the text, a permutation matrix consists of the rows of the identity matrix written in some order. So each row and each column of a permutation matrix has exactly one 1 in it, and the remaining entries in that row or column are zero. Thus there are n choices for the column containing the 1 in the first row; once that is chosen, there are $n-1$ choices for the column to contain the 1 in the second row. Next, there are $n-2$ choices in the third row. Continuing, we decrease the number of choices by one each time, until there is just one choice in the last row. Multiplying all these together gives the total number of choices, i.e., the number of $n \times n$ permutation matrices, which is $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$.

27. To solve $A\mathbf{x} = P^T LU\mathbf{x} = \mathbf{b}$, multiply both sides by P to get $LU\mathbf{x} = P\mathbf{b}$. Now compute $P\mathbf{b}$, which is a permutation of the rows of $\mathbf{b}$, and solve the resulting equation by the method outlined after Theorem 3.15:

$$P^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P\mathbf{b} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}.$$

Then

$$L\mathbf{y} = P\mathbf{b} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \Rightarrow \begin{array}{rcrcrcl} y_1 & & & & & = & 1 \\ & & y_2 & & & = & 1 \\ \frac{1}{2}y_1 & - & \frac{1}{2}y_2 & + & y_3 & = & 5 \end{array}$$

$$\Rightarrow \begin{aligned} y_1 &= 1 \\ y_2 &= 1 \\ y_3 &= 5 - \frac{1}{2}y_1 + \frac{1}{2}y_2 \end{aligned} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}.$$

Likewise,

$$U\mathbf{x} = \mathbf{y} \Rightarrow \begin{bmatrix} 2 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \Rightarrow \begin{array}{rcrcrcl} 2x_1 & + & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & x_3 & = & 1 \\ & & & - & \frac{5}{2}x_3 & = & 5 \end{array}$$

$$\Rightarrow \begin{aligned} x_3 &= -2 \\ x_2 &= x_3 + 1 \\ 2x_1 &= 1 - 3x_2 - 2x_3 \end{aligned} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}.$$

28. To solve $A\mathbf{x} = P^T LU\mathbf{x} = \mathbf{b}$, multiply both sides by P to get $LU\mathbf{x} = P\mathbf{b}$. Now compute $P\mathbf{b}$, which is a permutation of the rows of $\mathbf{b}$, and solve the resulting equation by the method outlined after Theorem 3.15:

$$P^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow P\mathbf{b} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 16 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 16 \end{bmatrix}.$$

Then

$$L\mathbf{y} = P\mathbf{b} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 16 \end{bmatrix} \Rightarrow \begin{aligned} y_1 \phantom{{}+y_2+y_3} &= -4 \\ y_1 + y_2 \phantom{{}+y_3} &= 4 \\ 2y_1 - y_2 + y_3 &= 16 \end{aligned}$$

$$\begin{aligned} & y_1 = -4 \\ \Rightarrow\ & y_2 = 4 - y_1 \\ & y_3 = 16 - 2y_1 + y_2 \end{aligned} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 32 \end{bmatrix}.$$

Likewise,

$$U\mathbf{x} = \mathbf{y} \Rightarrow \begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 32 \end{bmatrix} \Rightarrow \begin{aligned} 4x_1 + x_2 + 2x_3 &= -4 \\ - x_2 + x_3 &= 8 \\ 2x_3 &= 32 \end{aligned}$$

$$\begin{aligned} & x_3 = 16 \\ \Rightarrow\ & x_2 = x_3 - 8 \\ & 4x_1 = -x_2 - 2x_3 - 4 \end{aligned} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 8 \\ 16 \end{bmatrix}.$$

29. Let $A = (a_{ij})$ and $B = (b_{ij})$ be unit lower triangular $n \times n$ matrices, so that

$$a_{ij} = b_{ij} = \begin{cases} 0 & i < j \\ 1 & i = j \\ * & i > j \end{cases}$$

(where of course the $*$'s maybe different for A and B). Then if $C = (c_{ij}) = AB$, we have

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i(j-1)}b_{(j-1)j} + a_{ij}b_{jj} + \cdots + a_{in}b_{nj}.$$

If $i < j$, then all the terms up through $a_{i(j-1)}b_{(j-1)j}$ since the b's are zero, and all terms from the next term to the end are zero since the a's are zero.

If $i = j$, then all the terms up through $a_{i(j-1)}b_{(j-1)j}$ since the b's are zero, the $a_{ij}b_{jj} = a_{jj}b_{jj}$ term is 1, and all terms from the next term to the end are zero since the a's are zero.

Finally, if $i > j$, multiple terms in the sum may be nonzero.

Thus we get

$$c_{ij} = \begin{cases} 0 & i < j \\ 1 & i = j \\ * & i > j \end{cases}$$

so that C is lower unit triangular as well.

30. To invert L, we row-reduce $\begin{bmatrix} L & I \end{bmatrix}$:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 & 0 & 0 & \cdots & 1 \end{array}\right].$$

To do this, we first add multiples of the first row to each of the lower rows to make the first column zero except for the 1 in row 1. Note that the result of this is a unit lower triangular matrix on the right (since we added multiples of the leading 1 in the first row to the lower rows). Now do the same with row 2: add multiples of that row to each of the lower rows, clearing column 2 on the left except for the 1 in row 2. Again, for the same reason, the result on the right remains unit lower triangular. Continue this process for each row. The result is the identity matrix on the left, and a unit lower triangular matrix on the right. Thus L is invertible and its inverse is unit lower triangular.

31. We start by finding D. First, suppose D is a diagonal 2×2 matrix and A is any 2×2 matrix with 1's on the diagonal. Then

$$DA = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} A$$

has the effect of multiplying the first row of A by a and the second row of A by b, so results in a matrix with a and b in the diagonal entries. In our example, we want to write

$$U = \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix}$$

as unit upper triangular, say $U = DU_1$. So we must turn the -2 and 6 into 1's, and thus

$$U = \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = DU_1.$$

Thus

$$A = LU = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = LDU_1.$$

32. We start by finding D. First, suppose D is a diagonal 3×3 matrix and A is any 3×3 matrix with 1's on the diagonal. Then

$$DA = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} A$$

has the effect of multiplying the first row of A by a, the second row of A by b, and the third row of A by c, so the result is a matrix with a, b, and c on the diagonal. In our example, we want to write

$$U = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

as unit upper triangular, say $U = DU_1$. So we must turn the 2, 5, and 2 into 1's, and thus

$$U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 1 \end{bmatrix} = DU_1.$$

Thus

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 1 \end{bmatrix} = LDU_1.$$

33. We must show that if $A = LDU$ is both symmetric and invertible, then $U = L^T$. Since A is symmetric,

$$L(DU) = LDU = A = A^T = (LDU)^T = U^T D^T L^T = U^T(D^T L^T) = U^T(DL^T)$$

since the diagonal matrix D is symmetric. Since L is lower triangular, it follows that L^T is upper triangular. Since any diagonal matrix is also upper triangular, Exercise 29 in Section 3.2 tells us that

DL^T and DU are both upper triangular. Finally, since U is upper triangular, it follows that U^T is lower triangular. Thus we have $A = U^T(DL^T) = L(DU)$, giving two LU-decompositions of A. But since A is invertible, Theorem 3.16 tells us that the LU-decomposition is unique, so that $U^T = L$. Taking transposes gives $U = L^T$.

34. Suppose $A = L_1D_1L_1^T = L_2D_2L_2^T$ where L_1 and L_2 are unit lower triangular, D_1 and D_2 are diagonal, and A is symmetric and invertible. We want to show that $L_1 = L_2$ and $D_1 = D_2$. First, since

$$L_2(D_2L_2^T) = A = L_1(D_1L_1^T)$$

gives two LU-factorizations of an invertible matrix A, it follows from Theorem 3.16 that these two factorizations are the same, so that $L_1 = L_2$ and $D_2L_2^T = D_1L_1^T$. Since $L_1 = L_2$, this equation becomes $D_2L_2^T = D_1L_2^T$. From Exercise 47 in Section 3.3, since $A = L_2(D_2L_2^T)$ and A is invertible, we see that both L_2 and $D_2L_2^T$ are invertible; thus L_2^T is invertible as well ($(L_2^T)^{-1} = (L_2^{-1})^T$). So multiply $D_2L_2^T = D_1L_2^T$ on the right by $(L_2^T)^{-1}$, giving $D_2 = D_1$, and we are done.

3.5 Subspaces, Basis, Dimension, and Rank

1. Geometrically, $x = 0$ is a line through the origin in $\mathbb{R}^2$, so you might conclude from Example 3.37 that this is a subspace. And in fact, the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x = 0$ is

$$\begin{bmatrix} 0 \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since y is arbitrary, $S = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, and this is a subspace of $\mathbb{R}^2$ by Theorem 3.19.

2. This is not a subspace. For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in S, but $-\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is not in S. This violates property 3 of the definition of a subspace.

3. Geometrically, $y = 2x$ is a line through the origin in $\mathbb{R}^2$, so you might conclude from Example 3.37 that this is a subspace. And in fact, the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $y = 2x$ is

$$\begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Since y is arbitrary, $S = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$, and this is a subspace of $\mathbb{R}^2$ by Theorem 3.19.

4. This is not a subspace. For example, both $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ are in S, since $-3 \cdot (-1)$ and $2 \cdot 2$ are both nonnegative. But their sum,

$$\begin{bmatrix} -3 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

is not in S since $-1 \cdot 1 < 0$. So S is not a subspace (violates property 2 of the definition).

5. Geometrically, $x = y = z$ is a line through the origin in $\mathbb{R}^3$, so you could conclude from Exercise 9 in this section that this is a subspace. And in fact, the set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $x = y = z$ is

$$\begin{bmatrix} x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since x is arbitrary, $S = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$, and this is a subspace of $\mathbb{R}^3$ by Theorem 3.19.

6. Geometrically, this is a line in $\mathbb{R}^3$ lying in the xz-plane and passing through the origin, so Exercise 9 in this section implies that it is a subspace. The set of vectors $\begin{bmatrix}x\\y\\z\end{bmatrix}$ satisfying these conditions is

$$\begin{bmatrix}x\\0\\2x\end{bmatrix} = x\begin{bmatrix}1\\0\\2\end{bmatrix}.$$

Since x is arbitrary, $S = \operatorname{span}\left(\begin{bmatrix}1\\0\\2\end{bmatrix}\right)$, and this is a subspace of $\mathbb{R}^3$ by Theorem 3.19.

7. This is a plane in $\mathbb{R}^3$, but it does not pass through the origin since $(0,0,0)$ does not satisfy the equation. So condition 1 in the definition is violated, and this is not a subspace.

8. This is not a subspace. For example

$$\begin{bmatrix}1\\0\\1\end{bmatrix} \in S \text{ and } \begin{bmatrix}0\\1\\2\end{bmatrix} \in S$$

since for the first vector $|1-0| = |0-1|$ and for the second $|0-1| = |1-2|$. But their sum,

$$\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\3\end{bmatrix} \notin S$$

since $|1-1| \neq |1-3|$.

9. If a line ℓ passes through the origin, then it has the equation $\mathbf{x} = \mathbf{0} + t\mathbf{d} = t\mathbf{d}$, so that the set of points on ℓ is just $\operatorname{span}(\mathbf{d})$. Then Theorem 3.19 implies that ℓ is a subspace of $\mathbb{R}^3$. (Note that we did not use the assumption that the vectors lay in $\mathbb{R}^3$; in fact this conclusion holds in any $\mathbb{R}^n$ for $n \geq 1$.)

10. This is not a subspace of $\mathbb{R}^2$. For example, $(1,0) \in S$ and $(0,1) \in S$, but their sum, $(1,1)$, is not in S since it does not lie on either axis.

11. As in Example 3.41, $\mathbf{b}$ is in $\operatorname{col}(A)$ if and only if the augmented matrix

$$\begin{bmatrix}A & \mathbf{b}\end{bmatrix} = \left[\begin{array}{ccr|c}1 & 0 & -1 & 3\\1 & 1 & 1 & 2\end{array}\right]$$

is consistent as a linear system. So we row-reduce that augmented matrix:

$$\left[\begin{array}{ccr|c}1 & 0 & -1 & 3\\1 & 1 & 1 & 2\end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccr|r}1 & 0 & -1 & 3\\0 & 1 & 2 & -1\end{array}\right].$$

This matrix is in row echelon form and has no zero rows, so the system is consistent and thus $\mathbf{b} \in \operatorname{col}(A)$.

Using Example 3.41, $\mathbf{w}$ is in $\operatorname{row}(A)$ if and only if the matrix

$$\left[\begin{array}{c}A\\\hline\mathbf{w}\end{array}\right] = \left[\begin{array}{rrr}1 & 0 & -1\\1 & 1 & 1\\\hline -1 & 1 & 1\end{array}\right]$$

can be row-reduced to a matrix whose last row is zero by operations *excluding* row interchanges involving the last row. Thus

$$\left[\begin{array}{rrr} 1 & 0 & -1 \\ 1 & 1 & 1 \\ \hline -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-R_1} \left[\begin{array}{rrr} 1 & 0 & -1 \\ 0 & 1 & 2 \\ \hline 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{rrr} 1 & 0 & -1 \\ 0 & 1 & 2 \\ \hline 0 & 0 & -2 \end{array}\right]$$

This matrix is in row echelon form and has no zero rows, so we cannot make the last row zero. Thus $\mathbf{w} \notin \text{row}(A)$.

12. Using Example 3.41, $\mathbf{b}$ is in $\text{col}(A)$ if and only if the augmented matrix

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \left[\begin{array}{rrr|r} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & -4 & 0 \end{array}\right]$$

is consistent as a linear system. So we row-reduce that augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & -4 & 0 \end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{rrr|r} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & -1 & -1 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{rrr|r} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{rrr|r} 1 & 1 & -3 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The linear system has a solution (in fact, an infinite number of them), so is consistent. Thus $\mathbf{b}$ is in $\text{col}(A)$.

Using Example 3.41, $\mathbf{w}$ is in $\text{row}(A)$ if and only if the matrix

$$\left[\begin{array}{c} A \\ \hline \mathbf{w} \end{array}\right] = \left[\begin{array}{rrr} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \\ \hline 2 & 4 & -5 \end{array}\right]$$

can be row-reduced to a matrix whose last row is zero by operations *excluding* row interchanges involving the last row. Thus

$$\left[\begin{array}{rrr} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \\ \hline 2 & 4 & -5 \end{array}\right] \xrightarrow[R_4-2R_1]{R_3-R_1} \left[\begin{array}{rrr} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \\ \hline 0 & 2 & 1 \end{array}\right] \xrightarrow{R_4+R_3} \left[\begin{array}{rrr} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \\ \hline 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{w}$ is in $\text{row}(A)$.

13. As in the remarks following Example 3.41, we determine whether $\mathbf{w}$ is in $\text{row}(A)$ using $\text{col}(A^T)$. To say $\mathbf{w}$ is in $\text{row}(A)$ means that $\mathbf{w}^T$ is in $\text{col}(A)$, so we form the augmented matrix and row-reduce to see if the system has a solution:

$$\begin{bmatrix} A^T \mid \mathbf{w}^T \end{bmatrix} = \left[\begin{array}{rr|r} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{rr|r} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{array}\right] \xrightarrow{R_2-\frac{1}{2}R_3} \left[\begin{array}{rr|r} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{array}\right] \xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{rr|r} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

This matrix has a row that is all zeros except for the final column, so the system is inconsistent. Thus $\mathbf{w}^T \notin \text{col}(A^T)$ and thus $\mathbf{w} \notin \text{row}(A)$.

14. As in the remarks following Example 3.41, we determine whether $\mathbf{w}$ is in $\text{row}(A)$ using $\text{col}(A^T)$. To say $\mathbf{w}$ is in $\text{row}(A)$ means that $\mathbf{w}^T$ is in $\text{col}(A)$, so we form the augmented matrix and row-reduce to see if the system has a solution:

$$\left[A^T \mid \mathbf{w}^T\right] = \left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 1 & 2 & -1 & 4 \\ -3 & 1 & -4 & -5 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-R_1} \\ \xrightarrow{R_3+3R_1} \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This is in row reduced form, and the final row consists of all zeros. This is a consistent system, so that $\mathbf{w}^T \in \text{col}(A^T)$ and thus $\mathbf{w} \in \text{row}(A)$.

15. To check if $\mathbf{v}$ is in null(A), we need only multiply A by $\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0},$$

so that $\mathbf{v} \notin \text{null}(A)$.

16. To check if $\mathbf{v}$ is in null(A), we need only multiply A by $\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0},$$

so that $\mathbf{v} \in \text{null}(A)$.

17. We follow the comment after Example 3.48. Start by finding the reduced row echelon form of A:

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Thus a basis for row(A) is given by

$$\left\{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}\right\}.$$

A basis for col(A) is given by the column vectors of the original matrix A corresponding to the leading 1s in the row-reduced matrix, so a basis for col(A) is

$$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}.$$

Finally, from the row-reduced form, the solutions to $A\mathbf{x} = \mathbf{0}$ where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

can be found by noting that x_3 is a free variable; setting it to t and then solving for the other two variables gives $x_1 = t$ and $x_2 = -2t$. Thus a basis for the nullspace of A is given by

$$\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

18. We follow the comment after Example 3.48. Start by finding the reduced row echelon form of A:

$$\begin{bmatrix}1 & 1 & -3\\0 & 2 & 1\\1 & -1 & -4\end{bmatrix} \xrightarrow{R_3-R_1} \begin{bmatrix}1 & 1 & -3\\0 & 2 & 1\\0 & -2 & -1\end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix}1 & 1 & -3\\0 & 2 & 1\\0 & 0 & 0\end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1 & 1 & -3\\0 & 1 & \frac{1}{2}\\0 & 0 & 0\end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix}1 & 0 & -\frac{7}{2}\\0 & 1 & \frac{1}{2}\\0 & 0 & 0\end{bmatrix}$$

Thus a basis for row(A) is given by

$$\left\{\begin{bmatrix}1 & 0 & -\frac{7}{2}\end{bmatrix}, \begin{bmatrix}0 & 1 & \frac{1}{2}\end{bmatrix}\right\} \quad \text{or, clearing fractions,} \quad \left\{\begin{bmatrix}2 & 0 & -7\end{bmatrix}, \begin{bmatrix}0 & 2 & 1\end{bmatrix}\right\}.$$

Since the row-reduction did not involve any row exchanges, a basis for row(A) is also given by the rows of A corresponding to nonzero rows in the reduced matrix:

$$\left\{\begin{bmatrix}1 & 1 & -3\end{bmatrix}, \begin{bmatrix}0 & 2 & 1\end{bmatrix}\right\}.$$

A basis for col(A) is then given by the original column vectors of A corresponding to the columns containing the leading 1s in the row-reduced matrix; thus a basis for col(A) is given by

$$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$$

Finally, from the row-reduced form, the solutions of $A\mathbf{x} = \mathbf{0}$ where

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

can be found by noting that x_3 is a free variable; we set it to t, and then solving for the other two variables in terms of t to get $x_1 = \frac{7}{2}t$ and $x_2 = -\frac{1}{2}t$. Thus a basis for the nullspace of A is given by

$$\mathbf{x} = \begin{bmatrix}\frac{7}{2}\\-\frac{1}{2}\\1\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}7\\-1\\2\end{bmatrix}$$

19. We follow the comment after Example 3.48. Start by finding the reduced row echelon form of A:

$$\begin{bmatrix}1 & 1 & 0 & 1\\0 & 1 & -1 & 1\\0 & 1 & -1 & -1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1 & 1 & 0 & 1\\0 & 1 & -1 & 1\\0 & 0 & 0 & -2\end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix}1 & 1 & 0 & 1\\0 & 1 & -1 & 1\\0 & 0 & 0 & 1\end{bmatrix} \begin{matrix}\xrightarrow{R_1-R_3}\\ \xrightarrow{R_2-R_3}\end{matrix}$$

$$\begin{bmatrix}1 & 1 & 0 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Thus a basis for row(A) is given by

$$\left\{\begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}\right\}.$$

Since the row-reduction did not involve any row exchanges, a basis for row(A) is also given by the rows of A corresponding to nonzero rows in the reduced matrix; since all the rows are nonzero, the rows of A form a basis for row(A).

A basis for col(A) is then given by the original column vectors of A corresponding to the columns containing the leading 1s in the row-reduced matrix; thus a basis for col(A) is given by

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$$

Finally, from the row-reduced form, the solutions of $A\mathbf{x} = \mathbf{0}$ where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

can be found by noting that x_3 is a free variable; we set it to t. Also, $x_4 = 0$; solving for the other two variables in terms of t gives $x_1 = -t$ and $x_2 = t$. Thus a basis for the nullspace of A is given by

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

20. We follow the comment after Example 3.48. Start by finding the reduced row echelon form of A:

$$\begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ -1 & 2 & 1 & 2 & 3 \\ 2 & -4 & 0 & 2 & 1 \end{bmatrix} \begin{matrix} \xrightarrow{R_2+R_1} \\ \xrightarrow{R_3-2R_1} \end{matrix} \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & -2 & -6 & -7 \end{bmatrix} \xrightarrow{R_3+R_2}$$

$$\begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 1 & 3 & \frac{7}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & -2 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 3 & \frac{7}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for $\operatorname{row}(A)$ is given by the nonzero rows of the reduced matrix, which after clearing fractions are

$$\left\{ \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 & 6 & 7 \end{bmatrix} \right\}.$$

A basis for $\operatorname{col}(A)$ is then given by the original column vectors of A corresponding to the columns containing the leading 1s in the row-reduced matrix; thus a basis for $\operatorname{col}(A)$ is given by

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Finally, from the row-reduced form, the solutions of $A\mathbf{x} = \mathbf{0}$ where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

can be found by noting that $x_2 = s$, $x_4 = t$, and $x_5 = u$ are free variables. Solving for the other two variables gives $x_1 = 2s - t - \frac{1}{2}u$ and $x_3 = -3t - \frac{7}{2}u$. Thus the nullspace of A is given by

$$\left\{ \begin{bmatrix} 2s - t + \frac{1}{2}u \\ s \\ -3t - \frac{7}{2}u \\ t \\ u \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{7}{2} \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span} \left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{7}{2} \\ 0 \\ 1 \end{bmatrix} \right).$$

Clearing fractions, we get for a basis for the nullspace of A

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -7 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

21. Since $\text{row}(A) = \text{col}(A^T)$ and $\text{col}(A) = \text{row}(A^T)$, it suffices to find the row and column spaces of A^T. Then

$$A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

so that the column space of A^T is given by the columns of A^T corresponding to columns of this matrix containing a leading 1, so

$$\text{col}(A^T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow \text{row}(A) = \left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right\}.$$

The row space of A^T is given by the nonzero rows of the reduced matrix, so

$$\text{row}(A^T) = \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right\} \Rightarrow \text{col}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

22. Since $\text{row}(A) = \text{col}(A^T)$ and $\text{col}(A) = \text{row}(A^T)$, it suffices to find the row and column spaces of A^T. Then

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -3 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

so that the column space of A^T is given by the columns of A^T corresponding to columns of the reduced matrix containing a leading 1, so

$$\text{col}(A^T) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\} \Rightarrow \text{row}(A) = \left\{ \begin{bmatrix} 1 & 1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \right\}.$$

The row space of A^T is given by the nonzero rows of the reduced matrix, so

$$\text{row}(A^T) = \left\{ \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \right\} \Rightarrow \text{col}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

23. Since $\text{row}(A) = \text{col}(A^T)$ and $\text{col}(A) = \text{row}(A^T)$, it suffices to find the row and column spaces of A^T. Then

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so that the column space of A^T is given by the columns of A^T corresponding to columns of the reduced matrix containing a leading 1, so

$$\text{col}(A^T) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \Rightarrow \text{row}(A) = \left\{ \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & -1 \end{bmatrix} \right\}.$$

The row space of A^T is given by the nonzero rows of the reduced matrix, so

$$\text{row}(A^T) = \left\{ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\} \Rightarrow \text{col}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

24. Since $\operatorname{row}(A) = \operatorname{col}(A^T)$ and $\operatorname{col}(A) = \operatorname{row}(A^T)$, it suffices to find the row and column spaces of A^T. Then

$$A^T = \begin{bmatrix} 2 & -1 & 1 \\ -4 & 2 & -2 \\ 0 & 1 & 1 \\ 2 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so that the column space of A^T is given by the columns of A^T corresponding to columns of the reduced matrix containing a leading 1, so

$$\operatorname{col}(A^T) = \left\{ \begin{bmatrix} 2 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\} \Rightarrow \operatorname{row}(A) = \left\{ \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 & 1 & 2 & 3 \end{bmatrix} \right\}.$$

The row space of A^T is given by the nonzero rows of the reduced matrix, so

$$\operatorname{row}(A^T) = \left\{ \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \right\} \Rightarrow \operatorname{col}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

25. There is no reason to expect the bases for the row and column spaces found in Exercises 17 and 21 to be the same, since we used different methods to find them. What is important is that they should span the same subspaces. The basis for the row space found in Exercise 17 was $\left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \right\}$, and Exercise 21 gave $\left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right\}$. Since the first basis vector is the same, in order to show the subspaces are the same it suffices to see that $\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$ is in the span of the second basis, and that $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ is in the span of the first basis. But

$$\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = -\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}.$$

So the two bases span the same row space. Similarly, the basis for the two column spaces were

$$\text{Exercise 17: } \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad \text{Exercise 21: } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Here the second basis vectors are the same, and since

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

each basis vector is a linear combination of the vectors in the other basis, so they span the same column space.

26. There is no reason to expect the bases for the row and column spaces found in Exercises 18 and 22 to be the same, since we used different methods to find them. What is important is that they should span the same subspaces. The basis for the row space found in Exercise 18 was $\left\{ \begin{bmatrix} 2 & 0 & -7 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \right\}$, and Exercise 22 gave $\left\{ \begin{bmatrix} 1 & 1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \right\}$. Since the second basis vector is the same, in order to show the subspaces are the same it suffices to see that $\begin{bmatrix} 2 & 0 & -7 \end{bmatrix}$ is in the span of the second basis, and that $\begin{bmatrix} 1 & 1 & -3 \end{bmatrix}$ is in the span of the first basis. But

$$\begin{bmatrix} 2 & 0 & -7 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 & -3 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & -3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & -7 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 1 \end{bmatrix}.$$

So the two bases span the same row space. Similarly, the basis for the two column spaces were

$$\text{Exercise 18: } \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}, \qquad \text{Exercise 22: } \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Here the first basis vectors are the same, and since

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

each basis vector is a linear combination of the vectors in the other basis, so they span the same column space.

27. As in Example 3.46, a basis for the span of the given set of vectors is the same as a basis for the row space of the matrix that has rows equal to these vectors. Thus we form a matrix whose rows are $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ and row-reduce it:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{array}{c} \xrightarrow{R_1+R_2} \\ \\ \xrightarrow{R_3-R_2} \end{array} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the row space of the matrix, and thus a basis for the span of the given vectors, is therefore given by the nonzero rows of the reduced matrix:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Note that since no row interchanges were involved, the rows of the original matrix corresponding to nonzero rows in the reduced matrix also form a basis for the span of the given vectors.

28. As in Example 3.46, a basis for the span of the given set of vectors is the same as a basis for the row space of the matrix that has rows equal to these vectors. Thus we form a matrix whose rows are $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$, and $\begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$ and row-reduce it:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{array}{c} \xrightarrow{R_2-R_1} \\ \\ \xrightarrow{R_4-2R_1} \end{array} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{R_2\leftrightarrow R_4} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\begin{array}{c} \xrightarrow{R_3-R_2} \\ \xrightarrow{R_4-2R_2} \end{array} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{R_4+R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the row space of the matrix, and thus a basis for the span of the given vectors, is therefore given by

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(Note that this matrix could be further reduced, and we would get a different basis).

29. As in Example 3.46, a basis for the span of the given set of vectors is the same as a basis for the row space of the matrix that has rows equal to these vectors. Thus we form a matrix whose rows are

$\begin{bmatrix}2 & -3 & 1\end{bmatrix}$, $\begin{bmatrix}1 & -1 & 0\end{bmatrix}$, $\begin{bmatrix}4 & -4 & 1\end{bmatrix}$ and row-reduce it:

$$\begin{bmatrix}2 & -3 & 1\\1 & -1 & 0\\4 & -4 & 1\end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix}1 & -1 & 0\\2 & -3 & 1\\4 & -4 & 1\end{bmatrix} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{bmatrix}1 & -1 & 0\\0 & -1 & 1\\0 & -2 & 1\end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix}1 & -1 & 0\\0 & 1 & -1\\0 & -2 & 1\end{bmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{bmatrix}1 & -1 & 0\\0 & 1 & -1\\0 & 0 & -1\end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix}1 & -1 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix}1 & -1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow{R_1+R_2} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

A basis for the row space of the matrix, and thus a basis for the span of the given vectors, is therefore given by

$$\left\{\begin{bmatrix}1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\end{bmatrix}\right\}$$

(Note that as soon as we got to the first matrix on the second line above, we could have concluded that the reduced matrix would have no zero rows; since it was 3×3, it would thus be the identity matrix, so that we could immediately write down the basis above, or simply use the basis given by the rows of that matrix).

30. As in Example 3.46, a basis for the span of the given set of vectors is the same as a basis for the row space of the matrix that has rows equal to these vectors. Thus we form a matrix whose rows are $\begin{bmatrix}0 & 1 & -2 & 1\end{bmatrix}$, $\begin{bmatrix}3 & 1 & -1 & 0\end{bmatrix}$, $\begin{bmatrix}2 & 1 & 5 & 1\end{bmatrix}$ and row-reduce it:

$$\begin{bmatrix}0 & 1 & -2 & 1\\3 & 1 & -1 & 0\\2 & 1 & 5 & 1\end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix}3 & 1 & -1 & 0\\0 & 1 & -2 & 1\\2 & 1 & 5 & 1\end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix}1 & \frac{1}{3} & -\frac{1}{3} & 0\\0 & 1 & -2 & 1\\2 & 1 & 5 & 1\end{bmatrix}$$

$$\xrightarrow{R_3-2R_1} \begin{bmatrix}1 & \frac{1}{3} & -\frac{1}{3} & 0\\0 & 1 & -2 & 1\\0 & \frac{1}{3} & \frac{17}{3} & 1\end{bmatrix} \xrightarrow{R_3-\frac{1}{3}R_2} \begin{bmatrix}1 & \frac{1}{3} & -\frac{1}{3} & 0\\0 & 1 & -2 & 1\\0 & 0 & \frac{19}{3} & \frac{2}{3}\end{bmatrix}$$

$$\xrightarrow{\frac{3}{19}R_3} \begin{bmatrix}1 & \frac{1}{3} & -\frac{1}{3} & 0\\0 & 1 & -2 & 1\\0 & 0 & 1 & \frac{2}{19}\end{bmatrix}$$

Although this matrix could be reduced further, we already have a basis for the row space:

$$\begin{bmatrix}1 & \frac{1}{3} & -\frac{1}{3} & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & -2 & 1\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 1 & \frac{2}{19}\end{bmatrix},$$

or

$$\begin{bmatrix}3 & 1 & -1 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & -2 & 1\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 19 & 2\end{bmatrix}.$$

31. If, as in Exercise 29, we form the matrix whose rows are those vectors and row-reduce it, we get the identity matrix (see Exercise 29). Because there were no row exchanges, the rows of the original matrix corresponding to nonzero rows in the row-reduced matrix form a basis for the span of the vectors (because there were no row exchanges). Thus a basis consists of all three given vectors:

$$\begin{bmatrix}2 & -3 & 1\end{bmatrix}, \begin{bmatrix}1 & -1 & 0\end{bmatrix}, \begin{bmatrix}4 & -4 & 1\end{bmatrix}.$$

32. If, as in Exercise 30, we form the matrix whose rows are those vectors and row-reduce it, we get a matrix with no zero rows. Thus the rank of the matrix is 3, so the rows are linearly independent, so a basis consists of all three row vectors:

$$\begin{bmatrix}0 & 1 & -2 & 1\end{bmatrix}, \begin{bmatrix}3 & 1 & -1 & 0\end{bmatrix}, \begin{bmatrix}2 & 1 & 5 & 1\end{bmatrix}.$$

33. If R is a matrix in echelon form, then $\text{row}(R)$ is the span of the rows of R, by definition. But nonzero rows of R are linearly independent, since their first entries are in different columns, so those nonzero rows form a basis for $\text{row}(R)$ (they span, and are linearly independent).

34. $\text{col}(A)$ is the span of the columns of A by definition. If the columns of A are also linearly independent, then they form a basis for $\text{col}(A)$ by definition of "basis".

35. The rank of A is the number of vectors in a basis for either $\text{row}(A)$ or $\text{col}(A)$, by definition. From Exercise 17, $\text{row}(A)$ has two vectors in its basis, so $\text{rank}(A) = 2$. Then $\text{nullity}(A)$ is the number of columns minus the rank, or $3 - 2 = 1$.

36. The rank of A is the number of vectors in a basis for either $\text{row}(A)$ or $\text{col}(A)$, by definition. From Exercise 18, $\text{row}(A)$ has two vectors in its basis, so $\text{rank}(A) = 2$. Then $\text{nullity}(A)$ is the number of columns minus the rank, or $3 - 2 = 1$.

37. The rank of A is the number of vectors in a basis for either $\text{row}(A)$ or $\text{col}(A)$, by definition. From Exercise 19, $\text{row}(A)$ has three vectors in its basis, so $\text{rank}(A) = 3$. Then $\text{nullity}(A)$ is the number of columns minus the rank, or $4 - 3 = 1$.

38. The rank of A is the number of vectors in a basis for either $\text{row}(A)$ or $\text{col}(A)$, by definition. From Exercise 20, $\text{row}(A)$ has two vectors in its basis, so $\text{rank}(A) = 2$. Then $\text{nullity}(A)$ is the number of columns minus the rank, or $5 - 2 = 3$.

39. Since $\text{nullity}(A) > 0$, the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$, where at least one $c_i \neq 0$. Then $A\mathbf{x} = \sum c_i\mathbf{a}_i = \mathbf{0}$. Since at least one $c_i \neq 0$, this shows that the columns of A are linearly dependent.

40. Since $\text{rank}(A)$ is at most the number of columns of A and also at most the number of rows of A, in this case $\text{rank}(A) \leq 2$. Thus any row-reduced form of A will have at least $4 - 2 = 2$ zero rows, which means that some row of A is a linear combination of the other rows, so that the rows of A are linearly dependent.

41. Since $\text{rank}(A)$ is at most the number of columns of A and also at most the number of rows of A, in this case $\text{rank}(A) \leq 5$ and $\text{rank}(A) \leq 3$, so that $\text{rank}(A) \leq 3$ and its possible values are 0, 1, 2, and 3. Using the Rank Theorem, we know that

$$\text{nullity}(A) = n - \text{rank}(A) = 5 - \text{rank}(A), \text{ so that } \text{nullity}(A) = 5,\ 4,\ 3, \text{ or } 2.$$

42. Since $\text{rank}(A)$ is at most the number of columns of A and also at most the number of rows of A, in this case $\text{rank}(A) \leq 2$ and $\text{rank}(A) \leq 4$, so that $\text{rank}(A) \leq 2$ and its possible values are 0, 1, and 2. Using the Rank Theorem, we know that

$$\text{nullity}(A) = 2 - \text{rank}(A) = 1 - \text{rank}(A), \text{ so that } \text{nullity}(A) = 2,\ 1, \text{ or } 0.$$

43. First row-reduce A:

$$\begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix} \xrightarrow[R_3 - aR_1]{R_2 + 2R_1} \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2a+2 \\ 0 & -2a-2 & 1-a^2 \end{bmatrix} \xrightarrow{R_2 + \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2a+2 \\ 0 & 0 & -a^2+a+2 \end{bmatrix}$$

If $a = -1$, then we have

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ so that } \text{rank}(A) = 1.$$

If $a = 2$, then we have

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 12 & 6 \\ 0 & 0 & 0 \end{bmatrix} \text{ so that } \text{rank}(A) = 2.$$

Otherwise, the row-reduced matrix has all rows nonzero, so that $\text{rank}(A) = 3$.

44. First row-reduce A:

$$\begin{bmatrix} a & 2 & -1 \\ 3 & 3 & -2 \\ -2 & -1 & a \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -2 & -1 & a \\ 3 & 3 & -2 \\ a & 2 & -1 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 2 & a-2 \\ 3 & 3 & -2 \\ a & 2 & -1 \end{bmatrix}$$

$$\begin{array}{c} \xrightarrow{R_2-3R_1} \\ \xrightarrow{R_3-aR_2} \end{array} \begin{bmatrix} 1 & 2 & a-2 \\ 0 & -3 & 4-3a \\ 0 & 2-2a & -a^2+2a-1 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & a-2 \\ 0 & 1 & a-\frac{4}{3} \\ 0 & 2-2a & -a^2+2a-1 \end{bmatrix}$$

$$\xrightarrow{R_3-(2-2a)R_2} \begin{bmatrix} 1 & 2 & a-2 \\ 0 & 1 & a-\frac{4}{3} \\ 0 & 0 & (a-1)\left(a-\frac{5}{3}\right) \end{bmatrix}$$

The first two rows are always nonzero. If $a = 1$ or $a = \frac{5}{3}$, then the bottom row is zero, so that $\operatorname{rank}(A) = 2$. Otherwise, all three rows are nonzero and $\operatorname{rank}(A) = 3$.

45. As in example 3.52,

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$ if and only if the matrix with those vectors as its rows has rank 3. Row-reduce that matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since the matrix has rank 3 (it has no nonzero rows and is in echelon form), the vectors are linearly independent; since there are three of them, they form a basis for $\mathbb{R}^3$.

46. As in example 3.52,

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$ if and only if the matrix with those vectors as its rows has rank 3. Row-reduce that matrix:

$$\begin{bmatrix} 1 & -1 & 3 \\ -1 & 5 & 1 \\ 1 & -3 & 1 \end{bmatrix} \begin{array}{c} \xrightarrow{R_2+2R_1} \\ \xrightarrow{R_3-R_1} \end{array} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & 4 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{R_2+2R_3} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -2 \end{bmatrix}$$

Since the matrix does not have rank 3, it follows that these vectors do not form a basis for $\mathbb{R}^3$. (In fact,

$$-\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so the vectors are linearly dependent.)

47. As in example 3.52,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^4$ if and only if the matrix with those vectors as its rows has rank 4. Row-reduce that matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_4\leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$
$$\xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_4+R_3} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

This matrix is in echelon form and has no zero rows, so it has rank 4 and thus these vectors form a basis for $\mathbb{R}^4$.

48. As in example 3.52,

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for $\mathbb{R}^4$ if and only if the matrix with those vectors as its rows has rank 4. Row-reduce that matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_4+R_1} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_4+R_2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$
$$\xrightarrow{R_4+R_3} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the matrix does not have rank 4, these vectors do not form a basis for $\mathbb{R}^4$. (In fact, the sum of the four vectors is zero, so they are linearly dependent.)

49. As in example 3.52,

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{Z}_2^3$ if and only if the matrix with those vectors as its rows has rank 3. Row-reduce that matrix over $\mathbb{Z}_2$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the matrix does not have rank 3, these vectors do not form a basis for $\mathbb{Z}_2^3$.

50. As in example 3.52,

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{Z}_3^3$ if and only if the matrix with those vectors as its rows has rank 3. Row-reduce that matrix over $\mathbb{Z}_3$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since the matrix has rank 3, these vectors form a basis for $\mathbb{Z}_3^3$.

51. We want to solve

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix},$$

so set up the augmented matrix of the linear system and row-reduce to solve it:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-R_2} \\ \\ \xrightarrow{R_3+R_2} \end{array} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $c_1 = 3$ and $c_2 = -2$, so that

$$3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}.$$

Then the coordinate vector $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

52. We want to solve

$$c_1 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix},$$

so set up the augmented matrix of the linear system and row-reduce to solve it:

$$\left[\begin{array}{cc|c} 3 & 5 & 1 \\ 1 & 1 & 3 \\ 4 & 6 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & 5 & 1 \\ 4 & 6 & 4 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-3R_1} \\ \xrightarrow{R_3-4R_1} \end{array} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2 & -8 \\ 0 & 2 & -8 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -4 \\ 0 & 2 & -8 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-R_2} \\ \\ \xrightarrow{R_3-2R_2} \end{array} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $c_1 = 7$ and $c_2 = -4$, so that

$$7 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - 4 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix},$$

Then the coordinate vector $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$.

53. Row-reduce A over $\mathbb{Z}_2$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix is in echelon form with one zero row, so it has rank 2. Thus $\operatorname{nullity}(A) = 3 - \operatorname{rank}(A) = 3 - 2 = 1$.

54. Row-reduce A over $\mathbb{Z}_3$:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 0 & 0 \end{bmatrix} \begin{array}{c} \xrightarrow{R_2+R_1} \\ \xrightarrow{R_3+R_1} \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix is in echelon form with one zero row, so it has rank 2. Thus nullity$(A) = 3 - \operatorname{rank}(A) = 3 - 2 = 1$.

55. Row-reduce A over $\mathbb{Z}_5$:

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 4 & 0 \end{bmatrix} \xrightarrow[R_3+4R_1]{R_2+3R_1} \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{R_3+4R_2} \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

This matrix is in echelon form with no zero rows, so it has rank 3. Thus nullity$(A) = 4 - \operatorname{rank}(A) = 4 - 3 = 1$.

56. Row-reduce A over $\mathbb{Z}_7$:

$$A = \begin{bmatrix} 2 & 4 & 0 & 0 & 1 \\ 6 & 3 & 5 & 1 & 0 \\ 1 & 0 & 2 & 2 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\substack{R_2+4R_1 \\ R_3+3R_1 \\ R_4+3R_1}} \begin{bmatrix} 2 & 4 & 0 & 0 & 1 \\ 0 & 5 & 5 & 1 & 4 \\ 0 & 5 & 2 & 2 & 1 \\ 0 & 6 & 1 & 1 & 4 \end{bmatrix} \xrightarrow{\substack{R_3+6R_2 \\ R_4+3R_2}}$$

$$\begin{bmatrix} 2 & 4 & 0 & 0 & 1 \\ 0 & 5 & 5 & 1 & 4 \\ 0 & 0 & 4 & 1 & 4 \\ 0 & 0 & 2 & 4 & 2 \end{bmatrix} \xrightarrow{R_4+3R_3} \begin{bmatrix} 2 & 4 & 0 & 0 & 1 \\ 0 & 5 & 5 & 1 & 4 \\ 0 & 0 & 4 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is in echelon form with one zero row, so it has rank 3. Thus nullity$(A) = 5 - \operatorname{rank}(A) = 5 - 3 = 2$.

57. Since $\mathbf{x}$ is in null(A), we know that $A\mathbf{x} = \mathbf{0}$. But if $\mathbf{A}_i$ is the i^{th} row of A, then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \cdot \mathbf{x} \\ \mathbf{A}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{A}_m \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus $\mathbf{A}_i \cdot \mathbf{x} = 0$ for all i, so that every row of A is orthogonal to $\mathbf{x}$.

58. By the Fundamental Theorem, if A and B have rank n, then they are both invertible. Therefore, by Theorem 3.9(c), AB is invertible as well. Applying the Fundamental Theorem again shows that AB also has rank n.

59. **(a)** Let $\mathbf{b}_k$ be the k^{th} column of B. Then $A\mathbf{b}_k$ is the k^{th} column of AB. Now suppose that some set of columns of AB, say $A\mathbf{b}_{k_1}, A\mathbf{b}_{k_2}, \ldots, A\mathbf{b}_{k_r}$, are linearly independent. Then

$$\begin{gathered} c_{k_1}\mathbf{b}_{k_1} + c_{k_2}\mathbf{b}_{k_2} + \cdots + c_{k_r}\mathbf{b}_{k_r} = \mathbf{0} \Rightarrow \\ A\left(c_{k_1}\mathbf{b}_{k_1} + c_{k_2}\mathbf{b}_{k_2} + \cdots + c_{k_r}\mathbf{b}_{k_r}\right) = \mathbf{0} \Rightarrow \\ c_{k_1}(A\mathbf{b}_{k_1}) + c_{k_2}(A\mathbf{b}_{k_2}) + \cdots + c_{k_r}(A\mathbf{b}_{k_r} = \mathbf{0}). \end{gathered}$$

But the $A\mathbf{b}_{k_i}$ are linearly independent, so all the c_{k_i} are zero and thus the $\mathbf{b}_{k_i}$ are linearly independent. So the number of linearly independent columns in B is at least as large as the number of linearly independent columns of AB; it follows that rank$(AB) \le$ rank(B).

(b) There are many examples. For instance,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then B has rank 2, and

$$AB = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \text{ has rank 1.}$$

60. **(a)** Recall that by Theorem 3.25, if C is any matrix, then $\operatorname{rank}(C^T) = \operatorname{rank}(C)$. Thus

$$\operatorname{rank}(AB) = \operatorname{rank}((AB)^T) = \operatorname{rank}(B^TA^T) \le \operatorname{rank}(A^T) = \operatorname{rank}(A),$$

using Exercise 59(a) with A^T in place of B and B^T in place of A.

(b) For example, we can use the transposes of the matrices in part (b) of Exercise 59:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then A has rank 2 and

$$AB = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \text{ has rank 1.}$$

61. **(a)** Since U is invertible, we have $A = U^{-1}(UA)$, so that using Exercise 59(a) twice,

$$\operatorname{rank}(A) = \operatorname{rank}\left(U^{-1}(UA)\right) \le \operatorname{rank}(UA) \le \operatorname{rank}(A).$$

Thus $\operatorname{rank}(A) = \operatorname{rank}(UA)$.

(b) Since V is invertible, we have $A = (AV)V^{-1}$, so that using Exercise 60(a) twice,

$$\operatorname{rank}(A) = \operatorname{rank}\left((AV)V^{-1}\right) \le \operatorname{rank}(AV) \le \operatorname{rank}(A).$$

Thus $\operatorname{rank}(A) = \operatorname{rank}(AV)$.

62. Suppose first that A has rank 1. Then $\operatorname{col}(A) = \operatorname{span}(\mathbf{u})$ for some vector $\mathbf{u}$. It follows that if $\mathbf{a}_i$ is the i^{th} column of A, then for each i, we have $\mathbf{a}_i = c_i\mathbf{u}$. Let $\mathbf{v}^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$. Then $A = \mathbf{u}\mathbf{v}^T$. For the converse, suppose that $A = \mathbf{u}\mathbf{v}^T$. Suppose that $\mathbf{v}^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$. Then

$$A = \mathbf{u}\mathbf{v}^T = \begin{bmatrix} c_1\mathbf{u} & \cdots & c_n\mathbf{u} \end{bmatrix}.$$

Thus every column of A is a scalar multiple of $\mathbf{u}$, so that $\operatorname{col}(A) = \operatorname{span}(\mathbf{u})$ and thus $\operatorname{rank}(A) = 1$.

63. If $\operatorname{rank}(A) = r$, then a basis for $\operatorname{col}(A)$ consists of r vectors, say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r \in \mathbb{R}^m$. So if $\mathbf{a}_i$ is the i^{th} column of A, then for each i we have

$$\mathbf{a}_i = c_{i1}\mathbf{u}_1 + c_{i2}\mathbf{u}_2 + \cdots + c_{ir}\mathbf{u}_r, \; i = 1, 2, \ldots, n.$$

Now let $\mathbf{v}_i^T = \begin{bmatrix} c_{1i} & c_{2i} & \cdots & c_{ni} \end{bmatrix}$, for $i = 1, 2, \ldots, r$. Then so that

$$\begin{aligned} A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} &= \begin{bmatrix} c_{11}\mathbf{u}_1 + c_{12}\mathbf{u}_2 + \cdots + c_{1r}\mathbf{u}_r & \cdots & c_{n1}\mathbf{u}_1 + c_{n2}\mathbf{u}_2 + \cdots + c_{nr}\mathbf{u}_r \end{bmatrix} \\ &= \begin{bmatrix} c_{11}\mathbf{u}_1 & \cdots & c_{n1}\mathbf{u}_1 \end{bmatrix} + \begin{bmatrix} c_{12}\mathbf{u}_2 & \cdots & c_{n2}\mathbf{u}_2 \end{bmatrix} + \begin{bmatrix} c_{1r}\mathbf{u}_r & \cdots & c_{nr}\mathbf{u}_r \end{bmatrix} \\ &= \mathbf{u}_1 \cdot \mathbf{v}_1^T + \mathbf{u}_2 \cdot \mathbf{v}_2^T + \cdots + \mathbf{u}_r \cdot \mathbf{v}_r^T. \end{aligned}$$

By exercise 62, each $\mathbf{u}_i \cdot \mathbf{v}_i^T$ is a matrix of rank 1, so A is a sum of r matrices of rank 1.

64. Let $\mathcal{A} = \{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ be a basis for $\operatorname{col}(A)$ and $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ be a basis for $\operatorname{col}(B)$. Then $\operatorname{rank}(A) = r$ and $\operatorname{rank}(B) = s$. Let $\mathbf{a}_i$ and $\mathbf{b}_i$ be the columns of A and B respectively, for $i = 1, 2, \ldots, n$. Then if $x \in \operatorname{col}(A + B)$,

$$\mathbf{x} = \sum_{i=1}^n c_1(\mathbf{a}_i + \mathbf{b}_i) = \sum_{i=1}^n \left(\sum_{j=1}^r \alpha_{ij}\mathbf{u}_j + \sum_{k=1}^s \beta_{ik}\mathbf{v}_k \right) = \sum_{j=1}^r \left(\sum_{i=1}^n c_i\alpha_{ij} \right) \mathbf{u}_j + \sum_{k=1}^s \left(\sum_{i=1}^n c_i\beta_{ik} \right) \mathbf{v}_k,$$

so that $\mathbf{x}$ is a linear combination of $\mathcal{A}$ and $\mathcal{B}$. Thus $\mathcal{A} \cup \mathcal{B}$ spans $\operatorname{col}(A + B)$. But this implies that

$$\operatorname{col}(A + B) \subseteq \operatorname{span}(\mathcal{A} \cup \mathcal{B}) = \operatorname{span}(\mathcal{A}) + \operatorname{span}(\mathcal{B}) = \operatorname{col}(A) + \operatorname{col}(B).$$

Since $\operatorname{rank}(A + B)$ is the dimension of $\operatorname{col}(A + B)$ and $\operatorname{rank}(A) + \operatorname{rank}(B)$ is the dimension of $\operatorname{col}(A)$ plus the dimension of $\operatorname{col}(B)$, we are done.

65. We show that $\text{col}(A) \subseteq \text{null}(A)$ and then use the Rank Theorem. Suppose $\mathbf{x} = \sum c_i \mathbf{a}_i \in \text{col}(A)$. Let $\mathbf{a}_i$ be the i^{th} column of A; then $\mathbf{a}_i = A\mathbf{e}_i$ where the $\mathbf{e}_i$ are the standard basis vectors for $\mathbb{R}^n$. Then

$$A\mathbf{x} = A\left(\sum c_i\mathbf{a}_i\right) = \sum c_i\left(A\mathbf{a}_i\right) = \sum c_i\left(A\left(A\mathbf{e}_i\right)\right) = \sum c_i\left(A^2\mathbf{e}_i\right) = \sum c_i\left(O\mathbf{e}_i\right) = \mathbf{0}.$$

Thus any vector in the column space of A is taken to zero by A, so it is in the null space of A. Thus $\text{col}(A) \subseteq \text{null}(A)$. But then

$$\text{rank}(A) = \dim(\text{col}(A)) \le \dim(\text{null}(A)) = \text{nullity}(A),$$

so by the Rank Theorem,

$$\text{rank}(A) + \text{rank}(A) \le \text{rank}(A) + \text{nullity}(A) = n \quad\Rightarrow\quad 2\,\text{rank}(A) \le n \quad\Rightarrow\quad \text{rank}(A) \le \frac{n}{2}.$$

66. **(a)** Note first that since $\mathbf{x}$ is $n \times 1$, then $\mathbf{x}^T$ is $1 \times n$ and thus $\mathbf{x}^T A\mathbf{x}$ is 1×1. Thus $\left(\mathbf{x}^T A\mathbf{x}\right)^T = \mathbf{x}^T A\mathbf{x}$. Now since A is skew-symmetric,

$$\mathbf{x}^T A\mathbf{x} = \left(\mathbf{x}^T A\mathbf{x}\right)^T = \mathbf{x}^T A^T \left(\mathbf{x}^T\right)^T = \mathbf{x}^T\left(-A\right)\mathbf{x} = -\mathbf{x}^T A\mathbf{x}.$$

So $\mathbf{x}^T A\mathbf{x}$ is equal to its own negation, so it must be zero.

(b) By Theorem 3.27, to show $I + A$ is invertible it suffices to show that $\text{null}(I + A) = \mathbf{0}$. Choose $\mathbf{x}$ with $(I + A)\mathbf{x} = \mathbf{0}$. Then $\mathbf{x}^T(I + A)\mathbf{x} = \mathbf{x}^T\mathbf{0} = \mathbf{x}\cdot\mathbf{0} = 0$, so that

$$0 = \mathbf{x}^T(I + A)\mathbf{x} = \mathbf{x}^T I\mathbf{x} + \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T I\mathbf{x} + \mathbf{x}^T\mathbf{0} = \mathbf{x}^T I\mathbf{x} = \mathbf{x}^T\mathbf{x} = \mathbf{x}\cdot\mathbf{x}$$

since $A\mathbf{x} = \mathbf{0}$. But $\mathbf{x}\cdot\mathbf{x} = 0$ implies that $\mathbf{x} = \mathbf{0}$. So any vector in $\text{null}(I + A)$ is the zero vector and thus $I + A$ has a zero null space, so it is invertible.

3.6 Introduction to Linear Transformations

1. We must compute $T(\mathbf{u})$ and $T(\mathbf{v})$, so we take the product of the matrix A of T and each of these two vectors:

$$\begin{aligned} T(\mathbf{u}) = A\mathbf{u} &= \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix} \\ T(\mathbf{v}) = A\mathbf{v} &= \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}. \end{aligned}$$

2. We must compute $T(\mathbf{u})$ and $T(\mathbf{v})$, so we take the product of the matrix A of T and each of these two vectors:

$$\begin{aligned} T(\mathbf{u}) = A\mathbf{u} &= \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 - 1\cdot 2 \\ 1\cdot 1 + 2\cdot 2 \\ 1\cdot 1 + 4\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} \\ T(\mathbf{v}) = A\mathbf{v} &= \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3\cdot 3 - 1\cdot(-2) \\ 1\cdot 3 + 2\cdot(-2) \\ 1\cdot 3 + 4\cdot(-2) \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ -5 \end{bmatrix} \end{aligned}$$

3. Use the remark following Example 3.55. T is a linear transformation if and only if

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2), \text{ where } \mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

Then

$$\begin{aligned}
T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= T\left(c_1\begin{bmatrix}x_1\\y_1\end{bmatrix} + c_2\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)\\
&= T\begin{bmatrix}c_1x_1 + c_2x_2\\c_1y_1 + c_2y_2\end{bmatrix}\\
&= \begin{bmatrix}(c_1x_1 + c_2x_2) + (c_1y_1 + c_2y_2)\\(c_1x_1 + c_2x_2) - (c_1y_1 + c_2y_2)\end{bmatrix}\\
&= \begin{bmatrix}(c_1x_1 + c_1y_1) + (c_2x_2 + c_2y_2)\\(c_1x_1 - c_1y_1) + (c_2x_2 - c_2y_2)\end{bmatrix}\\
&= \begin{bmatrix}c_1x_1 + c_1y_1\\c_1x_1 - c_1y_1\end{bmatrix} + \begin{bmatrix}c_2x_2 + c_2y_2\\c_2x_2 - c_2y_2\end{bmatrix}\\
&= c_1\begin{bmatrix}x_1 + y_1\\x_1 - y_1\end{bmatrix} + c_2\begin{bmatrix}x_2 + y_2\\x_2 - y_2\end{bmatrix}\\
&= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)
\end{aligned}$$

and thus T is linear.

4. Use the remark following Example 3.55. T is a linear transformation if and only if

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2),\ \text{where } \mathbf{v}_1 = \begin{bmatrix}x_1\\y_1\end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix}x_2\\y_2\end{bmatrix}.$$

Then

$$\begin{aligned}
T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= T\left(c_1\begin{bmatrix}x_1\\y_1\end{bmatrix} + c_2\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)\\
&= T\begin{bmatrix}c_1x_1 + c_2x_2\\c_1y_1 + c_2y_2\end{bmatrix}\\
&= \begin{bmatrix}-(c_1y_1 + c_2y_2)\\(c_1x_1 + c_2x_2) + 2(c_1y_1 + c_2y_2)\\3(c_1x_1 + c_2x_2) - 4(c_1y_1 + c_2y_2)\end{bmatrix}\\
&= \begin{bmatrix}-c_1y_1\\c_1x_1 + 2c_1y_1\\3c_1x_1 - 4c_1y_1\end{bmatrix} + \begin{bmatrix}-c_2y_2\\c_2x_2 + 2c_2y_2\\3c_2x_2 - 4c_2y_2\end{bmatrix}\\
&= c_1\begin{bmatrix}-y_1\\x_1 + 2y_1\\3x_1 - 4y_1\end{bmatrix} + c_2\begin{bmatrix}-y_2\\x_2 + 2y_2\\3x_2 - 4y_2\end{bmatrix}\\
&= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)
\end{aligned}$$

and thus T is linear.

5. Use the remark following Example 3.55. T is a linear transformation if and only if

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2),\ \text{where } \mathbf{v}_1 = \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}.$$

Then

$$\begin{aligned}
T(c_1\mathbf{v}_1+c_2\mathbf{v}_2) &= T\left(c_1\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}+c_2\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}\right)\\
&= T\begin{bmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\\c_1z_1+c_2z_2\end{bmatrix}\\
&= \begin{bmatrix}(c_1x_1+c_2x_2)-(c_1y_1+c_2y_2)+(c_1z_1+c_2z_2)\\2(c_1x_1+c_2x_2)+(c_1y_1+c_2y_2)-3(c_1z_1+c_2z_2)\end{bmatrix}\\
&= \begin{bmatrix}(c_1x_1-c_1y_1+c_1z_1)+(c_2x_2-c_2y_2+c_2z_2)\\(2c_1x_1+c_1y_1-3c_1z_1)+(2c_2x_2+c_2y_2-3c_2z_2)\end{bmatrix}\\
&= \begin{bmatrix}c_1x_1-c_1y_1+c_1z_1\\2c_1x_1+c_1y_1-3c_1z_1\end{bmatrix}+\begin{bmatrix}c_2x_2-c_2y_2+c_2z_2\\2c_2x_2+c_2y_2-3c_2z_2\end{bmatrix}\\
&= c_1\begin{bmatrix}x_1-y_1+z_1\\2x_1+y_1-3z_1\end{bmatrix}+c_2\begin{bmatrix}x_2-y_2+z_2\\2x_2+y_2-3z_2\end{bmatrix}\\
&= c_1T(\mathbf{v}_1)+c_2T(\mathbf{v}_2)
\end{aligned}$$

and thus T is linear.

6. Use the remark following Example 3.55. T is a linear transformation if and only if

$$T(c_1\mathbf{v}_1+c_2\mathbf{v}_2)=c_1T(\mathbf{v}_1)+c_2T(\mathbf{v}_2),\ \text{where } \mathbf{v}_1=\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix},\ \mathbf{v}_2=\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}.$$

Then

$$\begin{aligned}
T(c_1\mathbf{v}_1+c_2\mathbf{v}_2) &= T\left(c_1\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}+c_2\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}\right)\\
&= T\begin{bmatrix}c_1x_1+c_2x_2\\c_1y_1+c_2y_2\\c_1z_1+c_2z_2\end{bmatrix}\\
&= \begin{bmatrix}(c_1x_1+c_2x_2)+(c_1z_1+c_2z_2)\\(c_1y_1+c_2y_2)+(c_1z_1+c_2z_2)\\(c_1x_1+c_2x_2)+(c_1y_1+c_2y_2)\end{bmatrix}\\
&= \begin{bmatrix}c_1x_1+c_1z_1\\c_1y_1+c_1z_1\\c_1x_1+c_1y_1\end{bmatrix}+\begin{bmatrix}c_2x_2+c_2z_2\\c_2y_2+c_2z_2\\c_2x_2+c_2y_2\end{bmatrix}\\
&= c_1\begin{bmatrix}x_1+z_1\\y_1+z_1\\x_1+y_1\end{bmatrix}+c_2\begin{bmatrix}x_2+z_2\\y_2+z_2\\x_2+y_2\end{bmatrix}\\
&= c_1T(\mathbf{v}_1)+c_2T(\mathbf{v}_2)
\end{aligned}$$

and thus T is linear.

7. For example,

$$T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1^2\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix},\ T\begin{bmatrix}2\\0\end{bmatrix}=\begin{bmatrix}0\\2^2\end{bmatrix}=\begin{bmatrix}0\\4\end{bmatrix},\ \text{but } T\begin{bmatrix}1+2\\0\end{bmatrix}=T\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}0\\3^2\end{bmatrix}=\begin{bmatrix}0\\9\end{bmatrix}\neq\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}0\\4\end{bmatrix}.$$

8. For example,

$$T\begin{bmatrix}-1\\0\end{bmatrix}=\begin{bmatrix}|-1|\\|0|\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\ T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}|1|\\|0|\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\ \text{but } T\begin{bmatrix}-1+1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}|0|\\|0|\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\neq\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}.$$

9. For example

$$T\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}2\cdot 2\\2+2\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix},\text{ but } T\begin{bmatrix}2+2\\2+2\end{bmatrix}=T\begin{bmatrix}4\\4\end{bmatrix}=\begin{bmatrix}4\cdot 4\\4+4\end{bmatrix}=\begin{bmatrix}16\\8\end{bmatrix}\neq\begin{bmatrix}4\\4\end{bmatrix}+\begin{bmatrix}4\\4\end{bmatrix}.$$

10. For example,

$$T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1+1\\0-1\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix},\text{ but } T\begin{bmatrix}1+1\\0+0\end{bmatrix}=T\begin{bmatrix}2\\0\end{bmatrix}=\begin{bmatrix}2+1\\0-1\end{bmatrix}=\begin{bmatrix}3\\-1\end{bmatrix}\neq\begin{bmatrix}2\\-1\end{bmatrix}+\begin{bmatrix}2\\-1\end{bmatrix}.$$

11. By Theorem 3.31, the standard matrix of T is the matrix A whose i^{th} column is $T(\mathbf{e}_i)$. Since

$$T(\mathbf{e}_1)=T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1+0\\1-0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix},\ T(\mathbf{e}_2)=T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0+1\\0-1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix},$$

the matrix of T is

$$A=\begin{bmatrix}1&1\\1&-1\end{bmatrix}.$$

12. By Theorem 3.31, the standard matrix of T is the matrix A whose i^{th} column is $T(\mathbf{e}_i)$. Since

$$T(\mathbf{e}_1)=T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-0\\1+2\cdot 0\\3\cdot 1-4\cdot 0\end{bmatrix}=\begin{bmatrix}0\\1\\3\end{bmatrix},\ T(\mathbf{e}_2)=T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\0+2\cdot 1\\3\cdot 0-4\cdot 1\end{bmatrix}=\begin{bmatrix}-1\\2\\-4\end{bmatrix},$$

the matrix of T is

$$A=\begin{bmatrix}0&-1\\1&2\\3&-4\end{bmatrix}.$$

13. By Theorem 3.31, the standard matrix of T is the matrix A whose i^{th} column is $T(\mathbf{e}_i)$. Since

$$T(\mathbf{e}_1)=T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1-0+0\\2\cdot 1+0-3\cdot 0\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},$$
$$T(\mathbf{e}_2)=T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0-1+0\\2\cdot 0+1-3\cdot 0\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix},$$
$$T(\mathbf{e}_3)=T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0-0+1\\2\cdot 0+0-3\cdot 1\end{bmatrix}=\begin{bmatrix}1\\-3\end{bmatrix},$$

the matrix of T is

$$A=\begin{bmatrix}1&-1&1\\2&1&-3\end{bmatrix}.$$

14. By Theorem 3.31, the standard matrix of T is the matrix A whose i^{th} column is $T(\mathbf{e}_i)$. Since

$$T(\mathbf{e}_1)=T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1+0\\0+0\\1+0\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix},$$
$$T(\mathbf{e}_2)=T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0+0\\1+0\\0+1\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix},$$
$$T(\mathbf{e}_3)=T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0+1\\0+1\\0+0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix},$$

the matrix of T is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

15. Reflecting a vector in the x-axis means negating the x-coordinate. So using the method of Example 3.56,

$$F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} = x\begin{bmatrix} -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and thus F is a matrix transformation with matrix

$$F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows that F is a linear transformation.

16. Example 3.58 shows that a rotation of 45° about the origin sends

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ to } \begin{bmatrix} \cos 45^\circ \\ \sin 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ to } \begin{bmatrix} \cos 135^\circ \\ \sin 135^\circ \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix},$$

so that R is a matrix transformation with matrix

$$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

It follows that R is a linear transformation.

17. Since D stretches by a factor of 2 in the x-direction and a factor of 3 in the y-direction, we see that

$$D\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus D is a matrix transformation with matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

It follows that D is a linear transformation.

18. Since $P = (x, y)$ projects onto the line whose direction vector is $\langle 1, 1\rangle$, it follows that the projection of P onto that line is

$$\operatorname{proj}_{\langle 1,1\rangle} \langle x, y\rangle = \left(\frac{\langle x, y\rangle \cdot \langle 1, 1\rangle}{\langle 1, 1\rangle \cdot \langle 1, 1\rangle}\right)\langle 1, 1\rangle = \left\langle \frac{x+y}{2}, \frac{x+y}{2} \right\rangle.$$

Thus

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix} = x\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + y\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus T is a matrix transformation with the matrix above, and it follows that T is a linear transformation.

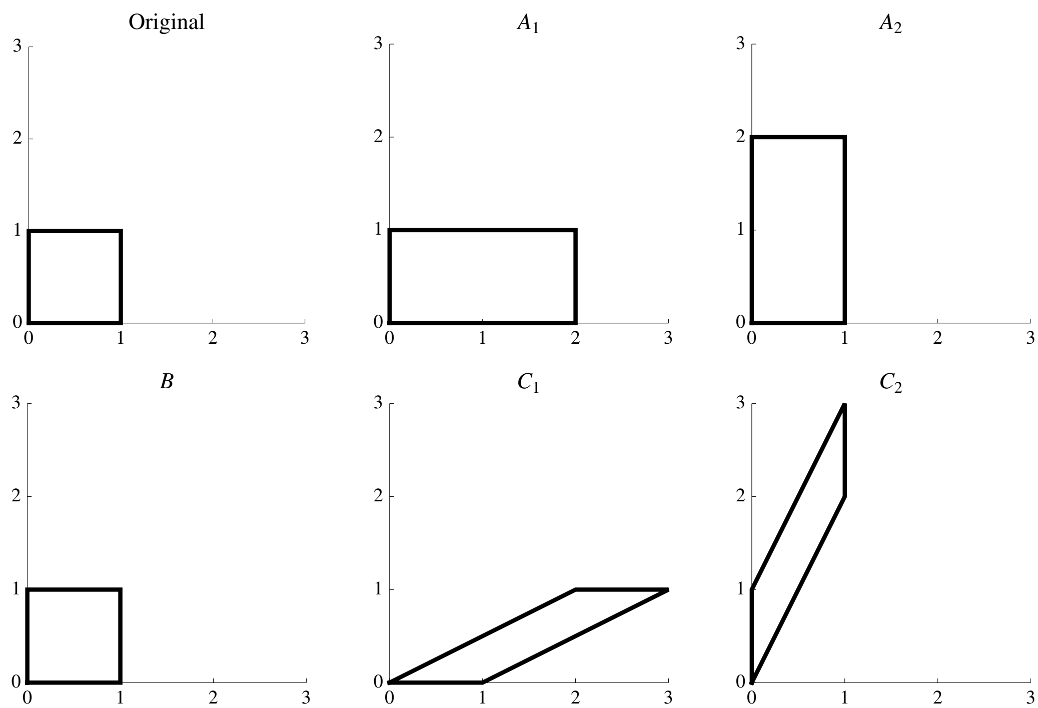
19. Let

$$A_1 = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Then

$$\begin{aligned}
A_1\begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}k & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}kx\\y\end{bmatrix}: \quad A_1 \text{ stretches vectors horizontally by a factor of } k\\
A_2\begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}1 & 0\\0 & k\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\ky\end{bmatrix}: \quad A_2 \text{ stretches vectors vertically by a factor of } k\\
B\begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}y\\x\end{bmatrix}: \quad B \text{ reflects vectors in the line } y = x\\
C_1\begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}1 & k\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+ky\\y\end{bmatrix}: \quad C_1 \text{ extends vectors horizontally by } ky\\
C_2\begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}1 & 0\\k & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y+kx\end{bmatrix}: \quad C_1 \text{ extends vectors vertically by } kx
\end{aligned}$$

(C_1 and C_2 are called *shears.*) The effect of each of these transformations, with $k = 2$, on the unit square is



Note that reflection across the line $y = x$ leaves the square unchanged, since it is symmetric about that line.

20. Using the formula from Example 3.58, we compute the matrix for a counterclockwise rotation of 120°:

$$R_{120^\circ}(\mathbf{e}_1) = \begin{bmatrix}\cos 120^\circ\\ \sin 120^\circ\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix} \text{ and } R_{120^\circ}(\mathbf{e}_2) = \begin{bmatrix}-\sin 120^\circ\\ \cos 120^\circ\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{3}}{2}\\ -\frac{1}{2}\end{bmatrix}.$$

So by Theorem 3.31, we have

$$A = \begin{bmatrix}R_{120^\circ}(\mathbf{e}_1) & R_{120^\circ}(\mathbf{e}_2)\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}.$$

21. A clockwise rotation of 30° is a rotation of -30°. Then using the formula from Example 3.58, we compute the matrix for that rotation:

$$R_{-30^\circ}(\mathbf{e}_1) = \begin{bmatrix} \cos(-30^\circ) \\ \sin(-30^\circ) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \text{ and } R_{120^\circ}(\mathbf{e}_2) = \begin{bmatrix} -\sin(-30^\circ) \\ \cos(-30^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}.$$

So by Theorem 3.31, we have

$$A = \begin{bmatrix} R_{-30^\circ}(\mathbf{e}_1) & R_{-30^\circ}(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

22. A direction vector of the line $y = 2x$ is $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so

$$P_\ell(\mathbf{e}_1) = \left(\frac{\mathbf{d}\cdot\mathbf{e}_1}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = \frac{1}{1+4}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix} \text{ and } P_\ell(\mathbf{e}_2) = \left(\frac{\mathbf{d}\cdot\mathbf{e}_2}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = \frac{2}{1+4}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \end{bmatrix}.$$

Thus by Theorem 3.31, we have

$$A = \begin{bmatrix} P_\ell(\mathbf{e}_1) & P_\ell(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}.$$

23. A direction vector of the line $y = -x$ is $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so

$$P_\ell(\mathbf{e}_1) = \left(\frac{\mathbf{d}\cdot\mathbf{e}_1}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = \frac{1}{1+1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \text{ and } P_\ell(\mathbf{e}_2) = \left(\frac{\mathbf{d}\cdot\mathbf{e}_2}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = \frac{-1}{1+2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Thus by Theorem 3.31, we have

$$A = \begin{bmatrix} P_\ell(\mathbf{e}_1) & P_\ell(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

24. Reflection in the line $y = x$ is given by

$$R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = x\begin{bmatrix} 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

so that the matrix of R is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

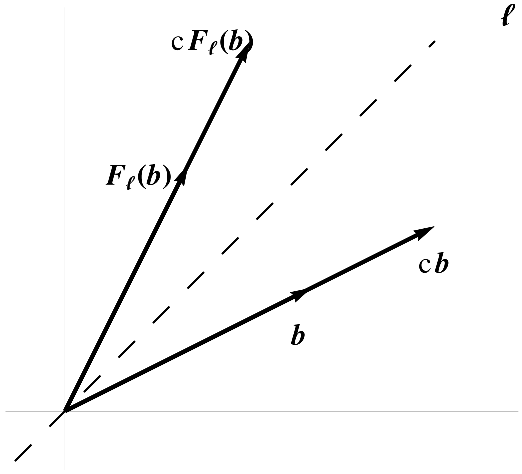
25. Reflection in the line $y = -x$ is given by

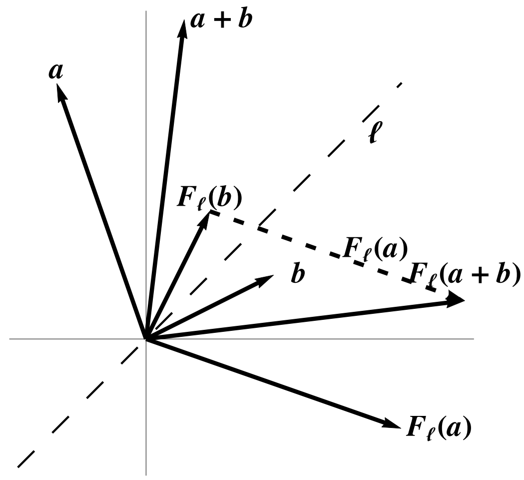
$$R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix} = x\begin{bmatrix} 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

so that the matrix of R is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

26. **(a)** Consider the following diagrams:





From the left-hand graph, we see that $F_\ell(c\mathbf{b}) = cF_\ell(\mathbf{b})$, and the right-hand graph shows that $F_\ell(\mathbf{a}+\mathbf{b}) = F_\ell(\mathbf{a}) + F_\ell(\mathbf{b})$.

(b) Let $P_\ell(x)$ be the projection of $\mathbf{x}$ on the line ℓ; this is the image of $\mathbf{x}$ under the linear transformation P_ℓ. Then from the figure in the text, since the diagonals of the parallelogram shown bisect each other, and the point furthest from the origin on ℓ is $\mathbf{x} + F_\ell(\mathbf{x})$, we have

$$\mathbf{x} + F_\ell(\mathbf{x}) = 2P_\ell(\mathbf{x}), \text{ so that } F_\ell(\mathbf{x}) = 2P_\ell(\mathbf{x}) - \mathbf{x}.$$

Now, Example 3.59 gives us the matrix of P_ℓ, where ℓ has direction vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, so we have

$$\begin{aligned}
F_\ell(\mathbf{x}) &= 2P_\ell(\mathbf{x}) - \mathbf{x} \\
&= \frac{2}{d_1^2+d_2^2}\begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix}\mathbf{x} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x} \\
&= \left(\frac{2}{d_1^2+d_2^2}\begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix} - \frac{1}{d_1^2+d_2^2}\begin{bmatrix} d_1^2+d_2^2 & 0 \\ 0 & d_1^2+d_2^2 \end{bmatrix}\right)\mathbf{x} \\
&= \frac{1}{d_1^2+d_2^2}\begin{bmatrix} d_1^2-d_2^2 & 2d_1d_2 \\ 2d_1d_2 & -d_1^2+d_2^2 \end{bmatrix}\mathbf{x},
\end{aligned}$$

which shows that the matrix of F_ℓ is as claimed in the exercise statement.

(c) If θ is the angle that ℓ forms with the positive x-axis, we may take

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

as the direction vector. Substituting those values into the formula for F_ℓ from part (b) gives

$$\begin{aligned}
F_\ell &= \frac{1}{d_1^2+d_2^2}\begin{bmatrix} d_1^2-d_2^2 & 2d_1d_2 \\ 2d_1d_2 & -d_1^2+d_2^2 \end{bmatrix} \\
&= \frac{1}{\cos^2\theta+\sin^2\theta}\begin{bmatrix} \cos^2\theta-\sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -\cos^2\theta+\sin^2\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta-\sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -\cos^2\theta+\sin^2\theta \end{bmatrix},
\end{aligned}$$

since $\cos^2\theta + \sin^2\theta = 1$. Now, $\cos^2\theta - \sin^2\theta = \cos 2\theta$, and $2\cos\theta\sin\theta = \sin 2\theta$, so the matrix above simplifies to

$$F_\ell = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

27. By part (b) of the previous exercise, since the line $y = 2x$ has direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we have for the standard matrix of this reflection

$$F_\ell(\mathbf{x}) = \frac{1}{1+4}\begin{bmatrix} 1-4 & 2\cdot 2 \\ 2\cdot 2 & -1+4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

28. The direction vector of this line is $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, so that $\frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}}{1} = \sqrt{3}$, and thus $\theta = \frac{\pi}{3} = 60^\circ$. So by part (c) of the previous exercise, the standard matrix of this reflection is

$$F_\ell(\mathbf{x}) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos 120^\circ & \sin 120^\circ \\ \sin 120^\circ & -\cos 120^\circ \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

29. Since

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{bmatrix}, \quad S\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2y_1 + y_3 \\ 3y_2 - y_3 \\ y_1 - y_2 \\ y_1 + y_2 + y_3 \end{bmatrix},$$

we substitute the result of T into the equation for S:

$$\begin{aligned}(S\circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= S\begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 + (3x_1 + 4x_2) \\ 3(2x_1 - x_2) - (3x_1 + 4x_2) \\ x_1 - (2x_1 - x_2) \\ x_1 + (2x_1 - x_2) + (3x_1 + 4x_2) \end{bmatrix} \\ &= \begin{bmatrix} 5x_1 + 4x_2 \\ 3x_1 - 7x_2 \\ -x_1 + x_2 \\ 6x_1 + 3x_2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 4 \\ 3 & -7 \\ -1 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.\end{aligned}$$

Thus the standard matrix of $S\circ T$ is

$$A = \begin{bmatrix} 5 & 4 \\ 3 & -7 \\ -1 & 1 \\ 6 & 3 \end{bmatrix}.$$

30. **(a)** By direct substitution, we have

$$(S\circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = S\left(T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = S\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2(x_1 - x_2) \\ -(x_1 + x_2) \end{bmatrix} = \begin{bmatrix} 2x_1 - 2x_2 \\ -x_1 - x_2 \end{bmatrix}.$$

Thus

$$[S\circ T] = \begin{bmatrix} 2 & -2 \\ -1 & -1 \end{bmatrix}.$$

(b) Using matrix multiplication, we first find the standard matrices for S and T, which are

$$[S] = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad [T] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$[S][T] = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & -1 \end{bmatrix}.$$

Then $[S \circ T] = [S][T]$.

31. **(a)** By direct substitution, we have

$$(S \circ T) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = S\left(T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = S \begin{bmatrix} x_1 + 2x_2 \\ -3x_1 + x_2 \end{bmatrix} = \begin{bmatrix} (x_1 + 2x_2) + 3(-3x_1 + x_2) \\ (x_1 + 2x_2) - (-3x_1 + x_2) \end{bmatrix} = \begin{bmatrix} -8x_1 + 5x_2 \\ 4x_1 + x_1 \end{bmatrix}.$$

Thus

$$[S \circ T] = \begin{bmatrix} -8 & 5 \\ 4 & 1 \end{bmatrix}.$$

(b) Using matrix multiplication, we first find the standard matrices for S and T, which are

$$[S] = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}, \quad [T] = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}.$$

Then

$$[S][T] = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 5 \\ 4 & 1 \end{bmatrix}.$$

Then $[S \circ T] = [S][T]$.

32. **(a)** By direct substitution, we have

$$(S \circ T) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = S\left(T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = S \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} x_2 + 3(-x_1) \\ 2x_2 - x_1 \\ x_2 - (-x_1) \end{bmatrix} = \begin{bmatrix} -3x_1 + x_2 \\ -x_1 + 2x_2 \\ x_1 + x_2 \end{bmatrix}.$$

Thus

$$[S \circ T] = \begin{bmatrix} -3 & 1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}.$$

(b) Using matrix multiplication, we first find the standard matrices for S and T, which are

$$[S] = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}, \quad [T] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$[S][T] = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Then $[S \circ T] = [S][T]$.

33. **(a)** By direct substitution, we have

$$(S \circ T) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = S\left(T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = S \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 - x_2 + x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 4(x_1 + x_2 - x_3) - 2(2x_1 - x_2 + x_3) \\ -(x_1 + x_2 - x_3) + (2x_1 - x_2 + x_3) \end{bmatrix} = \begin{bmatrix} 6x_2 - 6x_3 \\ x_1 - 2x_2 + 2x_3 \end{bmatrix}$$

Thus
$$[S\circ T]=\begin{bmatrix}0&6&-6\\1&-2&2\end{bmatrix}.$$

(b) Using matrix multiplication, we first find the standard matrices for S and T, which are
$$[S]=\begin{bmatrix}4&-2\\-1&1\end{bmatrix},\quad [T]=\begin{bmatrix}1&1&-1\\2&-1&1\end{bmatrix}.$$

Then
$$[S][T]=\begin{bmatrix}4&-2\\-1&1\end{bmatrix}\begin{bmatrix}1&1&-1\\2&-1&1\end{bmatrix}=\begin{bmatrix}0&6&-6\\1&-2&2\end{bmatrix}.$$

Then $[S\circ T]=[S][T]$.

34. **(a)** By direct substitution, we have
$$(S\circ T)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=S\left(T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=S\begin{bmatrix}x_1+2x_2\\2x_2-x_3\end{bmatrix}=\begin{bmatrix}(x_1+2x_2)-(2x_2-x_3)\\(x_1+2x_2)+(2x_2-x_3)\\-(x_1+2x_2)+(2x_2-x_3)\end{bmatrix}=\begin{bmatrix}x_1+x_3\\x_1+4x_2-x_3\\-x_1-x_3\end{bmatrix}$$

Thus
$$[S\circ T]=\begin{bmatrix}1&0&1\\1&4&-1\\-1&0&-1\end{bmatrix}.$$

(b) Using matrix multiplication, we first find the standard matrices for S and T, which are
$$[S]=\begin{bmatrix}1&-1\\1&1\\-1&1\end{bmatrix},\quad [T]=\begin{bmatrix}1&2&0\\0&2&-1\end{bmatrix}.$$

Then
$$[S][T]=\begin{bmatrix}1&-1\\1&1\\-1&1\end{bmatrix}\begin{bmatrix}1&2&0\\0&2&-1\end{bmatrix}=\begin{bmatrix}1&0&1\\1&4&-1\\-1&0&-1\end{bmatrix}.$$

Then $[S\circ T]=[S][T]$.

35. **(a)** By direct substitution, we have
$$(S\circ T)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=S\left(T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=S\begin{bmatrix}x_1+x_2\\x_2+x_3\\x_1+x_3\end{bmatrix}=\begin{bmatrix}(x_1+x_2)-(x_2+x_3)\\(x_2+x_3)-(x_1+x_3)\\-(x_1+x_2)+(x_1+x_3)\end{bmatrix}=\begin{bmatrix}x_1-x_3\\-x_1+x_2\\-x_2+x_3\end{bmatrix}.$$

Thus
$$[S\circ T]=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}.$$

(b) Using matrix multiplication, we first find the standard matrices for S and T, which are
$$[S]=\begin{bmatrix}1&-1&0\\0&1&-1\\-1&0&1\end{bmatrix},\quad [T]=\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}.$$

Then
$$[S][T]=\begin{bmatrix}1&-1&0\\0&1&-1\\-1&0&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}.$$

Then $[S\circ T]=[S][T]$.

36. A counterclockwise rotation T through 60° is given by (see Example 3.58)

$$[T] = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix},$$

and reflection S in $y = x$ is reflection in a line with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which has the matrix (see Exercise 26(b))

$$[S] = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus by Theorem 3.32, the composite transformation is given by

$$[S \circ T] = [S][T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

37. Reflection T in the y-axis is taking the negative of the x-coordinate, so it has matrix

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

while clockwise rotation S through 30° has matrix (see Example 3.58)

$$[S] = \begin{bmatrix} \cos(-30^\circ) & -\sin(-30^\circ) \\ \sin(-30^\circ) & \cos(-30^\circ) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Thus by Theorem 3.32, the composite transformation is given by

$$[S \circ T] = [S][T] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

38. Clockwise rotation T through 45° has matrix (see Example 3.58)

$$[T] = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

while projection S onto the y-axis has matrix (see Example 3.59, or note that this transformation simply sets the x-coordinate to zero)

$$[S] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus by Theorem 3.32, the composite transformation is given by

$$[T \circ S \circ T] = [T][S][T] = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

39. Reflection T in the line $y = x$, with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, is given by (see Exercise 26(b))

$$[T] = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Counterclockwise rotation S through 30° has matrix (see Example 3.58)

$$[S] = \begin{bmatrix} \cos 30^\circ & \sin 30^\circ \\ -\sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Finally, reflection R in the line $y = -x$, with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, is given by (see Exercise 26(b))

$$[R] = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Then by Theorem 3.32, the composite transformation is given by

$$[R \circ S \circ T] = [R][S][T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

40. Using the formula for rotation through an angle θ, from Example 3.58, we have

$$\begin{aligned}
[R_\alpha \circ R_\beta] = [R_\alpha][R_\beta] &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \\
&= R_{\alpha+\beta}.
\end{aligned}$$

41. Assume line ℓ has direction vector $\mathbf{l} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ and line m has direction vector $\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$. Let $\theta = \beta - \alpha$ be the angle from ℓ to m (note that if $\alpha > \beta$, then θ is negative). Then by Exercise 26(c),

$$\begin{aligned}
F_m \circ F_\ell &= \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix} \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\beta\cos 2\alpha + \sin 2\beta\sin 2\alpha & \cos 2\beta\sin 2\alpha - \sin 2\beta\cos 2\alpha \\ \sin 2\beta\cos 2\alpha - \cos 2\beta\sin 2\alpha & \cos 2\beta\cos 2\alpha + \sin 2\beta\sin 2\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos(2(\alpha-\beta)) & \sin(2(\alpha-\beta)) \\ -\sin(2(\alpha-\beta)) & \cos(2(\alpha-\beta)) \end{bmatrix} \\
&= \begin{bmatrix} \cos(2(\beta-\alpha)) & -\sin(2(\beta-\alpha)) \\ \sin(2(\beta-\alpha)) & \cos(2(\beta-\alpha)) \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \\
&= R_{2\theta}.
\end{aligned}$$

42. **(a)** Let P be the projection onto a line with direction vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. Then the standard matrix of P is

$$P = \frac{1}{d_1^2 + d_2^2}\begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix},$$

and thus

$$\begin{aligned}
P \circ P = P^2 &= \frac{1}{d_1^2 + d_2^2}\begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix} \frac{1}{d_1^2 + d_2^2}\begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix} \\
&= \frac{1}{(d_1^2 + d_2^2)^2}\begin{bmatrix} d_1^4 + d_1^2d_2^2 & d_1^3d_2 + d_1d_2^3 \\ d_1^3d_2 + d_1d_2^3 & d_1^2d_2^2 + d_2^4 \end{bmatrix} \\
&= \frac{1}{(d_1^2 + d_2^2)^2}\begin{bmatrix} d_1^2(d_1^2 + d_2^2) & d_1d_2(d_1^2 + d_2^2) \\ d_1d_2(d_1^2 + d_2^2) & d_2^2(d_1^2 + d_2^2) \end{bmatrix} \\
&= \frac{1}{d_1^2 + d_2^2}\begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix} \\
&= P.
\end{aligned}$$

Note that we have proven this fact before, in a different form. Since $P(\mathbf{v}) = \operatorname{proj}_{\mathbf{d}} \mathbf{v}$, then

$$(P \circ P)(\mathbf{v}) = \operatorname{proj}_{\mathbf{d}}(\operatorname{proj}_{\mathbf{d}} \mathbf{v}) = \operatorname{proj}_{\mathbf{d}} \mathbf{v} = P(\mathbf{v})$$

by Exercise 70(a) in Section 1.2.

(b) Let P be any projection matrix. Using the form of P from the previous exercise, we first prove that $P \neq I$. Suppose it were. Then $d_1 d_2 = 0$, so that either $d_1 = 0$ or $d_2 = 0$. But then either d_1^2 or d_2^2 would be zero, so that P could not be the identity matrix. So P is a non-identity matrix such that $P^2 = P$. Then $P^2 - P = P(P - I) = O$. Since $P \neq I$, we see that $P - I \neq O$, so there is some k such that $(P - I)\mathbf{e}_k \neq \mathbf{0}$ (otherwise, every row of $P - I$ would be orthogonal to every standard basis vector, so would be zero, so that $P - I$ would be the zero matrix). Let $\mathbf{x} = (P - I)\mathbf{e}_k$. But then

$$P\mathbf{x} = P(P - I)\mathbf{e}_k = O\mathbf{e}_k = \mathbf{0},$$

so that $\mathbf{x} \in \operatorname{null}(P)$. Since the null space of P contains a nonzero vector, P is not invertible by Theorem 3.27.

43. Let ℓ, m, and n be lines through the origin with angles from the x-axis of θ, α, and β respectively. Then

$$\begin{aligned}
F_n \circ F_m \circ F_\ell &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\theta \cos 2\alpha + \sin 2\theta \sin 2\alpha & \cos 2\theta \sin 2\alpha - \sin 2\theta \cos 2\alpha \\ \sin 2\theta \cos 2\alpha - \cos 2\theta \sin 2\alpha & \sin 2\theta \sin 2\alpha + \cos 2\theta \cos 2\alpha \end{bmatrix} \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos 2(\theta - \alpha) & -\sin 2(\theta - \alpha) \\ \sin 2(\theta - \alpha) & \cos 2(\theta - \alpha) \end{bmatrix} \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos 2(\theta - \alpha) \cos 2\beta - \sin 2(\theta - \alpha) \sin 2\beta & \cos 2(\theta - \alpha) \sin 2\beta + \sin 2(\theta - \alpha) \cos 2\beta \\ \sin 2(\theta - \alpha) \cos 2\beta + \cos 2(\theta - \alpha) \sin 2\beta & \sin 2(\theta - \alpha) \sin 2\beta - \cos 2(\theta - \alpha) \cos 2\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos 2(\theta - \alpha + \beta) & \sin 2(\theta - \alpha + \beta) \\ \sin 2(\theta - \alpha + \beta) & -\cos 2(\theta - \alpha + \beta) \end{bmatrix}.
\end{aligned}$$

This is a reflection through a line making an angle of $\theta - \alpha + \beta$ with the x-axis.

44. If ℓ is a line with direction vector $\mathbf{d}$ through the point P, then it has equation $\mathbf{x} = \mathbf{p} + t\mathbf{d}$. Since T is linear, we have

$$T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{d}) = T(\mathbf{p}) + tT(\mathbf{d}),$$

so that the equation of the resulting graph has equation $T(\mathbf{x}) = T(\mathbf{p}) + tT(\mathbf{d})$. If $T(\mathbf{d}) = \mathbf{0}$, then the image of ℓ has equation $T(\mathbf{x}) = T(\mathbf{p})$ so that it is the single point $T(\mathbf{p})$. Otherwise, the image of ℓ is the line ℓ' with direction vector $T(\mathbf{d})$ passing through $T(\mathbf{p})$.

45. Suppose that we start with two parallel lines

$$\ell : \ \mathbf{x} = \mathbf{p} + t\mathbf{d}, \qquad m : \ \mathbf{x} = \mathbf{q} + s\mathbf{d}.$$

Then the images of these lines have equations

$$\ell' : \ T(\mathbf{p}) + tT(\mathbf{d}), \qquad m' : \ T(\mathbf{q}) + sT(\mathbf{d}).$$

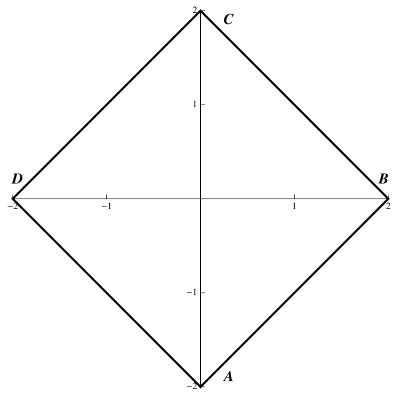
Then

- If $T(\mathbf{d}) \neq \mathbf{0}$ and $T(\mathbf{p}) \neq T(\mathbf{q})$, then ℓ' and m' are distinct parallel lines.
- If $T(\mathbf{d}) \neq \mathbf{0}$ and $T(\mathbf{p}) = T(\mathbf{q})$, then ℓ' and m' are the same line.
- If $T(\mathbf{d}) = \mathbf{0}$ and $T(\mathbf{p}) \neq T(\mathbf{q})$, then ℓ' and m' are distinct points $T(\mathbf{p})$ and $T(\mathbf{q})$.
- If $T(\mathbf{d}) = \mathbf{0}$ and $T(\mathbf{p}) \neq T(\mathbf{q})$, then ℓ' and m' are the single point $T(\mathbf{p})$.

46. Using the linear transformation T from Exercise 3, we have

$$T\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}0\\-2\end{bmatrix},\quad T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix},\quad T\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix},\quad T\begin{bmatrix}-1\\-1\end{bmatrix}=\begin{bmatrix}-2\\0\end{bmatrix}.$$

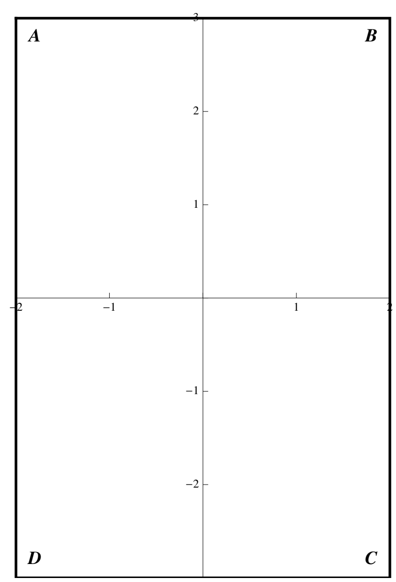
The image is the square below:



47. Using the linear transformation D from Exercise 17, we have

$$D\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}-2\\3\end{bmatrix},\quad D\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix},\quad D\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}2\\-3\end{bmatrix},\quad D\begin{bmatrix}-1\\-1\end{bmatrix}=\begin{bmatrix}-2\\-3\end{bmatrix}.$$

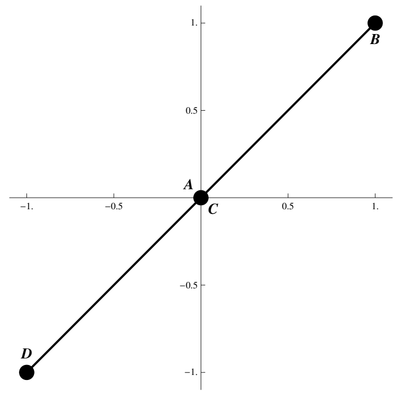
The image is the rectangle below:



48. Using the linear transformation P from Exercise 18, we have

$$P\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad P\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix},\quad P\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad P\begin{bmatrix}-1\\-1\end{bmatrix}=\begin{bmatrix}-1\\-1\end{bmatrix}.$$

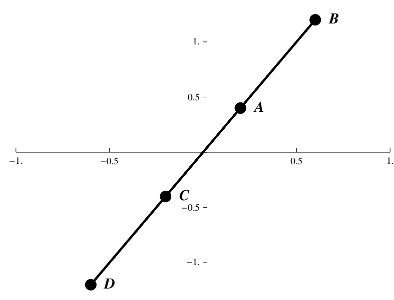
The image is the line below:



49. Using the projection P from Exercise 22, we have

$$P\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}\frac{1}{5}\\\frac{2}{5}\end{bmatrix},\quad P\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}\frac{3}{5}\\\frac{6}{5}\end{bmatrix},\quad P\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-\frac{1}{5}\\-\frac{2}{5}\end{bmatrix},\quad P\begin{bmatrix}-1\\-1\end{bmatrix}=\begin{bmatrix}-\frac{3}{5}\\-\frac{6}{5}\end{bmatrix}.$$

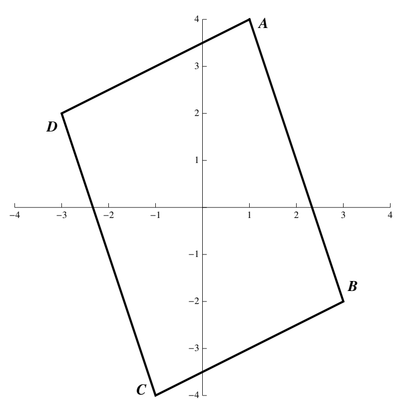
The image is the line below:



50. Using the transformation T from Exercise 31, with matrix $\begin{bmatrix}1&2\\-3&1\end{bmatrix}$, we have

$$T\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix},\quad T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3\\-2\end{bmatrix},\quad T\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\-4\end{bmatrix},\quad T\begin{bmatrix}-1\\-1\end{bmatrix}=\begin{bmatrix}-3\\2\end{bmatrix}.$$

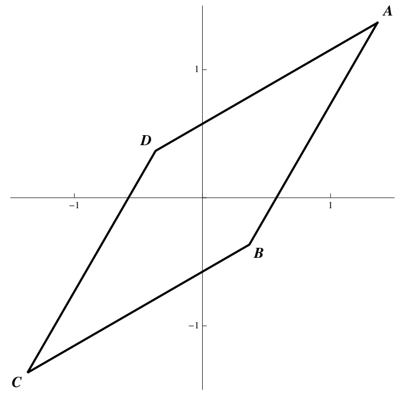
The image is the parallelogram below:



51. Using the transformation resulting from Exercise 37, which we call Q, we have

$$Q\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}+1}{2}\\\frac{\sqrt{3}+1}{2}\end{bmatrix},\quad Q\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}\frac{-\sqrt{3}+1}{2}\\\frac{\sqrt{3}-1}{2}\end{bmatrix},\quad Q\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}\frac{-\sqrt{3}-1}{2}\\\frac{-\sqrt{3}-1}{2}\end{bmatrix},\quad Q\begin{bmatrix}-1\\-1\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}-1}{2}\\\frac{-\sqrt{3}+1}{2}\end{bmatrix}.$$

The image is the parallelogram below:



52. If ℓ has direction vector $\mathbf{d}$, then using standard properties of the dot product and of scalar multiplication, we get

$$P_\ell(c\mathbf{v}) = \operatorname{proj}_{\mathbf{d}}(c\mathbf{v}) = \left(\frac{\mathbf{d}\cdot c\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = \left(\frac{c(\mathbf{d}\cdot\mathbf{v})}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = c\left(\left(\frac{\mathbf{d}\cdot\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d}\right) = cP_\ell(\mathbf{v}).$$

53. First suppose that T is linear. Then applying property 1 first, followed by property 2, we have

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = T(c_1\mathbf{v}_1) + T(c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2).$$

For the reverse direction, assume that $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$. To prove property 1, set $c_1 = c_2 = 1$; this gives

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2).$$

For property 2, set $c_1 = c$, $\mathbf{v}_1 = \mathbf{v}$, and $c_2 = 0$; this gives

$$T(c\mathbf{v}) = T(c\mathbf{v} + 0\mathbf{v}_2) = cT(\mathbf{v}) + 0T(\mathbf{v}_2) = cT(\mathbf{v}).$$

Thus T is linear.

54. Let range(T) be the range of T and col($[T]$) be the column space of $[T]$. We show that range(T) = col($[T]$) by showing that they are both equal to span($\{T(\mathbf{e}_i)\}$).

To see that range(T) = span($\{T(\mathbf{e}_i)\}$), note that $\mathbf{x} \in \operatorname{range}(T)$ if and only if $\mathbf{x} = T(\mathbf{v})$ for some $\mathbf{v} = \sum c_i\mathbf{e}_i$, so that

$$\mathbf{x} = T(\mathbf{v}) = T\left(\sum c_i\mathbf{e}_i\right) = \sum c_iT(\mathbf{e}_i),$$

which is the case if and only if $\mathbf{x} \in \operatorname{span}(\{T(\mathbf{e}_i)\})$. Thus range($T$) = span($\{T(\mathbf{e}_i)\}$).

Next, we show that col($[T]$) = span($\{T(\mathbf{e}_i)\}$). By Theorem 2, $[T] = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n)\end{bmatrix}$. Thus if $\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n\end{bmatrix}$, then

$$T(\mathbf{v}) = v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \cdots + v_nT(\mathbf{e}_n).$$

Then

$$\mathbf{x} \in \operatorname{col}([T]) \iff \mathbf{x} = T(\mathbf{v}) = T\left(\sum v_iT(\mathbf{e}_i)\right) = \sum v_iT(\mathbf{e}_i),$$

so that col($[T]$) = span($\{T(\mathbf{e}_i)\}$).

55. The Fundamental Theorem of Invertible Matrices implies that any invertible 2×2 matrix is a product of elementary matrices. The matrices in Exercise 19 represent the three types of elementary 2×2 matrices, so T_A must be a composition of the three types of transformations represented in Exercise 19.

3.7 Applications

1. $\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \quad \mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.38 \\ 0.62 \end{bmatrix}.$

2. Since there are two steps involved, the proportion we seek will be $[P^2]_{21}$ — the probability of a state 1 population member being in state 2 after two steps:

$$P^2 = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}^2 = \begin{bmatrix} 0.4 & 0.36 \\ 0.6 & 0.64 \end{bmatrix}.$$

The proportion of the state 1 population in state 2 after two steps is 0.6.

3. Since there are two steps involved, the proportion we seek will be $[P^2]_{22}$ — the probability of a state 2 population member remaining in state 2 after two steps:

$$P^2 = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}^2 = \begin{bmatrix} 0.4 & 0.36 \\ 0.6 & 0.64 \end{bmatrix}.$$

The proportion of the state 2 population in state 2 after two steps is 0.64.

4. The steady-state vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has the property that $P\mathbf{x} = \mathbf{x}$, or $(I - P)\mathbf{x} = \mathbf{0}$. So we form the augmented matrix of that system and row-reduce it:

$$\begin{bmatrix} I - P \mid 0 \end{bmatrix} : \left[\begin{array}{cc|c} 1 - 0.5 & -0.3 & 0 \\ -0.5 & 1 - 0.7 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0.5 & -0.3 & 0 \\ 0.5 & 0.3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -0.6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $x_2 = t$ is free and $x_1 = \frac{3}{5}t$. Since we also want $\mathbf{x}$ to be a probability vector, we also have $x_1 + x_2 = \frac{3}{5}t + t = \frac{8}{5}t = 1$, so that $t = \frac{5}{8}$. Thus the steady-state vector is

$$\mathbf{x} = \begin{bmatrix} \frac{3}{8} \\ \frac{5}{8} \end{bmatrix}.$$

5. $\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 120 \\ 180 \\ 90 \end{bmatrix} = \begin{bmatrix} 150 \\ 120 \\ 120 \end{bmatrix}, \quad \mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 150 \\ 120 \\ 120 \end{bmatrix} = \begin{bmatrix} 155 \\ 120 \\ 115 \end{bmatrix}.$

6. Since there are two steps involved, the proportion we seek will be $[P^2]_{11}$ — the probability of a state 1 population member being in state 1 after two steps:

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix}^2 = \begin{bmatrix} \frac{5}{12} & \frac{7}{18} & \frac{7}{18} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{9} \\ \frac{1}{4} & \frac{5}{18} & \frac{7}{18} \end{bmatrix}.$$

The proportion of the state 1 population in state 1 after two steps is $\frac{5}{12}$.

7. Since there are two steps involved, the proportion we seek will be $[P^2]_{32}$ — the probability of a state 2 population member being in state 3 after two steps:

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix}^2 = \begin{bmatrix} \frac{5}{12} & \frac{7}{18} & \frac{7}{18} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{9} \\ \frac{1}{4} & \frac{5}{18} & \frac{7}{18} \end{bmatrix}.$$

The proportion of the state 2 population in state 3 after two steps is $\frac{5}{18}$.

8. The steady-state vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ has the property that $P\mathbf{x} = \mathbf{x}$, or $(I - P)\mathbf{x} = \mathbf{0}$. So we form the augmented matrix of that system and row-reduce it:

$$\left[I - P \mid 0\right] : \left[\begin{array}{ccc|c} 1 - \frac{1}{2} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 - \frac{1}{3} & -\frac{2}{3} & 0 \\ -\frac{1}{2} & -\frac{1}{3} & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{2} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{2}{3} & -\frac{2}{3} & 0 \\ -\frac{1}{2} & -\frac{1}{3} & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $x_3 = t$ is free and $x_1 = \frac{4}{3}t$, $x_2 = t$. Since we want $\mathbf{x}$ to be a probability vector, we also have

$$x_1 + x_2 + x_3 = \frac{4}{3}t + t + t = \frac{10}{3}t = 1, \text{ so that } t = \frac{3}{10}.$$

Thus the steady-state vector is

$$\mathbf{x} = \begin{bmatrix} \frac{2}{5} \\ \frac{3}{10} \\ \frac{3}{10} \end{bmatrix}.$$

9. Let the first column and row represent dry days and the second column and row represent wet days.

(a) Given the data in the exercise, the transition matrix is

$$P = \begin{bmatrix} 0.750 & 0.338 \\ 0.250 & 0.662 \end{bmatrix}, \quad \text{with} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where x_1 and x_2 denote the probabilities of dry and wet days respectively.

(b) Since Monday is dry, we have $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since Wednesday is two days later, the probability vector for Wednesday is given by

$$\mathbf{x}_2 = P(P\mathbf{x}_0) = \begin{bmatrix} 0.750 & 0.338 \\ 0.250 & 0.662 \end{bmatrix} \begin{bmatrix} 0.750 & 0.338 \\ 0.250 & 0.662 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.647 \\ 0.353 \end{bmatrix}.$$

Thus the probability that Wednesday is wet is 0.353.

(c) We wish to find the steady state probability vector; that is, over the long run, the probability that a given day will be dry or wet. Thus we need to solve $P\mathbf{x} = \mathbf{x}$, or $(I - P)\mathbf{x} = \mathbf{0}$. So form the augmented matrix and row-reduce:

$$\left[I - P \mid 0\right] : \left[\begin{array}{cc|c} 1 - 0.750 & -0.338 & 0 \\ -0.750 & 1 - 0.662 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0.250 & -0.338 & 0 \\ -0.750 & 0.338 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1.352 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Let $x_2 = t$, so that $x_1 = 1.352t$; then we also want $x_1 + x_2 = 1$, so that

$$x_1 + x_2 = 2.352t = 1, \text{ so that } t = \frac{1}{2.352} \approx 0.425.$$

Thus the steady state probability vector is

$$\mathbf{x} = \begin{bmatrix} 1.352 \cdot 0.425 \\ 0.425 \end{bmatrix} \approx \begin{bmatrix} 0.575 \\ 0.425 \end{bmatrix}.$$

So in the long run, about 57.5% of the days will be dry and 42.5% will be wet.

10. Let the three rows (and columns) correspond to, in order, tall, medium, or short.

(a) From the given data, the transition matrix is

$$P = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}.$$

(b) Starting with a short person, we have $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. That person's grandchild is two generations later, so that

$$\mathbf{x}_2 = P(P\mathbf{x}_0) = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.24 \\ 0.48 \\ 0.28 \end{bmatrix}.$$

So the probability that a grandchild will be tall is 0.24.

(c) The current distribution of heights is $\mathbf{x}_0 = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix}$, and we want to find $\mathbf{x}_3$:

$$\mathbf{x}_2 = P^3\mathbf{x}_0 = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}^3 \begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.2457 \\ 0.5039 \\ 0.2504 \end{bmatrix}.$$

So roughly 24.6% of the population is tall, 50.4% medium, and 25.0% short.

(d) We want to solve $(I - P)\mathbf{x} = \mathbf{0}$, so form the augmented matrix and row-reduce:

$$\begin{bmatrix} I - P \mid 0 \end{bmatrix} : \left[\begin{array}{ccc|c} 1-0.6 & -0.1 & -0.2 & 0 \\ -0.2 & 1-0.7 & -0.4 & 0 \\ -0.2 & -0.2 & 1-0.4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0.4 & -0.1 & -0.2 & 0 \\ -0.2 & 0.3 & -0.4 & 0 \\ -0.2 & -0.2 & 0.6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $x_3 = t$ is free, and $x_1 = t$, $x_2 = 2t$. Since

$$x_1 + x_2 + x_3 = t + 2t + t = 4t = 1, \text{ we get } t = \frac{1}{4},$$

so that the steady state vector is

$$\mathbf{x} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}.$$

So in the long run, half the population is medium and one quarter each are tall and short.

11. Let the rows (and columns) represent good, fair, and poor crops, in order.

(a) From the data given, the transition matrix is

$$P = \begin{bmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{bmatrix}.$$

(b) We want to find the probability that starting from a good crop in 1940 we end up with a good crop in each of the five succeeding years, so we want $(P^i)_{11}$ for $i = 1, 2, 3, 4, 5$. Computing P^i and looking at the upper left entries, we get

$$\begin{aligned} 1941: &\quad P_{11} &&= 0.0800 \\ 1942: &\quad (P^2)_{11} &&= 0.1062 \\ 1943: &\quad (P^3)_{11} &&\approx 0.1057 \\ 1944: &\quad (P^4)_{11} &&\approx 0.1057 \\ 1945: &\quad (P^5)_{11} &&\approx 0.1057. \end{aligned}$$

(c) To find the long-run proportion, we want to solve $P\mathbf{x} = \mathbf{x}$. So row-reduce $\left[I - P \mid 0\right]$:

$$\left[I - P \mid 0\right] = \left[\begin{array}{ccc|c} 1 - 0.08 & -0.09 & -0.11 & 0 \\ -0.07 & 1 - 0.11 & -0.05 & 0 \\ -0.85 & -0.80 & 1 - 0.84 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -0.126 & 0 \\ 0 & 1 & -0.066 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $x_3 = t$ is free, and $x_1 = 0.126t$, $x_2 = 0.066t$. Since $x_1 + x_2 + x_3 = 1$, we have

$$x_1 + x_2 + x_3 = 0.126t + 0.066t + t = 1.192t = 1, \text{ so that } t = \frac{1}{1.192} \approx 0.839.$$

So the steady state vector is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \approx \begin{bmatrix} 0.126 \cdot 0.839 \\ 0.066 \cdot 0.839 \\ 0.839 \end{bmatrix} \approx \begin{bmatrix} 0.106 \\ 0.055 \\ 0.839 \end{bmatrix}.$$

In the long run, about 10.6% of the crops will be good, 5.5% will be fair, and 83.9% will be poor.

12. (a) Looking at the connectivity of the graph, we get for the transition matrix

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{1}{2} & 0 \end{bmatrix}.$$

Thus for example from state 2, there is a $\frac{1}{3}$ probability of getting to state 3, since there are three paths out of state 2 one of which leads to state 3. Thus $P_{32} = \frac{1}{3}$.

(b) First find the steady-state probability distribution by solving $P\mathbf{x} = \mathbf{x}$:

$$\left[I - P \mid 0\right] = \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & -\frac{1}{4} & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{4} & -\frac{1}{3} & 0 \\ -\frac{1}{2} & -\frac{1}{3} & 1 & -\frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{2} & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So x_4 is free; setting $x_4 = 3t$ to avoid fractions gives $x_1 = 2t$, $x_2 = 3t$, $x_3 = 4t$, and $x_4 = 3t$. Since this is a probability vector, we also have

$$x_1 + x_2 + x_3 + x_3 = 2t + 3t + 4t + 3t = 12t = 1, \text{ so that } t = \frac{1}{12}.$$

So the long run probability vector is

$$\mathbf{x} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{4} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}.$$

Since we initially started with 60 total robots, the steady-state distribution will be

$$60\mathbf{x} = \begin{bmatrix} 10 \\ 15 \\ 20 \\ 15 \end{bmatrix}.$$

13. First suppose that $P = \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_n \end{bmatrix}$ is a stochastic matrix (so that the elements of each $\mathbf{p}_i$ sum to 1). Let $\mathbf{j}$ be a row vector consisting entirely of 1's. Then

$$\mathbf{j}P = \begin{bmatrix} \mathbf{j} \cdot \mathbf{p}_1 & \cdots & \mathbf{j} \cdot \mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \sum(\mathbf{p}_1)_i & \cdots & \sum(\mathbf{p}_n)_i \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} = \mathbf{j}.$$

Conversely, suppose that $\mathbf{j}P = \mathbf{j}$. Then $\mathbf{j} \cdot \mathbf{p}_i = 1$ for each i, so that $\sum_j (\mathbf{p}_i)_j = 1$ for all i, showing that the column sums of P are 1 so that P is stochastic.

14. Let $P = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ and $Q = \begin{bmatrix} w_1 & z_1 \\ w_2 & z_2 \end{bmatrix}$ be stochastic matrices. Then

$$PQ = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} w_1 & z_1 \\ w_2 & z_2 \end{bmatrix} = \begin{bmatrix} x_1w_1 + y_1w_2 & x_1z_1 + y_1z_2 \\ x_2w_1 + y_2w_2 & x_2z_1 + y_2z_2 \end{bmatrix}.$$

Now, both P and Q are stochastic, so that $x_1 + x_2 = y_1 + y_2 = z_1 + z_2 = w_1 + w_2 = 1$. Then summing each column of PQ gives

$$\begin{aligned} x_1w_1 + y_1w_2 + x_2w_1 + y_2w_2 &= (x_1 + x_2)w_1 + (y_1 + y_2)w_2 = w_1 + w_2 = 1 \\ x_1z_1 + y_1z_2 + x_2z_1 + y_2z_2 &= (x_1 + x_2)z_1 + (y_1 + y_2)z_2 = z_1 + z_2 = 1. \end{aligned}$$

Thus each column of PQ sums to 1 so that PQ is also stochastic.

15. The transition matrix from Exercise 9 is

$$P = \begin{bmatrix} 0.750 & 0.338 \\ 0.250 & 0.662 \end{bmatrix}.$$

To find the expected number of days until a wet day, we delete the second row and column of P (corresponding to wet days), giving $Q = \begin{bmatrix} 0.750 \end{bmatrix}$. Then the expected number of days until a wet day is

$$(I - Q)^{-1} = (1 - 0.750)^{-1} = 4.$$

We expect four days until the next wet day.

16. The transition matrix from Exercise 10 is

$$P = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}.$$

We delete the row and column of this matrix corresponding to "tall", which is the first row and column, giving

$$Q = \begin{bmatrix} 0.7 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}.$$

Then

$$(I - Q)^{-1} = \begin{bmatrix} 0.3 & -0.4 \\ -0.2 & 0.6 \end{bmatrix}^{-1} = \begin{bmatrix} 6.0 & 4.0 \\ 2.0 & 3.0 \end{bmatrix}.$$

The column of this matrix corresponding to "short" is the second column, since this was the third column in the original matrix. Summing the values in the second column shows that the expected number of generations before a short person has a tall descendant is 7.

17. The transition matrix from Exercise 11 is

$$P = \begin{bmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{bmatrix}.$$

Since we want to know how long we expect it to be before a good crop, we remove the first row and column, leaving

$$Q = \begin{bmatrix} 0.11 & 0.05 \\ 0.80 & 0.84 \end{bmatrix}.$$

Then

$$(I-Q)^{-1} = \begin{bmatrix} 0.89 & -0.05 \\ -0.80 & 0.16 \end{bmatrix}^{-1} = \begin{bmatrix} 1.562 & 0.488 \\ 7.812 & 8.691 \end{bmatrix}.$$

Since we are starting from a fair crop, we sum the entries in the first column (which was the "fair" column in the original matrix), giving $1.562+7.812$, or 9 or 10 years expected until the next good crop.

18. The transition matrix from Exercise 12 is

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{1}{2} & 0 \end{bmatrix}.$$

The expected number of moves from any junction other than 4 to reach junction 4 is found by removing the fourth row and column of P, giving

$$Q = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix}.$$

Then

$$(I-Q)^{-1} = \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{3} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{22}{13} & \frac{10}{13} & \frac{8}{13} \\ \frac{15}{13} & \frac{21}{13} & \frac{9}{13} \\ \frac{16}{13} & \frac{12}{13} & \frac{20}{13} \end{bmatrix},$$

so that a robot is expected to reach junction 4 starting from junction 1 in $\frac{22}{13}+\frac{15}{13}+\frac{16}{13} = \frac{53}{13} \approx 4.08$ moves, from junction 2 in $\frac{10}{13}+\frac{21}{13}+\frac{12}{13} = \frac{43}{13} \approx 3.31$ moves, and from junction 3 in $\frac{8}{13}+\frac{9}{13}+\frac{20}{13} = \frac{37}{13} \approx 2.85$ moves.

19. Since the sums of the columns of the matrix E are both 1, and all numbers are nonnegative, this matrix is an exchange matrix. To find a solution to $E\mathbf{x} = \mathbf{x}$, or equivalently $(E-I)\mathbf{x} = \mathbf{0}$, we row-reduce the augmented matrix of that system:

$$\begin{bmatrix} E-I & \big| & 0 \end{bmatrix} : \left[\begin{array}{cc|c} -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus the solutions are of the form $x_1 = \frac{1}{2}t$, $x_2 = t$. Set $t = 2$ to avoid fractions, giving a price vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Of course, other values of t lead to equally valid price vectors.

20. Since neither column sum is 1, this is not an exchange matrix.

21. While the first column sums to 1, the second does not, so this is not an exchange matrix.

22. Since the sums of the columns of the matrix E are both 1, and all numbers are nonnegative, this matrix is an exchange matrix. To find a solution to $E\mathbf{x} = \mathbf{x}$, or equivalently $(E - I)\mathbf{x} = \mathbf{0}$, we row-reduce the augmented matrix of that system:

$$\left[E - I \mid 0\right] : \left[\begin{array}{rr|r} -0.9 & 0.6 & 0 \\ 0.9 & -0.6 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{rr|r} -0.9 & 0.6 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{10}{9}R_1} \left[\begin{array}{rr|r} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus the solutions are of the form $x_1 = \frac{2}{3}t$, $x_2 = t$. Set $t = 3$ to avoid fractions, giving a price vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Of course, other values of t lead to equally valid price vectors.

23. Since the matrix contains a negative entry, it is not an exchange matrix.

24. Since the sums of the columns of the matrix E are both 1, and all numbers are nonnegative, this matrix is an exchange matrix. To find a solution to $E\mathbf{x} = \mathbf{x}$, or equivalently $(E - I)\mathbf{x} = \mathbf{0}$, we row-reduce the augmented matrix of that system:

$$\left[E - I \mid 0\right] : \left[\begin{array}{rrr|r} -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{3} & 0 \end{array}\right] \xrightarrow[R_3+R_1]{-R_2} \left[\begin{array}{rrr|r} -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{array}\right] \xrightarrow[R_3-R_2]{R_1-R_2}$$

$$\left[\begin{array}{rrr|r} -\frac{1}{2} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_1} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the solutions are of the form $x_1 = \frac{2}{3}t$, $x_2 = \frac{1}{3}t$, $x_3 = t$. Set $t = 3$ to avoid fractions, giving a price vector $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. Of course, other values of t lead to equally valid price vectors.

25. Since the sums of the columns of the matrix E are both 1, and all numbers are nonnegative, this matrix is an exchange matrix. To find a solution to $E\mathbf{x} = \mathbf{x}$, or equivalently $(E - I)\mathbf{x} = \mathbf{0}$, we row-reduce the augmented matrix of that system:

$$\left[E - I \mid 0\right] : \left[\begin{array}{rrr|r} -0.7 & 0 & 0.2 & 0 \\ 0.3 & -0.5 & 0.3 & 0 \\ 0.4 & 0.5 & -0.5 & 0 \end{array}\right] \xrightarrow{-\frac{10}{7}R_1} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{2}{7} & 0 \\ 0.3 & -0.5 & 0.3 & 0 \\ 0.4 & 0.5 & -0.5 & 0 \end{array}\right]$$

$$\xrightarrow[R_3-0.4R_1]{R_2-0.3R_1} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{2}{7} & 0 \\ 0 & -\frac{1}{2} & \frac{27}{70} & 0 \\ 0 & \frac{1}{2} & -\frac{27}{70} & 0 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{2}{7} & 0 \\ 0 & -\frac{1}{2} & \frac{27}{70} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{2}{7} & 0 \\ 0 & 1 & -\frac{27}{35} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the solutions are of the form $x_1 = \frac{2}{7}t$, $x_2 = \frac{27}{35}t$, $x_3 = t$. Set $t = 35$ to avoid fractions, giving a price vector $\begin{bmatrix} 10 \\ 27 \\ 35 \end{bmatrix}$. Of course, other values of t lead to equally valid price vectors.

26. Since the sums of the columns of the matrix E are both 1, and all numbers are nonnegative, this matrix is an exchange matrix. To find a solution to $E\mathbf{x} = \mathbf{x}$, or equivalently $(E - I)\mathbf{x} = \mathbf{0}$, we row-reduce the augmented matrix of that system:

$$\left[E-I \mid 0\right]: \left[\begin{array}{rrr|r} -0.50 & 0.70 & 0.35 & 0 \\ 0.25 & -0.70 & 0.25 & 0 \\ 0.25 & 0 & -0.6 & 0 \end{array}\right] \xrightarrow{-2R_1} \left[\begin{array}{rrr|r} 1 & -1.40 & -0.70 & 0 \\ 0.25 & -0.70 & 0.25 & 0 \\ 0.25 & 0 & -0.6 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-0.25R_1} \\ \xrightarrow{R_3-0.25R_1} \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -1.40 & -0.70 & 0 \\ 0 & -0.35 & 0.425 & 0 \\ 0 & 0.35 & -0.425 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-4R_2} \\ \\ \xrightarrow{R_3+R_2} \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -2.40 & 0 \\ 0 & -0.35 & 0.425 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3.5}R_2} \left[\begin{array}{rrr|r} 1 & 0 & -2.40 & 0 \\ 0 & 1 & -1.214 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So one possible solution is $\begin{bmatrix} 2.4 \\ 1.214 \\ 1 \end{bmatrix}$. Multiplying through by 70 will roughly clear fractions, giving $\approx \begin{bmatrix} 168 \\ 85 \\ 70 \end{bmatrix}$ if an integral solution is desired.

27. This matrix is a consumption matrix since all of its entries are nonnegative. By Corollary 3.36, it is productive since the sum of the entries in each column is strictly less than 1. (Note that Corollary 3.35 does not apply since the sum of the entries in the second row exceeds 1).

28. This matrix is a consumption matrix since all of its entries are nonnegative. By Corollary 3.35, it is productive since the sum of the entries in each row is strictly less than 1. (Note that Corollary 3.36 does not apply since the sum of the entries in the third column exceeds 1).

29. This matrix is a consumption matrix since all of its entries are nonnegative. Since the entries in the second row as well as those in the second column sum to a number greater than 1, neither Corollary 3.35 nor 3.36 applies. So to determine if C is productive, we must use the definition, which involves seeing if $I-C$ is invertible and if its inverse has all positive entries.

$$I-C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.35 & 0.25 & 0 \\ 0.15 & 0.55 & 0.35 \\ 0.45 & 0.30 & 0.60 \end{bmatrix} = \begin{bmatrix} 0.65 & -0.25 & 0 \\ -0.15 & 0.45 & -0.35 \\ -0.45 & -0.30 & 0.40 \end{bmatrix}.$$

Using technology,

$$(I-C)^{-1} \approx \begin{bmatrix} -13.33 & -17.78 & -15.56 \\ -38.67 & -46.22 & -40.44 \\ -44.00 & -54.67 & -45.33 \end{bmatrix}.$$

Since this is not a positive matrix, it follows that C is not productive.

30. This matrix is a consumption matrix since all of its entries are nonnegative. Since the entries in the first row as well as those in the each column sum to a number at least equal to 1, neither Corollary 3.35 nor 3.36 applies. So to determine if C is productive, we must use the definition, which involves seeing if $I-C$ is invertible and if its inverse has all positive entries.

$$I-C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.4 & 0.1 & 0.4 \\ 0.3 & 0.2 & 0.2 & 0.1 \\ 0 & 0.4 & 0.5 & 0.3 \\ 0.5 & 0 & 0.2 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.4 & -0.1 & -0.4 \\ -0.3 & 0.8 & -0.2 & -0.1 \\ 0 & -0.4 & 0.5 & -0.3 \\ -0.5 & 0 & -0.2 & 0.8 \end{bmatrix}.$$

Using technology reveals that $I-C$ is not invertible, so that C is not productive.

31. To find a solution $\mathbf{x}$ to $(I-C)\mathbf{x} = \mathbf{d}$, we invert $I-C$ and compute $\mathbf{x} = (I-C)^{-1}\mathbf{d}$. Here

$$I-C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then $\det(I-C) = \frac{1}{2}\cdot\frac{1}{2} - \left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right) = \frac{1}{8}$, so that

$$(I-C)^{-1} = \frac{1}{1/8}\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & 4 \end{bmatrix}.$$

Thus a feasible production vector is

$$\mathbf{x} = (I-C)^{-1}\mathbf{d} = \begin{bmatrix} 4 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}.$$

32. To find a solution $\mathbf{x}$ to $(I-C)\mathbf{x} = \mathbf{d}$, we invert $I-C$ and compute $\mathbf{x} = (I-C)^{-1}\mathbf{d}$. Here

$$I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.9 & -0.4 \\ -0.3 & 0.8 \end{bmatrix}.$$

Then $\det(I-C) = 0.9\cdot 0.8 - (-0.4)(-0.3) = 0.60$, so that

$$(I-C)^{-1} = \frac{1}{0.60}\begin{bmatrix} 0.8 & 0.4 \\ 0.3 & 0.9 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

Thus a feasible production vector is

$$\mathbf{x} = (I-C)^{-1}\mathbf{d} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ \frac{5}{2} \end{bmatrix}.$$

33. To find a solution $\mathbf{x}$ to $(I-C)\mathbf{x} = \mathbf{d}$, we invert $I-C$ and compute $\mathbf{x} = (I-C)^{-1}\mathbf{d}$. Here

$$I - C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.2 & 0.1 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.2 & -0.1 \\ 0 & 0.6 & -0.8 \\ 0 & 0 & 0.5 \end{bmatrix}.$$

Then row-reduce $\left[I - C \mid I\right]$ to compute $(I-C)^{-1}$:

$$\left[I - C \mid I\right] = \left[\begin{array}{ccc|ccc} 0.5 & -0.2 & -0.1 & 1 & 0 & 0 \\ 0 & 0.6 & -0.8 & 0 & 1 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & \frac{2}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & \frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right].$$

Thus a feasible production vector is

$$\mathbf{x} = (I-C)^{-1}\mathbf{d} = \begin{bmatrix} 2 & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 8 \end{bmatrix}.$$

34. To find a solution $\mathbf{x}$ to $(I-C)\mathbf{x} = \mathbf{d}$, we invert $I-C$ and compute $\mathbf{x} = (I-C)^{-1}\mathbf{d}$. Here

$$I - C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.4 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.9 & -0.4 & -0.1 \\ 0 & 0.8 & -0.2 \\ -0.3 & -0.2 & 0.7 \end{bmatrix}.$$

Compute $(I-C)^{-1}$ using technology to get

$$(I-C)^{-1} \approx \begin{bmatrix} 1.238 & 0.714 & 0.381 \\ 0.143 & 1.429 & 0.429 \\ 0.571 & 0.714 & 1.714 \end{bmatrix}.$$

Thus a feasible production vector is

$$\mathbf{x} = (I-C)^{-1}\mathbf{d} = \begin{bmatrix} 1.238 & 0.714 & 0.381 \\ 0.143 & 1.429 & 0.429 \\ 0.571 & 0.714 & 1.714 \end{bmatrix}\begin{bmatrix} 1.1 \\ 3.5 \\ 2.0 \end{bmatrix} = \begin{bmatrix} 4.624 \\ 6.014 \\ 6.557 \end{bmatrix}.$$

35. Since $\mathbf{x} \geq 0$ and $A \geq O$, then $A\mathbf{x}$ has all nonnegative entries as well, so that $A\mathbf{x} \geq O\mathbf{x} = \mathbf{0}$. But then

$$\mathbf{x} > A\mathbf{x} \geq O\mathbf{x} = \mathbf{0},$$

so that $\mathbf{x} > \mathbf{0}$.

36. **(a)** Since all entries of all matrices are positive, it is clear that $AC \geq O$ and $BD \geq O$. To see that $AC \geq BD$, let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, and $D = (d_{ij})$. Then each $a_{ij} \geq b_{ij}$ and each $c_{ij} \geq d_{ij}$. So

$$(AC)_{ij} = \sum_{k=1}^{n} a_{ik}c_{kj} \geq \sum_{k=1}^{n} b_{ik}c_{kj} \geq \sum_{k=1}^{n} b_{ik}d_{kj} = (BD)_{ij}.$$

So every entry of AC is greater than or equal to the corresponding entry of BD, and thus $AC \geq BD$.

(b) Let $\mathbf{x}^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$. Then for any i and j, we know that $a_{ij}x_j \geq b_{ij}x_j$ since $a_{ij} > b_{ij}$ and $x_j \geq 0$. Since $\mathbf{x} \neq \mathbf{0}$, we know that some component of $\mathbf{x}$, say x_k, is strictly positive. Then $a_{ik}x_k > b_{ik}x_k$ since $x_k > 0$. But then

$$(A\mathbf{x})_i = a_{i1}x_1 + \cdots + a_{ik}x_k + \cdots + a_{in}x_n > b_{i1}x_1 + \cdots + b_{ik}x_k + \cdots + b_{in}x_n = (B\mathbf{x})_i.$$

37. Multiplying, we get

$$\begin{aligned}
\mathbf{x}_1 = L\mathbf{x}_0 &= \begin{bmatrix} 2 & 5 \\ 0.6 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 45 \\ 6 \end{bmatrix} \\
\mathbf{x}_2 = L\mathbf{x}_1 &= \begin{bmatrix} 2 & 5 \\ 0.6 & 0 \end{bmatrix} \begin{bmatrix} 45 \\ 6 \end{bmatrix} = \begin{bmatrix} 120 \\ 22.5 \end{bmatrix} \\
\mathbf{x}_3 = L\mathbf{x}_2 &= \begin{bmatrix} 2 & 5 \\ 0.6 & 0 \end{bmatrix} \begin{bmatrix} 120 \\ 27 \end{bmatrix} = \begin{bmatrix} 375 \\ 72 \end{bmatrix}
\end{aligned}$$

38. Multiplying, we get

$$\begin{aligned}
\mathbf{x}_1 = L\mathbf{x}_0 &= \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 2 \end{bmatrix} \\
\mathbf{x}_2 = L\mathbf{x}_1 &= \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \\
\mathbf{x}_3 = L\mathbf{x}_2 &= \begin{bmatrix} 0 & 1 & 2 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1.2 \\ 1 \end{bmatrix}
\end{aligned}$$

39. Multiplying, we get

$$\begin{aligned}
\mathbf{x}_1 = L\mathbf{x}_0 &= \begin{bmatrix} 1 & 1 & 3 \\ 0.7 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 500 \\ 70 \\ 50 \end{bmatrix} \\
\mathbf{x}_2 = L\mathbf{x}_1 &= \begin{bmatrix} 1 & 1 & 3 \\ 0.7 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 500 \\ 70 \\ 50 \end{bmatrix} = \begin{bmatrix} 720 \\ 350 \\ 35 \end{bmatrix} \\
\mathbf{x}_3 = L\mathbf{x}_2 &= \begin{bmatrix} 1 & 1 & 3 \\ 0.7 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 720 \\ 350 \\ 35 \end{bmatrix} = \begin{bmatrix} 1175 \\ 504 \\ 175 \end{bmatrix}.
\end{aligned}$$

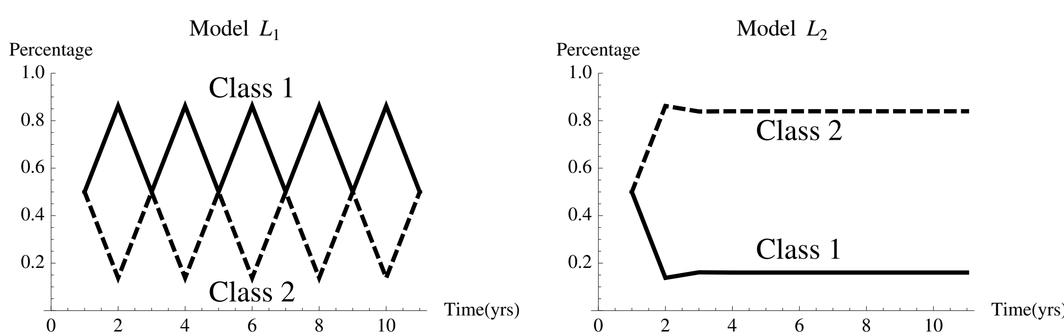
40. Multiplying, we get

$$\mathbf{x}_1 = L\mathbf{x}_0 = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 80 \\ 5 \\ 7 \\ 3 \end{bmatrix}$$
$$\mathbf{x}_2 = L\mathbf{x}_1 = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 5 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 34 \\ 40 \\ 3.5 \\ 2.1 \end{bmatrix}$$
$$\mathbf{x}_3 = L\mathbf{x}_2 = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 34 \\ 40 \\ 3.5 \\ 2.1 \end{bmatrix} = \begin{bmatrix} 57.5 \\ 17 \\ 28 \\ 1.05 \end{bmatrix}.$$

41. **(a)** The first ten generations of the species using each of the two models are as follows:

$$\begin{array}{ccccccc} & \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 \\ L_1 & \begin{bmatrix} 10 \\ 10 \end{bmatrix} & \begin{bmatrix} 50 \\ 8 \end{bmatrix} & \begin{bmatrix} 40 \\ 40 \end{bmatrix} & \begin{bmatrix} 200 \\ 32 \end{bmatrix} & \begin{bmatrix} 160 \\ 160 \end{bmatrix} & \begin{bmatrix} 800 \\ 128 \end{bmatrix} \\ L_2 & \begin{bmatrix} 10 \\ 10 \end{bmatrix} & \begin{bmatrix} 50 \\ 8 \end{bmatrix} & \begin{bmatrix} 208 \\ 40 \end{bmatrix} & \begin{bmatrix} 872 \\ 166.4 \end{bmatrix} & \begin{bmatrix} 3654.4 \\ 697.6 \end{bmatrix} & \begin{bmatrix} 15315.2 \\ 2923.52 \end{bmatrix} \end{array}$$
$$\begin{array}{cccccc} & \mathbf{x}_6 & \mathbf{x}_7 & \mathbf{x}_8 & \mathbf{x}_9 & \mathbf{x}_{10} \\ L_1 & \begin{bmatrix} 640 \\ 640 \end{bmatrix} & \begin{bmatrix} 3200 \\ 512 \end{bmatrix} & \begin{bmatrix} 2560 \\ 2560 \end{bmatrix} & \begin{bmatrix} 12800 \\ 2048 \end{bmatrix} & \begin{bmatrix} 10240 \\ 10240 \end{bmatrix} \\ L_2 & \begin{bmatrix} 64184.3 \\ 12252.2 \end{bmatrix} & \begin{bmatrix} 268989 \\ 51347.5 \end{bmatrix} & \begin{bmatrix} 1.127 \times 10^6 \\ 215192 \end{bmatrix} & \begin{bmatrix} 4.724 \times 10^6 \\ 901844 \end{bmatrix} & \begin{bmatrix} 1.980 \times 10^7 \\ 3.780 \times 10^6 \end{bmatrix} \end{array}.$$

(b) Graphs of the first ten generations using each model are:



The graphs suggest that the first model does not have a steady state, while the second Leslie matrix quickly approaches a steady state ratio of class 1 to class 2 population.

42. Start with $\mathbf{x}_0 = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$. Then

$$\mathbf{x}_1 = \begin{bmatrix} 0 & 0 & 20 \\ 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 400 \\ 2 \\ 10 \end{bmatrix}$$
$$\mathbf{x}_2 = \begin{bmatrix} 0 & 0 & 20 \\ 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 400 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 200 \\ 40 \\ 1 \end{bmatrix}$$
$$\mathbf{x}_3 = \begin{bmatrix} 0 & 0 & 20 \\ 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 200 \\ 40 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}.$$

Thus after three generations the population is back to where it started, so that the population is cyclic.

43. Start with $\mathbf{x}_0 = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$. Then

$$\mathbf{x}_1 = \begin{bmatrix} 0 & 0 & 20 \\ s & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 400 \\ 20s \\ 10 \end{bmatrix}$$
$$\mathbf{x}_2 = \begin{bmatrix} 0 & 0 & 20 \\ s & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 400 \\ 20s \\ 10 \end{bmatrix} = \begin{bmatrix} 200 \\ 400s \\ 10s \end{bmatrix}$$
$$\mathbf{x}_3 = \begin{bmatrix} 0 & 0 & 20 \\ s & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 200 \\ 400s \\ 10s \end{bmatrix} = \begin{bmatrix} 200s \\ 200s \\ 200s \end{bmatrix}.$$

Thus after three generations the population is still evenly distributed among the three classes, but the total population has been multiplied by $10s$. So

- If $s < 0.1$, then the overall population declines after three years;
- If $s = 0.1$, then the overall population is cyclic every three years;
- If $s > 0.1$, then the overall population increases after three years.

44. The table gives us seven separate age classes, each of duration two years; from the table, the Leslie matrix of the system is

$$L = \begin{bmatrix} 0 & 0.4 & 1.8 & 1.8 & 1.8 & 1.6 & 0.6 \\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \end{bmatrix}.$$

Since in 1990,

$$\mathbf{x}_0 = \begin{bmatrix} 10 \\ 2 \\ 8 \\ 5 \\ 12 \\ 0 \\ 1 \end{bmatrix}, \text{ so that } \mathbf{x}_1 = L\mathbf{x}_0 = \begin{bmatrix} 46.4 \\ 3 \\ 1.4 \\ 7.2 \\ 4.5 \\ 10.8 \\ 0 \end{bmatrix}, \text{ and } \mathbf{x}_2 = L\mathbf{x}_1 = \begin{bmatrix} 42.06 \\ 13.92 \\ 2.1 \\ 1.26 \\ 6.48 \\ 4.05 \\ 6.48 \end{bmatrix}.$$

Then $\mathbf{x}_1$ and $\mathbf{x}_2$ represent the population distribution in 1991 and 1992. To project the population for the years 2000 and 2010, note that $\mathbf{x}_n = L^n\mathbf{x}_0$. Thus

$$\mathbf{x}_{10} = L^{10}\mathbf{x}_0 \approx \begin{bmatrix} 69.12 \\ 18.61 \\ 12.24 \\ 10.59 \\ 8.29 \\ 6.51 \\ 3.57 \end{bmatrix}, \quad \mathbf{x}_{20} = L^{20}\mathbf{x}_0 \approx \begin{bmatrix} 167.80 \\ 46.08 \\ 29.54 \\ 24.35 \\ 20.06 \\ 16.48 \\ 9.05 \end{bmatrix}.$$

45. The adjacency matrix contains a 1 in row i, column j if there is an edge between v_i and v_j:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

46. The adjacency matrix contains a 1 in row i, column j if there is an edge between v_i and v_j:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

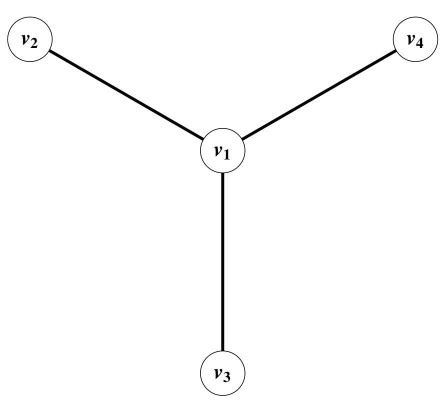
47. The adjacency matrix contains a 1 in row i, column j if there is an edge between v_i and v_j:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

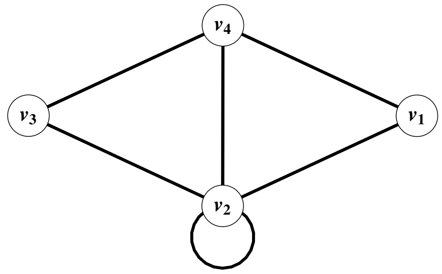
48. The adjacency matrix contains a 1 in row i, column j if there is an edge between v_i and v_j:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

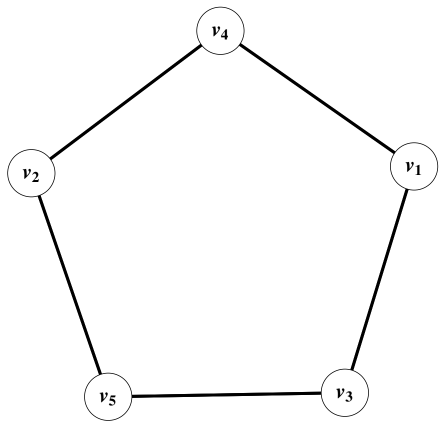
49.



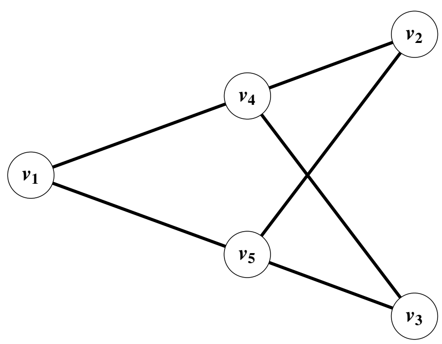
50.



51.



52.



53. The adjacency matrix contains a 1 in row i, column j if there is an edge from v_i to v_j:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

54. The adjacency matrix contains a 1 in row i, column j if there is an edge from v_i to v_j:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

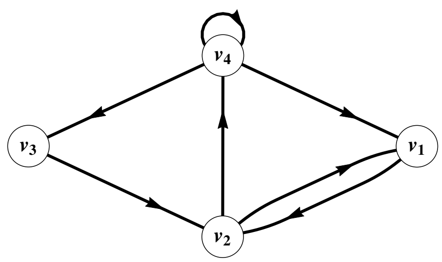
55. The adjacency matrix contains a 1 in row i, column j if there is an edge from v_i to v_j:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

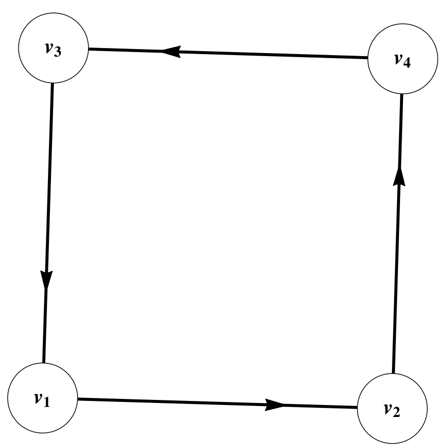
56. The adjacency matrix contains a 1 in row i, column j if there is an edge from v_i to v_j:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

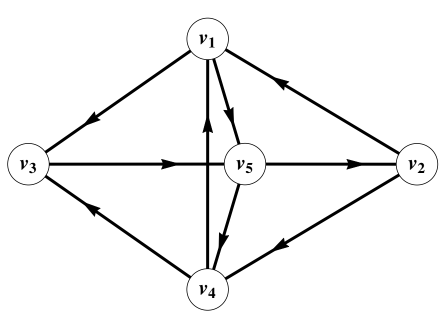
57.



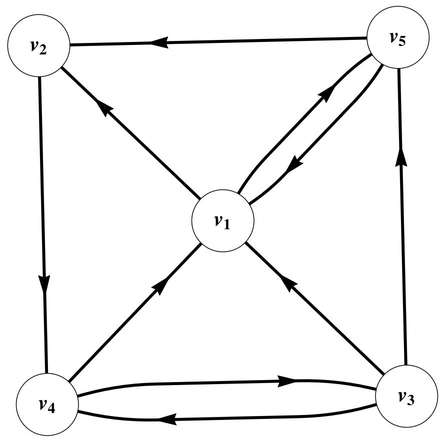
58.



59.



60.



61. In Exercise 50,

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \text{ so } A^2 = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 2 & 4 & 2 & 3 \\ 2 & 2 & 2 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix}.$$

The number of paths of length 2 between v_1 and v_2 is given by $\left(A^2\right)_{12} = 2$. So there are two paths of length 2 between v_1 and v_2.

62. In Exercise 52,

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \text{ so } A^2 = \begin{bmatrix} 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix}.$$

The number of paths of length 2 between v_1 and v_2 is given by $\left(A^2\right)_{12} = 2$. So there are two paths of length 2 between v_1 and v_2.

63. In Exercise 50,

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \text{ so } A^3 = \begin{bmatrix} 3 & 7 & 3 & 6 \\ 7 & 11 & 7 & 8 \\ 3 & 7 & 3 & 6 \\ 6 & 8 & 6 & 5 \end{bmatrix}.$$

The number of paths of length 3 between v_1 and v_3 is given by $\left(A^3\right)_{13} = 3$. So there are three paths of length 3 between v_1 and v_3.

64. In Exercise 52,

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \text{ so } A^4 = \begin{bmatrix} 12 & 12 & 12 & 0 & 0 \\ 12 & 12 & 12 & 0 & 0 \\ 12 & 12 & 12 & 0 & 0 \\ 0 & 0 & 0 & 18 & 18 \\ 0 & 0 & 0 & 18 & 18 \end{bmatrix}.$$

The number of paths of length 4 between v_2 and v_2 is given by $\left(A^4\right)_{22} = 12$. So there are twelve paths of length 4 between v_2 and itself.

65. In Exercise 57,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \text{ so } A^2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}.$$

The number of paths of length 2 from v_1 to v_3 is $\left(A^2\right)_{13} = 0$. There are no paths of length 2 from v_1 to v_3.

66. In Exercise 57,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \text{ so } A^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{bmatrix}.$$

The number of paths of length 3 from v_4 to v_1 is $\left(A^3\right)_{41} = 3$. There are three paths of length 3 from v_4 to v_1.

67. In Exercise 60,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \text{ so } A^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 3 & 0 & 3 & 3 \\ 3 & 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The number of paths of length 3 from v_4 to v_1 is $\left(A^3\right)_{41} = 3$. There are three paths of length 3 from v_4 to v_1.

68. In Exercise 60,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \text{ so } A^4 = \begin{bmatrix} 3 & 2 & 1 & 2 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 6 & 5 & 3 & 3 & 2 \\ 3 & 4 & 1 & 4 & 4 \\ 2 & 2 & 1 & 2 & 3 \end{bmatrix}.$$

The number of paths of length 4 from v_1 to v_4 is $\left(A^3\right)_{14} = 2$. There are two paths of length 4 from v_1 to v_4.

69. **(a)** If there are no ones in row i, then no edges join vertex i to any of the other vertices. Thus G is a disconnected graph, and vertex i is an isolated vertex.

(b) If there are no ones in column j, then no edges join vertex j to any of the other vertices. Thus G is a disconnected graph, and vertex j is an isolated vertex.

70. **(a)** If row i of A^2 is all zeros, then there are no paths of length 2 from vertex i to any other vertex.

(b) If column j of A^2 is all zeros, then there are no paths of length 2 from any vertex to vertex j.

71. The adjacency matrix for the digraph is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The number of wins that each player had is given by

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 3 \\ 1 \\ 3 \end{bmatrix},$$

so the ranking is: first, P_2; second, P_3, P_4, P_6 (tie); third, P_1, P_5 (tie). Ranking instead by combined wins and indirect wins, we compute

$$(A + A^2) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 3 & 0 & 2 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 3 & 1 \\ 2 & 1 & 1 & 0 & 3 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 8 \\ 9 \\ 4 \\ 10 \end{bmatrix},$$

so the ranking is: first, P_2; second, P_6; third, P_4; fourth, P_3; fifth, P_5; sixth, P_1.

72. Let vertices 1 through 7 correspond to, in order, rodent, plant, insect, bird, fish, fox, and bear. Then the adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

(a) The number of direct sources of food is the number of ones in each row, so it is

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

Thus birds and bears have the most direct sources of food, 3.

(b) The number of species for which a given species is a direct source of food is the number of ones in its column. Clearly plants, with a value of 4, are a direct source of food for the greatest number of species. Note the analogy of parts (a) and (b) with the previous exercise, where eating a given species is equivalent to winning that head-to-head match, and being eaten is equivalent to losing it.

(c) Since $\left(A^2\right)_{ij}$ gives the number of paths of length 2 from vertex i to vertex j, it follows that $\left(A^2\right)_{ij}$ is the number of ways in which species j is an indirect food source for species i. So summing the entries of the i^{th} row of A^2 gives the total number of indirect food sources:

$$A^2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 1 \\ 4 \\ 5 \end{bmatrix}.$$

Thus bears have the most indirect food sources. Combined direct and indirect food sources are given by

$$(A + A^2)\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\\6\\3\\6\\8\end{bmatrix}.$$

(d) The matrix A^* is the matrix A after removing row 2 and column 2:

$$A^* = \begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&1&0&0\\0&1&0&0&0&0\\1&0&1&0&0&0\\1&0&0&1&1&0\end{bmatrix}.$$

Then we have

$$A^*\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\2\\1\\2\\3\end{bmatrix},$$

so the species with the most direct food sources is the bear, with three. The species that is a direct food source for the most species is the species for which the column sum is the largest; three species, rodents, insects, and birds, each are prey for two species. Note that the column sums can be computed as

$$\begin{bmatrix}1&1&1&1&1&1\end{bmatrix} A^* \text{ or } (A^*)^T\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}.$$

The number of indirect food sources each species has is

$$(A^*)^2\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\\2\\3\end{bmatrix},$$

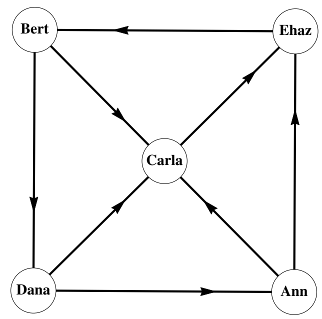
and the number of combined direct and indirect food sources is

$$\left(A^* + (A^*)^2\right)\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\3\\1\\4\\6\end{bmatrix}.$$

Thus birds have lost the most combined direct and indirect sources of food (5). The bear, fox, and fish species have each lost two combined sources of food, and rodents and insects have each lost one combined food source.

(e) Since $\left(A^*\right)^5$ is the zero matrix, after a while, no species will have a direct food source, so that the ecosystem will be wiped out.

73. (a) Using the relationships given, we get the digraph



Letting vertices 1 through 5 correspond to the people in alphabetical order, the adjacency matrix is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

(b) The element $\left[A^k\right]_{ij}$ is 1 if there is a path of length k from i to j, so there is a path of length at most k from i to j if $\left[A + A^2 + \cdots + A^k\right]_{ij} \geq 1$. If Bert hears a rumor and we want to know how many steps it takes before everyone else has heard the rumor, then we must calculate A, $A + A^2$, $A + A^2 + A^3$, and so forth, until every entry in row 2 (except for the one in column 2, which corresponds to Bert himself) is nonzero. This is not true for A, since for example $A_{21} = 0$. Next,

$$A + A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix},$$

so everyone else will have heard the rumor, either directly or indirectly, from Bert after two steps (Carla will have heard it twice).

(c) Using the same logic as in part (b), we must compute successive sums of powers of A until every entry in the first row, except for the entry in the first column corresponding to Ann herself, is nonzero. From part (b), we see that this is not the case for $A + A^2$ since $\left[A + A^2\right]_{14} = 0$. Next,

$$A + A^2 + A^3 = \begin{bmatrix} 0 & 2 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 0 & 3 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix},$$

so everyone else will have heard the rumor from Ann after three steps. In fact, everyone but Dana will have heard it twice.

(d) j is reachable from i by a path of length k if $\left[A^k\right]_{ij}$ is nonzero. Now, if j is reachable from i at all, it is reachable by a path of length at most n, where n is the number of vertices in the digraph, so we must compute

$$\left[A + A^2 + A^3 + \cdots + A^n\right]_{ij}$$

and see if it is nonzero. If it is, the two vertices are connected.

74. Let G be a graph with m vertices, and $A = (a_{ij})$ its adjacency matrix.

(a) The basis for the induction is $n = 1$, where the statement says that a_{ij} is the number of direct paths between vertices i and j. But this is just the definition of the adjacency matrix, since $a_{ij} = 1$ if i and j have an edge between them and is zero otherwise. Next, assume $[A^k]_{ij}$ is the number of paths of length k between i and j. Then $[A^{k+1}]_{ij} = \sum_{l=1}^{m}[A^k]_{il}[A]_{lj}$. For a given vertex l, $[A^k]_{il}$ gives the number of paths of length k from i to l, and $[A]_{lj}$ gives the number of paths of length 1 from l to j (this is either zero or one). So the product gives the number of paths of length $k+1$ from i to j passing through l. Summing over all vertices l in the graph gives all paths of length $k+1$ from i to j, proving the inductive step.

(b) If G is a digraph, the statement of the result must be modified to say that the (i, j) entry of A^n is equal to the number of n-paths *from* i *to* j; the proof is valid since the adjacency matrix of a digraph takes arrow directions into account.

75. $[AA^T]_{ij} = \sum_{k=1}^{m} a_{ik}a_{jk}$. So this matrix entry is the number of vertices k that have edges from *both* vertex i and vertex j.

76. From the diagram in Exercise 49, this graph is bipartite, with $U = \{v_1\}$ and $V = \{v_2, v_3, v_4\}$.

77. From the diagram in Exercise 52, this graph is bipartite, with $U = \{v_1, v_2, v_3\}$ and $V = \{v_4, v_5\}$.

78. The graph in Exercise 51 is not bipartite. For suppose $v_1 \in U$. Then since there are edges from v_1 to both v_3 and v_4, we must have $v_3, v_4 \in V$. Next, there are edges from v_3 to v_5, and from v_4 to v_2, so that $v_2, v_5 \in U$. But there is an edge from v_2 to v_5. So this graph cannot be bipartite.

79. Examining the matrix, we see that vertices 1, 2, and 4 are connected to vertices 3, 5, and 6 only, so that if we define $U = \{v_1, v_2, v_4\}$ and $V = \{v_3, v_5, v_6\}$, we have shown that this graph is bipartite.

80. Suppose that G has n vertices.

(a) First suppose that the adjacency matrix of G is of the form shown. Suppose that the zero blocks have p and $q = n - p$ rows respectively. Then any edges at vertices $v_1, v_2, \ldots, v_p$ have their other vertex in vertices $v_{p+1}, v_{p+2}, \ldots, v_n$, since the upper left $p \times p$ matrix is O. Similarly, any edges at vertices $v_{p+1}, v_{p+2}, \ldots, v_n$ have their other vertex in vertices $v_1, v_2, \ldots, v_p$, since the lower right $q \times q$ matrix is O. So G is bipartite, with $U = \{v_1, v_2, \ldots, v_p\}$ and $V = \{v_{p+1}, v_{p+2}, \ldots, v_n\}$. Next suppose that G is bipartite. Label the vertices so that $U = \{v_1, v_2, \ldots, v_p\}$ and $V = \{v_{p+1}, v_{p+2}, \ldots, v_n\}$. Now consider the adjacency matrix A of G. $a_{ij} = 0$ if $i, j \le p$, since then both v_i and v_j are in U; similarly, $a_{ij} = 0$ if $i, j > p$, since then both v_i and v_j are in V. So the upper left $p \times p$ submatrix as well as the bottom right $q \times q$ submatrix are the zero matrix. The remaining two submatrices are transposes of each other since this is an undirected graph, so that $a_{ij} = 1$ if and only if $a_{ji} = 1$. Thus A is of the form shown.

(b) Suppose G is bipartite and A its adjacency matrix, of the form given in part (a). It suffices to show that odd powers of A have zero blocks at upper left of size $p \times p$ and at lower right of size $q \times q$, since then there are no paths of odd length between any vertex and itself. We do this by induction. Clearly $A^1 = A$ is of that form. Now,

$$A^2 = \begin{bmatrix} OO + BB^T & OB + BO \\ B^TO + OB^T & B^TB + OO \end{bmatrix} = \begin{bmatrix} BB^T & O \\ O & B^TB \end{bmatrix} = \begin{bmatrix} D_1 & O \\ O & D_2 \end{bmatrix}$$

where $D_1 = BB^T$ and $D_2 = B^TB$. Suppose that A^{2k-1} has the required zero blocks. Then

$$\begin{aligned} A^{2k+1} = A^{2k-1}A^2 &= \begin{bmatrix} O & C_1 \\ C_2 & O \end{bmatrix} \begin{bmatrix} D_1 & O \\ O & D_2 \end{bmatrix} \\ &= \begin{bmatrix} OD_1 + C_1O & OO + C_1D_2 \\ C_2D_1 + OO & D_1O + OD_2 \end{bmatrix} \\ &= \begin{bmatrix} O & C_1D_2 \\ C_2D_1 & O \end{bmatrix}. \end{aligned}$$

Thus A^{2k+1} is of this form as well. So no odd power of A has any entries in the upper left or lower right block, so there are no odd paths beginning and ending at the same vertex. Thus all circuits must be of even length. (Note that in the matrix A^{2k+1} above, we must in fact have $(C_1D_2)^T = C_2D_1$ since powers of a symmetric matrix are symmetric, but we do not need this fact.)

Chapter Review

1. (a) True. See Exercise 34 in Section 3.2. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix. Then AA^T is defined and is an $m \times m$ matrix, and A^TA is also defined and is an $n \times n$ matrix.

(b) False. This is true only when A is invertible. For example, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{then} \quad AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

but $A \neq O$.

(c) False. What is true is that $XAA^{-1} = BA^{-1}$ so that $X = BA^{-1}$. Matrix multiplication is not commutative, so that $BA^{-1} \neq A^{-1}B$ in general.

(d) True. This is Theorem 3.11 in Section 3.3.

(e) True. If E performs kR_i, then E is a matrix with all zero entries off the diagonal, so it is its own transpose and thus E^T is also the elementary matrix that performs kR_i. If E performs $R_i + kR_j$, then the only off-diagonal entry in E is $[E]_{ij} = k$, so the only off-diagonal entry of E^T is $[E^T]_{ji} = k$ and thus E^T is the elementary matrix performing $R_j + kR_i$. Finally, if E performs $R_i \leftrightarrow R_j$, then $[E]_{ii} = [E]_{jj} = 0$ and $[E]_{ij} = [E]_{ji} = 1$, so that $E^T = E$ and thus E^T is also an elementary matrix exchanging R_i and R_j.

(f) False. Elementary matrices perform a single row operation, while their products can perform multiple row operations. There are many examples. For instance, consider the elementary matrices corresponding to $R_1 \leftrightarrow R_2$ and $2R_1$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix},$$

which is not an elementary matrix.

(g) True. See Theorem 3.21 in Section 3.5.

(h) False. For this to be true, the plane must pass through the origin.

(i) True, since

$$T(c_1\mathbf{u} + c_2\mathbf{v}) = -(c_1\mathbf{u} + c_2\mathbf{v}) = -c_1\mathbf{u} - c_2\mathbf{v} = c_1(-\mathbf{u}) + c_2(-\mathbf{v}) = c_1T(\mathbf{u}) + c_2T(\mathbf{v}).$$

(j) False; the matrix is 5×4, not 4×5. See Theorems 3.30 and 3.31 in Section 3.6.

2. Since A is 2×2, so is A^2. Since B is 2×3, the product makes sense. Then

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}, \text{ so that } A^2B = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 50 & -36 & 41 \\ 129 & -93 & 106 \end{bmatrix}.$$

3. B^2 is not defined, since the number of columns of B is not equal to the number of rows of B. So this matrix product does not make sense.

4. Since B^T is 3×2, A^{-1} is 2×2, and B is 2×3, this matrix product makes sense as long as A is invertible. Since $\det A = 1 \cdot 5 - 2 \cdot 3 = -1$, we have

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix},$$

so that

$$B^TA^{-1}B = \begin{bmatrix} 2 & 3 \\ 0 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -9 & 3 \\ 17 & -6 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 5 \\ -9 & -9 & 21 \\ 16 & 18 & -41 \end{bmatrix}.$$

5. Since BB^T is always defined (see exercise 34 in section 3.2) and is square, $(BB^T)^{-1}$ makes sense if BB^T is invertible. We have

$$BB^T = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 34 \end{bmatrix}.$$

Then $\det(BB^T) = 5 \cdot 34 - 2 \cdot 2 = 166$, so that

$$(BB^T)^{-1} = \frac{1}{166} \begin{bmatrix} 34 & -2 \\ -2 & 5 \end{bmatrix}.$$

6. Since B^TB is always defined (see exercise 34 in section 3.2) and is square, $(B^TB)^{-1}$ makes sense if B^TB is invertible. We have

$$B^TB = \begin{bmatrix} 2 & 3 \\ 0 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 13 & -9 & 10 \\ -9 & 9 & -12 \\ 10 & -12 & 17 \end{bmatrix}.$$

To compute the inverse, we row-reduce $\left[B^TB \mid I\right]$:

$$\left[B^TB \mid I\right] = \left[\begin{array}{ccc|ccc} 13 & -9 & 10 & 1 & 0 & 0 \\ -9 & 9 & -12 & 0 & 1 & 0 \\ 10 & -12 & 17 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - \frac{10}{13}R_1]{R_2 + \frac{9}{13}R_1} \left[\begin{array}{ccc|ccc} 13 & -9 & 10 & 1 & 0 & 0 \\ 0 & \frac{36}{13} & -\frac{66}{13} & \frac{9}{13} & 1 & 0 \\ 0 & -\frac{66}{13} & \frac{121}{13} & -\frac{10}{13} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 + \frac{66}{36}R_2} \left[\begin{array}{ccc|ccc} 13 & -9 & 10 & 1 & 0 & 0 \\ 0 & \frac{36}{13} & -\frac{66}{13} & \frac{9}{13} & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{11}{6} & 1 \end{array}\right].$$

Since the matrix on the left has a zero row, B^TB is not invertible, so $(B^TB)^{-1}$ does not exist.

7. The outer product expansion of AA^T is

$$AA^T = \mathbf{a}_1A_1^T + \mathbf{a}_2A_2^T = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 10 \\ 10 & 25 \end{bmatrix} = \begin{bmatrix} 5 & 13 \\ 13 & 34 \end{bmatrix}.$$

8. By Theorem 3.9 in Section 3.3, we know that $(A^{-1})^{-1} = A$, so we must find the inverse of the given matrix. Now, $\det A^{-1} = \frac{1}{2} \cdot 4 - (-1)\left(-\frac{3}{2}\right) = \frac{1}{2}$, so that

$$A = (A^{-1})^{-1} = \frac{1}{1/2} \begin{bmatrix} 4 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 1 \end{bmatrix}.$$

9. Since $AX = B = \begin{bmatrix} -1 & -3 \\ 5 & 0 \\ 3 & -2 \end{bmatrix}$, we have $X = A^{-1}B$. To find A^{-1}, we compute

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & \frac{3}{2} \end{array}\right].$$

Then

$$X = A^{-1}B = \begin{bmatrix} 2 & -\frac{1}{2} & \frac{3}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 5 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ 2 & 4 \\ 1 & -6 \end{bmatrix}.$$

10. Since $\det A = 1\cdot 6 - 2\cdot 4 = -2 \neq 0$, A is invertible, so by Theorem 3.12 in Section 3.3, it can be written as a product of elementary matrices. As in Example 3.29 in Section 3.3, we start by row-reducing A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So by applying

$$E_1 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & -2 & 0 & 1 \end{bmatrix}$$

in that order to A we get the identity matrix, so that $E_3E_2E_1A = I$. Since elementary matrices are invertible, this means that $A = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$. Since E_1 is $R_2 - 4R_1$, it follows that E_1^{-1} is $R_2 + 4R_1$. Since E_2 is $-\frac{1}{2}R_2$, it follows that E_2^{-1} is $-2R_2$. Since E_3 is $R_1 - 2R_2$, it follows that E_3^{-1} is $R_1 + 2R_2$. Thus

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

11. To show that $(I - A)^{-1} = I + A + A^2$, it suffices to show that $(I + A + A^2)(I - A) = I$. But

$$(I+A+A^2)(I-A) = II - IA + AI - A^2 + A^2I - A^3 = I - A + A - A^2 + A^2 - A^3 = I + A^3 = I + O = I.$$

12. Use the multiplier method. We want to reduce A to an upper triangular matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -3 & -1 \end{bmatrix} \xrightarrow{R_3 - \frac{3}{2}R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 2 \end{bmatrix} = U.$$

Then $\ell_{21} = 3$, $\ell_{31} = 2$, and $\ell_{32} = \frac{3}{2}$, so that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{3}{2} & 1 \end{bmatrix}.$$

13. We follow the comment after Example 3.48 in Section 3.5. Start by finding the reduced row echelon form of A:

$$\begin{bmatrix} 2 & -4 & 5 & 8 & 5 \\ 1 & -2 & 2 & 3 & 1 \\ 4 & -8 & 3 & 2 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 2 & 3 & 1 \\ 2 & -4 & 5 & 8 & 5 \\ 4 & -8 & 3 & 2 & 6 \end{bmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 + 5R_2} \begin{bmatrix} 1 & -2 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix} \xrightarrow{\frac{1}{8}R_3} \begin{bmatrix} 1 & -2 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 - 3R_3]{R_1 - R_3} \begin{bmatrix} 1 & -2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus a basis for row(A) is given by the nonzero rows of the matrix, or

$$\left\{\begin{bmatrix}1 & -2 & 0 & -1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1 & 2 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0 & 0 & 1\end{bmatrix}\right\}.$$

Note that since the matrix is rank 3, so all three rows are nonzero, the rows of the original matrix also provide a basis for row(A).

A basis for col(A) is given by the column vectors of the original matrix A corresponding to the leading 1s in the row-reduced matrix, so a basis for col(A) is

$$\left\{\begin{bmatrix}2\\1\\4\end{bmatrix}, \begin{bmatrix}5\\2\\3\end{bmatrix}, \begin{bmatrix}5\\1\\6\end{bmatrix}\right\}.$$

Finally, from the row-reduced form, the solutions to $A\mathbf{x} = \mathbf{0}$ where

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}$$

can be found by noting that $x_2 = s$ and $x_4 = t$ are free variables and that $x_1 = 2s + t$, $x_3 = -2t$, and $x_5 = 0$. Thus the nullspace of A is

$$\begin{bmatrix}2s+t\\s\\-2t\\t\\0\end{bmatrix} = s\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\-2\\1\\0\end{bmatrix},$$

so that a basis for the nullspace is given by

$$\left\{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-2\\1\\0\end{bmatrix}\right\}.$$

14. The row spaces will be the same since by Theorem 3.20 in Section 3.5, if A and B are row-equivalent, they have the same row spaces. However, the column spaces need not be the same. For example, let

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix} = B.$$

Then A and B are row-equivalent, but

$$\text{col}(A) = \text{span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right) \text{ but } \text{col}(B) = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right).$$

15. If A is invertible, then so is A^T. But by Theorem 3.27 in Section 3.5, invertible matrices have trivial null spaces, so that null$(A) = $ null$(A^T) = \mathbf{0}$. If A is not invertible, then the null spaces need not be the same. For example, let

$$A = \begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \text{ so that } A^T = \begin{bmatrix}1 & 0 & 0\\1 & 0 & 0\\1 & 0 & 0\end{bmatrix}.$$

Then A^T row reduces to

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \text{ so that } \operatorname{null}(A^T)=\operatorname{span}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right).$$

But

$$\operatorname{null}(A)=\operatorname{span}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right).$$

Since every vector in $\operatorname{null}(A^T)$ has a zero first component, neither of the basis vectors for $\operatorname{null}(A)$ is in $\operatorname{null}(A^T)$, so the spaces cannot be equal.

16. If the rows of A add up to zero, then the rows are linearly dependent, so by Theorem 3.27 in Section 3.5, A is not invertible.

17. Since A has n linearly independent columns, $\operatorname{rank}(A)=n$. Theorem 3.28 of Section 3.5 tells us that $\operatorname{rank}(A^TA)=\operatorname{rank}(A)=n$, so A^TA is invertible since it is an $n\times n$ matrix. For the case of AA^T, since m may be greater than n, there is no reason to believe that this matrix must be invertible. One counterexample is

$$A=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}, \quad A^T=\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}, \quad AA^T=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}\to\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}.$$

Thus AA^T is not invertible, since it row-reduces to a matrix with a zero row.

18. The matrix of T is $[T]=\begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2)\end{bmatrix}$. From the given data, we compute

$$\begin{aligned}T\begin{bmatrix}1\\0\end{bmatrix}&=T\left(\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}2\\3\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0\\5\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}\\ T\begin{bmatrix}0\\1\end{bmatrix}&=T\left(\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}2\\3\end{bmatrix}-\frac{1}{2}\begin{bmatrix}0\\5\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}.\end{aligned}$$

Thus

$$[T]=\begin{bmatrix}1&1\\4&-1\end{bmatrix}.$$

19. Let R be the rotation and P the projection. The given line has direction vector $\begin{bmatrix}1\\-2\end{bmatrix}$. Then from Examples 3.58 and 3.59 in Section 3.6,

$$\begin{aligned}[R]&=\begin{bmatrix}\cos 45^\circ & -\sin 45^\circ\\ \sin 45^\circ & \cos 45^\circ\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}\\ [P]&=\frac{1}{1^2+2^2}\begin{bmatrix}1^2 & 1\cdot(-2)\\ 1\cdot(-2) & (-2)^2\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & -\frac{2}{5}\\ -\frac{2}{5} & \frac{4}{5}\end{bmatrix}.\end{aligned}$$

So R followed by P is

$$[P\circ R]=[P][R]=\begin{bmatrix}\frac{1}{5} & -\frac{2}{5}\\ -\frac{2}{5} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}=\begin{bmatrix}-\frac{\sqrt{2}}{10} & -\frac{3\sqrt{2}}{10}\\ \frac{\sqrt{2}}{5} & \frac{3\sqrt{2}}{5}\end{bmatrix}.$$

20. Suppose that $c\mathbf{v}+dT(\mathbf{v})=\mathbf{0}$. We want to show that $c=d=0$. Since T is linear and $T^2(\mathbf{v})=\mathbf{0}$, applying T to both sides of this equation gives

$$T(c\mathbf{v}+T(dT(\mathbf{v})))=cT(\mathbf{v})+dT(T(\mathbf{v}))=cT(\mathbf{v})+dT^2(\mathbf{v})=cT(\mathbf{v})=\mathbf{0}.$$

Since $T(\mathbf{v})\neq\mathbf{0}$, it follows that $c=0$. But then $c\mathbf{v}+dT(\mathbf{v})=dT(\mathbf{v})=\mathbf{0}$. Again since $T(\mathbf{v})\neq\mathbf{0}$, it follows that $d=0$. Thus $c=d=0$ and $\mathbf{v}$ and $T(\mathbf{v})$ are linearly independent.

Chapter 4

Eigenvalues and Eigenvectors

4.1 Introduction to Eigenvalues and Eigenvectors

1. Following example 4.1, to see that $\mathbf{v}$ is an eigenvector of A, we show that $A\mathbf{v}$ is a multiple of $\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 3 \cdot 1 \\ 3 \cdot 1 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\mathbf{v}.$$

Thus $\mathbf{v}$ is an eigenvector of A with eigenvalue 3.

2. Following example 4.1, to see that $\mathbf{v}$ is an eigenvector of A, we show that $A\mathbf{v}$ is a multiple of $\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot (-3) \\ 2 \cdot 3 + 1 \cdot (-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = -\mathbf{v}$$

Thus $\mathbf{v}$ is an eigenvector of A with eigenvalue -1.

3. Following example 4.1, to see that $\mathbf{v}$ is an eigenvector of A, we show that $A\mathbf{v}$ is a multiple of $\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \cdot 1 + 1 \cdot (-2) \\ 6 \cdot 1 + 0 \cdot (-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -3\mathbf{v}.$$

Thus $\mathbf{v}$ is an eigenvector of A with eigenvalue -3.

4. Following example 4.1, to see that $\mathbf{v}$ is an eigenvector of A, we show that $A\mathbf{v}$ is a multiple of $\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot 4 - 2 \cdot 2 \\ 5 \cdot 4 - 7 \cdot 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} = 3\mathbf{v}$$

Thus $\mathbf{v}$ is an eigenvector of A with eigenvalue 3.

5. Following example 4.1, to see that $\mathbf{v}$ is an eigenvector of A, we show that $A\mathbf{v}$ is a multiple of $\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + 0 \cdot (-1) + 0 \cdot 1 \\ 0 \cdot 2 + 1 \cdot (-1) - 2 \cdot 1 \\ 1 \cdot 2 + 0 \cdot (-1) + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 3\mathbf{v}.$$

Thus $\mathbf{v}$ is an eigenvector of A with eigenvalue 3.

6. Following example 4.1, to see that $\mathbf{v}$ is an eigenvector of A, we show that $A\mathbf{v}$ is a multiple of $\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot (-2) + 1 \cdot 1 - 1 \cdot 1 \\ 1 \cdot (-2) + 1 \cdot 1 + 1 \cdot 1 \\ 1 \cdot (-2) + 2 \cdot 1 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{v}$$

Thus $\mathbf{v}$ is an eigenvector of A with eigenvalue 0.

7. We want to show that $\lambda = 3$ is an eigenvalue of

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

Following Example 4.2, that means we wish to find a vector $\mathbf{v}$ with $A\mathbf{v} = 3\mathbf{v}$, i.e. with $(A - 3I)\mathbf{v} = \mathbf{0}$. So we show that $(A - 3I)\mathbf{v} = \mathbf{0}$ has nontrivial solutions.

$$A - 3I = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}.$$

Row-reducing, we get

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

It follows that any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x = 2y$ is an eigenvector, so for example one eigenvector corresponding to $\lambda = 3$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

8. We want to show that $\lambda = -1$ is an eigenvalue of

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Following Example 4.2, that means we wish to find a vector $\mathbf{v}$ with $A\mathbf{v} = -\mathbf{v}$, i.e. with $(A - (-I))\mathbf{v} = (A + I)\mathbf{v} = \mathbf{0}$. So we show that $(A + I)\mathbf{v} = \mathbf{0}$ has nontrivial solutions.

$$A + I = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}.$$

Row-reducing, we get

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

It follows that any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x = -y$ is an eigenvector, so for example one eigenvector corresponding to $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

9. We want to show that $\lambda = 1$ is an eigenvalue of

$$A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$$

Following Example 4.2, that means we wish to find a vector $\mathbf{v}$ with $A\mathbf{v} = 1\mathbf{v}$, i.e. with $(A - I)\mathbf{v} = \mathbf{0}$. So we show that $(A - I)\mathbf{v} = \mathbf{0}$ has nontrivial solutions.

$$A - I = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 4 \end{bmatrix}.$$

Row-reducing, we get

$$\begin{bmatrix} -1 & 4 \\ -1 & 4 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -4 \\ -1 & 4 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}.$$

It follows that any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x = 4y$ is an eigenvector, so for example one eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

10. We want to show that $\lambda = -6$ is an eigenvalue of

$$A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$$

Following Example 4.2, that means we wish to find a vector $\mathbf{v}$ with $A\mathbf{v} = -6\mathbf{v}$, i.e. with $(A-(-6I))\mathbf{v} = (A+6I)\mathbf{v} = \mathbf{0}$. So we show that $(A+6I)\mathbf{v} = \mathbf{0}$ has nontrivial solutions.

$$A + 6I = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 10 & -2 \\ 5 & -1 \end{bmatrix}.$$

Row-reducing, we get

$$\begin{bmatrix} 10 & -2 \\ 5 & -1 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1} \begin{bmatrix} 10 & -2 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{10}R_1} \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 0 \end{bmatrix}$$

It follows that any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x = \frac{1}{5}y$ is an eigenvector, so for example one eigenvector corresponding to $\lambda = -6$ is $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

11. We want to show that $\lambda = -1$ is an eigenvalue of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

Following Example 4.2, that means we wish to find a vector $\mathbf{v}$ with $A\mathbf{v} = -1\mathbf{v}$, i.e. with $(A+I)\mathbf{v} = \mathbf{0}$. So we show that $(A+I)\mathbf{v} = \mathbf{0}$ has nontrivial solutions.

$$A + I = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}.$$

Row-reducing, we get

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{4}R_2} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $x = y = -z$ is an eigenvector, so for example one eigenvector corresponding to $\lambda = -1$ is $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

12. We want to show that $\lambda = 2$ is an eigenvalue of

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix}$$

Following example 4.2, that means we wish to find a vector $\mathbf{v}$ with $A\mathbf{v} = 2\mathbf{v}$, i.e. with $(A - 2I)\mathbf{v} = \mathbf{0}$. So we show that $(A - 2I)\mathbf{v} = \mathbf{0}$ has nontrivial solutions.

$$A - 2I = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$$

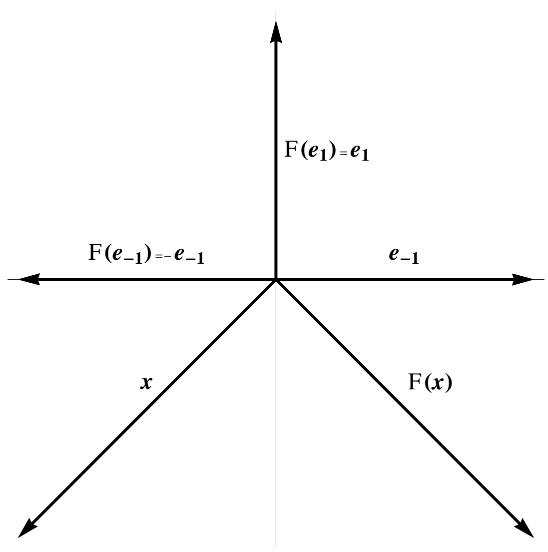
Row-reducing, we have

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix} \xrightarrow[R_3-4R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that any vector $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$ with $x = 0$ and $y = z$ is an eigenvector. Thus for example $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.

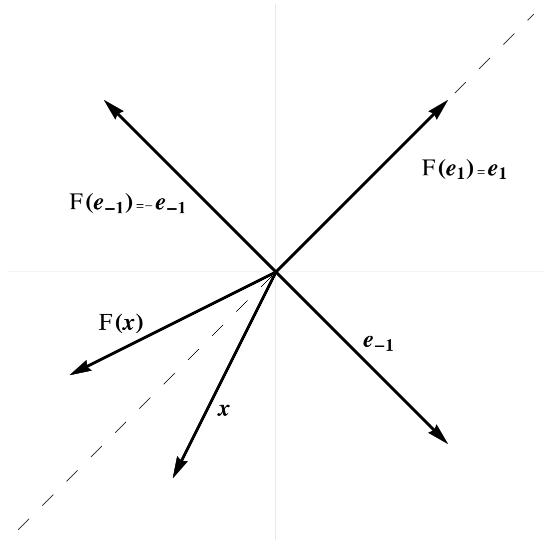
13. Since A is the reflection F in the y-axis, the only vectors that F maps parallel to themselves are vectors parallel to the x-axis (multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$), which are reversed (eigenvalue $\lambda = -1$) and vectors parallel to the y-axis (multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$), which are sent to themselves (eigenvalue $\lambda = 1$). So $\lambda = \pm 1$ are the eigenvalues of A, with eigenspaces

$$E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \qquad E_1 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$



14. The only vectors that the transformation F corresponding to A maps to themselves are vectors perpendicular to $y = x$, which are reversed (so correspond to the eigenvalue $\lambda = -1$) and vectors parallel to $y = x$, which are sent to themselves (so correspond to the eigenvalue $\lambda = 1$. Vectors perpendicular to $y = x$ are multiples of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; vectors parallel to $y = x$ are multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so the eigenspaces are

$$E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \qquad E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$



15. If $\mathbf{x}$ is an eigenvalue of A, then $A\mathbf{x}$ is a multiple of $\mathbf{x}$:

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Then clearly any vector parallel to the x-axis, i.e., of the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$, is an eigenvector for A with eigenvalue 1. But this projection also transforms vectors parallel to the y-axis to the zero vector, since the projection of such a vector on the x-axis is $\mathbf{0}$. So vectors of the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$ are eigenvectors for A with eigenvalue 0. Finally, a vector of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ where neither x nor y is zero is not an eigenvector: it is transformed into $\begin{bmatrix} x \\ 0 \end{bmatrix}$, and comparing the x-coordinates shows that λ would have to be 1, while comparing y-coordinates shows that it would have to be zero. Thus there is no such λ. So in summary,

$$\lambda = 1 \text{ is an eigenvalue with eigenspace } E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$
$$\lambda = 0 \text{ is an eigenvalue with eigenspace } E_0 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

16. This projection sends a vector to its projection on the given line. Then a vector parallel to the direction vector of that line will be sent to itself, since its projection on the line is the vector itself. A vector orthogonal to the direction vector of the line will be sent to the zero vector, since its projection on the line is the zero vector. Since the direction vector is $\begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$, or $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, there are two eigenspaces

$$\operatorname{span}\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right), \ \lambda = 1; \qquad \operatorname{span}\left(\begin{bmatrix} -3 \\ 4 \end{bmatrix}\right), \ \lambda = -1.$$

If a vector $\mathbf{v}$ is neither parallel to nor orthogonal to the direction vector of the line, then the projection transforms $\mathbf{v}$ to a vector parallel to the direction vector, thus not parallel to $\mathbf{v}$. Thus the image of $\mathbf{v}$ cannot be a multiple of $\mathbf{v}$. Hence the eigenvalues and eigenspaces above are the only ones.

17. Consider the action of A on a vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. If $y = 0$, then A multiplies $\mathbf{x}$ by 2, so this is an eigenspace for $\lambda = 2$. If $x = 0$, then A multiplies $\mathbf{x}$ by 3, so this is an eigenspace for $\lambda = 3$. If neither x

nor y is zero, then $\begin{bmatrix} x \\ y \end{bmatrix}$ is mapped to $\begin{bmatrix} 2x \\ 3y \end{bmatrix}$, which is not a multiple of $\begin{bmatrix} x \\ y \end{bmatrix}$. So the only eigenspaces are

$$\operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \ \lambda = 2; \qquad \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right), \ \lambda = 3.$$

18. Since A rotates the plane 90° counterclockwise around the origin, any nonzero vector is taken to a vector orthogonal to itself. So there are no nonzero vectors that are taken to vectors parallel to themselves (that is, to multiples of themselves). The only vector taken to a multiple of itself is the zero vector; this is not an eigenvector since eigenvectors are by definition nonzero. So this matrix has no eigenvectors and no eigenvalues. Analytically, the matrix of A is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

so that

$$A\mathbf{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

If this is a multiple of the original vector, then there is a constant a such that $x = -ay$ and $y = ax$. Substituting ax for y in the first equation gives $x = -a^2x$, which is impossible unless $x = 0$. But then the second equation forces $y = 0$. So again we see that there are no eigenvectors.

19. Since eigenvectors $\mathbf{x}$ are those taken to a multiple of themselves, i.e., to a vector parallel to themselves, by A, we look for unit vectors whose image under A is in the same direction. It appears that these vectors are vectors along the x and y-axes, so the eigenspaces are $E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $E_2 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. The length of the resultant vectors in the first eigenspace appears are same length as the original vector, so that $\lambda_1 = 1$. The length of the resultant vectors in the second eigenspace are twice the original length, so that $\lambda_2 = 2$.

20. Since eigenvectors $\mathbf{x}$ are those taken to a multiple of themselves, i.e., to a vector parallel to themselves, by A, we look for unit vectors whose image under A is in the same direction. The only vectors satisfying this criterion are those parallel to the line $y = x$, i.e., $\operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, so this is the only eigenspace. The terminal point of the line in the direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ appears to be the point $(2, 5)$, so that it has a length of $\sqrt{29}$. Since the original vector is a unit vector, the eigenvalue is $\sqrt{29}$.

21. Since eigenvectors $\mathbf{x}$ are those taken to a multiple of themselves, i.e., to a vector parallel to themselves, by A, we look for unit vectors whose image under A is in the same direction. The vectors parallel to the line $y = x$, i.e., $\operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ satisfy this criterion, so this is an eigenspace E_1. The vectors parallel to the line $y = -x$, i.e. $\operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$, also satisfy this criterion, so this is another eigenspace E_2. For E_1, the terminal point of the line along $y = x$ appears to be $(2, 2)$, so its length is $2\sqrt{2}$. Since the original vector is a unit vector, the corresponding eigenvalue is $\lambda_1 = 2\sqrt{2}$. For E_2, the resultant vectors are zero length, so $\lambda_2 = 0$.

22. Since eigenvectors $\mathbf{x}$ are those taken to a multiple of themselves, i.e., to a vector parallel to themselves, by A, we look for unit vectors whose image under A is in the same direction. There are none — every vector is "bent" to the left by the transformation. So this transformation has no eigenvectors.

23. Using the method of Example 4.5, we wish to find solutions to the equation

$$\begin{aligned} 0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= \det\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} \\ &= (4-\lambda)(1-\lambda) - 2\cdot(-1) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3). \end{aligned}$$

Thus the eigenvalues are 2 and 3.

To find the eigenspace corresponding to $\lambda = 2$, we must find the null space of $A - 2I$.

$$\left[A - 2I \mid 0\right] = \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} t \\ 2t \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$ to get $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. As a check,

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 - 1 \cdot 2 \\ 2 \cdot 1 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

To find the eigenspace corresponding to $\lambda = 3$, we must find the null space of $A - 3I$.

$$\left[A - 3I \mid 0\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

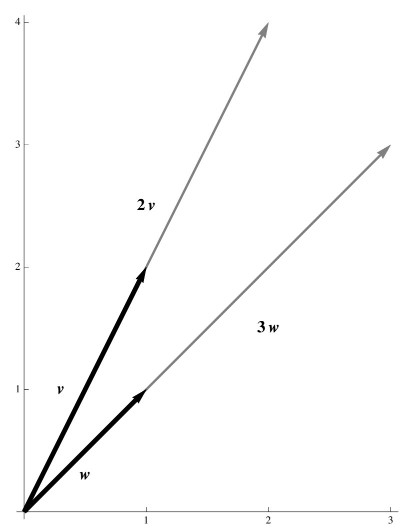
$$\begin{bmatrix} t \\ t \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$ to get $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. As a check,

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 - 1 \cdot 1 \\ 2 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The plot below shows the effect of A on the eigenvectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$



24. Using the method of Example 4.5, we wish to find solutions to the equation

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & 4 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 2-\lambda & 4 \\ 6 & -\lambda \end{bmatrix}$$
$$= (2-\lambda)(-\lambda) - 4\cdot 6 = \lambda^2 - 2\lambda - 24 = (\lambda - 6)(\lambda + 4).$$

Thus the eigenvalues are -4 and 6.

To find the eigenspace corresponding to $\lambda = -4$, we must find the null space of $A + 4I$.

$$\begin{bmatrix} A + 4I & \big| & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 6 & 4 & 0 \\ 6 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} -\frac{2}{3}t \\ t \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} -2t \\ 3t \end{bmatrix}$$

A basis for this eigenspace is given by choosing for example $t = 1$ to get $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$. As a check,

$$A\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\cdot(-2) + 4\cdot 3 \\ 6\cdot(-2) + 0\cdot 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \end{bmatrix} = -4\begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

To find the eigenspace corresponding to $\lambda = 6$, we must find the null space of $A - 6I$.

$$\begin{bmatrix} A - 6I & \big| & 0 \end{bmatrix} = \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 6 & -6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

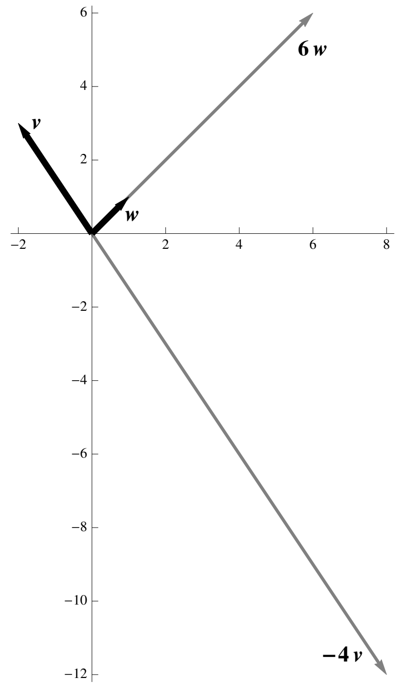
$$\begin{bmatrix} t \\ t \end{bmatrix}$$

A basis for this eigenspace is given by choosing for example $t = 1$ to get $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. As a check,

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 4\cdot 1 \\ 6\cdot 1 + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The plot below shows the effect of A on the eigenvectors

$$\mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$



25. Using the method of Example 4.5, we wish to find solutions to the equation

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det \begin{bmatrix} 2-\lambda & 5 \\ 0 & 2-\lambda \end{bmatrix}$$
$$= (2-\lambda)(2-\lambda) - 5 \cdot 0 = (\lambda - 2)^2$$

Thus the only eigenvalue is $\lambda = 2$.

To find the eigenspace corresponding to $\lambda = 2$, we must find the null space of $A - 2I$.

$$\left[A - 2I \mid 0\right] = \left[\begin{array}{cc|c} 0 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

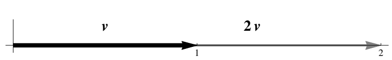
so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} t \\ 0 \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. As a check,

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 5 \cdot 0 \\ 0 \cdot 1 + 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The plot below shows the effect of A on the eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:



26. Using the method of Example 4.5, we wish to find solutions to the equation

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det \begin{bmatrix} 1-\lambda & 2 \\ -2 & 3-\lambda \end{bmatrix}$$
$$= (1-\lambda)(3-\lambda) + 4 = \lambda^2 - 4\lambda + 7$$

Since this quadratic has no (real) roots, A has no (real) eigenvalues.

27. Using the method of Example 4.5, we wish to find solutions to the equation

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}$$
$$= (1-\lambda)(1-\lambda) - 1\cdot(-1) = \lambda^2 - 2\lambda + 2.$$

Using the quadratic formula, this quadratic has roots

$$\lambda = \frac{2 \pm \sqrt{2^2 - 4\cdot 2}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Thus the eigenvalues are $1+i$ and $1-i$.

To find the eigenspace corresponding to $\lambda = 1+i$, we must find the null space of $A - (1+i)I$.

$$\left[A - (1+i)I \mid 0\right] = \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \xrightarrow{iR_1} \left[\begin{array}{cc|c} 1 & i & 0 \\ -1 & -i & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} -it \\ t \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda = 1-i$, we must find the null space of $A - (1-i)I$.

$$\left[A - (1-i)I \mid 0\right] = \left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow{-iR_1} \left[\begin{array}{cc|c} 1 & -i & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} it \\ t \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

28. We wish to find solutions to the equation

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 2-\lambda & -3 \\ 1 & -\lambda \end{bmatrix}$$
$$= (2-\lambda)(-\lambda) - (-3)\cdot 1 = \lambda^2 - 2\lambda + 3.$$

Using the quadratic formula, this has roots $\lambda = \frac{2\pm\sqrt{4-4\cdot 3}}{2} = 1 \pm i\sqrt{2}$, so these are the eigenvalues.

To find the eigenspace corresponding to $\lambda = 1 + i\sqrt{2}$, we must find the null space of $A - (1+i\sqrt{2})I$:

$$\left[A - (1+i\sqrt{2})I \mid 0\right] = \left[\begin{array}{cc|c} 1-i\sqrt{2} & -3 & 0 \\ 1 & -1-i\sqrt{2} & 0 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_2}$$
$$\left[\begin{array}{cc|c} 1 & -1-i\sqrt{2} & 0 \\ 1-i\sqrt{2} & -3 & 0 \end{array}\right] \xrightarrow{R_2-(1-i\sqrt{2})R_1} \left[\begin{array}{cc|c} 1 & -1-i\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} (1+i\sqrt{2})t \\ t \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix} 1+i\sqrt{2} \\ 1 \end{bmatrix}$. As a check,

$$A\begin{bmatrix} 1+i\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1+i\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 2(1+i\sqrt{2}) - 3\cdot 1 \\ (1+i\sqrt{2}) + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} -1+2i\sqrt{2} \\ 1+i\sqrt{2} \end{bmatrix} = (1+i\sqrt{2})\begin{bmatrix} 1+i\sqrt{2} \\ 1 \end{bmatrix}$$

To find the eigenspace corresponding to $\lambda = 1 - i\sqrt{2}$, we must find the null space of $A - (1 - i\sqrt{2})I$.

$$\left[A - (1-i\sqrt{2})I \mid 0\right] = \left[\begin{array}{cc|c} 1+i\sqrt{2} & -3 & 0 \\ 1 & -1+i\sqrt{2} & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{cc|c} 1 & -1+i\sqrt{2} & 0 \\ 1+i\sqrt{2} & -3 & 0 \end{array}\right] \xrightarrow{R_2-(1+i\sqrt{2})R_1} \left[\begin{array}{cc|c} 1 & -1+i\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} (1-i\sqrt{2})t \\ t \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix} 1-i\sqrt{2} \\ 1 \end{bmatrix}$. As a check,

$$A\begin{bmatrix} 1-i\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1-i\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 2(1-i\sqrt{2}) - 3\cdot 1 \\ (1-i\sqrt{2}) + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} -1-2i\sqrt{2} \\ 1-i\sqrt{2} \end{bmatrix} = (1-i\sqrt{2})\begin{bmatrix} 1-i\sqrt{2} \\ 1 \end{bmatrix}$$

29. Using the method of Example 4.5, we wish to find solutions to the equation

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & i \\ i & 1-\lambda \end{bmatrix}$$
$$= (1-\lambda)(1-\lambda) - i\cdot i = \lambda^2 - 2\lambda + 2.$$

Using the quadratic formula, this quadratic has roots

$$\lambda = \frac{2 \pm \sqrt{2^2 - 4\cdot 2}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Thus the eigenvalues are $1 + i$ and $1 - i$.

To find the eigenspace corresponding to $\lambda = 1 + i$, we must find the null space of $A - (1 + i)I$.

$$\left[A - (1+i)I \mid 0\right] = \left[\begin{array}{cc|c} -i & i & 0 \\ i & -i & 0 \end{array}\right] \xrightarrow{iR_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ i & -i & 0 \end{array}\right] \xrightarrow{R_2-iR_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} t \\ t \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda = 1 - i$, we must find the null space of $A - (1 - i)I$.

$$\left[A - (1-i)I \mid 0\right] = \left[\begin{array}{cc|c} i & i & 0 \\ i & i & 0 \end{array}\right] \xrightarrow{-iR_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ i & i & 0 \end{array}\right] \xrightarrow{R_2-iR_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} -t \\ t \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

30. Using the method of Example 4.5, we wish to find solutions to the equation

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 0 & 1+i \\ 1-i & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} -\lambda & 1+i \\ 1-i & 1-\lambda \end{bmatrix}$$
$$= (-\lambda)(1-\lambda) - (1+i)(1-i) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

Thus the eigenvalues are 2 and -1.

To find the eigenspace corresponding to $\lambda = 2$, we must find the null space of $A - 2I$.

$$\begin{bmatrix} A - 2I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -2 & 1+i & 0 \\ 1-i & -1 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2}(1+i) & 0 \\ 1-i & -1 & 0 \end{array}\right] \xrightarrow{R_2-(1-i)R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2}(1+i) & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} \frac{1}{2}(1+i)t \\ t \end{bmatrix}, \text{ or, clearing fractions, } \begin{bmatrix} (1+i)t \\ 2t \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda = -1$, we must find the null space of $A - (-I) = A + I$.

$$\begin{bmatrix} A + I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1+i & 0 \\ 1-i & 2 & 0 \end{array}\right] \xrightarrow{R_2-(1-i)R_1} \left[\begin{array}{cc|c} 1 & 1+i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix} -(1+i)t \\ t \end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$.

31. Using the method of Example 4.5, we wish to find solutions in $\mathbb{Z}_3$ to the equation

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix}$$
$$= (1-\lambda)(2-\lambda) - 0 \cdot 1 = (\lambda - 1)(\lambda - 2).$$

Thus the eigenvalues are 1 and 2.

32. Using the method of Example 4.5, we wish to find solutions in $\mathbb{Z}_3$ to the equation

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$
$$= (2-\lambda)(2-\lambda) - 1 \cdot 1 = \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 + 2\lambda = \lambda(\lambda + 2) = \lambda(\lambda - 1).$$

Thus the eigenvalues are 0 and 1.

33. Using the method of Example 4.5, we wish to find solutions in $\mathbb{Z}_5$ to the equation

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 3-\lambda & 1 \\ 4 & -\lambda \end{bmatrix}$$
$$= (3-\lambda)(-\lambda) - 1 \cdot 4 = \lambda^2 - 3\lambda - 4 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = (\lambda - 4)^2.$$

Thus the only eigenvalue is 4.

34. Using the method of Example 4.5, we wish to find solutions in $\mathbb{Z}_5$ to the equation

$$\begin{aligned} 0 = \det(A - \lambda I) &= \det\left(\begin{bmatrix} 1 & 4 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 4 \\ 4 & -\lambda \end{bmatrix} \\ &= (1-\lambda)(-\lambda) - 4\cdot 4 = \lambda^2 - \lambda - 16 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = (\lambda-3)^2. \end{aligned}$$

Thus the only eigenvalue is 3.

35. (a) To find the eigenvalues of A, find solutions to the equation

$$\begin{aligned} 0 = \det(A - \lambda I) &= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \\ &= (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A). \end{aligned}$$

(b) Using the quadratic formula to find the roots of this quadratic gives

$$\begin{aligned} \lambda &= \frac{\operatorname{tr}(A) \pm \sqrt{(\operatorname{tr}(A))^2 - 4\det(A)}}{2} \\ &= \frac{a+d \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4bc}}{2} \\ &= \frac{1}{2}\left(a + d \pm \sqrt{a^2 - 2ad + d^2 + 4bc}\right) \\ &= \frac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right). \end{aligned}$$

(c) Let

$$\lambda_1 = \frac{1}{2}\left(a + d + \sqrt{(a-d)^2 + 4bc}\right), \qquad \lambda_2 = \frac{1}{2}\left(a + d - \sqrt{(a-d)^2 + 4bc}\right)$$

be the two eigenvalues. Then

$$\begin{aligned} \lambda_1 + \lambda_2 &= \frac{1}{2}\left(a + d + \sqrt{(a-d)^2 + 4bc}\right) + \frac{1}{2}\left(a + d - \sqrt{(a-d)^2 + 4bc}\right) \\ &= \frac{1}{2}(a + d + a + d) \\ &= a + d = \operatorname{tr}(A), \\ \lambda_1\lambda_2 &= \left(\frac{1}{2}\left(a + d - \sqrt{(a-d)^2 + 4bc}\right)\right)\left(\frac{1}{2}\left(a + d - \sqrt{(a-d)^2 + 4bc}\right)\right) \\ &= \frac{1}{4}\left((a+d)^2 - \left((a-d)^2 + 4bc\right)\right) \\ &= \frac{1}{4}(a^2 + 2ad + d^2 - a^2 + 2ad - d^2 - 4bc) \\ &= \frac{1}{4}(4ad - 4bc) = \det(A). \end{aligned}$$

36. A quadratic equation will have two distinct real roots when its discriminant (the quantity under the square root sign in the quadratic formula) is positive, one real root when it is zero, and no real roots when it is negative. In Exercise 35, the discriminant is $(a-d)^2 + 4bc$.

(a) For A to have to distinct real eigenvalues, we must have $(a-d)^2 + 4bc > 0$.

(b) For A to have a single real eigenvalue, it must be the case that $(a-d)^2 + 4bc = 0$, so that $(a-d)^2 = -4bc$. Note that this implies that either $a = d$ and at least one of b or c is zero, or that $a \neq d$ and either b or c, but not both, is negative.

(c) For A to have no real eigenvalues, we must have $(a-c)^2 + 4bc < 0$.

37. From Exercise 35, since $c = 0$, the eigenvalues are

$$\frac{1}{2}\left(a+d\pm\sqrt{(a-d)^2+4b\cdot 0}\right) = \frac{1}{2}\left(a+d\pm|a-d|\right)$$
$$= \left\{\frac{1}{2}(a+d+a-d),\ \frac{1}{2}(a+d+d-a)\right\} = \{a,d\}.$$

To find the eigenspace corresponding to $\lambda = a$, we must find the null space of $A - aI$.

$$\begin{bmatrix}A-aI \mid 0\end{bmatrix} = \left[\begin{array}{cc|c}0 & b & 0\\ 0 & d-a & 0\end{array}\right].$$

If $b = 0$ and $a = d$, then this is the zero matrix, so that any vector is an eigenvector; a basis for the eigenspace is

$$\left\{\begin{bmatrix}1\\0\end{bmatrix},\ \begin{bmatrix}0\\1\end{bmatrix}\right\}.$$

The reason for this is that under these conditions, the matrix A is a multiple of the identity matrix, so it simply stretches any vector by a factor of $a = d$. Otherwise, either $b \neq 0$ or $d - a \neq 0$, so the above matrix row-reduces to

$$\left[\begin{array}{cc|c}0 & 1 & 0\\ 0 & 0 & 0\end{array}\right]$$

so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix}t\\0\end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix}1\\0\end{bmatrix}$. As a check,

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\cdot 1+b\cdot 0\\ 0\cdot 1+d\cdot 0\end{bmatrix} = \begin{bmatrix}a\\0\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix}.$$

To find the eigenspace corresponding to $\lambda = d$, we must find the null space of $A - dI$.

$$\begin{bmatrix}A-dI \mid 0\end{bmatrix} = \left[\begin{array}{cc|c}a-d & b & 0\\ 0 & 0 & 0\end{array}\right].$$

If $b = 0$ and $a = d$, then this is again the zero matrix, which we dealt with above. Otherwise, either $b \neq 0$ or $d - a \neq 0$, so that eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix}-bt\\(a-d)t\end{bmatrix},\ \text{or}\ \begin{bmatrix}bt\\(d-a)t\end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t = 1$, to get $\begin{bmatrix}b\\d-a\end{bmatrix}$. As a check,

$$A\begin{bmatrix}b\\d-a\end{bmatrix} = \begin{bmatrix}a\cdot b+b\cdot(d-a)\\ 0\cdot b+d\cdot(d-a)\end{bmatrix} = \begin{bmatrix}bd\\d(d-a)\end{bmatrix} = d\begin{bmatrix}b\\d-a\end{bmatrix}.$$

38. Using the method of Example 4.5, we wish to find solutions to the equation

$$0 = \det(A-\lambda I) = \det\left(\begin{bmatrix}a & b\\ -b & a\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) = \det\begin{bmatrix}a-\lambda & b\\ -b & a-\lambda\end{bmatrix}$$
$$= (a-\lambda)(a-\lambda) - 4\cdot(-b) = \lambda^2 - 2a\lambda + (a^2+4b).$$

Using Exercise 35, the eigenvalues are

$$\frac{1}{2}\left(a+a\pm\sqrt{(a-a)^2+4b\cdot(-b)}\right)=a\pm\frac{1}{2}\sqrt{-4b^2}=a\pm bi.$$

To find the eigenspace corresponding to $\lambda=a+bi$, we must find the null space of $A-(a+bi)I$.

$$\left[A-(a+bi)I \mid 0\right]=\left[\begin{array}{cc|c}-bi & b & 0\\ -b & -bi & 0\end{array}\right].$$

If $b=0$ then this is the zero matrix, so that any vector is an eigenvector; a basis for the eigenspace is

$$\left\{\begin{bmatrix}1\\0\end{bmatrix},\ \begin{bmatrix}0\\1\end{bmatrix}\right\}.$$

The reason for this is that under these conditions, the matrix A is a times the identity matrix, so it simply stretches any vector by a factor of a. Otherwise, $b\neq 0$, so we can row-reduce the matrix:

$$\left[\begin{array}{cc|c}-bi & b & 0\\ -b & -bi & 0\end{array}\right]\xrightarrow{\frac{1}{b}iR_1}\left[\begin{array}{cc|c}1 & i & 0\\ -b & -bi & 0\end{array}\right]\xrightarrow{R_2+bR_1}\left[\begin{array}{cc|c}1 & i & 0\\ 0 & 0 & 0\end{array}\right].$$

Thus eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix}-it\\t\end{bmatrix}=\begin{bmatrix}t\\it\end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t=1$, to get $\begin{bmatrix}1\\i\end{bmatrix}$. As a check,

$$A\begin{bmatrix}1\\i\end{bmatrix}=\begin{bmatrix}a+bi\\-b+ai\end{bmatrix}=(a+bi)\begin{bmatrix}1\\i\end{bmatrix}.$$

To find the eigenspace corresponding to $\lambda=a-bi$, we must find the null space of $A-(a-bi)I$.

$$\left[A-(a-bi)I \mid 0\right]=\left[\begin{array}{cc|c}bi & b & 0\\ -b & bi & 0\end{array}\right].$$

If $b=0$ then this is the zero matrix, which we dealt with above. Otherwise, $b\neq 0$, so we can row-reduce the matrix:

$$\left[\begin{array}{cc|c}bi & b & 0\\ -b & bi & 0\end{array}\right]\xrightarrow{-\frac{1}{b}iR_1}\left[\begin{array}{cc|c}1 & -i & 0\\ -b & bi & 0\end{array}\right]\xrightarrow{R_2+bR_1}\left[\begin{array}{cc|c}1 & -i & 0\\ 0 & 0 & 0\end{array}\right].$$

Thus eigenvectors corresponding to this eigenvalue have the form

$$\begin{bmatrix}it\\t\end{bmatrix}.$$

A basis for this eigenspace is given by choosing for example $t=1$, to get $\begin{bmatrix}i\\1\end{bmatrix}$. As a check,

$$A\begin{bmatrix}i\\1\end{bmatrix}=\begin{bmatrix}ai+b\\-bi+a\end{bmatrix}=\begin{bmatrix}b+ai\\a-bi\end{bmatrix}=(a-bi)\begin{bmatrix}i\\1\end{bmatrix}.$$

4.2 Determinants

1. Expanding along the first row gives

$$\begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 5 & 1 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} = 1(1\cdot 2 - 1\cdot 1) - 0 + 3(5\cdot 1 - 1\cdot 0) = 16.$$

Expanding along the first column gives

$$\begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 5\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 1(1\cdot 2 - 1\cdot 1) - 5(0\cdot 2 - 3\cdot 1) + 0 = 16.$$

2. Expanding along the first row gives

$$\begin{aligned} \begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & 3 & 0 \end{vmatrix} &= 0\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & -2 \\ -1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix} \\ &= 0 - 1(2\cdot 0 - (-2)\cdot(-1)) - 1(2\cdot 3 - 3\cdot(-1)) = -7. \end{aligned}$$

Expanding along the first column gives

$$\begin{aligned} \begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & 3 & 0 \end{vmatrix} &= 0\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} \\ &= 0 - 2(1\cdot 0 - (-1)\cdot 3) - 1(1\cdot(-2) - (-1)\cdot 3) = -7. \end{aligned}$$

3. Expanding along the first row gives

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1(0\cdot(-1) - 1\cdot 1) + 1(-1\cdot(-1) - 1\cdot 0) + 0 = 0.$$

Expanding along the first column gives

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} + 0\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1(0\cdot(-1) - 1\cdot 1) + 1(-1\cdot(-1) - 0\cdot 1) + 0 = 0.$$

4. Expanding along the first row gives

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1(0\cdot 1 - 1\cdot 1) - 1(1\cdot 1 - 1\cdot 0) + 0 = -2.$$

Expanding along the first column gives

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1(0\cdot 1 - 1\cdot 1) - 1(1\cdot 1 - 1\cdot 0) + 0 = -2.$$

5. Expanding along the first row gives

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 1(3\cdot 2 - 1\cdot 1) - 2(2\cdot 2 - 1\cdot 3) + 3(2\cdot 1 - 3\cdot 3) = -18.$$

Expanding along the first column gives

$$\begin{vmatrix}1&2&3\\2&3&1\\3&1&2\end{vmatrix} = 1\begin{vmatrix}3&1\\1&2\end{vmatrix} - 2\begin{vmatrix}2&1\\3&2\end{vmatrix} + 3\begin{vmatrix}2&3\\3&1\end{vmatrix} = 1(3\cdot 2 - 1\cdot 1) - 2(2\cdot 2 - 1\cdot 3) + 3(2\cdot 1 - 3\cdot 3) = -18.$$

6. Expanding along the first row gives

$$\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix} = 1\begin{vmatrix}5&6\\8&9\end{vmatrix} - 2\begin{vmatrix}4&6\\7&9\end{vmatrix} + 3\begin{vmatrix}4&5\\7&8\end{vmatrix} = 1(5\cdot 9 - 6\cdot 8) - 2(4\cdot 9 - 6\cdot 7) + 3(4\cdot 8 - 5\cdot 7) = 0.$$

Expanding along the first column gives

$$\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix} = 1\begin{vmatrix}5&6\\8&9\end{vmatrix} - 4\begin{vmatrix}2&3\\8&9\end{vmatrix} + 7\begin{vmatrix}2&3\\5&6\end{vmatrix} = 1(5\cdot 9 - 6\cdot 8) - 4(2\cdot 9 - 3\cdot 8) + 7(2\cdot 6 - 3\cdot 5) = 0.$$

7. Noting the pair of zeros in the third row, use cofactor expansion along the third row to get

$$\begin{vmatrix}5&2&2\\-1&1&2\\3&0&0\end{vmatrix} = 3\begin{vmatrix}2&2\\1&2\end{vmatrix} - 0\begin{vmatrix}5&2\\-1&2\end{vmatrix} + 0\begin{vmatrix}5&2\\-1&1\end{vmatrix} = 3(2\cdot 2 - 2\cdot 1) - 0 + 0 = 6.$$

8. Noting the zero in the second row, use cofactor expansion along the second row to get

$$\begin{vmatrix}1&1&-1\\2&0&1\\3&-2&1\end{vmatrix} = -2\begin{vmatrix}1&-1\\-2&1\end{vmatrix} - 1\begin{vmatrix}1&1\\3&-2\end{vmatrix} = -2(1\cdot 1 - (-1)\cdot(-2)) - 1(1\cdot(-2) - 1\cdot 3) = 7.$$

9. Noting the zero in the bottom right, use the third row, since the multipliers there are smaller:

$$\begin{vmatrix}-4&1&3\\2&-2&4\\1&-1&0\end{vmatrix} = 1\begin{vmatrix}1&3\\-2&4\end{vmatrix} - (-1)\begin{vmatrix}-4&3\\2&4\end{vmatrix} = 1(1\cdot 4 - 3\cdot(-2)) + 1(-4\cdot 4 - 3\cdot 2) = -12.$$

10. Noting that the first column has only one nonzero entry, we expand along the first column:

$$\begin{vmatrix}\cos\theta&\sin\theta&\tan\theta\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{vmatrix} = \cos\theta(\cos\theta\cdot\cos\theta - (-\sin\theta)\cdot\sin\theta) = \cos\theta(\cos^2\theta + \sin^2\theta) = \cos\theta.$$

11. Use the first column:

$$\begin{vmatrix}a&b&0\\0&a&b\\a&0&b\end{vmatrix} = a\begin{vmatrix}a&b\\0&b\end{vmatrix} + a\begin{vmatrix}b&0\\a&b\end{vmatrix} = a(a\cdot b - b\cdot 0) + a(b\cdot b - 0\cdot a) = a^2b + ab^2.$$

12. Use the first row (other choices are equally easy):

$$\begin{vmatrix}0&a&0\\b&c&d\\0&e&0\end{vmatrix} = -a\begin{vmatrix}b&d\\0&0\end{vmatrix} = -a(b\cdot 0 - d\cdot 0) = 0.$$

13. Use the third row, since it contains only one nonzero entry:

$$\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{vmatrix} = (-1)^{3+2} \cdot 1 \begin{vmatrix} 1 & 0 & 3 \\ 2 & 2 & 6 \\ 1 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 3 \\ 2 & 2 & 6 \\ 1 & 2 & 1 \end{vmatrix}.$$

To compute this determinant, use cofactor expansion along the first row:

$$-\begin{vmatrix} 1 & 0 & 3 \\ 2 & 2 & 6 \\ 1 & 2 & 1 \end{vmatrix} = -1\begin{vmatrix} 2 & 6 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -1(2 \cdot 1 - 6 \cdot 2) - 3(2 \cdot 2 - 2 \cdot 1) = 4.$$

14. Noting that the second column has only one nonzero entry, we expand along that column. The nonzero entry is in row 3, column 2, so it gets a minus sign, since $(-1)^{3+2} = -1$:

$$\begin{vmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{vmatrix} = -(-1)\begin{vmatrix} 2 & 3 & -1 \\ 1 & 2 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 2 & 2 \\ 2 & 1 & -3 \end{vmatrix}.$$

To evaluate the remaining determinant, we use expansion along the first row:

$$\begin{aligned} \begin{vmatrix} 2 & 3 & -1 \\ 1 & 2 & 2 \\ 2 & 1 & -3 \end{vmatrix} &= 2\begin{vmatrix} 2 & 2 \\ 1 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ &= 2(2 \cdot (-3) - 2 \cdot 1) - 3(1 \cdot (-3) - 2 \cdot 2) - 1(1 \cdot 1 - 2 \cdot 2) = 8. \end{aligned}$$

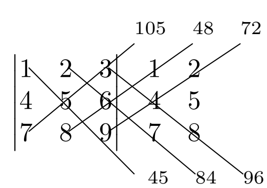
15. Expand along the first row:

$$\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & d & e & f \\ g & h & i & j \end{vmatrix} = -a\begin{vmatrix} 0 & 0 & b \\ 0 & d & e \\ g & h & i \end{vmatrix}.$$

Now expand along the first row again:

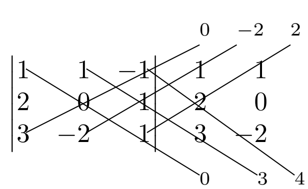
$$-a\begin{vmatrix} 0 & 0 & b \\ 0 & d & e \\ g & h & i \end{vmatrix} = -ab\begin{vmatrix} 0 & d \\ g & h \end{vmatrix} = -ab(0 \cdot h - dg) = abdg.$$

16. Using the method of Example 4.9, we get



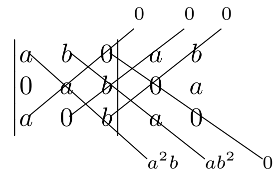
so that the determinant is $-(105 + 48 + 72) + (45 + 84 + 96) = 0$.

17. Using the method of Example 4.9,



so that the determinant is $-(0 + (-2) + 2) + (0 + 3 + 4) = 7$.

18. Using the method of Example 4.9, we get



so that the determinant is $-(0+0+0)+(a^2b+ab^2+0)=a^2b+ab^2$.

19. Consider (2) in the text. The three upward-pointing diagonals have product $a_{31}a_{22}a_{13}$, $a_{32}a_{23}a_{11}$, and $a_{33}a_{21}a_{12}$, while the downward-pointing diagonals have product $a_{11}a_{22}a_{33}$, $a_{12}a_{23}a_{31}$, and $a_{13}a_{21}a_{32}$, so the determinant by that method is

$$\det A = -(a_{31}a_{22}a_{13}+a_{32}a_{23}a_{11}+a_{33}a_{21}a_{12})+(a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}).$$

Expanding along the first row gives

$$\begin{aligned}\det A &= a_{11}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}\\ &= a_{11}a_{22}a_{33}-a_{11}a_{32}a_{23}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31},\end{aligned}$$

which, after rearranging, is the same as the first answer.

20. Consider the 2×2 matrix $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$. Then

$$C_{11}=(-1)^{1+1}\det[a_{22}]=a_{22}, \qquad C_{12}=(-1)^{1+2}\det[a_{21}]=-a_{21},$$

so that from definition (4)

$$\det A=\sum_{j=1}^{2}a_{1j}C_{1j}=a_{11}C_{11}+a_{12}C_{12}=a_{11}a_{12}+a_{12}\cdot(-a_{21})=a_{11}a_{22}-a_{12}a_{21},$$

which agrees with the definition of a 2×2 determinant.

21. If A is lower triangular, then A^T is upper triangular. Since by Theorem 4.10 $\det A=\det A^T$, if we prove the result for upper triangular matrices, we will have proven it for lower triangular matrices as well. So assume A is upper triangular. The basis for the induction is the case $n=1$, where $A=[a_{11}]$ and $\det A=a_{11}$, which is clearly the "product" of the entries on the diagonal. Now assume that the statement holds for $n\times n$ upper triangular matrices, and let A be an $(n+1)\times(n+1)$ upper triangular matrix. Using cofactor expansion along the last row, we get

$$\det A=(-1)^{(n+1)+(n+1)}a_{n+1,n+1}A_{n+1,n+1}=a_{n+1,n+1}A_{n+1,n+1}.$$

But $A_{n+1,n+1}$ is an $n\times n$ upper triangular matrix with diagonal entries $a_1, a_2, \ldots, a_n$, so applying the inductive hypothesis we get

$$\det A=a_{n+1,n+1}A_{n+1,n+1}=a_{n+1,n+1}(a_1a_2\cdots a_n)=a_1a_2\cdots a_na_{n+1},$$

proving the result for $(n+1)\times(n+1)$ matrices. So by induction, the statement is true for all $n\ge 1$.

22. Row-reducing the matrix in Exercise 1, and modifying the determinant as required by Theorem 4.3, gives

$$\begin{vmatrix}1 & 0 & 3\\ 5 & 1 & 1\\ 0 & 1 & 2\end{vmatrix}\xrightarrow{R_2-5R_1}\begin{vmatrix}1 & 0 & 3\\ 0 & 1 & -14\\ 0 & 1 & 2\end{vmatrix}\xrightarrow{R_3-R_2}\begin{vmatrix}1 & 0 & 3\\ 0 & 1 & -14\\ 0 & 0 & 16\end{vmatrix}$$

Note that we did not introduce any factors in front of the determinant since all of the row operations were of the form R_i+kR_j, which does not change the determinant. The final determinant is easy to evaluate since the matrix is upper triangular; the determinant of the original matrix is $1\cdot 1\cdot 16=16$.

23. Row-reducing the matrix in Exercise 9, and modifying the determinant as required by Theorem 4.3, gives

$$\begin{vmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} - \begin{vmatrix} 1 & -1 & 0 \\ 2 & -2 & 4 \\ -4 & 1 & 3 \end{vmatrix} \xrightarrow[R_3+4R_1]{R_2-2R_1} - \begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 4 \\ 0 & -3 & 3 \end{vmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{vmatrix} 1 & -1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 4 \end{vmatrix}.$$

There were two row interchanges, each of which introduced a factor of -1 in the determinant. The other row operations were of the form $R_i + kR_j$, which does not change the determinant. The final determinant is easy to evaluate since the matrix is upper triangular; the determinant of the original matrix is $1 \cdot (-3) \cdot 4 = -12$.

24. Row-reducing the matrix in Exercise 13, and modifying the determinant as required by Theorem 4.3, gives

$$\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{vmatrix} \xrightarrow[R_4-R_1]{R_2-2R_1} \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 7 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 2 & -2 \end{vmatrix} \xrightarrow{R_2 \leftrightarrow R_3} - \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 7 & 2 & 0 \\ 0 & 5 & 2 & -2 \end{vmatrix}$$

$$\xrightarrow[R_4-5R_2]{R_3-7R_2} - \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & -2 \end{vmatrix} \xrightarrow{R_4-R_3} - \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix}.$$

There was one row interchange, which introduced a factor of -1 in the determinant. The other row operations were of the form $R_i + kR_j$, which does not change the determinant. The final determinant is easy to evaluate since the matrix is upper triangular; the determinant of the original matrix is $-(1 \cdot 1 \cdot 2 \cdot (-2)) = 4$.

25. Row-reducing the matrix in Exercise 14, and modifying the determinant as required by Theorem 4.3, gives

$$\begin{vmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} - \begin{vmatrix} 1 & 0 & 2 & 2 \\ 2 & 0 & 3 & -1 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{vmatrix} \xrightarrow[R_4-2R_1]{R_2-2R_1} - \begin{vmatrix} 1 & 0 & 2 & 2 \\ 0 & 0 & -1 & -5 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -3 & -7 \end{vmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{vmatrix} 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & -3 & -7 \end{vmatrix} \xrightarrow{R_4-3R_3} \begin{vmatrix} 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 8 \end{vmatrix}.$$

There were two row interchanges, each of which introduced a factor of -1 in the determinant. The other row operations were of the form $R_i + kR_j$, which does not change the determinant. The final determinant is easy to evaluate since the matrix is upper triangular; the determinant of the original matrix is $1 \cdot (-1) \cdot (-1) \cdot 8 = 8$.

26. Since the third row is a multiple of the first row, the determinant is zero since we can use the row operation $R_3 - 2R_1$ to zero the third row, and doing so does not change the determinant.

27. This matrix is upper triangular, so its determinant is the product of the diagonal entries, which is $3 \cdot (-2) \cdot 4 = -24$.

28. If we interchange the first and third rows, the result is an upper triangular matrix with diagonal entries 3, 5, and 1. Since interchanging the rows negates the determinant, the determinant of the original matrix is $-(3 \cdot 5 \cdot 1) = -15$.

29. The third column is -2 times the first column, so the columns are not linearly independent and thus the matrix does not have rank 3, so is not invertible. By Theorem 4.6, its determinant is zero.

30. Since the third row is the sum of the first and second rows, the rows are not linearly independent and thus the matrix does not have rank 3, so is not invertible. By Theorem 4.6, its determinant is zero.

31. Since the first column is the sum of the second and third columns, the columns are not linearly independent and thus the matrix does not have rank 3, so is not invertible. By Theorem 4.6, its determinant is zero.

32. Interchanging the second and third rows gives the identity matrix, which has determinant 1. Since interchanging the rows introduces a minus sign, the determinant of the original matrix is -1.

33. Interchange the first and second rows, and also the third and fourth rows, to give a diagonal matrix with diagonal entries -3, 2, 1, and 4. Performing two row interchanges does not change the determinant, so the determinant of the original matrix is $-3 \cdot 2 \cdot 1 \cdot 4 = -24$.

34. The sum of the first and second rows is equal to the sum of the third and fourth rows, so the rows are not linearly independent and thus the matrix does not have rank 4, so is not invertible. By Theorem 4.6, its determinant is zero.

35. This matrix results from the original matrix by multiplying the first row by 2, so its determinant is $2 \cdot 4 = 8$.

36. This matrix results from the original matrix by multiplying the first column by 2, the second column by $\frac{1}{3}$, and the third column by -1. Thus the determinant of this matrix is $2 \cdot \frac{1}{3} \cdot (-1) = -\frac{2}{3}$ times the original determinant, so its value is $-\frac{2}{3} \cdot 4 = -\frac{8}{3}$.

37. This matrix results from the original matrix by interchanging the first and second rows. This multiplies the determinant by -1, so the determinant of this matrix is -4.

38. This matrix results from the original matrix by subtracting the third column from the first. This operation does not change the determinant, so the determinant of this matrix is also 4.

39. This matrix results from the original matrix by exchanging the first and third columns (which multiplies the determinant by -1) and then multiplying the first column by 2 (which multiplies the determinant by 2). So the determinant of this matrix is $2 \cdot (-1) \cdot 4 = -8$.

40. To get from the original matrix to this one, row 2 was multiplied by 3, and then twice row 3 was added to each of rows 1 and 2. The first of these operations multiplies the determinant by 3, while the additions do not change the determinant. Thus the determinant of this matrix is $3 \cdot 4 = 12$.

41. Suppose first that A has a zero row, say the i^{th} row is zero. Using cofactor expansion along that row gives

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} 0 C_{ij} = 0,$$

so the determinant is zero. Similarly, if A has a zero column, say the j^{th} column, then using cofactor expansion along that column gives

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} 0 C_{ij} = 0,$$

and again the determinant is zero. Alternatively, the column case can be proved from the row case by noting that if A has a zero column, then A^T has a zero row, and $\det A = \det A^T$ by Theorem 4.10.

42. Dealing first with rows, we want to show that if $A \xrightarrow{R_i+kR_j} B$ where $i \neq j$ then $\det B = \det A$. Let $\mathbf{A}_i$ be the i^{th} row of $\mathbf{A}$. Then

$$B = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_i + k\mathbf{A}_j \\ \vdots \\ \mathbf{A}_n \end{bmatrix}; \quad \text{let} \quad C = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ k\mathbf{A}_j \\ \vdots \\ \mathbf{A}_n \end{bmatrix}.$$

Then A, B, and C are identical except that the i^{th} row of B is the sum of the i^{th} rows of A and C, so that by part (a), $\det B = \det A + \det C$. However, since $i \neq j$, the i^{th} row of C, which is $k\mathbf{A}_j$, is a multiple of the j^{th} row, which is $\mathbf{A}_j$. By Theorem 4.3(d), multiplying the j^{th} row by k multiplies the determinant of C by k. However, the result is a matrix with two identical rows; both the i^{th} and j^{th} rows are $k\mathbf{A}_j$. So $\det C = 0$ by Theorem 4.3(c). Hence $\det B = \det A$ as desired.

If instead $A \xrightarrow{C_i+kC_j} B$ where $i \neq j$, then $A^T \xrightarrow{R_i+kR_j} B^T$, so that by the first part and Theorem 4.10

$$\det A = \det A^T = \det B^T = \det B.$$

43. If E interchanges two rows, then $\det E = -1$ by Theorem 4.4. Also, EB is the same as B but with two rows interchanged, so by Theorem 4.3(b), $\det(EB) = -\det B = \det(E)\det(B)$. Next, if E results from multiplying a row by a constant k, then $\det E = k$ by Theorem 4.4. But EB is the same as B except that one row has been multiplied by k, so that by Theorem 4.3(d), $\det(EB) = k\det(B) = \det(E)\det(B)$. Finally, if E results from adding a multiple of one row to another row, then $\det E = 1$ by Theorem 4.4. But in this case EB is obtained from B by adding a multiple of one row to another, so by Theorem 4.3(f), $\det(EB) = \det(B) = \det(E)\det(B)$.

44. Let the row vectors of A be $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$. Then the row vectors of kA are $k\mathbf{A}_1, k\mathbf{A}_2, \ldots, k\mathbf{A}_n$. Using Theorem 4.3(d) n times,

$$\det(kA) = \begin{vmatrix} k\mathbf{A}_1 \\ k\mathbf{A}_2 \\ \vdots \\ k\mathbf{A}_n \end{vmatrix} = k \begin{vmatrix} \mathbf{A}_1 \\ k\mathbf{A}_2 \\ \vdots \\ k\mathbf{A}_n \end{vmatrix} = k^2 \begin{vmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ k\mathbf{A}_n \end{vmatrix} = \cdots = k^n \begin{vmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_n \end{vmatrix} = k^n \det A.$$

45. By Theorem 4.6, A is invertible if and only if $\det A \neq 0$. To find $\det A$, expand along the first column:

$$\begin{aligned} \det A &= k \begin{vmatrix} k+1 & 1 \\ -8 & k-1 \end{vmatrix} + k \begin{vmatrix} -k & 3 \\ k+1 & 1 \end{vmatrix} \\ &= k((k+1)(k-1) - 1 \cdot (-8)) + k(-k \cdot 1 - 3(k+1)) = k^3 - 4k^2 + 4k = k(k-2)^2. \end{aligned}$$

Thus $\det A = 0$ when $k = 0$ or $k = 2$, so A is invertible for all k except $k = 0$ and $k = 2$.

46. By Theorem 4.6, A is invertible if and only if $\det A \neq 0$. To find $\det A$, expand along the first row:

$$\begin{aligned} \begin{vmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{vmatrix} &= k \begin{vmatrix} 2 & k \\ k & k \end{vmatrix} - k \begin{vmatrix} k^2 & k \\ 0 & k \end{vmatrix} \\ &= k(2k - k^2) - k(k^3 - 0) = k(2k - k^2 - k^3) = k^2(2 - k - k^2) = -k^2(k+2)(k-1). \end{aligned}$$

Since the matrix is invertible precisely when its determinant is nonzero, and since the determinant is zero when $k = 0$, $k = 1$, or $k = -2$, we see that the matrix is invertible for all values of k except for 0, 1, and -2.

47. By Theorem 4.8, $\det(AB) = \det(A)\det(B) = 3 \cdot (-2) = -6$.

48. By Theorem 4.8, $\det(A^2) = \det(A)\det(A) = 3 \cdot 3 = 9$.

49. By Theorem 4.9, $\det(B^{-1}) = \frac{1}{\det(B)}$, so that by Theorem 4.8

$$\det(B^{-1}A) = \det(B^{-1})\det(A) = \frac{\det(A)}{\det(B)} = -\frac{3}{2}.$$

50. By Theorem 4.7, $\det(2A) = 2^n \det(A) = 3 \cdot 2^n$.

51. By Theorem 4.10, $\det(B^T) = \det(B)$, so that by Theorem 4.7, $\det(3B^T) = 3^n \det(B^T) = -2 \cdot 3^n$.

52. By Theorem 4.10, $\det(A^T) = \det(A)$, so that by Theorem 4.8,

$$\det(AA^T) = \det(A)\det(A^T) = \det(A)^2 = 9.$$

53. By Theorem 4.8, $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.

54. If B is invertible, then $\det(B) \neq 0$ and $\det(B^{-1}) = \frac{1}{\det(B)}$, so that using Theorem 4.8

$$\begin{aligned}\det(B^{-1}AB) &= \det(B^{-1}(AB)) = \det(B^{-1})\det(AB)\\ &= \det(B^{-1})\det(A)\det(B) = \frac{1}{\det(B)}\det(A)\det(B) = \det(A).\end{aligned}$$

55. If $A^2 = A$, then $\det(A^2) = \det(A)$. But $\det(A^2) = \det(A)\det(A) = \det(A)^2$, so that $\det(A) = \det(A)^2$. Thus $\det(A)$ is either 0 or 1.

56. We first show that Theorem 4.8 extends to prove that if $m \geq 1$, then $\det(A^m) = \det(A)^m$. The case $m = 1$ is obvious, and $m = 2$ is Theorem 4.8. Now assume the statement is true for $m = k$. Then using Theorem 4.8 and the truth of the statement for $m = k$,

$$\det(A^{k+1}) = \det(AA^k) = \det(A)\det(A^k) = \det(A)\det(A)^k = \det(A)^{k+1}.$$

So the statement holds for all $m \geq 1$. Now, if $A^m = O$, then $\det(A^m) = \det(A)^m = \det(O) = 0$, so that $\det(A) = 0$.

57. The coefficient matrix and constant vector are

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

To apply Cramer's Rule, we compute

$$\begin{aligned}\det A &= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2\\ \det A_1(\mathbf{b}) &= \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3\\ \det A_2(\mathbf{b}) &= \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1.\end{aligned}$$

Then by Cramer's Rule,

$$x = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{3}{2}, \quad y = \frac{\det A_2(\mathbf{b})}{\det A} = -\frac{1}{2}.$$

58. The coefficient matrix and constant vector are

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

To apply Cramer's Rule, we compute

$$\det A = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 7$$

$$\det A_1(\mathbf{b}) = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} = 14$$

$$\det A_2(\mathbf{b}) = \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = -7.$$

Then by Cramer's Rule,

$$x = \frac{\det A_1(\mathbf{b})}{\det A} = 2, \quad y = \frac{\det A_2(\mathbf{b})}{\det A} = -1.$$

59. The coefficient matrix and constant vector are

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

To apply Cramer's Rule, we compute

$$\det A = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cdot 1 \cdot 1 = 2$$

$$\det A_1(\mathbf{b}) = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

$$\det A_2(\mathbf{b}) = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$\det A_3(\mathbf{b}) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 2.$$

Then by Cramer's Rule,

$$x = \frac{\det A_1(\mathbf{b})}{\det A} = -1, \quad y = \frac{\det A_2(\mathbf{b})}{\det A} = 0, \quad z = \frac{\det A_3(\mathbf{b})}{\det A} = 1.$$

60. The coefficient matrix and constant vector are

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

To apply Cramer's Rule, we compute

$$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 4$$

$$\det A_1(\mathbf{b}) = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix} = 3\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 9$$

$$\det A_2(\mathbf{b}) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -3$$

$$\det A_3(\mathbf{b}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2.$$

Then by Cramer's Rule,

$$x = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{9}{4}, \quad y = \frac{\det A_2(\mathbf{b})}{\det A} = -\frac{3}{4}, \quad z = \frac{\det A_3(\mathbf{b})}{\det A} = \frac{1}{2}.$$

61. We have $\det A = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$. The four cofactors are

$$C_{11} = (-1)^{1+1} \cdot (-1) = -1, \quad C_{12} = (-1)^{1+2} \cdot 1 = -1,$$
$$C_{21} = (-1)^{2+1} \cdot 1 = -1, \quad C_{22} = (-1)^{2+2} \cdot 1 = 1.$$

Then

$$\operatorname{adj} A = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \text{ so } A^{-1} = \frac{1}{\det A} \operatorname{adj} A = -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

62. We have $\det A = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 7$. The four cofactors are

$$C_{11} = (-1)^{1+1} \cdot 3 = 3, \quad C_{12} = (-1)^{1+2} \cdot 1 = -1,$$
$$C_{21} = (-1)^{2+1} \cdot (-1) = 1, \quad C_{22} = (-1)^{2+2} \cdot 2 = 2.$$

Then

$$\operatorname{adj} A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \text{ so } A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}.$$

63. We have $\det A = 2$ from Exercise 59. The nine cofactors are

$$C_{11} = (-1)^{1+1} \cdot 1 = 1, \quad C_{12} = (-1)^{1+2} \cdot 0 = 0, C_{13} = (-1)^{1+3} \cdot 0 = 0,$$
$$C_{21} = (-1)^{2+1} \cdot 1 = -1, \quad C_{22} = (-1)^{2+2} \cdot 2 = 2, C_{23} = (-1)^{2+3} \cdot 0 = 0,$$
$$C_{31} = (-1)^{3+1} \cdot (-2) = -2, \quad C_{32} = (-1)^{3+2} \cdot 2 = -2, C_{33} = (-1)^{3+3} \cdot 2 = 2.$$

Then

$$\operatorname{adj} A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & -2 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix},$$

so

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{2} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

64. We have $\det A = 4$ from Exercise 60. The nine cofactors are

$$C_{11} = (-1)^{1+1} \cdot 1 = 1, \quad C_{12} = (-1)^{1+2} \cdot (-1) = 1, C_{13} = (-1)^{1+3} \cdot (-2) = -2,$$
$$C_{21} = (-1)^{2+1} \cdot (-1) = 1, \quad C_{22} = (-1)^{2+2} \cdot 1 = 1, C_{23} = (-1)^{2+3} \cdot (-2) = 2,$$
$$C_{31} = (-1)^{3+1} \cdot 2 = 2, \quad C_{32} = (-1)^{3+2} \cdot 2 = -2, C_{33} = (-1)^{3+3} \cdot 0 = 0.$$

Then

$$\operatorname{adj} A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ 2 & -2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ -2 & 2 & 0 \end{bmatrix},$$

so

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{4} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

65. First, since $A^{-1}=\frac{1}{\det A}\operatorname{adj} A$, we have $\operatorname{adj} A=(\det A)A^{-1}$. Then

$$(\operatorname{adj} A)\cdot\frac{1}{\det A}A=\left((\det A)A^{-1}\right)\cdot\frac{1}{\det A}A=\det A\cdot\frac{1}{\det A}A^{-1}A=I.$$

Thus $\operatorname{adj} A$ times $\frac{1}{\det A}A$ is the identity, so that $\operatorname{adj} A$ is invertible and

$$(\operatorname{adj} A)^{-1}=\frac{1}{\det A}A.$$

This proves the first equality. Finally, substitute A^{-1} for A in the equation $\operatorname{adj} A=(\det A)A^{-1}$; this gives $\operatorname{adj}(A^{-1})=(\det A^{-1})A=\frac{1}{\det A}A$, proving the second equality.

66. By Exercise 65, $\det(\operatorname{adj} A)=\det((\det A)A^{-1})$; by Theorem 4.7, this is equal to $(\det A)^n\det(A^{-1})$. But $\det(A^{-1})=\frac{1}{\det A}$, so that $\det(\operatorname{adj} A)=(\det A)^n\frac{1}{\det A}=(\det A)^{n-1}$.

67. Note that exchanging R_{i-1} and R_i results in a matrix in which row i has been moved "up" one row and all other rows remain the same. Refer to the diagram at the end of the solution when reading what follows. Starting with $R_s\leftrightarrow R_{s-1}$, perform $s-r$ adjacent interchanges, each one moving the row that started as row s up one row. At the end of these interchanges, row s will have been moved "up" to row r, and all other rows will remain in the same order. The original row r is therefore at row $r+1$. We wish to move that row down to row s. Note that exchanging R_i and R_{i+1} results in a matrix in which row i has been moved "down" one row and all other rows remain the same. So starting with $R_{r+1}\leftrightarrow R_{r+2}$, perform $s-r-1$ adjacent interchanges, each one moving the row that started as row $r+1$ "down" one row. At the end of these interchanges, row $r+1$ will have been moved down $s-r-1$ rows, so it will be at row $s-r-1+r+1=s$, and all other rows will be as they were. Now, row $r+1$ originally started life as row r, so the end result is that row s has been moved to row r, and row r to row s. We used $s-r$ adjacent interchanges to move row s, and another $s-r-1$ adjacent interchanges to move row r, for a total of $2(s-r)-1$ adjacent interchanges.

$$\text{Original: }\begin{bmatrix}\mathbf{R}_1\\ \mathbf{R}_2\\ \vdots\\ \mathbf{R}_{r-1}\\ \mathbf{R}_r\\ \mathbf{R}_{r+1}\\ \vdots\\ \mathbf{R}_{s-1}\\ \mathbf{R}_s\\ \mathbf{R}_{s+1}\\ \vdots\\ \mathbf{R}_n\end{bmatrix}$$

$$\text{After initial } s-r \text{ exchanges: }\begin{bmatrix}\mathbf{R}_1\\ \mathbf{R}_2\\ \vdots\\ \mathbf{R}_{r-1}\\ \mathbf{R}_s\\ \mathbf{R}_r\\ \mathbf{R}_{r+1}\\ \vdots\\ \mathbf{R}_{s-1}\\ \mathbf{R}_{s+1}\\ \vdots\\ \mathbf{R}_n\end{bmatrix}\quad \text{After next } s-r-1 \text{ exchanges: }\begin{bmatrix}\mathbf{R}_1\\ \mathbf{R}_2\\ \vdots\\ \mathbf{R}_{r-1}\\ \mathbf{R}_s\\ \mathbf{R}_{r+1}\\ \vdots\\ \mathbf{R}_{s-1}\\ \mathbf{R}_r\\ \mathbf{R}_{s+1}\\ \vdots\\ \mathbf{R}_n\end{bmatrix}.$$

68. The proof is very similar to the proof given in the text for the case of expansion along a row. Let B be the matrix obtained by moving column j of matrix A to column 1:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2j} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}$$

$$B = \begin{bmatrix} a_{1j} & a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ a_{2j} & a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{nj} & a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}$$

By the discussion in Exercise 67, it requires $j-1$ adjacent column interchanges to move column j to column 1, so by Lemma 4.14, $\det B = (-1)^{j-1}\det A$, so that $\det A = (-1)^{j-1}\det B$. Then by Lemma 4.13, evaluating $\det B$ using cofactor expansion along the first column gives

$$\det A = (-1)^{j-1}\det B = (-1)^{j-1}\sum_{i=1}^{n} b_{i1}B_{i1} = (-1)^{j-1}\sum_{i=1}^{n} a_{ij}B_{i1},$$

where A_{ij} and B_{ij} are the cofactors of A and B. It remains only to understand how the cofactors of A and B are related. Clearly the matrix formed by removing row i and column j of A is the same matrix as is formed by removing row i and column 1 of B, so the only difference is the sign that we put on the determinant of that matrix. In A, the sign is $(-1)^{i+j}$; in B, it is $(-1)^{i+1}$. Thus

$$B_{i1} = (-1)^{j-1}A_{ij}.$$

Substituting that in the equation for $\det A$ above gives

$$\det A = (-1)^{j-1}\sum_{i=1}^{n} a_{ij}B_{i1} = (-1)^{j-1}\sum_{i=1}^{n}(-1)^{j-1}a_{ij}A_{ij} = (-1)^{2j-2}\sum_{i=1}^{n} a_{ij}A_{ij} = \sum_{i=1}^{n} a_{ij}A_{ij},$$

proving Laplace expansion along columns.

69. Since P and S are both square, suppose P is $n\times n$ and S is $m\times m$. Then O is $m\times n$ and Q is $n\times m$. Proceed by induction on n. If $n=1$, then

$$A = \begin{bmatrix} p_{11} & Q \\ O & S \end{bmatrix},$$

and Q is $1\times n$ while O is $n\times 1$. Compute the determinant by cofactor expansion along the first column. The only nonzero entry in the first column is p_{11}, so that $\det A = p_{11}A_{11}$. Since O has one column and Q one row, removing the first row and column of A leaves only S, so that $A_{11} = \det S$ and thus $\det A = p_{11}\det S = (\det P)\det S)$. This proves the case $n=1$ and provides the basis for the induction.

Now suppose that the statement is true when P is $k\times k$; we want to prove it when P is $(k+1)\times(k+1)$. So let P be a $(k+1)\times(k+1)$ matrix. Again compute $\det A$ using column 1:

$$\det A = \sum_{i=1}^{k+1+m} a_{i1}\left((-1)^{i+1}\det A_{i1}\right).$$

Now, $a_{i1} = p_{i1}$ for $i \le k+1$, and $a_{i1} = 0$ for $i > k+1$, so that

$$\det A = \sum_{i=1}^{k+1} p_{i1}\left((-1)^{i+1}\det A_{i1}\right).$$

A_{i1} is the submatrix formed by removing row i and column 1 of A; since i ranges from 1 to $k+1$, we see that this process removes row i and column 1 of P, but does not change S. So we are left with a matrix

$$A_{i1} = \begin{bmatrix} P_{i1} & Q_i \\ O & S \end{bmatrix}.$$

Now P_{i1} is $k \times k$, so we can apply the inductive hypothesis to get

$$\det A_{i1} = (\det P_{i1})(\det S).$$

Substitute that into the equation for $\det A$ above to get

$$\begin{aligned}
\det A &= \sum_{i=1}^{k+1} p_{i1}\left((-1)^{i+1}\det A_{i1}\right) \\
&= \sum_{i=1}^{k+1} p_{i1}\left((-1)^{i+1}(\det P_{i1})(\det S)\right) \\
&= (\det S)\left(\sum_{i=1}^{k+1} p_{i1}\left((-1)^{i+1}\det P_{i1}\right)\right).
\end{aligned}$$

But the sum is just cofactor expansion of P along the first column, so its value is $\det P$, and finally we get $\det A = (\det S)(\det P) = (\det P)(\det S)$ as desired.

70. (a) There are many examples. For instance, let

$$P = S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Since the second and third rows of A are identical, $\det A = 0$, but

$$(\det P)(\det S) - (\det Q)(\det R) = 1 \cdot 1 - 0 \cdot 0 = 1,$$

and the two are unequal.

(b) We have

$$\begin{aligned}
BA &= \begin{bmatrix} P^{-1} & O \\ -RP^{-1} & I \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} \\
&= \begin{bmatrix} P^{-1}P + OR & P^{-1}Q + OS \\ -RP^{-1}P + IR & -RP^{-1}Q + IS \end{bmatrix} \\
&= \begin{bmatrix} I & P^{-1}Q \\ -R + R & -RP^{-1}Q + S \end{bmatrix} \\
&= \begin{bmatrix} I & P^{-1}Q \\ O & S - RP^{-1}Q \end{bmatrix}.
\end{aligned}$$

It is clear that the proof in Exercise 69 can easily be modified to prove the same result for a block lower triangular matrix. That is, if A, P, and S are square and

$$A = \begin{bmatrix} P & O \\ Q & S \end{bmatrix},$$

then $\det A = (\det P)(\det S)$. Applying this to the matrix B gives

$$\det B = (\det P^{-1})(\det I) = \det P^{-1} = \frac{1}{\det P}.$$

Further, applying Exercise 69 to BA gives

$$\det(BA) = (\det I)(\det(S - RP^{-1}Q)) = \det(-SRP^{-1}Q).$$

So finally

$$\det A = \frac{1}{\det B}\det(BA) = (\det P)(\det(S - RP^{-1}Q)).$$

(c) If $PR = RP$, then

$$\begin{aligned}\det A &= (\det P)(\det(S - RP^{-1}Q)) = \det(P(S - RP^{-1}Q)) \\ &= \det(PS - (PR)P^{-1}Q)) = \det(PS - RPP^{-1}Q) = \det(PS - RQ).\end{aligned}$$

Exploration: Geometric Applications of Determinants

1. **(a)** We have

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 1\\ 3 & -1 & 2\end{vmatrix} = \mathbf{i}\begin{vmatrix}1 & 1\\ -1 & 2\end{vmatrix} - \mathbf{j}\begin{vmatrix}0 & 1\\ 3 & 2\end{vmatrix} + \mathbf{k}\begin{vmatrix}0 & 1\\ 3 & -1\end{vmatrix} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}.$$

(b) We have

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -1 & 2\\ 0 & 1 & 1\end{vmatrix} = \mathbf{i}\begin{vmatrix}-1 & 2\\ 1 & 1\end{vmatrix} - \mathbf{j}\begin{vmatrix}3 & 2\\ 0 & 1\end{vmatrix} + \mathbf{k}\begin{vmatrix}3 & -1\\ 0 & 1\end{vmatrix} = -3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}.$$

(c) We have

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 2 & 3\\ 2 & -4 & -6\end{vmatrix} = \mathbf{i}\begin{vmatrix}2 & 3\\ -4 & -6\end{vmatrix} - \mathbf{j}\begin{vmatrix}-1 & 3\\ 2 & -6\end{vmatrix} + \mathbf{k}\begin{vmatrix}-1 & 2\\ 2 & -4\end{vmatrix} = \mathbf{0}.$$

(d) We have

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 1\\ 1 & 2 & 3\end{vmatrix} = \mathbf{i}\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix} - \mathbf{j}\begin{vmatrix}1 & 1\\ 1 & 3\end{vmatrix} + \mathbf{k}\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

2. First,

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \mathbf{u}\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}\\ &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})\cdot(\mathbf{i}(v_2w_3 - v_3w_2) - \mathbf{j}(v_1w_3 - v_3w_1) + \mathbf{k}(v_1w_2 - v_2w_1))\\ &= (u_1v_2w_3 - u_1v_3w_2) - (u_2v_1w_3 - u_2v_3w_1) + (u_3v_1w_2 - u_3v_2w_1).\end{aligned}$$

Computing the determinant, we get

$$\begin{aligned}\begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix} &= u_1\begin{vmatrix}v_2 & v_3\\ w_2 & w_3\end{vmatrix} - u_2\begin{vmatrix}v_1 & v_3\\ w_1 & w_3\end{vmatrix} + u_3\begin{vmatrix}v_1 & v_2\\ w_1 & w_2\end{vmatrix}\\ &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1)\\ &= (u_1v_2w_3 - u_1v_3w_2) - (u_2v_1w_3 - u_2v_3w_1) + (u_3v_1w_2 - u_3v_2w_1),\end{aligned}$$

and the two expressions are equal.

3. With $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as in the previous exercise,

(a) We have

$$\mathbf{v}\times\mathbf{u}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ v_1 & v_2 & v_3\\ u_1 & u_2 & u_3\end{vmatrix}\xrightarrow{R_2\leftrightarrow R_3} = -\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix} = -(\mathbf{u}\times\mathbf{v}).$$

(b) We have

$$\mathbf{u}\times\mathbf{0}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ 0 & 0 & 0\end{vmatrix}.$$

This matrix has a zero row, so its determinant is zero by Theorem 4.3. Thus $\mathbf{u}\times\mathbf{0}=\mathbf{0}$.

(c) We have

$$\mathbf{u}\times\mathbf{0}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ u_1 & u_2 & u_3\end{vmatrix}.$$

This matrix has two equal rows, so its determinant is zero by Theorem 4.3. Thus $\mathbf{u}\times\mathbf{u}=\mathbf{0}$.

(d) We have by Theorem 4.3

$$\mathbf{u}\times k\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ kv_1 & kv_2 & kv_3\end{vmatrix} = k\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix} = k(\mathbf{u}\times\mathbf{v}).$$

(e) We have by Theorem 4.3

$$\mathbf{u}\times(\mathbf{v}+\mathbf{w})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1+w_1 & v_2+w_2 & v_3+w_3\end{vmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix} + \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ w_1 & w_2 & w_3\end{vmatrix} = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}.$$

(f) From Exercise 2 above,

$$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=\begin{vmatrix}u_1 & u_2 & u_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

$$\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})=\begin{vmatrix}v_1 & v_2 & v_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}.$$

But each of these matrices has two equal rows, so each has zero determinant. Thus

$$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})=0.$$

(g) From Exercise 2 above,

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix},\qquad (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})=\begin{vmatrix}w_1 & w_2 & w_3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}.$$

The second matrix is obtained from the first by exchanging rows 1 and 2 and then exchanging rows 1 and 3. Since each row exchange multiplies the determinant by -1, it remains unchanged so that the two products are equal.

4. The vectors $\mathbf{u}$ and $\mathbf{v}$, considered as vectors in $\mathbb{R}^3$, have their third coordinates equal to zero, so the area of the parallelogram with these edges is

$$\mathcal{A}=\|\mathbf{u}\times\mathbf{v}\| = \left\|\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & 0\\ v_1 & v_2 & 0\end{bmatrix}\right\| = \left\|\mathbf{k}\det\begin{bmatrix}u_1 & u_2\\ v_1 & v_2\end{bmatrix}\right\| = \left|\det\begin{bmatrix}u_1 & u_2\\ v_1 & v_2\end{bmatrix}\right| = \left|\det\begin{bmatrix}u_1 & v_1\\ u_2 & v_2\end{bmatrix}\right|.$$

5. The area of the rectangle is $(a+c)(b+d)$. The small rectangles at top left and bottom right each have area bc. The triangles at top and bottom each have base a and height b, so the area of each is $\frac{1}{2}ab$; altogether they have area ab. Finally, the smaller triangles at left and right each have base d and height c, so the area of each is $\frac{1}{2}cd$; altogether they have area cd. So in total, the area in the big rectangle that is not in the parallelogram is $bc+ab+cd$, so the area of the parallelogram is

$$(a+c)(b+d)-2bc-ab-cd = ab+ad+bc+cd-2bc-ab-cd = ad-bc.$$

For this parallelogram, $\mathbf{u}=\begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix} c \\ d \end{bmatrix}$, and indeed

$$\det \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = ad-bc.$$

As for the absolute value sign, note that the formula $\mathcal{A}=\|\mathbf{u}\times\mathbf{v}\|$ does not depend on which of the two sides we choose for $\mathbf{u}$ and which for $\mathbf{v}$, but the sign of the determinant does. In our particular case, when we choose $\mathbf{u}$ and $\mathbf{v}$ as above, we don't need the absolute value sign, as the area works out properly. It is present in the norm formula to represent the fact that we need not make particular choices for $\mathbf{u}$ and $\mathbf{v}$.

6. **(a)** By Exercise 5, the area is

$$\mathcal{A}=\left|\det\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\right| = |2\cdot 4-(-1)\cdot 3| = 11.$$

(b) By Exercise 5, the area is

$$\mathcal{A}=\left|\det\begin{bmatrix} 3 & 5 \\ 4 & 5 \end{bmatrix}\right| = |3\cdot 5-4\cdot 5| = 5.$$

7. A parallelepiped of height h with base area a has volume ha. This can be seen using calculus; the volume is $\int_0^h a\,dz = [az]_0^h = ah$. Alternatively, it can be seen intuitively, since the parallelepiped consists of "slices" each of area a, and the height of those slices is h. Now, for the parallelepiped in Figure 4.11, the height is clearly $\|\mathbf{u}\|\cos\theta$. The base is a parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$, so its area is $\|\mathbf{v}\times\mathbf{w}\|$. Thus the volume of the parallelepiped is $\|\mathbf{u}\|\,\|\mathbf{v}\times\mathbf{w}\|\cos\theta$. But

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \|\mathbf{u}\|\,\|\mathbf{v}\times\mathbf{w}\|\cos\theta,$$

so that the volume is $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ which, by Exercise 2, is equal to the absolute value of the determinant of the matrix $\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix}$, which is equal to the absolute value of the determinant of the transpose of that matrix, or $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$.

8. Compare Figure 4.12 to Figure 4.11. The base of the tetrahedron is a triangle that is one half of the area of the parallelogram in Figure 4.11. So from the hint, we have

$$\begin{aligned}
\mathcal{V} &= \frac{1}{3}(\text{area of the base})(\text{height}) \\
&= \frac{1}{6}(\text{area of the parallelogram})(\text{height}) \\
&= \frac{1}{6}(\text{volume of the parallelepiped}) \\
&= \frac{1}{6}\left|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\right|.
\end{aligned}$$

9. By problem 4, the area of the parallelogram determined by $A\mathbf{u}$ and $A\mathbf{v}$ is

$$T_A(P) = \left|\det\begin{bmatrix} A\mathbf{u} & A\mathbf{v} \end{bmatrix}\right| = \left|\det(A\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix})\right| = \left|(\det A)(\det\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix})\right| = |\det A|\,(\text{area of } P).$$

10. By Problem 7, the volume of the parallelepiped determined by $A\mathbf{u}$, $A\mathbf{v}$, and $A\mathbf{w}$ is

$$\begin{aligned} T_A(P) = \left|\det\begin{bmatrix} A\mathbf{u} & A\mathbf{v} & A\mathbf{w}\end{bmatrix}\right| &= \left|\det(A\begin{bmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{bmatrix})\right| \\ &= \left|(\det A)(\det\begin{bmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{bmatrix})\right| = \left|\det A\right|(\text{volume of } P). \end{aligned}$$

11. **(a)** The line through $(2,3)$ and $(-1,0)$ is given by

$$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ -1 & 0 & 1 \end{vmatrix}.$$

Evaluate by cofactor expansion along the third row:

$$-1\begin{vmatrix} y & 1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} x & y \\ 2 & 3 \end{vmatrix} = -1(y-3) + (3x-2y) = 3x - 3y + 3.$$

The equation of the line is $3x - 3y + 3 = 0$, or $x - y + 1 = 0$.

(b) The line through $(1,2)$ and $(4,3)$ is given by

$$\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}.$$

Evaluate by cofactor expansion along the first row:

$$x\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} - y\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -x + 3y - 5.$$

The equation of the line is $-x + 3y - 5 = 0$, or $x - 3y + 5 = 0$.

12. Suppose the three points satisfy

$$\begin{aligned} ax_1 + by_1 + c &= 0 \\ ax_2 + by_2 + c &= 0 \\ ax_3 + by_3 + c &= 0, \end{aligned}$$

which can be written as the matrix equation

$$XA = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = O.$$

Then $\det X \neq 0$ if and only if X is invertible, which implies that $X^{-1}XA = X^{-1}O = O$, so that $A = O$. Thus $A \neq O$ if and only if $\det X = O$; this says precisely that the three points lie on the line $ax + by + c = 0$ with a and b not both zero if and only if $\det X = 0$.

13. Following the discussion at the beginning of the section, since the three points are noncollinear, they determine a unique plane, say $ax + by + cz + d = 0$. Since all three points lie on the plane, their coordinates satisfy this equation, so that

$$\begin{aligned} ax_1 + by_1 + cz_1 + d &= 0 \\ ax_2 + by_2 + cz_2 + d &= 0 \\ ax_3 + by_3 + cz_3 + d &= 0. \end{aligned}$$

These three equations together with the equation of the plane then form a system of linear equations in the variables a, b, c, and d. Since we know that the plane exists, it follows that there are values of a, b, c, and d, not all zero, satisfying these equations. So the system has a nontrivial solution, and therefore its coefficient matrix, which is

$$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix}$$

cannot be invertible, by the Fundamental Theorem of Invertible Matrices. Thus its determinant must be zero by Theorem 4.6.

To see what happens if the three points are collinear, try row reducing the above coefficient matrix:

$$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} \begin{array}{c} \\ \\ \xrightarrow{R_3 - R_2} \\ \xrightarrow{R_4 - R_2} \end{array} \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 & 0 \end{bmatrix}.$$

If the points are collinear, then

$$\begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix} = k \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$$

for some $k \neq 0$. Then performing $R_4 - kR_3$ gives a matrix with a zero row. So its determinant is trivially zero and the system has an infinite number of solutions. This corresponds to the fact that there are an infinite number of planes passing through the line determined by the three collinear points.

14. Suppose the four points satisfy

$$\begin{aligned} ax_1 + by_1 + cz_1 + d &= 0 \\ ax_2 + by_2 + cz_2 + d &= 0 \\ ax_3 + by_3 + cz_3 + d &= 0 \\ ax_4 + by_4 + cz_4 + d &= 0, \end{aligned}$$

which can be written as the matrix equation

$$XA = \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = O.$$

Then $\det X \neq 0$ if and only if X is invertible, which implies that $X^{-1}XA = X^{-1}O = O$, so that $A = O$. Thus $A \neq O$ if and only if $\det X = O$; this says precisely that the four points lie on the plane $ax + by + cz + d = 0$ with a, b, and c not all zero if and only if $\det X = 0$.

15. Substituting the values $(x, y) = (-1, 10)$, $(0, 5)$ and $(3, 2)$ in turn into the equation gives the linear system in a, b, and c

$$\begin{aligned} a - b + c &= 10 \\ a \phantom{{}+3b+9c} &= 5 \\ a + 3b + 9c &= 2. \end{aligned}$$

The determinant of the coefficient matrix of this system is

$$X = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 3 & 9 \end{vmatrix} = -1 \begin{vmatrix} -1 & 1 \\ 3 & 9 \end{vmatrix} = 12 \neq 0,$$

Since X is has nonzero determinant, it is invertible by Theorem 4.6, so we have

$$X\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}10\\5\\2\end{bmatrix}\quad\Rightarrow\quad\begin{bmatrix}a\\b\\c\end{bmatrix}=X^{-1}\begin{bmatrix}10\\5\\2\end{bmatrix},$$

so that the system has a unique solution. To find that solution, we can either invert X or we can row-reduce the augmented matrix. Using the second method, and writing the second equation first in the augmented matrix, we have

$$\left[\begin{array}{rrr|r}1&0&0&5\\1&-1&1&10\\1&3&9&2\end{array}\right]\xrightarrow[R_3-R_1]{R_2-R_1}\left[\begin{array}{rrr|r}1&0&0&5\\0&-1&1&5\\0&3&9&-3\end{array}\right]\xrightarrow{-R_2}\left[\begin{array}{rrr|r}1&0&0&5\\0&1&-1&-5\\0&3&9&-3\end{array}\right]\xrightarrow{R_3-3R_2}$$

$$\left[\begin{array}{rrr|r}1&0&0&5\\0&1&-1&-5\\0&0&12&12\end{array}\right]\xrightarrow{\frac{1}{12}R_3}\left[\begin{array}{rrr|r}1&0&0&5\\0&1&-1&-5\\0&0&1&1\end{array}\right]\xrightarrow{R_2+R_3}\left[\begin{array}{rrr|r}1&0&0&5\\0&1&0&-4\\0&0&1&1\end{array}\right].$$

So the system has the unique solution $a=5$, $b=-4$, $c=1$ and the parabola through the three points is $y=5-4x+x^2$.

16. (a) The linear system in a, b, and c is

$$\begin{aligned}a+\ \ b+\ \ c&=-1\\a+2b+4c&=\ \ 4\\a+3b+9c&=\ \ 3,\end{aligned}$$

so row-reducing the augmented matrix gives

$$\left[\begin{array}{rrr|r}1&1&1&-1\\1&2&4&4\\1&3&9&3\end{array}\right]\xrightarrow[R_3-R_1]{R_2-R_1}\left[\begin{array}{rrr|r}1&1&1&-1\\0&1&3&5\\0&2&8&4\end{array}\right]\xrightarrow{R_3-2R_2}\left[\begin{array}{rrr|r}1&1&1&-1\\0&1&3&5\\0&0&2&-6\end{array}\right]\xrightarrow{\frac{1}{2}R_3}$$

$$\left[\begin{array}{rrr|r}1&1&1&-1\\0&1&3&5\\0&0&1&-3\end{array}\right]\xrightarrow[R_2-3R_3]{R_1-R_3}\left[\begin{array}{rrr|r}1&1&0&2\\0&1&0&14\\0&0&1&-3\end{array}\right]\xrightarrow{R_1-R_2}\left[\begin{array}{rrr|r}1&0&0&-12\\0&1&0&14\\0&0&1&-3\end{array}\right].$$

So the system has the unique solution $a=-12$, $b=14$, $c=-3$ and the parabola through the three points is $y=-12+14x-3x^2$.

(b) The linear system in a, b, and c is

$$\begin{aligned}a-\ \ b+\ \ c&=-3\\a+\ \ b+\ \ c&=-1\\a+3b+9c&=\ \ 1,\end{aligned}$$

so row-reducing the augmented matrix gives

$$\left[\begin{array}{rrr|r}1&-1&1&-3\\1&1&1&-1\\1&3&9&1\end{array}\right]\xrightarrow[R_3-R_1]{R_2-R_1}\left[\begin{array}{rrr|r}1&-1&1&-3\\0&2&0&2\\0&4&8&4\end{array}\right]\xrightarrow[\frac{1}{4}R_3]{\frac{1}{2}R_2}\left[\begin{array}{rrr|r}1&-1&1&-3\\0&1&0&1\\0&1&2&1\end{array}\right]\xrightarrow{R_3-R_2}$$

$$\left[\begin{array}{rrr|r}1&-1&1&-3\\0&1&0&1\\0&0&2&0\end{array}\right]\xrightarrow{\frac{1}{2}R_3}\left[\begin{array}{rrr|r}1&-1&1&-3\\0&1&0&1\\0&0&1&0\end{array}\right]\xrightarrow{R_1+R_2-R_3}\left[\begin{array}{rrr|r}1&0&0&-2\\0&1&0&1\\0&0&1&0\end{array}\right].$$

So the system has the unique solution $a=-2$, $b=1$, $c=0$. Thus the equation of the curve passing through these points is in fact a line, since $c=0$, and its equation is $y=-2+x$.

17. Computing the determinant, we get

$$\begin{aligned}\begin{vmatrix}1 & a_1 & a_1^2\\ 1 & a_2 & a_2^2\\ 1 & a_3 & a_3^2\end{vmatrix} &= 1\begin{vmatrix}a_2 & a_2^2\\ a_3 & a_3^2\end{vmatrix} - 1\begin{vmatrix}a_1 & a_1^2\\ a_3 & a_3^2\end{vmatrix} + 1\begin{vmatrix}a_1 & a_1^2\\ a_2 & a_2^2\end{vmatrix}\\ &= a_2a_3^2 - a_2^2a_3 - a_1a_3^2 + a_1^2a_3 + a_1a_2^2 - a_1^2a_2.\end{aligned}$$

Now compute the product on the right:

$$\begin{aligned}(a_2-a_1)(a_3-a_1)(a_3-a_2) &= (a_2a_3 - a_1a_3 - a_1a_2 + a_1^2)(a_3-a_2)\\ &= a_2a_3^2 - a_2^2a_3 - a_1a_3^2 + a_1a_2a_3 - a_1a_2a_3 + a_1a_2^2 + a_1^2a_3 - a_1^2a_2\\ &= a_2a_3^2 - a_2^2a_3 - a_1a_3^2 + a_1a_2^2 + a_1^2a_3 - a_1^2a_2.\end{aligned}$$

The two expressions are equal. Furthermore, the determinant is nonzero since it is the product of the three differences $a_2 - a_1$, $a_3 - a_1$, and $a_3 - a_2$, which are all nonzero because we are assuming that the a_i are distinct real numbers.

18. Recall that adding a multiple of one column to another does not change the determinant (see Theorem 4.3). So given the matrix

$$\begin{vmatrix}1 & a_1 & a_1^2 & a_1^3\\ 1 & a_2 & a_2^2 & a_2^3\\ 1 & a_3 & a_3^2 & a_3^3\\ 1 & a_4 & a_4^2 & a_4^3\end{vmatrix},$$

apply in order the column operations $C_4 - a_1C_3$, $C_3 - a_1C_2$, $C_2 - a_1C_1$ to get

$$\begin{vmatrix}1 & a_1-a_1 & a_1^2-a_1^2 & a_1^3-a_1^3\\ 1 & a_2-a_1 & a_2^2-a_2a_1 & a_2^3-a_2^2a_1\\ 1 & a_3-a_1 & a_3^2-a_3a_1 & a_3^3-a_3^2a_1\\ 1 & a_4-a_1 & a_4^2-a_4a_1 & a_4^3-a_4^2a_1\end{vmatrix} = (a_4-a_1)(a_3-a_1)(a_2-a_1)\begin{vmatrix}1 & 0 & 0 & 0\\ 1 & 1 & a_2 & a_2^2\\ 1 & 1 & a_3 & a_3^2\\ 1 & 1 & a_4 & a_4^2\end{vmatrix}.$$

Then expand along the first row, giving

$$(a_4-a_1)(a_3-a_1)(a_2-a_1)\begin{vmatrix}1 & a_2 & a_2^2\\ 1 & a_3 & a_3^2\\ 1 & a_4 & a_4^2\end{vmatrix}.$$

But we know how to evaluate this determinant from Exercise 17; doing so, we get for the original determinant

$$(a_4-a_1)(a_3-a_1)(a_2-a_1)(a_3-a_2)(a_4-a_2)(a_4-a_3)$$

which is the same as the expression given in the problem statement.

For the second part, we can interpret the matrix above as the coefficient matrix formed by starting with the cubic $y = a + bx + cx^2 + dx^3$, substituting the four given points for x, and regarding the resulting equations as a linear system in a, b, c, and d, just as in Exercise 15. Since the determinant is nonzero if the a_i are all distinct, it follows that the system has a unique solution, so that there is a unique polynomial of degree at most 3 (it may be less than a cubic; for example, see Exercise 16(b)) passing through the four points.

19. For the $n \times n$ determinant given, use induction on n (note that Exercise 18 was essentially an inductive computation, since it invoked Exercise 17 at the end). We have already proven the cases $n = 1$, $n = 2$, and $n = 3$. So assume that the statement holds for $n = k$, and consider the determinant

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^{k-1} & a_1^k \\ 1 & a_2 & \cdots & a_2^{k-1} & a_2^k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_k & \cdots & a_k^{k-1} & a_k^k \\ 1 & a_{k+1} & \cdots & a_{k+1}^{k-1} & a_{k+1}^k \end{vmatrix}.$$

apply in order the column operations $C_{k+1} - a_1C_k$, $C_k - a_1C_{k-1}, \ldots, C_2 - a_1C_1$ to get

$$\begin{vmatrix} 1 & a_1 - a_1 & \cdots & a_1^{k-1} - a_1^{k-1} & a_1^k - a_1^k \\ 1 & a_2 - a_1 & \cdots & a_2^{k-1} - a_1a_2^{k-2} & a_2^k - a_1a_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_k - a_1 & \cdots & a_k^{k-1} - a_1a_k^{k-2} & a_k^k - a_1a_k^{k-1} \\ 1 & a_{k+1} - a_1 & \cdots & a_{k+1}^{k-1} - a_1a_{k+1}^{k-2} & a_{k+1}^k - a_1a_{k+1}^{k-1} \end{vmatrix} = \prod_{j=2}^{k+1}(a_j - a_1) \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & a_2^{k-2} & a_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & a_k^{k-2} & a_k^{k-1} \\ 1 & 1 & \cdots & a_{k+1}^{k-2} & a_{k+1}^{k-1} \end{vmatrix}.$$

Now expand along the first row, giving

$$\prod_{j=2}^{k+1}(a_j - a_1) \begin{vmatrix} 1 & \cdots & a_2^{k-2} & a_2^{k-1} \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & a_k^{k-2} & a_k^{k-1} \\ 1 & \cdots & a_{k+1}^{k-2} & a_{k+1}^{k-1} \end{vmatrix}.$$

But the remaining determinant is again a Vandermonde determinant in the k variables $a_2, a_3, \ldots, a_{k+1}$, so we know its value from the inductive hypothesis. Substituting that value, we get for the determinant of the original matrix

$$\prod_{j=2}^{k+1}(a_j - a_1) \prod_{2 \le i < j \le k+1}(a_j - a_i) = \prod_{1 \le i < j \le k+1}(a_j - a_i)$$

as desired.

For the second part, we can interpret the matrix above as the coefficient matrix formed by starting with the $(n-1)^{\text{st}}$ degree polynomial $y = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$, substituting the n given points for x, and regarding the resulting equations as a linear system in the c_i, just as in Exercises 15 and 18. Since the determinant is nonzero if the x-coordinates of the points are all distinct (being a product of their differences), it follows that the system has a unique solution, so that there is a unique polynomial of degree at most $n - 1$ (it may be less; for example, see Exercise 16(b)) passing through the n points.

4.3 Eigenvalues and Eigenvectors of $n \times n$ Matrices

1. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) - (-2) \cdot 3 = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4)$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, which are $\lambda = 3$ and $\lambda = 4$.

(c) To find the eigenspace corresponding to the eigenvalue $\lambda = 3$ we must find the null space of

$$A - 3I = \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix}$$

Row reduce this matrix:

$$\left[A-3I \mid 0\right] = \left[\begin{array}{cc|c} -2 & 3 & 0 \\ -2 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 3$ are of the form $\begin{bmatrix} 3t \\ 2t \end{bmatrix}$, so that $E_3 = \operatorname{span}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$.

To find the eigenspace corresponding to the eigenvalue $\lambda = 4$ we must find the null space of

$$A - 4I = \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix}$$

Row reduce this matrix:

$$\left[A-4I \mid 0\right] = \left[\begin{array}{cc|c} -3 & 3 & 0 \\ -2 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 4$ are of the form $\begin{bmatrix} t \\ t \end{bmatrix}$, so that $E_4 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

(d) Since $\lambda = 3$ and $\lambda = 4$ are single roots of the characteristic polynomials, the algebraic multiplicity of each is 1. Since each eigenspace is one-dimensional, the geometric multiplicity of each eigenvalue is also 1.

2. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = (2-\lambda)(-\lambda) - 1 \cdot (-1) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

(b) The eigenvalues of A are the roots of the characteristic polynomial; this polynomial has as its only root $\lambda = 1$.

(c) To find the eigenspace corresponding to the eigenvalue we must find the null space of

$$A - I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Row reduce this matrix:

$$\left[A-I \mid 0\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 1$ are of the form $\begin{bmatrix} -t \\ t \end{bmatrix}$, which is $\operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$.

(d) Since $\lambda = 1$ is a double root of the characteristic polynomial, its algebraic multiplicity is 2. However, the eigenspace is one-dimensional, so its geometric multiplicity is 1.

3. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda)(3-\lambda) = -(\lambda-1)(\lambda+2)(\lambda-3).$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, which are $\lambda = 1$, $\lambda = -2$, and $\lambda = 3$.

(c) To find the eigenspace corresponding to the eigenvalue $\lambda = 1$ we must find the null space of

$$A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Row reduce this matrix:

$$\left[A - I \mid 0\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 1$ are of the form $\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$, so that $E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

To find the eigenspace corresponding to the eigenvalue $\lambda = -2$ we must find the null space of

$$A + 2I = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

Row reduce this matrix:

$$\left[A + 2I \mid 0\right] = \left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = -2$ are of the form $\begin{bmatrix} -\frac{1}{3}t \\ t \\ 0 \end{bmatrix}$, so that $E_{-2} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}\right)$.

To find the eigenspace corresponding to the eigenvalue $\lambda = 3$ we must find the null space of

$$A - 3I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Row reduce this matrix:

$$\left[A - 3I \mid 0\right] = \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{10} & 0 \\ 0 & 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 3$ are of the form

$$\begin{bmatrix} \frac{1}{10}t \\ \frac{1}{5}t \\ t \end{bmatrix}, \quad \text{or, clearing fractions,} \quad \begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}, \quad \text{so that} \quad E_{-2} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}\right).$$

(d) Since all three eigenvalues are single roots of the characteristic polynomials, the algebraic multiplicity of each is 1. Since each eigenspace is one-dimensional, the geometric multiplicity of each eigenvalue is also 1.

4. (a) The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & 1-\lambda \\ 1 & 1 \end{vmatrix} \\ &= (1-\lambda)(\lambda^2 - \lambda - 1) + 1(-1+\lambda) = -\lambda^3 + 2\lambda^2 + \lambda - 2 \\ &= -(\lambda+1)(\lambda-1)(\lambda-2). \end{aligned}$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, which are $\lambda_1 = -1$, $\lambda_2 = 1$, and $\lambda_3 = 2$.

(c) To find the eigenspace corresponding to $\lambda_1 = -1$ we must find the null space of

$$A + I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Row reduce this matrix:

$$\left[A + I \mid 0\right] = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to λ_1 are of the form $\begin{bmatrix} -\frac{1}{2}t \\ -\frac{1}{2}t \\ t \end{bmatrix}$, or $\begin{bmatrix} -t \\ -t \\ 2t \end{bmatrix}$, which is span $\left(\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\right)$.

To find the eigenspace corresponding to $\lambda_2 = 1$ we must find the null space of

$$A - I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Row reduce this matrix:

$$\left[A - I \mid 0\right] = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to λ_2 are of the form $\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$, which is span $\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$.

To find the eigenspace corresponding to $\lambda_3 = 2$ we must find the null space of

$$A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

Row reduce this matrix:

$$\left[A - 2I \mid 0\right] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to λ_3 are of the form $\begin{bmatrix} t \\ t \\ t \end{bmatrix}$, which is span $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

(d) Since each eigenvalue is a single root of the characteristic polynomial, each has algebraic multiplicity 1. Since each eigenspace is one-dimensional, each eigenvalue also has geometric multiplicity 1.

5. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 = -\lambda^2(\lambda - 1).$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, which are $\lambda = 0$ and $\lambda = 1$.

(c) To find the eigenspace corresponding to the eigenvalue $\lambda = 0$ we must find the null space of A. Row reduce this matrix:

$$\begin{bmatrix} A \mid 0 \end{bmatrix} = \left[\begin{array}{rrr|r} 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 0$ are of the form

$$\begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},$$

so that the eigenspace is $E_0 = \operatorname{span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$.

To find the eigenspace corresponding to the eigenvalue $\lambda = 1$ we must find the null space of

$$A - I = \begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Row reduce this matrix:

$$\begin{bmatrix} A - I \mid 0 \end{bmatrix} = \left[\begin{array}{rrr|r} 0 & 2 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 1$ are of the form

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix}, \quad \text{so that} \quad E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

(d) Since $\lambda = 0$ is a double root of the characteristic polynomial, it has algebraic multiplicity 2; since its eigenspace is one-dimensional, it has geometric multiplicity 1. Since $\lambda = 1$ is a single root of the characteristic polynomial, it has algebraic multiplicity 1; since its eigenspace is one-dimensional, it also has geometric multiplicity 1.

6. (a) The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 2 \\ 3 & -1-\lambda & 3 \\ 2 & 0 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} -1-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} 3 & -1-\lambda \\ 2 & 0 \end{vmatrix} \\ &= (1-\lambda)(-(1+\lambda)(1-\lambda) - 3 \cdot 0) + 2(3 \cdot 0 + 2(1+\lambda)) \\ &= (1-\lambda)(\lambda^2 - 1) + 4 + 4\lambda = -\lambda^3 + \lambda^2 + 5\lambda + 3 = -(\lambda+1)^2(\lambda - 3). \end{aligned}$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, which are $\lambda_1 = -1$ and $\lambda_2 = 3$.

(c) To find the eigenspace corresponding to $\lambda_1 = -1$ we must find the null space of

$$A + I = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 0 & 3 \\ 2 & 0 & 2 \end{bmatrix}.$$

Row reduce this matrix:

$$\left[A+I \mid 0\right] = \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right).$$

To find the eigenspace corresponding to $\lambda_2 = 3$ we must find the null space of

$$A - 3I = \begin{bmatrix} -2 & 0 & 2 \\ 3 & -4 & 3 \\ 2 & 0 & -2 \end{bmatrix}.$$

Row reduce this matrix:

$$\left[A-3I \mid 0\right] = \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 3 & -4 & 3 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_2 are of the form

$$\begin{bmatrix} t \\ \frac{3}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{3}{2} \\ 1 \end{bmatrix} = s \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \right).$$

(d) λ_1 is a double root of the characteristic polynomial, and its eigenspace has dimension 2, so its algebraic and geometric multiplicities are both equal to 2. λ_2 is a single root of the characteristic polynomial, and its eigenspace has dimension 1, so its algebraic and geometric multiplicities are both equal to 1.

7. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ -1 & 0 & 2-\lambda \end{vmatrix} = -\lambda^3 + 9\lambda^2 - 27\lambda + 27 = -(\lambda-3)^3.$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, so the only eigenvalue is $\lambda = 3$.

(c) To find the eigenspace corresponding to the eigenvalue $\lambda = 3$ we must find the null space of

$$A - 3I = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

Row reduce this matrix:

$$\left[A-3I \mid 0\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 3$ are of the form

$$\begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{so that} \quad E_3 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

(d) Since $\lambda = 3$ is a triple root of the characteristic polynomial, it has algebraic multiplicity 3; since its eigenspace is two-dimensional, it has geometric multiplicity 2.

8. (a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & -1 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & -1 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)((1-\lambda)^2 - (-1)\cdot(-1)) = (2-\lambda)(\lambda^2 - 2\lambda) \\ &= -\lambda(\lambda-2)^2\end{aligned}$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, which are $\lambda_1 = 0$ and $\lambda_2 = 2$.

(c) To find the eigenspace corresponding to $\lambda_1 = 0$ we must find the null space of

$$A + 0I = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

Row reduce this matrix:

$$\left[A + 0I \mid 0\right] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} s \\ 0 \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

To find the eigenspace corresponding to $\lambda_2 = 2$ we must find the null space of

$$A - 2I = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}.$$

Row reduce this matrix:

$$\left[A - 2I \mid 0\right] = \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right).$$

(d) Since $\lambda_1 = 0$ is a single root of the characteristic polynomial and its eigenspace has dimension 1, its algebraic and geometric multiplicities are both equal to 1. Since $\lambda_2 = 2$ is a double root of the characteristic polynomial and its eigenspace has dimension 2, its algebraic and geometric multiplicities are both equal to 2.

9. (a) The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 & 0 & 0 \\ -1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 4 \\ 0 & 0 & 1 & 1-\lambda \end{vmatrix} = \lambda^4 - 6\lambda^3 + 9\lambda^2 + 4\lambda - 12 = (\lambda+1)(\lambda-2)^2(\lambda-3)$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, so the eigenvalues are $\lambda = -1$, $\lambda = 2$, and $\lambda = 3$.

(c) To find the eigenspace corresponding to the eigenvalue $\lambda = -1$ we must find the null space of

$$A + I = \begin{bmatrix} 4 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Row reduce this matrix:

$$\left[A + I \mid 0\right] = \left[\begin{array}{cccc|c} 4 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = -1$ are of the form

$$\begin{bmatrix} 0 \\ 0 \\ -2t \\ t \end{bmatrix}, \quad \text{so that} \quad E_{-1} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right).$$

To find the eigenspace corresponding to the eigenvalue $\lambda = 2$ we must find the null space of

$$A - 2I = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Row reduce this matrix:

$$\left[A - 2I \mid 0\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 2$ are of the form

$$\begin{bmatrix} -s \\ s \\ 0 \\ 0 \end{bmatrix}, \quad \text{so that} \quad E_2 = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

To find the eigenspace corresponding to the eigenvalue $\lambda = 3$ we must find the null space of

$$A - 3I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Row reduce this matrix:

$$\left[A - 3I \mid 0\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 3$ are of the form

$$\begin{bmatrix} 0 \\ 0 \\ 2t \\ t \end{bmatrix}, \quad \text{so that} \quad E_3 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right).$$

(d) Since $\lambda = 3$ and $\lambda = -1$ are single roots of the characteristic polynomial, they each have algebraic multiplicity 1; their eigenspaces are each one-dimensional, so those two eigenvalues have geometric multiplicity 1 as well. Since $\lambda = 2$ is a double root of the characteristic polynomial, it has algebraic multiplicity 2; since its eigenspace is one-dimensional, it only has geometric multiplicity 1.

10. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 & 0 \\ 0 & 1-\lambda & 4 & 5 \\ 0 & 0 & 3-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix}$$

Since this is a triangular matrix, its determinant is the product of the diagonal entries by Theorem 4.2, so the characteristic polynomial is $(2-\lambda)^2(1-\lambda)(3-\lambda)$.

(b) The eigenvalues of A are the roots of the characteristic polynomial, which are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

(c) To find the eigenspace corresponding to $\lambda_1 = 1$ we must find the null space of

$$A - I = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Row reduce this matrix:

$$\begin{bmatrix} A - I \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} -t \\ t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

To find the eigenspace corresponding to $\lambda_2 = 2$ we must find the null space of

$$A - 2I = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 4 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Row reduce this matrix:

$$\begin{bmatrix} A - 2I \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_2 are of the form

$$\begin{bmatrix} s \\ t \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right).$$

To find the eigenspace corresponding to $\lambda_3 = 3$ we must find the null space of

$$A - 3I = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -2 & 4 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Row reduce this matrix:

$$\begin{bmatrix} A - 3I \mid 0 \end{bmatrix} = \left[\begin{array}{cccc|c} -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & 5 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} 3t \\ 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right).$$

(d) Since $\lambda_1 = 1$ is a single root of the characteristic polynomial and its eigenspace has dimension 1, its algebraic and geometric multiplicities are both equal to 1. Since $\lambda_2 = 2$ is a double root of the characteristic polynomial and its eigenspace has dimension 2, its algebraic and geometric multiplicities are both equal to 2. Since $\lambda_3 = 3$ is a single root of the characteristic polynomial and its eigenspace has dimension 1, its algebraic and geometric multiplicities are both equal to 1.

11. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 1 & 1 & 3-\lambda & 0 \\ -2 & 1 & 2 & -1-\lambda \end{vmatrix} = (\lambda - 1)^2(\lambda - 3)(\lambda + 1)$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, so the eigenvalues are $\lambda = -1$, $\lambda = 1$, and $\lambda = 3$.

(c) To find the eigenspace corresponding to the eigenvalue $\lambda = -1$ we must find the null space of

$$A + I = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 4 & 0 \\ -2 & 1 & 2 & 0 \end{bmatrix}$$

Row reduce this matrix:

$$\begin{bmatrix} A + I \mid 0 \end{bmatrix} = \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ -2 & 1 & 2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = -1$ are of the form

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix}, \quad \text{so that} \quad E_{-1} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

To find the eigenspace corresponding to the eigenvalue $\lambda = 1$ we must find the null space of

$$A - I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ -2 & 1 & 2 & -2 \end{bmatrix}$$

Row reduce this matrix:

$$\begin{bmatrix} A - I \mid 0 \end{bmatrix} = \left[\begin{array}{rrrr|r} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ -2 & 1 & 2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 2 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 1$ are of the form

$$\begin{bmatrix} -\frac{2}{3}t \\ -2s + \frac{2}{3}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}, \quad \text{so that} \quad E_1 = \operatorname{span}\left(\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 3 \end{bmatrix}\right).$$

To find the eigenspace corresponding to the eigenvalue $\lambda = 3$ we must find the null space of

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 1 & 2 & -4 \end{bmatrix}$$

Row reduce this matrix:

$$\begin{bmatrix} A - 3I \mid 0 \end{bmatrix} = \left[\begin{array}{rrrr|r} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -2 & 1 & 2 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to $\lambda = 3$ are of the form

$$\begin{bmatrix} 0 \\ 0 \\ 2t \\ t \end{bmatrix}, \quad \text{so that} \quad E_3 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right).$$

(d) Since $\lambda = 3$ and $\lambda = -1$ are single roots of the characteristic polynomial, they each have algebraic multiplicity 1; their eigenspaces are each one-dimensional, so those two eigenvalues have geometric multiplicity 1 as well. Since $\lambda = 1$ is a double root of the characteristic polynomial, it has algebraic multiplicity 2; since its eigenspace is two-dimensional, it also has geometric multiplicity 2.

12. (a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 4-\lambda & 0 & 1 & 0 \\ 0 & 4-\lambda & 1 & 1 \\ 0 & 0 & 1-\lambda & 2 \\ 0 & 0 & 3 & -\lambda \end{vmatrix} \\ &= -3\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 2 \end{vmatrix} + (-\lambda)\begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} \\ &= -6(4-\lambda)^2 - \lambda(1-\lambda)(4-\lambda)^2 \\ &= (4-\lambda)^2(\lambda^2-\lambda-6) = (4-\lambda)^2(\lambda-3)(\lambda+2)\end{aligned}$$

since each of the 3×3 matrices is upper triangular, so that by Theorem 4.2 their determinants are the product of their diagonal elements.

(b) The eigenvalues of A are the roots of the characteristic polynomial, which are $\lambda_1 = 4$, $\lambda_2 = 3$, and $\lambda_3 = -2$.

(c) To find the eigenspace corresponding to $\lambda_1 = 4$ we must find the null space of

$$A - 4I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

Row reduce this matrix:

$$\left[A - 4I \mid 0\right] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} s \\ t \\ 0 \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

To find the eigenspace corresponding to $\lambda_2 = 3$ we must find the null space of

$$A - 3I = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Row reduce this matrix:

$$\left[A - 3I \mid 0\right] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} -t \\ -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}\right).$$

To find the eigenspace corresponding to $\lambda_3 = -2$ we must find the null space of

$$A + 2I = \begin{bmatrix} 6 & 0 & 1 & 0 \\ 0 & 6 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 3 & 2 \end{bmatrix}.$$

Row reduce this matrix:

$$\left[\begin{array}{c|c} A+2I & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 6 & 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & 1 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{9} & 0 \\ 0 & 1 & 0 & \frac{1}{18} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} \frac{1}{9}t \\ -\frac{1}{18}t \\ -\frac{2}{3}t \\ t \end{bmatrix} = \begin{bmatrix} 2s \\ -s \\ -12s \\ 18s \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ -12 \\ 18 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 2 \\ -1 \\ -12 \\ 18 \end{bmatrix}\right).$$

(d) Since $\lambda_1 = 4$ is a double root of the characteristic polynomial and its eigenspace has dimension 2, its algebraic and geometric multiplicities are both equal to 2. Since $\lambda_2 = 3$ is a single root of the characteristic polynomial and its eigenspace has dimension 1, its algebraic and geometric multiplicities are both equal to 1. Since $\lambda_3 = -2$ is a single root of the characteristic polynomial and its eigenspace has dimension 1, its algebraic and geometric multiplicities are both equal to 1.

13. We must show that if A is invertible and $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x} = \lambda^{-1}\mathbf{x}$. First note that since A is invertible, 0 is not an eigenvalue, so that $\lambda \neq 0$ and this statement makes sense. Now,

$$\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}(\lambda\mathbf{x}) = \lambda(A^{-1}\mathbf{x}).$$

Dividing through by λ gives $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$ as required.

14. The remark referred to in Section 3.3 tells us that if A is invertible and $n > 0$ is an integer, then $A^{-n} = (A^{-1})^n = (A^n)^{-1}$. Now, part (a) gives us the result in part (c) when $n > 0$ is an integer. When $n = 0$, the statement says that $\lambda^0 = 1$ is an eigenvalue of $A^0 = I$, which is true. So it remains to prove the statement for $n < 0$. We use induction on n. For $n = -1$, this is the statement that if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$, which is Theorem 4.18(b). Now assume the statement holds for $n = -k$, and prove it for $n = -(k+1)$. Note that by the remark in Section 3.3,

$$A^{-(k+1)} = (A^{k+1})^{-1} = (A^kA)^{-1} = A^{-1}(A^k)^{-1} = A^{-1}A^{-k}.$$

Now

$$A^{-(k+1)}\mathbf{x} = A^{-1}(A^{-k}\mathbf{x}) \overset{\substack{\text{inductive}\\\text{hypothesis}}}{=} A^{-1}(\lambda^{-k}\mathbf{x}) = \lambda^{-k}A^{-1}\mathbf{x} \overset{n=-1}{=} \lambda^{-k}\lambda^{-1}\mathbf{x} = \lambda^{-(k+1)}\mathbf{x}$$

and we are done.

15. If we can express $\mathbf{x}$ as a linear combination of the eigenvectors, it will be easy to compute $A^{10}\mathbf{x}$ since we know what A^{10} does to the eigenvectors. To so express $\mathbf{x}$, we solve the system $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, which is

$$\begin{aligned} c_1 + c_2 &= 5 \\ -c_1 + c_2 &= 1. \end{aligned}$$

This has the solution $c_1 = 2$ and $c_2 = 3$, so that $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2$. But by Theorem 4.4(a), $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A^{10} with eigenvalues λ_1^{10} and λ_2^{10}. Thus

$$A^{10}\mathbf{x} = A^{10}(2\mathbf{v}_1 + 3\mathbf{v}_2) = 2A^{10}\mathbf{v}_1 + 3A^{10}\mathbf{v}_2 = 2\lambda_1^{10}\mathbf{v}_1 + 3\lambda_2^{10}\mathbf{v}_2 = \frac{1}{512}\mathbf{v}_1 + 3072\mathbf{v}_2.$$

Substituting for $\mathbf{v}_1$ and $\mathbf{v}_2$ gives

$$A^{10}\mathbf{x} = \begin{bmatrix} 3072 + \frac{1}{512} \\ 3072 - \frac{1}{512} \end{bmatrix}.$$

16. As in Exercise 15,

$$A^k\mathbf{x} = 2 \cdot \left(\frac{1}{2}\right)^k \mathbf{v}_1 + 3 \cdot 2^k \mathbf{v}_2 = \begin{bmatrix} 3 \cdot 2^k + 2^{1-k} \\ 3 \cdot 2^k - 2^{1-k} \end{bmatrix}.$$

As $k \to \infty$, the 2^{1-k} terms approach zero, so that $A^k\mathbf{x} \approx 3 \cdot 2^k \mathbf{v}_2$.

17. We want to express $\mathbf{x}$ as a linear combination of the eigenvectors, say $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, so we want a solution of the linear system

$$\begin{aligned} c_1 + c_2 + c_3 &= 2 \\ c_2 + c_3 &= 1 \\ c_3 &= 2. \end{aligned}$$

Working from the bottom and substituting gives $c_3 = 2$, $c_2 = -1$, and $c_1 = 1$, so that $\mathbf{x} = \mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3$. Then

$$\begin{aligned} A^{20}\mathbf{x} &= A^{20}(\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3) \\ &= A^{20}\mathbf{v}_1 - A^{20}\mathbf{v}_2 + 2A^{20}\mathbf{v}_3 \\ &= \lambda_1^{20}\mathbf{v}_1 - \lambda_2^{20}\mathbf{v}_2 + 2\lambda_3^{20}\mathbf{v}_3 \\ &= \left(-\frac{1}{3}\right)^{20} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1}{3}\right)^{20} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \cdot 1^{20} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 - \frac{1}{3^{20}} \\ 2 \end{bmatrix}. \end{aligned}$$

18. As in Exercise 17,

$$\begin{aligned} A^k\mathbf{x} &= A^k(\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3) \\ &= A^k\mathbf{v}_1 - A^k\mathbf{v}_2 + 2A^k\mathbf{v}_3 \\ &= \lambda_1^k\mathbf{v}_1 - \lambda_2^k\mathbf{v}_2 + 2\lambda_3^k\mathbf{v}_3 \\ &= \left(-\frac{1}{3}\right)^k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1}{3}\right)^k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \cdot 1^k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \pm\frac{1}{3^k} - \frac{1}{3^k} + 2 \\ -\frac{1}{3^k} + 2 \\ 2 \end{bmatrix}. \end{aligned}$$

As $k \to \infty$, the $\frac{1}{3^k}$ terms approach zero, and $A^k\mathbf{x} \to \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

19. **(a)** Since $(\lambda I)^T = \lambda I^T = \lambda I$, we have

$$A^T - \lambda I = A^T - (\lambda I)^T = (A - \lambda I)^T,$$

so that using Theorem 4.10,

$$\det(A^T - \lambda I) = \det\left((A - \lambda I)^T\right) = \det(A - \lambda I).$$

But the left-hand side is the characteristic polynomial of A^T, and the right-hand side is the characteristic polynomial of A. Thus the two matrices have the same characteristic polynomial and thus the same eigenvalues.

(b) There are many examples. For instance

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, with eigenspaces

$$E_1 = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right), \quad E_2 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

A^T has the same eigenvalues by part (a), but the eigenspaces are

$$E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \quad E_2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

20. By Theorem 4.18(b), if λ is an eigenvalue of A, then λ^m is an eigenvalue of $A^m = O$. But the only eigenvalue of O is zero, since A^m takes any vector to zero. Thus $\lambda^m = 0$, so that $\lambda = 0$.

21. By Theorem 4.18(a), if λ is an eigenvalue of A with eigenvector $\mathbf{x}$, then λ^2 is an eigenvalue of $A^2 = A$ with eigenvector $\mathbf{x}$. Since different eigenvalues have linearly independent eigenvectors by Theorem 4.20, we must have $\lambda^2 = \lambda$, so that $\lambda = 0$ or $\lambda = 1$.

22. If $A\mathbf{v} = \lambda\mathbf{v}$, then $A\mathbf{v} - cI\mathbf{v} = \lambda\mathbf{v} - cI\mathbf{v}$, so that $(A - cI)\mathbf{v} = (\lambda - c)\mathbf{v}$. But this equation means exactly that $\mathbf{v}$ is an eigenvector of $A - cI$ with eigenvalues $\lambda - c$.

23. **(a)** The characteristic polynomial of A is

$$\left|A - \lambda I\right| = \begin{vmatrix} 3-\lambda & 2 \\ 5 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2).$$

So the eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -2$. For λ_1, we have

$$A - 5I = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

and for λ_2, we have

$$A + 2I = \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 0 \end{bmatrix}.$$

It follows that

$$E_5 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \quad E_{-2} = \operatorname{span}\left(\begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 5 \end{bmatrix}\right).$$

(b) By Theorem 4.4(b), A^{-1} has eigenvalues $\frac{1}{5}$ and $-\frac{1}{2}$, with

$$E_{1/5} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \quad E_{-1/2} = \operatorname{span}\left(\begin{bmatrix} -2 \\ 5 \end{bmatrix}\right).$$

By Exercise 22, $A - 2I$ has eigenvalues $5 - 2 = 3$ and $-2 - 2 = -4$ with

$$E_3 = \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right), \quad E_{-4} = \operatorname{span}\left(\begin{bmatrix}-2\\5\end{bmatrix}\right).$$

Also by Exercise 22, $A + 2I = A - (-2I)$ has eigenvalues $5 + 2 = 7$ and $-2 + 2 = 0$ with

$$E_7 = \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right), \quad E_0 = \operatorname{span}\left(\begin{bmatrix}-2\\5\end{bmatrix}\right).$$

24. (a) There are many examples. For instance, let

$$A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \qquad B = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}.$$

Then the characteristic equation of A is $\lambda^2 - 1$, so that A has eigenvalues $\lambda = \pm 1$. The characteristic equation of B is $(\mu - 1)^2$, so that B has eigenvalue $\mu = 1$. But

$$A + B = \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}$$

has characteristic polynomial $\lambda^2 - 2\lambda - 1$, so it has eigenvalues $1 \pm \sqrt{2}$. The possible values of $\lambda + \mu$ are 0 and 2.

(b) Use the same A and B as above. Then

$$AB = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix},$$

with characteristic polynomial $\lambda^2 - \lambda - 1$ and eigenvalues $\frac{1}{2}(1 \pm \sqrt{5})$, which again does not match $\lambda + \mu$.

(c) If λ and μ both have eigenvector $\mathbf{x}$, then

$$(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x} = (\lambda + \mu)\mathbf{x},$$

which shows that $\lambda + \mu$ is an eigenvalue of $A + B$ with eigenvector $\mathbf{x}$. For the product, we have

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A(\mu\mathbf{x}) = \mu A\mathbf{x} = \mu\lambda\mathbf{x} = \lambda\mu\mathbf{x},$$

showing that $\lambda\mu$ is an eigenvalue of AB with eigenvector $\mathbf{x}$.

25. No, they do not. For example, take $B = I_2$ and let A be any 2×2 rank 2 matrix that has an eigenvalue other than 1. For example let $A = \begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}$, which has eigenvalues 2 and -2. Since A has rank 2, it row-reduces to I, so A and I are row-equivalent but do not have the same eigenvalues.

26. By definition, the companion matrix of $p(x) = x^2 - 7x + 12$ is

$$C(p) = \begin{bmatrix}-(-7) & -12\\1 & 0\end{bmatrix} = \begin{bmatrix}7 & -12\\1 & 0\end{bmatrix},$$

which has characteristic polynomial

$$\begin{vmatrix}7 - \lambda & -12\\1 & -\lambda\end{vmatrix} = (7 - \lambda)(-\lambda) + 12 = \lambda^2 - 7\lambda + 12.$$

27. By definition, the companion matrix of $p(x) = x^3 + 3x^2 - 4x + 12$ is

$$C(p) = \begin{bmatrix} -3 & -(-4) & -12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 4 & -12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

which has characteristic polynomial

$$\begin{aligned} \begin{vmatrix} -3-\lambda & 4 & -12 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} &= -1 \begin{vmatrix} -3-\lambda & -12 \\ 1 & 0 \end{vmatrix} - \lambda \begin{vmatrix} -3-\lambda & 4 \\ 1 & -\lambda \end{vmatrix} \\ &= -12 - \lambda((-3-\lambda)(-\lambda) - 4) \\ &= -\lambda^3 - 3\lambda^2 + 4\lambda - 12. \end{aligned}$$

28. (a) If $p(x) = x^2 + ax + b$, then

$$C(p) = \begin{bmatrix} -a & -b \\ 1 & 0 \end{bmatrix},$$

so the characteristic polynomial of $C(p)$ is

$$\begin{vmatrix} -a-\lambda & -b \\ 1 & -\lambda \end{vmatrix} = (-a-\lambda)(-\lambda) + b = \lambda^2 + a\lambda + b.$$

(b) Suppose that λ is an eigenvalue of $C(p)$, and that a corresponding eigenvector is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C(p) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ means that

$$\begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix} = \begin{bmatrix} -a & -b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ax_1 - bx_2 \\ x_1 \end{bmatrix},$$

so that $x_1 = \lambda x_2$. Setting $x_2 = 1$ gives an eigenvector $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$.

29. (a) If $p(x) = x^3 + ax^2 + bx + c$, then

$$C(p) = \begin{bmatrix} -a & -b & -c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

so the characteristic polynomial of $C(p)$ is

$$\begin{aligned} \begin{vmatrix} -a-\lambda & -b & -c \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} &= -1 \begin{vmatrix} -a-\lambda & -c \\ 1 & 0 \end{vmatrix} - \lambda \begin{vmatrix} -a-\lambda & -b \\ 1 & -\lambda \end{vmatrix} \\ &= -c - \lambda((-a-\lambda)(-\lambda) + b) \\ &= -\lambda^3 - a\lambda^2 - b\lambda - c. \end{aligned}$$

(b) Suppose that λ is an eigenvalue of $C(p)$, and that a corresponding eigenvector is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then

$\lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C(p) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ means that

$$\begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -ax_1 - bx_2 - cx_3 \\ x_1 \\ x_2 \end{bmatrix},$$

so that $x_1 = \lambda x_2$ and $x_2 = \lambda x_3$. Setting $x_3 = 1$ gives an eigenvector $\begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$.

30. Using Exercise 28, the characteristic polynomial of the matrix

$$C(p) = \begin{bmatrix} -a & -b \\ 1 & 0 \end{bmatrix}$$

is $\lambda^2 + a\lambda + b$. If we want this matrix to have eigenvalues 2 and 5, then $\lambda^2 + a\lambda + b = (\lambda - 2)(\lambda - 5) = \lambda^2 - 7\lambda + 10$, so we can take

$$A = \begin{bmatrix} 7 & -10 \\ 1 & 0 \end{bmatrix}.$$

31. Using Exercise 28, the characteristic polynomial of the matrix

$$C(p) = \begin{bmatrix} -a & -b & -c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is $-(\lambda^3 + a\lambda^2 + b\lambda + c)$. If we want this matrix to have eigenvalues -2, 1, and 3, then

$$-(\lambda^3 + a\lambda^2 + b\lambda + c) = -(\lambda + 2)(\lambda - 1)(\lambda - 3) = \lambda^3 - 2\lambda^2 - 5\lambda + 6,$$

so we can take

$$A = \begin{bmatrix} 2 & 5 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

32. (a) Exercises 28 and 29 provide a basis for the induction, proving the cases $n = 2$ and $n = 3$. So assume that the statement holds for $n = k$; that is, that

$$\begin{vmatrix} -a_{k-1} - \lambda & -a_{k-2} & \cdots & -a_1 & -a_0 \\ 1 & -\lambda & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix} = (-1)^k p(\lambda),$$

where $p(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$. Now suppose that $q(x) = x^{k+1} + b_k x^k + \cdots + b_1 x + b_0$ has degree $k + 1$; then

$$C(q) = \begin{vmatrix} -b_k - \lambda & -b_{k-1} & \cdots & -b_1 & -b_0 \\ 1 & -\lambda & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix}.$$

Evaluate this determinant by expanding along the last column, giving

$$C(q) = (-1)^k(-b_0) \begin{vmatrix} 1 & -\lambda & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & \cdots & -\lambda \\ 0 & 0 & \cdots & 1 \end{vmatrix} + (-\lambda) \begin{vmatrix} -b_k - \lambda & -b_{k-1} & \cdots & -b_2 & -b_1 \\ 1 & -\lambda & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix}.$$

The first determinant is of an upper triangular matrix, so it is the product of its diagonal entries, which is 1. The second determinant is the characteristic polynomial of $x^k + b_k x^{k-1} + \cdots + b_2 x + b_1$, which is a k^{th} degree polynomial. So by the inductive hypothesis, this determinant is $(-1)^k(\lambda^k + b_k\lambda^{k-1} + \cdots + b_2\lambda + b_1)$. So the entire expression becomes

$$\begin{aligned} C(q) &= (-1)^k(-b_0) - \lambda\left((-1)^k(\lambda^k + b_k\lambda^{k-1} + \cdots + b_2\lambda + b_1)\right) \\ &= (-1)^{k+1}b_0 + (-1)^{k+1}(\lambda^{k+1} + b_k\lambda^k + \cdots + b_2\lambda^2 + b_1\lambda) \\ &= (-1)^{k+1}(\lambda^{k+1} + b_k\lambda^k + \cdots + b_2\lambda^2 + b_1\lambda + b_0) \\ &= (-1)^{k+1}q(\lambda). \end{aligned}$$

(b) The method is quite similar to Exercises 28(b) and 29(b). Suppose that λ is an eigenvalue of $C(p)$ with eigenvector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}.$$

Then $\lambda\mathbf{x} = C(p)\mathbf{x}$, so that

$$\begin{aligned} \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_{n-1} \\ \lambda x_n \end{bmatrix} &= \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} -a_{n-1}x_1 - a_{n-2}x_2 - \cdots - a_1x_{n-1} - a_0x_n \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}. \end{aligned}$$

But this means that $x_i = \lambda x_{i+1}$ for $i = 1, 2, \ldots, n-1$. Setting $x_n = 1$ gives an eigenvector

$$\mathbf{x} = \begin{bmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}.$$

33. The characteristic polynomial of A is

$$c_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 = \lambda^2 - 4\lambda + 5.$$

Now,

$$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^2 = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix},$$

so that

$$\begin{aligned} c_A(A) &= A^2 - 4A + 5I_2 \\ &= \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix} - 4\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= O. \end{aligned}$$

34. The characteristic polynomial of A is

$$c_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2.$$

Now,

$$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 3 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix},$$

so that

$$\begin{aligned} c_A(A) &= -A^3 + 2A^2 + A - 2I_3 \\ &= -\begin{bmatrix} 3 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3+4+1-2 & -3+2+1-0 & -2+2+0-0 \\ -3+2+1-0 & -2+4+0-2 & -3+2+1-0 \\ -2+2+0-0 & -3+2+1-0 & -3+4+1-2 \end{bmatrix} \\ &= O. \end{aligned}$$

35. Since $A^2 - 4A + 5I = 0$, we have $A^2 = 4A - 5I$, and

$$\begin{aligned} A^3 &= A(4A - 5I) = 4A^2 - 5A = 4(4A - 5I) - 5A = 11A - 20I \\ A^4 &= (A^2)(A^2) = (4A - 5I)(4A - 5I) = 16A^2 - 40A + 25I = 16(4A - 5I) - 40A + 25I = 24A - 55I. \end{aligned}$$

36. Since $-A^3 + 2A^2 + A - 2I = 0$, we have

$$\begin{aligned} A^3 &= 2A^2 + A - 2I, \\ A^4 &= A(A^3) = A(2A^2 + A - 2I) = 2A^3 + A^2 - 2A = 2(2A^2 + A - 2I) + A^2 - 2A = 5A^2 - 4I. \end{aligned}$$

37. Since $A^2 - 4A + 5I = 0$, we get $A(A - 4I) = -5I$. Then solving for A^{-1} yields

$$\begin{aligned} A^{-1} &= -\frac{1}{5}A - \frac{-4}{5}I = -\frac{1}{5}A + \frac{4}{5}I \\ A^{-2} &= \left(A^{-1}\right)^2 = \left(-\frac{1}{5}A + \frac{4}{5}I\right)^2 = \frac{1}{25}A^2 - \frac{8}{25}A + \frac{16}{25}I^2 = \frac{1}{25}(4A - 5I) - \frac{8}{25}A + \frac{16}{25}I = -\frac{4}{25}A + \frac{11}{25}I. \end{aligned}$$

38. Since $-A^3 + 2A^2 + A - 2I = 0$, rearranging terms gives $A(-A^2 + 2A + I) = 2I$. Then solving for A^{-1} yields (using Exercise 36)

$$\begin{aligned} A^{-1} &= \frac{1}{2}I(-A^2 + 2A + I) = -\frac{1}{2}A^2 + A + \frac{1}{2}I \\ A^{-2} &= \left(A^{-1}\right)^2 = \left(-\frac{1}{2}A^2 + A + \frac{1}{2}I\right)^2 \\ &= \frac{1}{4}A^4 - A^3 + \frac{1}{2}A^2 + A + \frac{1}{4}I^2 \\ &= \frac{1}{4}(5A^2 - 4I) - (2A^2 + A - 2I) + \frac{1}{2}A^2 + A + \frac{1}{4}I \\ &= -\frac{1}{4}A^2 + \frac{5}{4}I. \end{aligned}$$

39. Apply Exercise 69 in Section 4.2 to the matrix $A - \lambda I$:

$$c_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} P - \lambda I & Q \\ O & S - \lambda I \end{bmatrix} = (\det(P - \lambda I))(\det(S - \lambda I)) = c_P(\lambda)c_S(\lambda).$$

(Note that neither O or Q is affected by subtracting an identity matrix, since all the 1's are along the diagonal, in either P or S.)

40. Start with the equation given in the problem statement:

$$\det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n).$$

This is an equation, so in particular it holds for $\lambda = 0$, and it becomes

$$\det A = \det(A - 0I) = (-1)^n(-\lambda_1)(-\lambda_2)\cdots(-\lambda_n) = (-1)^n(-1)^n\lambda_1\lambda_2\cdots\lambda_n = \lambda_1\lambda_2\cdots\lambda_n,$$

proving the first result. For the second result, we will find the coefficients of λ^{n-1} on the left and right sides. On the left side, the easiest method given the tools we have is induction on n. For $n = 2$,

$$\det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = \lambda^2 - (a_{11} + a_{22})\lambda - a_{12}a_{21},$$

so that the coefficient of $\lambda^{n-1} = \lambda$ is $-\operatorname{tr}(A)$. Claim that in general, for $n \ge 1$, the coefficient of λ^{n-1} in $\det(A - \lambda I)$ is $(-1)^{n-1}\operatorname{tr}(A)$. Suppose this statement holds for $n = k$. Then if A is $(k+1)\times(k+1)$, we have (expanding along the first column of $A - \lambda I$)

$$\det A = (a_{11} - \lambda)\det(A_{11} - \lambda I) + \sum_{i=2}^{k+1}(-1)^{i+1}a_{i1}\det(A_{i1} - \lambda I).$$

Now, the terms in the sum on the right each consist of a constant times the determinant of a $k \times k$ matrix derived by removing row i and column 1 from A. Since $i > 1$, this means we have removed both the entries $a_{11} - \lambda I$ and $a_{i1} - \lambda I$ from A, so that there are only $k - 1$ entries in A that contain λ. So the sum on the right cannot produce any terms with λ^k. For the other term, note that $A_{11} - \lambda I$ is a $k \times k$ matrix; using the inductive hypothesis we see that the coefficient of λ^{k-1} in $\det(A_{11} - \lambda I)$ is $(-1)^{k-1}\operatorname{tr}(A_{11})$. Now

$$\det A = (a_{11} - \lambda)((-1)^k\lambda^k + (-1)^{k-1}\operatorname{tr}(A_{11})\lambda^{k-1} + \dots) + \sum \dots,$$

so that the coefficient of λ^k in $\det A$ is

$$(-1)^k a_{11} - (-1)^{k-1}\operatorname{tr}(A_{11}) = (-1)^k(a_{11} + \operatorname{tr}(A_{11})) = (-1)^k(a_{11} + a_{22} + \cdots + a_{k+1,k+1}) = (-1)^k\operatorname{tr}(A).$$

Now look at the product on the right in the original expression

$$\det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n).$$

We can get a constant times λ^{n-1} by expanding the right-hand side when exactly $n - 1$ of the factors use λ and one uses λ_i for some i; that contribution to the λ^{n-1} term in the characteristic polynomial is $-\lambda_i\lambda^{n-1}$. Summing over all possible choices for i shows that the coefficient of the λ^{n-1} term is

$$(-1)^n(-\lambda_1 - \lambda_2 - \cdots - \lambda_n) = (-1)^{n+1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = (-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n).$$

Since the coefficient of λ^{n-1} on the left must be the same as that on the right, we have

$$(-1)^{n-1}\operatorname{tr}(A) = (-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n), \quad \text{or} \quad \operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

41. Suppose that the eigenvalues of A are $\alpha_1, \alpha_2, \ldots, \alpha_n$ (repetitions included) and for B are $\beta_1, \beta_2, \ldots, \beta_n$. Let the eigenvalues of $A+B$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then by Exercise 40,

$$\begin{aligned} \operatorname{tr}(A+B) &= \lambda_1 + \lambda_2 + \cdots + \lambda_n \\ \operatorname{tr}(A) + \operatorname{tr}(B) &= \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n. \end{aligned}$$

Since $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$ by Exercise 44(a) in Section 3.2, we get

$$\lambda_1 + \lambda_2 + \cdots + \lambda_2 = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n$$

as desired.

Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of AB. Then again by Exercise 40 we have

$$\mu_1\mu_2\cdots\mu_n = \det(AB) = \det(A)\det(B) = \alpha_1\alpha_2\cdots\alpha_n\beta_1\beta_2\cdots\beta_n$$

as desired.

42. Suppose $\mathbf{x} = \sum_{i=1}^n c_i\mathbf{v}_i$, where the $\mathbf{v}_i$ are eigenvectors of A with corresponding eigenvalues λ_i. Then Theorem 4.18 tells us that $\mathbf{v}_i$ are eigenvectors of A^k with corresponding eigenvalues λ_i^k, so that

$$A^k\mathbf{x} = A^k\left(\sum_{i=1}^n c_i\mathbf{v}_i\right) = \sum_{i=1}^n c_iA^k\mathbf{v}_i = \sum_{i=1}^n c_i\lambda_i^k\mathbf{v}_i.$$

4.4 Similarity and Diagonalization

1. The two characteristic polynomials are

$$\begin{aligned} c_A(\lambda) &= \begin{vmatrix} 4-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 3 = \lambda^2 - 5\lambda + 1 \\ c_B(\lambda) &= \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1. \end{aligned}$$

Since the two characteristic polynomials are unequal, the two matrices are not similar by Theorem 4.22.

2. The two characteristic polynomials are

$$\begin{aligned} c_A(\lambda) &= \begin{vmatrix} 2-\lambda & 1 \\ -4 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) + 4 = \lambda^2 - 8\lambda + 16 \\ c_B(\lambda) &= \begin{vmatrix} 3-\lambda & -1 \\ -5 & 7-\lambda \end{vmatrix} = (3-\lambda)(7-\lambda) - 5 = \lambda^2 - 10\lambda + 16. \end{aligned}$$

Since the two characteristic polynomials are unequal, the two matrices are not similar by Theorem 4.22.

3. Both A and B are triangular matrices, so by Theorem 4.15, A has eigenvalues 2 and 4, while B has eigenvalues 1 and 4. By Theorem 4.22, the two matrices are not similar.

4. The two characteristic polynomials are

$$\begin{aligned}
c_A(\lambda) &= \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 0 & -1 \\ 0 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)((1-\lambda)^2-1) = (1-\lambda)(\lambda^2-2\lambda) = -\lambda^3+3\lambda^2-2\lambda \\
c_B(\lambda) &= \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)((2-\lambda)(1-\lambda)-2) = (1-\lambda)(\lambda^2-3\lambda) = -\lambda^3+4\lambda^2-3\lambda.
\end{aligned}$$

Since the two characteristic polynomials are unequal, the two matrices are not similar by Theorem 4.22.

5. By Theorem 4.23, the columns of P are the eigenvectors of A corresponding to the eigenvalues, which are the diagonal entries of D. Thus

$$\begin{aligned}
\lambda_1 &= 4 \quad \text{with eigenspace} \quad \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\
\lambda_2 &= 3 \quad \text{with eigenspace} \quad \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).
\end{aligned}$$

6. By Theorem 4.23, the columns of P are the eigenvectors of A corresponding to the eigenvalues, which are the diagonal entries of D. Thus

$$\begin{aligned}
\lambda_1 &= 2 \quad \text{with eigenspace} \quad \operatorname{span}\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\right) \\
\lambda_2 &= 0 \quad \text{with eigenspace} \quad \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) \\
\lambda_3 &= -1 \quad \text{with eigenspace} \quad \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right).
\end{aligned}$$

7. By Theorem 4.23, the columns of P are the eigenvectors of A corresponding to the eigenvalues, which are the diagonal entries of D. Thus

$$\begin{aligned}
\lambda_1 &= 6 \quad \text{with eigenspace} \quad \operatorname{span}\left(\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}\right) \\
\lambda_2 &= -2 \quad \text{with eigenspace} \quad \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right).
\end{aligned}$$

8. To determine whether A is diagonalizable, we first compute its eigenvalues:

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = (5-\lambda)^2-4 = \lambda^2-10\lambda+21 = (\lambda-3)(\lambda-7).$$

Since A is a 2×2 matrix with 2 distinct eigenvalues, it is diagonalizable by Theorem 4.25. To find the diagonalizing matrix P, we find the eigenvectors of A. To find the eigenspace corresponding to $\lambda_1 = 3$ we must find the null space of

$$A - 3I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Row reduce this matrix:

$$\begin{bmatrix} A - 3I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right).$$

To find the eigenspace corresponding to $\lambda_2 = 7$ we must find the null space of

$$A - 7I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

Row reduce this matrix:

$$\begin{bmatrix} A - 7I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

Thus

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{so that} \quad P^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and so by Theorem 4.23,

$$P^{-1}AP = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}.$$

9. To determine whether A is diagonalizable, we first compute its eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = (-3-\lambda)(1-\lambda) - 4\cdot(-1) = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2.$$

So the only eigenvalue is $\lambda = -1$, with algebraic multiplicity 2. To see if A is diagonalizable, we must see if its geometric multiplicity is also 2 by finding its eigenvectors. To find the eigenspace corresponding to $\lambda = -1$, we must find the null space of

$$A + I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

by row-reducing it:

$$\begin{bmatrix} A + I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} 2t \\ t \end{bmatrix}, \text{ so a basis for the eigenspace is } \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Thus $\lambda = -1$ has geometric multiplicity 1, which is less than the algebraic multiplicity, so that A is not diagonalizable by Theorem 4.27.

10. To determine whether A is diagonalizable, we first compute its eigenvalues:

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^3$$

since the matrix is a triangular matrix. Thus the only eigenvalue is $\lambda = 3$, with algebraic multiplicity 3. To see if A is diagonalizable, we must compute the eigenspace corresponding to this eigenvalue. To find the eigenspace corresponding to $\lambda = 3$ we row-reduce $\begin{bmatrix} A-3I & 0 \end{bmatrix}$:

$$\begin{bmatrix} A-3I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

But this matrix is already row-reduced, and thus the eigenspace is

$$\begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

Thus the geometric multiplicity is only 1 while the algebraic multiplicity is 3. So by Theorem 4.27, the matrix is not diagonalizable.

11. To determine whether A is diagonalizable, we first compute its eigenvalues:

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2 = -(\lambda+1)(\lambda-1)(\lambda-2).$$

Since A is a 3×3 matrix with 3 distinct eigenvalues, it is diagonalizable by Theorem 4.25. To find the diagonalizing matrix P, we find the eigenvectors of A. To find the eigenspace corresponding to $\lambda = -1$, we must find the null space of

$$A + I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

by row-reducing it:

$$\begin{bmatrix} A+I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} -\frac{1}{2}t \\ -\frac{1}{2}t \\ t \end{bmatrix} \text{ or, clearing fractions } \begin{bmatrix} -t \\ -t \\ 2t \end{bmatrix}.$$

Thus a basis for this eigenspace is $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda = 1$, we must find the null space of

$$A - I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

by row-reducing it:

$$\left[A - I \mid 0\right] = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}, \text{ so a basis for the eigenspace is } \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

To find the eigenspace corresponding to $\lambda = 2$, we must find the null space of

$$A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

by row-reducing it:

$$\left[A - 2I \mid 0\right] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus eigenvectors corresponding to λ_1 are of the form

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix}, \text{ so a basis for the eigenspace is } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then a diagonalizing matrix P is the matrix whose columns are the eigenvectors of A:

$$P = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \text{ so that } P^{-1} = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

and so by Theorem 4.23,

$$P^{-1}AP = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

12. To determine whether A is diagonalizable, we first compute its eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & 1 \\ 3 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda).$$

So the eigenvalues are $\lambda_1 = 1$ with algebraic multiplicity 2, and $\lambda_2 = 2$ with algebraic multiplicity 1. To find the eigenspace corresponding to $\lambda_1 = 1$ we row-reduce $\left[A - I \mid 0\right]$:

$$\left[A - I \mid 0\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the eigenspace is

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right).$$

Thus the geometric multiplicity of this eigenvalue is only 1 while the algebraic multiplicity is 2. So by Theorem 4.27, the matrix is not diagonalizable.

13. To determine whether A is diagonalizable, we first compute its eigenvalues:

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda & 2 & 1\\ -1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{vmatrix}=\lambda^2-\lambda^3=-\lambda^2(\lambda-1).$$

So A has eigenvalues 0 and 1. Considering $\lambda=0$, we row-reduce $A-0I=A$:

$$\left[\begin{array}{rrr|r}1 & 2 & 1 & 0\\ -1 & 0 & 1 & 0\\ 1 & 1 & 0 & 0\end{array}\right]\to\left[\begin{array}{rrr|r}1 & 0 & -1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right],$$

so that the eigenspace is

$$\operatorname{span}\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right).$$

Since $\lambda=0$ has algebraic multiplicity 2 but geometric multiplicity 1, the matrix is not diagonalizable.

14. To determine whether A is diagonalizable, we first compute its eigenvalues:

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & 0 & 0 & 2\\ 0 & 3-\lambda & 2 & 1\\ 0 & 0 & 3-\lambda & 0\\ 0 & 0 & 0 & 1-\lambda\end{vmatrix}=(1-\lambda)(2-\lambda)(3-\lambda)^2$$

since the matrix is upper triangular. So the eigenvalues are $\lambda_1=1$ with algebraic multiplicity 1, $\lambda_2=2$ with algebraic multiplicity 1, and $\lambda_3=3$ with algebraic multiplicity 2. To find the eigenspace corresponding to $\lambda_3=3$ we row-reduce $\left[A-3I \mid 0\right]$:

$$\left[A-3I \mid 0\right]=\left[\begin{array}{rrrr|r}-1 & 0 & 0 & 2 & 0\\ 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 0\end{array}\right]\to\left[\begin{array}{rrrr|r}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right].$$

Thus the eigenspace is

$$\begin{bmatrix}0\\t\\0\\0\end{bmatrix}=t\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\operatorname{span}\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right).$$

Thus the geometric multiplicity of this eigenvalue is only 1 while the algebraic multiplicity is 2. So by Theorem 4.27, the matrix is not diagonalizable.

15. Since A is triangular, we can read off its eigenvalues, which are $\lambda_1=2$ and $\lambda_2=-2$, each of which has algebraic multiplicity 2. Computing eigenspaces, we have

$$\left[A-2I \mid 0\right]=\left[\begin{array}{rrrr|r}0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -4 & 0 & 0\\ 0 & 0 & 0 & -4 & 0\end{array}\right]\to\left[\begin{array}{rrrr|r}0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right],\text{ so } E_2 \text{ has basis }\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}$$

$$\left[A+2I \mid 0\right]\left[\begin{array}{rrrr|r}4 & 0 & 0 & 4 & 0\\ 0 & 4 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]\to\left[\begin{array}{rrrr|r}1 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right],\text{ so } E_{-2} \text{ has basis }\left\{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}.$$

Since each eigenvalue also has geometric multiplicity 2, the matrix is diagonalizable, and

$$P=\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix},\quad D=\begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$$

provides a solution to

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = D.$$

16. As in Example 4.29, we must find a matrix P such that

$$P^{-1}AP = P^{-1} \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} P$$

is diagonal. To do this, we find the eigenvalues and a basis for the eigenspaces of A. The eigenvalues of A are the roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = (-4-\lambda)(5-\lambda) + 18 = \lambda^2 - \lambda - 2.$$

This polynomial factors as $(\lambda - 2)(\lambda + 1)$, so the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$. The eigenspace associated with λ_1 can be found by finding the null space of $A + I$:

$$\begin{bmatrix} A + I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -3 & 6 & 0 \\ -3 & 6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus a basis for this eigenspace is given by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

The eigenspace associated with λ_2 can be found by finding the null space of $A - 2I$:

$$\begin{bmatrix} A - 2I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -6 & 6 & 0 \\ -3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus a basis for this eigenspace is given by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Set

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \text{so that } P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Then

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \text{ so that } A^5 = PD^5P^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -34 & 66 \\ -33 & 65 \end{bmatrix}$$

17. As in Example 4.29, we must first diagonalize A. To do this, we find the eigenvalues and a basis for the eigenspaces of A. The eigenvalues of A are the roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 6 \\ 1 & -\lambda \end{vmatrix} = (-1-\lambda)(-\lambda) - 6 = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2),$$

so the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$. To compute the eigenspaces, we row-reduce $\begin{bmatrix} A + 3I \mid 0 \end{bmatrix}$ and $\begin{bmatrix} A - 2I \mid 0 \end{bmatrix}$:

$$\begin{bmatrix} A + 3I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} 2 & 6 & 0 \\ 1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ so } E_{-3} \text{ has basis } \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} A - 2I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -3 & 6 & 0 \\ 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ so } E_2 \text{ has basis } \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

Thus

$$P = \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix} D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

satisfy $P^{-1}AP = D$, so that

$$\begin{aligned} \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}^{10} &= (PDP^{-1})^{10} = PD^{10}P^{=1} \\ &= \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}^{10} \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-3)^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}^{10} \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} 35{,}839 & -69{,}630 \\ -11{,}605 & 24{,}234 \end{bmatrix}. \end{aligned}$$

18. As in Example 4.29, we must first diagonalize A. To do this, we find the eigenvalues and a basis for the eigenspaces of A. The eigenvalues of A are the roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -3 \\ -1 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda-5)(\lambda-1),$$

so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 5$. To compute the eigenspaces, we row-reduce $\left[A - I \mid 0\right]$ and $\left[A - 5I \mid 0\right]$:

$$\begin{aligned} \left[A - I \mid 0\right] &= \left[\begin{array}{rr|r} 3 & -3 & 0 \\ -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ so that } E_1 \text{ has basis } \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}, \\ \left[A - 5I \mid 0\right] &= \left[\begin{array}{rr|r} -1 & -3 & 0 \\ -1 & -3 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ so that } E_5 \text{ has basis } \left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}. \end{aligned}$$

Set

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}, \quad \text{so that } P^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4}, \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

Then

$$A^{-6} = PD^{-6}P^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^{-6} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{3907}{5^6} & \frac{11718}{5^6} \\ \frac{3906}{5^6} & \frac{11719}{5^6} \end{bmatrix}.$$

19. As in Example 4.29, we must first diagonalize A. To do this, we find the eigenvalues and a basis for the eigenspaces of A. The eigenvalues of A are the roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = (-\lambda)(2-\lambda) - 3 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1),$$

so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$. To compute the eigenspaces, we row-reduce $\left[A - 3I \mid 0\right]$ and $\left[A + I \mid 0\right]$:

$$\begin{aligned} \left[A - 3I \mid 0\right] &= \left[\begin{array}{rr|r} -3 & 3 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ so that } E_3 \text{ has basis } \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}, \\ \left[A + I \mid 0\right] &= \left[\begin{array}{rr|r} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ so that } E_{-1} \text{ has basis } \left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}. \end{aligned}$$

Set

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}, \quad \text{so that } P^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4}, \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$\begin{aligned} A^k = PD^kP^{-1} &= \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3^k+3(-1)^k}{4} & \frac{3^{k+1}-3(-1)^k}{4} \\ \frac{3^k-(-1)^k}{4} & \frac{3^{k+1}+(-1)^k}{4} \end{bmatrix}. \end{aligned}$$

20. As in Example 4.29, we must first diagonalize A. To do this, we find the eigenvalues and a basis for the eigenspaces of A. The eigenvalues of A are the roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 5\lambda^2 - \lambda^3 = -\lambda^2(\lambda - 5),$$

so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 5$. To compute the eigenspaces we row-reduce $\left[A - 0I \mid 0\right]$ and $\left[A - 5I \mid 0\right]$:

$$\left[A \mid 0\right] = \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } E_0 \text{ has basis } \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\left[A - 5I \mid 0\right] = \left[\begin{array}{ccc|c} -3 & 1 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 1 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } E_5 \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Set

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{so that } P^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

Then

$$D = P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

so that

$$A^8 = PD^8P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5^8 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 156250 & 78125 & 156250 \\ 156250 & 78125 & 156250 \\ 156250 & 78125 & 156250 \end{bmatrix}.$$

21. As in Example 4.29, we must first diagonalize A. To do this, we find the eigenvalues and a basis for the eigenspaces of A. Since A is upper triangular, its eigenvalues are the diagonal entries, which are $\lambda_1 = -1$ and $\lambda_2 = 1$. To compute the eigenspaces, we row-reduce $\left[A + I \mid 0\right]$ and $\left[A - I \mid 0\right]$:

$$\left[A + I \mid 0\right] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } E_{-1} \text{ has basis } \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\},$$

$$\left[A - I \mid 0\right] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } E_1 \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Set

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad \text{so that } P^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then

$$D = P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that

$$\begin{aligned} A^{2015} = PD^{2015}P^{-1} &= \begin{bmatrix} -1 & -1 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} (-1)^{2015} & 0 & 0 \\ 0 & (-1)^{2015} & 0 \\ 0 & 0 & 1^{2015} \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A. \end{aligned}$$

22. As in Example 4.29, we must first diagonalize A. To do this, we find the eigenvalues and a basis for the eigenspaces of A. The eigenvalues of A are the roots of its characteristic polynomial

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)((2-\lambda)^2 - 1) = (1-\lambda)(\lambda^2 - 4\lambda + 3) = -(\lambda-1)^2(\lambda-3). \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. To compute the eigenspaces, we row-reduce $\begin{bmatrix} A - I \mid 0 \end{bmatrix}$ and $\begin{bmatrix} A - 3I \mid 0 \end{bmatrix}$.

$$\begin{aligned} \begin{bmatrix} A - I \mid 0 \end{bmatrix} &= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } E_1 \text{ has basis } \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \\ \begin{bmatrix} A - 3I \mid 0 \end{bmatrix} &= \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } E_3 \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

Set

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{so that } P^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that

$$\begin{aligned}
A^k = PD^kP^{-1} &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}(3^k+1) & 0 & \frac{1}{2}(3^k-1) \\ \frac{1}{2}(3^k-1) & 1 & \frac{1}{2}(3^k-1) \\ \frac{1}{2}(3^k-1) & 0 & \frac{1}{2}(3^k+1) \end{bmatrix}
\end{aligned}$$

23. As in Example 4.29, we must first diagonalize A. To do this, we find the eigenvalues and a basis for the eigenspaces of A. The eigenvalues of A are the roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & -2-\lambda & 2 \\ 0 & 1 & 1-\lambda \end{vmatrix} = -(\lambda-1)(\lambda-2)(\lambda+3).$$

Thus the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_2 = -3$. To compute the eigenspaces, we row-reduce $\begin{bmatrix} A - I \mid 0 \end{bmatrix}$, $\begin{bmatrix} A - 2I \mid 0 \end{bmatrix}$, and $\begin{bmatrix} A + 3I \mid 0 \end{bmatrix}$.

$$\begin{bmatrix} A - I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } E_1 \text{ has basis } \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} A - 2I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } E_2 \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} A + 3I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } E_3 \text{ has basis } \left\{ \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

Set

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -4 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{so that } P^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{10} & -\frac{1}{5} & \frac{1}{10} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

so that

$$\begin{aligned}
A^k = PD^kP^{-1} &= \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & (-3)^k \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{10} & -\frac{1}{5} & \frac{1}{10} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{10}(5 + (-3)^k + 2^{k+2}) & \frac{1}{5}(2^k - (-3)^k) & \frac{1}{10}(-5 + (-3)^k + 2^{k+2}) \\ -\frac{2}{5}((-3)^k - 2^k) & \frac{1}{5}(4(-3)^k + 2^k) & -\frac{2}{5}((-3)^k - 2^k) \\ \frac{1}{10}(-5 + (-3)^k + 2^{k+2}) & \frac{1}{5}(2^k - (-3)^k) & \frac{1}{10}(5 + (-3)^k + 2^{k+2}) \end{bmatrix}.
\end{aligned}$$

24. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & k-\lambda \end{vmatrix} = (\lambda-1)(\lambda-k),$$

so that if $k \neq 1$, then A is a 2×2 matrix with two distinct eigenvalues, so it is diagonalizable. If $k = 1$, then

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which gives a one-dimensional eigenspace with basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since $\lambda = 1$ has algebraic multiplicity 2 but geometric multiplicity 1, A is not diagonalizable. So A is diagonalizable exactly when $k \neq 1$.

25. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & k \\ 0 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2.$$

The only eigenvalue is 1, and then

$$A - I = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}.$$

If $k = 0$, then A is diagonal to start with, while if $k \neq 0$, then we get a one-dimensional eigenspace with basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since $\lambda = 1$ then has algebraic multiplicity 2 but geometric multiplicity 1, A is not diagonalizable. So A is diagonalizable exactly when $k = 0$.

26. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} k-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda(\lambda - k) - 1 = \lambda^2 - k\lambda - 1,$$

so that A has the eigenvalues $\lambda = \frac{k\pm\sqrt{k^2+4}}{2}$. Since $k^2 + 4$ is never zero, and always positive, the eigenvalues of A are real and distinct, so A is diagonalizable for all k.

27. Since A is triangular, the only eigenvalue is 1, with algebraic multiplicity 3. Then

$$A - I = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $k = 0$, then A is diagonal to start with, while if $k \neq 0$, then we get a one-dimensional eigenspace with basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$. Since $\lambda = 1$ then has algebraic multiplicity 3 but geometric multiplicity 2, A is not diagonalizable. So A is diagonalizable exactly when $k = 0$.

28. Since A is triangular, the eigenvalues are 1, with algebraic multiplicity 2, and 2, with algebraic multiplicity 1. Then

$$A - I = \begin{bmatrix} 0 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

regardless of the value of k, so $\lambda = 1$ has a two-dimensional eigenspace with basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$, i.e., with geometric multiplicity 2. Since this is equal to the algebraic multiplicity of λ, we see that A is diagonalizable for all k.

29. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & k \\ 1 & 1-\lambda & k \\ 1 & 1 & k-\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + k\lambda^2 = -\lambda^2(\lambda - (k+2)).$$

If $k = -2$, then A has only the eigenvalue 0, with algebraic multiplicity 3, and then

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so the eigenspace is two-dimensional and therefore the geometric multiplicity is not equal to the algebraic multiplicity, so that A is not diagonalizable. If $k \neq -2$, then 0 has algebraic multiplicity 2, and then

$$A = \begin{bmatrix} 1 & 1 & -k \\ 1 & 1 & -k \\ 1 & 1 & -k \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the eigenspace is again two-dimensional, so the geometric multiplicity equals the algebraic multiplicity and A is diagonalizable. Thus A is diagonalizable exactly when $k \neq -2$.

30. We are given that $A \sim B$, so that $B = P^{-1}AP$ for some invertible matrix P, and that $B \sim C$, so that $C = Q^{-1}BQ$ for some invertible matrix Q. But then

$$C = Q^{-1}BQ = Q^{-1}P^{-1}APQ = (PQ)^{-1}APQ,$$

so that C is $R^{-1}AR$ for the invertible matrix $R = PQ$.

31. A is invertible if and only if $\det A \neq 0$. But since A and B are similar, $\det A = \det B$ by part (a) of this theorem. Thus A is invertible if and only if $\det B \neq 0$, which happens if and only if B is invertible.

32. By Exercise 61 in Section 3.5, if U is invertible then $\operatorname{rank} A = \operatorname{rank}(UA) = \operatorname{rank}(AU)$. Now, since A and B are similar, we have $B = P^{-1}AP$ for some invertible matrix P, so that

$$\operatorname{rank} B = \operatorname{rank}(P^{-1}(AP)) = \operatorname{rank}(AP) = \operatorname{rank} A,$$

using the result above twice.

33. Suppose that $B = P^{-1}AP$ where P is invertible. This can be rewritten as $BP^{-1} = P^{-1}A$. Let λ be an eigenvalue of A with eigenvector $\mathbf{x}$. Then

$$B(P^{-1}\mathbf{x}) = (BP^{-1})\mathbf{x} = (P^{-1}A)\mathbf{x} = P^{-1}(A\mathbf{x}) = P^{-1}(\lambda\mathbf{x}) = \lambda(P^{-1}\mathbf{x}).$$

Thus $P^{-1}\mathbf{x}$ is an eigenvector for B, corresponding to λ. So every eigenvalue of A is also an eigenvalue of B. By writing the similarity equation as $A = Q^{-1}BQ$, we see in exactly the same way that every eigenvalue of B is an eigenvalue of A. So A and B have the same eigenvalues. (Note that we have also proven a relationship between the eigenvectors corresponding to a given eigenvalue).

34. Suppose that $A \sim B$, say $A = P^{-1}BP$. Then if $m = 0$, we have $A^m = B^m = I$, so that $A^0 \sim B^0$. If $m > 0$, then

$$A^m = \left(P^{-1}BP\right)^m = \underbrace{P^{-1}BP \cdot P^{-1}BP \cdots P^{-1}BP \cdot P^{-1}BP}_{m \text{ times}} = P^{-1}B^mP.$$

But $A^m = P^{-1}B^mP$ means that $A^m \sim B^m$.

35. From part (f), $A^m \sim B^m$ for $m \geq 0$. If $m < 0$, then

$$A^m = \left(A^{-m}\right)^{-1} = \left(P^{-1}B^{-m}P\right)^{-1}$$

since A^{-m} and B^{-m} are similar because $-m > 0$. But $\left(P^{-1}B^{-m}P\right)^{-1} = P^{-1}\left(B^{-m}\right)^{-1}\left(P^{-1}\right)^{-1} = P^{-1}B^mP$. Thus $A^m \sim B^m$ for $m < 0$, so they are similar for all integers m.

36. Let $P = B^{-1}$. Then

$$BA = BABB^{-1} = B(AB)B^{-1} = P^{-1}(AB)P,$$

showing that $AB \sim BA$.

37. Exercise 45 in Section 3.2 shows that if A and B are $n \times n$, then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. But then if $B = P^{-1}AP$, we have

$$\operatorname{tr} B = \operatorname{tr}(P^{-1}AP) = \operatorname{tr}((P^{-1}A)P) = \operatorname{tr}(P(P^{-1}A)) = \operatorname{tr}((PP^{-1})A) = \operatorname{tr} A.$$

38. A is triangular, so it has eigenvalues -1 and 3. For B, we have

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3).$$

So the eigenvalues of B are also -1 and 3, and thus

$$A \sim B \sim \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = D.$$

Reducing $A + I$ and $A - 3I$ shows that $E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right)$ and $E_3 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. Reducing $B + I$ and $B - 3I$ shows that $E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ and $E_3 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Thus

$$D = Q^{-1}AQ = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -4 & 0 \end{bmatrix}$$
$$D = R^{-1}BR = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

so that finally

$$B = RDR^{-1} = RQ^{-1}AQR^{-1} = (QR^{-1})^{-1}A(QR^{-1}).$$

Thus

$$P = QR^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}$$

is an invertible matrix such that $B = P^{-1}AP$.

39. We have

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -3 \\ 4 & -2-\lambda \end{vmatrix} = (5-\lambda)(-2-\lambda) + 12 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$$
$$\det(B - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ -6 & 4-\lambda \end{vmatrix} = (-1-\lambda)(4-\lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2).$$

So A and B both have eigenvalues 1 and 2, so that

$$A \sim B \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = D.$$

Reducing $A - I$ and $A - 2I$ shows that $E_1 = \operatorname{span}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$ and $E_2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Reducing $B - I$ and $B - 2I$ shows that $E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ and $E_3 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$. Thus

$$D = Q^{-1}AQ = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$$
$$D = R^{-1}BR = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix},$$

so that finally

$$B = RDR^{-1} = RQ^{-1}AQR^{-1} = (QR^{-1})^{-1}A(QR^{-1}).$$

Thus

$$P = QR^{-1} = \begin{bmatrix} 7 & -2 \\ 10 & -3 \end{bmatrix}$$

is an invertible matrix such that $B = P^{-1}AP$.

40. A is upper triangular, so it has eigenvalues -2, 1, and 2. For B, we have

$$\det(B - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & -5 \\ 1 & 2-\lambda & -1 \\ 2 & 2 & -4-\lambda \end{vmatrix} = -(\lambda+2)(\lambda-1)(\lambda-2),$$

So B also has eigenvalues -2, 1, and 2, so that

$$A \sim B \sim \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D.$$

Reducing $A + 2I$, $A - I$, and $A - 2I$ gives

$$E_{-2} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}\right), \quad E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}\right), \quad E_2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

Reducing $B + 2I$, $B - I$, and $B - 2I$ gives

$$E_{-2} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right), \quad E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right), \quad E_2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right).$$

Therefore

$$D = Q^{-1}AQ = \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{12} \\ 0 & 0 & -\frac{1}{3} \\ 1 & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -4 & -1 & 0 \\ 0 & -3 & 0 \end{bmatrix}$$

$$D = R^{-1}BR = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 2 & -5 \\ 1 & 2 & -1 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix},$$

so that finally

$$B = RDR^{-1} = RQ^{-1}AQR^{-1} = (QR^{-1})^{-1}A(QR^{-1}).$$

Thus

$$P = QR^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -5 \\ -3 & 0 & 3 \end{bmatrix}$$

is an invertible matrix such that $B = P^{-1}AP$.

41. We have

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & -1-\lambda & 1 \\ 2 & 0 & 1-\lambda \end{vmatrix} = -(\lambda+1)^2(\lambda-3)$$

$$\det(B - \lambda I) = \begin{vmatrix} -3-\lambda & -2 & 0 \\ 6 & 5-\lambda & 0 \\ 4 & 4 & -1-\lambda \end{vmatrix} = -(\lambda+1)^2(\lambda-3),$$

so that A and B both have eigenvalues -1 (of algebraic multiplicity 2) and 3. Therefore

$$A \sim B \sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D.$$

Reducing $A + I$ and $A - 3I$ gives

$$E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), \quad E_3 = \operatorname{span}\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right).$$

Reducing $B + I$ and $B - 3I$ gives

$$E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right), \quad E_3 = \operatorname{span}\left(\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}\right).$$

Therefore

$$D = Q^{-1}AQ = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{4} & 1 & -\frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

$$D = R^{-1}BR = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 0 \\ -1 & -1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} -3 & -2 & 0 \\ 6 & 5 & 0 \\ 4 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix},$$

so that finally

$$B = RDR^{-1} = RQ^{-1}AQR^{-1} = (QR^{-1})^{-1}A(QR^{-1}).$$

Thus

$$P = QR^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{5}{2} & -\frac{3}{2} & 0 \end{bmatrix}$$

is an invertible matrix such that $B = P^{-1}AP$.

42. Suppose that $B = P^{-1}AP$. This gives $PB = AP$, so that $(PB)^T = (AP)^T$. Expanding yields $B^TP^T = P^TA^T$, so that $B^T \sim A^T$ and thus $A^T \sim B^T$.

43. Suppose that $P^{-1}AP = D$, where D is diagonal. Then

$$D = D^T = (P^{-1}AP)^T = P^TA^T(P^T)^{-1}.$$

Now let $Q = (P^T)^{-1}$; the equation above becomes $D = Q^{-1}A^TQ$, so that A^T is diagonalizable.

44. If A is invertible, it has no zero eigenvalues. So if it is diagonalizable, the diagonal matrix D is also invertible since none of the diagonal entries are zero. Then D^{-1} is also diagonal, consisting of the inverses of each of the diagonal entries of D. Suppose that $D = P^{-1}AP$. Then

$$D^{-1} = \left(P^{-1}AP\right)^{-1} = P^{-1}A^{-1}\left(P^{-1}\right)^{-1} = P^{-1}A^{-1}P,$$

showing that A^{-1} is also diagonalizable (and, incidentally, showing again that the eigenvalues of A^{-1} are the inverses of the eigenvalues of A, by looking at the entries of D^{-1}.)

45. Suppose that A is diagonalizable, and that λ is its only eigenvalue. Then if $D = P^{-1}AP$ is diagonal, all of its diagonal entries must equal λ, so that $D = \lambda I$. Then $\lambda I = P^{-1}AP$ implies that $P(\lambda I)P^{-1} = PP^{-1}APP^{-1} = A$. However, $P(\lambda I)P^{-1} = \lambda PIP^{-1} = \lambda I$. Thus $A = \lambda I$.

46. Since A and B each have n distinct eigenvalues, they are both diagonalizable by Theorem 4.25. Recall that diagonal matrices commute: if D and E are both diagonal, then $DE = ED$. Now, if A and B have the same eigenvectors, then since the diagonalizing matrix P has columns equal to the eigenvectors, we see that for the *same* invertible matrix P, both $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$ are diagonal. But then

$$AB = (PD_AP^{-1})(PD_BP^{-1}) = PD_AD_BP^{-1} = PD_BD_AP^{-1} = (PD_BP^{-1})(PD_AP^{-1}) = BA.$$

Conversely, if $AB = BA$, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the distinct eigenvalues of A, and $\mathbf{p}_i$ the corresponding eigenvectors, which are also distinct since they are linearly independent. Then

$$A(B\mathbf{p}_i) = (AB)\mathbf{p}_i = (BA)\mathbf{p}_i = B(A\mathbf{p}_i) = B(\alpha_i\mathbf{P}_i) = \alpha_i B\mathbf{p}_i.$$

Thus $B\mathbf{p}_i$ is also an eigenvector of A corresponding to α_i, so it must be a multiple of $\mathbf{p}_i$. That is, $B\mathbf{p}_i = \beta_i\mathbf{p}_i$ for all i, so that the eigenvectors of B are also $\mathbf{p}_i$. So every eigenvector of A is an eigenvector of B. Since B also has n distinct eigenvectors, the set of eigenvectors must be the same.

47. If $A \sim B$, then A and B have the same characteristic polynomial, so the algebraic multiplicities of their eigenvalues are the same.

48. From the solution to Exercise 33, we know that if $B = P^{-1}AP$ and $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvalue of B with eigenvalue λ. Therefore, if λ is an eigenvalue of A with multiplicity m, its eigenspace has a basis consisting of m distinct vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, so that the vectors $P^{-1}\mathbf{v}_1, P^{-1}\mathbf{v}_2, \ldots, P^{-1}\mathbf{v}_m$ are in the eigenspace of B corresponding to λ. These vectors are linearly independent since P^{-1} is an invertible linear transformation. So each of the n eigenvectors of A corresponds to a distinct eigenvector of B; since each matrix has n eigenvectors, the reverse is true as well. Therefore the m eigenvalues of B found above are all of the eigenvalues of B corresponding to λ, so that the dimension of the eigenspace of λ for B is m as well. So the geometric multiplicities of the eigenvalues of A and B are the same.

49. If $D = P^{-1}AP$ and D is diagonal, then every entry of D is either 0 or 1, since every eigenvalue of A is either 0 or 1 by assumption. Since D^2 is computed by squaring each diagonal entry, we see that $D^2 = D$. But then $PDP^{-1} = A$, and

$$A^2 = (PDP^{-1})^2 = (PDP^{-1})(PDP^{-1}) = P^{-1}D^2P = P^{-1}DP = A,$$

so that A is idempotent.

50. Suppose that A is nilpotent and diagonalizable, so that $D = P^{-1}AP$ (so that $A = PDP^{-1}$) and $A^m = O$ for some $m > 1$. Then

$$D^m = (P^{-1}AP)^m = P^{-1}A^mP = P^{-1}OP = O.$$

Since D is diagonal, D^m is found by taking the m^{th} power of each diagonal entry. If this is the zero matrix, then all the diagonal entries must be zero, so that $D = O$. But then $A = POP^{-1} = O$, so that A is the zero matrix.

51. **(a)** Suppose we have three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ with

$$A\mathbf{v}_1 = \mathbf{v}_1, \quad A\mathbf{v}_2 = \mathbf{v}_2, \quad A\mathbf{v}_3 = \mathbf{v}_3.$$

Then all three vectors must lie in the eigenspace E_1 corresponding to $\lambda = 1$, since all three vectors are multiplied by 1 when A is applied. However, since the algebraic multiplicity of $\lambda = 1$ is 2, the geometric multiplicity is no greater than 2, by Lemma 4.26. But any three vectors in a two-dimensional space must be linearly independent, so that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

(b) By Theorem 4.27, A is diagonalizable if and only if the algebraic multiplicity of each eigenvalue equals its geometric multiplicity. So the dimensions of the eigenspaces must be $\dim E_{-1} = 1$, $\dim E_1 = 2$, $\dim E_2 = 3$, since the algebraic multiplicity of -1 is 1, that of 1 is 2, and that of 2 is 3.

52. (a) The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc) = \lambda^2 - \operatorname{tr}(A) + \det(A).$$

Therefore the eigenvalues are

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{a+d \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

So if $(a-d)^2 + 4bc > 0$, then the two eigenvalues of A are real and distinct and therefore A is diagonalizable. If $(a-d)^2 + 4bc < 0$, then A has no real eigenvalues and so is not diagonalizable.

(b) If $(a-d)^2 + 4bc = 0$, then A has one real eigenvalue, and A may or may not be diagonalizable depending on the geometric multiplicity of that eigenvalue. For example, the zero matrix O has $(a-d)^2 + 4bc = 0$, and it is already a diagonal matrix, so is diagonalizable. However,

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

satisfies $(a-d)^2 + 4bc = 0$; from the formula above, its only real eigenvalue is $\lambda = 0$, and it is easy to see by row-reducing A that $E_0 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. So the geometric multiplicity of $\lambda = 0$ is only 1 and thus A is not diagonalizable, by Theorem 4.27.

4.5 Iterative Methods for Computing Eigenvalues

1. (a) For the given value of $\mathbf{x}_5$, we estimate a dominant eigenvector of A to be

$$\begin{bmatrix} 1 \\ \frac{11{,}109}{4443} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 2.500 \end{bmatrix}.$$

The corresponding dominant eigenvalue is computed by finding

$$\mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 4443 \\ 11{,}109 \end{bmatrix} = \begin{bmatrix} 26{,}661 \\ 66{,}651 \end{bmatrix},$$

and $\frac{26{,}661}{4443} \approx 6.001$, $\frac{66{,}651}{11{,}109} \approx 6.000$.

(b) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 10 = \lambda^2 - 5\lambda - 6 = (\lambda+1)(\lambda-6)$, so the dominant eigenvalue is 6.

2. (a) For the given value of $\mathbf{x}_5$, we estimate a dominant eigenvector of A to be

$$\begin{bmatrix} 1 \\ -\frac{3904}{7811} \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.500 \end{bmatrix}.$$

The corresponding dominant eigenvalue is computed by finding

$$\mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 7811 \\ -3904 \end{bmatrix} = \begin{bmatrix} 39{,}061 \\ -19{,}529 \end{bmatrix},$$

and $\frac{39{,}061}{7811} \approx 5.001$, $\frac{-19{,}529}{-3904} \approx 5.002$.

(b) $\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{vmatrix} = (7-\lambda)(-1-\lambda) + 12 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$, so the dominant eigenvalue is 5.

3. **(a)** For the given value of $\mathbf{x}_5$, we estimate a dominant eigenvector of A to be

$$\begin{bmatrix} 1 \\ \frac{89}{144} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0.618 \end{bmatrix}.$$

The corresponding dominant eigenvalue is computed by finding

$$\mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 144 \\ 89 \end{bmatrix} = \begin{bmatrix} 377 \\ 233 \end{bmatrix},$$

and $\frac{377}{144} \approx 2.618$, $\frac{233}{89} \approx 2.618$.

(b) $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1$, which has roots $\lambda = \frac{1}{2}(3 \pm \sqrt{5})$. So the dominant eigenvalue is $\frac{1}{2}(3 + \sqrt{5}) \approx 2.618$.

4. **(a)** For the given value of $\mathbf{x}_5$, we estimate a dominant eigenvector of A to be

$$\begin{bmatrix} 1 \\ \frac{239.500}{60.625} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 3.951 \end{bmatrix}.$$

The corresponding dominant eigenvalue is computed by finding

$$\mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} 1.5 & 0.5 \\ 2.0 & 3.0 \end{bmatrix} \begin{bmatrix} 60.625 \\ 239.500 \end{bmatrix} = \begin{bmatrix} 210.688 \\ 839.75 \end{bmatrix},$$

and $\frac{210.688}{60.625} \approx 3.475$, $\frac{839.75}{239.500} \approx 3.506$.

(b) $\det(A - \lambda I) = \begin{vmatrix} 1.5-\lambda & 0.5 \\ 2.0 & 3.0-\lambda \end{vmatrix} = (1.5-\lambda)(3.0-\lambda) - 1.0 = \lambda^2 - 4.5\lambda + 3.5$, which has roots $\lambda = 1$ and $\lambda = 3.5$. So the dominant eigenvalue is 3.5.

5. **(a)** The entry of largest magnitude in $\mathbf{x}_5$ is $m_5 = 11.001$, so that

$$\mathbf{y}_5 = \frac{1}{11.001}\mathbf{x}_5 = \begin{bmatrix} -\frac{3.667}{11.001} \\ 1 \end{bmatrix} = \begin{bmatrix} -0.333 \\ 1 \end{bmatrix}.$$

So the dominant eigenvalue is about 11 and $\mathbf{y}_5$ approximates the dominant eigenvector.

(b) We have

$$A\mathbf{y}_5 = \begin{bmatrix} 2 & -3 \\ -3 & 10 \end{bmatrix} \begin{bmatrix} -0.333 \\ 1 \end{bmatrix} = \begin{bmatrix} -3.666 \\ 10.999 \end{bmatrix}$$

$$\lambda_1\mathbf{y}_5 = 11.001 \begin{bmatrix} -0.333 \\ 1 \end{bmatrix} = \begin{bmatrix} -3.663 \\ 11.001 \end{bmatrix},$$

so we have indeed approximated an eigenvalue and an eigenvector of A.

6. **(a)** The entry of largest magnitude in $\mathbf{x}_{10}$ is $m_{10} = 5.530$, so that

$$\mathbf{y}_{10} = \frac{1}{5.530}\mathbf{x}_{10} = \begin{bmatrix} 1 \\ \frac{1.470}{5.530} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0.266 \end{bmatrix}.$$

So the dominant eigenvalue is about 5.53 and $\mathbf{y}_{10}$ approximates the dominant eigenvector.

(b) We have

$$A\mathbf{y}_{10} = \begin{bmatrix} 5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.266 \end{bmatrix} = \begin{bmatrix} 5.532 \\ 1.468 \end{bmatrix}$$

$$\lambda_1 \mathbf{y}_{10} = 5.530 \begin{bmatrix} 1 \\ 0.266 \end{bmatrix} = \begin{bmatrix} 5.530 \\ 1.471 \end{bmatrix},$$

so we have indeed approximated an eigenvalue and an eigenvector of A.

7. (a) The entry of largest magnitude in $\mathbf{x}_8$ is $m_8 = 10$, so that

$$\mathbf{y}_8 = \frac{1}{10}\mathbf{x}_8 = \begin{bmatrix} 1 \\ \frac{0.001}{10.0} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0.0001 \\ 1 \end{bmatrix}.$$

So the dominant eigenvalue is about 10 and $\mathbf{y}_8$ approximates the dominant eigenvector.

(b) We have

$$A\mathbf{y}_8 = \begin{bmatrix} 4 & 0 & 6 \\ -1 & 3 & 1 \\ 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0001 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 0.0003 \\ 10 \end{bmatrix}$$

$$\lambda_1 \mathbf{y}_8 = 10 \begin{bmatrix} 1 \\ 0.0001 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 0.001 \\ 10 \end{bmatrix},$$

so we have indeed approximated an eigenvalue and an eigenvector of A, though the approximation in the middle component is not great.

8. (a) The entry of largest magnitude in $\mathbf{x}_{10}$ is $m_{10} = 3.415$, so that

$$\mathbf{y}_{10} = \frac{1}{3.415}\mathbf{x}_{10} = \begin{bmatrix} 1 \\ \frac{2.914}{3.415} \\ -\frac{1.207}{3.415} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0.853 \\ -0.353 \end{bmatrix}.$$

So the dominant eigenvalue is about 3.415 and $\mathbf{y}_{10}$ approximates the dominant eigenvector.

(b) We have

$$A\mathbf{y}_{10} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & -3 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.853 \\ -0.353 \end{bmatrix} = \begin{bmatrix} 3.412 \\ 2.912 \\ -1.206 \end{bmatrix}$$

$$\lambda_1 \mathbf{y}_{10} = 3.415 \begin{bmatrix} 1 \\ 0.853 \\ -0.353 \end{bmatrix} = \begin{bmatrix} 3.415 \\ 2.913 \\ -1.205 \end{bmatrix},$$

so we have indeed approximated an eigenvalue and an eigenvector of A.

9. With the given A and $\mathbf{x}_0$, we get the values below:

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 26 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 17.692 \\ 5.923 \end{bmatrix}$	$\begin{bmatrix} 18.017 \\ 6.004 \end{bmatrix}$	$\begin{bmatrix} 17.999 \\ 6.000 \end{bmatrix}$	$\begin{bmatrix} 18.000 \\ 6.000 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1.000 \\ 0.308 \end{bmatrix}$	$\begin{bmatrix} 1.000 \\ 0.335 \end{bmatrix}$	$\begin{bmatrix} 1.000 \\ 0.333 \end{bmatrix}$	$\begin{bmatrix} 1.000 \\ 0.333 \end{bmatrix}$	$\begin{bmatrix} 1.000 \\ 0.333 \end{bmatrix}$
m_k	1	26	17.692	18.017	17.999	18

We conclude that the dominant eigenvalue of A is 18, with eigenvector $\begin{bmatrix}3\\1\end{bmatrix}$.

10. With the given A and $\mathbf{x}_0$, we get the values below:

$$\begin{array}{c|ccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathbf{x}_k & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}-6\\8\end{bmatrix} & \begin{bmatrix}8.500\\8.000\end{bmatrix} & \begin{bmatrix}-9.765\\9.882\end{bmatrix} & \begin{bmatrix}9.929\\-9.905\end{bmatrix} & \begin{bmatrix}-9.990\\9.995\end{bmatrix} & \begin{bmatrix}9.997\\-9.996\end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}-0.75\\1.00\end{bmatrix} & \begin{bmatrix}1.000\\-0.941\end{bmatrix} & \begin{bmatrix}-0.988\\1.000\end{bmatrix} & \begin{bmatrix}1.000\\-0.998\end{bmatrix} & \begin{bmatrix}-1.000\\1.000\end{bmatrix} & \begin{bmatrix}1.000\\-1.000\end{bmatrix} \\ m_k & 1 & 8 & 8.5 & 9.882 & 9.929 & 9.995 & 9.997 \end{array}$$

We conclude that the dominant eigenvalue of A is -10 (not 10, since at each step the sign of each component of $\mathbf{y}_k$ is reversed) , with eigenvector $\begin{bmatrix}1\\-1\end{bmatrix}$.

11. With the given A and $\mathbf{x}_0$, we get the values below:

$$\begin{array}{c|ccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathbf{x}_k & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}7\\2\end{bmatrix} & \begin{bmatrix}7.571\\2.857\end{bmatrix} & \begin{bmatrix}7.755\\3.132\end{bmatrix} & \begin{bmatrix}7.808\\3.212\end{bmatrix} & \begin{bmatrix}7.823\\3.234\end{bmatrix} & \begin{bmatrix}7.827\\3.240\end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}1.000\\0.286\end{bmatrix} & \begin{bmatrix}1.000\\0.377\end{bmatrix} & \begin{bmatrix}1.000\\0.404\end{bmatrix} & \begin{bmatrix}1.000\\0.411\end{bmatrix} & \begin{bmatrix}-1.000\\0.412\end{bmatrix} & \begin{bmatrix}1.000\\0.414\end{bmatrix} \\ m_k & 1 & 7 & 7.571 & 7.755 & 7.808 & 7.823 & 7.827 \end{array}$$

We conclude that the dominant eigenvalue of A is about 7.83, with eigenvector about $\begin{bmatrix}1\\0.414\end{bmatrix}$. The true values are $\lambda = 5 + 2\sqrt{2}$ with eigenvector $\begin{bmatrix}1\\\sqrt{2}-1\end{bmatrix}$.

12. With the given A and $\mathbf{x}_0$, we get the values below:

$$\begin{array}{c|ccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathbf{x}_k & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}3.5\\1.5\end{bmatrix} & \begin{bmatrix}4.143\\1.286\end{bmatrix} & \begin{bmatrix}3.966\\1.345\end{bmatrix} & \begin{bmatrix}4.009\\1.330\end{bmatrix} & \begin{bmatrix}3.998\\1.334\end{bmatrix} & \begin{bmatrix}4.001\\1.333\end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}1.000\\0.429\end{bmatrix} & \begin{bmatrix}1.000\\0.310\end{bmatrix} & \begin{bmatrix}1.000\\0.339\end{bmatrix} & \begin{bmatrix}1.000\\0.332\end{bmatrix} & \begin{bmatrix}-1.000\\0.334\end{bmatrix} & \begin{bmatrix}1.000\\0.333\end{bmatrix} \\ m_k & 1 & 3.5 & 4.143 & 3.966 & 4.009 & 3.998 & 4.001 \end{array}$$

We conclude that the dominant eigenvalue of A is 4, with eigenvector $\begin{bmatrix}3\\1\end{bmatrix}$.

13. With the given A and $\mathbf{x}_0$, we get the values below:

$$\begin{array}{c|cccccc} k & 0 & 1 & 2 & 3 & 4 & 5 \\ \mathbf{x}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}21\\15\\13\end{bmatrix} & \begin{bmatrix}16.810\\12.238\\10.714\end{bmatrix} & \begin{bmatrix}17.011\\12.371\\10.824\end{bmatrix} & \begin{bmatrix}16.999\\12.363\\10.818\end{bmatrix} & \begin{bmatrix}17.000\\12.264\\10.818\end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}1.000\\0.714\\0.619\end{bmatrix} & \begin{bmatrix}1.000\\0.728\\0.637\end{bmatrix} & \begin{bmatrix}1.000\\0.727\\0.636\end{bmatrix} & \begin{bmatrix}1.000\\0.727\\0.636\end{bmatrix} & \begin{bmatrix}1.000\\0.727\\0.636\end{bmatrix} \\ m_k & 1 & 21 & 16.81 & 17.011 & 16.999 & 17 \end{array}$$

We conclude that the dominant eigenvalue of A is 17, with corresponding eigenvector about $\begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$.

In fact the eigenspace corresponding to this eigenvalue is $\operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right)$, and we have found one eigenvector in this space.

14. With the given A and $\mathbf{x}_0$, we get the values below:

$$
\begin{array}{c|ccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix} & \begin{bmatrix} 3.4 \\ 4.6 \\ 3.4 \end{bmatrix} & \begin{bmatrix} 3.217 \\ 4.478 \\ 3.217 \end{bmatrix} & \begin{bmatrix} 3.155 \\ 4.437 \\ 3.155 \end{bmatrix} & \begin{bmatrix} 3.133 \\ 4.422 \\ 3.133 \end{bmatrix} & \begin{bmatrix} 3.126 \\ 4.417 \\ 3.126 \end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.8 \\ 1.0 \\ 0.8 \end{bmatrix} & \begin{bmatrix} 0.739 \\ 1.000 \\ 0.739 \end{bmatrix} & \begin{bmatrix} 0.718 \\ 1.000 \\ 0.718 \end{bmatrix} & \begin{bmatrix} 0.711 \\ 1.000 \\ 0.711 \end{bmatrix} & \begin{bmatrix} 0.709 \\ 1.000 \\ 0.709 \end{bmatrix} & \begin{bmatrix} 0.708 \\ 1.000 \\ 0.708 \end{bmatrix} \\
m_k & 1 & 5 & 4.6 & 4.478 & 4.437 & 4.422 & 4.417
\end{array}
$$

We conclude that the dominant eigenvalue of A is ≈ 4.4, with corresponding eigenvector $\approx \begin{bmatrix} 0.708 \\ 1.000 \\ 0.708 \end{bmatrix}$.

In fact the dominant eigenvalue is $3 + \sqrt{2}$ with eigenspace spanned by $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$.

15. With the given matrix, and using $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we get

$$
\begin{array}{c|cccccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 8 \\ 2 \\ 4 \end{bmatrix} & \begin{bmatrix} 5.75 \\ 0.50 \\ 2.25 \end{bmatrix} & \begin{bmatrix} 5.26 \\ 0.17 \\ 1.87 \end{bmatrix} & \begin{bmatrix} 5.10 \\ 0.07 \\ 1.74 \end{bmatrix} & \begin{bmatrix} 5.04 \\ 0.03 \\ 1.70 \end{bmatrix} & \begin{bmatrix} 5.02 \\ 0.01 \\ 1.68 \end{bmatrix} & \begin{bmatrix} 5.01 \\ 0 \\ 1.67 \end{bmatrix} & \begin{bmatrix} 5 \\ 0 \\ 1.67 \end{bmatrix} & \begin{bmatrix} 5 \\ 0 \\ 1.67 \end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.25 \\ 0.50 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.09 \\ 0.39 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.03 \\ 0.36 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.01 \\ 0.34 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.01 \\ 0.34 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0.33 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0.33 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0.33 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0.33 \end{bmatrix} \\
m_k & 1 & 8 & 5.75 & 5.26 & 5.10 & 5.04 & 5.02 & 5.01 & 5 & 5
\end{array}
$$

We conclude that the dominant eigenvalue of A is 5, with corresponding eigenvector $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

16. With the given matrix, and using $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we get

$$
\begin{array}{c|cccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 \\
\mathbf{x}_k & \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 24 \\ 2 \\ 6 \end{bmatrix} & \begin{bmatrix} 11.0 \\ 1.5 \\ -2.5 \end{bmatrix} & \begin{bmatrix} 14.18 \\ 2.45 \\ -7.91 \end{bmatrix} & \begin{bmatrix} 16.38 \\ 3.12 \\ -11.65 \end{bmatrix} & \begin{bmatrix} 16.38 \\ 3.12 \\ -11.65 \end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.08 \\ 0.25 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.14 \\ -0.23 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.17 \\ -0.56 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.19 \\ -0.71 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.20 \\ -0.77 \end{bmatrix} \\
m_k & 2 & 24 & 11 & 14.18 & 16.38 & 17.41
\end{array}
$$

$$\begin{array}{c|cccccc} k & 6 & 7 & 8 & 9 & 10 & 11 \\ \mathbf{x}_k & \begin{bmatrix} 17.80 \\ 3.54 \\ -14.05 \end{bmatrix} & \begin{bmatrix} 17.93 \\ 3.58 \\ -14.28 \end{bmatrix} & \begin{bmatrix} 17.98 \\ 3.59 \\ -14.36 \end{bmatrix} & \begin{bmatrix} 17.99 \\ 3.60 \\ -14.39 \end{bmatrix} & \begin{bmatrix} 18.00 \\ 3.60 \\ -14.40 \end{bmatrix} & \begin{bmatrix} 18.00 \\ 3.60 \\ -14.40 \end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix} 1 \\ 0.20 \\ -0.79 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.2 \\ -0.8 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.2 \\ -0.8 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.2 \\ -0.8 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.2 \\ -0.8 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.2 \\ -0.8 \end{bmatrix} \\ m_k & 17.8 & 17.93 & 17.98 & 17.99 & 18 & 18 \end{array}$$

We conclude that the dominant eigenvalue of A is 18, with corresponding eigenvector $\begin{bmatrix} 1 \\ 0.2 \\ -0.8 \end{bmatrix}$, or $\begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$.

17. With the given A and $\mathbf{x}_0$, we get the values below:

$$\begin{array}{c|cccc} k & 0 & 1 & 2 & 3 \\ \mathbf{x}_k & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 7 \\ 2 \end{bmatrix} & \begin{bmatrix} 7.571 \\ 2.857 \end{bmatrix} & \begin{bmatrix} 7.755 \\ 3.132 \end{bmatrix} \\ R(\mathbf{x}_k) & 7 & 7.755 & 7.823 & 7.828 \end{array}$$

The Rayleigh quotient converges to the dominant eigenvalue, to three decimal places, after three iterations.

18. With the given A and $\mathbf{x}_0$, we get the values below:

$$\begin{array}{c|cccc} k & 0 & 1 & 2 & 3 \\ \mathbf{x}_k & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 3.5 \\ 1.5 \end{bmatrix} & \begin{bmatrix} 4.143 \\ 1.286 \end{bmatrix} & \begin{bmatrix} 3.966 \\ 1.345 \end{bmatrix} \\ R(\mathbf{x}_k) & 3.5 & 3.966 & 3.998 & 4 \end{array}$$

The Rayleigh quotient converges to the dominant eigenvalue after three iterations.

19. With the given A and $\mathbf{x}_0$, we get the values below:

$$\begin{array}{c|ccc} k & 0 & 1 & 2 \\ \mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 21 \\ 15 \\ 13 \end{bmatrix} & \begin{bmatrix} 16.810 \\ 12.238 \\ 10.714 \end{bmatrix} \\ R(\mathbf{x}_k) & 16.333 & 16.998 & 17 \end{array}$$

The Rayleigh quotient converges to the dominant eigenvalue after two iterations.

20. With the given A and $\mathbf{x}_0$, we get the values below:

$$\begin{array}{c|cccc} k & 0 & 1 & 2 & 3 \\ \mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix} & \begin{bmatrix} 3.4 \\ 4.6 \\ 3.4 \end{bmatrix} & \begin{bmatrix} 3.217 \\ 4.478 \\ 3.217 \end{bmatrix} \\ R(\mathbf{x}_k) & 4.333 & 4.404 & 4.413 & 4.414 \end{array}$$

The Rayleigh quotient converges to the dominant eigenvalue, to three decimal places, after three iterations.

21. With the given matrix, and using $\mathbf{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}$, we get

$$\begin{array}{c|ccccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\mathbf{x}_k & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}5\\4\end{bmatrix} & \begin{bmatrix}4.8\\3.2\end{bmatrix} & \begin{bmatrix}4.67\\2.67\end{bmatrix} & \begin{bmatrix}4.57\\2.29\end{bmatrix} & \begin{bmatrix}4.50\\2.00\end{bmatrix} & \begin{bmatrix}4.44\\1.78\end{bmatrix} & \begin{bmatrix}4.40\\1.60\end{bmatrix} & \begin{bmatrix}4.36\\1.45\end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}1\\0.8\end{bmatrix} & \begin{bmatrix}1\\0.67\end{bmatrix} & \begin{bmatrix}1\\0.57\end{bmatrix} & \begin{bmatrix}1\\0.5\end{bmatrix} & \begin{bmatrix}1\\0.44\end{bmatrix} & \begin{bmatrix}1\\0.40\end{bmatrix} & \begin{bmatrix}1\\0.36\end{bmatrix} & \begin{bmatrix}1\\0.33\end{bmatrix} \\
m_k & 4.5 & 4.49 & 4.46 & 4.43 & 4.4 & 4.37 & 4.34 & 4.32 & 4.3
\end{array}$$

The characteristic polynomial is

$$\det(A - \lambda I) = \begin{bmatrix}4-\lambda & 1\\0 & 4-\lambda\end{bmatrix} = (\lambda - 4)^2,$$

so that A has the double eigenvalue 4. To find the corresponding eigenspace,

$$\begin{bmatrix}A - 4I \mid 0\end{bmatrix} = \left[\begin{array}{cc|c}0 & 1 & 0\\0 & 0 & 0\end{array}\right],$$

so that the eigenspace has basis $\begin{bmatrix}1\\0\end{bmatrix}$. The values m_k appear to be converging to 4, but very slowly; the vectors $\mathbf{y}_k$ may also be converging to the eigenvector, but very slowly if at all.

22. With the given matrix, and using $\mathbf{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}$, we get

$$\begin{array}{c|ccccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\mathbf{x}_k & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}4\\0\end{bmatrix} & \begin{bmatrix}3\\-1\end{bmatrix} & \begin{bmatrix}2.67\\-1.33\end{bmatrix} & \begin{bmatrix}2.50\\-1.50\end{bmatrix} & \begin{bmatrix}2.40\\-1.60\end{bmatrix} & \begin{bmatrix}2.33\\-1.67\end{bmatrix} & \begin{bmatrix}2.29\\-1.71\end{bmatrix} & \begin{bmatrix}2.25\\-1.70\end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}1\\-0.33\end{bmatrix} & \begin{bmatrix}1\\-0.5\end{bmatrix} & \begin{bmatrix}1\\-0.6\end{bmatrix} & \begin{bmatrix}1\\-0.67\end{bmatrix} & \begin{bmatrix}1\\-0.71\end{bmatrix} & \begin{bmatrix}1\\-0.75\end{bmatrix} & \begin{bmatrix}1\\-0.78\end{bmatrix} \\
m_k & 2 & 3 & 2.8 & 2.6 & 2.47 & 2.38 & 2.32 & 2.28 & 2.25
\end{array}$$

The characteristic polynomial is

$$\det(A - \lambda I) = \begin{bmatrix}3-\lambda & 1\\-1 & 1-\lambda\end{bmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2,$$

so that A has the double eigenvalue 2. To find the corresponding eigenspace,

$$\begin{bmatrix}A - 2I \mid 0\end{bmatrix} = \left[\begin{array}{cc|c}1 & 1 & 0\\-1 & -1 & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & 1 & 0\\0 & 0 & 0\end{array}\right]$$

so that the eigenspace has basis $\begin{bmatrix}1\\-1\end{bmatrix}$. The values m_k appear to be converging to 2, but very slowly; the vectors $\mathbf{y}_k$ may also be converging to the eigenvector, but very slowly.

23. With the given matrix, and using $\mathbf{x}_0 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, we get

$$\begin{array}{c|ccccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\mathbf{x}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}5\\4\\1\end{bmatrix} & \begin{bmatrix}4.2\\3.2\\0.2\end{bmatrix} & \begin{bmatrix}4.05\\3.05\\0.05\end{bmatrix} & \begin{bmatrix}4.01\\3.01\\0.01\end{bmatrix} & \begin{bmatrix}4\\3\\0\end{bmatrix} & \begin{bmatrix}4\\3\\0\end{bmatrix} & \begin{bmatrix}4\\3\\0\end{bmatrix} & \begin{bmatrix}4\\3\\0\end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}1\\0.8\\0.2\end{bmatrix} & \begin{bmatrix}1\\0.76\\0.05\end{bmatrix} & \begin{bmatrix}1\\0.75\\0.01\end{bmatrix} & \begin{bmatrix}1\\0.75\\0\end{bmatrix} & \begin{bmatrix}1\\0.75\\0\end{bmatrix} & \begin{bmatrix}1\\0.75\\0\end{bmatrix} & \begin{bmatrix}1\\0.75\\0\end{bmatrix} & \begin{bmatrix}1\\0.75\\0\end{bmatrix} \\
m_k & 3.33 & 4.05 & 4.03 & 4.01 & 4 & 4 & 4 & 4 & 4
\end{array}$$

The matrix is upper triangular, so that A has an eigenvalue 1 and a double eigenvalue 4. To find the eigenspace for $\lambda = 4$,

$$\left[A - 4I \mid 0\right] = \left[\begin{array}{ccc|c}0 & 0 & 1 & 0\\0 & 0 & 0 & 0\\0 & 0 & -3 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}0 & 0 & 1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

so that the eigenspace has basis $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$. The values m_k converge to 4, while $\mathbf{y}_k$ converge apparently to $\begin{bmatrix}1\\0.75\\0\end{bmatrix}$, which is in the eigenspace.

24. With the given matrix, and using $\mathbf{x}_0 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, we get

$$\begin{array}{c|ccccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\mathbf{x}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}0\\6\\5\end{bmatrix} & \begin{bmatrix}0\\5.83\\4.17\end{bmatrix} & \begin{bmatrix}0\\5.71\\3.57\end{bmatrix} & \begin{bmatrix}0\\5.62\\3.12\end{bmatrix} & \begin{bmatrix}0\\5.56\\2.78\end{bmatrix} & \begin{bmatrix}0\\5.50\\2.50\end{bmatrix} & \begin{bmatrix}0\\5.45\\2.27\end{bmatrix} & \begin{bmatrix}0\\5.42\\2.08\end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}0\\1\\0.83\end{bmatrix} & \begin{bmatrix}0\\1\\0.71\end{bmatrix} & \begin{bmatrix}0\\1\\0.62\end{bmatrix} & \begin{bmatrix}0\\1\\0.56\end{bmatrix} & \begin{bmatrix}0\\1\\0.50\end{bmatrix} & \begin{bmatrix}0\\1\\0.45\end{bmatrix} & \begin{bmatrix}0\\1\\0.42\end{bmatrix} & \begin{bmatrix}0\\1\\0.38\end{bmatrix} \\
m_k & 3.67 & 5.49 & 5.47 & 5.45 & 5.42 & 5.4 & 5.38 & 5.36 & 5.34
\end{array}$$

The matrix is upper triangular, so that A has an eigenvalue 0 and a double eigenvalue 5. To find the eigenspace for $\lambda = 5$,

$$\left[A - 5I \mid 0\right] = \left[\begin{array}{ccc|c}-5 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

so that the eigenspace has basis $\begin{bmatrix}0\\1\\0\end{bmatrix}$. The values m_k appear to converge slowly to 5, while $\mathbf{y}_k$ also appear to converge slowly to $\begin{bmatrix}0\\1\\0\end{bmatrix}$.

25. With the given matrix, and using $\mathbf{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}$, we get

$$\begin{array}{c|ccccc} k & 0 & 1 & 2 & 3 & 4 \\ \mathbf{x}_k & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}-1\\-1\end{bmatrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}-1\\-1\end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}1\\1\end{bmatrix} \\ m_k & 1 & 1 & -1 & 1 & -1 \end{array}$$

The characteristic polynomial is

$$\det(A - \lambda I) = \begin{bmatrix}-1-\lambda & 2\\ -1 & 1-\lambda\end{bmatrix} = \lambda^2 + 1,$$

so that A has no real eigenvalues, so the power method cannot succeed.

26. With the given matrix, and using $\mathbf{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}$, we get

$$\begin{array}{c|cccc} k & 0 & 1 & 2 & 3 \\ \mathbf{x}_k & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}3\\3\end{bmatrix} & \begin{bmatrix}3\\3\end{bmatrix} & \begin{bmatrix}3\\3\end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}1\\1\end{bmatrix} \\ m_k & 1 & 3 & 3 & 3 \end{array}$$

The characteristic polynomial is

$$\det(A - \lambda I) = \begin{bmatrix}2-\lambda & 1\\ -2 & 5-\lambda\end{bmatrix} = \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4)$$

so that the power method converges to a non-dominant eigenvalue. This happened because we chose $\begin{bmatrix}1\\1\end{bmatrix}$ as the initial value, which is in fact an eigenvector for $\lambda = 3$:

$$\begin{bmatrix}2 & 1\\ -2 & 5\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix}.$$

27. With the given matrix, and using $\mathbf{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}$, we get

$$\begin{array}{c|ccccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \mathbf{x}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}3\\4\\3\end{bmatrix} & \begin{bmatrix}2.5\\4.0\\2.5\end{bmatrix} & \begin{bmatrix}2.25\\4\\2.25\end{bmatrix} & \begin{bmatrix}2.12\\4\\2.12\end{bmatrix} & \begin{bmatrix}2.06\\4\\2.06\end{bmatrix} & \begin{bmatrix}2.03\\4\\2.03\end{bmatrix} & \begin{bmatrix}2.02\\4\\2.02\end{bmatrix} & \begin{bmatrix}2.01\\4\\2.01\end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}0.75\\1\\0.75\end{bmatrix} & \begin{bmatrix}0.62\\1\\0.62\end{bmatrix} & \begin{bmatrix}0.56\\1\\0.56\end{bmatrix} & \begin{bmatrix}0.53\\1\\0.53\end{bmatrix} & \begin{bmatrix}0.52\\1\\0.52\end{bmatrix} & \begin{bmatrix}0.51\\1\\0.51\end{bmatrix} & \begin{bmatrix}0.5\\1\\0.5\end{bmatrix} & \begin{bmatrix}0.5\\1\\0.5\end{bmatrix} \\ m_k & 1 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{array}$$

The characteristic polynomial is

$$\det(A - \lambda I) = \begin{bmatrix}-5-\lambda & 1 & 7\\ 0 & 4-\lambda & 0\\ 7 & 1 & -5-\lambda\end{bmatrix} = (\lambda + 12)(\lambda - 4)(\lambda - 2)$$

so that the power method converges to a non-dominant eigenvalue. This happened because we chose $\begin{bmatrix}1\\1\\1\end{bmatrix}$ as the initial value. This vector is orthogonal to the eigenvector corresponding to the dominant eigenvalue $\lambda = -12$, which is $\begin{bmatrix}1\\0\\-1\end{bmatrix}$:

$$\begin{bmatrix}-5 & 1 & 7\\0 & 4 & 0\\7 & 1 & -5\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}-12\\0\\12\end{bmatrix} = -12\begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

28. With the given matrix, and using $\mathbf{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}$, we get

k	0	1	2	3	4	5	6	7	8
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0\\2\\1\end{bmatrix}$	$\begin{bmatrix}-1\\1\\-0.5\end{bmatrix}$	$\begin{bmatrix}-2\\0\\-2.5\end{bmatrix}$	$\begin{bmatrix}0.8\\0.8\\1.8\end{bmatrix}$	$\begin{bmatrix}0\\0.89\\1\end{bmatrix}$	$\begin{bmatrix}-0.89\\0.89\\0.11\end{bmatrix}$	$\begin{bmatrix}-2\\0\\-1.88\end{bmatrix}$	$\begin{bmatrix}1\\1\\1.94\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\\0.5\end{bmatrix}$	$\begin{bmatrix}-1\\1\\-0.5\end{bmatrix}$	$\begin{bmatrix}0.8\\0\\1\end{bmatrix}$	$\begin{bmatrix}0.44\\0.44\\1\end{bmatrix}$	$\begin{bmatrix}0\\0.89\\1\end{bmatrix}$	$\begin{bmatrix}-1\\1\\0.12\end{bmatrix}$	$\begin{bmatrix}1\\0\\0.94\end{bmatrix}$	$\begin{bmatrix}0.52\\0.52\\1\end{bmatrix}$
m_k	1	0.6	1.44	1.49	1	0.5	0.88	1.5	1

Neither m_k nor $\mathbf{y}_k$ shows clear signs of converging to anything. The characteristic polynomial is

$$\det(A - \lambda I) = \begin{bmatrix}1-\lambda & -1 & 0\\1 & 1-\lambda & 0\\1 & -1 & 1-\lambda\end{bmatrix} = (\lambda - 1)(\lambda^2 - 2x + 2)$$

so that A has eigenvalues 1 and $1 \pm i$. The absolute value of each of the complex eigenvectors is 2, so 1 is not the eigenvalue of largest magnitude, and the power method cannot succeed.

29. In Exercise 9, we found that $\lambda_1 = 18$. To find λ_2, we apply the power method to

$$A - 18I = \begin{bmatrix}-4 & 12\\5 & -15\end{bmatrix} \text{ with } \mathbf{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}.$$

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}8\\-10\end{bmatrix}$	$\begin{bmatrix}15.2\\-19.0\end{bmatrix}$	$\begin{bmatrix}15.2\\-19.0\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}-0.8\\1\end{bmatrix}$	$\begin{bmatrix}-0.8\\1\end{bmatrix}$	$\begin{bmatrix}-0.8\\1\end{bmatrix}$
m_k	1	-10	-19	-19

Thus $\lambda_2 - \lambda_1 = -19$, so that $\lambda_2 = -1$ is the second eigenvalue of A.

30. In Exercise 10, we found that $\lambda_1 = -10$, so we apply the power method to

$$A + 10I = \begin{bmatrix}4 & 4\\8 & 8\end{bmatrix} \text{ with } \mathbf{x}_0 = \begin{bmatrix}1\\1\end{bmatrix}:$$

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}8\\16\end{bmatrix}$	$\begin{bmatrix}6\\12\end{bmatrix}$	$\begin{bmatrix}6\\12\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}0.5\\1\end{bmatrix}$	$\begin{bmatrix}0.5\\1\end{bmatrix}$	$\begin{bmatrix}0.5\\1\end{bmatrix}$
m_k	1	16	12	12

Thus $\lambda_2 - \lambda_1 = 12$, so that $\lambda_2 = 2$ is the second eigenvalue of A.

31. In Exercise 13, we found that $\lambda_1 = 17$, so we apply the power method to

$$A - 17I = \begin{bmatrix} -8 & 4 & 8 \\ 4 & -2 & -4 \\ 8 & -4 & -8 \end{bmatrix} \text{ with } \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\begin{array}{c|cccc} k & 0 & 1 & 2 & 3 \\ \mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 4 \\ -2 \\ -4 \end{bmatrix} & \begin{bmatrix} 18 \\ -9 \\ -18 \end{bmatrix} & \begin{bmatrix} 18 \\ -9 \\ -18 \end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix} \\ m_k & 1 & -4 & -18 & -18 \end{array}$$

Thus $\lambda_2 - \lambda_1 = -18$, so that $\lambda_2 = -1$ is the second eigenvalue of A.

32. In Exercise 14, we found that $\lambda_1 \approx 4.4$, so we apply the power method to

$$A - 4.4I = \begin{bmatrix} -1.4 & 1 & 0 \\ 1 & -1.4 & 1 \\ 0 & 1 & -1.4 \end{bmatrix} \text{ with } \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\begin{array}{c|cccccc} k & 0 & 1 & 2 & 3 & 4 & 5 \\ \mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.4 \\ 0.6 \\ -0.4 \end{bmatrix} & \begin{bmatrix} 1.933 \\ -2.733 \\ 1.933 \end{bmatrix} & \begin{bmatrix} 1.990 \\ -2.815 \\ 1.990 \end{bmatrix} & \begin{bmatrix} 1.990 \\ -2.814 \\ 1.990 \end{bmatrix} & \begin{bmatrix} 1.990 \\ -2.814 \\ 1.990 \end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.667 \\ 1 \\ -0.667 \end{bmatrix} & \begin{bmatrix} -0.707 \\ 1 \\ -0.707 \end{bmatrix} & \begin{bmatrix} -0.707 \\ 1 \\ -0.707 \end{bmatrix} & \begin{bmatrix} -0.707 \\ 1 \\ -0.707 \end{bmatrix} & \begin{bmatrix} -0.707 \\ 1 \\ -0.707 \end{bmatrix} \\ m_k & 1 & 0.6 & -2.733 & -2.815 & -2.814 & -2.814 \end{array}$$

Thus $\lambda_2 - \lambda_1 \approx -2.81$, so that $\lambda_2 \approx 1.59$ is the second eigenvalue of A (in fact, $\lambda_2 = 3 - \sqrt{2}$).

33. With A and $\mathbf{x}_0$ as in Exercise 9, proceed to calculate x_k by solving $Ax_k = y_{k-1}$ at each step to apply the inverse power method. We get

$$\begin{array}{c|cccccc} k & 0 & 1 & 2 & 3 & 4 & 5 \\ \mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} & \begin{bmatrix} 0.833 \\ -1.056 \end{bmatrix} & \begin{bmatrix} 0.798 \\ -0.997 \end{bmatrix} & \begin{bmatrix} 0.8 \\ -1 \end{bmatrix} & \begin{bmatrix} 0.8 \\ -1 \end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.789 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.801 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.8 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.8 \\ 1 \end{bmatrix} \\ m_k & 1 & -0.5 & -1.056 & -0.997 & -1 & -1 \end{array}$$

We conclude that the eigenvalue of A of smallest magnitude is $\frac{1}{-1} = -1$.

34. With A and $\mathbf{x}_0$ as in Exercise 10, proceed to calculate x_k by solving $Ax_k = y_{k-1}$ at each step to apply the inverse power method. We get

$$\begin{array}{c|ccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix} & \begin{bmatrix} 0.225 \\ 0.400 \end{bmatrix} & \begin{bmatrix} 0.256 \\ 0.525 \end{bmatrix} & \begin{bmatrix} 0.249 \\ 0.495 \end{bmatrix} & \begin{bmatrix} 0.250 \\ 0.501 \end{bmatrix} & \begin{bmatrix} 0.25 \\ 0.50 \end{bmatrix} \\ \mathbf{y}_k & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.25 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.562 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.488 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.502 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \\ m_k & 1 & 0.4 & 0.4 & 0.525 & 0.495 & 0.501 & 0.5 \end{array}$$

We conclude that the eigenvalue of A of smallest magnitude is $\frac{1}{0.5} = 2$.

35. With A as in Exercise 7, and $\mathbf{x}_0$ as given, proceed to calculate x_k by solving $Ax_k = y_{k-1}$ at each step to apply the inverse power method. We get

$$\begin{array}{c|ccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} & \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \end{bmatrix} & \begin{bmatrix} 0.500 \\ 0.333 \\ -0.500 \end{bmatrix} & \begin{bmatrix} 0.500 \\ 0.111 \\ -0.500 \end{bmatrix} & \begin{bmatrix} 0.500 \\ 0.259 \\ -0.500 \end{bmatrix} & \begin{bmatrix} 0.50 \\ 0.16 \\ -0.50 \end{bmatrix} & \\
\mathbf{y}_k & \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ -0.667 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ -0.222 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ -0.519 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ -0.321 \\ 1 \end{bmatrix} & \\
m_k & -1 & 0.5 & -0.5 & -0.5 & -0.5 & -0.5 &
\end{array}$$

We conclude that the eigenvalue of A of smallest magnitude is $\frac{1}{-0.5} = -2$.

36. With A and $\mathbf{x}_0$ as in Exercise 14, proceed to calculate x_k by solving $Ax_k = y_{k-1}$ at each step to apply the inverse power method. We get

$$\begin{array}{c|ccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.286 \\ 0.143 \\ 0.286 \end{bmatrix} & \begin{bmatrix} 0.357 \\ -0.071 \\ 0.357 \end{bmatrix} & \begin{bmatrix} 0.457 \\ -0.371 \\ 0.457 \end{bmatrix} & \begin{bmatrix} 0.545 \\ -0.634 \\ 0.545 \end{bmatrix} & \begin{bmatrix} -0.511 \\ 0.674 \\ -0.511 \end{bmatrix} & \begin{bmatrix} -0.468 \\ 0.645 \\ -0.468 \end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ -0.2 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ -0.812 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.859 \\ 1 \\ -0.859 \end{bmatrix} & \begin{bmatrix} -0.758 \\ 1 \\ -0.758 \end{bmatrix} & \begin{bmatrix} -0.725 \\ 1 \\ -0.725 \end{bmatrix} \\
m_k & 1 & 0.286 & 0.357 & 0.457 & -0.634 & 0.674 & 0.645
\end{array}$$

We conclude that the eigenvalue of A of smallest magnitude is $\approx \frac{1}{0.645} \approx 1.55$. In fact the eigenvalue is $3 - \sqrt{2}$.

37. With $\alpha = 0$, we are just using the inverse power method, so the solution to Exercise 33 provides the eigenvalue closest to zero.

38. Since $\alpha = 0$, there is no shift required, so this is the inverse power method. Take $\mathbf{x}_0 = \mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and proceed to calculate x_k by solving $Ax_k = y_{k-1}$ at each step to apply the inverse power method. We get

$$\begin{array}{c|ccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\mathbf{x}_k & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.125 \\ 0.375 \end{bmatrix} & \begin{bmatrix} 0.417 \\ -0.750 \end{bmatrix} & \begin{bmatrix} 0.306 \\ -1.083 \end{bmatrix} & \begin{bmatrix} 0.340 \\ -0.981 \end{bmatrix} & \begin{bmatrix} 0.332 \\ -1.005 \end{bmatrix} & \begin{bmatrix} 0.334 \\ -0.999 \end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0.333 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.556 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.282 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.346 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.33 \\ 1 \end{bmatrix} & \begin{bmatrix} -0.334 \\ 1 \end{bmatrix} \\
m_k & 1 & 0.375 & -0.75 & -1.083 & -0.981 & -1.005 & -0.999
\end{array}$$

We conclude that the eigenvalue of A closest to 0 (i.e., the eigenvalue smallest in magnitude) is $\frac{1}{-1} = -1$.

39. We first shift A:

$$A - 5I = \begin{bmatrix} -1 & 0 & 6 \\ -1 & -2 & 1 \\ 6 & 0 & -1 \end{bmatrix}.$$

Now take $\mathbf{x}_0 = \mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and proceed to calculate x_k by solving $Ax_k = y_{k-1}$ at each step to apply the inverse power method. We get

$$\begin{array}{c|cccc}
k & 0 & 1 & 2 & 3 \\
\mathbf{x}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}0.2\\-0.5\\0.2\end{bmatrix} & \begin{bmatrix}-0.08\\-0.50\\-0.08\end{bmatrix} & \begin{bmatrix}0.32\\-0.50\\0.32\end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}-0.4\\1\\-0.4\end{bmatrix} & \begin{bmatrix}0.16\\1\\0.16\end{bmatrix} & \begin{bmatrix}-0.064\\1\\-0.064\end{bmatrix} \\
m_k & 1 & -0.5 & -0.5 & -0.5
\end{array}$$

We conclude that the eigenvalue of A closest to 5 is $5 + \frac{1}{-0.5} = 3$.

40. We first shift A:

$$A + 2I = \begin{bmatrix}11 & 4 & 7\\4 & 17 & -4\\8 & -4 & 11\end{bmatrix}.$$

Now take $\mathbf{x}_0 = \mathbf{y}_0 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, and proceed to calculate x_k by solving $Ax_k = y_{k-1}$ at each step to apply the inverse power method. We get

$$\begin{array}{c|cccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 \\
\mathbf{x}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}-0.158\\0.158\\0.263\end{bmatrix} & \begin{bmatrix}-0.832\\0.432\\0.853\end{bmatrix} & \begin{bmatrix}-0.990\\0.496\\0.991\end{bmatrix} & \begin{bmatrix}-0.999\\0.500\\1.000\end{bmatrix} & \begin{bmatrix}-1\\0.5\\1\end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}-0.6\\0.6\\1\end{bmatrix} & \begin{bmatrix}-0.975\\0.506\\1\end{bmatrix} & \begin{bmatrix}-0.999\\0.5\\1\end{bmatrix} & \begin{bmatrix}-1\\0.5\\1\end{bmatrix} & \begin{bmatrix}-1\\0.5\\1\end{bmatrix} \\
m_k & 1 & 0.263 & 0.853 & 0.991 & 1 & 1
\end{array}$$

We conclude that the eigenvalue of A closest to -2 is $-2 + \frac{1}{1} = -1$.

41. The companion matrix of $p(x) = x^2 + 2x - 2$ is

$$C(p) = \begin{bmatrix}-2 & 2\\1 & 0\end{bmatrix},$$

so we apply the inverse power method with $\mathbf{x}_0 = \mathbf{y}_0 = \begin{bmatrix}1\\1\end{bmatrix}$, calculating x_k by solving $Ax_k = y_{k-1}$ at each step. We get

$$\begin{array}{c|ccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\mathbf{x}_k & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}1\\1.5\end{bmatrix} & \begin{bmatrix}1\\1.333\end{bmatrix} & \begin{bmatrix}1\\1.375\end{bmatrix} & \begin{bmatrix}1\\1.364\end{bmatrix} & \begin{bmatrix}1\\1.367\end{bmatrix} & \begin{bmatrix}1\\1.366\end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}0.667\\1\end{bmatrix} & \begin{bmatrix}0.75\\1\end{bmatrix} & \begin{bmatrix}0.727\\1\end{bmatrix} & \begin{bmatrix}0.733\\1\end{bmatrix} & \begin{bmatrix}0.732\\1\end{bmatrix} & \begin{bmatrix}0.732\\1\end{bmatrix} \\
m_k & 1 & 1.5 & 1.333 & 1.375 & 1.364 & 1.367 & 1.366
\end{array}$$

So the root of $p(x)$ closest to 0 is $\approx \frac{1}{1.366} \approx 0.732$. The exact value is $-1 + \sqrt{3}$.

42. The companion matrix of $p(x) = x^2 - x - 3$ is

$$C(p) = \begin{bmatrix}1 & 3\\1 & 0\end{bmatrix},$$

so we apply the inverse power method to

$$C(p) - 2I = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

with $\mathbf{x}_0 = \mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, calculating x_k by solving $Ax_k = y_{k-1}$ at each step. We get

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 3.2 \\ 1.4 \end{bmatrix}$	$\begin{bmatrix} 3.312 \\ 1.438 \end{bmatrix}$	$\begin{bmatrix} 3.302 \\ 1.434 \end{bmatrix}$	$\begin{bmatrix} 3.303 \\ 1.434 \end{bmatrix}$	$\begin{bmatrix} 3.303 \\ 1.434 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.438 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.434 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.434 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.434 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.434 \end{bmatrix}$
m_k	1	5	3.2	3.312	3.302	3.303	3.303

So the root of $p(x)$ closest to 2 is $\approx 2 + \frac{1}{3.303} \approx 2.303$. The exact value is $\frac{1}{2}(1 + \sqrt{13})$.

43. The companion matrix of $p(x) = x^3 - 2x^2 + 1$ is

$$C(p) = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

so we apply the inverse power method to $C(p)$. If we first start with $\mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we find that this is an eigenvector of $C(p)$ with eigenvalue 1, but this may not be the closest root to 0, so we start instead with with $\mathbf{x}_0 = \mathbf{y}_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, calculating x_k by solving $Ax_k = y_{k-1}$ at each step. We get

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$	$\begin{bmatrix} -0.5 \\ 1 \\ 0.5 \end{bmatrix}$	$\begin{bmatrix} -0.667 \\ 1 \\ -1.667 \end{bmatrix}$	$\begin{bmatrix} -0.6 \\ 1 \\ -1.6 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.333 \\ -0.667 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.4 \\ -0.6 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.375 \\ -0.625 \\ 1 \end{bmatrix}$
m_k	1	-1	-2	-1.6	-1.667	-1.6

k	6	7	8	9
$\mathbf{x}_k$	$\begin{bmatrix} -0.625 \\ 1 \\ -1.625 \end{bmatrix}$	$\begin{bmatrix} -0.615 \\ 1 \\ -1.615 \end{bmatrix}$	$\begin{bmatrix} -0.619 \\ 1 \\ 1.619 \end{bmatrix}$	$\begin{bmatrix} -0.618 \\ 1 \\ -1.618 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 0.385 \\ -0.615 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.381 \\ -0.619 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.382 \\ -0.618 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.382 \\ -0.618 \\ 1 \end{bmatrix}$
m_k	-1.625	-1.615	-1.619	-1.618

We conclude that the root of $p(x)$ closest to zero is $\frac{1}{-1.618} \approx= -0.618$. The actual root is $\frac{1}{2}(1 - \sqrt{5})$.

44. The companion matrix of $p(x) = x^3 - 2x^2 + 1$ is

$$C(p) = \begin{bmatrix} 5 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

so we apply the inverse power method to $C(p) - 5I$ starting with $\mathbf{x}_0 = \mathbf{y}_0 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, calculating x_k by solving $Ax_k = y_{k-1}$ at each step. We get

$$\begin{array}{c|ccccccc}
k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\mathbf{x}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}-2.333\\-0.667\\-0.333\end{bmatrix} & \begin{bmatrix}-3.762\\-0.810\\-0.190\end{bmatrix} & \begin{bmatrix}-3.909\\-0.825\\-0.175\end{bmatrix} & \begin{bmatrix}-3.918\\-0.826\\-0.174\end{bmatrix} & \begin{bmatrix}-3.919\\-0.826\\-0.174\end{bmatrix} & \begin{bmatrix}-3.919\\-0.826\\-0.174\end{bmatrix} \\
\mathbf{y}_k & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}1\\0.286\\0.143\end{bmatrix} & \begin{bmatrix}1\\0.215\\0.051\end{bmatrix} & \begin{bmatrix}1\\0.211\\0.045\end{bmatrix} & \begin{bmatrix}1\\0.211\\0.044\end{bmatrix} & \begin{bmatrix}1\\0.211\\0.044\end{bmatrix} & \begin{bmatrix}1\\0.211\\0.044\end{bmatrix} \\
m_k & 1 & -2.333 & -3.762 & -3.909 & -3.918 & -3.919 & -3.919
\end{array}$$

We conclude that the root of $p(x)$ closest to zero is $5 + \frac{1}{-3.919} \approx = 4.745$.

45. First, $A-\alpha I$ is invertible, since α is an eigenvalue of A precisely when $A-\alpha I$ is *not* invertible. Let $\mathbf{x}$ be an eigenvector for λ. Then $A\mathbf{x} = \lambda\mathbf{x}$, so that $A\mathbf{x} - \alpha\mathbf{x} = \lambda\mathbf{x} - \alpha\mathbf{x}$, and therefore $(A-\alpha I)\mathbf{x} = (\lambda - \alpha)\mathbf{x}$. Now left multiply both sides of this equation by $(A-\alpha I)^{-1}$, to get

$$\mathbf{x} = (A-\alpha I)^{-1}(\lambda - \alpha)\mathbf{x}, \text{ so that } \frac{1}{\lambda - \alpha}\mathbf{x} = (A - \alpha I)^{-1}\mathbf{x}.$$

But the last equation says exactly that $\lambda - \alpha$ is an eigenvalue of $(A - \alpha I)^{-1}$ with eigenvector $\mathbf{x}$.

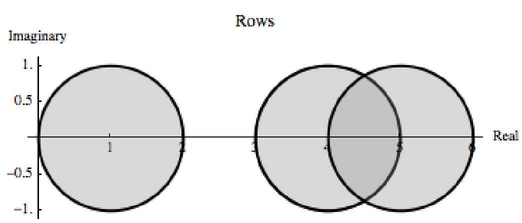
46. If λ is dominant, then $|\lambda| > |\gamma|$ for all other eigenvalues γ. But this means that the algebraic multiplicity of λ is 1, since it appears only once in the list of eigenvalues listed with multiplicity, so its geometric multiplicity is 1 and thus its eigenspace is one-dimensional.

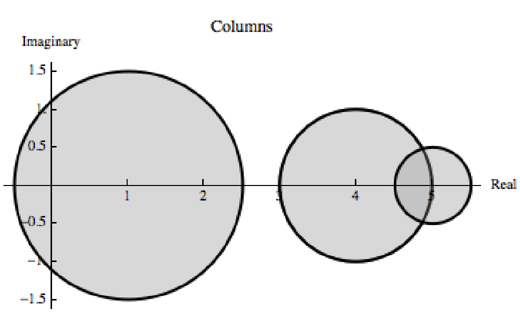
47. Using rows, the Gerschgorin disks are:

$$\begin{aligned}
&\text{Center } 1, \quad \text{radius } 1 + 0 = 1,\\
&\text{Center } 4, \quad \text{radius } \frac{1}{2} + \frac{1}{2} = 1,\\
&\text{Center } 5, \quad \text{radius } 1 + 0 = 1.
\end{aligned}$$

Using columns, they are

$$\begin{aligned}
&\text{Center } 1, \quad \text{radius } \frac{1}{2} + 1 = \frac{3}{2},\\
&\text{Center } 4, \quad \text{radius } 1 + 0 = 1,\\
&\text{Center } 5, \quad \text{radius } 0 + \frac{1}{2} = \frac{1}{2}:
\end{aligned}$$





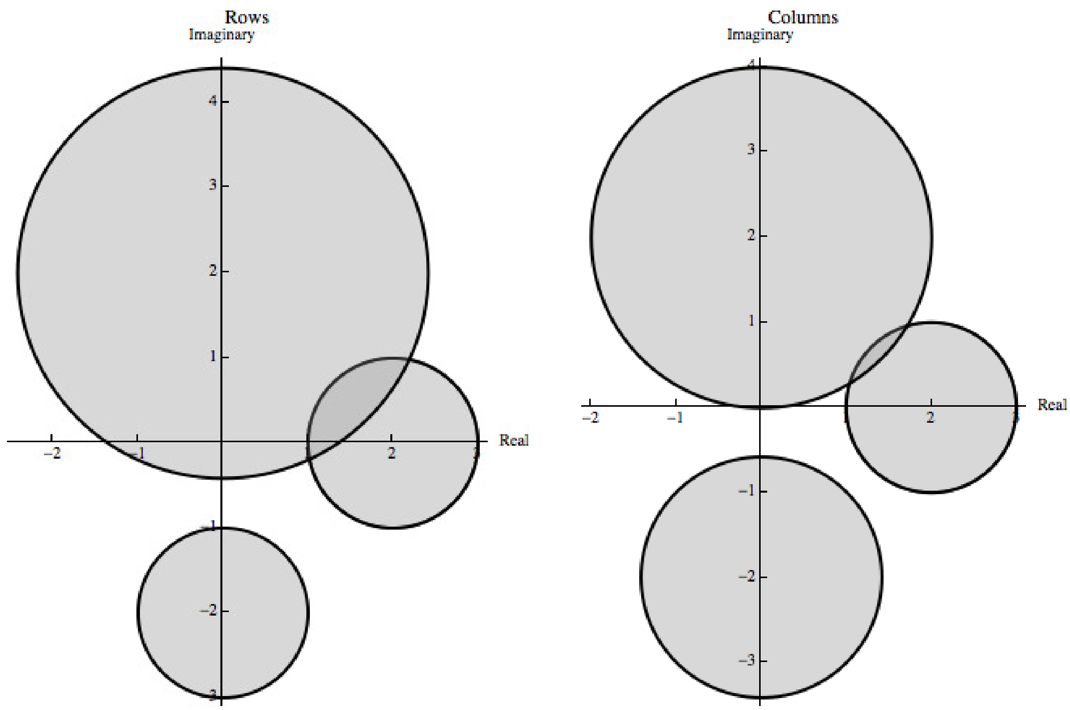
From this diagram, we see that A has a real eigenvalue between 0 and 2.

48. Using rows, the Gerschgorin disks are:

$$\begin{aligned}\text{Center } 2, &\quad \text{radius } |-i| + 0 = 1,\\ \text{Center } 2i, &\quad \text{radius } |1| + |1+i| = 1 + \sqrt{2},\\ \text{Center } -2i, &\quad \text{radius } 0 + 1 = 1.\end{aligned}$$

Using columns, they are

$$\begin{aligned}\text{Center } 2, &\quad \text{radius } 1 + 0 = 1,\\ \text{Center } 2i, &\quad \text{radius } |-i| + 1 = 2,\\ \text{Center } -2i, &\quad \text{radius } 0 + |1+i| = \sqrt{2}:\end{aligned}$$



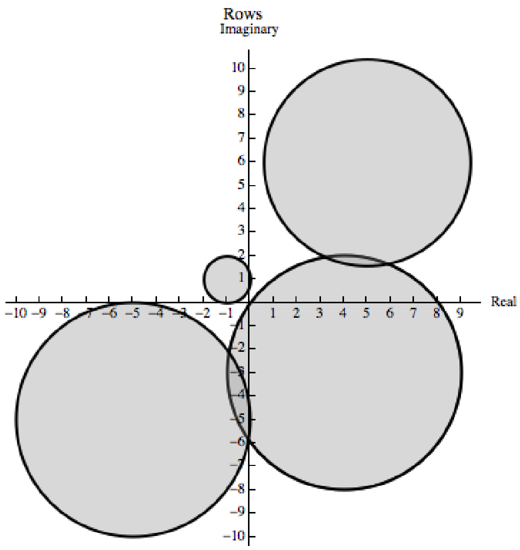
From this diagram, we conclude that A has an eigenvalue within one unit of $-2i$.

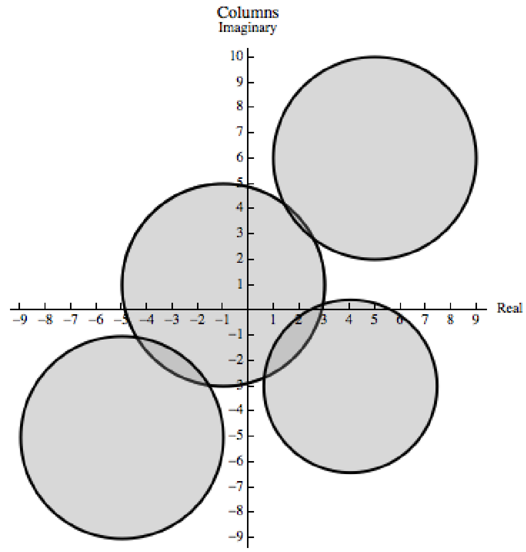
49. Using rows, the Gerschgorin disks are:

$$\begin{aligned}\text{Center } 4-3i, &\quad \text{radius } |i| + 2 + 2 = 5,\\ \text{Center } -1+i, &\quad \text{radius } |i| + 0 + 0 = 1,\\ \text{Center } 5+6i, &\quad \text{radius } |1+i| + |-i| + |2i| = 3 + \sqrt{2}\\ \text{Center } -5-5i, &\quad \text{radius } 1 + |-2i| + |2i| = 5.\end{aligned}$$

Using columns, they are

$$\begin{aligned}\text{Center } 4-3i, &\quad \text{radius } |i| + |1+i| + 1 = 2 + \sqrt{2},\\ \text{Center } -1+i, &\quad \text{radius } |i| + |-i| + |-2i| = 4,\\ \text{Center } 5+6i, &\quad \text{radius } 2 + 0 + |2i| = 4\\ \text{Center } -5-5i, &\quad \text{radius } |-2| + 0 + |2i| = 4.\end{aligned}$$





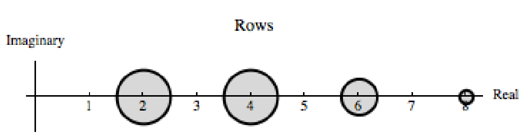
From this diagram, we conclude that A has an eigenvalue within 1 unit of $-1+i$.

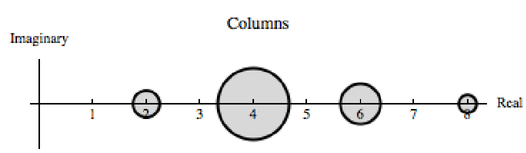
50. Using rows, the Gerschgorin disks are:

$$\begin{aligned}
&\text{Center } 2, \quad \text{radius } \frac{1}{2},\\
&\text{Center } 4, \quad \text{radius } \frac{1}{4}+\frac{1}{4}=\frac{1}{2},\\
&\text{Center } 6, \quad \text{radius } \frac{1}{6}+\frac{1}{6}=\frac{2}{3}\\
&\text{Center } 8, \quad \text{radius } \frac{1}{8}.
\end{aligned}$$

Using columns, they are

$$\begin{aligned}
&\text{Center } 2, \quad \text{radius } \frac{1}{4},\\
&\text{Center } 4, \quad \text{radius } \frac{1}{2}+\frac{1}{6}=\frac{1}{3},\\
&\text{Center } 6, \quad \text{radius } \frac{1}{4}+\frac{1}{8}=\frac{3}{8}\\
&\text{Center } 8, \quad \text{radius } \frac{1}{6}.
\end{aligned}$$





Since the circles are all disjoint, we conclude that A has four real eigenvalues, near 2, 4, 6, and 8.

51. Let A be a strictly diagonally dominant square matrix. Then in particular, none of the diagonal entries of A can be zero, since each row has the property that $|a_{ii}| > \sum_{j\neq i}|a_{ij}|$. Therefore each Gerschgorin disk is centered at a nonzero point, and its radius is less than the absolute value of its center, so the disk does not contain the origin. Since every eigenvalue is contained in some Gerschgorin disk, it follows that 0 is not an eigenvalue, so that the matrix is invertible by Theorem 4.17.

52. Let λ be any eigenvalue of A. Then by Theorem 4.29, λ is contained in the Gerschgorin disk for some row i, so that

$$|\lambda - a_{ii}| \le \sum_{j\neq i} |a_{ij}|.$$

By the Triangle Inequality, $|\lambda| \le |\lambda - a_{ii}| + |a_{ii}|$, so that

$$|\lambda| \le |\lambda - a_{ii}| + |a_{ii}| \le |a_{ii}| + \sum_{j\neq i} |a_{ij}| = \sum_{j=1}^{n} |a_{ij}| \le \|A\|,$$

since $\|A\|$ is the *maximum* of the possible row sums.

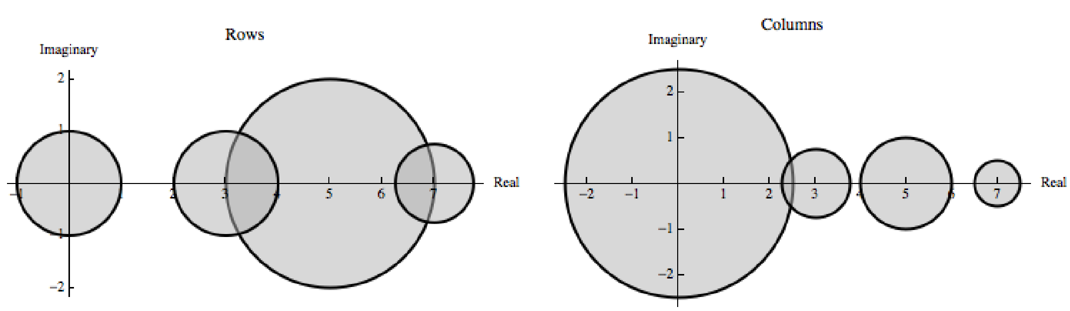
53. A stochastic matrix is a square matrix whose entries are probability vectors, which means that the sum of the entries in any column of A is 1, so the sum of the entries of any row in A^T is 1, so that $\left\|A^T\right\| = 1$. By Exercise 52, any eigenvalue of A^T must satisfy $|\lambda| \le \left\|A^T\right\| = 1$. But by Exercise 19 in Section 4.3, the eigenvalues of A and of A^T are the same, so that if λ is any eigenvalue of A, then $|\lambda| \le 1$.

54. Using rows, the Gerschgorin disks are:

$$\begin{aligned}
&\text{Center } 0, \quad \text{radius } 1,\\
&\text{Center } 5, \quad \text{radius } 2,\\
&\text{Center } 3, \quad \text{radius } \frac{1}{2} + \frac{1}{2} = 1\\
&\text{Center } 7, \quad \text{radius } \frac{3}{4}.
\end{aligned}$$

Using columns, they are

$$\begin{aligned}
&\text{Center } 0, \quad \text{radius } 2 + \frac{1}{2} = \frac{5}{2},\\
&\text{Center } 5, \quad \text{radius } 1,\\
&\text{Center } 3, \quad \text{radius } \frac{3}{4}\\
&\text{Center } 7, \quad \text{radius } \frac{1}{2}.
\end{aligned}$$



From the row diagram, we see that there is exactly one eigenvalue in the circle of radius 1 around 0, so it must be real and therefore there is exactly one eigenvalue in the interval $[-1, 1]$. From the column diagram, we similarly conclude that there is exactly one eigenvalue in the interval $[4, 6]$ and exactly one in $\left[\frac{13}{2}, \frac{15}{2}\right]$.

Since the matrix has real entries, the fourth eigenvalue must be real as well. From the row diagram, since the circle around zero contains exactly one eigenvalue, the fourth eigenvalue must be in the interval $\left[2, 7\,\frac{3}{4}\right]$. But it cannot be larger than $3\,\frac{3}{4}$ since then, from the column diagram, it would either not be in a disk or would be in an isolated disk already containing an eigenvalue. So it must lie in the range $\left[2, 3\,\frac{3}{4}\right]$.

In summary, the four eigenvalues are each in a different one of the four intervals

$$[-1, 1], \qquad [2, 3.75], \qquad [4, 6], \qquad [6.5, 7.5].$$

The actual values are ≈ -0.372, ≈ 2.908, ≈ 5.372, and ≈ 7.092.

4.6 Applications and the Perron-Frobenius Theorem

1. Since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, it follows that any power of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ contains zeroes. So this matrix is not regular.

2. Since the given matrix is upper triangular, any of its powers is upper triangular as well, so will contain a zero. Thus this matrix is not regular.

3. Since

$$\begin{bmatrix} \frac{1}{3} & 1 \\ \frac{2}{3} & 0 \end{bmatrix}^2 = \begin{bmatrix} \frac{7}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{2}{3} \end{bmatrix},$$

the square of the given matrix has all positive entries, so this matrix is regular.

4. We have

$$\begin{bmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{8} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{9}{16} & \frac{1}{4} & \frac{5}{8} \\ \frac{5}{16} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

so that the fourth power of this matrix contains all positive entries, so this matrix is regular.

5. If A and B are any 3×3 matrices with $a_{12} = a_{32} = b_{12} = b_{32} = 0$, then

$$(AB)_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 0 + 0 + 0 = 0, \quad (AB)_{32} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} = 0 + 0 + 0 = 0,$$

so that AB has the same property. Thus any power of the given matrix A has this property, so no power of A consists of positive entries, so A is not regular.

6. Since the third row of the matrix is all zero, the third row of any power will also be all zero, so this matrix is not regular.

7. The transition matrix P has characteristic equation

$$\det(P - \lambda I) = \begin{vmatrix} \frac{1}{3} - \lambda & \frac{1}{6} \\ \frac{2}{3} & \frac{5}{6} - \lambda \end{vmatrix} = \frac{1}{6}(\lambda - 1)(6\lambda - 1),$$

so its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{6}$. Thus P has two distinct eigenvalues, so is diagonalizable. Now, this matrix is regular, so the remark at the end of the proof of Theorem 4.33 shows that the vector $\mathbf{x}$ which forms the columns of L is a probability vector parallel to an eigenvector for $\lambda = 1$. To compute this, we row-reduce $P - I$:

$$\left[P - I \mid 0\right] = \left[\begin{array}{cc|c} -\frac{2}{3} & \frac{1}{6} & 0 \\ \frac{2}{3} & -\frac{1}{6} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that the corresponding eigenspace is

$$E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right).$$

Thus the columns of L are of the form $\begin{bmatrix} t \\ 4t \end{bmatrix}$ where $t + 4t = 5t = 1$, so that $t = \frac{1}{5}$. Therefore

$$L = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{4}{5} \end{bmatrix}.$$

8. The transition matrix P has characteristic equation

$$\det(P - \lambda I) = \begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{2} - \lambda & \frac{1}{3} \\ 0 & \frac{1}{6} & \frac{1}{2} - \lambda \end{vmatrix} = -\frac{1}{36}(\lambda - 1)(36\lambda^2 - 18\lambda + 1).$$

So one eigenvalue is 1; the roots of the quadratic are distinct, so that P has three distinct eigenvalues, so is diagonalizable. Now, this matrix is regular (its square has all positive entries), so the remark at the end of the proof of Theorem 4.33 shows that the vector $\mathbf{x}$ which forms the columns of L is a probability vector parallel to an eigenvector for $\lambda = 1$. To compute this, we row-reduce $P - I$:

$$\left[P - I \mid 0\right] = \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{6} & -\frac{1}{2} & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{3} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that the corresponding eigenspace is

$$E_1 = \operatorname{span}\left(\begin{bmatrix} 7 \\ 9 \\ 3 \end{bmatrix}\right).$$

Thus the columns of L are of the form $\begin{bmatrix} 7t \\ 9t \\ 3t \end{bmatrix}$ where $7t + 9t + 3t = 19t = 1$, so that $t = \frac{1}{19}$. Therefore

$$L = \frac{1}{19}\begin{bmatrix} 7 & 7 & 7 \\ 9 & 9 & 9 \\ 3 & 3 & 3 \end{bmatrix}.$$

9. The transition matrix P has characteristic equation

$$\det(P - \lambda I) = \begin{vmatrix} 0.2 - \lambda & 0.3 & 0.4 \\ 0.6 & 0.1 - \lambda & 0.4 \\ 0.2 & 0.6 & 0.2 - \lambda \end{vmatrix} = -(\lambda - 1)(\lambda^2 + 0.5\lambda + 0.08)$$

So one eigenvalue is 1; the roots of the quadratic are distinct (and complex), so that P has three distinct eigenvalues, so is diagonalizable. Now, this matrix is regular, so the remark at the end of the proof of Theorem 4.33 shows that the vector $\mathbf{x}$ which forms the columns of L is a probability vector parallel to an eigenvector for $\lambda = 1$. To compute this, we row-reduce $P - I$:

$$\begin{bmatrix} P - I & \mid & 0 \end{bmatrix} = \left[\begin{array}{rrr|r} -0.8 & 0.3 & 0.4 & 0 \\ 0.6 & -0.9 & 0.4 & 0 \\ 0.2 & 0.6 & -0.8 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -0.889 & 0 \\ 0 & 1 & -1.037 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that the corresponding eigenspace is

$$E_1 = \operatorname{span}\left(\begin{bmatrix} 0.889 \\ 1.037 \\ 1 \end{bmatrix}\right).$$

Thus the columns of L are of the form $\begin{bmatrix} 0.889t \\ 1.037t \\ 1t \end{bmatrix}$ where $0.889t + 1.037t + t = 2.926t = 1$, so that $t = \frac{1}{2.926}$. Therefore

$$L = \begin{bmatrix} 0.304 & 0.304 & 0.304 \\ 0.354 & 0.354 & 0.354 \\ 0.341 & 0.341 & 0.341 \end{bmatrix}.$$

(If the matrix is converted to fractions and the computation done that way, the resulting eigenvector will be $\begin{bmatrix} 24 \\ 28 \\ 27 \end{bmatrix}$, which must then be converted to a unit vector.)

10. Suppose that the sequence of iterates $\mathbf{x}_k$ approach both $\mathbf{u}$ and $\mathbf{v}$, and choose $\epsilon > 0$. Choose k such that $|\mathbf{x}_k - \mathbf{u}| < \frac{\epsilon}{2}$ and $|\mathbf{x}_k - \mathbf{v}| < \frac{\epsilon}{2}$. Then

$$|\mathbf{u} - \mathbf{v}| = |(\mathbf{u} - \mathbf{x}_k) + (\mathbf{x}_k - \mathbf{v})| \le |\mathbf{u} - \mathbf{x}_k| + |\mathbf{x}_k - \mathbf{v}| < \epsilon,$$

so that $|\mathbf{u} - \mathbf{v}|$ is less than any positive ϵ. So it must be zero; that is, we must have $\mathbf{u} = \mathbf{v}$.

11. The characteristic polynomial is

$$\det(L - \lambda I) = \begin{vmatrix} -\lambda & 2 \\ 0.5 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1),$$

so the positive eigenvalue is 1. To find a corresponding eigenvector, row-reduce

$$\begin{bmatrix} L - I & \mid & 0 \end{bmatrix} = \left[\begin{array}{rr|r} -1 & 2 & 0 \\ 0.5 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus $E_1 = \operatorname{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$.

12. The characteristic polynomial is

$$\det(L - \lambda I) = \begin{vmatrix} 1 - \lambda & 1.5 \\ 0.5 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 0.75 = (\lambda - 1.5)(\lambda + 0.5),$$

so the positive eigenvalue is 1.5. To find a corresponding eigenvector, row-reduce

$$\begin{bmatrix} L - 1.5I & \mid & 0 \end{bmatrix} = \left[\begin{array}{rr|r} -0.5 & 1.5 & 0 \\ 0.5 & -1.5 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus $E_{1.5} = \operatorname{span}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$.

13. The characteristic polynomial is

$$\det(L-\lambda I)=\begin{vmatrix}-\lambda & 7 & 4\\ 0.5 & -\lambda & 0\\ 0 & 0.5 & -\lambda\end{vmatrix}=-\lambda^3+3.5\lambda+1=-\frac{1}{2}(\lambda-2)(2\lambda^2+4\lambda+1),$$

so 2 is a positive eigenvalue. The other two roots are

$$\frac{-4\pm\sqrt{8}}{4},$$

both of which are negative. To find an eigenvector for 2, row-reduce

$$\begin{bmatrix}L-2I \mid 0\end{bmatrix}=\left[\begin{array}{ccc|c}-2 & 7 & 4 & 0\\ 0.5 & -2 & 0 & 0\\ 0 & 0.5 & -2 & 0\end{array}\right]\to\left[\begin{array}{ccc|c}1 & 0 & -16 & 0\\ 0 & 1 & -4 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

Thus $E_2=\operatorname{span}\left(\begin{bmatrix}16\\4\\1\end{bmatrix}\right)$.

14. The characteristic polynomial is

$$\det(L-\lambda I)=\begin{vmatrix}1-\lambda & 5 & 3\\ \frac{1}{3} & -\lambda & 0\\ 0 & \frac{2}{3} & -\lambda\end{vmatrix}=-x^3+x^2+\frac{5}{3}x+\frac{2}{3}=-\frac{1}{3}(x-2)(3x^2+3x+1)$$

so 2 is a positive eigenvalue. The other two eigenvalues are complex. To find an eigenvector for 2, row-reduce

$$\begin{bmatrix}L-2I \mid 0\end{bmatrix}=\left[\begin{array}{ccc|c}-1 & 5 & 3 & 0\\ \frac{1}{3} & -2 & 0 & 0\\ 0 & \frac{2}{3} & -2 & 0\end{array}\right]\to\left[\begin{array}{ccc|c}1 & 0 & -18 & 0\\ 0 & 1 & -3 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

Thus $E_2=\operatorname{span}\left(\begin{bmatrix}18\\3\\1\end{bmatrix}\right)$.

15. Since the eigenvector corresponding to the positive eigenvalue λ_1 for the Leslie matrix represents a stable ratio of population classes, one that proportion is reached, if $\lambda_1>1$, the population will grow without bound; if $\lambda_1=1$, it will be stable, and if $\lambda_1<1$ it will decrease and eventually die out.

16. From the Leslie matrix in Equation (3), we have

$$\det(L-\lambda I)=\begin{vmatrix}b_1-\lambda & b_2 & b_3 & \cdots & b_{n-1} & b_n\\ s_1 & -\lambda & 0 & \cdots & 0 & 0\\ 0 & s_2 & -\lambda & \cdots & 0 & 0\\ 0 & 0 & s_3 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & s_{n-1} & -\lambda\end{vmatrix}.$$

For $n=1$, this is just $b_1-\lambda$, which is equal to

$$c_L(\lambda)=(-1)^1(\lambda^1-b_1\lambda^0)=b_1-\lambda.$$

This forms the basis for the induction. Now assume the statement holds for $n-1$, and we prove it for n. Expand the above determinant along the last column, giving

$$\det(L-\lambda I) = (-1)^{n+1}b_n \begin{vmatrix} s_1 & -\lambda & 0 & \cdots & 0 \\ 0 & s_2 & -\lambda & \cdots & 0 \\ 0 & 0 & s_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-1} \end{vmatrix} + (-\lambda) \begin{vmatrix} b_1-\lambda & b_2 & b_3 & \cdots & b_{n-2} & b_{n-1} \\ s_1 & -\lambda & 0 & \cdots & 0 & 0 \\ 0 & s_2 & -\lambda & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-2} & 0 \end{vmatrix}.$$

The first matrix is upper triangular, so its determinant is the product of its diagonal entries; for the second, we use the inductive hypothesis. So

$$\begin{aligned} \det(L-\lambda I) &= (-1)^{n+1}b_n(s_1s_2\cdots s_{n-1}) \\ &\qquad - \lambda\left((-1)^{n-1}(\lambda^{n-1} - b_1\lambda^{n-2} - b_2s_1\lambda^{n-3} - \cdots\right. \\ &\qquad\qquad \left. - b_{n-2}s_1s_2\cdots s_{n-3}\lambda - b_{n-1}s_1s_2\cdots s_{n-2}\right) \\ &= (-1)^{n+1}b_n(s_1s_2\cdots s_{n-1}) \\ &\qquad + (-1)^n(\lambda^n - b_1\lambda^{n-1} - b_2s_1\lambda^{n-2} - \cdots \\ &\qquad\qquad - b_{n-2}s_1s_2\cdots s_{n-3}\lambda^2 - b_{n-1}s_1s_2\cdots s_{n-2}\lambda) \\ &= (-1)^n(\lambda^n - b_1\lambda^{n-1} - b_2s_1\lambda^{n-2} - \cdots \\ &\qquad - b_{n-2}s_1s_2\cdots s_{n-3}\lambda^2 - b_{n-1}s_1s_2\cdots s_{n-2}\lambda - b_ns_1s_2\cdots s_{n-1}), \end{aligned}$$

proving the result.

17. P is a diagonal matrix, so inverting it consists of inverting each of the diagonal entries. Now, multiplying any matrix on the left by a diagonal matrix multiplies each row by the corresponding diagonal entry, and multiplying on the right by a diagonal matrix multiplies each column by the corresponding diagonal entry, so

$$P^{-1}L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{s_1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{s_1s_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{s_1s_2\cdots s_{n-2}} & 0 \end{bmatrix}$$

and then

$$P^{-1}LP = \begin{bmatrix} b_1 & b_2s_1 & b_3s_1s_2 & \cdots & b_{n-1}s_1s_2\cdots s_{n-2} & b_ns_1s_2\cdots s_{n-1} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

But this is the companion matrix of the polynomial

$$p(x) = x^n - b_1x^{n-1} - b_2s_1x^{n-2} - \cdots - b_{n-1}s_1s_2\cdots s_{n-2}x - b_ns_1s_2\cdots s_{n-1},$$

and by Exercise 32 in Section 4.3, the characteristic polynomial of $P^{-1}LP$ is $(-1)^np(\lambda)$. Finally, the characteristic polynomials of L and $P^{-1}LP$ are the same since they have the same eigenvalues, so the characteristic polynomial of L is

$$\begin{aligned} \det(L-\lambda I) &= \det(P^{-1}LP - \lambda I) = (-1)^np(\lambda) \\ &= (-1)^n(\lambda^n - b_1\lambda^{n-1} - b_2s_1\lambda^{n-2} - \cdots - b_{n-1}s_1s_2\cdots s_{n-2}\lambda - b_ns_1s_2\cdots s_{n-1}) \end{aligned}$$

18. By Exercise 46 in Section 4.4, the eigenvectors of $P^{-1}LP$ are of the form $P^{-1}\mathbf{x}$ where $\mathbf{x}$ is an eigenvalue of L, and by Exercise 32 in Section 4.3, an eigenvector of $P^{-1}LP$ for λ_1 is

$$\begin{bmatrix} \lambda_1^{n-1} \\ \lambda_1^{n-2} \\ \lambda_1^{n-3} \\ \vdots \\ \lambda_1 \\ 1 \end{bmatrix}.$$

So an eigenvector of L corresponding to λ_1 is

$$P\begin{bmatrix} \lambda_1^{n-1} \\ \lambda_1^{n-2} \\ \lambda_1^{n-3} \\ \vdots \\ \lambda_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1^{n-1} \\ s_1\lambda_1^{n-2} \\ s_1s_2\lambda_1^{n-3} \\ \vdots \\ s_1s_2\cdots s_{n-2}\lambda_1 \\ s_1s_2\cdots s_{n-2}s_{n-1} \end{bmatrix}.$$

Finally, we can scale this vector by dividing each entry by λ_1^{n-1} to get

$$\begin{bmatrix} 1 \\ s_1/\lambda_1 \\ s_1s_2/\lambda_1^2 \\ \vdots \\ s_1s_2\cdots s_{n-2}/\lambda_1^{n-2} \\ s_1s_2\cdots s_{n-2}s_{n-1}/\lambda_1^{n-1} \end{bmatrix}.$$

19. Using a CAS, we find the positive eigenvalue of

$$L = \begin{bmatrix} 1 & 1 & 3 \\ 0.7 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

to be $\lambda \approx 1.7456$, so this is the steady state growth rate of the population with this Leslie matrix. From Exercise 18, a corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 0.7/1.7456 \\ 0.7\cdot 0.5/(1.7456)^2 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0.401 \\ 0.115 \end{bmatrix}.$$

So the long-term proportions of the age classes are in the ratios $1 : 0.401 : 0.115$.

20. Using a CAS, we find the positive eigenvalue of

$$L = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix}$$

to be $\lambda \approx 1.2023$, so this is the steady state growth rate of the population with this Leslie matrix. From Exercise 18, a corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 0.5/1.2023 \\ 0.5\cdot 0.7/(1.2023)^2 \\ 0.5\cdot 0.7\cdot 0.3/(1.2023)^3 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0.416 \\ 0.242 \\ 0.060 \end{bmatrix}$$

So the long-term proportions of the age classes are in the ratios $1 : 0.416 : 0.242 : 0.06$.

21. Using a CAS, we find the positive eigenvalue of

$$L = \begin{bmatrix} 0 & 0.4 & 1.8 & 1.8 & 1.8 & 1.6 & 0.6 \\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \end{bmatrix}.$$

to be $\lambda \approx 1.0924$, so this is the steady state growth rate of the population with this Leslie matrix. From Exercise 18, a corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 0.3/1.0924 \\ 0.3 \cdot 0.7/(1.0924)^2 \\ 0.3 \cdot 0.7 \cdot 0.9/(1.0924)^3 \\ 0.3 \cdot 0.7 \cdot 0.9 \cdot 0.9/(1.0924)^4 \\ 0.3 \cdot 0.7 \cdot 0.9 \cdot 0.9 \cdot 0.9/(1.0924)^5 \\ 0.3 \cdot 0.7 \cdot 0.9 \cdot 0.9 \cdot 0.9 \cdot 0.6/(1.0924)^6 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0.275 \\ 0.176 \\ 0.145 \\ 0.119 \\ 0.098 \\ 0.054 \end{bmatrix}.$$

The entries of this vector give the long-term proportions of the age classes.

22. **(a)** The table gives us 13 separate age classes, each of duration two years; from the table, the Leslie matrix of the system is

$$L = \begin{bmatrix} 0 & 0.02 & 0.70 & 1.53 & 1.67 & 1.65 & 1.56 & 1.45 & 1.22 & 0.91 & 0.70 & 0.22 & 0 \\ 0.91 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.74 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.59 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.49 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.38 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.27 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.15 & 0 \end{bmatrix}.$$

Using a CAS, this matrix has positive eigenvalue ≈ 1.333, with a corresponding eigenvector (normalized to add to 100)

$$\begin{bmatrix} 36.1 \\ 24.65 \\ 16.27 \\ 10.38 \\ 6.23 \\ 3.46 \\ 1.74 \\ 0.77 \\ 0.28 \\ 0.08 \\ 0.016 \\ 0.0021 \\ 0.00023 \end{bmatrix}.$$

(b) In the long run, the percentage of seals in each age group is given by the entries in the eigenvector in part (a), and the growth rate will be about 1.333.

23. **(a)** Each term in r is the number of daughters born to a female in a particular age class times the probability of a given individual surviving to be in that age class, so it represents the expected number of daughters produced by a given individual while she is in that age class. So the sum of the terms gives the expected number of daughters ever born to a particular female over her lifetime.

(b) Define $g(\lambda)$ as suggested. Then $g(\lambda) = 1$ if and only if

$$\begin{aligned} \lambda^n g(\lambda) = b_1\lambda^{n-1} + b_2s_1\lambda^{n-2} + \cdots + b_ns_1s_2\cdots s_{n-1} = \lambda^n \quad &\Rightarrow \\ \lambda^n - b_1\lambda^{n-1} - b_2s_1\lambda^{n-2} - \cdots - b_ns_1s_2\cdots s_{n-1} = 0.& \end{aligned}$$

But from Exercise 16, this is true if and only if λ is a zero of the characteristic polynomial of the Leslie matrix, indicating that λ is an eigenvalue of L. So $r = g(1) = 1$ if and only if $\lambda = 1$ is an eigenvalue of L, as desired.

(c) For given values of b_i and s_i, it is clear that $g(\lambda)$ is a decreasing function of λ. Also, $r = g(1)$ and $g(\lambda_1) = 1$. So if $\lambda_1 > 1$, then $r = g(1) > g(\lambda_1) = 1$ so that $r > 1$. On the other hand, if $\lambda_1 < 1$, then $r = g(1) < g(\lambda_1) = 1$, so that $r < 1$. So we have shown that if the population is decreasing, then $r < 1$; if the population is increasing, then $r > 1$, and (part (b)) if the population is stable, then $r = 1$. Since this exhausts the possibilities, we have proven the converses as well. Thus $r < 1$ if and only if the population is decreasing and $r > 1$ if and only if the population is increasing. Note that this corresponds to our intuition, since if the net reproduction rate is less than one, we would expect the population in the next generation to be smaller.

24. If λ_1 is the unique positive eigenvalue, then $\lambda_1\mathbf{x} = L\mathbf{x}$ for the corresponding eigenvector $\mathbf{x}$. If h is the sustainable harvest ratio, then $(1-h)L\mathbf{x} = x$, which means that $L\mathbf{x} = \frac{1}{1-h}\mathbf{x}$. Thus $\lambda_1 = \frac{1}{1-h}$, so that $h = 1 - \frac{1}{\lambda_1}$.

25. **(a)** Exercise 21 found the positive eigenvalue of L to be $\lambda_1 \approx 1.0924$, so from Exercise 24, $h = 1 - \frac{1}{\lambda_1} \approx 1 - \frac{1}{1.0924} \approx 0.0846$.

(b) Reducing the initial population levels by 0.0846 gives

$$\mathbf{x}_0' = (1-0.0846)\begin{bmatrix}10\\2\\8\\5\\12\\0\\1\end{bmatrix} = \begin{bmatrix}9.15\\1.831\\7.323\\4.577\\10.985\\0\\0.915\end{bmatrix}.$$

Then

$$\mathbf{x}_1' = L\mathbf{x}_0' = \begin{bmatrix}0 & 0.4 & 1.8 & 1.8 & 1.8 & 1.6 & 0.6\\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.7 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0.9 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0\end{bmatrix}\begin{bmatrix}9.15\\1.831\\7.323\\4.577\\10.985\\0\\0.915\end{bmatrix} = \begin{bmatrix}42.47\\2.75\\1.28\\6.59\\4.12\\9.89\\0\end{bmatrix}.$$

The population has still grown substantially, because we did not start with a stable population vector. If we repeat with the stable population ratios computed in Exercise 21, we get

$$\mathbf{x}_0' = (1-0.0846)\begin{bmatrix}1\\0.275\\0.176\\0.145\\0.119\\0.098\\0.054\end{bmatrix} = \begin{bmatrix}0.915\\0.252\\0.161\\0.133\\0.109\\0.090\\0.049\end{bmatrix},$$

and

$$\mathbf{x}_1' = L\mathbf{x}_0' = \begin{bmatrix}0 & 0.4 & 1.8 & 1.8 & 1.8 & 1.6 & 0.6\\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.7 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0.9 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0\end{bmatrix}\begin{bmatrix}0.915\\0.252\\0.161\\0.133\\0.109\\0.090\\0.049\end{bmatrix} = \begin{bmatrix}0.999\\0.275\\0.176\\0.145\\0.119\\0.098\\0.053\end{bmatrix},$$

which is essentially $\mathbf{x}_0$.

26. Since $\lambda_1 \approx 1.333$, we have $h = 1 - \frac{1}{\lambda_1} = 1 - \frac{1}{1.333} \approx 0.25$, which is about one quarter of the population.

27. Suppose $\lambda_1 = r_1(\cos 0 + i\sin 0)$; then $r_1 = \lambda_1 > 0$ since λ_1 is a positive eigenvalue. Let $\lambda = r(\cos\theta + i\sin\theta)$ be some other eigenvalue. Then $g(\lambda) = g(\lambda_1) = 1$. Computing $g(\lambda)$ gives

$$\begin{aligned}1 = g(\lambda) &= \frac{b_1}{r(\cos\theta + i\sin\theta)} + \frac{b_2 s_1}{r^2(\cos\theta + i\sin\theta)^2} + \cdots + \frac{b_n s_1 s_2\cdots s_{n-1}}{r^n(\cos\theta+i\sin\theta)^n}\\ &= \frac{b_1}{r}(\cos\theta - i\sin\theta) + \frac{b_2 s_1}{r^2}(\cos\theta - i\sin\theta)^2 + \cdots + \frac{b_2 s_1 s_2\cdots s_{n-1}}{r^n}(\cos\theta - i\sin\theta)^n,\end{aligned}$$

where we get to the final equation by multiplying each fraction by $\frac{(\cos\theta - i\sin\theta)^k}{(\cos\theta - i\sin\theta)^k} = 1$ and using the identity $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$. Now apply DeMoivre's theorem to get

$$1 = g(\lambda) = \frac{b_1}{r}(\cos\theta - i\sin\theta) + \frac{b_2 s_1}{r^2}(\cos 2\theta - i\sin 2\theta) + \cdots + \frac{b_2 s_1 s_2\cdots s_{n-1}}{r^n}(\cos n\theta - i\sin n\theta).$$

Since $g(\lambda) = 1$, the imaginary parts must all cancel, so we are left with

$$1 = g(\lambda) = \frac{b_1}{r}\cos\theta + \frac{b_2 s_1}{r^2}\cos 2\theta + \cdots + \frac{b_2 s_1 s_2\cdots s_{n-1}}{r^n}\cos n\theta.$$

But $\cos k\theta \le 1$ for all k, and b_i, s_i, and r are all nonnegative, so

$$1 = g(\lambda) \le \frac{b_1}{r} + \frac{b_2 s_1}{r^2} + \cdots + \frac{b_2 s_1 s_2\cdots s_{n-1}}{r^n}.$$

Since $g(\lambda_1) = 1$ as well, we get

$$\frac{b_1}{r_1} + \frac{b_2 s_1}{r_1^2} + \cdots + \frac{b_2 s_1 s_2\cdots s_{n-1}}{r_1^n} \le \frac{b_1}{r} + \frac{b_2 s_1}{r^2} + \cdots + \frac{b_2 s_1 s_2\cdots s_{n-1}}{r^n}.$$

Then since $r_1 > 0$, we get $r_1 \ge r$.

28. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix}2-\lambda & 0\\ 1 & 1-\lambda\end{vmatrix} = (\lambda - 2)(\lambda - 1),$$

so that the Perron root is $\lambda_1 = 2$. To find its Perron eigenvector,

$$\begin{bmatrix} A - 2I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 2$. The entries of the Perron eigenvector must sum to 1, however, so dividing through by 2 gives for the Perron eigenvector $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

29. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 6 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2),$$

so that the Perron root is $\lambda_1 = 3$. To find its Perron eigenvector,

$$\begin{bmatrix} A - 3I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so that $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 3$. The entries of the Perron eigenvector must sum to 1, however, so dividing through by 5 gives for the Perron eigenvector $\begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$.

30. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(\lambda + 1)^2(\lambda - 2),$$

so that the Perron root is $\lambda_1 = 2$. To find its Perron eigenvector,

$$\begin{bmatrix} A - 2I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 2$. The entries of the Perron eigenvector must sum to 1, however, so dividing through by 3 gives for the Perron eigenvector $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

31. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = -\lambda(\lambda - 1)(\lambda - 3),$$

so that the Perron root is $\lambda_1 = 3$. To find its Perron eigenvector,

$$\begin{bmatrix} A - 3I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so that $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 3$. The entries of the Perron eigenvector must sum to 1, however, so dividing through by 4 gives for the Perron eigenvector $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$.

32. Here $n = 4$, and

$$(I + A)^{n-1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 3 & 1 & 1 & 3 \\ 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 \end{bmatrix} > O,$$

so A is irreducible.

33. Here $n = 4$, and

$$(I + A)^{n-1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 4 & 0 & 4 & 0 \\ 6 & 4 & 6 & 4 \\ 4 & 0 & 4 & 0 \\ 6 & 4 & 6 & 4 \end{bmatrix},$$

so A is reducible. Interchanging rows 1 and 4 and then columns 1 and 4 produces a matrix in the required form, so that

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

34. Here $n = 5$, and

$$(I + A)^{n-1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}^4 = \begin{bmatrix} 11 & 6 & 11 & 0 & 11 \\ 22 & 11 & 17 & 0 & 17 \\ 28 & 16 & 23 & 0 & 22 \\ 23 & 7 & 33 & 16 & 18 \\ 6 & 6 & 5 & 0 & 6 \end{bmatrix},$$

so that A is reducible. Exchanging rows 1 and 4 and then exchanging columns 1 and 4 produces a matrix in the required form, so that

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and B is a 1×1 matrix while D is 4×4.

35. Here $n = 5$, and

$$(I + A)^{n-1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}^4 = \begin{bmatrix} 1 & 4 & 1 & 5 & 11 \\ 1 & 1 & 5 & 11 & 16 \\ 5 & 6 & 6 & 12 & 21 \\ 6 & 4 & 5 & 6 & 12 \\ 5 & 1 & 11 & 16 & 22 \end{bmatrix},$$

so that A is irreducible.

36. (a) Let G be a graph and A its adjacency matrix. Let G' be the graph obtained from G by adding an edge from each vertex to itself if no such edge exists in G. Then G' is connected if and only if G is, since every vertex is always connected to itself. The adjacency matrix A' of G results from A by replacing every 0 on the diagonal by a 1. Now consider the matrix $A + I$. This matrix is identical to A' except that some diagonal entries of A' equal to 1 may be equal to 2 in $A + I$ (if the vertex in question had an edge from it to itself in G). But since all entries in $A + I$ and A' are nonnegative, $(A')^{n-1} > O$ if and only $(A + I)^{n-1} > O$ — whether those diagonal entries are 1 or 2 does not affect whether a particular entry in the power is zero or not. So G is connected if and only if G' is connected, which is the case if and only if $(A')^{n-1}$ of its adjacency matrix is $> O$, so that there is a path between any two vertices of length $n - 1$ (note that any shorter path can be made into a path of length $n-1$ by adding traverses of the edge from the vertex to itself). But $(A')^{n-1} > O$ then means that $(A + I)^{n-1} > O$, so that A is irreducible.

(b) Since all the graphs in Section 4.0 are connected, they all have irreducible adjacency matrices. The graph in Figure 4.1 has a primitive adjacency matrix, since

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} > O.$$

So does the graph in Figure 4.2:

$$A^4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}^4 = \begin{bmatrix} 15 & 4 & 9 & 9 & 4 & 9 & 4 & 9 & 9 & 9 \\ 4 & 15 & 4 & 9 & 9 & 9 & 9 & 4 & 9 & 9 \\ 9 & 4 & 15 & 4 & 9 & 9 & 9 & 9 & 4 & 9 \\ 9 & 9 & 4 & 15 & 4 & 9 & 9 & 9 & 9 & 4 \\ 4 & 9 & 9 & 4 & 15 & 4 & 9 & 9 & 9 & 9 \\ 9 & 9 & 9 & 9 & 4 & 15 & 9 & 4 & 4 & 9 \\ 4 & 9 & 9 & 9 & 9 & 9 & 15 & 9 & 4 & 4 \\ 9 & 4 & 9 & 9 & 9 & 4 & 9 & 15 & 9 & 4 \\ 9 & 9 & 4 & 9 & 9 & 4 & 4 & 9 & 15 & 9 \\ 9 & 9 & 9 & 4 & 9 & 9 & 4 & 4 & 9 & 15 \end{bmatrix} > O$$

The graph in Figure 4.3 has a primitive adjacency matrix:

$$A^4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 6 & 1 & 4 & 4 & 1 \\ 1 & 6 & 1 & 4 & 4 \\ 4 & 1 & 6 & 1 & 4 \\ 4 & 4 & 1 & 6 & 1 \\ 1 & 4 & 4 & 1 & 6 \end{bmatrix} > O.$$

However, the adjacency matrix for the graph is Figure 4.4 is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and any power of A will always have a 3×3 block of zeroes in either the upper left or upper right, so A is not primitive.

37. (a) Suppose G is bipartite with disjoint vertex sets U and V and A is its adjacency matrix. Recall that $(A^k)_{ij}$ gives the number of paths between vertices i and j. Also, from Exercise 80 in Section 3.7, if $u \in U$ and $v \in V$, then any path from u to itself will have even length, and any path from

u to v will have odd length, since in any path, each successive edge moves you from U to V and back. So odd powers of A will have zero entries along the diagonal, while even powers of A will have zero entries in any cell corresponding to a vertex in U and a vertex in V. Since every power of A has a zero entry, A is not primitive.

(b) Number the vertices so that $v_1, v_2, \ldots, v_k$ are in U and $v_{k+1}, v_{k+2}, \ldots, v_n$ are in V. Then the adjacency matrix of G is

$$A = \begin{bmatrix} O & B \\ B^T & O \end{bmatrix}.$$

Write the eigenvector $\mathbf{v}$ corresponding to λ as $\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}$, where $\mathbf{v}_1$ is a k-element column vector and $\mathbf{v}_2$ is an $(n-k)$-element column vector. Then

$$A\mathbf{v} = \lambda\mathbf{v} \iff \begin{bmatrix} B\mathbf{v}_2 & B^T\mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{v}_1 \\ \lambda\mathbf{v}_2 \end{bmatrix},$$

so that $B\mathbf{v}_2 = \lambda\mathbf{v}_1$ and $B^T\mathbf{v}_1 = \lambda\mathbf{v}_2$. Then

$$-\lambda \begin{bmatrix} \mathbf{v}_1 \\ -\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -\lambda\mathbf{v}_1 \\ \lambda\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -B\mathbf{v}_2 \\ B^T\mathbf{v}_1 \end{bmatrix} = A \begin{bmatrix} -\mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}.$$

This shows that $-\lambda$ is an eigenvalue of A with corresponding eigenvector $\begin{bmatrix} -\mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}$.

38. (a) Since each vertex of G connects to k other vertices, we see that each column of the adjacency matrix A of G sums to k. Thus $A = kP$, where P is a matrix each of whose columns sums to 1. So P is a transition matrix, and by Theorem 4.30, 1 is an eigenvalue of P, so that k is an eigenvalue of $kP = A$.

(b) If A is primitive, then $A^r > O$ for some r, and therefore $P^r > O$ as well. Therefore P is regular, so by Theorem 4.31, every other eigenvalue of P has absolute value strictly less than 1, so that every other eigenvalue of $A = kP$ has absolute value strictly less than k.

39. This exercise is left to the reader after doing the exploration suggested in Section 4.0.

40. (a) Let $A = [a_{ij}]$. Then $cA = [ca_{ij}]$, so that

$$|cA| = [|ca_{ij}|] = [|c| \cdot |a_{ij}|] = |c| \, [|a_{ij}|] = |c| \cdot |A|.$$

(b) Using the Triangle Inequality,

$$|A+B| = [|a_{ij} + b_{ij}|] \le [|a_{ij}| + |b_{ij}|] = [|a_{ij}|] + [|b_{ij}|] = |A| + |B|.$$

(c) $A\mathbf{x}$ is the $n \times 1$ matrix whose i^{th} entry is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n.$$

Thus the i^{th} entry of $|A\mathbf{x}|$ is

$$\begin{aligned} |(A\mathbf{x})_i| &= |a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n| \\ &\le |a_{i1}x_1| + |a_{i2}x_2| + \cdots + |a_{in}x_n| \\ &= |a_{i1}| \, |x_1| + |a_{i2}| \, |x_2| + \cdots + |a_{in}| \, |x_n|. \end{aligned}$$

But this is just the i^{th} entry of $|A| \, |\mathbf{x}|$.

(d) The ij entry of $|AB|$ is

$$\begin{aligned} |AB|_{ij} &= |a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}| \\ &\le |a_{i1}b_{1j}| + |a_{i2}b_{2j}| + \cdots + |a_{in}b_{nj}| \\ &= |a_{i1}| \, |b_{1j}| + |a_{i2}| \, |b_{2j}| + \cdots + |a_{in}| \, |b_{nj}|, \end{aligned}$$

which is the ij entry of $|A| \, |B|$.

41. A matrix is reducible if, after permuting rows and columns according to the same permutation, it can be written

$$\begin{bmatrix} B & C \\ O & D \end{bmatrix},$$

where B and D are square and O is the zero matrix. If A is 2×2, then B, C, O, and D must all be 1×1 matrices (which are therefore just the entries of A). If A is in the above form after applying no permutations, leaving A alone, then $a_{21}=0$. If A is in this form after applying the only other possible permutation — exchanging the two rows and the two columns — then the resulting matrix is

$$\begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix},$$

so that $a_{12}=0$. This proves the result.

42. (a) If λ_1 and $\mathbf{v}_1$ are the Perron eigenvalue and eigenvector respectively, then by Theorem 4.37, $\lambda_1 > 0$ and $\mathbf{v}_1 > \mathbf{0}$. By Exercise 22 in Section 4.3, $\lambda_1 - 1$ is an eigenvalue, with eigenvector $\mathbf{v}_1$, for $A - I$, and therefore $1-\lambda_1$ is an eigenvalue for $I - A$, again with eigenvector $\mathbf{v}_1$. Since $I - A$ is invertible, Theorem 4.18 says that $\frac{1}{1-\lambda_1}$ is an eigenvalue for $(I-A)^{-1}$, again with eigenvector $\mathbf{v}_1$. But $(I-A)^{-1} \geq 0$ and $\mathbf{v}_1 \geq 0$, so that

$$\frac{1}{1-\lambda_1}\mathbf{v}_1 = (I-A)^{-1}\mathbf{v}_1 \geq 0.$$

It follows that $\frac{1}{1-\lambda_1} \geq 0$ and thus $\lambda_1 < 1$. Since we also have $\lambda_1 > 0$, the result follows: $0 < \lambda_1 < 1$.

(b) Since $0<\lambda_1<1$, $\mathbf{v}_1 > \lambda_1\mathbf{v}_1 = A\mathbf{v}_1$.

43. $x_0 = 1$, $x_1 = 2x_0 = 2$, $x_2 = 2x_1 = 4$, $x_3 = 2x_2 = 8$, $x_4 = 2x_3 = 16$, $x_5 = 2x_4 = 32$.

44. $a_1 = 128$, $a_2 = \frac{a_1}{2} = 64$, $a_3 = \frac{a_2}{2} = 32$, $a_4 = \frac{a_3}{2} = 16$, $a_5 = \frac{a_4}{2} = 8$.

45. $y_0 = 0$, $y_1 = 1$, $y_2 = y_1 - y_0 = 1$, $y_3 = y_2 - y_1 = 0$, $y_4 = y_3 - y_2 = -1$, $y_5 = y_4 - y_3 = -1$.

46. $b_0 = 1$, $b_1 = 1$, $b_2 = 2b_1 + b_0 = 3$, $b_3 = 2b_2 + b_1 = 7$, $b_4 = 2b_3 + b_2 = 17$, $b_5 = 2b_4 + b_3 = 41$.

47. The recurrence is $x_n - 3x_{n-1} - 4x_{n-2} = 0$, so the characteristic equation is $\lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1) = 0$. Thus the system has eigenvalues -1 and 4, so $x_n = c_1(-1)^n + c_2\cdot 4^n$, where c_1 and c_2 are constants. Using the initial conditions, we have

$$\begin{aligned} 0 = x_0 &= c_1(-1)^0 + c_2\cdot 4^0 = c_1 + c_2, \\ 5 = x_1 &= c_1(-1)^1 + c_2\cdot 4^1 = -c_1 + 4c_2. \end{aligned}$$

Add the two equations to solve for c_2, then substitute back in to get $c_1 = -1$ and $c_2 = 1$, so that $x_n = -(-1)^n + 4^n = 4^n + (-1)^{n-1}$.

48. The recurrence is $x_n - 4x_{n-1} + 3x_{n-2} = 0$, so the characteristic equation is $\lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1) = 0$. Thus the system has eigenvalues 1 and 3, so $x_n = c_1\cdot 1^n + c_2\cdot 3^n$, where c_1 and c_2 are constants. Using the initial conditions, we have

$$\begin{aligned} 0 = x_0 &= c_1\cdot 1^0 + c_2\cdot 3^0 = c_1 + c_2, \\ 1 = x_1 &= c_1\cdot 1^1 + c_2\cdot 3^1 = c_1 + 3c_2. \end{aligned}$$

Subtract the two equations to solve for c_2, then substitute back in to get $c_1 = -\frac{1}{2}$ and $c_2 = \frac{1}{2}$, so that $x_n = -\frac{1}{2}\cdot 1^n + \frac{1}{2}\cdot 3^n = \frac{1}{2}(3^n - 1)$.

49. The recurrence is $y_n - 4y_{n-1} + 4y_{n-2} = 0$, so the characteristic equation is $\lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$. Thus the system has the double eigenvalue 2, so $y_n = c_1\cdot 2^n + c_2\cdot n2^n$, where c_1 and c_2 are constants. Using the initial conditions, we have

$$\begin{aligned} 1 = y_1 &= c_1\cdot 2^1 + c_2\cdot 1\cdot 2^1 = 2c_1 + 2c_2, \\ 6 = y_2 &= c_1\cdot 2^2 + c_2\cdot 2\cdot 2^2 = 4c_1 + 8c_2. \end{aligned}$$

Solving the system gives $c_1 = -\frac{1}{2}$ and $c_2 = 1$, so that $y_n = -\frac{1}{2}\cdot 2^n + n\cdot 2^n = n2^n - 2^{n-1}$.

50. The recurrence is $a_n - a_{n-1} + \frac{1}{4}a_{n-2} = 0$, so the characteristic equation is $\frac{1}{4}(4\lambda^2 - 4\lambda + 1) = \frac{1}{4}(2\lambda - 1)^2 = 0$. Thus the system has the double eigenvalue $\frac{1}{2} = 2^{-1}$, so $a_n = c_1 \cdot 2^{-n} + c_2 \cdot n2^{-n}$, where c_1 and c_2 are constants. Using the initial conditions, we have

$$\begin{aligned}
4 = a_0 &= c_1 \cdot 2^{-0} + c_2 \cdot 0 \cdot 2^{-0} = c_1,\\
1 = a_1 &= c_1 \cdot 2^{-1} + c_2 \cdot 1 \cdot 2^{-1} = \frac{1}{2}c_1 + \frac{1}{2}c_2.
\end{aligned}$$

Solving the system gives $c_1 = 4$ and $c_2 = -2$, so that $a_n = 4 \cdot 2^{-n} - 2n \cdot 2^{-n} = 2^{2-n} - n2^{1-n}$.

51. The recurrence is $b_n - 2b_{n-1} - 2b_{n-2} = 0$, so the characteristic equation is $\lambda^2 - 2\lambda - 2 = 0$. Using the quadratic formula, this equation has roots $\lambda = 1 \pm \sqrt{3}$, which are therefore the eigenvalues. So $b_n = c_1 \cdot (1+\sqrt{3})^n + c_2 \cdot (1-\sqrt{3})^n$, where c_1 and c_2 are constants. Using the initial conditions, we have

$$\begin{aligned}
0 = b_0 &= c_1 \cdot (1+\sqrt{3})^0 + c_2 \cdot (1-\sqrt{3})^0 = c_1 + c_2,\\
1 = b_1 &= c_1 \cdot (1+\sqrt{3})^1 + c_2 \cdot (1-\sqrt{3})^1 = c_1 + c_2 + (c_1 - c_2)\sqrt{3}.
\end{aligned}$$

From the first equation, $c_1 + c_2 = 0$, so the second becomes $1 = (c_1 - c_2)\sqrt{3}$, so that $c_1 - c_2 = \frac{\sqrt{3}}{3}$. Since $c_2 = -c_1$, we get $c_1 = \frac{\sqrt{3}}{6}$ and $c_2 = -\frac{\sqrt{3}}{6}$, so that the recurrence is

$$b_n = \frac{\sqrt{3}}{6}(1+\sqrt{3})^n - \frac{\sqrt{3}}{6}(1-\sqrt{3})^n = \frac{\sqrt{3}}{6}\left((1+\sqrt{3})^n - (1-\sqrt{3})^n\right).$$

52. The recurrence is $y_n - y_{n-1} + y_{n-2} = 0$, so the characteristic equation is $\lambda^2 - \lambda + 1 = 0$. Using the quadratic formula, this equation has roots $\lambda = \frac{1}{2}(1 \pm \sqrt{3}\,i)$, which are therefore the eigenvalues. So $y_n = c_1 \cdot \left(\frac{1}{2}(1+\sqrt{3}\,i)\right)^n + c_2 \cdot \left(\frac{1}{2}(1-\sqrt{3}\,i)\right)^n$, where c_1 and c_2 are constants. Using the initial conditions, we have

$$\begin{aligned}
0 = y_0 &= c_1 \cdot \left(\frac{1}{2}(1+\sqrt{3}\,i)\right)^0 + c_2 \cdot \left(\frac{1}{2}(1-\sqrt{3}\,i)\right)^0 = c_1 + c_2,\\
1 = y_1 &= c_1 \cdot \left(\frac{1}{2}(1+\sqrt{3}\,i)\right)^1 + c_2 \cdot \left(\frac{1}{2}(1-\sqrt{3}\,i)\right)^1 = \frac{1}{2}c_1(1+\sqrt{3}\,i) + \frac{1}{2}c_2(1-\sqrt{3}\,i).
\end{aligned}$$

From the first equation, $c_1 + c_2 = 0$, so the second becomes $2 = (c_1 - c_2)\sqrt{3}\,i$, so that $c_1 - c_2 = -\frac{2\sqrt{3}}{3}i$. Since $c_2 = -c_1$, we get $c_1 = -\frac{\sqrt{3}}{3}i$ and $c_2 = \frac{\sqrt{3}}{6}$, so that the recurrence is

$$y_n = -\frac{\sqrt{3}}{3}i\left(\frac{1}{2}(1+\sqrt{3}\,i)\right)^n + \frac{\sqrt{3}}{3}i\left(\frac{1}{2}(1-\sqrt{3}\,i)\right)^n.$$

53. Working with the matrix A from the Theorem, since it has distinct eigenvalues $\lambda_1 \neq \lambda_2$, it can be diagonalized, so that there is some invertible P such that

$$P = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, \qquad P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

First consider the case where one of the eigenvalues, say λ_2, is zero. In this case the matrix must be of the form

$$A = \begin{bmatrix} a & 0 \\ 1 & 0 \end{bmatrix},$$

so that the other eigenvalue is a. In this case, the recurrence relation is $x_n = ax_{n-1} + 0x_{n-2} = ax_{n-1}$, so if we are given x_j and x_{j-1} as initial conditions, we have $x_n = a^{n-j}x_j$. So take $c_1 = x_j a^{-j}$ and $c_2 = 0$; then $x_n = c_1\lambda_1^n + c_2\lambda_2^n$ as desired. Now suppose that neither eigenvalue is zero. Then

$$A^k = PD^kP^{-1} = \frac{1}{eh - fg}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\begin{bmatrix} h & -f \\ -g & e \end{bmatrix} = \frac{1}{eh - fg}\begin{bmatrix} eh\lambda_1^k - fg\lambda_2^k & -ef\lambda_1^k + ef\lambda_2^k \\ gh\lambda_1^k - gh\lambda_2^k & -fg\lambda_1^k + eh\lambda_2^k \end{bmatrix}.$$

If we are given two terms of the sequence, say x_{j-1} and x_j, then for any n we have

$$\begin{aligned}
\mathbf{x}_n = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} &= A^{n-j}\mathbf{x}_j \\
&= \frac{1}{eh-fg}\begin{bmatrix} eh\lambda_1^{n-j} - fg\lambda_2^{n-j} & -ef\lambda_1^{n-j} + ef\lambda_2^{n-j} \\ gh\lambda_1^{n-j} - gh\lambda_2^{n-j} & -fg\lambda_1^{n-j} + eh\lambda_2^{n-j} \end{bmatrix}\begin{bmatrix} x_j \\ x_{j-1} \end{bmatrix} \\
&= \frac{1}{eh-fg}\begin{bmatrix} \left(eh\lambda_1^{n-j} - fg\lambda_2^{n-j}\right)x_j + \left(-ef\lambda_1^{n-j} + ef\lambda_2^{n-j}\right)x_{j-1} \\ \left(gh\lambda_1^{n-j} - gh\lambda_2^{n-j}\right)x_j + \left(-fg\lambda_1^{n-j} + eh\lambda_2^{n-j}\right)x_{j-1} \end{bmatrix} \\
&= \frac{1}{eh-fg}\begin{bmatrix} \left((ehx_j - efx_{j-1})\lambda_1^{-j}\right)\lambda_1^n + \left((-fgx_j + efx_{j-1})\lambda_2^{-j}\right)\lambda_2^n \\ \left((ghx_j - fgx_{j-1})\lambda_1^{-j}\right)\lambda_1^n + \left((-ghx_j + ehx_{j-1})\lambda_2^{-j}\right)\lambda_2^n \end{bmatrix}.
\end{aligned}$$

Looking at the first row, we see that for

$$c_1 = \frac{ehx_j - efx_{j-1}}{eh-fg}\lambda_1^{-j}, \qquad c_2 = \frac{-fgx_j + efx_{j-1}}{eh-fg}\lambda_2^{-j},$$

c_1 and c_2 are independent of n, and we have $x_n = c_1\lambda_1^n + c_2\lambda_2^n$, as required.

54. (a) First suppose that the eigenvalues are distinct, i.e., that $\lambda_1 \neq \lambda_2$. Then Exercise 53 gives one formula for c_1 and c_2 involving the diagonalizing matrix P. But since we now know that c_1 and c_2 exist, we can derive a simpler formula. We know the solutions are of the form $x_n = c_1\lambda_1^n + c_2\lambda_2^n$, valid for all n. Since $x_0 = r$ and $x_1 = s$, we get the following linear system in c_1 and c_2:

$$\begin{aligned} \lambda_1^0 c_1 + \lambda_2^0 c_2 &= r \\ \lambda_1^1 c_1 + \lambda_2^1 c_2 &= s \end{aligned} \qquad \text{or} \qquad \begin{aligned} c_1 + \quad c_2 &= r \\ \lambda_1 c_1 + \lambda_2 c_2 &= s. \end{aligned}$$

Solving this system, for example by setting up the augmented matrix and row-reducing, gives a unique solution

$$c_1 = r - \frac{s - \lambda_1 r}{\lambda_2 - \lambda_1}, \qquad c_2 = \frac{s - \lambda_1 r}{\lambda_2 - \lambda_1}.$$

So the system has a unique solution in terms of $x_0 = r$, $x_1 = s$, λ_1, and λ_2. Note that the system has a unique solution (and these expressions make sense) since $\lambda_1 \neq \lambda_2$.

Next suppose that the eigenvalues are the same, say $\lambda_1 = \lambda_2 = \lambda$. Then the solutions are of the form $x_n = c_1\lambda^n + c_2 n\lambda^n$. Note that we must have $\lambda \neq 0$, since otherwise the characteristic polynomial of the recurrence matrix is λ^2, so the recurrence relation is $x_n = 0$. With $x_0 = r$ and $x_1 = s$, we get the following linear system in c_1 and c_2:

$$\begin{aligned} \lambda^0 c_1 + 0 \cdot \lambda^0 c_2 &= r \\ \lambda^1 c_1 + 1 \cdot \lambda^1 c_2 &= s \end{aligned} \qquad \text{or} \qquad \begin{aligned} c_1 \qquad &= r \\ \lambda c_1 + \lambda c_2 &= s. \end{aligned}$$

Thus $c_1 = r$, and substituting into the second equation gives (since $\lambda \neq 0$) $c_2 = \frac{s-\lambda r}{\lambda}$. So in this case as well we can find the scalars c_1 and c_2 uniquely. Finally, note that 0 is an eigenvalue of the recurrence matrix only when $b = 0$ in the recurrence relation $x_n = ax_{n-1} + bx_{n-2}$.

(b) From part (a), the general solution to the recurrence is

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n = \left(r - \frac{s - \lambda_1 r}{\lambda_2 - \lambda_1}\right)\lambda_1^n + \left(\frac{s - \lambda_1 r}{\lambda_2 - \lambda_1}\right)\lambda_2^n.$$

Substituting $x_0 = r = 0$ and $x_1 = s = 1$ gives

$$x_n = \left(0 - \frac{1 - \lambda_1 \cdot 0}{\lambda_2 - \lambda_1}\right)\lambda_1^n + \left(\frac{1 - \lambda_1 \cdot 0}{\lambda_2 - \lambda_1}\right)\lambda_2^n = \frac{1}{\lambda_2 - \lambda_1}(-\lambda_1^n + \lambda_2^n) = \frac{1}{\lambda_1 - \lambda_2}(\lambda_1^n - \lambda_2^n).$$

55. (a) For $n = 1$, since $f_0 = 0$ and $f_1 = f_2 = 1$, the equation holds:

$$A^1 = A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix}.$$

Now assume that the equation is true for $n = k$, and consider A^{k+1}:

$$A^{k+1} = A^k A = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{k+1} + f_k & f_{k+1} \\ f_k + f_{k-1} & f_k \end{bmatrix} = \begin{bmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix}$$

as desired.

(b) Since $\det A = -1$, it follows that $\det(A^n) = (\det A)^n = (-1)^n$. But from part (a),

$$\det(A^n) = f_{n+2} f_n - f_{n+1}^2 = (-1)^n.$$

(c) If we look at the 5×13 "rectangle" as being composed of two "triangles", we see that in fact they are not triangles: the "diagonal" of the "rectangle" is not straight because the 8×3 triangular pieces are not similar to the entire triangles, since $\frac{5}{13} \neq \frac{3}{8}$. So there is empty space along the diagonal of the figure that adds up to the missing 1 square unit.

56. (a) $t_1 = 1$ (the only way is using the 1×1 rectangle); $t_2 = 3$ (two 1×1 rectangles or either of the 1×2 rectangles); $t_3 = 5$; $t_4 = 11$; $t_5 = 21$. For t_0, we can think of that as the number of ways to tile an empty rectangle; there is only one way — using no rectangles.

(b) The rightmost rectangle in a tiling of a $1 \times n$ rectangle is either the 1×1 or one of the 1×2 rectangles. So the number of tilings of a $1 \times n$ rectangle is the number of tilings of a $1 \times (n-1)$ rectangle (adding the 1×1) plus twice the number of tilings of a $1 \times (n-2)$ rectangle (adding either 1×2). So the recurrence relation is

$$t_n = t_{n-1} + 2t_{n-2}.$$

(c) The characteristic equation is $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$, so the eigenvalues are -1 and 2. To find the constants, we solve

$$\begin{aligned} t_1 = 1 &= c_1(-1)^1 + c_2 \cdot 2^1 = -c_1 + 2c_2 \\ t_2 = 3 &= c_1(-1)^2 + c_2 \cdot 2^2 = c_1 + 4c_2. \end{aligned}$$

This gives $c_1 = \frac{1}{3}$ and $c_2 = \frac{2}{3}$, so that

$$t_n = \frac{1}{3}(-1)^n + \frac{2}{3} \cdot 2^n = \frac{1}{3}\left((-1)^n + 2^{n+1}\right).$$

Evaluating for various values of t gives

$$\begin{aligned} t_0 &= \frac{1}{3}\left((-1)^0 + 2\right) = 1, & t_1 &= \frac{1}{3}\left((-1)^1 + 2^2\right) = 1, \\ t_2 &= \frac{1}{3}\left((-1)^2 + 2^3\right) = 3, & t_3 &= \frac{1}{3}\left((-1)^3 + 2^4\right) = 5, \\ t_4 &= \frac{1}{3}\left((-1)^4 + 2^5\right) = 11, & t_5 &= \frac{1}{3}\left((-1)^5 + 2^6\right) = 21. \end{aligned}$$

These agree with part (a).

57. (a) $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $d_4 = 5$, $d_5 = 8$. As in Exercise 56, d_0 can be interpreted as the number of ways to cover a 2×0 rectangle, which is one — use no dominos.

(b) The right end of any tiling of a $2 \times n$ rectangle either consists of a domino laid vertically, or of two dominos each laid horizontally. Removing those leaves either a $2 \times (n-1)$ or a $2 \times (n-2)$ rectangle. So the number of tilings of a $2 \times n$ rectangle is equal to the number of tilings of a $2 \times (n-1)$ rectangle plus the number of tilings of a $2 \times (n-2)$ rectangle, or

$$d_n = d_{n-1} + d_{n-2}.$$

(c) We recognize the recurrence as the recurrence for the Fibonacci sequence; the initial values shift it by 1, so we get

$$d_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

Computing values using this formula gives the same values as in part (a), including the value for d_0.

58. Row-reducing gives

$$\begin{aligned}\left[A - \tfrac{1+\sqrt{5}}{2}I \;\middle|\; 0\right] &= \begin{bmatrix} 1-\frac{1+\sqrt{5}}{2} & 0 \\ 1 & -\frac{1+\sqrt{5}}{2}\end{bmatrix} \to \begin{bmatrix} 1 & \frac{-1-\sqrt{5}}{2} \\ 0 & 0\end{bmatrix} \\ \left[A - \tfrac{1-\sqrt{5}}{2}I \;\middle|\; 0\right] &= \begin{bmatrix} 1-\frac{1-\sqrt{5}}{2} & 0 \\ 1 & -\frac{1-\sqrt{5}}{2}\end{bmatrix} \to \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ 0 & 0\end{bmatrix}.\end{aligned}$$

So the eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix}\frac{1+\sqrt{5}}{2} \\ 1\end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix}\frac{1-\sqrt{5}}{2} \\ 1\end{bmatrix}.$$

Then

$$\begin{aligned}\lim_{k\to\infty}\frac{\mathbf{x}_k}{\lambda_1^k} &= \lim_{k\to\infty}\frac{\begin{bmatrix}\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k \\ \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\end{bmatrix}}{\left(\frac{1+\sqrt{5}}{2}\right)^k} \\ &= \lim_{k\to\infty}\begin{bmatrix}\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^k \\ \frac{1}{\sqrt{5}}\cdot\frac{2}{1+\sqrt{5}} - \frac{1}{\sqrt{5}}\cdot\frac{1-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^k\end{bmatrix}.\end{aligned}$$

Since $\left|\frac{1-\sqrt{5}}{1+\sqrt{5}}\right| \approx 0.381 < 1$, the terms involving k^{th} powers of this fraction go to zero as $k \to \infty$, leaving us with (in the limit)

$$\begin{bmatrix}\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}(1+\sqrt{5})}\end{bmatrix} = \frac{2}{\sqrt{5}(1+\sqrt{5})}\begin{bmatrix}\frac{1+\sqrt{5}}{2} \\ 1\end{bmatrix} = \frac{2}{\sqrt{5}(1+\sqrt{5})}\mathbf{v}_1 = \frac{5-\sqrt{5}}{10}\mathbf{v}_1,$$

which is a constant times $\mathbf{v}_1$.

59. The coefficient matrix has characteristic polynomial

$$\begin{bmatrix}1-\lambda & 3 \\ 2 & 2-\lambda\end{bmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1),$$

so that the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$. The corresponding eigenvectors are found by row-reduction:

$$\begin{aligned}\left[A - 4I \mid 0\right] &= \left[\begin{array}{rr|r} -3 & 3 & 0 \\ 2 & -2 & 0\end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0\end{array}\right] \quad\Rightarrow\quad \mathbf{v}_1 = \begin{bmatrix}1 \\ 1\end{bmatrix} \\ \left[A + I \mid 0\right] &= \left[\begin{array}{rr|r} 2 & 3 & 0 \\ 2 & 3 & 0\end{array}\right] \to \left[\begin{array}{rr|r} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0\end{array}\right] \quad\Rightarrow\quad \mathbf{v}_2 = \begin{bmatrix}3 \\ -2\end{bmatrix}.\end{aligned}$$

Then by Theorem 4.40,

$$\mathbf{x} = C_1e^{4t}\begin{bmatrix}1 \\ 1\end{bmatrix} + C_2e^{-t}\begin{bmatrix}3 \\ -2\end{bmatrix}.$$

Substitute $t = 0$ to determine the constants:

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix} = C_1 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^0 \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} C_1 + 3C_2 \\ C_1 - 2C_2 \end{bmatrix}.$$

These two equations give $C_1 = 3$ and $C_2 = -1$, so that the solution is

$$\mathbf{x} = 3e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-t} \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \text{ or } \begin{aligned} x(t) &= 3e^{4t} - 3e^{-t} \\ y(t) &= 3e^{4t} + 2e^{-t}. \end{aligned}$$

60. The coefficient matrix has characteristic polynomial

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3),$$

so that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. The corresponding eigenvectors are found by row-reduction:

$$\begin{aligned} \left[A - I \mid 0\right] &= \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \left[A - 3I \mid 0\right] &= \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Then by Theorem 4.40,

$$\mathbf{x} = C_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Substitute $t = 0$ to determine the constants:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} C_1 + C_2 \\ C_1 - C_2 \end{bmatrix}.$$

These two equations give $C_1 = 1$ and $C_2 = 0$, so that the solution is

$$\mathbf{x} = e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \cdot e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{ or } \begin{aligned} x(t) &= e^{t} \\ y(t) &= e^{t}. \end{aligned}$$

61. The coefficient matrix has characteristic polynomial

$$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) - 1 = \lambda^2 - 2 = (\lambda - \sqrt{2}\,)(\lambda + \sqrt{2}\,),$$

so that the eigenvalues are $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{2}$. The corresponding eigenvectors are found by row-reduction:

$$\begin{aligned} \left[A - \sqrt{2}I \mid 0\right] &= \left[\begin{array}{cc|c} 1-\sqrt{2} & 1 & 0 \\ 1 & -1-\sqrt{2} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1-\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} \\ \left[A + \sqrt{2}I \mid 0\right] &= \left[\begin{array}{cc|c} 1+\sqrt{2} & 1 & 0 \\ 1 & -1+\sqrt{2} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1+\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}. \end{aligned}$$

Then by Theorem 4.40,

$$\mathbf{x} = C_1 e^{\sqrt{2}\,t} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} + C_2 e^{-\sqrt{2}\,t} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}.$$

Substitute $t = 0$ to determine the constants:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = C_1 e^0 \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} + C_2 e^0 \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} C_1(1+\sqrt{2}) + C_2(1-\sqrt{2}) \\ C_1 + C_2 \end{bmatrix}.$$

The second equation gives $C_1 + C_2 = 0$; substituting into the first equation gives $(C_1 - C_2)\sqrt{2} = 1$, so that $C_1 - C_2 = \frac{\sqrt{2}}{2}$. It follows that $C_1 = \frac{\sqrt{2}}{4}$ and $C_2 = -\frac{\sqrt{2}}{4}$, so that the solution is

$$\mathbf{x} = \frac{\sqrt{2}}{4}e^{\sqrt{2}\,t}\begin{bmatrix}1+\sqrt{2}\\1\end{bmatrix} - \frac{\sqrt{2}}{4}e^{-\sqrt{2}\,t}\begin{bmatrix}1-\sqrt{2}\\1\end{bmatrix}, \text{ or } \begin{aligned} x_1(t) &= \frac{2+\sqrt{2}}{4}e^{\sqrt{2}\,t} + \frac{2-\sqrt{2}}{4}e^{-\sqrt{2}\,t} \\ x_2(t) &= \frac{\sqrt{2}}{4}\left(e^{\sqrt{2}\,t} - e^{-\sqrt{2}\,t}\right). \end{aligned}$$

62. The coefficient matrix has characteristic polynomial

$$\begin{bmatrix}1-\lambda & -1\\ 1 & 1-\lambda\end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2,$$

so that the eigenvalues are $\lambda_1 = 1+i$ and $\lambda_2 = 1-i$. The corresponding eigenvectors are found by row-reduction:

$$\begin{aligned} \left[A-(1+i)I \mid 0\right] &= \left[\begin{array}{cc|c} -i & -1 & 0\\ 1 & -i & 0\end{array}\right] \to \left[\begin{array}{cc|c} 1 & -i & 0\\ 0 & 0 & 0\end{array}\right] \quad\Rightarrow\quad \mathbf{v}_1 = \begin{bmatrix}i\\1\end{bmatrix} \\ \left[A-(1-i)I \mid 0\right] &= \left[\begin{array}{cc|c} i & -1 & 0\\ 1 & i & 0\end{array}\right] \to \left[\begin{array}{cc|c} 1 & i & 0\\ 0 & 0 & 0\end{array}\right] \quad\Rightarrow\quad \mathbf{v}_2 = \begin{bmatrix}-i\\1\end{bmatrix}. \end{aligned}$$

Then by Theorem 4.40,

$$\mathbf{y} = C_1e^{(1+i)t}\begin{bmatrix}i\\1\end{bmatrix} + C_2e^{(1-i)t}\begin{bmatrix}-i\\1\end{bmatrix}.$$

Substitute $t = 0$ to determine the constants:

$$\mathbf{y}(0) = \begin{bmatrix}1\\1\end{bmatrix} = C_1e^0\begin{bmatrix}i\\1\end{bmatrix} + C_2e^0\begin{bmatrix}-i\\1\end{bmatrix} = \begin{bmatrix}(C_1-C_2)i\\ C_1+C_2\end{bmatrix}.$$

Solving gives $C_1 = \frac{1-i}{2}$, $C_1 = \frac{1+i}{2}$, so that the solution is

$$\mathbf{y} = \frac{1-i}{2}e^{(1+i)t}\begin{bmatrix}i\\1\end{bmatrix} + \frac{1+i}{2}e^{(1-i)t}\begin{bmatrix}-i\\1\end{bmatrix},$$

or

$$\begin{aligned} y_1(t) &= \frac{1-i}{2}e^{(1+i)t}i + \frac{1+i}{2}e^{(1-i)t}\cdot(-i) = \frac{i+1}{2}e^{(1+i)t} + \frac{1-i}{2}e^{(1-i)t} \\ y_2(t) &= \frac{1-i}{2}e^{(1+i)t} + \frac{1+i}{2}e^{(1-i)t}. \end{aligned}$$

Now substitute $\cos t + i\sin t$ for e^{it}, giving

$$\begin{aligned} y_1(t) &= \frac{i+1}{2}e^{(1+i)t} + \frac{1-i}{2}e^{(1-i)t} \\ &= \frac{i+1}{2}e^t(\cos t + i\sin t) + \frac{1-i}{2}e^t(\cos t - i\sin t) \\ &= e^t\cos t + \left(\frac{i-1}{2} + \frac{-i-1}{2}\right)e^t\sin t \\ &= e^t(\cos t - \sin t) \\ y_2(t) &= \frac{1-i}{2}e^{(1+i)t} + \frac{1+i}{2}e^{(1-i)t} \\ &= \frac{1-i}{2}e^t(\cos t + i\sin t) + \frac{1+i}{2}e^t(\cos t - i\sin t) \\ &= e^t\cos t + \left(\frac{1+i}{2} + \frac{1-i}{2}\right)e^t\sin t \\ &= e^t(\cos t + \sin t). \end{aligned}$$

63. The coefficient matrix has characteristic polynomial

$$\begin{bmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = -\lambda^3 + \lambda = -\lambda(1-\lambda)(1+\lambda),$$

so that the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = -1$. The corresponding eigenvectors are found by row-reduction:

$$\begin{aligned}
\left[A - 0I \mid 0\right] &= \left[\begin{array}{rrr|r} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \\
\left[A - I \mid 0\right] &= \left[\begin{array}{rrr|r} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
\left[A + I \mid 0\right] &= \left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.
\end{aligned}$$

Then by Theorem 4.40,

$$\mathbf{x} = C_1 e^{0t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Substitute $t = 0$ to determine the constants:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = C_1 e^0 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + C_3 e^0 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -C_1 - C_3 \\ C_1 + C_2 + C_3 \\ C_1 + C_2 \end{bmatrix}$$

Solving gives $C_1 = -2$, $C_3 = 1$, $C_2 = 1$, so that the solution is

$$\mathbf{x} = -2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} x(t) &= 2 - e^{-t} \\ y(t) &= -2 + e^t + e^{-t} \\ z(t) &= -2 + e^t. \end{aligned}$$

64. The coefficient matrix has characteristic polynomial

$$\begin{bmatrix} 1-\lambda & 0 & 3 \\ 1 & -2-\lambda & 1 \\ 3 & 0 & -\lambda \end{bmatrix} = -(\lambda - 4)(\lambda + 2)^2$$

so that the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -2$ (with algebraic multiplicity 2). The corresponding eigenvectors are found by row-reduction:

$$\begin{aligned}
\left[A - 4I \mid 0\right] &= \left[\begin{array}{rrr|r} -3 & 0 & 3 & 0 \\ 1 & -6 & 1 & 0 \\ 3 & 0 & -3 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \\
\left[A + 2I \mid 0\right] &= \left[\begin{array}{rrr|r} 3 & 0 & 3 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 3 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Then by Theorem 4.40,

$$\mathbf{x} = C_1 e^{4t} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Substitute $t = 0$ to determine the constants:

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = C_1 e^0 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + C_2 e^0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_3 e^0 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3C_1 - C_3 \\ C_1 + C_2 \\ 3C_1 + C_3 \end{bmatrix}$$

Solving gives $C_1 = 1$, $C_2 = 2$, $C_3 = 1$, so that the solution is

$$\mathbf{x} = e^{4t} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + 2e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

or

$$\begin{aligned} x(t) &= 3e^{4t} - e^{-2t} \\ y(t) &= e^{4t} + 2e^{-2t} \\ z(t) &= 3e^{4t} + e^{-2t}. \end{aligned}$$

65. **(a)** The coefficient matrix has characteristic polynomial

$$\begin{bmatrix} 1.2 - \lambda & -0.2 \\ -0.2 & 1.5 - \lambda \end{bmatrix} = (1.2 - \lambda)(1.5 - \lambda) - 0.04 = \lambda^2 - 2.7\lambda + 1.76 = (\lambda - 1.1)(\lambda - 1.6)$$

so that the eigenvalues are $\lambda_1 = 1.1$ and $\lambda_2 = 1.6$. The corresponding eigenvectors are found by row-reduction:

$$\begin{aligned} \left[A - 1.1I \mid 0\right] &= \left[\begin{array}{cc|c} 0.1 & -0.2 & 0 \\ 0.2 & 0.4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \left[A - 1.6I \mid 0\right] &= \left[\begin{array}{cc|c} -0.4 & -0.2 & 0 \\ -0.2 & -0.1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}. \end{aligned}$$

Then by Theorem 4.40,

$$\mathbf{x} = C_1 e^{1.1t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{1.6t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Substitute $t = 0$ to determine the constants:

$$\mathbf{x}(0) = \begin{bmatrix} 400 \\ 500 \end{bmatrix} = C_1 e^0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^0 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2C_1 - C_2 \\ C_1 + 2C_2 \end{bmatrix}.$$

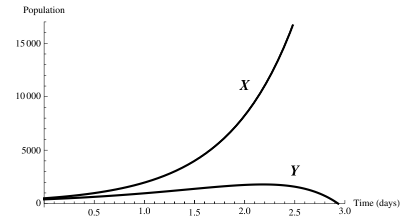
Solving gives $C_1 = 260$, $C_2 = 120$, so that the solution is

$$\mathbf{x} = 260e^{1.1t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 120e^{1.6t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

or

$$\begin{aligned} x(t) &= 520e^{1.1t} - 120e^{1.6t} \\ y(t) &= 260e^{1.1t} + 240e^{1.6t}. \end{aligned}$$

A plot of the populations over time is



From the graph, the population of Y dies out after about three days.

(b) Substituting a and b for 400 and 500 in the initial value computation above gives

$$\mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix} = C_1e^0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2e^0 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2C_1 - C_2 \\ C_1 + 2C_2 \end{bmatrix}.$$

Solving gives $C_1 = \frac{1}{5}(2a+b)$, $C_2 = \frac{1}{5}(-a+2b)$, so that the solution is

$$\mathbf{x} = \frac{1}{5}(2a+b)e^{1.1t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{5}(-a+2b)e^{1.6t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

or

$$\begin{aligned} x(t) &= \frac{2}{5}(2a+b)e^{1.1t} - \frac{1}{5}(-a+2b)e^{1.6t} \\ y(t) &= \frac{1}{5}(2a+b)e^{1.1t} + \frac{2}{5}(-a+2b)e^{1.6t}. \end{aligned}$$

In the long run, the $e^{1.6t}$ term will dominate, so if $a > 2b$, then $-a+2b < 0$, so the coefficient of $e^{1.6t}$ in $x(t)$ is positive and in $y(t)$ is negative, so that Y will die out and X will dominate. If $a < 2b$, then $-a+2b > 0$, so the reverse will happen — X will die out and Y will dominate. Finally, if $a = 2b$, then the $e^{1.6t}$ term disappears in the solutions, and both populations will grow at a rate proportional to $e^{1.1t}$, though one will grow twice as fast as the other.

66. The coefficient matrix has characteristic polynomial

$$\begin{bmatrix} -0.8-\lambda & 0.4 \\ 0.4 & -0.2-\lambda \end{bmatrix} = (-0.8-\lambda)(-0.2-\lambda) - 0.16 = \lambda^2 + \lambda = \lambda(\lambda+1)$$

so that the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -1$. The corresponding eigenvectors are found by row-reduction:

$$\begin{aligned} \begin{bmatrix} A - 0I \mid 0 \end{bmatrix} &= \left[\begin{array}{cc|c} -0.8 & 0.4 & 0 \\ 0.4 & -0.2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} A + I \mid 0 \end{bmatrix} &= \left[\begin{array}{cc|c} 0.2 & 0.4 & 0 \\ 0.4 & 0.8 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \end{aligned}$$

Then by Theorem 4.40,

$$\mathbf{x} = C_1e^{0t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Substitute $t = 0$ to determine the constants:

$$\mathbf{x}(0) = \begin{bmatrix} 15 \\ 10 \end{bmatrix} = C_1e^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2e^0 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 - 2C_2 \\ 2C_1 + C_2 \end{bmatrix}.$$

Solving gives $C_1 = 7$, $C_2 = -4$, so that the solution is

$$\mathbf{x} = 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

or

$$\begin{aligned} x(t) &= 7 + 8e^{-t} \\ y(t) &= 14 - 4e^{-t}. \end{aligned}$$

As $t \to \infty$, the exponential terms go to zero, and the populations stabilize at 7 of X and 14 of Y.

67. We want to convert $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ into an equation $\mathbf{u}' = A\mathbf{u}$. If we could write $\mathbf{b} = A\mathbf{b}'$, then letting $\mathbf{u} = \mathbf{x} + \mathbf{b}'$ would do the trick. To find $\mathbf{b}'$, we invert A:

$$\mathbf{b}' = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -30 \\ -10 \end{bmatrix} = \begin{bmatrix} -10 \\ -20 \end{bmatrix}.$$

Then let $\mathbf{u} = \mathbf{x} + \begin{bmatrix} -10 \\ -20 \end{bmatrix}$, so that

$$\mathbf{u}' = \left(\mathbf{x} + \begin{bmatrix} -10 \\ -20 \end{bmatrix}\right)' = \mathbf{x}' = A\mathbf{x} + \mathbf{b} = A\mathbf{x} + AA^{-1}\begin{bmatrix} -10 \\ -20 \end{bmatrix} = A\left(\mathbf{x} + A^{-1}\begin{bmatrix} -10 \\ -20 \end{bmatrix}\right) = A\mathbf{u}.$$

So now we want to solve this equation. A has characteristic polynomial

$$\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2,$$

so the eigenvalues are $1+i$ and $1-i$. Eigenvectors are found by row-reducing:

$$\begin{aligned} \left[A - (1+i)I \mid 0\right] &= \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \\ \left[A - (1-i)I \mid 0\right] &= \left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}. \end{aligned}$$

Then by Theorem 4.40,

$$\mathbf{u} = C_1 e^{(1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 e^{(1-i)t}\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Substitute $t = 0$ to determine the constants:

$$\mathbf{u}(0) = \mathbf{x}(0) + \begin{bmatrix} -10 \\ -20 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} = C_1 e^0 \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 e^0 \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 + iC_2 \\ iC_1 + C_2 \end{bmatrix}.$$

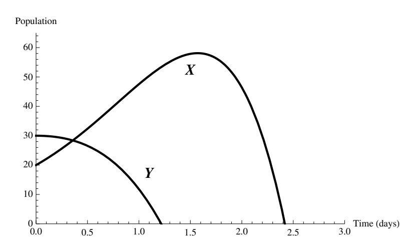
Solving gives $C_1 = C_2 = 5 - 5i$, so that the solution is

$$\begin{aligned} \mathbf{u} &= (5-5i)e^{(1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + (5-5i)e^{(1-i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= (5-5i)e^t(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix} + (5-5i)e^t(\cos t - i\sin t)\begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= e^t\begin{bmatrix} (5-5i)\cos t + (5+5i)\sin t + (5+5i)\cos t + (5-5i)\sin t \\ (5+5i)\cos t + (-5+5i)\sin t + (5-5i)\cos t + (-5-5i)\sin t \end{bmatrix} \\ &= e^t\begin{bmatrix} 10\cos t + 10\sin t \\ 10\cos t - 10\sin t \end{bmatrix}. \end{aligned}$$

Rewriting in terms of the original variables gives

$$\mathbf{x} = e^t\begin{bmatrix} 10\cos t + 10\sin t \\ 10\cos t - 10\sin t \end{bmatrix} - \begin{bmatrix} -10 \\ -20 \end{bmatrix} = e^t\begin{bmatrix} 10\cos t + 10\sin t \\ 10\cos t - 10\sin t \end{bmatrix} + \begin{bmatrix} 10 \\ 20 \end{bmatrix}.$$

A plot of the populations is below. Both populations die out; species Y after about 1.2 time units and species X after about 2.4.



68. We want to convert $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ into an equation $\mathbf{u}' = A\mathbf{u}$. If we could write $\mathbf{b} = A\mathbf{b}'$, then letting $\mathbf{u} = \mathbf{x} + \mathbf{b}'$ would do the trick. To find $\mathbf{b}'$, we invert A:

$$\mathbf{b}' = A^{-1}\mathbf{b} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 40 \end{bmatrix} = \begin{bmatrix} -20 \\ -20 \end{bmatrix}.$$

Then let $\mathbf{u} = \mathbf{x} + \begin{bmatrix} -20 \\ -20 \end{bmatrix}$, so that

$$\mathbf{u}' = \left(\mathbf{x} + \begin{bmatrix} -20 \\ -20 \end{bmatrix}\right)' = \mathbf{x}' = A\mathbf{x} + \mathbf{b} = A\mathbf{x} + AA^{-1}\begin{bmatrix} 0 \\ 40 \end{bmatrix} = A\left(\mathbf{x} + A^{-1}\begin{bmatrix} 0 \\ 40 \end{bmatrix}\right) = A\mathbf{u}.$$

So now we want to solve this equation. A has characteristic polynomial

$$\begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} = (-1-\lambda)^2 + 1 = \lambda^2 + 2\lambda + 2,$$

so the eigenvalues are $-1+i$ and $-1-i$. Eigenvectors are found by row-reducing:

$$\begin{aligned} \begin{bmatrix} A - (-1+i)I \mid 0 \end{bmatrix} &= \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \\ \begin{bmatrix} A - (-1-i)I \mid 0 \end{bmatrix} &= \left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}. \end{aligned}$$

Then by Theorem 4.40,

$$\mathbf{u} = C_1 e^{(-1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 e^{(-1-i)t}\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Substitute $t = 0$ to determine the constants:

$$\mathbf{u}(0) = \mathbf{x}(0) + \begin{bmatrix} -20 \\ -20 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \end{bmatrix} = C_1 e^0 \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 e^0 \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 + iC_2 \\ iC_1 + C_2 \end{bmatrix}.$$

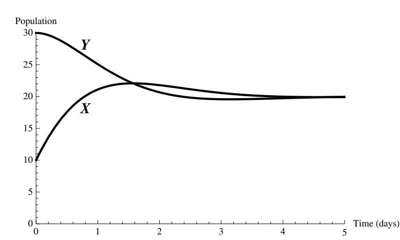
Solving gives $C_1 = -5 - 5i$ and $C_2 = 5 + 5i$, so that the solution is

$$\begin{aligned} \mathbf{u} &= (-5-5i)e^{(-1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + (5+5i)e^{(-1-i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= (-5-5i)e^{-t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix} + (5+5i)e^{-t}(\cos t - i\sin t)\begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= e^{-t}\begin{bmatrix} (-5-5i)\cos t + (5-5i)\sin t + (-5+5i)\cos t + (5+5i)\sin t \\ (5-5i)\cos t + (5+5i)\sin t + (5+5i)\cos t + (5-5i)\sin t \end{bmatrix} \\ &= e^{-t}\begin{bmatrix} -10\cos t + 10\sin t \\ 10\cos t + 10\sin t \end{bmatrix}. \end{aligned}$$

Rewriting in terms of the original variables gives

$$\mathbf{x} = e^{-t}\begin{bmatrix} -10\cos t + 10\sin t \\ 10\cos t + 10\sin t \end{bmatrix} - \begin{bmatrix} -20 \\ -20 \end{bmatrix} = e^{-t}\begin{bmatrix} -10\cos t + 10\sin t \\ 10\cos t + 10\sin t \end{bmatrix} + \begin{bmatrix} 20 \\ 20 \end{bmatrix}.$$

A plot of the populations is below; both populations stabilize at 20 as the e^{-t} term goes to zero.



69. **(a)** Making the change of variables $y = x'$ and $z = x$, the original equation $x'' + ax' + bx = 0$ becomes $y' + ay + bz = 0$; since $z = x$, we have $z' = x' = y$. So we get the system

$$\begin{aligned} y' &= -ay - bz \\ z' &= \quad\; y \qquad . \end{aligned}$$

(b) The coefficient matrix of the system in (a) is

$$\begin{bmatrix} -a & -b \\ 1 & 0 \end{bmatrix},$$

whose characteristic polynomial is $(-a - \lambda)(-\lambda) + b = \lambda^2 + a\lambda + b$.

70. Make the substitutions $y_1 = x^{(n-1)}$, $y_2 = x^{(n-2)}, \ldots, y_{n-1} = x'$, $y_n = x$. Then the original equation

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1x' + a_0 = 0$$

can be written as

$$y_1' + a_{n-1}y_1 + a_{n-2}y_2 + \cdots + a_2y_{n-2} + a_1y_{n-1} + a_0y_n = 0,$$

and for $i = 2, 3, \ldots, n$, we have $y_i' = y_{i-1}$. So we get the n equations

$$\begin{aligned} y_1' &= a_{n-1}y_1 - a_{n-2}y_2 - \cdots - a_2y_{n-2} - a_1y_{n-1} - a_0y_n \\ y_2' &= y_1 \\ y_3' &= y_2 \\ &\;\vdots \\ y_n' &= y_{n-1} \end{aligned}$$

whose coefficient matrix is

$$\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

This is the companion matrix of $p(\lambda)$.

71. From Exercise 69, we know that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, so the eigenvalues are 2 and 3. Therefore the general solution is $x(t) = C_1e^{2t} + C_2e^{3t}$.

72. From Exercise 69, we know that the characteristic polynomial is $\lambda^2 + 4\lambda + 3$, so the eigenvalues are -1 and -3. Therefore the general solution is $x(t) = C_1e^{-t} + C_2e^{-3t}$.

73. From Exercise 59, the matrix of the system has eigenvalues 4 and -1 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$. So by Theorem 4.41, the system $\mathbf{x}' = A\mathbf{x}$ has solution $\mathbf{x} = e^{At}\mathbf{x}(0)$. In this case $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$, we get

$$e^{AT} = P(e^t)^DP^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{5}e^{4t} + \frac{3}{5}e^{-t} & \frac{3}{5}e^{4t} - \frac{3}{5}e^{-t} \\ \frac{2}{5}e^{4t} - \frac{2}{5}e^{-t} & \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t} \end{bmatrix}.$$

Thus the solution is

$$\begin{bmatrix} \frac{2}{5}e^{4t} + \frac{3}{5}e^{-t} & \frac{3}{5}e^{4t} - \frac{3}{5}e^{-t} \\ \frac{2}{5}e^{4t} - \frac{2}{5}e^{-t} & \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3e^{4t} - 3e^{-t} \\ 3e^{4t} + 2e^{-t} \end{bmatrix},$$

which matches the result from Exercise 59.

74. From Exercise 60, the matrix of the system has eigenvalues 1 and 3 with corresponding eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$. So by Theorem 4.41, the system $\mathbf{x}' = A\mathbf{x}$ has solution $\mathbf{x} = e^{At}\mathbf{x}(0)$. In this case $\mathbf{x}(0) = \begin{bmatrix}1\\1\end{bmatrix}$, we get

$$e^{AT} = P(e^t)^D P^{-1} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}e^t & 0\\0 & e^{3t}\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}^{-1} = \begin{bmatrix}\frac{1}{2}e^t + \frac{1}{2}e^{3t} & \frac{1}{2}e^t - \frac{1}{2}e^{3t}\\ \frac{1}{2}e^t - \frac{1}{2}e^{3t} & \frac{1}{2}e^t + \frac{1}{2}e^{3t}\end{bmatrix}.$$

Thus the solution is

$$\begin{bmatrix}\frac{1}{2}e^t + \frac{1}{2}e^{3t} & \frac{1}{2}e^t - \frac{1}{2}e^{3t}\\ \frac{1}{2}e^t - \frac{1}{2}e^{3t} & \frac{1}{2}e^t + \frac{1}{2}e^{3t}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}e^t\\e^t\end{bmatrix},$$

which matches the result from Exercise 60.

75. From Exercise 63, the matrix of the system has eigenvalues 0, 1, and -1 with corresponding eigenvectors $\begin{bmatrix}-1\\1\\1\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}-1\\1\\0\end{bmatrix}$. So by Theorem 4.41, the system $\mathbf{x}' = A\mathbf{x}$ has solution $\mathbf{x} = e^{At}\mathbf{x}(0)$. In this case $\mathbf{x}(0) = \begin{bmatrix}1\\0\\-1\end{bmatrix}$, and we get

$$\begin{aligned}e^{AT} = P(e^t)^D P^{-1} &= \begin{bmatrix}-1 & 0 & -1\\1 & 1 & 1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & e^t & 0\\0 & 0 & e^{-t}\end{bmatrix}\begin{bmatrix}-1 & 0 & -1\\1 & 1 & 1\\1 & 1 & 0\end{bmatrix}^{-1}\\ &= \begin{bmatrix}1 & 1-e^{-t} & -1+e^{-t}\\ -1+e^{-t} & -1+e^{-t}+e^t & 1-e^{-t}\\ -1+e^t & -1+e^t & 1\end{bmatrix}.\end{aligned}$$

Thus the solution is

$$\begin{bmatrix}1 & 1-e^{-t} & -1+e^{-t}\\ -1+e^{-t} & -1+e^{-t}+e^t & 1-e^{-t}\\ -1+e^t & -1+e^t & 1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}2-e^{-t}\\ -2+e^t+e^{-t}\\ -2+e^t\end{bmatrix},$$

which matches the result from Exercise 63.

76. From Exercise 64, the matrix of the system has eigenvalues 4 and -2 (with algebraic multiplicity 2) with corresponding eigenvectors $\begin{bmatrix}3\\1\\3\end{bmatrix}$ and $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$. So by Theorem 4.41, the system $\mathbf{x}' = A\mathbf{x}$ has solution $\mathbf{x} = e^{At}\mathbf{x}(0)$. In this case $\mathbf{x}(0) = \begin{bmatrix}2\\3\\4\end{bmatrix}$, and we get

$$\begin{aligned}e^{AT} = P(e^t)^D P^{-1} &= \begin{bmatrix}3 & 0 & -1\\1 & 1 & 0\\3 & 0 & 1\end{bmatrix}\begin{bmatrix}e^{4t} & 0 & 0\\0 & e^{-2t} & 0\\0 & 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}3 & 0 & -1\\1 & 1 & 0\\3 & 0 & 1\end{bmatrix}^{-1}\\ &= \begin{bmatrix}\frac{1}{2}e^{4t} + \frac{1}{2}e^{-2t} & 0 & \frac{1}{2}e^{4t} - \frac{1}{2}e^{-2t}\\ \frac{1}{6}e^{4t} - \frac{1}{6}e^{-2t} & e^{-2t} & \frac{1}{6}e^{4t} - \frac{1}{6}e^{-2t}\\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{-2t} & 0 & \frac{1}{2}e^{4t} + \frac{1}{2}e^{-2t}\end{bmatrix}.\end{aligned}$$

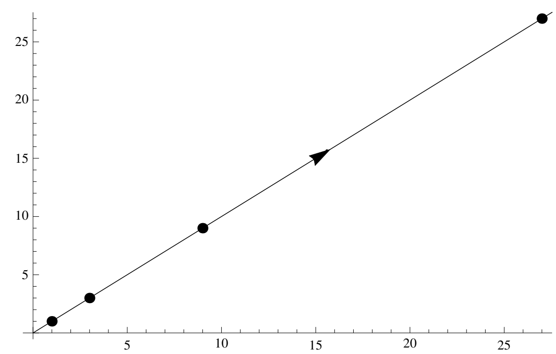
Thus the solution is

$$\begin{bmatrix} \frac{1}{2}e^{4t}+\frac{1}{2}e^{-2t} & 0 & \frac{1}{2}e^{4t}-\frac{1}{2}e^{-2t} \\ \frac{1}{6}e^{4t}-\frac{1}{6}e^{-2t} & e^{-2t} & \frac{1}{6}e^{4t}-\frac{1}{6}e^{-2t} \\ \frac{1}{2}e^{4t}-\frac{1}{2}e^{-2t} & 0 & \frac{1}{2}e^{4t}+\frac{1}{2}e^{-2t} \end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}3e^{4t}-e^{-2t}\\ e^{4t}+2e^{-2t}\\ 3e^{4t}+e^{-2t}\end{bmatrix},$$

which matches the result from Exercise 64.

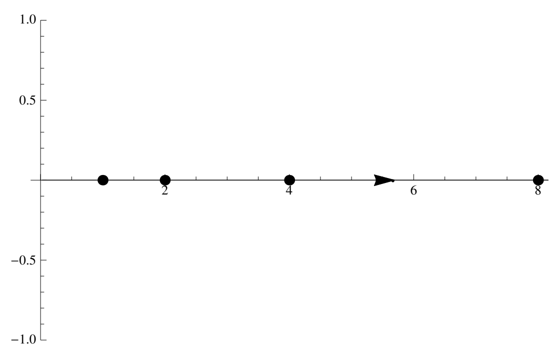
77. **(a)** With $A = \begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix}9\\9\end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix}27\\27\end{bmatrix}.$$



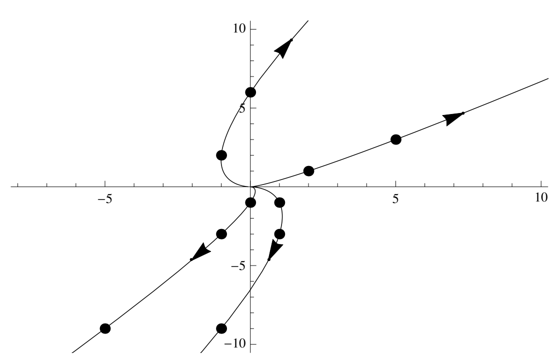
(b) With $A = \begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix}4\\0\end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix}8\\0\end{bmatrix}.$$



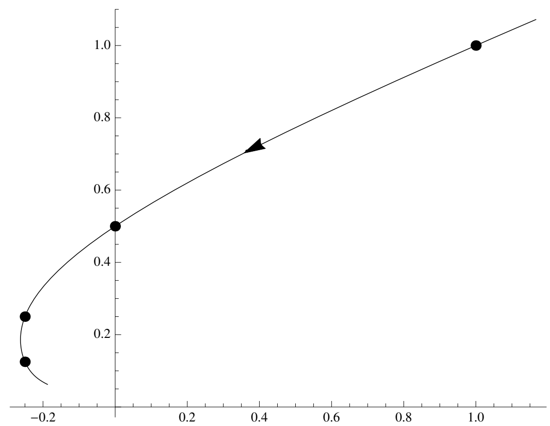
(c) Since the matrix is upper triangular, we see that its eigenvalues are 2 and 3; since these are both greater than 1 in magnitude, the origin is a repeller.

(d) Some other trajectories are



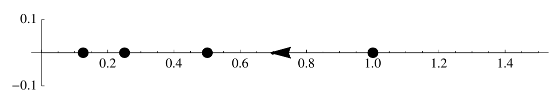
78. (a) With $A = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.5 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0.5\end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix}-0.25\\0.25\end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix}-0.25\\0.125\end{bmatrix}.$$



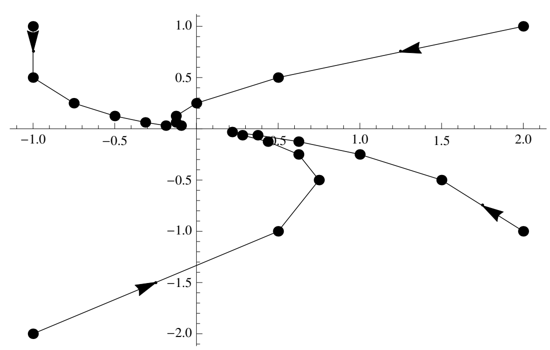
(b) With $A = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.5 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0.5\\0\end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix}0.25\\0\end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix}0.125\\0\end{bmatrix}.$$



(c) Since the matrix is upper triangular, we see that its only eigenvalue is 0.5; since this is less than 1 in magnitude, the origin is an attractor.

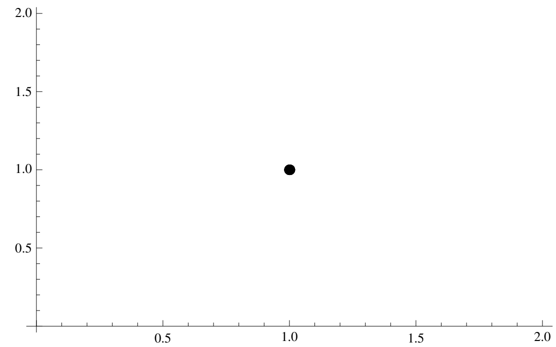
(d) Some other trajectories are



79. (a) With $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, we have

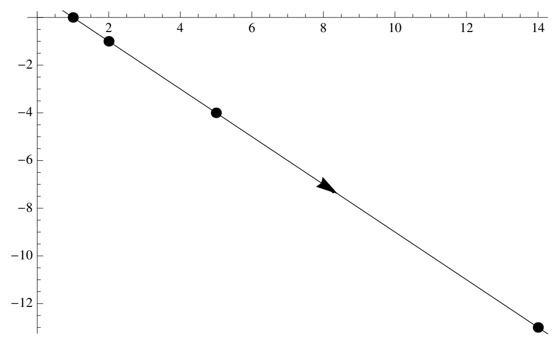
$$\mathbf{x}_1 = A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix},$$

so that $\begin{bmatrix}1\\1\end{bmatrix}$ is a fixed point:



(b) With $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 5 \\ -4 \end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 14 \\ -13 \end{bmatrix}.$$

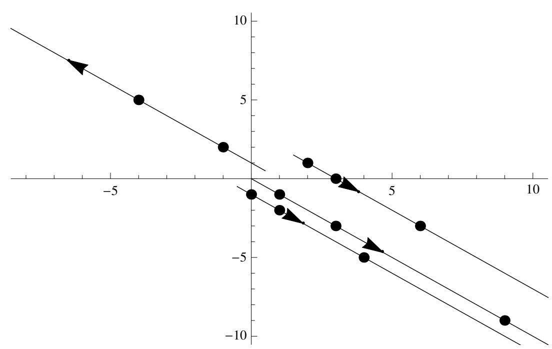


(c) The characteristic polynomial of the matrix is

$$\det(A - \lambda I) = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

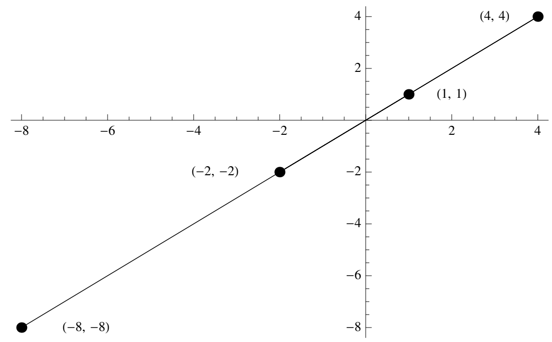
Thus one eigenvalue is greater than 1 in magnitude and the other is equal to 1. So the origin is not a repeller, an attractor, or a saddle point; there are no nonzero values of $\mathbf{x}_0$ for which the trajectory of $\mathbf{x}_0$ approaches the origin.

(d) Some other trajectories are



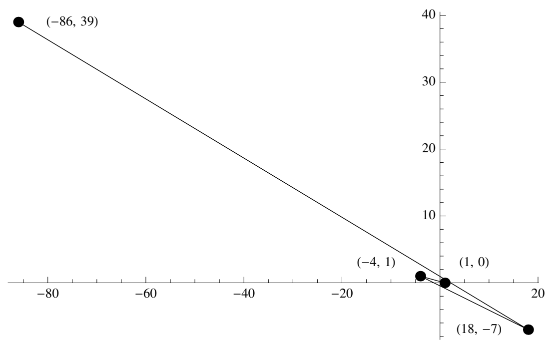
80. **(a)** With $A = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} -8 \\ -8 \end{bmatrix}.$$



(b) With $A = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 18 \\ -7 \end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} -86 \\ 39 \end{bmatrix}.$$

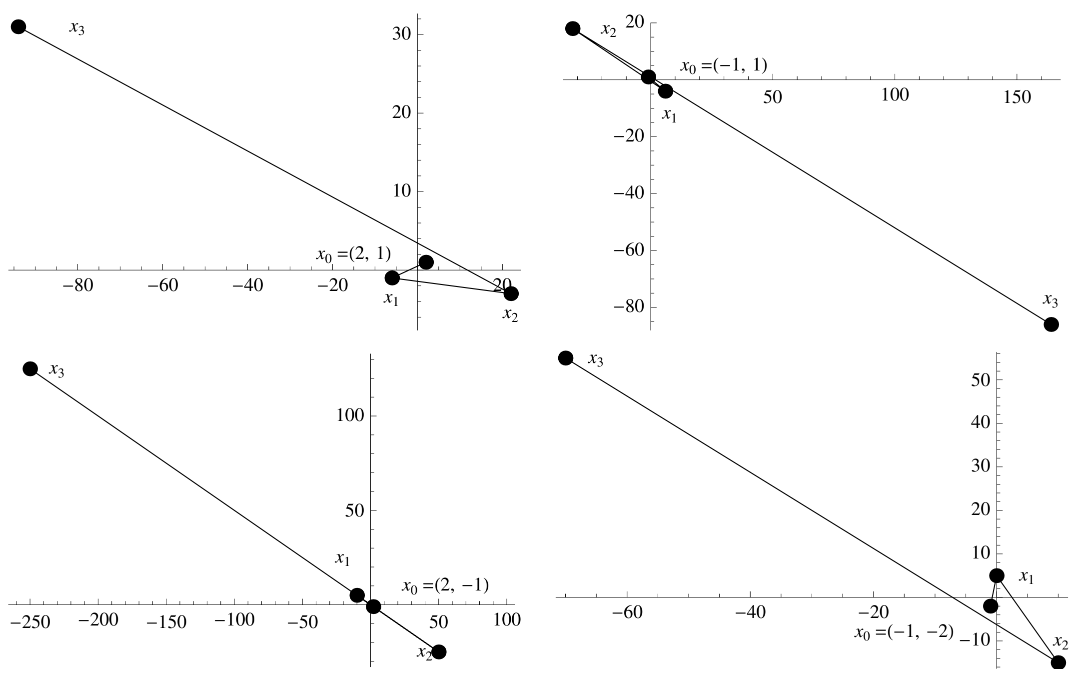


(c) The characteristic polynomial of the matrix is

$$\det(A - \lambda I) = \begin{bmatrix} -4-\lambda & 2 \\ 1 & -3-\lambda \end{bmatrix} = \lambda^2 + 7\lambda + 10 = (\lambda+5)(\lambda+2).$$

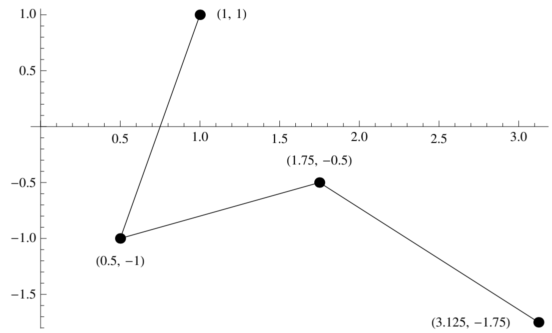
Since both eigenvalues, -5 and -2, are greater than 1 in magnitude, the origin is a repeller.

(d) Some other trajectories are



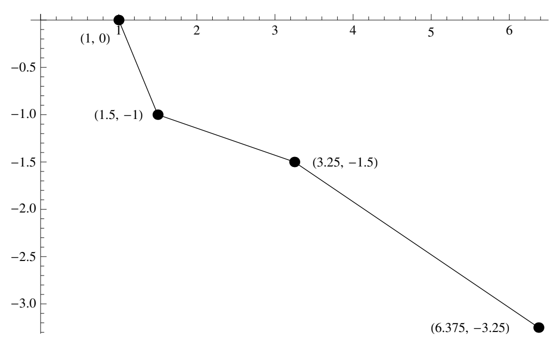
81. (a) With $A = \begin{bmatrix} 1.5 & -1 \\ -1 & 0 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} -8 \\ -8 \end{bmatrix}.$$



(b) With $A = \begin{bmatrix} 1.5 & -1 \\ -1 & 0 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 18 \\ -7 \end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} -86 \\ 39 \end{bmatrix}.$$

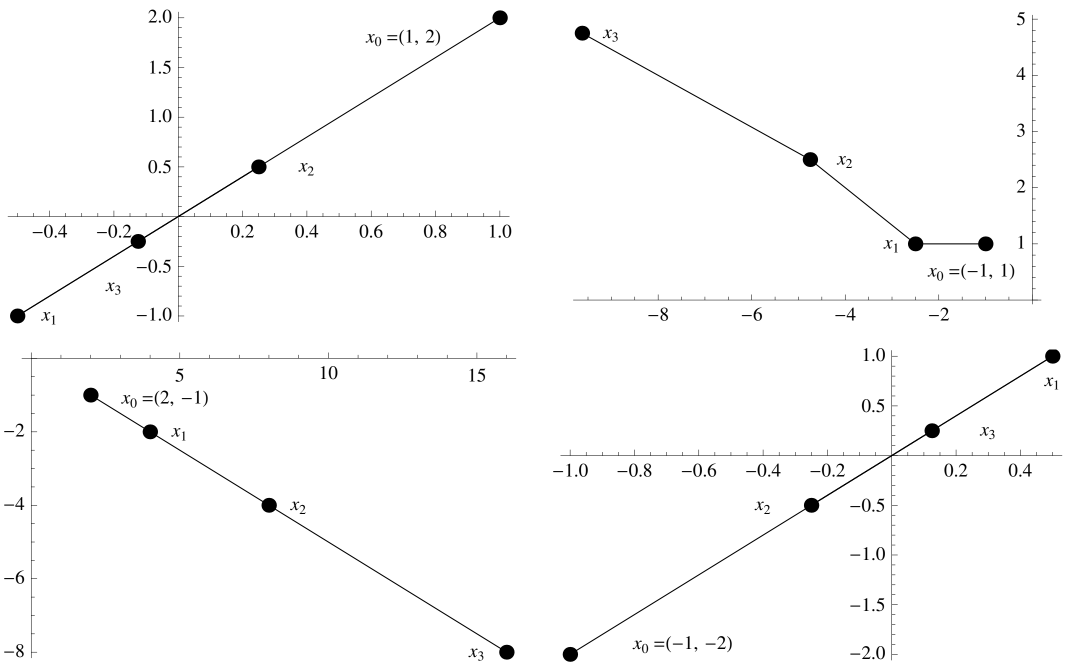


(c) The characteristic polynomial of the matrix is

$$\det(A - \lambda I) = \begin{bmatrix} 1.5 - \lambda & -1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 - 1.5\lambda + 1 = (\lambda - 2)(\lambda + 0.5)$$

Since one eigenvalue is greater than 1 in magnitude and the other is less, the origin is a saddle point. Eigenvectors for $\lambda = -0.5$ are attracted to the origin, and other vectors are repelled.

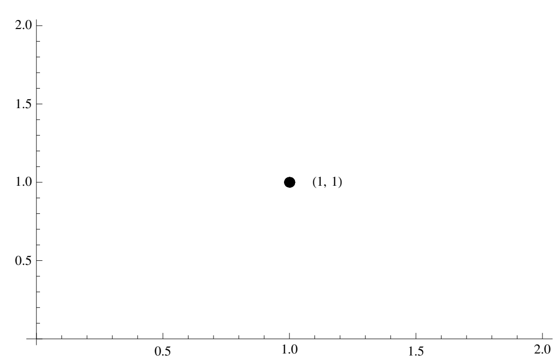
(d) Some other trajectories are



82. **(a)** With $A = \begin{bmatrix} 0.1 & 0.9 \\ 0.5 & 0.5 \end{bmatrix}$, we have

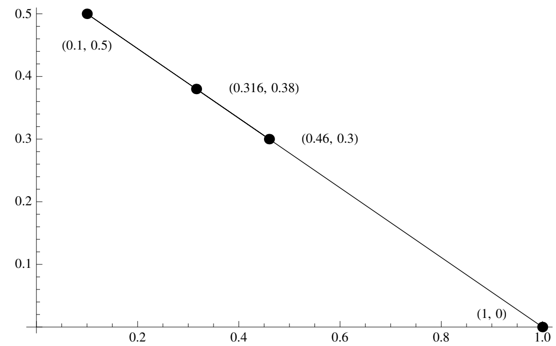
$$\mathbf{x}_1 = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a fixed point - it is an eigenvector for the eigenvalue $\lambda = 1$.



(b) With $A = \begin{bmatrix} 0.1 & 0.9 \\ 0.5 & 0.5 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0.46 \\ 0.30 \end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 0.316 \\ 0.38 \end{bmatrix}.$$

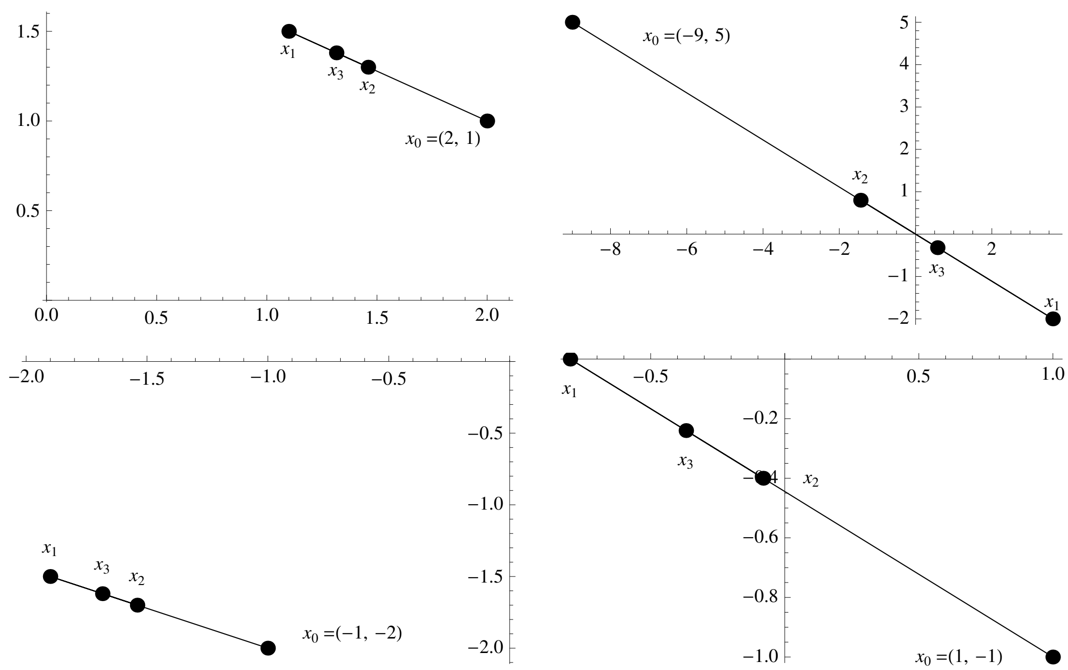


(c) The characteristic polynomial of the matrix is

$$\det(A - \lambda I) = \begin{bmatrix} 0.1-\lambda & 0.9 \\ 0.5 & 0.5-\lambda \end{bmatrix} = \lambda^2 - 0.6\lambda - 0.45 = (\lambda - 1)(\lambda + 0.4).$$

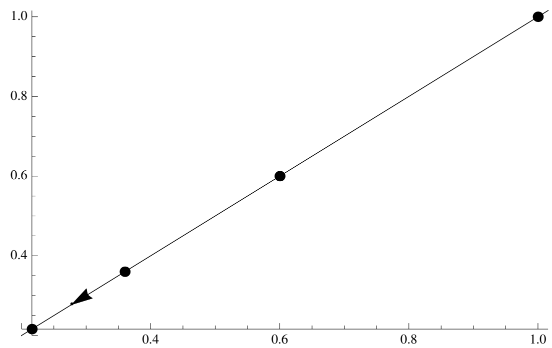
Since one eigenvalue is less than 1 in magnitude and the other is equal to 1, the origin is neither an attractor, a repeller, nor a saddle point.

(d) Some other trajectories are



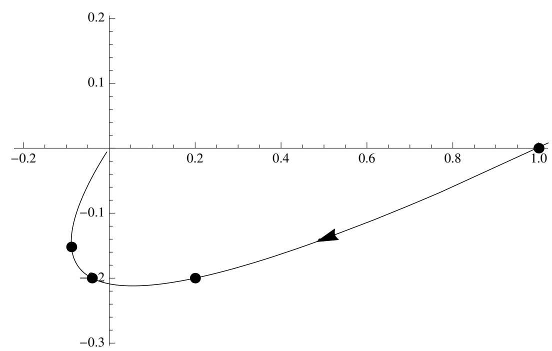
83. **(a)** With $A = \begin{bmatrix} 0.2 & 0.4 \\ -0.2 & 0.8 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0.36 \\ 0.36 \end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_2 = \begin{bmatrix} 0.216 \\ 0.216 \end{bmatrix}.$$



(b) With $A = \begin{bmatrix} 0.2 & 0.4 \\ -0.2 & 0.8 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0.2\\-0.2\end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix}-0.04\\-0.20\end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix}-0.088\\-0.152\end{bmatrix}.$$

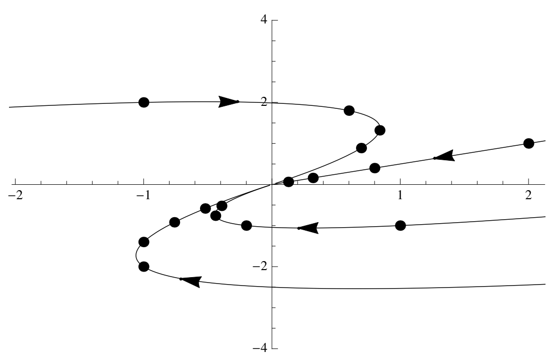


(c) The characteristic polynomial of the matrix is

$$\det(A - \lambda I) = \begin{bmatrix} 0.2-\lambda & 0.4 \\ -0.2 & 0.8-\lambda \end{bmatrix} = \lambda^2 - \lambda + 0.24 = (\lambda - 0.4)(\lambda - 0.6).$$

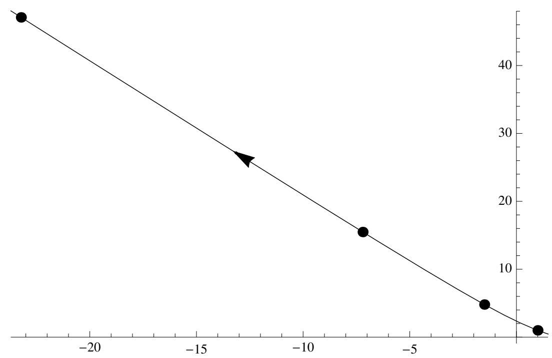
Since both eigenvalues are smaller than 1 in magnitude, the origin is an attractor.

(d) Some other trajectories are



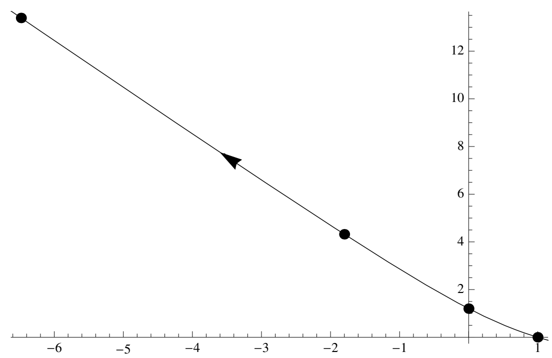
84. **(a)** With $A = \begin{bmatrix} 0 & -1.5 \\ 1.2 & 3.6 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1.5\\4.8\end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix}-7.2\\15.48\end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_2 = \begin{bmatrix}-23.22\\47.088\end{bmatrix}.$$



(b) With $A = \begin{bmatrix} 0 & -1.5 \\ 1.2 & 3.6 \end{bmatrix}$, we have

$$\mathbf{x}_1 = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} -1.8 \\ 4.32 \end{bmatrix}, \quad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} -6.48 \\ 13.392 \end{bmatrix}.$$

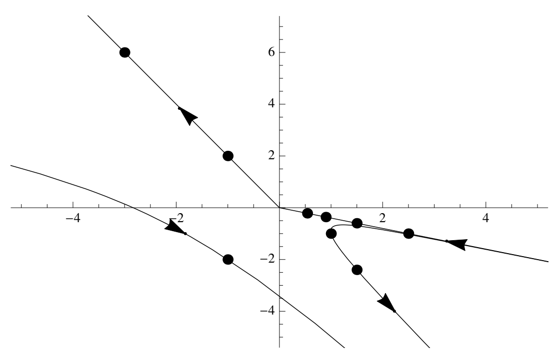


(c) The characteristic polynomial of the matrix is

$$\det(A - \lambda I) = \begin{bmatrix} -\lambda & -1.5 \\ 1.2 & 3.6 - \lambda \end{bmatrix} = \lambda^2 - 3.6\lambda + 1.8 = (\lambda - 3)(\lambda - 0.6).$$

Since one eigenvalue is smaller than 1 in magnitude and the other is greater, the origin is a saddle point; the origin attracts some initial points and repels others.

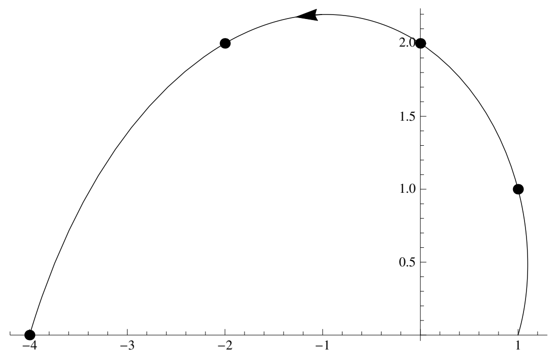
(d) Some other trajectories are



85. With the given matrix A, we have $r = \sqrt{a^2 + b^2} = \sqrt{2}$ and $\theta = \frac{\pi}{4}$, so that

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

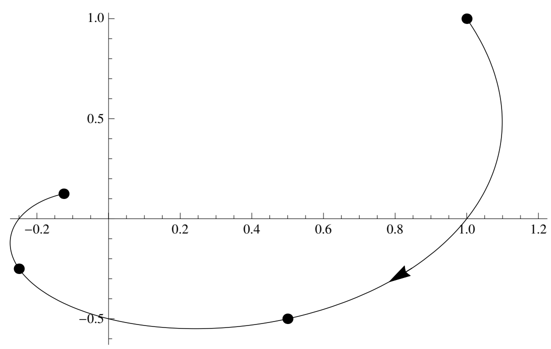
Since $r = |\lambda| = \sqrt{2} > 1$, the origin is a spiral repeller:



86. With the given matrix A, we have $r = \sqrt{a^2+b^2} = 0.5$ and $\theta = \frac{3\pi}{4}$, so that

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

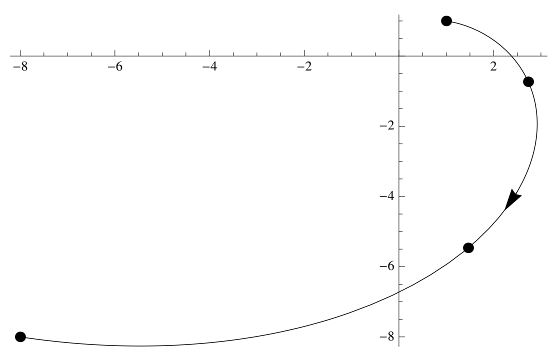
Since $r = |\lambda| = 0.5 < 1$, the origin is a spiral attractor:



87. With the given matrix A, we have $r = \sqrt{a^2+b^2} = 2$ and $\theta = \frac{5\pi}{3}$, so that

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

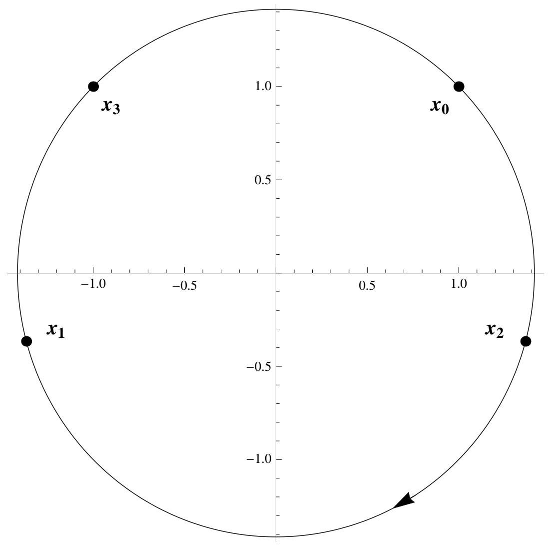
Since $r = |\lambda| = 2 > 1$, the origin is a spiral repeller:



88. With the given matrix A, we have $r = \sqrt{a^2+b^2} = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$ and $\theta = \frac{5\pi}{6}$, so that

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

Since $r = |\lambda| = 1$, the origin is the orbital center:



89. The characteristic polynomial of A is

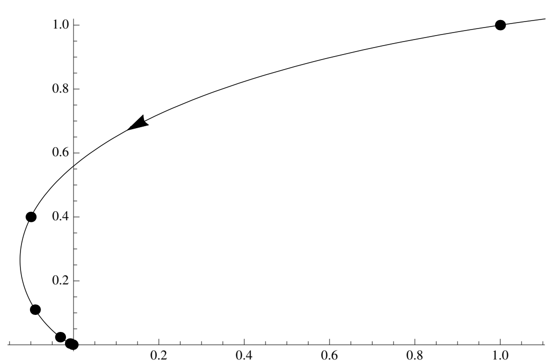
$$\det(A - \lambda I) = \begin{bmatrix} 0.1-\lambda & -0.2 \\ 0.1 & 0.3-\lambda \end{bmatrix} = \lambda^2 - 0.4\lambda + 0.05;$$

solving this quadratic gives $\lambda = 0.2 \pm 0.1i$ for the eigenvalues. Row-reducing gives $\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ for an eigenvector. Thus $P = \begin{bmatrix} \operatorname{Re}\mathbf{x} & \operatorname{Im}\mathbf{x} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, and so

$$C = P^{-1}AP = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0.1 & -0.2 \\ 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.2 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}.$$

Since $|\lambda| = \sqrt{0.05} < 1$, the origin is a spiral attractor. The first six points are

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} -0.1 \\ 0.4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -0.09 \\ 0.11 \end{bmatrix},$$
$$\mathbf{x}_3 = \begin{bmatrix} -0.031 \\ 0.024 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} -0.0079 \\ 0.0041 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} -0.00161 \\ 0.00044 \end{bmatrix}.$$



90. The characteristic polynomial of A is

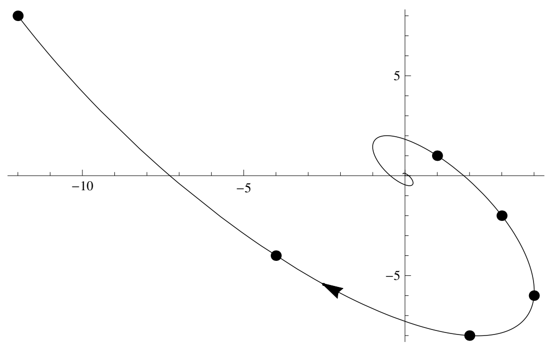
$$\det(A - \lambda I) = \begin{bmatrix} 2-\lambda & 1 \\ -2 & -\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 2;$$

solving this quadratic gives $\lambda = 1 \pm i$ for the eigenvalues. Row-reducing gives $\begin{bmatrix} -1-i \\ 2 \end{bmatrix}$ for an eigenvector. Thus $P = \begin{bmatrix} \operatorname{Re}\mathbf{x} & \operatorname{Im}\mathbf{x} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$, and so

$$C = P^{-1}AP = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Since $|\lambda| = \sqrt{2} > 1$, the origin is a spiral repeller. The first six points are

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ -6 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} -12 \\ 8 \end{bmatrix}.$$



91. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{bmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda + 1;$$

solving this quadratic gives $\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i$ for the eigenvalues. Row-reducing gives $\begin{bmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2} i \\ 1 \end{bmatrix}$ for an eigenvector. Thus

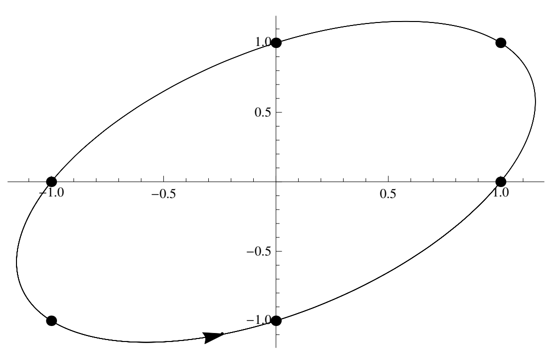
$$P = \begin{bmatrix} \operatorname{Re}\mathbf{x} & \operatorname{Im}\mathbf{x} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & 0 \end{bmatrix},$$

and so

$$C = P^{-1}AP = \begin{bmatrix} 0 & 1 \\ -\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Since $|\lambda| = 1$, the origin is an orbital center. The first six points are

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$



92. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{bmatrix} -\lambda & -1 \\ 1 & \sqrt{3} - \lambda \end{bmatrix} = \lambda^2 - \sqrt{3}\lambda + 1;$$

solving this quadratic gives $\lambda = \frac{\sqrt{3}}{2} \pm \frac{1}{2}\, i$ for the eigenvalues. Row-reducing gives $\begin{bmatrix} -\frac{\sqrt{3}}{2} - \frac{1}{2}i \\ 1 \end{bmatrix}$ for an eigenvector. Thus

$$P = \begin{bmatrix} \operatorname{Re}\mathbf{x} & \operatorname{Im}\mathbf{x} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix},$$

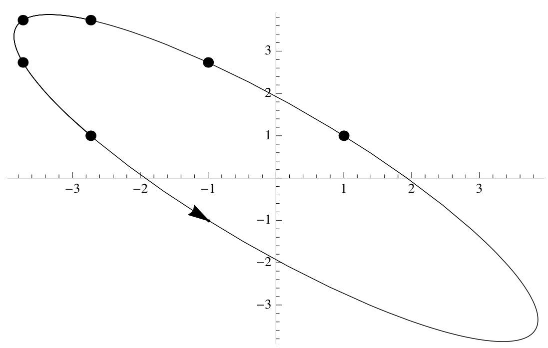
and so

$$C = P^{-1}AP = \begin{bmatrix} 0 & 1 \\ -2 & -\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Since $|\lambda| = 1$, the origin is an orbital center. The first six points are

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 + \sqrt{3} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 - \sqrt{3} \\ 2 + \sqrt{3} \end{bmatrix},$$

$$\mathbf{x}_3 = \begin{bmatrix} -2 - \sqrt{3} \\ 2 + \sqrt{3} \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} -2 - \sqrt{3} \\ 1 + \sqrt{3} \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} -1 - \sqrt{3} \\ 1 \end{bmatrix}.$$



Chapter Review

1. **(a)** False. What is true is that if A is an $n \times n$ matrix, then $\det(-A) = (-1)^n \det A$. This is because multiplying *any* row by -1 multiplies the determinant by -1, and $-A$ multiplies *every* row by -1. See Theorem 4.7.

(b) True, since $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$. See Theorem 4.8.

(c) False. It depends on what order the columns are in. Swapping any two columns of A multiplies the determinant by -1. See Theorem 4.4.

(d) False. While $\det A^T = \det A$ by Theorem 4.10, $\det A^{-1} = \frac{1}{\det A}$ by Theorem 4.9. So unless $\det A = \pm 1$, the statement is false.

(e) False. For example, the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a characteristic polynomial of 0, so 0 is its only eigenvalue.

(f) False. For example, the identity matrix has only $\lambda = 1$ as an eigenvalue, but $\mathbf{e}_i$ is an eigenvector for every i.

(g) True. See Theorem 4.25 in Section 4.4.

(h) False. Any diagonal matrix with two identical entries on the diagonal (such as the identity matrix) provides a counterexample.

(i) False. If A and B are similar, then they have the same eigenvalues, but not necessarily the same eigenvectors. Exercise 48 in Section 4.4 shows that if $B = P^{-1}AP$ and $\mathbf{v}$ is an eigenvector of A, then $P^{-1}\mathbf{v}$ is an eigenvector for B for the same eigenvalue.

(j) False. For example, if A is invertible, then its reduced row echelon form is the identity matrix, which has only the eigenvalue 1. So if A has an eigenvalue other than 1, it cannot be similar to the identity matrix.

2. **(a)** For example, expand along the first row:

$$\begin{aligned}\begin{vmatrix}1 & 3 & 5\\ 3 & 5 & 7\\ 7 & 9 & 11\end{vmatrix} &= 1\begin{vmatrix}5 & 7\\ 9 & 11\end{vmatrix} - 3\begin{vmatrix}3 & 7\\ 7 & 11\end{vmatrix} + 5\begin{vmatrix}3 & 5\\ 7 & 9\end{vmatrix}\\ &= 1(5\cdot 11 - 7\cdot 9) - 3(3\cdot 11 - 7\cdot 7) + 5(3\cdot 9 - 5\cdot 7)\\ &= 0.\end{aligned}$$

(b) Apply row operations to diagonalize the matrix:

$$\begin{bmatrix}1 & 3 & 5\\ 3 & 5 & 7\\ 7 & 9 & 11\end{bmatrix} \xrightarrow[R_3-7R_1]{R_2-3R_1} \begin{bmatrix}1 & 3 & 5\\ 0 & -4 & -8\\ 0 & -12 & -24\end{bmatrix} \xrightarrow{R_3-3R_2} \begin{bmatrix}1 & 3 & 5\\ 0 & -4 & -8\\ 0 & 0 & 0\end{bmatrix}.$$

Then $\det A$ is the product of the diagonal entries, which is zero.

3. To get from the first matrix to the second:

- Multiply the first column by 3; this multiplies the determinant by 3.
- Multiply the second column by 2; this multiplies the determinant by 2.
- Subtract 4 times the third column from the second; this does not change the determinant.
- Interchange the first and second rows; this multiplies the determinant by -1.

So the determinant of the resulting matrix is $3\cdot 3\cdot 2\cdot(-1) = -18$.

4. **(a)** We have

$$\det C = \det\left((AB)^{-1}\right) = \frac{1}{\det(AB)} = \frac{1}{\det A\det B} = \frac{1}{2\cdot\left(-\frac{1}{4}\right)} = -2.$$

(b) We have

$$\begin{aligned}\det C = \det\left(A^2B(3A^T)\right) &= \det A\cdot\det A\cdot\det B\cdot\det(3A^T)\\ &= -\det(3A^T) = -3^4\det(A^T) = -3^4\det A = -162.\end{aligned}$$

5. For any $n\times n$ matrix, $\det A = \det(A^T)$, and $\det(-A) = (-1)^n\det A$. So if A is skew-symmetric, then $A^T = -A$ and

$$\det A = \det(A^T) = \det(-A) = (-1)^n\det A = -\det A$$

since n is odd. But $\det A = -\det A$ means that $\det A = 0$.

6. Compute the determinant by expanding along the first row:

$$\begin{aligned}\begin{vmatrix}1 & -1 & 2\\ 1 & 1 & k\\ 2 & 4 & k^2\end{vmatrix} &= 1\begin{vmatrix}1 & k\\ 4 & k^2\end{vmatrix} - (-1)\begin{vmatrix}1 & k\\ 2 & k^2\end{vmatrix} + 2\begin{vmatrix}1 & 1\\ 2 & 4\end{vmatrix}\\ &= 1(k^2-4k) + 1(k^2-2k) + 2(4-2)\\ &= 2k^2 - 6k + 4 = 2(k-2)(k-1).\end{aligned}$$

So the determinant is zero when $k = 1$ or $k = 2$.

7. Since

$$A\mathbf{x} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

we see that $\mathbf{x}$ is indeed an eigenvector, with corresponding eigenvalue 5.

8. Since

$$A\mathbf{x} = \begin{bmatrix} 13 & -60 & -45 \\ -5 & 18 & 15 \\ 10 & -40 & -32 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \cdot 13 - 1 \cdot (-60) + 2 \cdot (-45) \\ 3 \cdot (-5) - 1 \cdot 18 + 2 \cdot 15 \\ 3 \cdot 10 - 1 \cdot (-40) + 2 \cdot (-32) \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix},$$

we see that $\mathbf{x}$ is indeed an eigenvector, with corresponding eigenvalue 3.

9. **(a)** The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -5-\lambda & -6 & 3 \\ 3 & 4-\lambda & -3 \\ 0 & 0 & -2-\lambda \end{vmatrix} \\ &= (-2-\lambda) \begin{vmatrix} -5-\lambda & -6 \\ 3 & 4-\lambda \end{vmatrix} \\ &= -(\lambda+2)(\lambda^2+\lambda-2) = -(\lambda+2)(\lambda+2)(\lambda-1). \end{aligned}$$

(b) The eigenvalues of A are the roots of the characteristic polynomial, which are $\lambda_1 = 1$ and $\lambda_2 = -2$ (with algebraic multiplicity 2).

(c) We row-reduce $A - \lambda I$ for each eigenvalue λ:

$$\left[\begin{array}{c|c} A - I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -6 & -6 & 3 & 0 \\ 3 & 3 & -3 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 3 & 3 & -3 & 0 \\ -6 & -6 & 3 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{\frac{1}{3}R_1} \\ \\ \xrightarrow{-\frac{1}{3}R_3} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -6 & -6 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+6R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{R_1+R_2} \\ \\ \xrightarrow{R_3+3R_1} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{c|c} A + 2I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -3 & -6 & 3 & 0 \\ 3 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+R_2} \left[\begin{array}{ccc|c} -3 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the eigenspaces are

$$E_1 = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right), \quad E_{-2} = \left\{\begin{bmatrix} -2s+t \\ s \\ t \end{bmatrix}\right\} = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

(d) Even though A does not have three distinct eigenvalues, it is diagonalizable since the eigenvalue of algebraic multiplicity 2, which is $\lambda_2 = -2$, also has geometric multiplicity 2. A diagonalizing matrix is a matrix P whose columns are the eigenvectors above, so

$$P = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$P^{-1}AP = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & -6 & 3 \\ 3 & 4 & -3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = D.$$

10. Since A is diagonalizable, $A \sim D$ where D is a diagonal matrix whose diagonal entries are its eigenvalues. Since $A \sim D$, we know that A and D have the same eigenvalues, so the diagonal entries of D are -2, 3, and 4 and thus $\det D = -24$. But $A \sim D$ also implies that $\det D = \det(P^{-1}AP) = \det(P^{-1}) \det A \det P = \det A$, so that $\det A = -24$.

11. Since

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

we know by Theorem 4.19 that

$$A^{-5} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = 5 \left(\frac{1}{2}\right)^{-5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2(-1)^{-5} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 160 \\ 160 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 162 \\ 158 \end{bmatrix}.$$

12. Since A is diagonalizable, it has n linearly independent eigenvectors, which therefore form a basis for $\mathbb{R}^n$. So if $\mathbf{x}$ is any vector, then $\mathbf{x} = \sum c_i \mathbf{v}_i$ for some constants c_i, so that by Theorem 4.19, and using the Triangle Inequality,

$$\|A^n \mathbf{x}\| = \left\| A^n \left(\sum_{i=1}^{n} c_i \mathbf{v}_i \right) \right\| = \left\| \sum c_i \lambda_i^n \mathbf{v}_i \right\| \le \sum |c_i| \, |\lambda_i|^n \, \|\mathbf{v}_i\|.$$

Then $|\lambda_i|^n \to 0$ as $n \to \infty$ since $|\lambda_i| < 1$, so each term of this sum goes to zero, so that $\|A^n \mathbf{x}\| \to 0$ as $n \to \infty$. Since this holds for all vectors $\mathbf{x}$, it follows that $A^n \to O$ as $n \to \infty$.

13. The two characteristic polynomials are

$$p_A = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 6 = \lambda^2 - 5\lambda - 2$$

$$p_B = \begin{vmatrix} 2-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 6 = \lambda^2 - 4\lambda - 2.$$

Since the characteristic polynomials are different, the matrices are not similar.

14. Since both matrices are diagonal, we can read off their eigenvalues; both matrices have eigenvalues 2 and 3. To convert A to B, we want to reverse the rows, and reverse the columns, so choose

$$P = P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$P^{-1}AP = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = B.$$

15. These matrices are not similar. If there were, then there would be some 3×3 matrix

$$P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

such that $P^{-1}AP = B$, or $AP = PB$. But

$$AP = \begin{bmatrix} a+d & b+e & c+f \\ d+g & e+h & f+i \\ g & h & i \end{bmatrix} \qquad PB = \begin{bmatrix} a & a+b & c \\ d & d+e & f \\ g & g+h & i \end{bmatrix}.$$

If these matrices are to be equal, then comparing entries in the last row and column gives $f = i = g = 0$. The upper left entries also give $d = 0$. The two middle entries are $e+h$ and $d+e = e$, so that $h = 0$ as well. Thus the entire bottom row of P is zero, so P cannot be invertible and A and B are not similar. Note that this is a counterexample to the converse of Theorem 4.22. These two matrices satisfy parts (a) through (g) of the conclusion, yet they are not similar.

16. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & k \\ 1 & -\lambda \end{vmatrix} = (2-\lambda)(-\lambda) - k = \lambda^2 - 2\lambda - k.$$

(a) A has eigenvalues 3 and -1 if $\lambda^2 - 2\lambda - k = (\lambda - 3)(\lambda + 1)$, so when $k = 3$.

(b) A has an eigenvalue with algebraic multiplicity 2 when its characteristic polynomial has a double root, which occurs when its discriminant is 0. But its discriminant is $(-2)^2 - 4 \cdot 1 \cdot (-k) = 4 + 4k$, so we must have $k = -1$.

(c) A has no real eigenvalues when the discriminant $4+4k$ of its characteristic polynomial is negative, so for $k < -1$.

17. If λ is an eigenvalue of A with eigenvector $\mathbf{x}$, then $A\mathbf{x} = \lambda\mathbf{x}$, so that

$$A^3\mathbf{x} = A(A(A(\mathbf{x}))) = A(A(\lambda\mathbf{x})) = A(\lambda^2\mathbf{x}) = \lambda^3\mathbf{x}.$$

If $A^3 = A$, then $A\mathbf{x} = \lambda^3\mathbf{x}$, so that $\lambda = \lambda^3$ and then $\lambda = 0$ or $\lambda = \pm 1$.

18. If A has two identical rows, then Theorem 4.3(c) says that $\det A = 0$, so that A is not invertible. But then by Theorem 4.17, 0 must be an eigenvalue of A.

19. If $\mathbf{x}$ is an eigenvalue of A with eigenvalue 3, then

$$(A^2 - 5A + 2I)\mathbf{x} = A^2\mathbf{x} - 5A\mathbf{x} + 2I\mathbf{x} = 9\mathbf{x} - 5 \cdot 3\mathbf{x} + 2\mathbf{x} = -4\mathbf{x}.$$

This shows that $\mathbf{x}$ is also an eigenvector of $A^2 - 5A + 2I$, with eigenvalue -4.

20. Suppose that $B = P^{-1}AP$ where P is invertible. This can be rewritten as $BP^{-1} = P^{-1}A$. Let λ be an eigenvalue of A with eigenvector $\mathbf{x}$. Then

$$B(P^{-1}\mathbf{x}) = (BP^{-1})\mathbf{x} = (P^{-1}A)\mathbf{x} = P^{-1}(A\mathbf{x}) = P^{-1}(\lambda\mathbf{x}) = \lambda(P^{-1}\mathbf{x}).$$

Thus $P^{-1}\mathbf{x}$ is an eigenvector for B, corresponding to λ.

Chapter 5

Orthogonality

5.1 Orthogonality in $\mathbb{R}^n$

1. Since

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= (-3)\cdot 2 + 1\cdot 4 + 2\cdot 1 = 0,\\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= (-3)\cdot 1 + 1\cdot(-1) + 2\cdot 2 = 0,\\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= 2\cdot 1 + 4\cdot(-1) + 1\cdot 2 = 0,\end{aligned}$$

this set of vectors is orthogonal.

2. Since

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= 4\cdot(-1) + 2\cdot 2 - 5\cdot 0 = 0,\\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= 4\cdot 2 + 2\cdot 1 - 5\cdot 2 = 0,\\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= -1\cdot 2 + 2\cdot 1 + 0\cdot 2 = 0,\end{aligned}$$

this set of vectors is orthogonal.

3. Since $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3\cdot(-1) + 1\cdot 2 - 1\cdot 1 = -2 \neq 0$, this set of vectors is not orthogonal. There is no need to check the other two dot products; all three must be zero for the set to be orthogonal.

4. Since $\mathbf{v}_1 \cdot \mathbf{v}_3 = 5\cdot 3 + 3\cdot 1 + 1\cdot(-1) = 17 \neq 0$, this set of vectors is not orthogonal. There is no need to check the other two dot products; all three must be zero for the set to be orthogonal.

5. Since

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= 2\cdot(-2) + 3\cdot 1 - 1\cdot(-1) + 4\cdot 0 = 0,\\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= 2\cdot(-4) + 3\cdot(-6) - 1\cdot 2 + 4\cdot 7 = 0,\\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= -2\cdot(-4) + 1\cdot(-6) - 1\cdot 2 + 0\cdot 7 = 0,\end{aligned}$$

this set of vectors is orthogonal.

6. Since $\mathbf{v}_2 \cdot \mathbf{v}_4 = -1\cdot 0 + 0\cdot(-1) + 1\cdot 1 + 2\cdot 1 = 3 \neq 0$, this set of vectors is not orthogonal. There is no need to check the other five dot products (they are all zero); all dot products must be zero for the set to be orthogonal.

7. Since the vectors are orthogonal:

$$\begin{bmatrix} 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 4\cdot 1 - 2\cdot 2 = 0,$$

they are linearly independent by Theorem 5.1. Since there are two vectors, they form an orthogonal basis for $\mathbb{R}^2$. By Theorem 5.2, we have

$$\mathbf{w} = \frac{\mathbf{v}_1 \cdot \mathbf{w}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{w}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{4 \cdot 1 - 2 \cdot (-3)}{4 \cdot 4 - 2 \cdot (-2)}\mathbf{v}_1 + \frac{1 \cdot 1 + 2 \cdot (-3)}{1 \cdot 1 + 2 \cdot 2}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1 - \mathbf{v}_2,$$

so that

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}.$$

8. Since the vectors are orthogonal:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 2 \end{bmatrix} = 1 \cdot (-6) + 3 \cdot 2 = 0,$$

they are linearly independent by Theorem 5.1. Since there are two vectors, they form an orthogonal basis for $\mathbb{R}^2$. By Theorem 5.2, we have

$$\mathbf{w} = \frac{\mathbf{v}_1 \cdot \mathbf{w}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{w}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{1 \cdot 1 + 3 \cdot 1}{1 \cdot 1 + 3 \cdot 3}\mathbf{v}_1 + \frac{-6 \cdot 1 + 2 \cdot 1}{-6 \cdot (-6) + 2 \cdot 2}\mathbf{v}_2 = \frac{2}{5}\mathbf{v}_1 - \frac{1}{10}\mathbf{v}_2,$$

so that

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} \frac{2}{5} \\ -\frac{1}{10} \end{bmatrix}.$$

9. Since the vectors are orthogonal:

$$\begin{aligned}
\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} &= 1 \cdot 1 + 0 \cdot 2 - 1 \cdot 1 = 0 \\
\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} &= 1 \cdot 1 + 0 \cdot (-1) - 1 \cdot 1 = 0 \\
\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} &= 1 \cdot 1 + 2 \cdot (-1) + 1 \cdot 1 = 0,
\end{aligned}$$

they are linearly independent by Theorem 5.1. Since there are three vectors, they form an orthogonal basis for $\mathbb{R}^3$. By Theorem 5.2, we have

$$\begin{aligned}
\mathbf{w} &= \frac{\mathbf{v}_1 \cdot \mathbf{w}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{w}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{v}_3 \cdot \mathbf{w}}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 \\
&= \frac{1 \cdot 1 + 0 \cdot 1 - 1 \cdot 1}{1 \cdot 1 + 0 \cdot 0 - 1 \cdot (-1)}\mathbf{v}_1 + \frac{1 \cdot 1 + 2 \cdot 1 + 1 \cdot 1}{1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1}\mathbf{v}_2 + \frac{1 \cdot 1 - 1 \cdot 1 + 1 \cdot 1}{1 \cdot 1 - 1 \cdot (-1) + 1 \cdot 1}\mathbf{v}_3 \\
&= \frac{2}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3,
\end{aligned}$$

so that

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

10. Since the vectors are orthogonal:

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}=1\cdot 1+1\cdot(-1)+1\cdot 0=0$$
$$\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}=1\cdot 1+1\cdot 1+1\cdot(-2)=0$$
$$\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\-2\end{bmatrix}=1\cdot 1-1\cdot 1+0\cdot(-2)=0,$$

they are linearly independent by Theorem 5.1. Since there are three vectors, they form an orthogonal basis for $\mathbb{R}^3$. By Theorem 5.2, we have

$$\begin{aligned}\mathbf{w}&=\frac{\mathbf{v}_1\cdot\mathbf{w}}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1+\frac{\mathbf{v}_2\cdot\mathbf{w}}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2+\frac{\mathbf{v}_3\cdot\mathbf{w}}{\mathbf{v}_3\cdot\mathbf{v}_3}\mathbf{v}_3\\&=\frac{1\cdot 1+1\cdot 2+1\cdot 3}{1\cdot 1+1\cdot 1+1\cdot 1}\mathbf{v}_1+\frac{1\cdot 1-1\cdot 2+0\cdot 3}{1\cdot 1-1\cdot(-1)+0\cdot 0}\mathbf{v}_2+\frac{1\cdot 1+1\cdot 2-2\cdot 3}{1\cdot 1+1\cdot 1-2\cdot(-2)}\mathbf{v}_3\\&=2\mathbf{v}_1-\frac{1}{2}\mathbf{v}_2-\frac{1}{2}\mathbf{v}_3,\end{aligned}$$

so that

$$[\mathbf{w}]_{\mathcal{B}}=\begin{bmatrix}2\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}.$$

11. Since $\|\mathbf{v}_1\|=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}=1$ and $\|\mathbf{v}_2\|=\sqrt{\left(-\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}=1$, the given set of vectors is orthonormal.

12. Since $\|\mathbf{v}_1\|=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{\sqrt{2}}\neq 1$ and $\|\mathbf{v}_2\|=\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2}=\frac{1}{\sqrt{2}}\neq 1$, the given set of vectors is not orthonormal. To normalize the vectors, we divide each by its norm to get

$$\left\{\frac{1}{1/\sqrt{2}}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix},\ \frac{1}{1/\sqrt{2}}\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}\right\}=\left\{\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix},\ \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}\right\}.$$

13. Taking norms, we get

$$\begin{aligned}\|\mathbf{v}_1\|&=\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2}=1\\\|\mathbf{v}_2\|&=\sqrt{\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2+0^2}=\frac{\sqrt{5}}{3}\\\|\mathbf{v}_3\|&=\sqrt{1^2+2^2+\left(\frac{5}{2}\right)^2}=\frac{3\sqrt{5}}{2}.\end{aligned}$$

So the first vector is already normalized; we must normalize the other two by dividing by their norms, to get

$$\left\{\begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix},\ \frac{1}{\sqrt{5}/3}\begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\0\end{bmatrix},\ \frac{1}{3\sqrt{5}/2}\begin{bmatrix}1\\2\\-\frac{5}{2}\end{bmatrix}\right\}=\left\{\begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix},\ \begin{bmatrix}\frac{2}{\sqrt{5}}\\-\frac{1}{\sqrt{5}}\\0\end{bmatrix},\ \begin{bmatrix}\frac{2}{3\sqrt{5}}\\\frac{4}{3\sqrt{5}}\\-\frac{5}{3\sqrt{5}}\end{bmatrix}\right\}.$$

14. Taking norms, we get

$$\begin{aligned}
\|\mathbf{v}_1\| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1\\
\|\mathbf{v}_2\| &= \sqrt{0^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + 0^2} = \frac{\sqrt{6}}{3}\\
\|\mathbf{v}_3\| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(-\frac{1}{6}\right)^2} = \frac{\sqrt{3}}{3}.
\end{aligned}$$

So the first vector is already normalized; we must normalize the other two by dividing by their norms, to get

$$\left\{\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}, \frac{1}{\sqrt{6}/3}\begin{bmatrix}0\\ \frac{1}{3}\\ \frac{2}{3}\\ \frac{1}{3}\end{bmatrix}, \frac{1}{\sqrt{3}/3}\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{6}\\ \frac{1}{6}\\ -\frac{1}{6}\end{bmatrix}\right\} = \left\{\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}, \begin{bmatrix}0\\ \frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{bmatrix}, \begin{bmatrix}\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{6}\\ \frac{\sqrt{3}}{6}\\ -\frac{\sqrt{3}}{6}\end{bmatrix}\right\}.$$

15. Taking norms, we get

$$\begin{aligned}
\|\mathbf{v}_1\| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1\\
\|\mathbf{v}_2\| &= \sqrt{0^2 + \left(\frac{\sqrt{6}}{3}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(-\frac{1}{\sqrt{6}}\right)^2} = 1\\
\|\mathbf{v}_3\| &= \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{\sqrt{3}}{6}\right)^2 + \left(\frac{\sqrt{3}}{6}\right)^2 + \left(-\frac{\sqrt{3}}{6}\right)^2} = 1\\
\|\mathbf{v}_4\| &= \sqrt{0^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1.
\end{aligned}$$

So these vectors are already orthonormal.

16. The columns of the matrix are orthogonal:

$$\begin{bmatrix}0\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\0\end{bmatrix} = 0 \cdot (-1) + 1 \cdot 0 = 0.$$

Further, they are orthonormal since

$$\begin{bmatrix}0\\1\end{bmatrix} \cdot \begin{bmatrix}0\\1\end{bmatrix} = 0 \cdot 0 + 1 \cdot 1 = 1, \qquad \begin{bmatrix}-1\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\0\end{bmatrix} = -1 \cdot (-1) + 0 \cdot 0 = 1.$$

Thus the matrix is an orthogonal matrix, so by Theorem 5.5, the inverse of this matrix is its transpose, or

$$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

17. The columns of the matrix are orthogonal:

$$\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 0.$$

Further, they are orthonormal since

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot \left(-\frac{1}{\sqrt{2}}\right) = 1$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 1.$$

Thus the matrix is an orthogonal matrix, so by Theorem 5.5, the inverse of this matrix is its transpose, or

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

18. This is not an orthogonal matrix since none of the columns have norm 1:

$$\mathbf{q}_1 \cdot \mathbf{q}_1 = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}, \qquad \mathbf{q}_2 \cdot \mathbf{q}_2 = \frac{1}{4} + \frac{1}{4} + 0 = \frac{1}{2}, \qquad \mathbf{q}_3 \cdot \mathbf{q}_3 = \frac{1}{25} + \frac{1}{25} + \frac{4}{25} = \frac{6}{25}.$$

19. This matrix Q is orthogonal by Theorem 5.4 since $QQ^T = I$:

$$\begin{aligned} QQ^T &= \begin{bmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta\sin^2\theta+\cos^2\theta+\sin^4\theta & \cos^3\theta\sin\theta-\sin\theta\cos\theta+\cos\theta\sin^3\theta & \cos\theta\sin^2\theta-\cos\theta\sin^2\theta \\ \cos^3\theta\sin\theta-\sin\theta\cos\theta+\cos\theta\sin^3\theta & \cos^4\theta+\sin^2\theta+\cos^2\theta\sin^2\theta & \cos^2\theta\sin\theta-\cos^2\theta\sin\theta \\ \sin^2\theta\cos\theta-\sin^2\theta\cos\theta & \sin\theta\cos^2\theta-\cos^2\theta\sin\theta & \sin^2\theta+\cos^2\theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus by Theorem 5.5 $Q^{-1} = Q^T$.

20. The columns are all orthogonal since each dot product consists of two terms of $\frac{1}{2} \cdot \frac{1}{2}$ and two of $-\frac{1}{2} \cdot \frac{1}{2}$. The columns are orthonormal since the norm of each column is

$$\sqrt{\left(\pm\frac{1}{2}\right)^2 + \left(\pm\frac{1}{2}\right)^2 + \left(\pm\frac{1}{2}\right)^2 + \left(\pm\frac{1}{2}\right)^2} = 1.$$

So the matrix is orthonormal, so by Theorem 5.5 its inverse is equal to its transpose,

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

21. The dot product of the first and fourth columns is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 1 \cdot \frac{1}{\sqrt{6}} \neq 0,$$

so the matrix is not orthonormal.

22. By Theorem 5.5, Q^{-1} is orthogonal if and only if $\left(Q^{-1}\right)^T = \left(Q^{-1}\right)^{-1}$. But since Q is orthogonal, $Q^{-1} = Q^T$, so that

$$\left(Q^{-1}\right)^T = \left(Q^T\right)^{-1} = \left(Q^{-1}\right)^{-1},$$

and Q^{-1} is orthogonal.

23. Since $\det Q = \det Q^T$, we have

$$1 = \det(I) = \det(QQ^{-1}) = \det(QQ^T) = (\det Q)(\det Q^T) = (\det Q)^2.$$

Since $(\det Q)^2 = 1$, we must have $\det Q = \pm 1$.

24. By Theorem 5.5, it suffices to show that $(Q_1Q_2)^{-1} = (Q_1Q_2)^T$. But using the fact that $Q_1^{-1} = Q_1^T$ and $Q_2^{-1} = Q_2^T$, we have

$$(Q_1Q_2)^{-1} = Q_2^{-1}Q_1^{-1} = Q_2^TQ_1^T = (Q_1Q_2)^T.$$

Thus Q_1Q_2 is orthogonal.

25. By Theorem 3.17, if P is a permutation matrix, then $P^{-1} = P^T$. So by Theorem 5.5, P is orthogonal.

26. Reordering the rows of Q is the same as multiplying it on the left by a permutation matrix P. By the previous exercise, P is orthogonal, and then by Exercise 24, PQ is also orthogonal. So reordering the rows of an orthogonal matrix results in an orthogonal matrix.

27. Let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$, and ϕ the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$. By Theorem 5.6, $\|Q\mathbf{x}\| = \|\mathbf{x}\|$, $\|Q\mathbf{y}\| = \|\mathbf{y}\|$, and $Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$, so that

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \frac{Q\mathbf{x} \cdot Q\mathbf{y}}{\|Q\mathbf{x}\|\,\|Q\mathbf{y}\|} = \cos\phi.$$

Since $0 \le \theta, \phi \le \pi$, the fact that their cosines are equal implies that $\theta = \phi$.

28. **(a)** Suppose that $Q = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is an orthogonal matrix. Then

$$QQ^T = Q^TQ = I \quad \Rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$, and thus both columns of Q are unit vectors. Also, $ac + bd = 0$, which implies that $ac = -bd$ and therefore $a^2c^2 = b^2d^2$. But $c^2 = 1 - d^2$ and $b^2 = 1 - a^2$; substituting gives

$$a^2(1 - d^2) = (1 - a^2)d^2 \quad \Rightarrow \quad a^2 - a^2d^2 = d^2 - a^2d^2 \quad \Rightarrow \quad a^2 = d^2 \quad \Rightarrow \quad a = \pm d.$$

If $a = d = 0$, then $b = c = \pm 1$, so we get one of the matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

all of which are of one of the required forms. Otherwise, if $a = d \neq 0$, then $c^2 = 1 - d^2 = 1 - a^2 = b^2$, so that $b = \pm c$ and we get

$$\begin{bmatrix} a & \pm b \\ b & a \end{bmatrix},$$

and we must choose the minus sign in order that the columns be orthogonal. So the result is

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

which is also of the required form. Finally, if $a = -d \neq 0$, then again $b^2 = c^2$ and we get

$$\begin{bmatrix} a & \pm b \\ b & -a \end{bmatrix};$$

this time we must choose the plus sign for the columns to be orthogonal, resulting in

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

which is again of the required form.

(b) Since $\begin{bmatrix} a \\ b \end{bmatrix}$ is a unit vector, we have $a^2+b^2 = 1$, so that (a, b) is a point on the unit circle. Therefore there is some θ with $(a, b) = (\cos\theta, \sin\theta)$. Substituting into the forms from part (a) gives the desired matrices.

(c) Let

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad R = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

From Example 3.58 in Section 3.6, Q is the standard matrix of a rotation through angle θ. From Exercise 26(c) in Section 3.6, R is a reflection through the line ℓ that makes an angle of $\frac{\theta}{2}$ with the x-axis, which is the line $y = \left(\tan\frac{\theta}{2}\right)x$ (or $x = 0$ if $\theta = \pi$).

(d) From part (c), we have $\det Q = \cos^2\theta + \sin^2\theta = 1$ and $\det R = -\cos^2\theta - \sin^2\theta = -1$.

29. Since $\det Q = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right) = 1$, this matrix represents a rotation. To find the angle, we have $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$, so that $\theta = \frac{\pi}{4} = 45°$.

30. Since $\det Q = -\frac{1}{2} \cdot \left(-\frac{1}{2}\right) - \frac{\sqrt{3}}{2} \cdot \left(-\frac{\sqrt{3}}{2}\right) = 1$, this matrix represents a rotation. To find the angle, we have $\cos\theta = -\frac{1}{2}$ and $\sin\theta = -\frac{\sqrt{3}}{2}$, so that $\theta = \frac{4\pi}{3} = 240°$.

31. Since $\det Q = -\frac{1}{2} \cdot \left(\frac{1}{2}\right) - \frac{\sqrt{3}}{2} \cdot \left(\frac{\sqrt{3}}{2}\right) = -1$, this matrix represents a reflection. If $\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$, then $\theta = \frac{2\pi}{3}$, so that by part (d) of the previous exercise the reflection occurs through the line $y = \left(\tan\frac{\pi}{3}\right)x = \sqrt{3}\,x$.

32. Since $\det Q = -\frac{3}{5} \cdot \frac{3}{5} - \left(-\frac{4}{5}\right) \cdot \left(-\frac{4}{5}\right) = -1$, this matrix represents a reflection. Let θ be the (third quadrant) angle with $\cos\theta = -\frac{3}{5}$ and $\sin\theta = -\frac{4}{5}$; then by part (d) of the previous exercise, the reflection occurs through the line $y = \left(\tan\frac{\theta}{2}\right)x$.

33. **(a)** Since A and B are orthogonal matrices, we know that $A^T = A^{-1}$ and $B^T = B^{-1}$, so that

$$A(A^T + B^T)B = A\left(A^{-1} + B^{-1}\right)B = AA^{-1}B + AB^{-1}B = B + A = A + B.$$

(b) Since A and B are orthogonal matrices, we have $\det A = \pm 1$ and $\det B = \pm 1$. If $\det A + \det B = 0$, then one of them must be 1 and the other must be -1, so that $(\det A)(\det B) = -1$. But by part (a),

$$\begin{aligned} \det(A+B) &= \det\left(A(A^T+B^T)B\right) = (\det A)\left(\det(A^T+B^T)\right)(\det B) \\ &= (\det A)\left(\det((A+B)^T)\right)(\det B) = (\det A)(\det B)\left(\det(A+B)\right) = -\det(A+B). \end{aligned}$$

Since $\det(A+B) = -\det(A+B)$, we conclude that $\det(A+B) = 0$ so that $A+B$ is not invertible.

34. It suffices to show that $QQ^T = I$. Now,

$$QQ^T = \begin{bmatrix} x_1 & \mathbf{y}^T \\ \mathbf{y} & I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^t \end{bmatrix} \begin{bmatrix} x_1 & \mathbf{y}^T \\ \mathbf{y} & I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^t \end{bmatrix}.$$

Also, $x_1^2 + \mathbf{y}\mathbf{y}^T = x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, since $\mathbf{x}$ is a unit vector. Rearranging gives $\mathbf{y}\mathbf{y}^T = 1 - x_1^2$. Further, we have

$$\begin{aligned}
\mathbf{y}^T\left(I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right) &= \mathbf{y}^T - \left(\frac{1}{1-x_1}\right)\mathbf{y}^T(1-x_1)^2 = \mathbf{y}^T - (1+x_1)\mathbf{y}^T \\
&= -x_1\mathbf{y}^T \\
\left(I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right)\mathbf{y} &= \mathbf{y} - \left(\frac{1}{1-x_1}\right)(1-x_1^2)\mathbf{y} = \mathbf{y} - (1+x_1)\mathbf{y} \\
&= -x_1\mathbf{y} \\
\left(I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right)\left(I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right) &= I^2 - 2\left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T + \left(\frac{1}{1-x_1}\right)^2\mathbf{y}\mathbf{y}^T\mathbf{y}\mathbf{y}^T \\
&= I - \frac{2(1-x_1)}{(1-x_1)^2}\mathbf{y}\mathbf{y}^T + \frac{1-x_1^2}{(1-x_1)^2}\mathbf{y}\mathbf{y}^T \\
&= I - \frac{1-2x_1+x_1^2}{(1-x_1)^2}\mathbf{y}\mathbf{y}^T \\
&= I - \mathbf{y}\mathbf{y}^T.
\end{aligned}$$

Then evaluating QQ^T gives

$$\begin{aligned}
QQ^T &= \begin{bmatrix} x_1^2 + \mathbf{y}^T\mathbf{y} & x_1\mathbf{y}^T + \mathbf{y}^T\left(I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right) \\ x_1\mathbf{y} + \left(I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right)\mathbf{y} & \mathbf{y}\mathbf{y}^T + \left(I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right)\left(I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right) \end{bmatrix} \\
&= \begin{bmatrix} 1 & x_1\mathbf{y}^T - x_1\mathbf{y}^T \\ x_1\mathbf{y} - x_1\mathbf{y} & \mathbf{y}\mathbf{y}^T + (I - \mathbf{y}\mathbf{y}^T) \end{bmatrix} \\
&= \begin{bmatrix} 1 & O \\ O & I \end{bmatrix} \\
&= I.
\end{aligned}$$

Thus Q is orthogonal.

35. Let Q be orthogonal and upper triangular. Then the columns $\mathbf{q}_i$ of Q are orthonormal; that is,

$$\mathbf{q}_i \cdot \mathbf{q}_j = \sum_{k=1}^n q_{ki}q_{kj} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

Since Q is upper triangular, we know that $q_{ij} = 0$ when $i > j$. We prove by induction on the rows that each row has only one nonzero entry, along the diagonal. Note that $q_{11} \neq 0$, since all other entries in the first column are zero and the norm of the first column vector must be 1. Now the dot product of $\mathbf{q}_1$ with $\mathbf{q}_j$ for $j > 1$ is

$$\mathbf{q}_1 \cdot \mathbf{q}_j = \sum_{k=1}^n q_{k1}q_{kj} = q_{11}q_{1j}.$$

Since these columns must be orthogonal, this product is zero, so that $q_{1j} = 0$ for $j > 1$ and therefore q_{11} is the only nonzero entry in the first row. Now assume that $q_{11}, q_{22}, \ldots, q_{ii}$ are the only nonzero entries in their rows, and consider $q_{i+1,i+1}$. It is the only nonzero entry in its column, since entries in lower-numbered rows are zero by the inductive hypothesis and entries in higher-numbered rows are zero since Q is upper triangular. Therefore, $q_{i+1,i+1} \neq 0$ since the $(i+1)^{\text{st}}$ column must have norm 1. Now consider the dot product of $\mathbf{q}_{i+1}$ with $\mathbf{q}_j$ for $j > i+1$:

$$\mathbf{q}_{i+1} \cdot \mathbf{q}_j = \sum_{k=1}^n q_{k,i+1}q_{kj} = q_{i+1,i+1}q_{i+1,j}.$$

Since these columns must be orthogonal, this product is zero, so that $q_{i+1,j} = 0$ for $j > i+1$. So all entries to the right of the $(i+1)^{\text{st}}$ column are zero. But also entries to the left of the $(i+1)^{\text{st}}$ column are zero since Q is upper triangular. Thus $q_{i+1,i+1}$ is the only nonzero entry in its row. So inductively, q_{kk} is the only nonzero entry in the k^{th} row for $k = 1, 2, \ldots, n$, so that Q is diagonal.

36. If $n > m$, then the maximum rank of A is m. By the Rank Theorem (Theorem 3.26, in Section 3.5), we know that $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$; since $n > m$, we have $\operatorname{nullity}(A) = n - m > 0$. So there is some nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. But then $\|\mathbf{x}\| \neq 0$ while $\|A\mathbf{x}\| = \|\mathbf{0}\| = 0$.

37. (a) Since the $\{\mathbf{v}_i\}$ are a basis, we may choose α_i, β_i such that

$$\mathbf{x} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k, \quad \mathbf{y} = \sum_{k=1}^{n} \beta_k \mathbf{v}_k.$$

Because $\{\mathbf{v}_i\}$ form an orthonormal basis, we know that $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$. Then

$$\mathbf{x} \cdot \mathbf{y} = \left(\sum_{k=1}^{n} \alpha_k \mathbf{v}_k\right) \cdot \left(\sum_{k=1}^{n} \beta_k \mathbf{v}_k\right) = \sum_{i,j=1}^{n} \alpha_i \beta_j \mathbf{v}_i \cdot \mathbf{v}_j = \sum_{i=1}^{n} \alpha_i \beta_i \mathbf{v}_i \cdot \mathbf{v}_i = \sum_{i=1}^{n} \alpha_i \beta_i.$$

But $\mathbf{x} \cdot \mathbf{v}_i = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k \cdot \mathbf{v}_i = \alpha_i$ and similarly $\mathbf{y} \cdot \mathbf{v}_i = \beta_i$. Therefore the sum at the end of the displayed equation above is

$$\sum_{i=1}^{n} \alpha_i \beta_i = \sum_{i=1}^{n} (\mathbf{x} \cdot \mathbf{v}_i)(\mathbf{y} \cdot \mathbf{v}_i) = (\mathbf{x} \cdot \mathbf{v}_1)(\mathbf{y} \cdot \mathbf{v}_1) + (\mathbf{x} \cdot \mathbf{v}_2)(\mathbf{y} \cdot \mathbf{v}_2) + \cdots + (\mathbf{x} \cdot \mathbf{v}_n)(\mathbf{y} \cdot \mathbf{v}_n),$$

proving Parseval's Identity.

(b) Since $[\mathbf{x}]_{\mathcal{B}} = [\alpha_k]$ and $[\mathbf{y}]_{\mathcal{B}} = [\beta_k]$, Parseval's Identity says that

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^{n} \alpha_k \beta_k = [\mathbf{x}]_{\mathcal{B}} \cdot [\mathbf{y}]_{\mathcal{B}}.$$

In other words, the dot product is invariant under a change of orthonormal basis.

5.2 Orthogonal Complements and Orthogonal Projections

1. W is the column space of $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so that $W^\perp$ is the null space of $A^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$. The augmented matrix is already row-reduced:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \end{array}\right],$$

so that a basis for the null space, and thus for $W^\perp$, is given by

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

2. W is the column space of $A = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$, so that $W^\perp$ is the null space of $A^T = \begin{bmatrix} -4 & 3 \end{bmatrix}$. Row-reducing gives

$$\left[\begin{array}{cc|c} -4 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{3}{4} & 0 \end{array}\right],$$

so that a basis for the null space, and thus for $W^\perp$, is given by

$$\begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}, \qquad \text{or} \qquad \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

3. W consists of

$$\begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ so that } \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\} \text{ is a basis for } W.$$

Then W is the column space of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},$$

so that $W^\perp$ is the null space of A^T. The augmented matrix is already row-reduced:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

so that the null space, which is $W^\perp$, is the set

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right).$$

This vector forms a basis for $W^\perp$.

4. W consists of

$$\begin{bmatrix} x \\ 2x+3z \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \text{ so that } \left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}\right\} \text{ is a basis for } W.$$

Then W is the column space of

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 0 & 1 \end{bmatrix},$$

so that $W^\perp$ is the null space of A^T. Row-reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \end{array}\right]$$

so that the null space, which is $W^\perp$, is the set

$$\begin{bmatrix} \frac{2}{3}t \\ -\frac{1}{3}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right).$$

This vector forms a basis for $W^\perp$.

5. W is the column space of $A = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, so that $W^\perp$ is the null space of $A^T = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}$. The augmented matrix is already row-reduced:

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \end{array}\right],$$

so that the null space, which is $W^\perp$, is the set

$$\begin{bmatrix} s-3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)$$

These two vectors form a basis for $W^\perp$.

6. W is the column space of $A = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{bmatrix}$, so that $W^\perp$ is the null space of $A^T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 2 \end{bmatrix}$. Row-reducing gives

$$\left[\begin{array}{ccc|c} \frac{1}{2} & -\frac{1}{2} & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \end{array}\right],$$

so the null space, which is $W^\perp$, is the set

$$\begin{bmatrix} s-4t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}\right)$$

These two vectors form a basis for $W^\perp$.

7. To find the required bases, we row-reduce $\left[\begin{array}{c|c} A & 0 \end{array}\right]$:

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 0 \\ 5 & 2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore, a basis for the row space of A is the nonzero rows of the reduced matrix, or

$$\mathbf{u}_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix}.$$

The null space null(A) is the set of solutions of the augmented matrix, which are given by

$$\begin{bmatrix} -s \\ 2s \\ s \end{bmatrix}.$$

Thus a basis for null(A) is

$$\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Every vector in row(A) is orthogonal to every vector in null(A):

$$\mathbf{u}_1 \cdot \mathbf{v} = 1 \cdot (-1) + 0 \cdot 2 + 1 \cdot 1 = 0, \qquad \mathbf{u}_2 \cdot \mathbf{v} = 0 \cdot (-1) + 1 \cdot 2 - 2 \cdot 1 = 0.$$

8. To find the required bases, we row-reduce $\left[\begin{array}{c|c} A & 0 \end{array}\right]$:

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -1 & 0 & 2 & 0 \\ -2 & 0 & 2 & 4 & 4 & 0 \\ 2 & 2 & -2 & 0 & 1 & 0 \\ -3 & -1 & 3 & 4 & 5 & 0 \end{array}\right] \to \left[\begin{array}{rrrrr|r} 1 & 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus a basis for the row space of A is the nonzero rows of the reduced matrix, or

$$\mathbf{u}_1 = \begin{bmatrix} 1 & 0 & -1 & -2 & 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The null space null(A) is the set of solutions of the augmented matrix, which are given by

$$\begin{bmatrix} s+2t \\ -2t \\ s \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right).$$

Thus a basis for null(A) is

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix}2\\-2\\0\\1\\0\end{bmatrix}.$$

Every vector in row(A) is orthogonal to every vector in null(A):

$$\begin{aligned}
\mathbf{u}_1 \cdot \mathbf{v}_1 &= 1 \cdot 1 + 0 \cdot 0 - 1 \cdot 1 - 2 \cdot 0 + 0 \cdot 0 = 0, & \mathbf{u}_1 \cdot \mathbf{v}_2 &= 1 \cdot 2 + 0 \cdot (-2) - 1 \cdot 0 - 2 \cdot 1 + 0 \cdot 0 = 0,\\
\mathbf{u}_2 \cdot \mathbf{v}_1 &= 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 + 2 \cdot 0 + 0 \cdot 0 = 0, & \mathbf{u}_2 \cdot \mathbf{v}_2 &= 0 \cdot 2 + 1 \cdot (-2) + 0 \cdot 0 + 2 \cdot 1 + 0 \cdot 0 = 0,\\
\mathbf{u}_3 \cdot \mathbf{v}_1 &= 0 \cdot 1 + 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 = 0, & \mathbf{u}_3 \cdot \mathbf{v}_2 &= 0 \cdot 2 + 0 \cdot (-2) + 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 = 0.
\end{aligned}$$

9. Using the row reduction from Exercise 7, we see that a basis for the column space of A is

$$\mathbf{u}_1 = \begin{bmatrix}1\\5\\0\\-1\end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix}-1\\2\\1\\-1\end{bmatrix}.$$

To find the null space of A^T, we must row-reduce $\left[A^T \mid 0\right]$:

$$\left[\begin{array}{rrrr|r} 1 & 5 & 0 & -1 & 0\\ -1 & 2 & 1 & -1 & 0\\ 3 & 1 & -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & -\frac{5}{7} & \frac{3}{7} & 0\\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The null space null(A^T) is the set of solutions of the augmented matrix, which are given by

$$\begin{bmatrix}\frac{5}{7}s - \frac{3}{7}t\\ -\frac{1}{7}s + \frac{2}{7}t\\ s\\ t\end{bmatrix} = s\begin{bmatrix}\frac{5}{7}\\-\frac{1}{7}\\1\\0\end{bmatrix} + t\begin{bmatrix}-\frac{3}{7}\\\frac{2}{7}\\0\\1\end{bmatrix} = \operatorname{span}\left(\begin{bmatrix}5\\-1\\7\\0\end{bmatrix}, \begin{bmatrix}-3\\2\\0\\7\end{bmatrix}\right).$$

Thus a basis for null(A^T) is

$$\mathbf{v}_1 = \begin{bmatrix}5\\-1\\7\\0\end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix}-3\\2\\0\\7\end{bmatrix}.$$

Every vector in col(A) is orthogonal to every vector in null(A^T):

$$\begin{aligned}
\mathbf{u}_1 \cdot \mathbf{v}_1 &= 1 \cdot 5 + 5 \cdot (-1) + 0 \cdot 7 - 1 \cdot 0 = 0, & \mathbf{u}_1 \cdot \mathbf{v}_2 &= 1 \cdot (-3) + 5 \cdot 2 + 0 \cdot 0 - 1 \cdot 7 = 0,\\
\mathbf{u}_2 \cdot \mathbf{v}_1 &= -1 \cdot 5 + 2 \cdot (-1) + 1 \cdot 7 - 1 \cdot 0 = 0, & \mathbf{u}_2 \cdot \mathbf{v}_2 &= -1 \cdot (-3) + 2 \cdot 2 + 1 \cdot 0 - 1 \cdot 7 = 0.
\end{aligned}$$

10. Using the row reduction from Exercise 8, we see that a basis for the column space of A is

$$\mathbf{u}_1 = \begin{bmatrix}1\\-2\\2\\-3\end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix}1\\0\\2\\-1\end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix}2\\4\\1\\5\end{bmatrix}$$

To find the null space of A^T, we must row-reduce $\left[A^T \mid 0\right]$:

$$\left[\begin{array}{rrrr|r} 1 & -2 & 2 & -3 & 0\\ 1 & 0 & 2 & -1 & 0\\ -1 & 2 & -2 & 3 & 0\\ 0 & 4 & 0 & 4 & 0\\ 2 & 4 & 1 & 5 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The null space $\operatorname{null}(A^T)$ is the set of solutions of the augmented matrix, which are given by

$$\begin{bmatrix} -s \\ -s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus a basis for $\operatorname{null}(A^T)$ is

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Every vector in $\operatorname{col}(A)$ is orthogonal to every vector in $\operatorname{null}(A^T)$:

$$\mathbf{u}_1 \cdot \mathbf{v}_1 = 1 \cdot (-1) - 2 \cdot (-1) + 2 \cdot 1 - 3 \cdot 1 = 0, \quad \mathbf{u}_2 \cdot \mathbf{v}_1 = 1 \cdot (-1) + 0 \cdot (-1) + 2 \cdot 1 - 1 \cdot 1 = 0,$$
$$\mathbf{u}_3 \cdot \mathbf{v}_1 = 2 \cdot (-1) + 4 \cdot (-1) + 1 \cdot 1 + 5 \cdot 1 = 0.$$

11. Let

$$A = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Then the subspace W spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$ is the column space of A, so that by Theorem 5.10, $W^\perp = (\operatorname{col}(A))^\perp = \operatorname{null}(A^T)$. To compute $\operatorname{null}(A^T)$, we row-reduce

$$\begin{bmatrix} A^T \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 4 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{5}{2} & 0 \end{array}\right].$$

Thus $\operatorname{null}(A^T)$ is the set of solutions of this augmented matrix, which is

$$\begin{bmatrix} -\frac{1}{4}t \\ \frac{5}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ \frac{5}{2} \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 10 \\ 4 \end{bmatrix}\right)$$

so this vector forms a basis for $W^\perp$.

12. Let

$$A = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 3 & -2 \\ -2 & 1 \end{bmatrix}.$$

Then the subspace W spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$ is the column space of A, so that by Theorem 5.10, $W^\perp = (\operatorname{col}(A))^\perp = \operatorname{null}(A^T)$. To compute $\operatorname{null}(A^T)$, we row-reduce

$$\begin{bmatrix} A^T \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & -1 & 3 & -2 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right].$$

Thus $\operatorname{null}(A^T)$ is the set of solutions of this augmented matrix, which is

$$\begin{bmatrix} -s+t \\ 2s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right),$$

so these two vectors are a basis for $W^\perp$.

13. Let

$$A = \begin{bmatrix} \mathbf{w}_1 \mid \mathbf{w}_2 \mid \mathbf{w}_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 5 \\ 6 & -3 & 6 \\ 3 & -2 & 1 \end{bmatrix}.$$

Then the subspace W spanned by $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ is the column space of A, so that by Theorem 5.10, $W^\perp = (\mathrm{col}(A))^\perp = \mathrm{null}(A^T)$. To compute $\mathrm{null}(A^T)$, we row-reduce

$$\begin{bmatrix} A^T \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{rrrr|r} 2 & -1 & 6 & 3 & 0 \\ -1 & 2 & -3 & -2 & 0 \\ 2 & 5 & 6 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 3 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus $\mathrm{null}(A^T)$ is the set of solutions of this augmented matrix, which is

$$\begin{bmatrix} -3s - \frac{4}{3}t \\ \frac{1}{3}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}\right),$$

so these two vectors form a basis for $W^\perp$.

14. Let

$$A = \begin{bmatrix} \mathbf{v}_1 \mid \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 2 & 2 \\ -1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & -3 & 2 \end{bmatrix}.$$

Then the subspace W spanned by $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ is the column space of A, so that by Theorem 5.10, $W^\perp = (\mathrm{col}(A))^\perp = \mathrm{null}(A^T)$. To compute $\mathrm{null}(A^T)$, we row-reduce

$$\begin{bmatrix} A^T \mid \mathbf{0} \end{bmatrix} = \left[\begin{array}{rrrrr|r} 4 & 6 & -1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 & -3 & 0 \\ 2 & 2 & 2 & -1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & -2 & 7 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & -5 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right].$$

Thus $\mathrm{null}(A^T)$ is the set of solutions of this augmented matrix, which is

$$\begin{bmatrix} 2s - 7t \\ -\frac{3}{2}s + 5t \\ t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ -\frac{3}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -7 \\ 5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Clearing fractions, we get

$$\begin{bmatrix} 4 \\ -3 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -7 \\ 5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

as a basis for $W^\perp$.

15. We have

$$\mathrm{proj}_W(\mathbf{v}) = \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 = \frac{1 \cdot 7 + 1 \cdot (-4)}{1 \cdot 1 + 1 \cdot 1}\mathbf{u}_1 = \frac{3}{2}\mathbf{u}_1 = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

16. We have

$$\begin{aligned}
\operatorname{proj}_W(\mathbf{v}) &= \left(\frac{\mathbf{u}_1\cdot\mathbf{v}}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{u}_2\cdot\mathbf{v}}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2 \\
&= \frac{1\cdot 3+1\cdot 1+1\cdot(-2)}{1\cdot 1+1\cdot 1+1\cdot 1}\mathbf{u}_1 + \frac{1\cdot 3-1\cdot 1+0\cdot(-2)}{1\cdot 1-1\cdot(-1)+0\cdot 0}\mathbf{u}_2 \\
&= \frac{2}{3}\mathbf{u}_1+\mathbf{u}_2 \\
&= \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}\frac{5}{3}\\-\frac{1}{3}\\\frac{2}{3}\end{bmatrix}
\end{aligned}$$

17. We have

$$\begin{aligned}
\operatorname{proj}_W(\mathbf{v}) &= \left(\frac{\mathbf{u}_1\cdot\mathbf{v}}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{u}_2\cdot\mathbf{v}}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2 \\
&= \frac{2\cdot 1-2\cdot 2+1\cdot 3}{2\cdot 2-2\cdot(-2)+1\cdot 1}\mathbf{u}_1 + \frac{-1\cdot 1+2\cdot 2+4\cdot 3}{-1\cdot(-1)+1\cdot 1+4\cdot 4}\mathbf{u}_2 \\
&= \frac{1}{9}\mathbf{u}_1+\frac{13}{18}\mathbf{u}_2 \\
&= \begin{bmatrix}\frac{2}{9}\\-\frac{2}{9}\\\frac{1}{9}\end{bmatrix} + \begin{bmatrix}-\frac{13}{18}\\\frac{13}{18}\\\frac{52}{18}\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\3\end{bmatrix}.
\end{aligned}$$

18. We have

$$\begin{aligned}
\operatorname{proj}_W(\mathbf{v}) &= \left(\frac{\mathbf{u}_1\cdot\mathbf{v}}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{u}_2\cdot\mathbf{v}}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2 + \left(\frac{\mathbf{u}_3\cdot\mathbf{v}}{\mathbf{u}_3\cdot\mathbf{u}_3}\right)\mathbf{u}_3 \\
&= \frac{1\cdot 3+1\cdot(-2)+0\cdot 4+0\cdot(-3)}{1\cdot 1+1\cdot 1+0\cdot 0+0\cdot 0}\mathbf{u}_1 + \frac{1\cdot 3-1\cdot(-2)-1\cdot 4+1\cdot(-3)}{1\cdot 1-1\cdot(-1)-1\cdot(-1)+1\cdot 1}\mathbf{u}_2 + \\
&\qquad \frac{0\cdot 3+0\cdot(-2)+1\cdot 4+1\cdot(-3)}{0\cdot 0+0\cdot 0+1\cdot 1+1\cdot 1}\mathbf{u}_3 \\
&= \frac{1}{2}\mathbf{u}_1-\frac{1}{2}\mathbf{u}_2+\frac{1}{2}\mathbf{u}_3 \\
&= \frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}.
\end{aligned}$$

19. Since W is spanned by the vector $\mathbf{u}_1 = \begin{bmatrix}1\\3\end{bmatrix}$, we have

$$\operatorname{proj}_W(\mathbf{v}) = \left(\frac{\mathbf{u}_1\cdot\mathbf{v}}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 = \frac{1\cdot 2+3\cdot(-2)}{1\cdot 1+3\cdot 3}\mathbf{u}_1 = -\frac{2}{5}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}-\frac{2}{5}\\-\frac{6}{5}\end{bmatrix}.$$

The component of $\mathbf{v}$ orthogonal to W is thus

$$\operatorname{proj}_{W^\perp}(\mathbf{v}) = \mathbf{v}-\operatorname{proj}_W(\mathbf{v}) = \begin{bmatrix}\frac{12}{5}\\-\frac{4}{5}\end{bmatrix}.$$

20. Since W is spanned by the vector $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, we have

$$\operatorname{proj}_W(\mathbf{v}) = \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 = \frac{1 \cdot 3 + 1 \cdot 2 + 1 \cdot (-1)}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1}\mathbf{u}_1 = \frac{4}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{4}{3}\\\frac{4}{3}\\\frac{4}{3}\end{bmatrix}.$$

The component of $\mathbf{v}$ orthogonal to W is thus

$$\operatorname{proj}_{W^\perp}(\mathbf{v}) = \mathbf{v} - \operatorname{proj}_W(\mathbf{v}) = \begin{bmatrix}\frac{5}{3}\\\frac{2}{3}\\-\frac{7}{3}\end{bmatrix}.$$

21. Since W is spanned by

$$\mathbf{u}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix},$$

we have

$$\begin{aligned}\operatorname{proj}_W(\mathbf{v}) &= \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2\\ &= \frac{1 \cdot 4 + 2 \cdot (-2) + 1 \cdot 3}{1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1}\mathbf{u}_1 + \frac{1 \cdot 4 - 1 \cdot (-2) + 1 \cdot 3}{1 \cdot 1 - 1 \cdot (-1) + 1 \cdot 1}\mathbf{u}_2\\ &= \frac{1}{2}\mathbf{u}_1 + 3\mathbf{u}_2\\ &= \begin{bmatrix}\frac{1}{2}\\1\\\frac{1}{2}\end{bmatrix} + \begin{bmatrix}3\\-3\\3\end{bmatrix} = \begin{bmatrix}\frac{7}{2}\\-2\\\frac{7}{2}\end{bmatrix}.\end{aligned}$$

The component of $\mathbf{v}$ orthogonal to W is thus

$$\operatorname{proj}_{W^\perp}(\mathbf{v}) = \mathbf{v} - \operatorname{proj}_W(\mathbf{v}) = \begin{bmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{bmatrix}.$$

22. Since W is spanned by

$$\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix}1\\0\\-1\\1\end{bmatrix},$$

we have

$$\begin{aligned}\operatorname{proj}_W(\mathbf{v}) &= \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2\\ &= \frac{1 \cdot 2 + 1 \cdot (-1) + 1 \cdot 5 + 0 \cdot 6}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0}\mathbf{u}_1 + \frac{1 \cdot 2 + 0 \cdot (-1) - 1 \cdot 5 + 1 \cdot 6}{1 \cdot 1 + 0 \cdot 0 - 1 \cdot (-1) + 1 \cdot 1}\mathbf{u}_2\\ &= 2\mathbf{u}_1 + \mathbf{u}_2\\ &= 2\begin{bmatrix}1\\1\\1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} = \begin{bmatrix}3\\2\\1\\1\end{bmatrix}.\end{aligned}$$

The component of $\mathbf{v}$ orthogonal to W is thus

$$\operatorname{proj}_{W^\perp}(\mathbf{v}) = \mathbf{v} - \operatorname{proj}_W(\mathbf{v}) = \begin{bmatrix} -1 \\ -3 \\ 4 \\ 5 \end{bmatrix}.$$

23. $W^\perp$ is defined to be the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W. So if $\mathbf{w} \in W \cap W^\perp$, then $\mathbf{w} \cdot \mathbf{w} = 0$. But this means that $\mathbf{w} = \mathbf{0}$. Thus $\mathbf{0}$ is the only element of $W \cap W^\perp$.

24. If $\mathbf{v} \in W^\perp$, then since $\mathbf{v}$ is orthogonal to every vector in W, it is certainly orthogonal to $\mathbf{w}_i$, so that $\mathbf{v} \cdot \mathbf{w}_i = 0$ for $i = 1, 2, \ldots, k$. For the converse, suppose $\mathbf{w} \in W$; then since $W = \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, there are scalars a_i such that $\mathbf{w} = \sum_{i=1}^k a_i \mathbf{w}_i$. Then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \sum_{i=1}^k a_i \mathbf{w}_i = \sum_{i=1}^k a_i \mathbf{v} \cdot \mathbf{w}_i = 0$$

since $\mathbf{v}$ is orthogonal to each of the $\mathbf{w}_i$. But this means that $\mathbf{v}$ is orthogonal to every vector in W, so that $\mathbf{v} \in W^\perp$.

25. No, it is not true. For example, let $W \subset \mathbb{R}^3$ be the xy-plane, spanned by $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then $W^\perp = \operatorname{span}(\mathbf{e}_3) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$. Now let $\mathbf{w} = \mathbf{e}_1$ and $\mathbf{w}' = \mathbf{e}_2$; then $\mathbf{w} \cdot \mathbf{w}' = 0$ but $\mathbf{w}' \notin W^\perp$.

26. Yes, this is true. Let $V = \operatorname{span}(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$; we want to show that $V = W^\perp$. By Theorem 5.9(d), all we must show is that for every $\mathbf{v} \in V$, $\mathbf{v}_i \cdot \mathbf{v} = 0$ for $i = 1, 2, \ldots, k$. But $\mathbf{v} = \sum_{j=k+1}^n c_j \mathbf{v}_j$, so that if $i \le k$, since the $\mathbf{v}_i$ are an orthogonal basis for $\mathbb{R}^n$ we get

$$\mathbf{v}_i \cdot \sum_{j=k+1}^n c_j \mathbf{v}_j = \sum_{j=k+1}^n c_j \mathbf{v}_j \cdot \mathbf{v}_i = 0.$$

27. Let $\{\mathbf{u}_k\}$ be an orthonormal basis for W. If

$$\mathbf{x} = \operatorname{proj}_W(\mathbf{x}) = \sum_{k=1}^n \left(\frac{\mathbf{u}_k \cdot \mathbf{x}}{\mathbf{u}_k \cdot \mathbf{u}_k}\right) \mathbf{u}_k = \sum_{k=1}^n (\mathbf{u}_k \cdot \mathbf{x}) \mathbf{u}_k,$$

then $\mathbf{x}$ is in the span of the $\mathbf{u}_k$, which is W. For the converse, suppose that $\mathbf{x} \in W$. Then $\mathbf{x} = \sum_{k=1}^n a_k \mathbf{u}_k$. Since the $\mathbf{u}_k$ are an orthogonal basis for W, we get

$$\mathbf{x} \cdot \mathbf{u}_i = \sum_{k=1}^n a_k \mathbf{u}_k \cdot \mathbf{u}_i = a_i \mathbf{u}_i \cdot \mathbf{u}_i = a_i.$$

Therefore

$$\mathbf{x} = \sum_{k=1}^n a_k \mathbf{u}_k = \sum_{k=1}^n (\mathbf{x} \cdot \mathbf{u}_k) \mathbf{u}_k = \operatorname{proj}_W(\mathbf{x}).$$

28. Let $\{\mathbf{u}_k\}$ be an orthonormal basis for W. Then

$$\operatorname{proj}_W(\mathbf{x}) = \sum_{k=1}^n \left(\frac{\mathbf{u}_k \cdot \mathbf{x}}{\mathbf{u}_k \cdot \mathbf{u}_k}\right) \mathbf{u}_k = \sum_{k=1}^n (\mathbf{u}_k \cdot \mathbf{x}) \mathbf{u}_k.$$

So if $\operatorname{proj}_W(\mathbf{x}) = \mathbf{0}$, then $\mathbf{u}_k \cdot \mathbf{x} = 0$ for all k since the $\mathbf{u}_k$ are linearly independent. But then by Theorem 4.9(d), $\mathbf{x} \in W^\perp$, so that $\mathbf{x}$ is orthogonal to W. Conversely, if $\mathbf{x}$ is orthogonal to W, then $\mathbf{u}_k \cdot \mathbf{x} = 0$ for all k, so the sum above is zero and thus $\operatorname{proj}_W(\mathbf{x}) = \mathbf{0}$.

29. If $\mathbf{u}_k$ is an orthonormal basis for W, then

$$\operatorname{proj}_W(\mathbf{x}) = \sum_{k=1}^{n} (\mathbf{u}_k \cdot \mathbf{x})\mathbf{u}_k \in W.$$

By Exercise 27, $x \in W$ if and only if $\operatorname{proj}_W(\mathbf{x}) = \mathbf{x}$. Applying this to $\operatorname{proj}_W(\mathbf{x}) \in W$ gives

$$\operatorname{proj}_W(\operatorname{proj}_W(\mathbf{x})) = \operatorname{proj}_W(\mathbf{x})$$

as desired.

30. **(a)** Let $W = \operatorname{span}(\mathcal{S}) = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, and let $\mathcal{T} = \{\mathbf{b}_1, \ldots, \mathbf{b}_j\}$ be an orthonormal basis for $W^\perp$. By Theorem 5.13, $\dim W + \dim W^\perp = k + j = n$. Therefore $\mathcal{S} \cup \mathcal{T}$ is a basis for $\mathbb{R}^n$. Further, it is an orthonormal basis: any two distinct members of $\mathcal{S}$ are orthogonal since $\mathcal{S}$ is an orthonormal set; similarly, any two distinct members of $\mathcal{T}$ are orthogonal. Also, $\mathbf{v}_i \cdot \mathbf{b}_j = 0$ since one of the vectors is in W and the other is in $W^\perp$. Finally, $\mathbf{v}_i \cdot \mathbf{v}_i = \mathbf{b}_j \cdot \mathbf{b}_j = 1$ since both sets are orthonormal sets.

Choose $\mathbf{x} \in \mathbb{R}^n$; then there are α_r, β_s such that

$$\mathbf{x} = \sum_{r=1}^{k} \alpha_r \mathbf{v}_r + \sum_{s=1}^{j} \beta_s \mathbf{b}_s.$$

Then using orthonormality of $\mathcal{S} \cup \mathcal{T}$, we have

$$\mathbf{x} \cdot \mathbf{v}_i = \left(\sum_{r=1}^{k} \alpha_r \mathbf{v}_r + \sum_{s=1}^{j} \beta_s \mathbf{b}_s \right) \cdot \mathbf{v}_i = \alpha_i \text{ and } \mathbf{x} \cdot \mathbf{b}_i = \left(\sum_{r=1}^{k} \alpha_r \mathbf{v}_r + \sum_{s=1}^{j} \beta_s \mathbf{b}_s \right) \cdot \mathbf{b}_i = \beta_i.$$

But then (again using orthonormality)

$$\begin{aligned}
\|\mathbf{x}\|^2 &= \mathbf{x} \cdot \mathbf{x} \\
&= \left(\sum_{r=1}^{k} \alpha_r \mathbf{v}_r + \sum_{s=1}^{j} \beta_s \mathbf{b}_s \right) \left(\sum_{r=1}^{k} \alpha_r \mathbf{v}_r + \sum_{s=1}^{j} \beta_s \mathbf{b}_s \right) \\
&= \sum_{r=1}^{k} \alpha_r^2 + \sum_{s=1}^{j} \beta_s^2 \\
&= \sum_{r=1}^{k} |\mathbf{x} \cdot \mathbf{v}_r|^2 + \sum_{s=1}^{j} |\mathbf{x} \cdot \mathbf{b}_s|^2 \\
&\geq \sum_{r=1}^{k} |\mathbf{x} \cdot \mathbf{v}_r|^2,
\end{aligned}$$

as required.

(b) Looking at the string of equalities and inequalities above, we see that equality holds if and only if $\sum_{s=1}^{j} |\mathbf{x} \cdot \mathbf{b}_s|^2 = 0$. But this means that $\mathbf{x} \cdot \mathbf{b}_s = 0$ for all s, which in turn means that $\mathbf{x} \in \operatorname{span}\{\mathbf{v}_j\} = W$.

5.3 The Gram-Schmidt Process and the QR Factorization

1. First obtain an orthogonal basis:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1 \cdot 1 + 1 \cdot 2}{1 \cdot 1 + 1 \cdot 1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Normalizing each of the vectors gives

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}, \qquad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{1/\sqrt{2}}\begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}.$$

2. First obtain an orthogonal basis:

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{x}_1 = \begin{bmatrix}3\\-3\end{bmatrix}\\ \mathbf{v}_2 &= \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 = \begin{bmatrix}3\\1\end{bmatrix} - \frac{3\cdot 3 - 3\cdot 1}{3\cdot 3 - 3\cdot(-3)}\begin{bmatrix}3\\-3\end{bmatrix}\\ &= \begin{bmatrix}3\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}3\\-3\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}.\end{aligned}$$

Normalizing each of the vectors gives

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{18}}\begin{bmatrix}3\\-3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}, \qquad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{8}}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

3. First obtain an orthogonal basis:

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{x}_1 = \begin{bmatrix}1\\-1\\-1\end{bmatrix}\\ \mathbf{v}_2 &= \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 = \begin{bmatrix}0\\3\\3\end{bmatrix} - \frac{1\cdot 0 - 1\cdot 3 - 1\cdot 3}{1\cdot 1 - 1\cdot(-1) - 1\cdot(-1)}\begin{bmatrix}1\\-1\\-1\end{bmatrix} = \begin{bmatrix}0\\3\\3\end{bmatrix} + \begin{bmatrix}2\\-2\\-2\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix}\\ \mathbf{v}_3 &= \mathbf{x}_3 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_3)\mathbf{v}_1 - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{x}_3)\mathbf{v}_2\\ &= \begin{bmatrix}3\\2\\4\end{bmatrix} - \frac{1\cdot 3 - 1\cdot 2 - 1\cdot 4}{1\cdot 1 - 1\cdot(-1) - 1\cdot(-1)}\begin{bmatrix}1\\-1\\-1\end{bmatrix} - \frac{2\cdot 3 + 1\cdot 2 + 1\cdot 4}{2\cdot 2 + 1\cdot 1 + 1\cdot 1}\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}3\\2\\4\end{bmatrix} + \begin{bmatrix}1\\-1\\-1\end{bmatrix} - 2\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}.\end{aligned}$$

Normalizing each of the vectors gives

$$\begin{aligned}\mathbf{q}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\end{bmatrix},\\ \mathbf{q}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{bmatrix},\\ \mathbf{q}_3 &= \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}.\end{aligned}$$

4. First obtain an orthogonal basis:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1\cdot 1 + 1\cdot 1 + 1\cdot 0}{1\cdot 1 + 1\cdot 1 + 1\cdot 1}\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\\ -\frac{2}{3}\end{bmatrix}$$

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{x}_3 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_3)\mathbf{v}_1 - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{x}_3)\mathbf{v}_2\\
&= \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1\cdot 1 + 1\cdot 0 + 1\cdot 0}{1\cdot 1 + 1\cdot 1 + 1\cdot 1}\mathbf{v}_1 - \frac{\frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0 - \frac{2}{3}\cdot 0}{\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} - \frac{2}{3}\cdot\left(-\frac{2}{3}\right)}\mathbf{v}_2\\
&= \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\\ -\frac{2}{3}\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ 0\end{bmatrix}.\end{aligned}$$

Normalizing each of the vectors gives

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}, \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{2/3}}\begin{bmatrix}\frac{1}{3}\\ \frac{1}{2}\\ -\frac{2}{3}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ -\frac{2}{\sqrt{6}}\end{bmatrix},$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{1/\sqrt{2}}\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\\ 0\end{bmatrix}$$

5. Using Gram-Schmidt, we get

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 = \begin{bmatrix}3\\4\\2\end{bmatrix} - \frac{1\cdot 3 + 1\cdot 4 + 0\cdot 2}{1\cdot 1 + 1\cdot 1 + 0\cdot 0}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}3\\4\\2\end{bmatrix} - \frac{7}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 2\end{bmatrix}.$$

6. Using Gram-Schmidt, we get

$$\begin{aligned}
\mathbf{v}_1 = \mathbf{x}_1 &= \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} \\
\mathbf{v}_2 = \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 &= \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix} - \frac{2\cdot 3 - 1\cdot(-1) + 1\cdot 0 + 2\cdot 4}{2\cdot 2 - 1\cdot(-1) + 1\cdot 1 + 2\cdot 2}\mathbf{v}_1 \\
&= \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix} \\
\mathbf{v}_3 = \mathbf{x}_3 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_3)\mathbf{v}_1 - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{x}_3)\mathbf{v}_2 & \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2\cdot 1 - 1\cdot 1 + 1\cdot 1 + 2\cdot 1}{2\cdot 2 - 1\cdot(-1) + 1\cdot 1 + 2\cdot 2}\mathbf{v}_1 - \frac{0\cdot 1 + \frac{1}{2}\cdot 1 - \frac{3}{2}\cdot 1 + 1\cdot 1}{0\cdot 0 + \frac{1}{2}\cdot\frac{1}{2} - \frac{3}{2}\cdot\left(-\frac{3}{2}\right) + 1\cdot 1}\mathbf{v}_2 \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{7}{5} \\ \frac{3}{5} \\ \frac{1}{5} \end{bmatrix}.
\end{aligned}$$

7. First compute the projection of $\mathbf{v}$ on each of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ from Exercise 5:

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}_1}\mathbf{v} &= \frac{\mathbf{v}_1\cdot\mathbf{v}}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 = \frac{1\cdot 4 + 1\cdot(-4) + 0\cdot 3}{1\cdot 1 + 1\cdot 1 + 0\cdot 0}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\operatorname{proj}_{\mathbf{v}_2}\mathbf{v} &= \frac{\mathbf{v}_2\cdot\mathbf{v}}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \frac{-\frac{1}{2}\cdot 4 + \frac{1}{2}\cdot(-4) + 2\cdot 3}{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 2^2}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix} = \frac{4}{9}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} \\ \frac{2}{9} \\ \frac{8}{9} \end{bmatrix}.
\end{aligned}$$

Then the component of $\mathbf{v}$ orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$\begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{2}{9} \\ \frac{2}{9} \\ \frac{8}{9} \end{bmatrix} = \begin{bmatrix} \frac{38}{9} \\ -\frac{38}{9} \\ \frac{19}{9} \end{bmatrix},$$

so that

$$\mathbf{v} = \begin{bmatrix} \frac{38}{9} \\ -\frac{38}{9} \\ \frac{19}{9} \end{bmatrix} + \frac{4}{9}\mathbf{v}_2.$$

8. First compute the projection of $\mathbf{v}$ on each of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ from Exercise 6:

$$\operatorname{proj}_{\mathbf{v}_1}\mathbf{v} = \frac{\mathbf{v}_1\cdot\mathbf{v}}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 = \frac{2\cdot 1 - 1\cdot 4 + 1\cdot 0 + 2\cdot 2}{2^2+1^2+1^2+2^2}\begin{bmatrix}2\\-1\\1\\2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}2\\-1\\1\\2\end{bmatrix} = \begin{bmatrix}\frac{2}{5}\\-\frac{1}{5}\\\frac{1}{5}\\\frac{2}{5}\end{bmatrix}$$

$$\operatorname{proj}_{\mathbf{v}_2}\mathbf{v} = \frac{\mathbf{v}_2\cdot\mathbf{v}}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \frac{0\cdot 1 + frac12\cdot 4 - \frac{3}{2}\cdot 0 + 1\cdot 2}{0^2+\left(\frac{1}{2}\right)^2+\left(-\frac{3}{2}\right)^2+1^2}\begin{bmatrix}0\\\frac{1}{2}\\-\frac{3}{2}\\1\end{bmatrix} = \frac{8}{7}\begin{bmatrix}0\\\frac{1}{2}\\-\frac{3}{2}\\1\end{bmatrix} = \begin{bmatrix}0\\\frac{4}{7}\\-\frac{12}{7}\\\frac{8}{7}\end{bmatrix}$$

$$\operatorname{proj}_{\mathbf{v}_3}\mathbf{v} = \frac{\mathbf{v}_3\cdot\mathbf{v}}{\mathbf{v}_3\cdot\mathbf{v}_3}\mathbf{v}_3 = \frac{\frac{1}{5}\cdot 1 + \frac{7}{5}\cdot 4 + \frac{3}{5}\cdot 0 + \frac{1}{5}\cdot 2}{\left(\frac{1}{5}\right)^2+\left(\frac{7}{5}\right)^2+\left(\frac{3}{5}\right)^2+\left(\frac{1}{5}\right)^2}\begin{bmatrix}\frac{1}{5}\\\frac{7}{5}\\\frac{3}{5}\\\frac{1}{5}\end{bmatrix} = \frac{31}{12}\begin{bmatrix}\frac{1}{5}\\\frac{7}{5}\\\frac{3}{5}\\\frac{1}{5}\end{bmatrix} = \begin{bmatrix}\frac{31}{60}\\\frac{217}{60}\\\frac{93}{60}\\\frac{31}{60}\end{bmatrix}.$$

Then the component of $\mathbf{v}$ orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$\begin{bmatrix}4\\-4\\3\end{bmatrix} - \begin{bmatrix}\frac{2}{5}\\-\frac{1}{5}\\\frac{1}{5}\\\frac{2}{5}\end{bmatrix} - \begin{bmatrix}0\\\frac{4}{7}\\-\frac{12}{7}\\\frac{8}{7}\end{bmatrix} - \begin{bmatrix}\frac{31}{60}\\\frac{217}{60}\\\frac{93}{60}\\\frac{31}{60}\end{bmatrix} = \begin{bmatrix}\frac{1}{12}\\\frac{1}{84}\\-\frac{1}{28}\\-\frac{5}{84}\end{bmatrix},$$

so that

$$\mathbf{v} = \begin{bmatrix}\frac{1}{12}\\\frac{1}{84}\\-\frac{1}{28}\\-\frac{5}{84}\end{bmatrix} + \frac{1}{5}\mathbf{v}_1 + \frac{8}{7}\mathbf{v}_2 + \frac{31}{12}\mathbf{v}_3.$$

9. To find the column space of the matrix, row-reduce it:

$$\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

It follows that, since there are 1s in all three columns, the columns of the original matrix are linearly independent, so they form a basis for the column space. We must orthogonalize those three vectors.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{0\cdot 1 + 1\cdot 0 + 1\cdot 1}{0\cdot 0 + 1\cdot 1 + 1\cdot 1}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{x}_3 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_3)\mathbf{v}_1 - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{x}_3)\mathbf{v}_2\\ &= \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{0\cdot 1 + 1\cdot 1 + 1\cdot 0}{0\cdot 0 + 1\cdot 1 + 1\cdot 1}\begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1\cdot 1 - \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0}{1\cdot 1 - \frac{1}{2}\cdot\left(-\frac{1}{2}\right) + \frac{1}{2}\cdot\frac{1}{2}}\begin{bmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\\ &= \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}.\end{aligned}$$

10. To find the column space of the matrix, row-reduce it:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that, since there are 1s in all three columns, the columns of the original matrix are linearly independent, so they form a basis for the column space. We must orthogonalize those three vectors.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \mathrm{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix} - \frac{1\cdot 1 + 1\cdot(-1) - 1\cdot 1 + 1\cdot 5}{1\cdot 1 + 1\cdot 1 - 1\cdot(-1) + 1\cdot 1} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{x}_3 - \mathrm{proj}_{\mathbf{v}_1}(\mathbf{x}_3)\mathbf{v}_1 - \mathrm{proj}_{\mathbf{v}_2}(\mathbf{x}_3)\mathbf{v}_2 \\ &= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1\cdot 1 + 1\cdot 2 - 1\cdot 0 + 1\cdot 1}{1\cdot 1 + 1\cdot 1 - 1\cdot(-1) + 1\cdot 1} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{0\cdot 1 - 2\cdot 2 + 2\cdot 0 + 4\cdot 1}{0\cdot 0 - 2\cdot(-2) + 2\cdot 2 + 4\cdot 4} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \end{aligned}$$

11. The given vector, $\mathbf{x}_1$, together with $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$; we want to orthogonalize that basis.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \mathrm{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{3\cdot 0 + 1\cdot 1 + 5\cdot 0}{3^2 + 1^2 + 5^2} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{35} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{35} \\ \frac{34}{35} \\ -\frac{1}{7} \end{bmatrix}$$

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{x}_3 - \mathrm{proj}_{\mathbf{v}_1}(\mathbf{x}_3)\mathbf{v}_1 - \mathrm{proj}_{\mathbf{v}_2}(\mathbf{x}_3)\mathbf{v}_2 \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{3\cdot 0 + 1\cdot 0 + 5\cdot 1}{3^2 + 1^2 + 5^2} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} - \frac{-\frac{3}{35}\cdot 0 + \frac{34}{35}\cdot 0 - \frac{1}{7}\cdot 1}{\left(-\frac{3}{35}\right)^2 + \left(\frac{34}{35}\right)^2 + \left(\frac{1}{7}\right)^2} \begin{bmatrix} -\frac{3}{35} \\ \frac{34}{35} \\ -\frac{1}{7} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} + \frac{5}{34} \begin{bmatrix} -\frac{3}{35} \\ \frac{34}{35} \\ -\frac{1}{7} \end{bmatrix} = \begin{bmatrix} -\frac{15}{34} \\ 0 \\ \frac{9}{34} \end{bmatrix}. \end{aligned}$$

12. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

Note that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Now let

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

To see that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ form a basis for $\mathbb{R}^4$, row-reduce the matrix that has these vectors as its columns:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since there are 1s in all four columns, the column vectors are linearly independent, so they form a basis for $\mathbb{R}^4$. Next we need to orthogonalize that basis. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are already orthogonal, we start with $\mathbf{x}_3$:

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{11}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{66} \\ \frac{1}{3} \\ \frac{49}{66} \\ -\frac{3}{11} \end{bmatrix}$$

$$\begin{aligned} \mathbf{v}_4 &= \mathbf{x}_4 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_4}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_4}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{x}_4}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right)\mathbf{v}_3 \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{11}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} + \frac{18}{49}\begin{bmatrix} \frac{5}{66} \\ \frac{1}{3} \\ \frac{49}{66} \\ -\frac{3}{11} \end{bmatrix} = \begin{bmatrix} -\frac{12}{49} \\ \frac{6}{49} \\ 0 \\ \frac{4}{49} \end{bmatrix}. \end{aligned}$$

13. To start, we let $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ be the third column of the matrix. Then the determinant of this matrix is (for example, expanding along the second row) $\frac{1}{\sqrt{6}} \neq 0$, so the resulting matrix is invertible and therefore the three column vectors form a basis for $\mathbb{R}^3$. The first two vectors are orthogonal and each is of unit length, so we must orthogonalize the third vector as well:

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{q}_1 \cdot \mathbf{x}_3}{\mathbf{q}_1 \cdot \mathbf{q}_1}\right)\mathbf{q}_1 - \left(\frac{\mathbf{q}_2 \cdot \mathbf{x}_3}{\mathbf{q}_2 \cdot \mathbf{q}_2}\right)\mathbf{q}_2 \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{-\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} - \frac{\frac{1}{\sqrt{3}}}{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2}\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} - \frac{1}{\sqrt{3}}\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \end{bmatrix}. \end{aligned}$$

Then $\mathbf{v}_3$ is orthogonal to the other two columns of the matrix, but we must normalize $\mathbf{v}_3$ to convert it to a unit vector.

$$\|\mathbf{v}_3\| = \sqrt{\left(\frac{1}{6}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)^2} = \frac{1}{\sqrt{6}},$$

so that

$$\mathbf{q}_3 = \frac{1}{1/\sqrt{6}}\mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix},$$

so that the completed matrix Q is

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

14. To simplify the calculations, we will start with the multiples $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix}$ of the given basis vectors (column vectors of Q). We will find an orthogonal basis containing those two vectors and then normalize all four at the very end. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are already orthogonal. If we choose

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

then calculating the determinant of $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{bmatrix}$, for example by expanding along the last column, shows that the matrix is invertible, so that these four vectors form a basis for $\mathbb{R}^4$. We now need to orthogonalize the last two vectors:

$$\begin{aligned}
\mathbf{p}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 0}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 - 3 \cdot 0}{2 \cdot 2 + 1 \cdot 1 + 0 \cdot 0 - 3 \cdot (-3)}\begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.
\end{aligned}$$

Now let $\mathbf{v}_3 = 4\mathbf{p}_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$, again avoiding fractions. Next,

$$\begin{aligned}
\mathbf{v}_4 &= \mathbf{x}_4 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_4}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_4}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{x}_4}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right)\mathbf{v}_3 \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1\cdot 0 + 1\cdot 0 + 1\cdot 0 + 1\cdot 1}{1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2\cdot 0 + 1\cdot 0 + 0\cdot 0 - 3\cdot 1}{2\cdot 2 + 1\cdot 1 + 0\cdot 0 - 3\cdot(-3)}\begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix} \\
&\qquad - \frac{-1\cdot 0 - 1\cdot 0 + 3\cdot 0 - 1\cdot 1}{-1\cdot(-1) - 1\cdot(-1) + 3\cdot 3 - 1\cdot(-1)}\begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{14}\begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix} + \frac{1}{12}\begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{21} \\ -\frac{5}{42} \\ 0 \\ \frac{1}{42} \end{bmatrix}.
\end{aligned}$$

We now normalize $\mathbf{v}_1$ through $\mathbf{v}_4$; note that we already have normalized forms for the first two.

$$\begin{aligned}
\|\mathbf{v}_3\| &= \sqrt{1^2 + 1^2 + 3^2 + 1^2} = 2\sqrt{3} \\
\|\mathbf{v}_4\| &= \sqrt{\left(\frac{2}{21}\right)^2 + \left(-\frac{5}{42}\right)^2 + 0^2 + \left(\frac{1}{42}\right)^2} = \frac{1}{\sqrt{42}},
\end{aligned}$$

so that the matrix is

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{2}{\sqrt{14}} & -\frac{\sqrt{3}}{6} & \frac{4}{\sqrt{42}} \\ \frac{1}{2} & \frac{1}{\sqrt{14}} & -\frac{\sqrt{3}}{6} & -\frac{5}{\sqrt{42}} \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & -\frac{3}{\sqrt{14}} & -\frac{\sqrt{3}}{6} & \frac{1}{\sqrt{42}} \end{bmatrix}.$$

15. From Exercise 9, $\|\mathbf{v}_1\| = \sqrt{2}$, $\|\mathbf{v}_2\| = \frac{\sqrt{6}}{2}$, and $\|\mathbf{v}_3\| = \frac{2\sqrt{3}}{3}$, so that

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

Therefore

$$R = Q^T A = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}.$$

16. From Exercise 10, $\|\mathbf{v}_1\| = 2$, $\|\mathbf{v}_2\| = 2\sqrt{6}$, and $\|\mathbf{v}_3\| = \sqrt{2}$, so that

$$Q = \begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3\end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{6}}{6} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{6}}{6} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{\sqrt{6}}{3} & 0 \end{bmatrix}.$$

Therefore

$$R = Q^TA = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2\sqrt{6} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

17. We have

$$R = Q^TA = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & 8 & 2 \\ 1 & 7 & -1 \\ -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 9 & \frac{1}{3} \\ 0 & 6 & \frac{2}{3} \\ 0 & 0 & \frac{7}{3} \end{bmatrix}$$

18. We have

$$R = Q^TA = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{6} & 2\sqrt{6} \\ 0 & \sqrt{3} \end{bmatrix}.$$

19. If A is already orthogonal, let $Q = A$ and $R = I$, which is upper triangular; then $A = QR$.

20. First, if A is invertible, then A has linearly independent columns, so that by Theorem 5.16, $A = QR$ where R is invertible. Since R is upper triangular, its determinant is the product of its diagonal entries, so that R invertible implies that all of its diagonal entries are nonzero. For the converse, if R is upper triangular with all diagonal entries nonzero, then it has nonzero determinant; since Q is orthogonal, it is invertible so has nonzero determinant as well. But then $\det A = \det(QR) = (\det Q)(\det R) \neq 0$, so that A is invertible. Note that when this occurs,

$$A^{-1} = (QR)^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T.$$

21. From Exercise 20, $A^{-1} = R^{-1}Q^T$. To compute R^{-1}, we row-reduce $\begin{bmatrix}R \mid I\end{bmatrix}$ using Exercise 15:

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{6} \\ 0 & 1 & 0 & 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{\sqrt{3}}{2} \end{array}\right].$$

Thus

$$A^{-1} = R^{-1}Q^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{6} \\ 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{6} \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

22. From Exercise 20, $A^{-1} = R^{-1}Q^T$. To compute R^{-1}, we row-reduce $\begin{bmatrix}R \mid I\end{bmatrix}$ using Exercise 17:

$$\left[\begin{array}{ccc|ccc} 3 & 9 & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 6 & \frac{2}{3} & 0 & 1 & 0 \\ 0 & 0 & \frac{7}{3} & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{2}{21} \\ 0 & 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{21} \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{7} \end{array}\right].$$

Thus

$$A^{-1} = R^{-1}Q^T = \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{2}{21} \\ 0 & \frac{1}{6} & -\frac{1}{21} \\ 0 & 0 & \frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{42} & -\frac{2}{7} & -\frac{11}{21} \\ \frac{1}{42} & \frac{1}{7} & \frac{2}{21} \\ \frac{2}{7} & -\frac{2}{7} & \frac{1}{7} \end{bmatrix}.$$

23. If $A = QR$ is a QR-factorization of A, then we want to show that $R\mathbf{x} = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$ so that R is invertible by part (c) of the Fundamental Theorem. Note that since Q is orthogonal, so is Q^T, so that by Theorem 5.6(b), $\left\|Q^T\mathbf{x}\right\| = \|\mathbf{x}\|$ for any vector $\mathbf{x}$. Then

$$R\mathbf{x} = \mathbf{0} \Rightarrow \|R\mathbf{x}\| = 0 \Rightarrow \left\|(Q^TA)\mathbf{x}\right\| = \left\|Q^T(A\mathbf{x})\right\| = \|A\mathbf{x}\| = 0 \Rightarrow A\mathbf{x} = \mathbf{0}.$$

If $\mathbf{a}_i$ are the columns of A, then $A\mathbf{x} = \mathbf{0}$ means that $\sum \mathbf{a}_i x_i = \mathbf{0}$. But this means that all of the x_i are zero since the columns of A are linearly independent. Thus $\mathbf{x} = \mathbf{0}$. By part (c) of the Fundamental Theorem, we conclude that R is invertible.

24. Recall that $\operatorname{row}(A) = \operatorname{row}(B)$ if and only if there is an invertible matrix M such that $A = MB$. Then $A = QR$ implies that $A^T = R^TQ^T$; since A has linearly independent columns, R is invertible, so that R^T is as well. Thus $\operatorname{row}(A^T) = \operatorname{row}(Q^T)$, so that $\operatorname{col}(A) = \operatorname{col}(Q)$.

Exploration: The Modified QR Process

1. $I - 2\mathbf{u}\mathbf{u}^T = I - 2\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\begin{bmatrix} d_1 & d_2 \end{bmatrix} = I - \begin{bmatrix} 2d_1^2 & 2d_1d_2 \\ 2d_1d_2 & 2d_2^2 \end{bmatrix} = \begin{bmatrix} 1-2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1-2d_2^2 \end{bmatrix}.$

2. **(a)** From Exercise 1,

$$Q = \begin{bmatrix} 1-2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1-2d_2^2 \end{bmatrix} = \begin{bmatrix} 1-2\cdot\frac{9}{25} & -2\cdot\frac{3}{5}\cdot\frac{4}{5} \\ -2\cdot\frac{3}{5}\cdot\frac{4}{5} & 1-2\cdot\frac{16}{25} \end{bmatrix} = \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} \\ -\frac{24}{25} & -\frac{7}{25} \end{bmatrix}.$$

(b) Since $\|\mathbf{x} - \mathbf{y}\| = \sqrt{(5-1)^2 + (5-7)^2} = 2\sqrt{5}$, we have

$$\mathbf{u} = \frac{1}{2\sqrt{5}}(\mathbf{x} - \mathbf{y}) = \begin{bmatrix} \frac{4}{2\sqrt{5}} \\ -\frac{2}{2\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}.$$

Then from Exercise 1,

$$Q = \begin{bmatrix} 1-2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1-2d_2^2 \end{bmatrix} = \begin{bmatrix} 1-2\cdot\frac{4}{5} & -2\cdot\frac{2}{\sqrt{5}}\cdot\left(-\frac{1}{\sqrt{5}}\right) \\ -2\cdot\frac{2}{\sqrt{5}}\cdot\left(-\frac{1}{\sqrt{5}}\right) & 1-2\cdot\frac{1}{5} \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

3. **(a)** $Q^T = (I - 2\mathbf{u}\mathbf{u}^T)^T = I - 2\left(\mathbf{u}\mathbf{u}^T\right)^T = I - 2\left(\mathbf{u}^T\right)^T\mathbf{u}^T = I - 2\mathbf{u}\mathbf{u}^T = Q$, so that Q is symmetric.

(b) Start with

$$QQ^T = QQ = (I - 2\mathbf{u}\mathbf{u}^T)(I - 2\mathbf{u}\mathbf{u}^T) = I - 4\mathbf{u}\mathbf{u}^T + \left(-2\mathbf{u}\mathbf{u}^T\right)^2 = I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T.$$

Now, $\mathbf{u}^T\mathbf{u} = \mathbf{u}\cdot\mathbf{u} = 1$ since $\mathbf{u}$ is a unit vector, so the equation above becomes

$$QQ^T = I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T = I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T = I.$$

Thus $Q^T = Q^{-1}$ and it follows that Q is orthogonal.

(c) This follows from parts (a) and (b): $Q^2 = QQ = QQ^T = QQ^{-1} = I$.

4. First suppose that $\mathbf{v}$ is in the span of $\mathbf{u}$; say $\mathbf{v} = c\mathbf{u}$. Then again using the fact that $\mathbf{u}^T\mathbf{u} = \mathbf{u}\cdot\mathbf{u} = 1$,

$$Q\mathbf{v} = Q(c\mathbf{u}) = (I - 2\mathbf{u}\mathbf{u}^T)(c\mathbf{u}) = c(\mathbf{u} - 2\mathbf{u}\mathbf{u}^T\mathbf{u}) = c(\mathbf{u} - 2\mathbf{u}) = -c\mathbf{u} = -\mathbf{v}.$$

If $\mathbf{v}\cdot\mathbf{u} = \mathbf{u}^T\mathbf{v} = 0$, then

$$Q\mathbf{v} = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{v} = \mathbf{v} - 2\mathbf{u}\mathbf{u}^T\mathbf{v} = \mathbf{v}.$$

5. First normalize the given vector to get

$$\mathbf{u} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}.$$

Then

$$\mathbf{u}\mathbf{u}^T = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}, \text{ so that } Q = I - 2\mathbf{u}\mathbf{u}^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

For Exercise 3, clearly Q is symmetric, and each column has norm 1. A short computation shows that each pair of columns is orthogonal. Therefore Q is orthogonal. Finally,

$$Q^2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = I.$$

For Exercise 4, first suppose that $\mathbf{v} = c\mathbf{u}$. Then

$$Q\mathbf{v} = Q(c\mathbf{u}) = cQ\mathbf{u} = c\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} = c\begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix} = -c\mathbf{u} = -\mathbf{v}.$$

If $\mathbf{v}\cdot\mathbf{u} = \mathbf{u}^T\mathbf{v} = 0$, then

$$\begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\sqrt{6}}x_1 - \frac{1}{\sqrt{6}}x_2 + \frac{2}{\sqrt{6}}x_3 = 0,$$

so that $x_1 - x_2 + 2x_3 = 0$ and therefore $x_2 = x_1 + 2x_3$. Thus if $\mathbf{v}\cdot\mathbf{u} = 0$, then $\mathbf{v} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. But then

$$\begin{aligned} Q\mathbf{v} &= Q\left(s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right) \\ &= sQ\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + tQ\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \\ &= s\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \\ &= s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \mathbf{v}. \end{aligned}$$

6. $\mathbf{x}-\mathbf{y}$ is a multiple of $\mathbf{u}$, so that $Q(\mathbf{x}-\mathbf{y})=-(\mathbf{x}-\mathbf{y})$ by Exercise 4. Next,

$$(\mathbf{x}+\mathbf{y})\cdot\mathbf{u}=\frac{1}{\|\mathbf{x}-\mathbf{y}\|}(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}).$$

But by Exercise 61 in Section 1.2, this dot product is zero because $\|\mathbf{x}\|=\|\mathbf{y}\|$. Thus $\mathbf{x}+\mathbf{y}$ is orthogonal to $\mathbf{u}$, so that $Q(\mathbf{x}+\mathbf{y})=\mathbf{x}+\mathbf{y}$. So we have

$$\begin{array}{l}Q(\mathbf{x}-\mathbf{y})=-(\mathbf{x}-\mathbf{y})\\ Q(\mathbf{x}+\mathbf{y})=\mathbf{x}+\mathbf{y}\end{array}\;\Rightarrow\;\begin{array}{l}Q\mathbf{x}-Q\mathbf{y}=-\mathbf{x}+\mathbf{y}\\ Q\mathbf{x}+Q\mathbf{y}=\mathbf{x}+\mathbf{y}\end{array}\;\Rightarrow\;2Q\mathbf{x}=2\mathbf{y}\;\Rightarrow\;Q\mathbf{x}=\mathbf{y}.$$

7. We have

$$\mathbf{u}=\frac{1}{\|\mathbf{x}-\mathbf{y}\|}(\mathbf{x}-\mathbf{y})=\frac{1}{2\sqrt{3}}\begin{bmatrix}-2\\2\\2\end{bmatrix},$$

so that

$$\mathbf{u}\mathbf{u}^T=\begin{bmatrix}\frac13&-\frac13&-\frac13\\-\frac13&\frac13&\frac13\\-\frac13&\frac13&\frac13\end{bmatrix}\;\Rightarrow\;Q=I-2\mathbf{u}\mathbf{u}^T=\begin{bmatrix}\frac13&\frac23&\frac23\\\frac23&\frac13&-\frac23\\\frac23&-\frac23&\frac13\end{bmatrix}.$$

Then

$$Q\mathbf{x}=\begin{bmatrix}\frac13&\frac23&\frac23\\\frac23&\frac13&-\frac23\\\frac23&-\frac23&\frac13\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}3\\0\\0\end{bmatrix}=\mathbf{y}.$$

8. Since $\|\mathbf{y}\|=\sqrt{\|\mathbf{x}\|^2}=\|\mathbf{x}\|$, if $\mathbf{a}_i$ is the i^{th} column vector of A, we can apply Exercise 6 to get

$$\begin{aligned}Q_1A=\begin{bmatrix}Q_1\mathbf{a}_1&Q_1\mathbf{a}_2&\dots&Q_1\mathbf{a}_n\end{bmatrix}&=\begin{bmatrix}Q_1\mathbf{x}&Q_1\mathbf{a}_2&\dots&Q_1\mathbf{a}_n\end{bmatrix}\\&=\begin{bmatrix}\mathbf{y}&Q_1\mathbf{a}_2&\dots&Q_1\mathbf{a}_n\end{bmatrix}=\begin{bmatrix}*&*\\\mathbf{0}&A_1\end{bmatrix},\end{aligned}$$

since the last $m-1$ entries in $\mathbf{y}$ are all zero. Here A_1 is $(m-1)\times(n-1)$, the left-hand $*$ is 1×1, and the right-hand $*$ is $1\times(n-1)$.

9. Since P_2 is a Householder matrix, it is orthogonal, so that

$$Q_2Q_2^T=\begin{bmatrix}1&\mathbf{0}\\\mathbf{0}&P_2\end{bmatrix}\begin{bmatrix}1&\mathbf{0}\\\mathbf{0}&P_2^T\end{bmatrix}=I,$$

so that Q_2 is orthogonal as well. Further,

$$Q_2Q_1A=\begin{bmatrix}1&\mathbf{0}\\\mathbf{0}&P_2\end{bmatrix}\begin{bmatrix}*&*\\\mathbf{0}&A_1\end{bmatrix}=\begin{bmatrix}*&*\\\mathbf{0}&P_2A_1\end{bmatrix}=\begin{bmatrix}*&*&*\\0&*&*\\\mathbf{0}&\mathbf{0}&A_2\end{bmatrix}.$$

10. Using induction, suppose that

$$Q_kQ_{k-1}\cdots Q_1A=\begin{bmatrix}*&*&*\\\mathbf{0}&*&*\\\mathbf{0}&\mathbf{0}&A_k\end{bmatrix}.$$

Repeat Exercise 8 on the matrix A_k, giving a Householder matrix P_{k+1} such that

$$P_{k+1}A_k=\begin{bmatrix}*&*\\\mathbf{0}&A_{k+1}\end{bmatrix}.$$

Let $Q_{k+1} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P_{k+1} \end{bmatrix}$. Then using the fact that P_{k+1} is orthogonal, we again conclude that Q_{k+1} is orthogonal, and that

$$Q_{k+1}Q_kQ_{k-1}\cdots Q_1A = \begin{bmatrix} * & * & * \\ \mathbf{0} & * & * \\ \mathbf{0} & \mathbf{0} & A_{k+1} \end{bmatrix}.$$

The result then follows by induction. The resulting matrix $R = Q_{m-1}\cdots Q_2Q_1A$ is upper triangular by construction.

11. If $Q = Q_1Q_2\cdots Q_{m-1}$, then Q is orthogonal since each Q_i is orthogonal, and

$$Q^T = (Q_1Q_2\cdots Q_{m-1})^T = Q_{m-1}^T\cdots Q_2^TQ_1^T = Q_{m-1}\cdots Q_2Q_1.$$

Thus $Q^TA = Q_{m-1}\cdots Q_2Q_1A = R$, so that $QQ^TA = A = QR$, since $QQ^T = I$.

12. (a) We start with $\mathbf{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, so that $\mathbf{y} = \begin{bmatrix} \sqrt{3^2 + (-4)^2} = 5 \\ 0 \end{bmatrix}$. Then

$$\mathbf{u} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \Rightarrow \mathbf{u}\mathbf{u}^T = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}.$$

Then

$$Q_1 = I - 2\mathbf{u}\mathbf{u}^T = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix} \Rightarrow Q_1A = \begin{bmatrix} 5 & 3 & -1 \\ 0 & -9 & -2 \end{bmatrix} = R,$$

and $A = Q_1^TR$.

(b) We start with $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, so that $\mathbf{y} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$. Then

$$\mathbf{u} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \Rightarrow \mathbf{u}\mathbf{u}^T = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Thus

$$Q_1 = I - 2\mathbf{u}\mathbf{u}^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \Rightarrow Q_1A = \begin{bmatrix} 3 & -5 & 1 & \frac{8}{3} \\ 0 & 4 & 3 & \frac{1}{3} \\ 0 & 3 & 1 & \frac{4}{3} \end{bmatrix}.$$

Then

$$A_1 = \begin{bmatrix} 4 & 3 & \frac{1}{3} \\ 3 & 1 & \frac{4}{3} \end{bmatrix},$$

so that in the next iteration

$$\mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \Rightarrow \mathbf{y} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \Rightarrow \mathbf{u} = \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}.$$

Then

$$\mathbf{u}\mathbf{u}^T = \begin{bmatrix} \frac{1}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{9}{10} \end{bmatrix}.$$

So

$$P_2 = I - 2\mathbf{u}\mathbf{u}^T = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{bmatrix} \Rightarrow Q_2 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{bmatrix}.$$

Finally,

$$Q_2Q_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{bmatrix} \begin{bmatrix} 3 & -5 & 1 & \frac{8}{3} \\ 0 & 4 & 3 & \frac{1}{3} \\ 0 & 3 & 1 & \frac{4}{3} \end{bmatrix} = \begin{bmatrix} 3 & -5 & 1 & \frac{8}{3} \\ 0 & 5 & 3 & \frac{16}{15} \\ 0 & 0 & 1 & -\frac{13}{15} \end{bmatrix} = R,$$

and $Q_1Q_2R = A$.

Exploration: Approximating Eigenvalues with the QR Algorithm

1. We have

$$A_1 = RQ = (Q^{-1}Q)RQ = Q^{-1}(QR)Q = Q^{-1}AQ,$$

so that $A_1 \sim A$. Thus by Theorem 4.22(e), A_1 and A have the same eigenvalues.

2. Using Gram-Schmidt, we have

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \frac{1 \cdot 0 + 1 \cdot 3}{1 \cdot 1 + 1 \cdot 1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}.$$

Normalizing these vectors gives

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \text{ so that } R = Q^TA = \begin{bmatrix} \sqrt{2} & \frac{3\sqrt{2}}{2} \\ 0 & -\frac{3\sqrt{2}}{2} \end{bmatrix}.$$

Thus

$$A_1 = RQ = \begin{bmatrix} \sqrt{2} & \frac{3\sqrt{2}}{2} \\ 0 & -\frac{3\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix}.$$

Then

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)$$

$$\det(A_1 - \lambda I) = \begin{vmatrix} \frac{5}{2}-\lambda & -\frac{1}{2} \\ -\frac{3}{2} & \frac{3}{2}-\lambda \end{vmatrix} = \left(\frac{5}{2}-\lambda\right)\left(\frac{3}{2}-\lambda\right) - \frac{3}{4} = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3).$$

So A and A_1 do indeed have the same eigenvalues.

3. The proof is by induction on k. We know that $A_1 \sim A$. Now assume $A_{n-1} \sim A$. Then

$$A_n = R_{n-1}Q_{n-1} = (Q_{n-1}^{-1}Q_{n-1})R_{n-1}Q_{n-1} = Q_{n-1}^{-1}(Q_{n-1}R_{n-1})Q_{n-1} = Q_{n-1}^{-1}A_{n-1}Q_{n-1},$$

so that $A_n \sim A_{n-1} \sim A$.

4. Converting $A_1 = Q_1R_1$ to decimals gives, using technology

$$A_1 = Q_1R_1 \approx \begin{bmatrix} 0.86 & 0.51 \\ -0.51 & 0.86 \end{bmatrix} \begin{bmatrix} 2.92 & -1.20 \\ 0 & 1.04 \end{bmatrix} \Rightarrow A_2 = R_1Q_1 \approx \begin{bmatrix} 3.12 & 0.47 \\ -0.53 & 0.88 \end{bmatrix}$$

$$A_2 = Q_2R_2 \approx \begin{bmatrix} -0.99 & 0.17 \\ 0.17 & 0.99 \end{bmatrix} \begin{bmatrix} -3.16 & -0.32 \\ 0 & 0.95 \end{bmatrix} \Rightarrow A_3 = R_2Q_2 \approx \begin{bmatrix} 3.06 & -0.84 \\ 0.16 & 0.93 \end{bmatrix}$$

$$A_3 = Q_3R_3 \approx \begin{bmatrix} -1.00 & -0.05 \\ -0.05 & 1.00 \end{bmatrix} \begin{bmatrix} -3.06 & 0.79 \\ 0 & 0.97 \end{bmatrix} \Rightarrow A_4 = R_3Q_3 \approx \begin{bmatrix} 3.02 & 0.94 \\ -0.05 & 0.97 \end{bmatrix}$$

$$A_4 = Q_4R_4 \approx \begin{bmatrix} -1.00 & 0.02 \\ 0.02 & 1.00 \end{bmatrix} \begin{bmatrix} -3.02 & -0.92 \\ 0 & 0.99 \end{bmatrix} \Rightarrow A_5 = R_4Q_4 \approx \begin{bmatrix} 3.00 & -0.98 \\ 0.02 & 0.99 \end{bmatrix}.$$

The A_k are approaching an upper triangular matrix U.

5. Since $A_k \sim A$ for all k, the eigenvalues of A and A_k are the same, so in the limit, the upper triangular matrix U will have eigenvalues equal to those of A; but the eigenvalues of a triangular matrix are its diagonal entries. Thus the diagonal entries of U will be the eigenvalues of A.

6. **(a)** Follow the same process as in Exercise 4.

$$A = QR \approx \begin{bmatrix} 0.71 & 0.71 \\ 0.71 & -0.71 \end{bmatrix} \begin{bmatrix} 2.83 & 2.83 \\ 0 & 1.41 \end{bmatrix} \Rightarrow A_1 = RQ \approx \begin{bmatrix} 4.02 & 0.00 \\ 1.00 & -1.00 \end{bmatrix}$$

$$A_1 = Q_1R_1 \approx \begin{bmatrix} -0.97 & -0.24 \\ -0.24 & 0.97 \end{bmatrix} \begin{bmatrix} -4.14 & 0.24 \\ 0 & -0.97 \end{bmatrix} \Rightarrow A_2 = R_1Q_1 \approx \begin{bmatrix} 3.96 & 1.23 \\ 0.23 & -0.94 \end{bmatrix}$$

$$A_2 = Q_2R_2 \approx \begin{bmatrix} -1.00 & -0.06 \\ -0.06 & 1.00 \end{bmatrix} \begin{bmatrix} -3.97 & -1.17 \\ 0 & -1.01 \end{bmatrix} \Rightarrow A_3 = R_2Q_2 \approx \begin{bmatrix} 4.04 & -0.93 \\ 0.06 & -1.01 \end{bmatrix}$$

$$A_3 = Q_3R_3 \approx \begin{bmatrix} -1.00 & -0.01 \\ -0.01 & 1.00 \end{bmatrix} \begin{bmatrix} -4.04 & 0.94 \\ 0 & -1.00 \end{bmatrix} \Rightarrow A_4 = R_3Q_3 \approx \begin{bmatrix} 4.03 & 0.98 \\ 0.01 & -1.00 \end{bmatrix}$$

$$A_4 = Q_4R_4 \approx \begin{bmatrix} -1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \begin{bmatrix} -4.03 & -0.98 \\ 0 & -1.00 \end{bmatrix} \Rightarrow A_5 = R_4Q_4 \approx \begin{bmatrix} 4.03 & -0.98 \\ 0.00 & -1.00 \end{bmatrix}.$$

The eigenvalues of A are approximately 4.03 and -1.00.

(b) Follow the same process as in Exercise 4.

$$A = QR \approx \begin{bmatrix} 0.45 & 0.89 \\ 0.89 & -0.45 \end{bmatrix} \begin{bmatrix} 2.24 & 1.34 \\ 0 & 0.45 \end{bmatrix} \Rightarrow A_1 = RQ \approx \begin{bmatrix} 2.20 & 1.40 \\ 0.40 & -0.20 \end{bmatrix}$$

$$A_1 = Q_1R_1 \approx \begin{bmatrix} -0.98 & -0.18 \\ -0.18 & 0.98 \end{bmatrix} \begin{bmatrix} -2.24 & -1.34 \\ 0 & -0.45 \end{bmatrix} \Rightarrow A_2 = R_1Q_1 \approx \begin{bmatrix} 2.44 & -0.92 \\ 0.08 & -0.44 \end{bmatrix}$$

$$A_2 = Q_2R_2 \approx \begin{bmatrix} -1.00 & -0.03 \\ -0.03 & 1.00 \end{bmatrix} \begin{bmatrix} -2.44 & 0.93 \\ 0 & -0.41 \end{bmatrix} \Rightarrow A_3 = R_2Q_2 \approx \begin{bmatrix} 2.41 & 1.01 \\ 0.01 & -0.41 \end{bmatrix}$$

$$A_3 = Q_3R_3 \approx \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2.41 & -1.01 \\ 0 & -0.42 \end{bmatrix} \Rightarrow A_4 = R_3Q_3 \approx \begin{bmatrix} 2.42 & -1 \\ 0 & -0.42 \end{bmatrix}$$

$$A_4 = Q_4R_4 \approx \begin{bmatrix} -1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \begin{bmatrix} -2.42 & 1 \\ 0 & -0.41 \end{bmatrix} \Rightarrow A_5 = R_4Q_4 \approx \begin{bmatrix} 2.41 & 1 \\ 0 & -0.41 \end{bmatrix}.$$

The eigenvalues of A are approximately 2.41 and -0.41 (the true values are $1 \pm \sqrt{2}$).

(c) Follow the same process as in Exercise 4.

$$A = QR \approx \begin{bmatrix} 0.24 & -0.06 & -0.97 \\ 0.24 & 0.97 & 0 \\ -0.94 & 0.23 & -0.24 \end{bmatrix} \begin{bmatrix} 4.24 & 0.47 & -0.94 \\ 0 & 1.94 & 1.26 \\ 0 & 0 & 0.73 \end{bmatrix} \Rightarrow$$

$$A_1 = RQ \approx \begin{bmatrix} 2.00 & 0 & -3.89 \\ -0.73 & 2.18 & -0.31 \\ -0.69 & 0.17 & -0.18 \end{bmatrix}$$

$$A_1 = Q_1R_1 \approx \begin{bmatrix} -0.89 & -0.33 & 0.30 \\ 0.33 & -0.94 & -0.07 \\ 0.31 & 0.03 & 0.95 \end{bmatrix} \begin{bmatrix} -2.24 & 0.76 & 3.32 \\ 0 & -2.05 & 1.57 \\ 0 & 0 & -1.31 \end{bmatrix} \Rightarrow$$

$$A_2 = R_1Q_1 \approx \begin{bmatrix} 3.27 & 0.13 & 2.44 \\ -0.18 & 1.98 & 1.64 \\ -0.40 & -0.04 & -1.25 \end{bmatrix}$$

$$A_2 = Q_2R_2 \approx \begin{bmatrix} -0.99 & -0.05 & 0.12 \\ 0.06 & -1.00 & 0.01 \\ 0.12 & 0.02 & 0.99 \end{bmatrix} \begin{bmatrix} -3.3 & -0.03 & -2.47 \\ 0 & -1.99 & -1.8 \\ 0 & 0 & -1.31 \end{bmatrix} \Rightarrow$$

$$A_3 = R_2Q_2 \approx \begin{bmatrix} 2.96 & 0.16 & -2.86 \\ -0.33 & 1.95 & -1.81 \\ -0.11 & -0.02 & -0.91 \end{bmatrix}$$

$$A_3 = Q_3R_3 \approx \begin{bmatrix} -0.99 & -0.11 & 0.04 \\ 0.11 & -0.99 & 0.01 \\ 0.04 & 0.01 & 1.00 \end{bmatrix} \begin{bmatrix} -2.98 & 0.06 & 2.61 \\ 0 & -1.95 & 2.10 \\ 0 & 0 & -1.03 \end{bmatrix} \Rightarrow$$

$$A_4 = R_3Q_3 \approx \begin{bmatrix} 3.07 & 0.30 & 2.49 \\ -0.14 & 1.96 & 2.09 \\ -0.04 & -0.01 & -1.03 \end{bmatrix}$$

$$A_4 = Q_4R_4 \approx \begin{bmatrix} -1 & -0.04 & 0.01 \\ 0.04 & -1 & 0 \\ 0.01 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3.07 & -0.21 & -2.41 \\ 0 & -1.97 & -2.20 \\ 0 & 0 & -0.99 \end{bmatrix} \Rightarrow$$

$$A_5 = R_4Q_4 \approx \begin{bmatrix} 3.03 & 0.34 & -2.45 \\ -0.12 & 1.96 & -2.21 \\ -0.01 & 0 & -0.99 \end{bmatrix}.$$

The eigenvalues of A are approximately 3.03, 1.96, and -0.99. (The true values are 3, 2, and -1).

(d) Follow the same process as in Exercise 4.

$$A = QR \approx \begin{bmatrix} 0.45 & 0.72 \\ 0 & 0.60 \\ -0.89 & 0.36 \end{bmatrix} \begin{bmatrix} 2.24 & -3.13 & -2.24 \\ 0 & 3.35 & 0 \end{bmatrix} \Rightarrow A_1 = RQ \approx \begin{bmatrix} 3.00 & -1.07 \\ 0 & 2.00 \end{bmatrix}$$

$$A_1 = Q_1R_1 \approx \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \begin{bmatrix} 3.00 & -1.07 \\ 0 & 2.00 \end{bmatrix} \Rightarrow A_2 = R_1Q_1 \approx \begin{bmatrix} 3.00 & -1.07 \\ 0 & 2.00 \end{bmatrix}.$$

Since Q_1 is the identity matrix, we get $Q_2 = I$ and $R_2 = R_1$, so that successive A_i are equal to A_1. Thus the eigenvalues of A are approximately 3 and 2 (in fact, they are 3, 2, and 0).

7. Following the process in Exercise 4,

$$A = QR = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \frac{8}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \Rightarrow A_1 = RQ = \begin{bmatrix} \frac{2}{5} & -\frac{21}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

$$A_1 = Q_1R_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{8}{\sqrt{5}} \\ 0 & \sqrt{5} \end{bmatrix} \Rightarrow A_2 = R_1Q_1 \approx \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = A.$$

Here $A_2 = A$ ($Q_1 = Q^{-1}$ and $R_1 = R^{-1}$); the algorithm fails because the eigenvalues of A, which are ± 1, do not have distinct absolute values.

8. Set

$$B = A + 0.9I = \begin{bmatrix} 2.9 & 3 \\ -1 & -1.1 \end{bmatrix}.$$

Then

$$B = QR \approx \begin{bmatrix} -0.95 & 0.33 \\ 0.33 & 0.95 \end{bmatrix} \begin{bmatrix} -3.07 & -3.19 \\ 0 & -0.06 \end{bmatrix} \Rightarrow B_1 = RQ \approx \begin{bmatrix} 1.86 & -4.02 \\ -0.02 & -0.06 \end{bmatrix}$$

$$B_1 = Q_1R_1 \approx \begin{bmatrix} -1 & 0.01 \\ 0.01 & 1 \end{bmatrix} \begin{bmatrix} -1.86 & 4.02 \\ 0 & -0.1 \end{bmatrix} \Rightarrow B_2 = R_1Q_1 \approx \begin{bmatrix} 1.9 & 4 \\ 0 & -0.1 \end{bmatrix}$$

$$B_2 = Q_2R_2 \approx \begin{bmatrix} -1.00 & 0 \\ 0 & 1.00 \end{bmatrix} \begin{bmatrix} -1.9 & -4 \\ 0 & -0.1 \end{bmatrix} \Rightarrow B_3 = R_2Q_2 \approx \begin{bmatrix} 1.9 & -4 \\ 0 & -0.1 \end{bmatrix}.$$

Further iterations will simply reverse the sign of the upper right entry of B_i. The eigenvalues of B are approximately 1.9 and -0.1, so the eigenvalues of A are approximately $1.9 - 0.9 = 1.0$ and $-0.1 - 0.9 = -1.0$. These are correct (see Exercise 7).

9. We prove the first equality using induction on k. For $k = 1$, it says that $Q_0A_1 = AQ_0$, or $QA_1 = AQ$. Now, $A_1 = RQ$ and $A = QR$, so that $QA_1 = QRQ = (QR)Q = AQ$. This proves the basis of the induction. Assume the equality holds for $k = n$; we prove it for $k = n + 1$:

$$\begin{aligned} Q_0Q_1Q_2\cdots Q_nA_{n+1} &= Q_0Q_1Q_2\cdots Q_{n-1}(Q_nA_{n+1}) \\ &= Q_0Q_1Q_2\cdots Q_{n-1}(Q_n(R_nQ_n)) \\ &= Q_0Q_1Q_2\cdots Q_{n-1}(Q_nR_n)Q_n \\ &= (Q_0Q_1Q_2\cdots Q_{n-1}A_n)Q_n \\ &= (AQ_0Q_1Q_2\cdots Q_{n-1})Q_n \\ &= AQ_0Q_1Q_2\cdots Q_n. \end{aligned}$$

For the second equality, note that $A_k = Q_kR_k$, so that

$$Q_0Q_1Q_2\cdots Q_{k-1}A_k = Q_0Q_1Q_2\cdots Q_{k-1}Q_kR_k = AQ_0Q_1Q_2\cdots Q_{k-1}.$$

Multiply both sides by $R_{k-1}\cdots R_2R_1$ to get

$$(Q_0Q_1Q_2\cdots Q_{k-1}Q_k)(R_kR_{k-1}\cdots R_2R_1) = A(Q_0Q_1Q_2\cdots Q_{k-1})(R_{k-1}\cdots R_2R_1),$$

as desired.

For the QR-decomposition of A^{k+1}, we again use induction. For $k = 0$, the equation says that $Q_0R_0 = A^{0+1} = A$, which is true. Now assume the statement holds for $k = n$. Then using the first equation in this exercise together with the fact that $A_k = Q_kR_k$ and the inductive hypothesis, we get

$$\begin{aligned} A^{n+1} &= A\cdot A^n \\ &= A(Q_0Q_1\cdots Q_{n-1})(R_{n-1}\cdots R_1R_0) \\ &= Q_0Q_1\cdots Q_{n-1}A_nR_{n-1}\cdots R_1R_0 \\ &= Q_0Q_1\cdots Q_{n-1}Q_nR_nR_{n-1}\cdots R_1R_0 \\ &= (Q_0Q_1\cdots Q_n)(R_nR_{n-1}\cdots R_1R_0), \end{aligned}$$

as desired.

5.4 Orthogonal Diagonalization of Symmetric Matrices

1. The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 15 = (\lambda-5)(\lambda-3)$$

Thus the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 5$. To find the eigenspace corresponding to $\lambda_1 = 3$, we row-reduce

$$\begin{bmatrix} A-3I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda_2 = 5$, we row-reduce

$$\begin{bmatrix} A-5I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

These eigenvectors are orthogonal, as is guaranteed by Theorem 5.19, so to turn them into orthonormal vectors, we normalize them. The norm of each vector is $\sqrt{2}$, so setting

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

we see that Q is an orthonormal matrix whose columns are the normalized eigenvectors for A, so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = D.$$

2. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 3 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 8 = (\lambda+4)(\lambda-2).$$

Thus the eigenvalues of A are $\lambda_1 = -4$ and $\lambda_2 = 2$. To find the eigenspace corresponding to $\lambda_1 = -4$, we row-reduce

$$\begin{bmatrix} A + 4I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda_2 = 2$, we row-reduce

$$\begin{bmatrix} A - 2I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

These eigenvectors are orthogonal, as is guaranteed by Theorem 5.19, so to turn them into orthonormal vectors, we normalize them. The norm of each vector is $\sqrt{2}$, so setting

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

we see that Q is an orthonormal matrix whose columns are the normalized eigenvectors for A, so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} = D.$$

3. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & \sqrt{2} \\ \sqrt{2} & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

Thus the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$. To find the eigenspace corresponding to $\lambda_1 = -1$, we row-reduce

$$\begin{bmatrix} A + I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda_2 = 2$, we row-reduce

$$\begin{bmatrix} A - 2I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -1 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$.

These eigenvectors are orthogonal, as is guaranteed by Theorem 5.19, so to turn them into orthonormal vectors, we normalize them. The norm of each vector is $\sqrt{3}$, so setting

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{3} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

we see that Q is an orthonormal matrix whose columns are the normalized eigenvectors for A, so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = D.$$

4. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} = \lambda^2 - 15\lambda + 50 = (\lambda - 5)(\lambda - 10).$$

Thus the eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = 10$. To find the eigenspace corresponding to $\lambda_1 = 5$, we row-reduce

$$\left[A - 5I \mid 0\right] = \left[\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda_2 = 10$, we row-reduce

$$\left[A - 10I \mid 0\right] = \left[\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

These eigenvectors are orthogonal, as is guaranteed by Theorem 5.19, so to turn them into orthonormal vectors, we normalize them. The norm of each vector is $\sqrt{5}$, so setting

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

we see that Q is an orthonormal matrix whose columns are the normalized eigenvectors for A, so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} = D.$$

5. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 3 & 1-\lambda \end{vmatrix} = -\lambda^3 + 7\lambda^2 - 2\lambda - 40 = -(\lambda - 4)(\lambda - 5)(\lambda + 2).$$

Thus the eigenvalues of A are $\lambda_1 = 4$, $\lambda_2 = 5$, and $\lambda_3 = -2$. To find the eigenspace corresponding to $\lambda_1 = 4$, we row-reduce

$$\left[A - 4I \mid 0\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

To find the eigenspace corresponding to $\lambda_2 = 5$, we row-reduce

$$\begin{bmatrix}A-5I \mid 0\end{bmatrix} = \left[\begin{array}{rrr|r}0 & 0 & 0 & 0\\ 0 & -4 & 3 & 0\\ 0 & 3 & -4 & 0\end{array}\right] \to \left[\begin{array}{rrr|r}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

To find the eigenspace corresponding to $\lambda_3 = -2$, we row-reduce

$$\begin{bmatrix}A+2I \mid 0\end{bmatrix} = \left[\begin{array}{rrr|r}7 & 0 & 0 & 0\\ 0 & 3 & 3 & 0\\ 0 & 3 & 3 & 0\end{array}\right] \to \left[\begin{array}{rrr|r}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix}0\\-1\\1\end{bmatrix}$.

These eigenvectors are orthogonal, as is guaranteed by Theorem 5.19, so to turn them into orthonormal vectors, we normalize them. The norm of the first and third vectors is $\sqrt{2}$, while the second is already a unit vector, so setting

$$Q = \begin{bmatrix}0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}$$

we see that Q is an orthonormal matrix whose columns are the normalized eigenvectors for A, so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix}4 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -2\end{bmatrix} = D.$$

6. The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix}2-\lambda & 3 & 0\\ 3 & 2-\lambda & 4\\ 0 & 4 & 2-\lambda\end{vmatrix} = -\lambda^3 + 6\lambda^2 + 13\lambda - 42 = -(\lambda+3)(\lambda-2)(\lambda-7).$$

Thus the eigenvalues of A are $\lambda_1 = -3$, $\lambda_2 = 2$, and $\lambda_3 = 7$. To find the eigenspace corresponding to $\lambda_1 = -3$, we row-reduce

$$\begin{bmatrix}A+3I \mid 0\end{bmatrix} = \left[\begin{array}{rrr|r}5 & 3 & 0 & 0\\ 3 & 5 & 4 & 0\\ 0 & 4 & 5 & 0\end{array}\right] \to \left[\begin{array}{rrr|r}1 & 0 & -\frac{3}{4} & 0\\ 0 & 1 & \frac{5}{4} & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

A basis for the eigenspace is given (after clearing fractions) by $\begin{bmatrix}3\\-5\\4\end{bmatrix}$.

To find the eigenspace corresponding to $\lambda_2 = 2$, we row-reduce

$$\begin{bmatrix}A-2I \mid 0\end{bmatrix} = \left[\begin{array}{rrr|r}0 & 3 & 0 & 0\\ 3 & 0 & 4 & 0\\ 0 & 4 & 0 & 0\end{array}\right] \to \left[\begin{array}{rrr|r}1 & 0 & \frac{4}{3} & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

A basis for the eigenspace is given (after clearing fractions) by $\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda_3 = 7$, we row-reduce

$$\begin{bmatrix} A - 7I \mid 0 \end{bmatrix} = \left[\begin{array}{rrr|r} -5 & 3 & 0 & 0 \\ 3 & -5 & 4 & 0 \\ 0 & 4 & -5 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{3}{4} & 0 \\ 0 & 1 & -\frac{5}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given (after clearing fractions) by $\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$.

These eigenvectors are orthogonal, as is guaranteed by Theorem 5.19, so to turn them into orthonormal vectors, we normalize them. The norms of these vectors are respectively $5\sqrt{2}$, 5, and $5\sqrt{2}$, so setting

$$Q = \begin{bmatrix} \frac{3}{5\sqrt{2}} & -\frac{4}{5} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{3}{5} & \frac{4}{5\sqrt{2}} \end{bmatrix}$$

we see that Q is an orthonormal matrix whose columns are the normalized eigenvectors for A, so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix} = D.$$

7. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda-1)(\lambda-2).$$

Thus the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$. To find the eigenspace corresponding to $\lambda_1 = 0$, we row-reduce

$$\begin{bmatrix} A - 0I \mid 0 \end{bmatrix} = \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda_2 = 1$, we row-reduce

$$\begin{bmatrix} A - I \mid 0 \end{bmatrix} = \left[\begin{array}{rrr|r} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

To find the eigenspace corresponding to $\lambda_3 = 2$, we row-reduce

$$\begin{bmatrix} A - 2I \mid 0 \end{bmatrix} = \left[\begin{array}{rrr|r} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

A basis for the eigenspace is given by $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

These eigenvectors are orthogonal, as is guaranteed by Theorem 5.19, so to turn them into orthonormal vectors, we normalize them. The norm of the first and third vectors is $\sqrt{2}$, while the second is already a unit vector, so setting

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

we see that Q is an orthonormal matrix whose columns are the normalized eigenvectors for A, so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D.$$

8. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = -(\lambda+1)^2(\lambda-5)$$

Thus the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 5$. To find the eigenspace E_{-1} corresponding to $\lambda_1 = -1$, we row-reduce

$$\begin{bmatrix} A + I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the eigenspace E_{-1} is

$$\begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \operatorname{span}\left(\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

To find the eigenspace E_5 corresponding to $\lambda_2 = 5$, we row-reduce

$$\begin{bmatrix} A - 5I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus a basis for E_5 is $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

E_5 is orthogonal to E_{-1} as guaranteed by Theorem 10.19, so to turn this basis into an orthogonal basis we must orthogonalize the basis for E_{-1} above. Use Gram-Schmidt:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

We double $\mathbf{v}_2$ to remove fractions, so we get the orthogonal basis

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The norms of these vectors are, respectively, $\sqrt{2}$, $\sqrt{6}$, and $\sqrt{3}$, so normalizing each vector and setting Q equal to the matrix whose columns are the normalized eigenvectors gives

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D.$$

9. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 1 & 1-\lambda \end{vmatrix} = \lambda^4 - 4\lambda^3 + 4\lambda^2 = \lambda^2(\lambda-2)^2.$$

Thus the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 2$. To find the eigenspace E_0 corresponding to $\lambda_1 = 0$, we row-reduce

$$\left[A + 0I \mid 0\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the eigenspace E_0 is

$$\begin{bmatrix} -s \\ s \\ -t \\ t \end{bmatrix} = \operatorname{span}\left(\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right).$$

To find the eigenspace E_2 corresponding to $\lambda_2 = 2$, we row-reduce

$$\left[A - 2I \mid 0\right] = \left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the eigenspace E_2 is

$$\begin{bmatrix} s \\ s \\ t \\ t \end{bmatrix} = \operatorname{span}\left(\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right).$$

E_0 is orthogonal to E_2 as guaranteed by Theorem 10.19, and $\mathbf{x}_1$ and $\mathbf{x}_2$ are also orthogonal, as are $\mathbf{x}_3$ and $\mathbf{x}_4$. So these four vectors form an orthogonal basis. The norm of each vector is $\sqrt{2}$, so normalizing each vector and setting Q equal to the matrix whose columns are the normalized eigenvectors gives

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = D.$$

10. The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & 2-\lambda \end{vmatrix} = \lambda^4 - 6\lambda^3 + 12\lambda^2 - 10\lambda + 3 = (\lambda - 1)^3(\lambda - 3)$$

Thus the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. To find the eigenspace E_1 corresponding to $\lambda_1 = 1$, we row-reduce

$$\begin{bmatrix} AI \mid 0 \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the eigenspace E_1 is

$$\begin{bmatrix} -s \\ t \\ u \\ s \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right).$$

Note that these vectors are mutually orthogonal.

To find the eigenspace E_3 corresponding to $\lambda_2 = 3$, we row-reduce

$$\begin{bmatrix} A - 3I \mid 0 \end{bmatrix} = \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus a basis for E_3 is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Since the eigenvectors of E_1 are orthogonal to those of E_3 by Theorem 10.19, and the eigenvectors of E_3 are already mutually orthogonal, all we need to do is to normalize the vectors. The norms are, respectively, $\sqrt{2}$, 1, 1, and $\sqrt{2}$, so setting Q equal to the matrix whose columns are the normalized eigenvectors gives

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix},$$

so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = D.$$

11. The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (a - \lambda)^2 - b^2.$$

This is zero when $a - \lambda = \pm b$, so the eigenvalues are $a + b$ and $a - b$. To find the corresponding eigenspaces, row-reduce:

$$\begin{aligned}\left[A-(a+b)I \mid 0\right] &= \left[\begin{array}{cc|c} -b & b & 0 \\ b & -b & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ \left[A-(a-b)I \mid 0\right] &= \left[\begin{array}{cc|c} b & b & 0 \\ b & b & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].\end{aligned}$$

So bases for the two eigenspaces (which are orthogonal by Theorem 5.19) are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Each of these vectors has norm $\sqrt{2}$, so normalizing each gives

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}.$$

12. The characteristic polynomial is (expanding along the second row)

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & 0 & b \\ 0 & a-\lambda & 0 \\ b & 0 & a-\lambda \end{vmatrix} = (a-\lambda)\left((a-\lambda)^2 - b^2\right).$$

This is zero when $a = \lambda$ or when $a - \lambda = \pm b$, so the eigenvalues are a, $a + b$ and $a - b$. To find the corresponding eigenspaces, row-reduce:

$$\begin{aligned}\left[A-aI \mid 0\right] &= \left[\begin{array}{ccc|c} 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \left[A-(a+b)I \mid 0\right] &= \left[\begin{array}{ccc|c} -b & 0 & b & 0 \\ 0 & -b & 0 & 0 \\ b & 0 & -b & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \left[A-(a-b)I \mid 0\right] &= \left[\begin{array}{ccc|c} b & 0 & b & 0 \\ 0 & b & 0 & 0 \\ b & 0 & b & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].\end{aligned}$$

So bases for the three eigenspaces (which are orthogonal by Theorem 5.19) are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Normalizing the second and third vectors gives

$$Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

so that

$$Q^{-1}AQ = Q^TAQ = \begin{bmatrix} a & 0 & 0 \\ 0 & a+b & 0 \\ 0 & 0 & a-b \end{bmatrix}.$$

13. (a) Since A and B are each orthogonally diagonalizable, they are each symmetric by the Spectral Theorem. Therefore $A + B$ is also symmetric, so it too is orthogonally diagonalizable by the Spectral Theorem.

(b) Since A is orthogonally diagonalizable, it is symmetric by the Spectral Theorem. Therefore cA is also symmetric, so it too is orthogonally diagonalizable by the Spectral Theorem.

(c) Since A is orthogonally diagonalizable, it is symmetric by the Spectral Theorem. Therefore $A^2 = AA$ is also symmetric, so it too is orthogonally diagonalizable by the Spectral Theorem.

14. If A is invertible and orthogonally diagonalizable, then $Q^TAQ = D$ for some invertible diagonal matrix D and orthogonal matrix Q, so that

$$D^{-1} = \left(Q^TAQ\right)^{-1} = Q^{-1}A^{-1}\left(Q^T\right)^{-1} = Q^TA^{-1}\left(Q^T\right)^T = Q^TA^{-1}Q.$$

Thus A^{-1} is also orthogonally diagonalizable (and by the same orthogonal matrix).

15. Since A and B are orthogonally diagonalizable, they are both symmetric. Since $AB = BA$, Exercise 36 in Section 3.2 implies that AB is also symmetric, so it is orthogonally diagonalizable as well.

16. Let A be a symmetric matrix. Then it is orthogonally diagonalizable, say $D = Q^TAQ$. Then the diagonal entries of D are the eigenvalues of A. First, if every eigenvalue of A is nonnegative, let

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \text{ and define } \sqrt{D} = \begin{bmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}.$$

Let $B = Q\sqrt{D}Q^T$. Then B is diagonally orthogonalizable by construction, so it is symmetric. Further, since Q is orthogonal,

$$B^2 = Q\sqrt{D}Q^TQ\sqrt{D}Q^T = Q\sqrt{D}Q^{-1}Q\sqrt{D}Q^T = Q\sqrt{D}\sqrt{D}Q^T = QDQ^T = A.$$

Conversely, suppose $A = B^2$ for some symmetric matrix B. Then B is orthogonally diagonalizable, say $Q^TBQ = D$, so that

$$Q^TAQ = Q^TB^2Q = (Q^TBQ)(Q^TBQ) = D^2.$$

But D^2 is a diagonal matrix whose diagonal entries are the squares of the diagonal entries of D, so they are all nonnegative. Thus the eigenvalues of A are all nonnegative.

17. From Exercise 1 we have

$$\lambda_1 = 3, \quad \mathbf{q}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \lambda_2 = 5, \quad \mathbf{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Therefore

$$A = \lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T = 3\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 5\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

18. From Exercise 2 we have

$$\lambda_1 = -4, \quad \mathbf{q}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \lambda_2 = 2, \quad \mathbf{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Therefore

$$A = \lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T = -4\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 2\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

19. From Exercise 5 we have

$$\lambda_1 = 4, \quad \mathbf{q}_1 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \lambda_2 = 5, \quad \mathbf{q}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \lambda_3 = -2, \quad \mathbf{q}_2 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Therefore

$$A = \lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T + \lambda_3\mathbf{q}_3\mathbf{q}_3^T = 4\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

20. From Exercise 8 we have

$$\lambda_1 = -1, \quad \mathbf{q}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \qquad \lambda_2 = 5, \quad \mathbf{q}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Therefore

$$A = \lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_1\mathbf{q}_2\mathbf{q}_2^T + \lambda_2\mathbf{q}_3\mathbf{q}_3^T = -\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} + 5\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

21. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are already orthogonal, normalize each of them to get

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Now let Q be the matrix whose columns are $\mathbf{q}_1$ and $\mathbf{q}_2$; then the desired matrix is

$$Q\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}Q^{-1} = Q\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

22. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are already orthogonal, normalize each of them to get

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \qquad \mathbf{q}_2 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}.$$

Now let Q be the matrix whose columns are $\mathbf{q}_1$ and $\mathbf{q}_2$; then the desired matrix is

$$Q\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}Q^{-1} = Q\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}Q^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\frac{9}{5} & \frac{12}{5} \\ \frac{12}{5} & \frac{9}{5} \end{bmatrix}.$$

23. Note that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are mutually orthogonal, with norms $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$ respectively. Let Q be the matrix whose columns are $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$, and $\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$; then the desired matrix is

$$\begin{aligned} Q\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}Q^{-1} &= Q\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}Q^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{5}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{8}{3} \end{bmatrix}. \end{aligned}$$

24. Note that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are mutually orthogonal, with norms $\sqrt{42}$, $\sqrt{3}$, and $\sqrt{14}$ respectively. Let Q be the matrix whose columns are $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$, and $\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$; then the desired matrix is

$$\begin{aligned}
Q\begin{bmatrix}1 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{bmatrix}Q^{-1} &= Q\begin{bmatrix}1 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{bmatrix}Q^T\\
&= \begin{bmatrix}\frac{4}{\sqrt{42}} & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}}\\ \frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{14}}\\ -\frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{14}}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{bmatrix}\begin{bmatrix}\frac{4}{\sqrt{42}} & \frac{5}{\sqrt{42}} & -\frac{1}{\sqrt{42}}\\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{14}} & \frac{3}{\sqrt{14}}\end{bmatrix}\\
&= \begin{bmatrix}-\frac{44}{21} & \frac{50}{21} & -\frac{10}{21}\\ \frac{50}{21} & -\frac{43}{42} & -\frac{25}{42}\\ -\frac{10}{21} & -\frac{25}{42} & -\frac{163}{42}\end{bmatrix}.
\end{aligned}$$

25. Since $\mathbf{q}$ is a unit vector, $\mathbf{q}\cdot\mathbf{q} = 1$, so that

$$\operatorname{proj}_W \mathbf{v} = \left(\frac{\mathbf{q}\cdot\mathbf{v}}{\mathbf{q}\cdot\mathbf{q}}\right)\mathbf{q} = (\mathbf{q}\cdot\mathbf{v})\mathbf{q} = \mathbf{q}(\mathbf{q}\cdot\mathbf{v}) = \mathbf{q}(\mathbf{q}^T\mathbf{v}) = (\mathbf{q}\mathbf{q}^T)\mathbf{v}.$$

26. **(a)** By definition (see Section 5.2), the projection of $\mathbf{v}$ onto W is

$$\begin{aligned}
\operatorname{proj}_W \mathbf{v} &= \left(\frac{\mathbf{q}_1\cdot\mathbf{v}}{\mathbf{q}_1\cdot\mathbf{q}_1}\right)\mathbf{q}_1 + \cdots + \left(\frac{\mathbf{q}_k\cdot\mathbf{v}}{\mathbf{q}_k\cdot\mathbf{q}_k}\right)\mathbf{q}_k\\
&= (\mathbf{q}_1\cdot\mathbf{v})\mathbf{q}_1 + \cdots + (\mathbf{q}_k\cdot\mathbf{v})\mathbf{q}_k\\
&= \mathbf{q}_1(\mathbf{q}_1\cdot\mathbf{v}) + \cdots + \mathbf{q}_k(\mathbf{q}_k\cdot\mathbf{v})\\
&= \mathbf{q}_1(\mathbf{q}_1^T\mathbf{v}) + \cdots + \mathbf{q}_k(\mathbf{q}_k^T\cdot\mathbf{v})\\
&= (\mathbf{q}_1\mathbf{q}_1^T + \cdots + \mathbf{q}_k\mathbf{q}_k^T)\mathbf{v}.
\end{aligned}$$

Therefore the matrix of the projection is indeed $\mathbf{q}_1\mathbf{q}_1^T + \cdots + \mathbf{q}_k\mathbf{q}_k^T$.

(b) First,

$$\begin{aligned}
P^T = (\mathbf{q}_1\mathbf{q}_1^T + \cdots + \mathbf{q}_k\mathbf{q}_k^T)^T &= \left(\mathbf{q}_1\mathbf{q}_1^T\right)^T + \cdots + \left(\mathbf{q}_k\mathbf{q}_k^T\right)^T\\
&= \left(\mathbf{q}_1^T\right)^T\mathbf{q}_1^T + \cdots + \left(\mathbf{q}_k^T\right)^T\mathbf{q}_k = \mathbf{q}_1\mathbf{q}_1^T + \cdots + \mathbf{q}_k\mathbf{q}_k^T = P.
\end{aligned}$$

Next, since $\mathbf{q}_i\cdot\mathbf{q}_j = 0$ for $i \neq j$, and $\mathbf{q}_i\cdot\mathbf{q}_i = \mathbf{q}_i^T\mathbf{q}_i = 1$, we have

$$\begin{aligned}
P^2 &= (\mathbf{q}_1\mathbf{q}_1^T)(\mathbf{q}_1\mathbf{q}_1^T) + \cdots + (\mathbf{q}_k\mathbf{q}_k^T)(\mathbf{q}_k\mathbf{q}_k^T)\\
&= \mathbf{q}_1\left(\mathbf{q}_1^T\mathbf{q}_1\right)\mathbf{q}_1^T + \cdots + \mathbf{q}_k\left(\mathbf{q}_k^T\mathbf{q}_k\right)\mathbf{q}_k^T = \mathbf{q}_1\mathbf{q}_1^T + \cdots \mathbf{q}_k\mathbf{q}_k^T = P.
\end{aligned}$$

(c) We have

$$QQ^T = \begin{bmatrix}\mathbf{q}_1 & \cdots & \mathbf{q}_k\end{bmatrix}\begin{bmatrix}\mathbf{q}_1^T\\ \vdots\\ \mathbf{q}_k^T\end{bmatrix} = \mathbf{q}_1\mathbf{q}_1^T + \cdots \mathbf{q}_k\mathbf{q}_k^T = P.$$

Thus $\operatorname{rank}(P) = \operatorname{rank}(QQ^T) = \operatorname{rank}(Q)$ by Theorem 3.28 in Section 3.5. But if $\mathbf{q}_1, \ldots, \mathbf{q}_k$ is an orthonormal basis, then $\operatorname{rank}(Q) = k$, so that $\operatorname{rank}(P) = k$ as well.

27. Suppose the eigenvalues are $\lambda_1, \ldots, \lambda_n$ and that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a corresponding orthonormal set of eigenvectors. Let $Q_1 = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$. Then

$$Q_1^T A Q_1 = \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} A \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \begin{bmatrix} A\mathbf{v}_1 & \cdots & A\mathbf{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \begin{bmatrix} \lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & * \\ \mathbf{0} & A_1 \end{bmatrix} = T,$$

where A_1 is an $(n-1) \times (n-1)$ matrix. The result now follows by induction since T is block upper triangular.

28. We know that if A is nilpotent, then its only eigenvalue is 0. So if A is nilpotent, then all of its eigenvalues are real, so that by Exercise 27 there is an orthogonal matrix Q such that Q^TAQ is upper triangular. But A and Q^TAQ have the same eigenvalues; since the eigenvalues of Q^TAQ are the diagonal entries, all the diagonal entries must be zero.

5.5 Applications

1. $f(\mathbf{x}) = \mathbf{x}^TA\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} = 2x^2 + 6xy + 4y^2.$

2. $f(\mathbf{x}) = \mathbf{x}^TA\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = 5x_1^2 + 2x_1x_2 - x_2^2.$

3. $f(\mathbf{x}) = \mathbf{x}^TA\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = 3 \cdot 1^2 - 4 \cdot 1 \cdot 6 + 4 \cdot 6^2 = 123.$

4. $f(\mathbf{x}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 1 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = 1x^2 + 2y^2 + 3z^2 + 0xy - 6xz + 2yz = x^2 + 2y^2 + 3z^2 - 6xz + 2yz.$

5. From Exercise 4,

$$f\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) = 2^2 + 2 \cdot (-1)^2 + 3 \cdot 1^2 - 6 \cdot 2 \cdot 1 + 2 \cdot (-1) \cdot 1 = -5.$$

6. $f(\mathbf{x}) = \mathbf{x}^TA\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = 2 \cdot 1^2 + 0 \cdot 2^2 + 1 \cdot 3^2 + 4 \cdot 1 \cdot 2 + 0 \cdot 1 \cdot 3 + 2 \cdot 2 \cdot 3 = 31.$

7. The diagonal entries are 1 and 2, and the corners are each $\frac{1}{2} \cdot 6 = 3$, so $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$.

8. The diagonal entries are both zero, and the corners are each $\frac{1}{2} \cdot 1 = \frac{1}{2}$, so $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$.

9. The diagonal entries are 3 and -1, and the corners are each $\frac{1}{2} \cdot (-3) = -\frac{3}{2}$, so $A = \begin{bmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix}$.

10. The diagonal entries are 1, 0, and -1, while the off-diagonal entries are $\frac{1}{2} \cdot 8 = 4$, 0, and $\frac{1}{2} \cdot (-6) = -3$. Thus

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix}.$$

11. The diagonal entries are 5, -1, and 2, while the off-diagonal entries are $\frac{1}{2}\cdot 2 = 1$, $\frac{1}{2}\cdot(-4) = -2$, and $\frac{1}{2}\cdot 4 = 2$. Thus

$$A = \begin{bmatrix} 5 & 1 & -2 \\ 1 & -1 & 2 \\ -2 & 2 & 2 \end{bmatrix}.$$

12. The diagonal entries are 2, -3, and 1, while the off-diagonal entries are $\frac{1}{2}\cdot 0 = 0$, $\frac{1}{2}\cdot(-4) = -2$, and $\frac{1}{2}\cdot 0 = 0$. Thus

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & -3 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

13. The matrix of this form is $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6).$$

So the eigenvalues are 1 and 6; finding the eigenvectors by the usual method gives corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Normalizing these and constructing Q gives

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}.$$

The new quadratic form is

$$f(\mathbf{y}) = \mathbf{y}^T D\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1^2 + 6y_2^2.$$

14. The matrix of this form is $A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3).$$

So the eigenvalues are 5 and -3; finding the eigenvectors by the usual method gives corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Normalizing these and constructing Q gives

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

The new quadratic form is

$$f(\mathbf{y}) = \mathbf{y}^T D\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 5y_1^2 - 3y_2^2.$$

15. The matrix of this form is $A = \begin{bmatrix} 7 & 4 & 4 \\ 4 & 1 & -8 \\ 4 & -8 & 1 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 7-\lambda & 4 & 4 \\ 4 & 1-\lambda & -8 \\ 4 & -8 & 1-\lambda \end{vmatrix} = (\lambda - 9)^2(\lambda + 9).$$

So the eigenvalues are 9 (of algebraic multiplicity 2) and -9; finding the eigenvectors by the usual method gives corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ for $\lambda = 9$ and $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ for $\lambda = -9$. Orthogonalizing $\mathbf{v}_1$ and $\mathbf{v}_2$ gives

$$\mathbf{v}_2' = \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ 1 \\ -\frac{4}{5} \end{bmatrix}.$$

Normalizing these and constructing Q gives

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & -\frac{1}{3} \\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3} \end{bmatrix}.$$

The new quadratic form is

$$f(\mathbf{y}) = \mathbf{y}^T D\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 9y_1^2 + 9y_2^2 - 9y_3^2.$$

16. The matrix of this form is $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (\lambda - 3)^2(\lambda + 1).$$

So the eigenvalues are 3 (of algebraic multiplicity 2) and -1; finding the eigenvectors by the usual method gives corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for $\lambda = 3$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ for $\lambda = -1$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are already orthogonal, we normalize all three vectors and construct Q to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}.$$

The new quadratic form is

$$f(\mathbf{y}) = \mathbf{y}^T D\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 + 3y_2^2 - y_3^2.$$

17. The matrix of this form is $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 1).$$

So the eigenvalues are 2, 1, and -1; finding the eigenvectors by the usual method gives corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Normalize all three vectors and construct Q to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}.$$

The new quadratic form is

$$f(\mathbf{x}') = (\mathbf{x}')^T D\mathbf{x}' = \begin{bmatrix} x' & y' & z' \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = 2(x')^2 + (y')^2 - (z')^2.$$

18. The matrix of this form is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (\lambda - 2)(\lambda + 1)^2.$$

So the eigenvalues are 2 and -1; finding the eigenvectors by the usual method gives corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for $\lambda = 2$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ for $\lambda = -1$. The latter two are already orthogonal, so simply normalize all three vectors and construct Q to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

The new quadratic form is

$$f(\mathbf{x}') = (\mathbf{x}')^T D\mathbf{x}' = \begin{bmatrix} x' & y' & z' \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = 2(x')^2 - (y')^2 - (z')^2.$$

19. The matrix of this form is $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda).$$

Since the eigenvalues, 1 and 2, are both positive, this is a positive definite form.

20. The matrix of this form is $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda(\lambda - 2).$$

Since the eigenvalues, 0 and 2, are both nonnegative, but one is zero, this is a positive semidefinite form.

21. The matrix of this form is $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+3)(\lambda+1).$$

Since the eigenvalues, -1 and -3, are both negative, this is a negative definite form.

22. The matrix of this form is $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda-3)(\lambda+1).$$

Since the eigenvalues, -1 and 3, have opposite signs, this is an indefinite form.

23. The matrix of this form is $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = -(\lambda-1)^2(\lambda-4).$$

Since both eigenvalues, 1 and 4, are positive, this is a positive definite form.

24. The matrix of this form is $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = -\lambda(\lambda-1)(\lambda-2).$$

Since the eigenvalues, 0, 1, and 2, are all nonnegative, but one of them is zero, this is a positive semidefinite form.

25. The matrix of this form is $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = -(\lambda+1)^2(\lambda-3).$$

Since the eigenvalues, -1 and 3, have opposite signs, this is an indefinite form.

26. The matrix of this form is $A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$, so that its characteristic polynomial is

$$\begin{vmatrix} -1-\lambda & -1 & -1 \\ -1 & -1-\lambda & -1 \\ -1 & -1 & -1-\lambda \end{vmatrix} = -\lambda^2(\lambda+3)$$

Since the eigenvalues, 0 and -3, are both nonpositive but one of them is zero, this is a negative semidefinite form.

27. By the Principal Axes Theorem, if A is the matrix of $f(\mathbf{x})$, then there is an orthogonal matrix Q such that $Q^TAQ = D$ is a diagonal matrix, and then if $\mathbf{y} = Q^T\mathbf{x}$, we have

$$f(\mathbf{x}) = \mathbf{x}^TA\mathbf{x} = \mathbf{y}^TD\mathbf{y} = \lambda_1y_1^2 + \cdots + \lambda_ny_n^2.$$

- For 5.22(a), $f(\mathbf{x})$ is positive definite if and only if $\lambda_1y_1^2 + \cdots + \lambda_ny_n^2 > 0$ for all $y_1, \ldots, y_n$. But this happens if and only if $\lambda_i > 0$ for all i, since if some $\lambda_i \le 0$, take $y_i = 1$ and all other $y_j = 0$ to get a value for $f(\mathbf{x})$ that is not positive.
- For 5.22(b), $f(\mathbf{x})$ is positive semidefinite if and only if $\lambda_1y_1^2 + \cdots + \lambda_ny_n^2 \ge 0$ for all $y_1, \ldots, y_n$. But this happens if and only if $\lambda_i \ge 0$ for all i, since if some $\lambda_i < 0$, take $y_i = 1$ and all other $y_j = 0$ to get a value for $f(\mathbf{x})$ that is negative.
- For 5.22(c), $f(\mathbf{x})$ is negative semidefinite if and only if $\lambda_1y_1^2 + \cdots + \lambda_ny_n^2 < 0$ for all $y_1, \ldots, y_n$. But this happens if and only if $\lambda_i < 0$ for all i, since if some $\lambda_i \ge 0$, take $y_i = 1$ and all other $y_j = 0$ to get a value for $f(\mathbf{x})$ that is nonnegative.
- For 5.22(d), $f(\mathbf{x})$ is negative semidefinite if and only if $\lambda_1y_1^2 + \cdots + \lambda_ny_n^2 \le 0$ for all $y_1, \ldots, y_n$. But this happens if and only if $\lambda_i \le 0$ for all i, since if some $\lambda_i > 0$, take $y_i = 1$ and all other $y_j = 0$ to get a value for $f(\mathbf{x})$ that is positive.
- For 5.22(e), $f(\mathbf{x})$ is indefinite if and only if $\lambda_1y_1^2 + \cdots + \lambda_ny_n^2$ can be either negative or positive. But from the arguments above, we see that this can happen if and only if some λ_i is positive and some λ_j is negative.

28. The given matrix represents the form $ax^2 + 2bxy + dy^2$, which can be written in the form given in the hint:

$$ax^2 + 2bxy + dy^2 = a\left(x + \frac{b}{a}y\right)^2 + \left(d - \frac{b^2}{a}\right)y^2.$$

This is a positive definite form if and only if the right-hand side is positive for all values of x and y not both zero. First suppose that $a > 0$ and $\det A = ad - b^2 > 0$. Then $ad > b^2$ so that $d > \frac{b^2}{a}$ (using the fact that $a > 0$), and thus $d - \frac{b^2}{a} > 0$. Therefore the first term on the right-hand side is always positive if either x or y is nonzero since $a > 0$ and nonzero squares are always positive, while the second term is always nonnegative since the constant factor is positive. Thus the form is positive definite. Next assume the form is positive definite; we must show $a > 0$ and $\det A > 0$. By way of contradiction, assume $a \le 0$; then setting $x = 1$ and $y = 0$ we get $a \le 0$ for the value of the form, which contradicts the assumption that it is positive definite. Thus $a > 0$. Next assume that $\det A = ad - b^2 \le 0$; then (since $a > 0$) $d \le \frac{b^2}{a}$. Now setting $x = -\frac{b}{a}$ and $y = 1$ gives $d - \frac{b^2}{a} \le 0$ for the value of the form, which is again a contradiction. Thus $a > 0$ and $\det A > 0$.

29. Since $A = B^TB$, we have $\mathbf{x}^TA\mathbf{x} = \mathbf{x}^TB^TB\mathbf{x} = (B\mathbf{x})^T(B\mathbf{x}) = \|B\mathbf{x}\|^2 \ge 0$. Now, if $\mathbf{x}^TA\mathbf{x} = 0$, then $\|B\mathbf{x}\|^2 = 0$. But since B is invertible, this implies that $\mathbf{x} = \mathbf{0}$. Therefore $\mathbf{x}^TA\mathbf{x} > 0$ for all $\mathbf{x} \ne \mathbf{0}$, so that A is positive definite.

30. Since A is symmetric and positive definite, we can find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$, where

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \text{ where } \lambda_i > 0 \text{ for all } i.$$

Let

$$C = C^T = \begin{bmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix};$$

then $D = C^2 = C^TC$. This gives

$$A = QDQ^T = QC^TCQ^T = (CQ^T)^TCQ^T.$$

Since all the λ_i are positive, C is invertible; since Q is orthogonal, it is invertible. Thus $B = CQ^T$ is invertible and we are done.

31. **(a)** $\mathbf{x}^T(cA)\mathbf{x} = c\left(\mathbf{x}^T A\mathbf{x}\right) > 0$ because $c > 0$ and A is positive definite.

(b) $\mathbf{x}^T A^2\mathbf{x} = \left(\mathbf{x}^T A\mathbf{x}\right)\left(\mathbf{x}^T A\mathbf{x}\right) = \left(\mathbf{x}^T A\mathbf{x}\right)^2 > 0.$

(c) $\mathbf{x}^T(A+B)\mathbf{x} = \left(\mathbf{x}^T A\mathbf{x}\right) + \left(\mathbf{x}^T B\mathbf{x}\right) > 0$ since both A and B are positive definite.

(d) First note that A is invertible by Exercise 30, since it factors as $A = B^T B$ where B is invertible. Since A is positive definite, its eigenvalues $\{\lambda_i\}$ are positive; the eigenvalues of A^{-1} are $\left\{\frac{1}{\lambda_1}\right\}$, which are also positive. Thus A^{-1} is positive definite.

32. From Exercise 30, we have $A = QC^TCQ^T$, where Q is orthogonal and C is a diagonal matrix. But then $C^T = C$, so that $A = QCCQ^T = QCQ^TQCQ^T = (QCQ^T)(QCQ^T) = B^2$. B is symmetric since $B = QCQ^T$ is orthogonally diagonalizable, and it is positive definite since its eigenvalues are the diagonal entries of C, which are all positive.

33. The eigenvalues of this form, from Exercise 20, are $\lambda_1 = 2$ and $\lambda_2 = 0$, so that by Theorem 5.23, the maximum value of f subject to the constraint $\|\mathbf{x}\| = 1$ is 2 and the minimum value is 0. These occur at the unit eigenvectors of A, which are found by row-reducing:

$$\begin{aligned}\left[A - 2I \mid 0\right] &= \left[\begin{array}{rr|r} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ \left[A - 0I \mid 0\right] &= \left[\begin{array}{rr|r} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].\end{aligned}$$

So normalized eigenvectors are

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Therefore the maximum value of 2 occurs at $\mathbf{x} = \pm\mathbf{v}_1$, and the minimum of 0 occurs at $\mathbf{x} = \pm\mathbf{v}_2$.

34. The eigenvalues of this form, from Exercise 22, are $\lambda_1 = 3$ and $\lambda_2 = -1$, so that by Theorem 5.23, the maximum value of f subject to the constraint $\|\mathbf{x}\| = 1$ is 3 and the minimum value is -1. These occur at the unit eigenvectors of A, which are found by row-reducing:

$$\begin{aligned}\left[A - 3I \mid 0\right] &= \left[\begin{array}{rr|r} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ \left[A + I \mid 0\right] &= \left[\begin{array}{rr|r} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].\end{aligned}$$

So normalized eigenvectors are

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Therefore the maximum value of 3 occurs at $\mathbf{x} = \pm\mathbf{v}_1$, and the minimum of -1 occurs at $\mathbf{x} = \pm\mathbf{v}_2$.

35. The eigenvalues of this form, from Exercise 23, are $\lambda_1 = 4$ and $\lambda_2 = 1$, so that by Theorem 5.23, the maximum value of f subject to the constraint $\|\mathbf{x}\| = 1$ is 4 and the minimum value is 1. These occur at the unit eigenvectors of A, which are found by row-reducing:

$$\begin{aligned}\left[A - 4I \mid 0\right] &= \left[\begin{array}{rrr|r} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \left[A - I \mid 0\right] &= \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].\end{aligned}$$

So normalized eigenvectors are

$$\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix},$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\\ 0\end{bmatrix}, \qquad \mathbf{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{bmatrix}.$$

Therefore the maximum value of 4 occurs at $\mathbf{x} = \pm\mathbf{v}_1$, and the minimum of 1 occurs at $\mathbf{x} = \pm\mathbf{v}_2$ as well as at $\mathbf{x} = \pm\mathbf{v}_3$.

36. The eigenvalues of this form, from Exercise 24, are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = 0$, so that by Theorem 5.23, the maximum value of f subject to the constraint $\|\mathbf{x}\| = 1$ is 2 and the minimum value is 0. These occur at the corresponding unit eigenvectors of A, which are found by row-reducing:

$$\begin{bmatrix}A - 2I \mid 0\end{bmatrix} = \left[\begin{array}{ccc|c}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\begin{bmatrix}A - 0I \mid 0\end{bmatrix} = \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

So normalized eigenvectors are

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\end{bmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{bmatrix}.$$

Therefore the maximum value of 2 occurs at $\mathbf{x} = \pm\mathbf{v}_1$, and the minimum of 0 occurs at $\mathbf{x} = \pm\mathbf{v}_2$.

37. Since $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, we have

$$f(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \ge \lambda_n(y_1^2 + y_2^2 + \cdots + y_n^2) = \lambda_n \|\mathbf{y}\|^2 = \lambda_n$$

since we are assuming the constraint $\|\mathbf{y}\| = 1$.

38. If $\mathbf{q}_n$ is a unit eigenvector corresponding to λ_n, then $A\mathbf{q}_n = \lambda_n\mathbf{q}_n$, so that

$$f(\mathbf{q}_n) = \mathbf{q}_n^T A\mathbf{q}_n = \mathbf{q}_n^T \lambda_n \mathbf{q}_n = \lambda_n\left(\mathbf{q}_n^T\mathbf{q}_n\right) = \lambda_n.$$

Thus the quadratic form actually takes the value λ_n, so by property (a) of Theorem 5.23, it must be the minimum value of $f(\mathbf{x})$, and it occurs when $\mathbf{x} = \mathbf{q}_n$.

39. Divide this equation through by 25 to get

$$\frac{x^2}{25} + \frac{y^2}{5} = 1,$$

which is an ellipse, from Figure 5.15.

40. Divide through by 4 and rearrange terms to get

$$\frac{x^2}{4} - \frac{y^2}{4} = 1,$$

which is a hyperbola, from Figure 5.15.

41. Rearrange terms to get

$$y = x^2 - 1,$$

which is a parabola, from Figure 5.15.

42. Divide through by 8 and rearrange terms to get

$$\frac{x^2}{4} + \frac{y^2}{8} = 1,$$

which is an ellipse, from Figure 5.15.

43. Rearrange terms to get

$$y^2 - 3x^2 = 1,$$

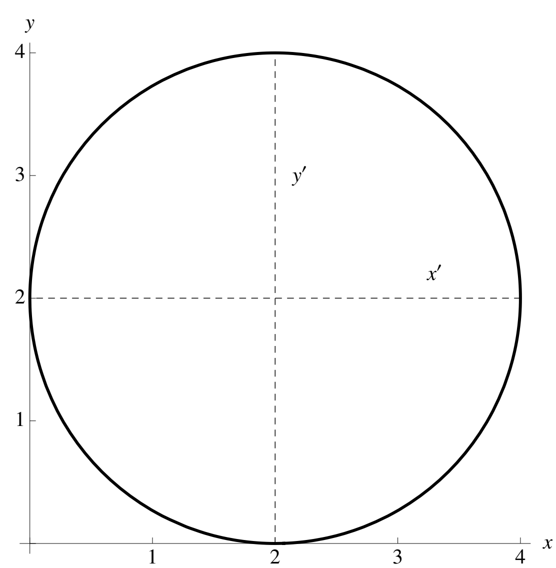
which is a hyperbola, from Figure 5.15.

44. This is a parabola, from Figure 5.15.

45. Group the x and y terms and then complete the square, giving

$$(x^2 - 4x + 4) + (y^2 - 4x + 4) = -4 + 4 + 4 \quad \Rightarrow (x-2)^2 + (y-2)^2 = 4.$$

This is a circle of radius 2 centered at $(2, 2)$. Letting $x' = x-2$ and $y' = y-2$, we have $(x')^2 + (y')^2 = 4$.



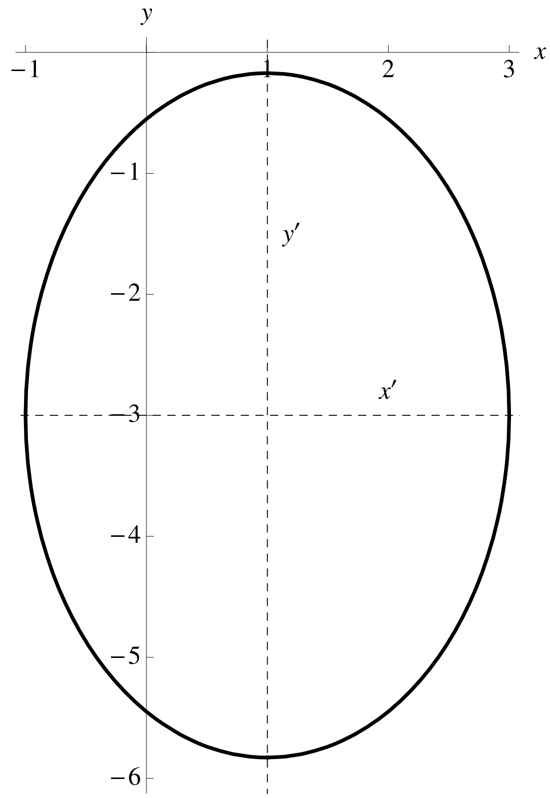
46. Group the x and y terms and simplify:

$$(4x^2 - 8x) + (2y^2 + 12y) = -6 \quad \Rightarrow \quad 4(x^2 - 2x) + 2(y^2 + 6y) = -6.$$

Now complete the square and simplify again:

$$\begin{gathered} 4(x^2 - 2x + 1) + 2(y^2 + 6y + 9) = -6 + 4 + 18 \quad \Rightarrow \\ 4(x-1)^2 + 2(y+3)^2 = 16 \quad \Rightarrow \\ \frac{(x-1)^2}{4} + \frac{(y+3)^2}{8} = 1. \end{gathered}$$

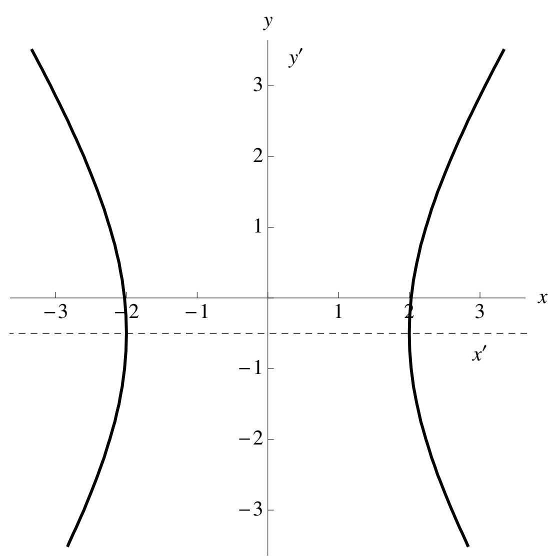
This is an ellipse centered at $(1, -3)$. Letting $x' = x - 1$ and $y' = y + 3$, we have $\frac{(x')^2}{4} + \frac{(y')^2}{8} = 1$.



47. Group the x and y terms to get $9x^2 - 4(y^2 + y) = 37$. Now complete the square and simplify:

$$9x^2 - 4\left(y^2 + y + \frac{1}{4}\right) = 37 - 1 \quad \Rightarrow \quad 9x^2 - 4\left(y + \frac{1}{2}\right)^2 = 36 \quad \Rightarrow \quad \frac{x^2}{4} - \frac{\left(y + \frac{1}{2}\right)^2}{9} = 1.$$

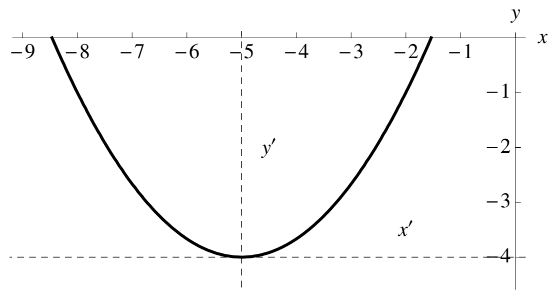
This is a hyperbola centered at $\left(0, -\frac{1}{2}\right)$. Letting $x' = x$ and $y' = y + \frac{1}{2}$, we have $\frac{(x')^2}{4} - \frac{(y')^2}{9} = 1$.



48. Rearrange terms to get $3y - 13 = x^2 + 10x$; then complete the square:

$$3y - 13 + 25 = x^2 + 10x + 25 = (x+5)^2 \quad \Rightarrow \quad 3y + 12 = (x+5)^2 \quad \Rightarrow \quad y + 4 = \frac{1}{3}(x+5)^2.$$

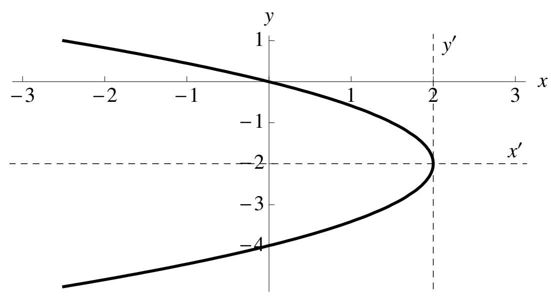
This is a parabola. Letting $x' = x + 5$ and $y' = y + 4$, we have $y' = \frac{1}{3}(x')^2$.



49. Rearrange terms to get $4x = -2y^2 - 8y$; then divide by 2, giving $2x = -y^2 - 4y$. Finally, complete the square:

$$2x - 4 = -(y^2 + 4y + 4) = -(y + 2)^2 \quad \Rightarrow \quad x - 2 = -\frac{1}{2}(y + 2)^2.$$

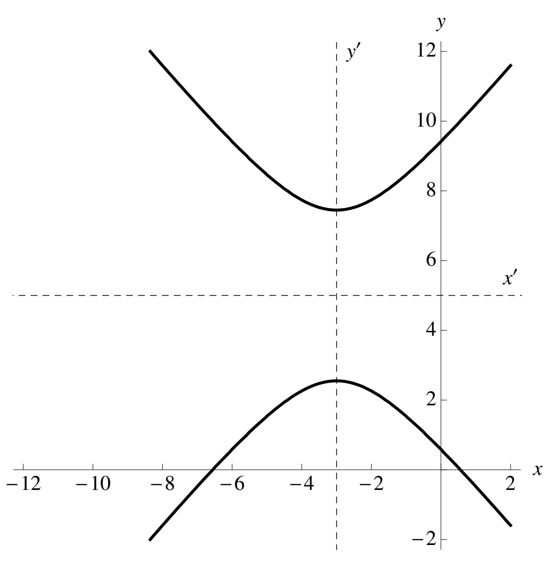
This is a parabola. Letting $x' = x - 2$ and $y' = y + 2$, we have $x' = -\frac{1}{2}(y')^2$.



50. Complete the square and simplify:

$$2y^2 - 3x^2 - 18x - 20y + 11 = 0 \quad \Rightarrow$$
$$2(y^2 - 10y + 25) - 3(x^2 + 6x + 9) + 11 - 50 + 27 = 0 \quad \Rightarrow$$
$$2(y - 5)^2 - 3(x + 3)^2 = 12 \quad \Rightarrow$$
$$\frac{(y - 5)^2}{6} - \frac{(x + 3)^2}{4} = 1.$$

This is a hyperbola. Letting $x' = x + 3$ and $y' = y - 5$, we have $\frac{(y')^2}{6} - \frac{(x')^2}{4} = 1$.



51. The matrix of this form is

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix},$$

which has characteristic polynomial

$$(1-\lambda)^2 - \frac{1}{4} = \lambda^2 - 2\lambda + \frac{3}{4} = \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{3}{2}\right).$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

so that the equation becomes

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}')^T D\mathbf{x}' = \frac{3}{2}(x')^2 + \frac{1}{2}(y')^2 = 6.$$

Dividing through by 6 to simplify gives

$$\frac{(x')^2}{4} + \frac{(y')^2}{12} = 1,$$

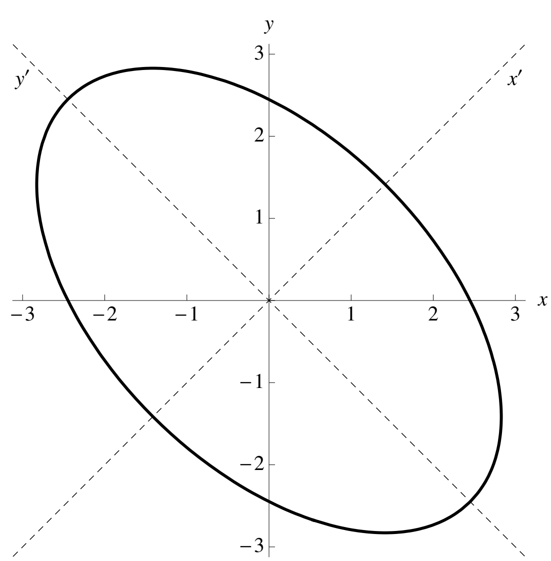
which is an ellipse. To determine the rotation matrix Q, find unit eigenvectors corresponding to $\lambda_1 = \frac{3}{2}$ and $\lambda_2 = \frac{1}{2}$ by row-reduction to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then

$$\mathbf{e}_1 \to Q\mathbf{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{e}_2 \to Q\mathbf{e}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

so that the axis rotation is through a 45° angle.



52. The matrix of this form is

$$A = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix},$$

which has characteristic polynomial

$$(4-\lambda)^2 - 25 = \lambda^2 - 8\lambda - 9 = (\lambda - 9)(\lambda + 1).$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix},$$

so that the equation becomes

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}')^T D\mathbf{x}' = 9\,(x')^2 - (y')^2 = 9.$$

Dividing through by 9 to simplify gives

$$(x')^2 - \frac{(y')^2}{9} = 1,$$

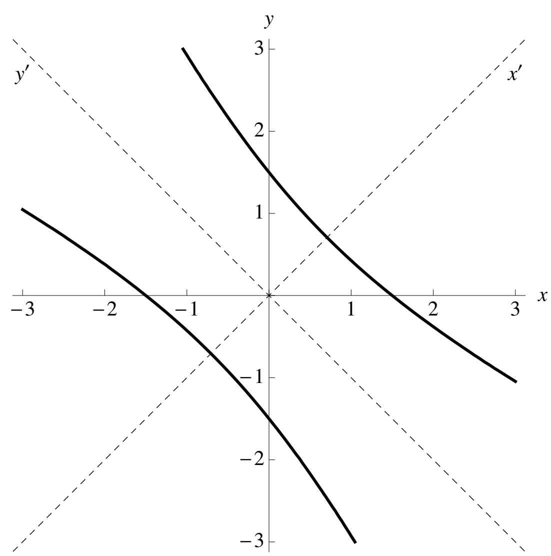
which is a hyperbola. To determine the rotation matrix Q, find unit eigenvectors corresponding to $\lambda_1 = 9$ and $\lambda_2 = 1$ by row-reduction to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then

$$\mathbf{e}_1 \to Q\mathbf{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{e}_2 \to Q\mathbf{e}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

so that the axis rotation is through a 45° angle.



53. The matrix of this form is

$$A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix},$$

which has characteristic polynomial

$$(4-\lambda)(-4-\lambda) - 9 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix},$$

so that the equation becomes

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}')^T D\mathbf{x}' = 5(x')^2 - 5(y')^2 = 5.$$

Dividing through by 5 to simplify gives

$$(x')^2 - (y')^2 = 1,$$

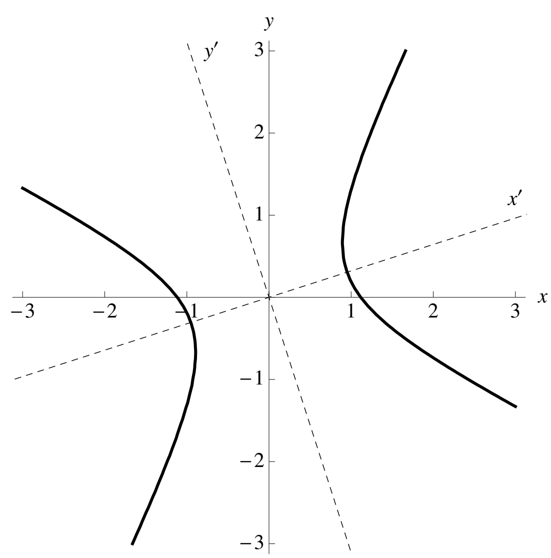
which is a hyperbola. To determine the rotation matrix Q, find unit eigenvectors corresponding to $\lambda_1 = 9$ and $\lambda_2 = 1$ by row-reduction to get

$$Q = \begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}.$$

Then

$$\mathbf{e}_1 \to Q\mathbf{e}_1 = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}, \qquad \mathbf{e}_2 \to Q\mathbf{e}_2 = \begin{bmatrix} -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix},$$

so that the axis rotation is through an angle whose tangent is $\frac{1/\sqrt{10}}{3/\sqrt{10}} = \frac{1}{3}$, or about 18.435°.



54. The matrix of this form is

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix},$$

which has characteristic polynomial

$$(3-\lambda)(3-\lambda) - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2).$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix},$$

so that the equation becomes

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}')^T D\mathbf{x}' = 4(x')^2 + 2(y')^2 = 8.$$

Dividing through by 8 to simplify gives

$$\frac{(x')^2}{2} + \frac{(y')^2}{4} = 1,$$

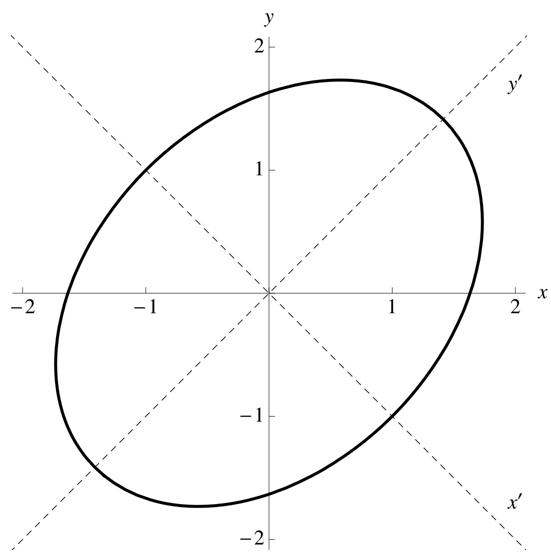
which is an ellipse. To determine the rotation matrix Q, find unit eigenvectors corresponding to $\lambda_1 = 4$ and $\lambda_2 = 2$ by row-reduction to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then

$$\mathbf{e}_1 \to Q\mathbf{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{e}_2 \to Q\mathbf{e}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

so that the axis rotation is through a $-45°$ angle.



55. Writing the equation as $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0$, we have

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -28\sqrt{2} & 22\sqrt{2} \end{bmatrix}, \qquad C = 84.$$

A has characteristic polynomial

$$(3-\lambda)(3-\lambda) - 4 = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1).$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

To determine the rotation matrix Q, find unit eigenvectors corresponding to $\lambda_1 = 5$ and $\lambda_2 = 1$ by row-reduction to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus

$$BQ = \begin{bmatrix} -28\sqrt{2} & 22\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -50 & -6 \end{bmatrix},$$

so the equation becomes

$$\begin{aligned} 5\left(x'\right)^2 - 50x' + \left(y'\right)^2 - 6y' &= -84 \quad \Rightarrow \\ 5\left(\left(x'\right)^2 - 10x' + 25\right) + \left(\left(y'\right)^2 - 6y' + 9\right) &= -84 + 125 + 9 \quad \Rightarrow \\ 5(x'-5)^2 + (y'-3)^2 &= 50 \quad \Rightarrow \\ \frac{(x'-5)^2}{10} + \frac{(y'-3)^2}{50} &= 1. \end{aligned}$$

Letting $x'' = x' - 5$ and $y'' = y' - 3$ gives

$$\frac{\left(x''\right)^2}{10} + \frac{\left(y''\right)^2}{50} = 1,$$

which is an ellipse.

56. Writing the equation as $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0$, we have

$$A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} -20 & -10 \end{bmatrix}, \qquad C = -5.$$

A has characteristic polynomial

$$(6-\lambda)(9-\lambda) - 4 = \lambda^2 - 15\lambda + 50 = (\lambda - 10)(\lambda - 15).$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

To determine the rotation matrix Q, find unit eigenvectors corresponding to $\lambda_1 = 10$ and $\lambda_2 = 5$ by row-reduction to get

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

Thus

$$BQ = \begin{bmatrix} -20 & -10 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 0 & -10\sqrt{5} \end{bmatrix},$$

so the equation becomes

$$\begin{aligned} 10\left(x'\right)^2 + 5\left(y'\right)^2 - 10\sqrt{5}y' &= 5 \quad \Rightarrow \\ 10\left(x'\right)^2 + 5\left(\left(y'\right)^2 - 2\sqrt{5}\,y' + 5\right) &= 5 + 25 \quad \Rightarrow \\ 10\left(x'\right)^2 + 5\left(y' - \sqrt{5}\right)^2 &= 30 \quad \Rightarrow \\ \frac{\left(x'\right)^2}{3} + \frac{\left(y' - \sqrt{5}\right)^2}{6} &= 1. \end{aligned}$$

Letting $x'' = x'$ and $y'' = y' - \sqrt{5}$ gives

$$\frac{\left(x''\right)^2}{3} + \frac{\left(y''\right)^2}{6} = 1,$$

which is an ellipse.

57. Writing the equation as $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0$, we have

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2\sqrt{2} & 0 \end{bmatrix}, \qquad C = -1.$$

A has characteristic polynomial

$$(-\lambda)(-\lambda) - 1 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

To determine the rotation matrix Q, find unit eigenvectors corresponding to $\lambda_1 = 1$ and $\lambda_2 = -1$ by row-reduction to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus

$$BQ = \begin{bmatrix} 2\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & 2 \end{bmatrix},$$

so the equation becomes

$$\begin{gathered} (x')^2 - (y')^2 + 2x' + 2y' = 1 \quad \Rightarrow \\ \left((x')^2 + 2x' + 1\right) - \left((y')^2 - 2y' + 1\right) = 1 + 1 - 1 \quad \Rightarrow \\ (x' + 1)^2 - (y' - 1)^2 = 1. \end{gathered}$$

Letting $x'' = x' + 1$ and $y'' = y' - 1$ gives

$$(x'')^2 - (y'')^2 = 1,$$

which is a hyperbola.

58. Writing the equation as $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0$, we have

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 4\sqrt{2} & 0 \end{bmatrix}, \qquad C = -4.$$

A has characteristic polynomial

$$(1 - \lambda)(1 - \lambda) - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2).$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

To determine the rotation matrix Q, find unit eigenvectors corresponding to $\lambda_1 = 2$ and $\lambda_2 = 0$ by row-reduction to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus

$$BQ = \begin{bmatrix} 4\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 4 & 4 \end{bmatrix},$$

so the equation becomes

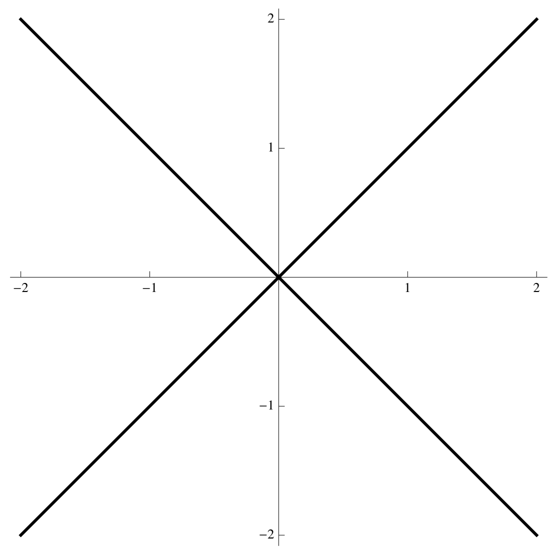
$$2\left(x'\right)^2 + 4x' + 4y' = 4 \quad \Rightarrow$$
$$2y' = -\left(x'\right)^2 - 2x' + 2 \quad \Rightarrow$$
$$2y' - 1 = -\left(\left(x'\right)^2 + 2x' + 1\right) \quad \Rightarrow$$
$$2\left(y' - \frac{1}{2}\right) = -\left(x' + 1\right)^2 \quad \Rightarrow$$
$$y' - \frac{1}{2} = -\frac{1}{2}\left(x' + 1\right)^2.$$

Letting $x'' = x' + 1$ and $y'' = y' - \frac{1}{2}$ gives

$$y'' = -\frac{1}{2}\left(x''\right)^2,$$

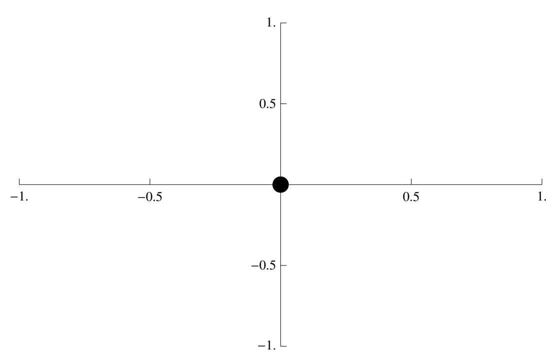
which is a parabola.

59. $x^2 - y^2 = (x+y)(x-y) = 0$ means that $x = y$ or $x = -y$, so the graph of this degenerate conic is the pair of lines $y = x$ and $y = -x$.

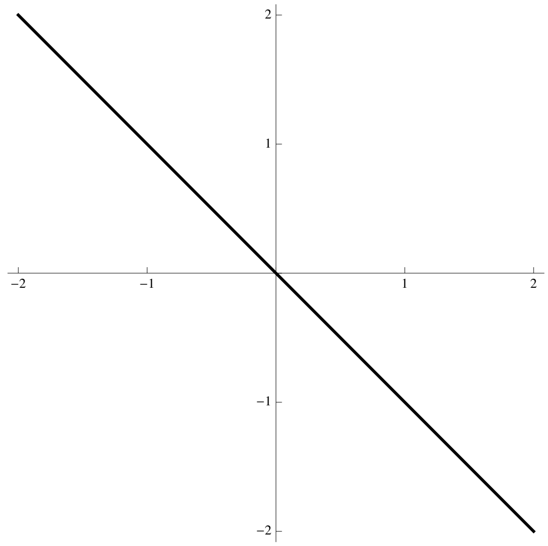


60. Rearranging terms gives $x^2 + 2y^2 = -2$, which has no solutions since the left-hand side is a sum of two squares, so is always nonnegative. Therefore this is an imaginary conic.

61. The left-hand side is a sum of two squares, so it equals zero if and only if $x = y = 0$. The graph of this degenerate conic is the single point $(0, 0)$.



62. $x^2 + 2xy + y^2 = (x + y)^2$. Then $(x + y)^2 = 0$ is satisfied only when $x + y = 0$, so the graph of this degenerate conic is the line $y = -x$.



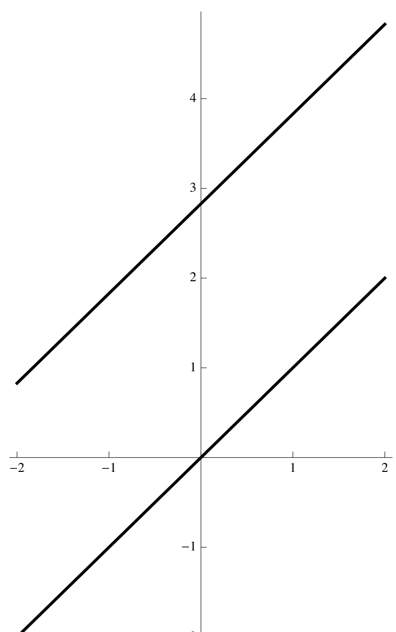
63. Rearranging terms gives

$$x^2 - 2xy + y^2 = 2\sqrt{2}(y - x), \text{ or } (x - y)^2 = 2\sqrt{2}(y - x).$$

This is satisfied when $y = x$. If $y \neq x$, then we can divide through by $y - x$ to get

$$y - x = 2\sqrt{2}, \text{ or } y = x + 2\sqrt{2}.$$

So the graph of this degenerate conic is the union of the lines $y = x$ and $y = x + 2\sqrt{2}$.



64. Writing the equation as $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0$, we have

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2\sqrt{2} & -2\sqrt{2} \end{bmatrix}, \qquad C = 6.$$

A has characteristic polynomial

$$(2-\lambda)(2-\lambda)-1=\lambda^2-4\lambda+3=(\lambda-3)(\lambda-1).$$

Therefore the associated diagonal matrix is

$$D=\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}.$$

To determine the rotation matrix Q, find unit eigenvectors corresponding to $\lambda_1=3$ and $\lambda_2=1$ by row-reduction to get

$$Q=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}.$$

Thus

$$BQ=\begin{bmatrix}2\sqrt{2} & -2\sqrt{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}0 & 4\end{bmatrix},$$

so the equation becomes

$$\begin{aligned}3\left(x'\right)^2+\left(y'\right)^2+4y'&=-6 \quad\Rightarrow\\ 3\left(x'\right)^2+\left(\left(y'\right)^2+4y'+4\right)&=-6+4 \quad\Rightarrow\\ 3\left(x'\right)^2+\left(y'+2\right)^2&=-2.\end{aligned}$$

The left-hand side is always nonnegative, so this equation has no solutions and this is an imaginary conic.

65. Let the eigenvalues of A be λ_1 and λ_2. Then there is an orthogonal matrix Q such that

$$Q^TAQ=D=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}.$$

Then by the Principal Axes Theorem, $\mathbf{x}^TA\mathbf{x}=\mathbf{x}'^TD\mathbf{x}'=\lambda_1(x')^2+\lambda_2(y')^2=k$. If $k\neq 0$, then we may divide through by k to get the standard equation

$$\frac{\lambda_1}{k}(x')^2+\frac{\lambda_2}{k}(y')^2=1.$$

(a) Since $\det A=\det D=\lambda_1\lambda_2<0$, the coefficients of $(x')^2$ and $(y')^2$ in the standard equation have opposite signs, so this curve is a hyperbola.

(b) Since $\det A=\det D=\lambda_1\lambda_2>0$, the coefficients of $(x')^2$ and $(y')^2$ in the standard equation have the same sign. If they are both positive, the curve is an ellipse; if they are positive and equal, then the ellipse is in fact a circle. If they are both negative, then the equation has no solutions so is an imaginary conic.

(c) Since $\det A=\det D=\lambda_1\lambda_2=0$, then at least one of the coefficients of $(x')^2$ and $(y')^2$ in the standard equation is zero. If the remaining coefficient is positive, then we get the two parallel lines $x'=\pm\sqrt{\frac{k}{\lambda_i}}$, while if it is zero or negative, the equation has no solutions and we get an imaginary conic.

(d) If $k=0$ but $\det A=\det D=\lambda_1\lambda_2\neq 0$, then the equation is

$$\lambda_1(x')^2+\lambda_2(y')^2=0,$$

and both λ_1 and λ_2 are nonzero. Then solving for y' gives

$$y'=\pm\sqrt{-\frac{\lambda_1}{\lambda_2}}x'.$$

If λ_1 and λ_2 have the same sign, then this equation has only the solution $x'=y'=0$, so the graph is just the origin. If they have opposite signs, then the graph is two straight lines whose equations are above.

(e) If $k = 0$ and $\det A = \det D = \lambda_1\lambda_2 = 0$, then the equation is

$$\lambda_1(x')^2 + \lambda_2(y')^2 = 0,$$

and at least one of λ_1 and λ_2 is zero. If $\lambda_1 = 0$, we get $x' = 0$, so the graph is the y'-axis, a straight line. Similarly, if $\lambda_2 = 0$, we get $y' = 0$, so the graph is the x'-axis, also a straight line (if both λ_1 and λ_2 are zero, then A was the zero matrix to start, so it does not correspond to a quadratic form).

66. The matrix of this form is

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix},$$

which has characteristic polynomial

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = -(\lambda - 2)^2(\lambda - 8)$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

so that the equation becomes

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}')^T D\mathbf{x}' = 8(x')^2 + 2(y')^2 + 2(z')^2 = 8, \text{ or } (x')^2 + \frac{(y')^2}{4} + \frac{(z')^2}{4} = 1.$$

This is an ellipsoid.

67. The matrix of this form is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix},$$

which has characteristic polynomial

$$-\lambda^3 + 3\lambda^2 + \lambda - 3 = -(x + 1)(x - 1)(x - 3)$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

so that the equation becomes

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}')^T D\mathbf{x}' = -(x')^2 + (y')^2 + 3(z')^2 = 1,$$

which is a hyperboloid of one sheet.

68. The matrix of this form is

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix},$$

which has characteristic polynomial

$$-\lambda^3 - 3\lambda^2 + 9\lambda + 27 = -(x + 3)^2(x - 3)$$

Therefore the associated diagonal matrix is

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

so that the equation becomes

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}')^T D\mathbf{x}' = -3\,(x')^2 - 3\,(y')^2 + 3\,(z')^2 = 12, \text{ or } \frac{(x')^2}{4} + \frac{(y')^2}{4} - \frac{(z')^2}{4} = -1.$$

This is a hyperboloid of two sheets.

69. Writing the form as $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0$, we have

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad C = 0.$$

The characteristic polynomial of A is $\lambda - \lambda^3 = -\lambda(\lambda-1)(\lambda+1)$. Computing unit eigenvectors for each eigenspace gives

$$\mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \qquad \mathbf{v}_{-1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

Then setting

$$Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

we get

$$BQ = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Then the form becomes

$$\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0 \;\Rightarrow\; \mathbf{x}'^T D\mathbf{x}' + BQ\mathbf{x}' = 0 \;\Rightarrow\; (y')^2 - (z')^2 + x' = 0 \;\Rightarrow\; x' = (z')^2 - (y')^2.$$

This is a hyperbolic paraboloid.

70. Writing the form as $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0$, we have

$$A = \begin{bmatrix} 16 & 0 & -12 \\ 0 & 100 & 0 \\ -12 & 0 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -60 & 0 & -80 \end{bmatrix}, \quad C = 0.$$

The characteristic polynomial of A is

$$-\lambda^3 + 125\lambda^2 - 2500\lambda = -\lambda(\lambda-25)(\lambda-100).$$

Computing unit eigenvectors for each eigenspace gives

$$\mathbf{v}_0 = \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix}, \qquad \mathbf{v}_{25} = \begin{bmatrix} -\frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}, \qquad \mathbf{v}_{100} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then setting

$$Q = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 100 \end{bmatrix}$$

we get

$$BQ = \begin{bmatrix} -100 & 0 & 0 \end{bmatrix}.$$

Then the form becomes

$$\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0 \;\Rightarrow\; \mathbf{x}'^T D\mathbf{x}' + BQ\mathbf{x}' = 0 \;\Rightarrow\; 25\,(y')^2 + 100\,(z')^2 - 100x' = 0 \;\Rightarrow\; x' = \frac{(y')^2}{4} + (z')^2.$$

This is an elliptic paraboloid.

71. Writing the form as $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0$, we have

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}, \quad C = 0.$$

The characteristic polynomial of A is

$$-\lambda^3 + 9\lambda = -\lambda(\lambda - 3)(\lambda + 3).$$

Computing unit eigenvectors for each eigenspace gives

$$\mathbf{v}_0 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \qquad \mathbf{v}_{-3} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}.$$

Then setting

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

we get

$$BQ = \begin{bmatrix} \sqrt{3} & 0 & 0 \end{bmatrix}.$$

Then the form becomes

$$\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C = 0 \;\Rightarrow\; \mathbf{x}'^T D\mathbf{x}' + BQ\mathbf{x}' = 0 \;\Rightarrow\; 3\,(y')^2 - 3\,(z')^2 + \sqrt{3}\,x' = 0 \;\Rightarrow\; x' = \frac{(y')^2}{1/\sqrt{3}} - \frac{(z')^2}{1/\sqrt{3}}$$

This is a hyperbolic paraboloid.

72. Writing the form as $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} = 15$, we have

$$A = \begin{bmatrix} 10 & 0 & -20 \\ 0 & 25 & 0 \\ -20 & 0 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 20\sqrt{2} & 50 & 20\sqrt{2} \end{bmatrix}.$$

The characteristic polynomial of A is

$$-\lambda^3 + 45\lambda^2 - 200\lambda - 7500 = -(\lambda + 10)(\lambda - 25)(\lambda - 30).$$

Computing unit eigenvectors for each eigenspace gives

$$\mathbf{v}_{-10} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{v}_{25} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_{30} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then setting

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} -10 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 30 \end{bmatrix}$$

we get

$$BQ = \begin{bmatrix} 40 & 50 & 0 \end{bmatrix}.$$

Then the form becomes

$$\begin{gathered}
\mathbf{x}^T A\mathbf{x} + B\mathbf{x} = 15 \;\Rightarrow \\
\mathbf{x}'^T D\mathbf{x}' + BQ\mathbf{x}' = 15 \;\Rightarrow \\
-10\,(x')^2 + 25\,(y')^2 + 30\,(z')^2 + 40x' + 50y' = 15 \;\Rightarrow \\
-10\left((x')^2 + 4x'\right) + 25\left((y')^2 + 2y'\right) + 30\,(z')^2 = 15 \Rightarrow \\
-10\,(x'+2)^2 + 25\,(y'+1)^2 + 30\,(z')^2 = 0 \Rightarrow \\
(x'+2)^2 = \frac{(y'+1)^2}{2/5} + \frac{(z')^2}{1/3}.
\end{gathered}$$

Letting $x'' = x' + 2$, $y'' = y' + 1$, $z'' = z'$ gives

$$(x'')^2 = \frac{(y'')^2}{2/5} + \frac{(z'')^2}{1/3}.$$

This is an elliptic cone.

73. Writing the form as $\mathbf{x}^T A\mathbf{x} + B\mathbf{x} = 6$, we have

$$A = \begin{bmatrix} 11 & 1 & 4 \\ 1 & 11 & -4 \\ 4 & -4 & 14 \end{bmatrix}, \quad B = \begin{bmatrix} -12 & 12 & 12 \end{bmatrix}.$$

The characteristic polynomial of A is

$$-\lambda^3 + 36\lambda^2 - 396\lambda + 1296 = -(\lambda - 6)(\lambda - 12)(\lambda - 18).$$

Computing unit eigenvectors for each eigenspace gives

$$\mathbf{v}_6 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \qquad \mathbf{v}_{12} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \qquad \mathbf{v}_{18} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}.$$

Then setting

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}, \quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

we get

$$BQ = \begin{bmatrix} 12\sqrt{3} & 0 & 0 \end{bmatrix}.$$

Then the form becomes

$$\begin{gathered}
\mathbf{x}^T A\mathbf{x} + B\mathbf{x} = 6 \;\Rightarrow \\
\mathbf{x}'^T D\mathbf{x}' + BQ\mathbf{x}' = 6 \;\Rightarrow \\
6\left(x'\right)^2 + 12\left(y'\right)^2 + 18\left(z'\right)^2 + 12\sqrt{3}x'6 \;\Rightarrow \\
6\left(\left(x'\right)^2 + 2\sqrt{3}\,x'\right) + 12\left(y'\right)^2 + 18\left(z'\right)^2 = 6 \Rightarrow \\
6\left(\left(x'\right)^2 + 2\sqrt{3}\,x' + 3\right) + 12\left(y'\right)^2 + 18\left(z'\right)^2 = 24 \Rightarrow \\
\frac{\left(x' + \sqrt{3}\right)^2}{4} + \frac{\left(y'\right)^2}{2} + \frac{\left(z'\right)^2}{4/3} = 1.
\end{gathered}$$

Letting $x'' = x' + \sqrt{3}$, $y'' = y'$, $z'' = z'$ gives

$$\frac{\left(x''\right)^2}{4} + \frac{\left(y''\right)^2}{2} + \frac{\left(z''\right)^2}{4/3} = 1.$$

This is an ellipsoid.

74. The strategy is to reduce the problem to one in which we can apply part (b) of Exercise 65. Using the hint, let $\mathbf{v}$ be an eigenvector corresponding to $\lambda = a - bi$, and let $P = \begin{bmatrix}\operatorname{Re}\mathbf{v} & \operatorname{Im}\mathbf{v}\end{bmatrix}$. Set $B = \left(PP^T\right)^{-1}$. Now, B is a real matrix; it is also symmetric since

$$B^T = \left(\left(PP^T\right)^{-1}\right)^T = \left(\left(PP^T\right)^T\right)^{-1} = \left(\left(P^T\right)^T P^T\right)^{-1} = \left(PP^T\right)^{-1} = B.$$

Also,

$$\det B = \det\left(\left(PP^T\right)^{-1}\right) = \frac{1}{\det\left(PP^T\right)} = \frac{1}{\det P \det P^T} = \frac{1}{\left(\det P\right)^2} > 0.$$

Thus Exercise 65(b) applies to B, so that $\mathbf{x}^T B\mathbf{x} = k$ is an ellipse if both eigenvalues of B are positive (since $k > 0$). But by Exercise 29 in this section, B^{-1} is positive definite, so that B is as well (Exercise 31(d)), and therefore has positive eigenvalues. Hence the trajectories of $\mathbf{x}^T B$ are ellipses.

Next, let

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

so that by Theorem 4.43, $A = PCP^{-1}$. Note that

$$C^TC = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ab-ab \\ ab-ab & a^2+b^2 \end{bmatrix} = \begin{bmatrix} |\lambda|^2 & 0 \\ 0 & |\lambda| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Now,

$$\begin{aligned}
(A\mathbf{x})^T B(A\mathbf{x}) &= \mathbf{x}^T A^T \left(PP^T\right)^{-1}(A\mathbf{x}) \\
&= \mathbf{x}^T \left(PCP^{-1}\right)^T \left(P^T\right)^{-1} P^{-1}PCP^{-1}\mathbf{x} \\
&= \mathbf{x}^T \left(P^{-1}\right)^T C^T P^T \left(P^T\right)^{-1} CP^{-1}\mathbf{x} \\
&= \mathbf{x}^T \left(P^T\right)^{-1} C^T C P^{-1}\mathbf{x} \\
&= \mathbf{x}^T \left(P^T\right)^{-1} P^{-1}\mathbf{x} \\
&= \mathbf{x}^T \left(PP^T\right)^{-1}\mathbf{x} \\
&= \mathbf{x}^T B\mathbf{x}.
\end{aligned}$$

Thus the quadratics $\mathbf{x}^T B\mathbf{x} = k$ and $(A\mathbf{x})^T B(A\mathbf{x}) = k$ are the same, so that the trajectories of $A\mathbf{x}$ are ellipses as well.

Chapter Review

1. (a) True. See Theorem 5.1 and the definition preceding it.

(b) True. See Theorem 5.15; the Gram-Schmidt process allows us to turn any basis into an orthonormal basis, showing that any subspace has such a basis.

(c) True. See the definition preceding Theorem 5.5 in Section 5.1.

(d) True. See Theorem 5.5.

(e) False. If Q is an orthogonal matrix, then $|\det Q| = 1$. However, there are many matrices with determinant 1 that are not orthogonal. For example,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.$$

(f) True. By Theorem 5.10, $(\text{row}(A))^{\perp} = \text{null}(A)$. But if $(\text{row}(A))^{\perp} = \mathbb{R}^n$, then $\text{null}(A) = \mathbb{R}^n$ so that $A\mathbf{x} = \mathbf{0}$ for every $\mathbf{x}$, and therefore $A = O$ (since $A\mathbf{e}_i = \mathbf{0}$ for every i, showing that every row of A is zero).

(g) False. For example, let W be the subset of $\mathbb{R}^2$ spanned by $\mathbf{e}_1$, and let $\mathbf{v} = \mathbf{e}_2$. Then $\text{proj}_W(\mathbf{v}) = \mathbf{0}$, but $\mathbf{v} \neq \mathbf{0}$.

(h) True. If A is orthogonal, then $A^{-1} = A^T$; but if A is also symmetric, then $A^T = A$. Thus $A^{-1} = A$, so that $A^2 = AA^{-1} = I$.

(i) False. For example, any diagonal matrix is orthogonally diagonalizable, since it is already diagonal. But if it has a zero entry on the diagonal, it is not invertible.

(j) True. For example, the diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal has this property.

2. The first pair of vectors is already orthogonal, since $1 \cdot 4 + 2 \cdot 1 + 3 \cdot (-2) = 0$. The other two pairs are orthogonal if and only if

$$\begin{aligned} 1 \cdot a + 2 \cdot b + 3 \cdot 3 = 0 \\ 4 \cdot a + 1 \cdot b - 2 \cdot 3 = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a + 2b &= -9 \\ 4a + b &= 6\,. \end{aligned}$$

Computing and row-reducing the augmented matrix gives

$$\left[\begin{array}{cc|c} 1 & 2 & -9 \\ 4 & 1 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -6 \end{array}\right]$$

So the unique solution is $a = 3$ and $b = -6$; these are the only values of a and b for which these vectors form an orthogonal set.

3. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Note that these vectors form an orthogonal set. Therefore using Theorem 5.2, we write

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \\ &= \left(\frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{v}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\right)\mathbf{u}_2 + \left(\frac{\mathbf{v}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\right)\mathbf{u}_3 \\ &= \frac{7\cdot 1 - 3\cdot 0 + 2\cdot 1}{1\cdot 1 + 0\cdot 0 + 1\cdot 1}\mathbf{u}_1 + \frac{7\cdot 1 - 3\cdot 1 + 2\cdot(-1)}{1\cdot 1 + 1\cdot 1 - 1\cdot(-1)}\mathbf{u}_2 + \frac{7\cdot(-1) - 3\cdot 2 + 2\cdot 1}{-1\cdot(-1) + 2\cdot 2 + 1\cdot 1}\mathbf{u}_3 \\ &= \frac{9}{2}\mathbf{u}_1 + \frac{2}{3}\mathbf{u}_2 - \frac{11}{6}\mathbf{u}_3. \end{aligned}$$

So the coordinate vector is

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \frac{9}{2} \\ \frac{2}{3} \\ -\frac{11}{6} \end{bmatrix}.$$

4. We first find $\mathbf{v}_2$. Since $\mathbf{v}_1$ and $\mathbf{v}_2 = \begin{bmatrix} x \\ y \end{bmatrix}$ are orthogonal, we have

$$\frac{3}{5}x + \frac{4}{5}y = 0, \quad \text{or} \quad x = -\frac{4}{3}y.$$

But also $\mathbf{v}_2$ is a unit vector, so that $x^2 + y^2 = \frac{16}{9}y^2 + y^2 = \frac{25}{9}y^2 = 1$. Therefore $y = \pm\frac{3}{5}$ and $x = \mp\frac{4}{5}$.

- If $\mathbf{v}_2 = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$, then

$$\mathbf{v} = -3\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 = -3\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} -\frac{11}{5} \\ -\frac{21}{10} \end{bmatrix}.$$

- If $\mathbf{v}_2 = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$, then

$$\mathbf{v} = -3\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 = -3\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix} = \begin{bmatrix} -\frac{7}{5} \\ -\frac{27}{10} \end{bmatrix}.$$

5. Using Theorem 5.5, we show that the matrix is orthogonal by showing that that $QQ^T = I$:

$$QQ^T = \begin{bmatrix} \frac{6}{7} & \frac{2}{7} & \frac{3}{7} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{4}{7\sqrt{5}} & -\frac{15}{7\sqrt{5}} & \frac{2}{7\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{6}{7} & -\frac{1}{\sqrt{5}} & \frac{4}{7\sqrt{5}} \\ \frac{2}{7} & 0 & -\frac{15}{7\sqrt{5}} \\ \frac{3}{7} & \frac{2}{\sqrt{5}} & \frac{2}{7\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

6. Let Q be the given matrix. If Q is to be orthogonal, then by Theorem 5.5, we must have

$$QQ^T = \begin{bmatrix} \frac{1}{2} & a \\ b & c \end{bmatrix} \begin{bmatrix} \frac{1}{2} & b \\ a & c \end{bmatrix} = \begin{bmatrix} \frac{1}{4} + a^2 & \frac{1}{2}b + ac \\ \frac{1}{2}b + ac & b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore we must have

$$a^2 + \frac{1}{4} = 1, \qquad ac + \frac{1}{2}b = 0, \qquad b^2 + c^2 = 1.$$

The first equation gives $a = \pm\frac{\sqrt{3}}{2}$. Substituting this into the second equation gives

$$\pm\frac{\sqrt{3}}{2}c + \frac{1}{2}b = 0, \qquad \Rightarrow \qquad b = \mp\sqrt{3}\,c \qquad \Rightarrow \qquad b^2 = 3c^2.$$

But $b^2 + c^2 = 1$ and $b^2 = 3c^2$ means that $4c^2 = 1$, so that $c = \pm\frac{1}{2}$. So there are two possibilities:

- If $a = \frac{\sqrt{3}}{2}$, then $b = -\sqrt{3}\,c$, so we get the two matrices

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

- If $a = -\frac{\sqrt{3}}{2}$, then $b = \sqrt{3}\,c$, so we get the two matrices

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

7. We must show that $Q\mathbf{v}_i \cdot Q\mathbf{v}_j = 0$ for $i \neq j$ and that $Q\mathbf{v}_i \cdot Q\mathbf{v}_i = 1$. Since Q is orthogonal, we know that $QQ^T = Q^TQ = I$. Then for any i and j (possibly equal) we have

$$Q\mathbf{v}_i \cdot Q\mathbf{v}_j = (Q\mathbf{v}_i)^T Q\mathbf{v}_j = \mathbf{v}_i^T \left(Q^TQ\right)\mathbf{v}_j = \mathbf{v}_i^T\mathbf{v}_j = \mathbf{v}_i \cdot \mathbf{v}_j.$$

Thus if $i \neq j$, the dot product is zero since $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$, and if $i = j$, then the dot product is 1 since $\mathbf{v}_i \cdot \mathbf{v}_i = 1$.

8. The given condition means that for all vectors $\mathbf{x}$ and $\mathbf{y}$,

$$\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \frac{Q\mathbf{x}\cdot Q\mathbf{y}}{\|Q\mathbf{x}\|\,\|Q\mathbf{y}\|}.$$

Let $\mathbf{q}_i$ be the i^{th} column of Q. Then $\mathbf{q}_i = Q\mathbf{e}_i$. Since the $\mathbf{e}_i$ are orthonormal and the $\mathbf{q}_i$ are unit vectors, we have

$$\mathbf{q}_i\cdot\mathbf{q}_j = \frac{\mathbf{q}_i\cdot\mathbf{q}_j}{\|\mathbf{q}_i\|\,\|\mathbf{q}_j\|} = \frac{\mathbf{e}_i\cdot\mathbf{e}_j}{\|\mathbf{e}_i\|\,\|\mathbf{e}_j\|} = \mathbf{e}_i\cdot\mathbf{e}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

It follows that Q is an orthogonal matrix.

9. W is the column space of $A = \begin{bmatrix}5\\2\end{bmatrix}$, so that $W^\perp$ is the null space of $A^T = \begin{bmatrix}5 & 2\end{bmatrix}$. Row-reduce the augmented matrix:

$$\left[\begin{array}{cc|c}5 & 2 & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & \frac{2}{5} & 0\end{array}\right],$$

so that a basis for the null space, and thus for $W^\perp$, is given by

$$\begin{bmatrix}-\frac{2}{5}\\1\end{bmatrix}, \text{ or } \begin{bmatrix}-2\\5\end{bmatrix}.$$

10. W is the column space of $A = \begin{bmatrix}1\\2\\-1\end{bmatrix}$, so that $W^\perp$ is the null space of $A^T = \begin{bmatrix}1 & 2 & -1\end{bmatrix}$. The augmented matrix is already row-reduced:

$$\left[\begin{array}{ccc|c}1 & 2 & -1 & 0\end{array}\right],$$

so that the null space, which is $W^\perp$, is the set

$$\begin{bmatrix}-2s+t\\s\\t\end{bmatrix} = s\begin{bmatrix}-2\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\1\end{bmatrix} = \operatorname{span}\left(\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right).$$

These two vectors form a basis for $W^\perp$.

11. Let A be the matrix whose columns are the given basis of W. Then by Theorem 5.10, $W^\perp = (\operatorname{col}(A))^\perp = \operatorname{null}(A^T)$. Row-reducing A^T, we get

$$\left[\begin{array}{ccc|c}1 & -1 & 4 & 0\\0 & 1 & -3 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & -3 & 0\end{array}\right],$$

so that a basis for $W^\perp$ is

$$\begin{bmatrix}-1\\3\\1\end{bmatrix}.$$

12. Let A be the matrix whose columns are the given basis of W. Then by Theorem 5.10, $W^{\perp} = (\text{col}(A))^{\perp} = \text{null}(A^T)$. Row-reducing A^T, we get

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 \end{array}\right],$$

so that

$$W^{\perp} = \left\{\begin{bmatrix} -s - \frac{3}{2}t \\ \frac{1}{2}t \\ s \\ t \end{bmatrix}\right\} = \left\{s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \end{bmatrix}\right\}.$$

13. Row-reducing A gives

$$A \to \begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so that a basis for row(A) is given by the nonzero rows of the reduced matrix, or

$$\left\{\begin{bmatrix} 1 & 0 & 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 2 & 1 \end{bmatrix}\right\}.$$

A basis for col(A) is given by the columns of A corresponding to leading 1s in the reduced matrix, so one basis is

$$\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -5 \end{bmatrix}\right\}.$$

A basis for null(A) is given by the solution set of the matrix, which, from the reduced form, is

$$\left\{\begin{bmatrix} -2s - 3t - 4u \\ -2t - u \\ s \\ t \\ u \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Finally, to determine null(A^T), we row-reduce A^T:

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 2 & 1 & -5 \\ 2 & -2 & 4 & 6 \\ 1 & 1 & 8 & -1 \\ 3 & -2 & 9 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\text{null}(A^T) = \left\{\begin{bmatrix} -5s - t \\ -3s + 2t \\ s \\ t \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

14. The orthogonal decomposition of $\mathbf{v}$ with respect to W is

$$\mathbf{v} = \text{proj}_W(\mathbf{v}) + (\mathbf{v} - \text{proj}_W(\mathbf{v})),$$

so we must find $\operatorname{proj}_W(\mathbf{v})$. Let the three given vectors in W be $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$. Then

$$\begin{aligned}
\operatorname{proj}_W(\mathbf{v}) &= \left(\frac{\mathbf{w}_1\cdot\mathbf{v}}{\mathbf{w}_1\cdot\mathbf{w}_1}\right)\mathbf{w}_1 + \left(\frac{\mathbf{w}_2\cdot\mathbf{v}}{\mathbf{w}_2\cdot\mathbf{w}_2}\right)\mathbf{w}_2 + \left(\frac{\mathbf{w}_3\cdot\mathbf{v}}{\mathbf{w}_3\cdot\mathbf{w}_3}\right)\mathbf{w}_3 \\
&= \frac{0\cdot 1+1\cdot 0+1\cdot(-1)+1\cdot 2}{0\cdot 0+1\cdot 1+1\cdot 1+1\cdot 1}\begin{bmatrix}0\\1\\1\\1\end{bmatrix} + \frac{1\cdot 1+0\cdot 0+1\cdot(-1)-1\cdot 2}{1\cdot 1+0\cdot 0+1\cdot 1-1\cdot(-1)}\begin{bmatrix}1\\0\\1\\-1\end{bmatrix} \\
&\qquad + \frac{3\cdot 1+1\cdot 0-2\cdot(-1)+1\cdot 2}{3\cdot 3+1\cdot 1-2\cdot(-2)+1\cdot 1}\begin{bmatrix}3\\1\\-2\\1\end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\0\\1\\-1\end{bmatrix} + \frac{7}{15}\begin{bmatrix}3\\1\\-2\\1\end{bmatrix} \\
&= \begin{bmatrix}\frac{11}{15}\\ \frac{4}{5}\\ -\frac{19}{15}\\ \frac{22}{15}\end{bmatrix}.
\end{aligned}$$

Then

$$\mathbf{v} - \operatorname{proj}_W(\mathbf{v}) = \begin{bmatrix}1\\0\\-1\\2\end{bmatrix} - \begin{bmatrix}\frac{11}{15}\\ \frac{4}{5}\\ -\frac{19}{15}\\ \frac{22}{15}\end{bmatrix} = \begin{bmatrix}\frac{4}{15}\\ -\frac{4}{5}\\ \frac{4}{15}\\ \frac{8}{15}\end{bmatrix}.$$

15. **(a)** Using Gram-Schmidt, we get

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\begin{aligned}
\mathbf{v}_2 = \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 &= \begin{bmatrix}1\\1\\1\\0\end{bmatrix} - -\frac{1\cdot 1+1\cdot 1+1\cdot 1+1\cdot 0}{1\cdot 1+1\cdot 1+1\cdot 1+1\cdot 1}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \\
&= \begin{bmatrix}1\\1\\1\\0\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\\ -\frac{3}{4}\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_3 &= \mathbf{x}_3 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_3)\mathbf{v}_1 - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{x}_3)\mathbf{v}_2 \\
&= \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{1\cdot 0+1\cdot 1+1\cdot 1+1\cdot 1}{1\cdot 1+1\cdot 1+1\cdot 1+1\cdot 1}\mathbf{v}_1 - \frac{\frac{1}{4}\cdot 0+\frac{1}{4}\cdot 1+\frac{1}{4}\cdot 1-\frac{3}{4}\cdot 1}{\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{9}{16}}\mathbf{v}_2 \\
&= \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}\frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\\ -\frac{3}{4}\end{bmatrix} = \begin{bmatrix}-\frac{2}{3}\\ \frac{1}{3}\\ \frac{1}{3}\\ 0\end{bmatrix}.
\end{aligned}$$

(b) To find a QR factorization, we normalize the vectors from (a) and use those as the column vectors of Q. Then we set $R = Q^T A$. Normalizing $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ yields

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{2\sqrt{3}}{3}\begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\\-\frac{3}{4}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{6}\\\frac{\sqrt{3}}{6}\\\frac{\sqrt{3}}{6}\\-\frac{\sqrt{3}}{2}\end{bmatrix}$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\sqrt{6}}{2}\begin{bmatrix}-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\\0\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{6}}{3}\\\frac{\sqrt{6}}{6}\\\frac{\sqrt{6}}{6}\\0\end{bmatrix}.$$

Then

$$Q = \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{3}\\ \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{6}\\ \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{6}\\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0\end{bmatrix},$$

and

$$R = Q^T A = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\\1 & 1 & 1\\1 & 0 & 1\end{bmatrix} = \begin{bmatrix}2 & \frac{3}{2} & \frac{3}{2}\\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{6}\\ 0 & 0 & \frac{\sqrt{6}}{3}\end{bmatrix}.$$

Then $A = QR$ is a QR factorization of A.

16. Note that the two given vectors,

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\2\\2\end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\\-1\end{bmatrix},$$

are indeed orthogonal. We extend these to a basis by adding $\mathbf{e}_3$ and $\mathbf{e}_4$. We can verify that this is a basis by computing the determinant of the matrix whose column vectors are the $\mathbf{v}_i$; this determinant is 1, so the matrix is invertible and therefore the vectors are linearly independent, so they do in fact form a basis for $\mathbb{R}^4$. We apply Gram-Schmidt to $\mathbf{v}_3$ and $\mathbf{v}_4$:

$$\begin{aligned}
\mathbf{v}_3 &= \mathbf{x}_3 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_3)\mathbf{v}_1 - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{x}_3)\mathbf{v}_2 \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{0\cdot 1 + 0\cdot 0 + 1\cdot 2 + 0\cdot 2}{1^2+0^2+2^2+2^2}\mathbf{v}_1 - \frac{0\cdot 0 + 0\cdot 1 + 1\cdot 1 + 0\cdot(-1)}{0^2+1^2+1^2+(-1)^2}\mathbf{v}_2 \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{9}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} \\ -\frac{1}{3} \\ \frac{2}{9} \\ -\frac{1}{9} \end{bmatrix} \\
\mathbf{v}_4 &= \mathbf{x}_4 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_4)\mathbf{v}_1 - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{x}_4)\mathbf{v}_2 - \operatorname{proj}_{\mathbf{v}_3}(\mathbf{x}_4)\mathbf{v}_3 \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{0\cdot 1 + 0\cdot 0 + 0\cdot 2 + 1\cdot 2}{1^2+0^2+2^2+2^2}\mathbf{v}_1 - \frac{0\cdot 0 + 0\cdot 1 + 0\cdot 1 + 0\cdot(-1)}{0^2+1^2+1^2+(-1)^2}\mathbf{v}_2 \\
&\qquad - \frac{0\cdot\left(-\frac{2}{9}\right) + 0\cdot\left(-\frac{1}{3}\right) + 0\cdot\frac{2}{9} + 1\cdot\left(-\frac{1}{9}\right)}{\frac{4}{81}+\frac{1}{9}+\frac{4}{81}+\frac{1}{81}}\mathbf{v}_3 \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{9}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -\frac{2}{9} \\ -\frac{1}{3} \\ \frac{2}{9} \\ -\frac{1}{9} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{6} \\ 0 \\ \frac{1}{6} \end{bmatrix}.
\end{aligned}$$

17. If $x_1 + x_2 + x_3 + x_4 = 0$, then $x_4 = -x_1 - x_2 - x_3$, so that a basis for W is given by

$$\mathcal{B} = \left\{ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},\ \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix},\ \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Then using Gram-Schmidt,

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \\
\mathbf{v}_2 &= \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_2)\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} - -\frac{1\cdot 0 + 0\cdot 1 + 0\cdot 0 - 1\cdot(-1)}{1^2+0^2+0^2+(-1)^2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_3 &= \mathbf{x}_3 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_3)\mathbf{v}_1 - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{x}_3)\mathbf{v}_2 \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1\cdot 0 + 0\cdot 0 + 0\cdot 1 - 1\cdot(-1)}{1^2+0^2+0^2+(-1)^2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{-\frac{1}{2}\cdot 0 + 1\cdot 0 + 0\cdot 1 - \frac{1}{2}\cdot(-1)}{\frac{1}{4}+1+0+\frac{1}{4}}\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ -\frac{1}{3} \end{bmatrix}.
\end{aligned}$$

18. (a) The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 9\lambda = -\lambda(\lambda-3)^2,$$

so the eigenvalues are 0 and 3. To find the eigenspaces, we row-reduce $\left[A - \lambda I \mid 0\right]$:

$$\begin{aligned}
\left[A - 0I \mid 0\right] &= \left[\begin{array}{rrr|r} 2 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
\left[A - 3I \mid 0\right] &= \left[\begin{array}{rrr|r} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].
\end{aligned}$$

Thus the eigenspaces are

$$E_0 = \operatorname{span}\left\{\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}, \qquad E_3 = \operatorname{span}\left\{\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

The vectors in E_0 are orthogonal to those in E_3 since they correspond to different eigenvalues, but the basis vectors above in E_3 are not orthogonal, so we use Gram-Schmidt to convert the second basis vector into one orthogonal to the first:

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1\cdot(-1) + 1\cdot 0 + 0\cdot 1}{1\cdot 1 + 1\cdot 1 + 0\cdot 0}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

Finally, normalize $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ and let Q be the matrix whose columns are those vectors; then Q is the orthogonal matrix

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

Finally, an orthogonal diagonalization of A is given by

$$Q^TAQ = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b) From part (a), we have $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 3$, with

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \qquad \mathbf{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \qquad \mathbf{q}_3 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}.$$

The spectral decomposition of A is

$$\begin{aligned} A &= \lambda_1 \mathbf{q}_1\mathbf{q}_1^T + \lambda_2 \mathbf{q}_2\mathbf{q}_2^T + \lambda_3 \mathbf{q}_3\mathbf{q}_3^T \\ &= 3\mathbf{q}_2\mathbf{q}_2^T + 3\mathbf{q}_3\mathbf{q}_3^T \\ &= 3\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} + 3\begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \\ &= 3\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + 3\begin{bmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}. \end{aligned}$$

19. See Example 5.20, or Exercises 23 and 24 in Section 5.4. The basis vector for E_{-2}, which we name $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, is orthogonal to the other two since they correspond to different eigenvalues. However, we must orthogonalize the basis for E_0 using Gram-Schmidt:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1\cdot 1 + 1\cdot 1 + 1\cdot 0}{1^2+1^2+0^2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Now $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are mutually orthogonal, with norms $\sqrt{2}$, $\sqrt{2}$, and 1 respectively. Let Q be the matrix whose columns are $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$, and $\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$; then the desired matrix is

$$\begin{aligned} Q\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} Q^{-1} &= Q\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} Q^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

20. $\mathbf{v}_i\mathbf{v}_i^T$ is symmetric since

$$\left(\mathbf{v}_i\mathbf{v}_i^T\right)^T = \left(\mathbf{v}_i^T\right)^T \mathbf{v}_i^T = \mathbf{v}_i^T\mathbf{v}_i,$$

so it is equal to its own transpose. Since a scalar times a symmetric matrix is a symmetric matrix, and the sum of symmetric matrices is symmetric, it follows that A is symmetric. Next,

$$A\mathbf{v}_j = \left(\sum_{i=1}^n c_i\mathbf{v}_i\mathbf{v}_i^T\right)\mathbf{v}_j = \sum_{i=1}^n c_i\mathbf{v}_i\left(\mathbf{v}_i^T\mathbf{v}_j\right) = \sum_{i=1}^n c_i\mathbf{v}_i\left(\mathbf{v}_i\cdot\mathbf{v}_j\right) = c_j\mathbf{v}_j$$

since $\{\mathbf{v}_i\}$ is a set of orthonormal vectors. This shows that each c_j is an eigenvalue for A with corresponding eigenvector $\mathbf{v}_j$. Since there are n of them, these are all the eigenvalues and eigenvectors.

Chapter 6

Vector Spaces

6.1 Vector Spaces and Subspaces

1. Let $V = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \right\}$. V is the set of all vectors in $\mathbb{R}^2$ whose first and second components are the same. We verify all ten axioms of a vector space:

1. If $\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix} \in V$ then $\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix} \in V$ since its first and second components are the same.

2. $\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix} = \begin{bmatrix} y+x \\ y+x \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} + \begin{bmatrix} x \\ x \end{bmatrix}$.

3. $\left(\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} \right) + \begin{bmatrix} z \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix} + \begin{bmatrix} z \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ x+y+z \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y+z \\ y+z \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} + \left(\begin{bmatrix} y \\ y \end{bmatrix} + \begin{bmatrix} z \\ z \end{bmatrix} \right)$.

4. $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$, and $\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x+0 \\ x+0 \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} \in V$.

5. If $\begin{bmatrix} x \\ x \end{bmatrix} \in V$, then $\begin{bmatrix} -x \\ -x \end{bmatrix} \in V$, and $\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} -x \\ -x \end{bmatrix} = \begin{bmatrix} x-x \\ x-x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

6. If $\begin{bmatrix} x \\ x \end{bmatrix} \in V$, then $c \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} cx \\ cx \end{bmatrix} \in V$ since its first and second components are the same.

7. $c \left(\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} \right) = c \begin{bmatrix} x+y \\ x+y \end{bmatrix} = \begin{bmatrix} c(x+y) \\ c(x+y) \end{bmatrix} = \begin{bmatrix} c(y+x) \\ c(y+x) \end{bmatrix} = c \begin{bmatrix} y+x \\ y+x \end{bmatrix} = c \left(\begin{bmatrix} y \\ y \end{bmatrix} + \begin{bmatrix} x \\ x \end{bmatrix} \right)$.

8. $(c+d) \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} (c+d)x \\ (c+d)x \end{bmatrix} = \begin{bmatrix} cx+dx \\ cx+dx \end{bmatrix} = \begin{bmatrix} cx \\ cx \end{bmatrix} + \begin{bmatrix} dx \\ dx \end{bmatrix} = c \begin{bmatrix} x \\ x \end{bmatrix} + d \begin{bmatrix} x \\ x \end{bmatrix}$.

9. $c \left(d \begin{bmatrix} x \\ x \end{bmatrix} \right) = c \begin{bmatrix} dx \\ dx \end{bmatrix} = \begin{bmatrix} c(dx) \\ c(dx) \end{bmatrix} = \begin{bmatrix} (cd)x \\ (cd)x \end{bmatrix} = (cd) \begin{bmatrix} x \\ x \end{bmatrix}$.

10. $1 \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 1x \\ 1x \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}$.

2. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \ : \ x, y \geq 0 \right\}$ fails to satisfy Axioms 5 and 6, so it is not a vector space:

5. If $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$, choose $c < 0$; then $c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} \notin V$ since at least one of cx and cy is negative.

6. If $\begin{bmatrix} x \\ x \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$, then $\begin{bmatrix} -x \\ -x \end{bmatrix} \notin V$ since $-x < 0$.

3. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;:\; xy \geq 0 \right\}$ fails to satisfy Axiom 1, so it is not a vector space:

1. Choose $x \neq y$ with $xy \geq 0$. Then $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} -y \\ -x \end{bmatrix}$ are both in V, but

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -y \\ -x \end{bmatrix} = \begin{bmatrix} x-y \\ y-x \end{bmatrix} = \begin{bmatrix} x-y \\ -(x-y) \end{bmatrix} \notin V$$

since $(x-y)(-(x-y)) = -(x-y)^2 < 0$ because $x \neq y$.

4. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;:\; x \geq y \right\}$ fails to satisfy Axioms 5 and 6, so it is not a vector space:

5. If $\begin{bmatrix} x \\ y \end{bmatrix} \in V$ and $x \neq y$, then $x > y$, so that $-x < -y$. Therefore $\begin{bmatrix} -x \\ -y \end{bmatrix} \notin V$.

6. Again let $\begin{bmatrix} x \\ y \end{bmatrix} \in V$ and $x \neq y$, and choose $c = -1$. Then $\begin{bmatrix} cx \\ cy \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} \notin V$ as above.

.

5. $\mathbb{R}^2$ with the given scalar multiplication fails to satisfy Axiom 8, so it is not a vector space:

8. Choose $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ with $y \neq 0$, and let c and d be nonzero scalars. Then

$$(c+d)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (c+d)x \\ y \end{bmatrix} = \begin{bmatrix} cx+dx \\ y \end{bmatrix}, \text{ while } c\begin{bmatrix} x \\ y \end{bmatrix} + d\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ y \end{bmatrix} + \begin{bmatrix} dx \\ y \end{bmatrix} = \begin{bmatrix} cx+dx \\ 2y \end{bmatrix}.$$

These two are unequal since we chose $y \neq 0$.

6. $\mathbb{R}^2$ with the given addition fails to satisfy Axiom 7, so it is not a vector space:

8. Choose $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ and let c be a scalar not equal 1. Then

$$\begin{aligned} c\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) &= c\begin{bmatrix} x_1+x_2+1 \\ y_1+y_2+1 \end{bmatrix} = \begin{bmatrix} cx_1+cx_2+c \\ cy_1+cy_2+c \end{bmatrix} \\ c\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix} + \begin{bmatrix} cx_2 \\ cy_2 \end{bmatrix} = \begin{bmatrix} cx_1+cx_2+1 \\ cy_1+cy_2+1 \end{bmatrix}. \end{aligned}$$

These two are unequal since we chose $c \neq 1$.

7. All axioms apply, so that this set is a vector space. Let x, y, and z by positive real numbers, so that x, y, $z \in V$, and c and d be any real numbers. Then

1. $x \oplus y = xy > 0$, so that $x \oplus y \in V$.

2. $x \oplus y = xy = yx = y \oplus x$.

3. $(x \oplus y) \oplus z = (xy) \oplus z = (xy)z = x(yz) = x \oplus (yz) = x \oplus (y \oplus z)$.

4. $\mathbf{0} = 1$, since $x \oplus 0 = x1 = 1x = 0 \oplus x = x$.

5. $-x$ is the positive real number $\frac{1}{x}$, since $x \oplus (-x) = x \cdot \frac{1}{x} = 1 = \mathbf{0}$.

6. $c \odot x = x^c > 0$, so that $c \odot x \in V$.

7. $c \odot (x \oplus y) = (xy)^c = x^c y^c = x^c \oplus y^c = (c \odot x) \oplus (c \odot y)$.

8. $(c+d) \odot x = x^{c+d} = x^c x^d = x^c \oplus x^d = (c \odot x) \oplus (c \odot y)$.

9. $c \odot (d \odot x) = c \odot x^d = \left(x^d\right)^c = x^{cd} = (x^c)^d = d \odot x^c = d \odot (c \odot x)$.

10. $1 \odot x = x^x = x$.

8. This is not a vector space, since Axiom 6 fails. Choose any rational number q, and let $c = \pi$. Then $q\pi$ is not rational, so that V is not closed under scalar multiplication.

9. All axioms apply, so that this is a vector space. Let x, y, z, u, v, w, c, and d be real numbers. Then $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ and $\begin{bmatrix} u & v \\ 0 & w \end{bmatrix} \in V$, and

1. The sum of upper triangular matrices is again upper triangular.

2. Matrix addition is commutative.

3. Matrix addition is associative.

4. The zero matrix O is upper triangular, and $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} + O = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$.

5. $-\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ is the matrix $\begin{bmatrix} -x & -y \\ 0 & -z \end{bmatrix}$.

6. $c\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} cx & cy \\ 0 & cz \end{bmatrix} \in V$ since it is upper triangular.

7.

$$\begin{aligned} c\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} + \begin{bmatrix} u & v \\ 0 & w \end{bmatrix}\right) &= c\left(\begin{bmatrix} x+u & y+v \\ 0 & z+w \end{bmatrix}\right) \\ &= \begin{bmatrix} c(x+u) & c(y+v) \\ 0 & c(z+w) \end{bmatrix} \\ &= \begin{bmatrix} cx+cu & cy+cv \\ 0 & cz+cw \end{bmatrix} \\ &= \begin{bmatrix} cx & cy \\ 0 & cz \end{bmatrix} + \begin{bmatrix} cu & cv \\ 0 & cw \end{bmatrix} \\ &= c\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} + c\begin{bmatrix} u & v \\ 0 & w \end{bmatrix}. \end{aligned}$$

8.

$$\begin{aligned} (c+d)\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} &= \begin{bmatrix} (c+d)x & (c+d)y \\ 0 & (c+d)z \end{bmatrix} \\ &= \begin{bmatrix} cx+dx & cy+dy \\ 0 & cz+dz \end{bmatrix} \\ &= \begin{bmatrix} cx & cy \\ 0 & cz \end{bmatrix} + \begin{bmatrix} dx & dy \\ 0 & dz \end{bmatrix} \\ &= c\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} + d\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}. \end{aligned}$$

9. $c\left(d\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = c\begin{bmatrix} dx & dy \\ 0 & dz \end{bmatrix} = \begin{bmatrix} c(dx) & c(dy) \\ 0 & c(dz) \end{bmatrix} = \begin{bmatrix} (cd)x & (cd)y \\ 0 & (cd)z \end{bmatrix} = (cd)\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$.

10. $1\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$.

10. This set V is not a vector space, since it is not closed under addition. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in V, \text{ but } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin V$$

since the product of the diagonal entries is not zero.

11. This set V is a vector space. Let A be a skew-symmetric matrix and c a constant. Then

1. The sum of skew-symmetric matrices is skew-symmetric.

2. Matrix addition is commutative.

3. Matrix addition is associative.

4. The zero matrix O is skew-symmetric, and $A + O = A$.

5. If A is skew-symmetric, then so is $-A$, since $(-A)_{ij} = -a_{ij} = a_{ji} = (A)_{ji} = -(-A)_{ji}$.

6. If A is skew-symmetric, then so is cA, since $(cA)_{ij} = c(A)_{ij} = -c(A)_{ji} = -(cA)_{ji}$.

7. Scalar multiplication distributes over matrix addition.

8, 9. These properties hold for general matrices, and skew-symmetric matrices are closed under these operations, so then in V.

10. $1A = A$ is skew-symmetric.

12. Axioms 1, 2, and 6 were verified in Example 6.3. For the remaining examples, using the notation from Example 3, and letting $r(x) = c_0 + c_1x + c_2x^2$,

3.

$$\begin{aligned}(p(x) + q(x)) + r(x) &= ((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2) + r(x) \\ &= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)x + (a_2 + b_2 + c_2)x^2 \\ &= (a_0 + a_1x + a_2x^2) + ((b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2) \\ &= p(x) + (q(x) + r(x)).\end{aligned}$$

4. The zero polynomial $0 + 0x + 0x^2$ is the zero vector, since

$$\mathbf{0} + p(x) + (0 + a_0) + (0 + a_1)x + (0 + a_2)x^2 = a_0 + a_1x + a_2x^2 = p(x).$$

5. $-p(x)$ is the polynomial $-a_0 - a_1x - a_2x^2$.

7.

$$\begin{aligned}c(p(x) + q(x)) &= c\left((a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)\right) \\ &= c\left((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2\right) \\ &= c(a_0 + b_0) + c(a_1 + b_1)x + c(a_2 + b_2)x^2 \\ &= (ca_0 + cb_0) + (ca_1 + cb_1)x + (ca_2 + cb_2)x^2 \\ &= \left(ca_0 + ca_1x + ca_2x^2\right) + \left(cb_0 + cb_1x + cb_2x^2\right) \\ &= c\left(a_0 + a_1x + a_2x^2\right) + c\left(b_0 + b_1x + b_2x^2\right) \\ &= cp(x) + cq(x).\end{aligned}$$

8.

$$\begin{aligned}(c + d)p(x) &= (c + d)\left(a_0 + a_1x + a_2x^2\right) \\ &= (c + d)a_0 + (c + d)a_1x + (c + d)a_2x^2 \\ &= (ca_0 + da_0) + (ca_1 + da_1)x + (ca_2 + da_2)x^2 \\ &= (ca_0 + ca_1x + ca_2x^2) + (da_0 + da_1x + da_2x^2) \\ &= c\left(a_0 + a_1x + a_2x^2\right) + d\left(a_0 + a_1x + a_2x^2\right) \\ &= cp(x) + dp(x).\end{aligned}$$

9.

$$\begin{aligned} c(dp(x)) &= c\left(da_0 + da_1x + da_2x^2\right) \\ &= c(da_0) + c(da_1)x + c(da_2)x^2 \\ &= (cd)a_0 + (cd)a_1x + (cd)a_2x^2 \\ &= (cd)\left(a_0 + a_1x + a_2x^2\right) \\ &= (cd)p(x). \end{aligned}$$

10. $1p(x) = 1a_0 + 1a_1x + 1a_2x^2 = a_0 + a_1x + a_2x^2 = p(x)$.

13. Axioms 1 and 6 were verified in Example 6.4. For the rest, using the notation from Example 4,

2, 3. Addition of functions is commutative and associative.

4. The zero function $f(x) = 0$ for all x is the zero vector, since $(\mathbf{0} + f)(x) = \mathbf{0}(x) + f(x) = f(x)$.

5. The function $-f$ such that $(-f)(x) = -f(x)$ is such that $(-f + f)(x) = (-f)(x) + f(x) = -f(x) + f(x) = 0$, so their sum is the zero function.

7. The function $c(f+g)$ is defined by $(c(f+g))(x) = c((f+g)(x))$ by the definition of scalar multiplication. But $(f+g)(x) = f(x)+g(x)$, so we get $(c(f+g))(x) = c(f(x)+g(x)) = cf(x)+cg(x) = (cf+cg)(x)$.

8. Again using the definition of scalar multiplication, the function $(c+d)f$ is defined by $((c+d)f)(x) = (c+d)(f(x)) = cf(x) + df(x) = (cf + df)(x)$.

9. The function df is defined by $(df)(x) = df(x)$, so $c(df)$ is defined by $(c(df))(x) = c((df)(x)) = cdf(x) = ((cd)f)(x)$.

10. $(1f)(x) = 1f(x) = f(x)$.

14. This is not a complex vector space, since Axiom 6 fails. Let $z \in \mathbb{C}$ be any nonzero complex number, and let $c \in \mathbb{C}$ be any complex number with nonzero real and imaginary parts. Then

$$c\begin{bmatrix} z \\ \bar{z} \end{bmatrix} = \begin{bmatrix} cz \\ c\bar{z} \end{bmatrix} \neq \begin{bmatrix} cz \\ \overline{cz} \end{bmatrix}.$$

15. This is a vector space over $\mathbb{C}$; the proof is identical to the proof that $M_{mn}(\mathbb{R})$ is a real vector space.

16. This is a complex vector space. Let $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{C}^2$, and let $c, d \in \mathbb{C}$. Then

1, 2, 3, 4, 5. These follow since addition is the same as in the usual vector space $\mathbb{C}^2$.

6. $c\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{c}z_1 \\ \bar{c}z_2 \end{bmatrix} \in \mathbb{C}^2$.

7.

$$\begin{aligned} c\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = c\begin{bmatrix} z_1 + w + 1 \\ z_2 + w_2 \end{bmatrix} = \begin{bmatrix} \bar{c}(z_1 + w_1) \\ \bar{c}(z_2 + w_2) \end{bmatrix} &= \begin{bmatrix} \bar{c}z_1 + \bar{c}w_1 \\ \bar{c}z_2 + \bar{c}w_2 \end{bmatrix} \\ &= \begin{bmatrix} \bar{c}z_1 \\ \bar{c}z_2 \end{bmatrix} + \begin{bmatrix} \bar{c}w_1 \\ \bar{c}w_2 \end{bmatrix} = c\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + c\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \end{aligned}$$

8. $(c+d)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \overline{(c+d)}z_1 \\ \overline{(c+d)}z_2 \end{bmatrix} = \begin{bmatrix} (\bar{c}+\bar{d})z_1 \\ (\bar{c}+\bar{d})z_2 \end{bmatrix} = \begin{bmatrix} \bar{c}z_1 \\ \bar{c}z_2 \end{bmatrix} + \begin{bmatrix} \bar{d}z_1 \\ \bar{d}z_2 \end{bmatrix} = c\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + d\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

9. $(cd)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \overline{(cd)}z_1 \\ \overline{(cd)}z_2 \end{bmatrix} = \begin{bmatrix} \bar{c}\bar{d}z_1 \\ \bar{c}\bar{d}z_2 \end{bmatrix} = c\begin{bmatrix} \bar{d}z_1 \\ \bar{d}z_2 \end{bmatrix} = c\left(d\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right)$.

10. $1\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{1}z_1 \\ \bar{1}z_2 \end{bmatrix} = \begin{bmatrix} 1z_1 \\ 1z_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.

17. This is not a complex vector space, since Axiom 6 fails. Let $x \in \mathbb{R}^n$ be any nonzero vector, and let $c \in \mathbb{C}$ be any complex number with nonzero imaginary part. Then $cx \notin \mathbb{R}^n$, so that $\mathbb{R}^n$ is not closed under scalar multiplication.

18. This set V is a vector space over $\mathbb{Z}_2$. Let x, $y \in \mathbb{Z}_2^n$ each have an even number of 1s, say x has $2r$ ones and y has $2s$ ones.

1. $x+y$ has a 1 where either x or y has a 1, except that if *both* x and y have a 1, $x+y$ has a zero. Say this happens in k different places. Then the total number of 1s in $x+y$ is $2r+2s-2k = 2(r+s-k)$, which is even, so V is closed under addition.

2. Since addition in $\mathbb{Z}_2$ is commutative, we see that $x + y = y + x$ since we are simply adding corresponding components.

3. Since addition in $\mathbb{Z}_2$ is associative, we see that $(x + y) + z = x + (y + z)$ since we are simply adding corresponding components.

4. $\mathbf{0} \in \mathbb{Z}_2^n$ has an even number (zero) of ones, so $\mathbf{0} \in V$, and $x + \mathbf{0} = x$.

5. Since $1 = -1$ in $\mathbb{Z}_2$, we can take $-x = x$; then $-x + x = x + x = \mathbf{0}$.

6. If $c \in \mathbb{Z}_2$, then cx is equal to either $\mathbf{0}$ or x depending on whether $c = 0$ or $c = 1$. In either case, $x \in V$ implies that $cx \in V$.

7, 8. Since multiplication distributes over addition in $\mathbb{Z}_2$, these axioms follow since operations are component by component.

9. Since multiplication in $\mathbb{Z}_2$ is distributive, these axioms follow since operations are component by component.

10. $1x = x$ since multiplication is component by component.

19. This set V is not a vector space:

1. For example, let x and y be vectors with exactly one 1. Then $x + y$ has either 0 or 2 ones (it has zero if the locations of the 1s in x and y are the same), so $x + y \notin V$ and V is not closed under addition.

4. $\mathbf{0} \notin V$ since $\mathbf{0}$ does not have an odd number of 1s.

6. If $c = 0$, then $cx = \mathbf{0} \notin V$ if $x \in V$, so that V is not closed under scalar multiplication.

20. This is a vector space over $\mathbb{Z}_p$; the proof is identical to the proof that $M_{mn}(\mathbb{R})$ is a real vector space.

21. This is not a vector space. Axioms 1 through 7 are all satisfied, as is Axiom 10, but Axioms 8 and 9 fail since addition and multiplication are not the same in $\mathbb{Z}_6$ as they are in $\mathbb{Z}_3$. For example:

8. Let $c = 2$, $d = 1$, and $\mathbf{u} = 1$. Then $(c + d)\mathbf{u} = (2 + 1)\mathbf{u} = 0\mathbf{u} = 0$ since addition of scalars is performed in $\mathbb{Z}_3$, while $c\mathbf{u} + d\mathbf{u} = 2 \cdot 1 + 1 \cdot 1 = 2 + 1 = 3$ since here the addition is performed in $\mathbb{Z}_6$.

9. Let $c = d = 2$ and $\mathbf{u} = 1$. Then $(cd)\mathbf{u} = 1 \cdot 1 = 1$ since the scalar multiplication is done in $\mathbb{Z}_3$, while $c(d\mathbf{u}) = 2(2 \cdot 1) = 2 \cdot 2 = 4$ since here the multiplication is done in $\mathbb{Z}_6$.

22. Since $0 = 0 + 0$, we have

$$0\mathbf{u} = (0 + 0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}$$

by Axiom 8. By Axiom 5, we add $-0\mathbf{u}$ to both sides of this equation, giving

$$-0\mathbf{u} + 0\mathbf{u} = (-0\mathbf{u} + 0\mathbf{u}) + 0\mathbf{u}, \text{ so that by Axiom 5 } \mathbf{0} = \mathbf{0} + 0\mathbf{u}.$$

Finally, $\mathbf{0} + 0\mathbf{u} = 0\mathbf{u}$ by Axiom 4, so that $\mathbf{0} = 0\mathbf{u}$.

23. To show that $(-1)\mathbf{u} = -\mathbf{u}$, it suffices to show that $(-1)\mathbf{u} + \mathbf{u} = \mathbf{0}$, by Axiom 5. But $(-1)\mathbf{u} + \mathbf{u} = (-1)\mathbf{u} + 1\mathbf{u}$ by Axiom 10. Then use Axiom 8 to rewrite this as $(-1+1)\mathbf{u} = 0\mathbf{u}$. By part (a) of Theorem 6.1, $0\mathbf{u} = \mathbf{0}$ and we are done.

24. This is a subspace. We check each of the conditions in Theorem 6.2:

1. $\begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ 0 \\ a+b \end{bmatrix} \in W$ since its first and third components are equal and its second component is zero.

2. $c \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ cz \end{bmatrix} \in W$.

25. This is a subspace. We check each of the conditions in Theorem 6.2:

1. $\begin{bmatrix} a \\ -a \\ 2a \end{bmatrix} + \begin{bmatrix} b \\ -b \\ 2b \end{bmatrix} = \begin{bmatrix} a+b \\ -a-b \\ 2a+2b \end{bmatrix} = \begin{bmatrix} (a+b) \\ -(a+b) \\ 2(a+b) \end{bmatrix} \in W$.

2. $c \begin{bmatrix} a \\ -a \\ 2a \end{bmatrix} = \begin{bmatrix} ac \\ -ac \\ 2ac \end{bmatrix} = \begin{bmatrix} (ac) \\ -(ac) \\ 2(ac) \end{bmatrix} \in W$.

26. This is not a subspace. It fails both conditions in Theorem 6.2:

1. $\begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} + \begin{bmatrix} c \\ d \\ c+d+1 \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \\ (a+c)+(b+d)+2 \end{bmatrix}$, which is not in W.

2. $c \begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} = \begin{bmatrix} ca \\ cb \\ c(a+b+1) \end{bmatrix} = \begin{bmatrix} ca \\ cb \\ ca+cb+c \end{bmatrix}$, which is not in W if $c \neq 1$.

27. This is not a subspace. It fails both conditions in Theorem 6.2:

1. $\begin{bmatrix} a \\ b \\ |a| \end{bmatrix} + \begin{bmatrix} c \\ d \\ |c| \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \\ |a|+|c| \end{bmatrix}$. If a and c do not have the same sign, then $|a|+|c| \neq |a+c|$, so that the sum is not in W.

2. $c \begin{bmatrix} a \\ b \\ |a| \end{bmatrix} = \begin{bmatrix} ca \\ cb \\ c\,|a| \end{bmatrix}$. If $c < 0$, then $c\,|a| \neq |ca|$, so that the result is not in W.

28. This is a subspace. We check each of the conditions in Theorem 6.2:

1. $\begin{bmatrix} a & b \\ b & 2a \end{bmatrix} + \begin{bmatrix} c & d \\ d & 2c \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ b+d & 2a+2c \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ b+d & 2(a+c) \end{bmatrix} \in W$.

2. $c \begin{bmatrix} a & b \\ b & 2a \end{bmatrix} = \begin{bmatrix} ca & cb \\ cb & 2ac \end{bmatrix} = \begin{bmatrix} ac & bc \\ bc & 2(ac) \end{bmatrix} \in W$.

29. This is not a subspace; condition 1 in Theorem 6.2 fails. For example, if $a, d \geq 0$, then

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \begin{bmatrix} -a & a \\ d & -d \end{bmatrix} \in W,$$

but

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} -a & a \\ d & -d \end{bmatrix} = \begin{bmatrix} 0 & a \\ d & 0 \end{bmatrix} \notin W$$

since $0 < ad$.

30. This is not a subspace; both conditions of Theorem 6.2 fail. For example:

Let $A = B = I$; then $\det A = \det B = 1$, so that $A, B \in W$, but $\det(A + B) = \det 2I = 2$, so $A + B \notin W$.

Let $A = I$ and c be any constant other than ± 1. Then $\det(cA) = c^n \det A \neq \det A = 1$, so that $cA \notin W$.

31. Since the sum of two diagonal matrices is again diagonal, and a scalar times a diagonal matrix is diagonal, this subset satisfies both conditions of Theorem 6.2, so it is a subspace.

32. This is not a subspace. For example, it fails the second condition: let A be idempotent and c be any constant other than 0 or 1. Then

$$(cA)^2 = c^2 A^2 = c^2 A \neq cA$$

so that cA is not idempotent, so is not in W.

33. This is a subspace. We check each of the conditions in Theorem 6.2:

1. Suppose that $A, C \in W$. Then $(A+C)B = AB+CB = BA+BC = B(A+C)$, so that $A+C \in W$.

2. Suppose that $A \in W$ and c is any scalar. Then $(cA)B = c(AB) = c(BA) = B(cA)$, so that $cA \in W$.

34. This is a subspace. We check each of the conditions in Theorem 6.2:

1. Let $\mathbf{v} = bx + cx^2$, $\mathbf{v}' = b'x + c'x^2$. Then

$$\mathbf{v} + \mathbf{v}' = bx + cx^2 + b'x + c'x^2 = (b + b')x + (c + c')x^2 \in W.$$

2. Let $\mathbf{v} = bx + cx^2$ and d be any scalar. Then $d\mathbf{v} = d(bx + cx^2) = (db)x + (dc)x^2 \in W$.

35. This is a subspace. We check each of the conditions in Theorem 6.2:

1. Let $\mathbf{v} = a + bx + cx^2$, $\mathbf{v}' = a' + b'x + c'x^2 \in W$. Then

$$\mathbf{v} + \mathbf{v}' = a + bx + cx^2 + a' + b'x + c'x^2 = (a + a') + (b + b')x + (c + c')x^2.$$

Then $(a + a') + (b + b') + (c + c') = (a + b + c) + (a' + b' + c') = 0$, so that $\mathbf{v} + \mathbf{v}' \in W$.

2. Let $\mathbf{v} = a + bx + cx^2$ and d be any scalar. Then $d\mathbf{v} = d(a + bx + cx^2) = (da) + (db)x + (dc)x^2$, and $da + db + dc = d(a + b + c) = d \cdot 0 = 0$, so that $d\mathbf{v} \in W$.

36. This is not a subspace; it fails condition 1 in Theorem 6.2. For example, let $\mathbf{v} = 1 + x$ and $\mathbf{v}' = x^2$. Then both $\mathbf{v}$ and $\mathbf{v}'$ are in V, but $\mathbf{v} + \mathbf{v}' = 1 + x + x^2 \notin V$ since the product of its coefficients is not zero.

37. This is not a subspace; it fails both conditions of Theorem 6.2.

1. For example let $\mathbf{v} = x^3$ and $\mathbf{v}' = -x^3$. Then $\mathbf{v} + \mathbf{v}' = 0$, which is not a degree 3 polynomial.

2. Let $c = 0$; then $c\mathbf{v} = 0$ for any $\mathbf{v}$, but $0 \notin W$, so that this condition fails.

38. This is a subspace. We check each of the conditions in Theorem 6.2:

1. Let $f, g \in W$. Then $(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$, so that $f + g \in W$.

2. Let $f \in W$ and c be any scalar. Then $(cf)(-x) = cf(-x) = cf(x) = (cf)(x)$, so that $cf \in W$.

39. This is a subspace. We check each of the conditions in Theorem 6.2:

1. Let $f, g \in W$. Then $(f+g)(-x) = f(-x)+g(-x) = -f(x)-g(x) = -(f(x)+g(x)) = -(f+g)(x)$, so that $f + g \in W$.

2. Let $f \in W$ and c be any scalar. Then $(cf)(-x) = cf(-x) = -cf(x) = -(cf)(x)$, so that $cf \in W$.

40. This is not a subspace; it fails both conditions in Theorem 6.2. For example, if $f, g \in W$, then $(f+g)(0) = f(0) + g(0) = 1 + 1 \neq 1$, so that $f + g \notin W$. Similarly, if c is a scalar not equal to 1, then $(cf)(0) = cf(0) = c \neq 1$, so that $cf \notin W$.

41. This is a subspace. We check each of the conditions in Theorem 6.2:

1. Let $f, g \in W$. Then $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$, so that $f + g \in W$.

2. Let $f \in W$ and c be any scalar. Then $(cf)(0) = cf(0) = c \cdot 0 = 0$, so that $cf \in W$.

42. If f and $g \in W$ are integrable and c is a constant, then

$$\int (f+g) = \int f + \int g, \qquad \int (cf) = c \int f,$$

so that both $f + g$ and $cf \in W$. Thus W is a subspace.

43. This is not a subspace; it fails condition 2 in Theorem 6.2. For example, let $f \in W$ be such that $f'(x) > 0$ everywhere, and choose $c = -1$; then $(cf)'(x) = cf'(x) < 0$, so that $cf \notin W$.

44. If $f, g \in W$ and c is a constant, then

$$(f+g)'' = f'' + g'', \qquad (cf)'' = cf'',$$

so that both $f + g$ and cf have continuous second derivatives, so they are both in W.

45. This is not a subspace; it fails both conditions in Theorem 6.2.

1. $f(x) = \frac{1}{x^2} \in W$, but $-f(x) = -\frac{1}{x^2} \notin W$, since $\lim_{x\to 0}(-f(x)) = -\infty$.

2. Take $c = 0$ and $f \in W$; then $\lim_{x\to 0}(cf)(x) = \lim_{x\to 0} 0 = 0$, so that $cf \notin W$.

46. Suppose that $\mathbf{v}, \mathbf{v}' \in U \cap W$, and let c be a scalar. Then $\mathbf{v}, \mathbf{v}' \in U$, so that $\mathbf{v} + \mathbf{v}'$ and $c\mathbf{v} \in U$. Similarly, $\mathbf{v}, \mathbf{v}' \in W$, so that $\mathbf{v} + \mathbf{v}'$ and $c\mathbf{v} \in W$. Therefore $\mathbf{v} + \mathbf{v}' \in U \cap W$ and $c\mathbf{v} \in U \cap W$, so that $U \cap W$ is a subspace.

47. For example, let $U = \left\{\begin{bmatrix} x \\ 0 \end{bmatrix}\right\}$, the x-axis, and let $W = \left\{\begin{bmatrix} 0 \\ y \end{bmatrix}\right\}$, the y-axis. Then $U \cup W$ consists of all vectors in $\mathbb{R}^2$ in which at least one component is zero. Choose $a, b \neq 0$; then

$$\begin{bmatrix} a \\ 0 \end{bmatrix} \in U, \ \begin{bmatrix} 0 \\ b \end{bmatrix} \in W, \quad \text{but} \quad \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \notin U \cup W$$

since neither a nor b is zero.

48. **(a)** $U + W$ is the xy-plane, since

$$U = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \ W = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix},$$

so that any point in the xy-plane is

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \in U + W.$$

However, a point not in the xy-plane has nonzero z-coordinate, so it cannot be a sum of an element of U and an element of W. Thus $U + W$ is exactly the xy-plane.

(b) Suppose $\mathbf{x}, \mathbf{x}' \in U + W$; say $\mathbf{x} = \mathbf{u} + \mathbf{w}$ and $\mathbf{x}' = \mathbf{u}' + \mathbf{w}'$, where $\mathbf{u}, \mathbf{u}' \in U$ and $\mathbf{w}, \mathbf{w}' \in W$. Then

$$\mathbf{x} + \mathbf{x}' = \mathbf{u} + \mathbf{w} + \mathbf{u}' + \mathbf{w}' = (\mathbf{u} + \mathbf{u}') + (\mathbf{w} + \mathbf{w}') \in U + W$$

since both U and W are subspaces. If c is a scalar, then

$$c\mathbf{x} = c(\mathbf{u} + \mathbf{w}) = c\mathbf{u} + c\mathbf{w} \in U + W,$$

again since both U and W are subspaces.

49. We verify all ten axioms of a vector space. Note that both U and V are vector spaces, so they satisfy all the vector space axioms. These facts will be used without comment below. Let $(\mathbf{u}_1, \mathbf{v}_1)$, $(\mathbf{u}_2, \mathbf{v}_2)$, and $(\mathbf{u}_3, \mathbf{v}_3)$ be elements of $U \times V$, and let c and d be scalars. Then

1. $(\mathbf{u}_1, \mathbf{v}_1) + (\mathbf{u}_2, \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2) \in U \times V$.

2. $(\mathbf{u}_1, \mathbf{v}_1) + (\mathbf{u}_2, \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2) = (\mathbf{u}_2 + \mathbf{u}_1, \mathbf{v}_2 + \mathbf{v}_1) = (\mathbf{u}_2, \mathbf{v}_2) + (\mathbf{u}_1, \mathbf{v}_1)$.

3.

$$\begin{aligned}((\mathbf{u}_1, \mathbf{v}_1) + (\mathbf{u}_2, \mathbf{v}_2)) + (\mathbf{u}_3, \mathbf{v}_3) &= (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{u}_3, \mathbf{v}_3) \\ &= ((\mathbf{u}_1 + \mathbf{u}_2) + \mathbf{u}_3, (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3) \\ &= (\mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3), \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)) \\ &= (\mathbf{u}_1, \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{u}_3, \mathbf{v}_2 + \mathbf{v}_3) \\ &= (\mathbf{u}_1, \mathbf{v}_1) + ((\mathbf{u}_2, \mathbf{v}_2) + (\mathbf{u}_3, \mathbf{v}_3)).\end{aligned}$$

4. $(\mathbf{0}, \mathbf{0}) \in U \times V$, and $(\mathbf{u}_1, \mathbf{v}_1) + (\mathbf{0}, \mathbf{0}) = (\mathbf{u}_1 + \mathbf{0}, \mathbf{v}_1 + \mathbf{0}) = (\mathbf{u}_1, \mathbf{v}_1)$.

5. $(\mathbf{u}_1, \mathbf{v}_1) + (-\mathbf{u}_1, -\mathbf{v}_1) = (\mathbf{u}_1 - \mathbf{u}_1, \mathbf{v}_1 - \mathbf{v}_1) = (\mathbf{0}, \mathbf{0})$.

6. $c(\mathbf{u}_1, \mathbf{v}_1) = (c\mathbf{u}_1, c\mathbf{v}_1) \in U \times V$.

7.

$$\begin{aligned}c\,((\mathbf{u}_1, \mathbf{v}_1) + (\mathbf{u}_2, \mathbf{v}_2)) &= c(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2) \\ &= (c(\mathbf{u}_1 + \mathbf{u}_2), c(\mathbf{v}_1 + \mathbf{v}_2)) \\ &= (c\mathbf{u}_1 + c\mathbf{u}_2, c\mathbf{v}_1 + c\mathbf{v}_2) \\ &= (c\mathbf{u}_1, c\mathbf{v}_1) + (c\mathbf{u}_2, c\mathbf{v}_2) \\ &= c(\mathbf{u}_1, \mathbf{v}_1) + c(\mathbf{u}_2, \mathbf{v}_2).\end{aligned}$$

8.

$$\begin{aligned}(c + d)(\mathbf{u}_1, \mathbf{v}_1) &= ((c + d)\mathbf{u}_1, (c + d)\mathbf{v}_1) \\ &= (c\mathbf{u}_1 + d\mathbf{u}_1, c\mathbf{v}_1 + d\mathbf{v}_1) \\ &= (c\mathbf{u}_1, c\mathbf{v}_1) + (d\mathbf{u}_1, d\mathbf{v}_1) \\ &= c(\mathbf{u}_1, \mathbf{v}_1) + d(\mathbf{u}_1, \mathbf{v}_1).\end{aligned}$$

9. $c\,(d(\mathbf{u}_1, \mathbf{v}_1)) = c(d\mathbf{u}_1, d\mathbf{v}_1) = (c(d\mathbf{u}_1), c(d\mathbf{u}_2)) = ((cd)\mathbf{u}_1, (cd)\mathbf{v}_1) = (cd)(\mathbf{u}_1, \mathbf{v}_1)$.

10. $1(\mathbf{u}_1, \mathbf{v}_1) = (1\mathbf{u}_1, 1\mathbf{v}_1) = (\mathbf{u}_1, \mathbf{v}_1)$.

50. Suppose that $(\mathbf{w}, \mathbf{w}), (\mathbf{w}', \mathbf{w}') \in \Delta$. Then

$$(\mathbf{w}, \mathbf{w}) + (\mathbf{w}', \mathbf{w}') = (\mathbf{w} + \mathbf{w}', \mathbf{w} + \mathbf{w}'),$$

which is in Δ since its two components are the same. Let c be a scalar. Then

$$c(\mathbf{w}, \mathbf{w}) = (c\mathbf{w}, c\mathbf{w}),$$

which is also in Δ since its components are the same. Therefore Δ satisfies both conditions of Theorem 6.2, so is a subspace.

51. If $aA + bB = C$, then

$$a\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a+b & a-b \\ b-a & a \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then $a - b = 2$ and $b - a = 3$, which is impossible. Therefore C is not in the span of A and B.

52. If $aA + bB = C$, then

$$a\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a+b & a-b \\ b-a & a \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 5 & -1 \end{bmatrix}.$$

Then $a = -1$ from the lower right. The upper left then gives $-1+b = 3$, so that $b = 4$. Then $a-b = -5$ and $b - a = 5$, so that $-A + 4B = C$.

53. Suppose that

$$a(1-2x) + b(x - x^2) + c(-2 + 3x + x^2) = (a - 2c) + (-2a + b + 3c)x + (-b + c)x^2 = 3 - 5x - x^2.$$

Then

$$\begin{aligned} a \qquad - 2c &= 3 \\ -2a + b + 3c &= -5 \\ -b + c &= -1. \end{aligned}$$

The third equation gives $b = c + 1$; substituting this in the second equation gives $-2a + 4c = -6$, which is -2 times the first equation. So there is a family of solutions, which can be parametrized as $a = 2t + 3$, $b = t + 1$, $c = t$. For example, with $t = 0$ we get

$$s(x) = 3p(x) + q(x),$$

and $s(x) \in \operatorname{span}(p(x), q(x), r(x))$.

54. Suppose that

$$a(1-2x) + b(x - x^2) + c(-2 + 3x + x^2) = (a - 2c) + (-2a + b + 3c)x + (-b + c)x^2 = 1 + x + x^2.$$

Then

$$\begin{aligned} a \qquad - 2c &= 1 \\ -2a + b + 3c &= 1 \\ -b + c &= 1. \end{aligned}$$

The third equation gives $b = c - 1$; substituting into the second equation gives $-2a + 4c - 1 = 1$, or $a - 2c = -1$. This conflicts with the first equation, so the system has no solution. Thus $s(x) \notin \operatorname{span}(p(x), q(x), r(x))$.

55. Since $\sin^2 x + \cos^2 x = 1$, we have $1 = f(x) + g(x) \in \operatorname{span}(\sin^2 x, \cos^2 x)$.

56. Since $\cos 2x = \cos^2 x - \sin^2 x = g(x) - f(x)$, we have $\cos 2x = -f(x) + g(x) \in \operatorname{span}(\sin^2 x, \cos^2 x)$.

57. Suppose that $\sin 2x = af(x) + bg(x) = a\sin^2 x + b\cos^2 x$ for all x. Then

$$\begin{aligned} &\text{Let } x = 0;\text{ then } 0 = a\sin^2 0 + b\cos^2 0 = b, \text{ so that } b = 0. \\ &\text{Let } x = \frac{\pi}{2};\text{ then } \sin\left(2\cdot\frac{\pi}{2}\right) = \sin\pi = 0 = a\sin^2\frac{\pi}{2} + b\cos^2\frac{\pi}{2} = a, \text{ so that } a = 0. \end{aligned}$$

Thus the only solution is $a = b = 0$, but this is not a valid solution since $\sin 2x \neq 0$. So $\sin 2x \notin \operatorname{span}(\sin^2 x, \cos^2 x)$.

58. Suppose that $\sin x = af(x) + bg(x) = a\sin^2 x + b\cos^2 x$ for all x. Then

$$\text{Let } x = 0; \text{ then } 0 = a\sin^2 0 + b\cos^2 0 = b, \text{ so that } b = 0.$$
$$\text{Let } x = \frac{\pi}{2}; \text{ then } \sin\frac{\pi}{2} = 1 = a\sin^2\frac{\pi}{2} + b\cos^2\frac{\pi}{2} = a, \text{ so that } a = 1.$$

Thus the only possibility is $\sin x = \sin^2 x$. But this is not a valid equation; for example, if $x = \frac{3\pi}{2}$, then the left-hand side is -1 while the right-hand side is 1. So $\sin x \notin \operatorname{span}(\sin^2 x, \cos^2 x)$.

59. Let

$$V_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad V_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Now, $V_4 = V_3 - V_1$, so that $\operatorname{span}(V_1, V_2, V_3, V_4) = \operatorname{span}(V_1, V_2, V_3)$. If $\operatorname{span}(V_1, V_2, V_3) = M_{22}$, then every matrix is in the span, so in particular E_{11} is. But

$$aV_1 + bV_2 + cV_3 = E_{11} \quad \Rightarrow \quad \begin{bmatrix} a+c & a+b \\ b+c & a+c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now the diagonal entries give $a + c = 0$ and $a + c = 1$, which has no solution. So E_{11} is not in the span and thus $\operatorname{span}(V_1, V_2, V_3) \neq M_{22}$.

60. Let

$$V_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad V_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Solving the four independent systems

$$\begin{aligned} E_{11} &= aV_1 + bV_2 + cV_3 + dV_4 & \qquad E_{12} &= aV_1 + bV_2 + cV_3 + dV_4 \\ E_{21} &= aV_1 + bV_2 + cV_3 + dV_4 & \qquad E_{22} &= aV_1 + bV_2 + cV_3 + dV_4 \end{aligned}$$

gives

$$\begin{aligned} E_{11} &= 2V_1 - V_2 - V_4 & \qquad E_{12} &= -V_1 + V_2 \\ E_{21} &= -V_1 + V_2 + V_4 & \qquad E_{22} &= -V_2 + V_3. \end{aligned}$$

Therefore $\operatorname{span}(V_1, V_2, V_3, V_4) \supseteq \operatorname{span}(E_{11}, E_{12}, E_{21}, E_{22}) = M_{22}$.

61. Let $p(x) = 1 + x$, $q(x) = x + x^2$, and $r(x) = 1 + x^2$. Then

$$1 = ap(x) + bq(x) + cs(x) \;\Rightarrow\; \begin{array}{rcrcrcl} a & & & + & c & = & 1 \\ a & + & b & & & = & 0 \\ & & b & + & c & = & 0 \end{array} \;\Rightarrow\; a = \frac{1}{2},\; b = -\frac{1}{2},\; c = \frac{1}{2}$$

$$x = ap(x) + bq(x) + cs(x) \;\Rightarrow\; \begin{array}{rcrcrcl} a & & & + & c & = & 0 \\ a & + & b & & & = & 1 \\ & & b & + & c & = & 0 \end{array} \;\Rightarrow\; a = \frac{1}{2},\; b = \frac{1}{2},\; c = -\frac{1}{2}$$

$$x^2 = ap(x) + bq(x) + cs(x) \;\Rightarrow\; \begin{array}{rcrcrcl} a & & & + & c & = & 0 \\ a & + & b & & & = & 0 \\ & & b & + & c & = & 1 \end{array} \;\Rightarrow\; a = -\frac{1}{2},\; b = \frac{1}{2},\; c = \frac{1}{2}.$$

Thus $\operatorname{span}(p(x), q(x), r(x)) \supseteq \operatorname{span}(1, x, x^2) = \mathscr{P}_2$.

62. Let $p(x) = 1 + x + 2x^2$, $q(x) = 2 + x + 2x^2$, and $r(x) = -1 + x + 2x^2$. Then

$$x = ap(x) + bq(x) + cs(x) \;\Rightarrow\; \begin{array}{rcrcrcl} a & + & 2b & - & c & = & 0 \\ a & + & b & + & c & = & 1 \\ 2a & + & 2b & + & 2c & = & 0\,. \end{array}$$

But the second pair of equations are inconsistent, so that $x \notin \operatorname{span}(p(x), q(x), r(x))$. Therefore $\operatorname{span}(p(x), q(x), r(x)) \neq \mathscr{P}_2$.

63. Suppose $\mathbf{0}$ and $\mathbf{0}'$ satisfy Axiom 4. Then $\mathbf{0}+\mathbf{0}'=\mathbf{0}$ since $\mathbf{0}'$ satisfies Axiom 4, and also $\mathbf{0}+\mathbf{0}'=\mathbf{0}'$ since $\mathbf{0}$ satisfies Axiom 4. Putting these two equalities together gives $\mathbf{0}=\mathbf{0}'$, so that $\mathbf{0}$ is unique.

64. Suppose $\mathbf{v}'$ and $\mathbf{v}''$ satisfy Axiom 5 for $\mathbf{v}$. Then using Axiom 5,

$$\mathbf{v}''=\mathbf{v}''+\mathbf{0}=\mathbf{v}''+(\mathbf{v}+\mathbf{v}')=(\mathbf{v}''+\mathbf{v})+\mathbf{v}'=\mathbf{0}+\mathbf{v}'=\mathbf{v}'.$$

6.2 Linear Independence, Basis, and Dimension

1. We wish to determine if there are constants c_1, c_2, and c_3, not all zero, such that

$$c_1\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}+c_2\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}+c_3\begin{bmatrix}1 & 0\\ 3 & 2\end{bmatrix}=0.$$

This is equivalent to asking for solutions to the linear system

$$\begin{aligned} c_1+c_2+\ \ c_3&=0\\ c_1-c_2\qquad\ &=0\\ c_2+3c_3&=0\\ -c_1\qquad +2c_3&=0.\end{aligned}$$

Row-reducing the corresponding augmented matrix, we get

$$\left[\begin{array}{rrr|r}1 & 1 & 1 & 0\\ 1 & -1 & 0 & 0\\ 0 & 1 & 3 & 0\\ -1 & 0 & 2 & 0\end{array}\right]\to\left[\begin{array}{rrr|r}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

This system has only the trivial solution $c_1=c_2=c_3=0$, so that these matrices are linearly independent.

2. We wish to determine if there are constants c_1, c_2, and c_3, not all zero, such that

$$c_1\begin{bmatrix}2 & -3\\ 4 & 2\end{bmatrix}+c_2\begin{bmatrix}1 & -1\\ 3 & 3\end{bmatrix}+c_3\begin{bmatrix}-1 & 3\\ 1 & 5\end{bmatrix}=0.$$

This is equivalent to asking for solutions to the linear system

$$\begin{aligned} 2c_1+\ \ c_2-\ \ c_3&=0\\ -3c_1-\ \ c_2+3c_3&=0\\ 4c_1+3c_2+\ \ c_3&=0\\ 2c_1+3c_2+5c_3&=0.\end{aligned}$$

Row-reducing the corresponding augmented matrix, we get

$$\left[\begin{array}{rrr|r}2 & 1 & -1 & 0\\ -3 & -1 & 3 & 0\\ 4 & 3 & 1 & 0\\ 2 & 3 & 5 & 0\end{array}\right]\to\left[\begin{array}{rrr|r}1 & 0 & -2 & 0\\ 0 & 1 & 3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

This system has a nontrivial solution $c_1=2t$, $c_2=-3t$, $c_3=t$; setting $t=1$ for example gives

$$2\begin{bmatrix}2 & -3\\ 4 & 2\end{bmatrix}-3\begin{bmatrix}1 & -1\\ 3 & 3\end{bmatrix}+\begin{bmatrix}-1 & 3\\ 1 & 5\end{bmatrix}=0.$$

so that the matrices are linearly dependent.

3. We wish to determine if there are constants c_1, c_2, c_3, and c_4, not all zero, such that

$$c_1 \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix} + c_4 \begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix} = 0.$$

This is equivalent to asking for solutions to the linear system

$$\begin{aligned} -c_1 + 3c_2 - c_4 &= 0 \\ c_1 + 2c_3 &= 0 \\ -2c_1 + c_2 - 3c_3 - c_4 &= 0 \\ 2c_1 + c_2 + c_3 + 7c_4 &= 0. \end{aligned}$$

Row-reducing the corresponding augmented matrix, we get

$$\left[\begin{array}{cccc|c} -1 & 3 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ -2 & 1 & -3 & -1 & 0 \\ 2 & 1 & 1 & 7 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This system has a nontrivial solution $c_1 = -4t$, $c_2 = -t$, $c_3 = 2t$, $c_4 = t$; setting $t = 1$ for example gives

$$-4 \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix} = 0.$$

so that the matrices are linearly dependent.

4. We wish to determine if there are constants c_1, c_2, c_3, and c_4, not all zero, such that

$$c_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = 0.$$

This is equivalent to asking for solutions to the linear system

$$\begin{aligned} c_1 + c_2 + c_4 &= 0 \\ c_1 + c_3 + c_4 &= 0 \\ c_2 + c_3 + c_4 &= 0 \\ c_1 + c_2 + c_3 &= 0. \end{aligned}$$

Row-reducing the corresponding augmented matrix, we get

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Since the only solution is the trivial solution, it follows that these four matrices are linearly independent over M_{22}.

5. Following, for instance, Example 6.26, we wish to determine if there are constants c_1 and c_2, not both zero, such that

$$c_1(x) + c_2(1 + x) = 0.$$

This is equivalent to asking for solutions to the linear system

$$\begin{aligned} c_2 &= 0 \\ c_1 + c_2 &= 0. \end{aligned}$$

By forward substitution, we see that the only solution is the trivial solution, $c_1 = c_2 = 0$. It follows that these polynomials are linearly independent over $\mathscr{P}_2$.

6. Following, for instance, Example 6.26, we wish to determine if there are constants c_1, c_2, and c_3, not all zero, such that

$$c_1(1+x)+c_2(1+x^2)+c_3(1-x+x^2)=(c_1+c_2+c_3)+(c_1-c_3)x+(c_2+c_3)x^2=0.$$

This is equivalent to asking for solutions to the linear system

$$\begin{aligned} c_1+c_2+c_3&=0\\ c_1\qquad -c_3&=0\\ c_2+c_3&=0. \end{aligned}$$

Row-reducing the corresponding augmented matrix, we get

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0\\ 1 & 0 & -1 & 0\\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 \end{array}\right].$$

Since the only solution is the trivial solution, it follows that these three polynomials are linearly independent over $\mathscr{P}_2$.

7. Following, for instance, Example 6.26, we wish to determine if there are constants c_1, c_2, and c_3, not all zero, such that

$$c_1(x)+c_2(2x-x^2)+c_3(3x+2x^2)=(c_1+2c_2+3c_3)x+(-c_2+2c_3)x^2=0.$$

This is a homogeneous system of two equations in three unknowns, so we know it has a nontrivial solution and these polynomials are linearly dependent. To find an expression showing the linear dependence, we must find solutions to the linear system

$$\begin{aligned} c_1+2c_2+3c_3&=0\\ -\ c_2+2c_3&=0. \end{aligned}$$

Row-reducing the corresponding augmented matrix, we get

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 0\\ 0 & -1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 7 & 0\\ 0 & 1 & -2 & 0 \end{array}\right].$$

The general solution is $c_1=-7t$, $c_2=2t$, $c_3=t$. Setting $t=1$, for example, we get

$$-7(x)+2(2x-x^2)+(3x+2x^2)=0.$$

8. Following, for instance, Example 6.26, we wish to determine if there are constants c_1, c_2, c_3, and c_4, not all zero, such that

$$c_1(2x)+c_2(x-x^2)+c_3(1+x^3)+c_4(2-x^2+x^3)=(c_3+2c_4)+(2c_1+c_2)x+(-c_2-c_4)x^2+(c_3+c_4)x^3=0.$$

This is equivalent to asking for solutions to the linear system

$$\begin{aligned} c_3+2c_4&=0\\ 2c_1+c_2\qquad\qquad&=0\\ -c_2\quad-\ c_4&=0\\ c_3+\ c_4&=0. \end{aligned}$$

Row-reducing the corresponding augmented matrix, we get

$$\left[\begin{array}{rrrr|r} 0 & 0 & 1 & 2 & 0\\ 2 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & -1 & 0\\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Since the only solution is the trivial solution, it follows that these four polynomials are linearly independent over $\mathscr{P}_3$.

9. Following, for instance, Example 6.26, we wish to determine if there are constants c_1, c_2, c_3, and c_4, not all zero, such that

$$\begin{aligned}c_1(1-2x)&+c_2(3x+x^2-x^3)+c_3(1+x^2+2x^3)+c_4(3+2x+3x^3)\\&=(c_1+c_3+3c_4)+(-2c_1+3c_2+2c_4)x+(c_2+c_3)x^2+(-c_2+2c_3+3c_4)x^3=0.\end{aligned}$$

This is equivalent to asking for solutions to the linear system

$$\begin{aligned}c_1 \qquad\quad + c_3 + 3c_4 &= 0\\ -2c_1 + 3c_2 \qquad\quad + 2c_4 &= 0\\ c_2 + c_3 \qquad\quad &= 0\\ -c_2 + 2c_3 + 3c_4 &= 0.\end{aligned}$$

Row-reducing the corresponding augmented matrix, we get

$$\left[\begin{array}{cccc|c}1 & 0 & 1 & 3 & 0\\ -2 & 3 & 0 & 2 & 0\\ 0 & 1 & 1 & 0 & 0\\ 0 & -1 & 2 & 3 & 0\end{array}\right] \to \left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right].$$

Since the only solution is the trivial solution, it follows that these four polynomials are linearly independent over $\mathscr{P}_3$.

10. We wish to find solutions to $a+b\sin x+c\cos x=0$ that hold for all x. Setting $x=0$ gives $a+c=0$, so that $a=-c$; setting $x=\pi$ gives $a-c=0$, so that $a=c$. This forces $a=c=0$; then setting $x=\frac{\pi}{2}$ gives $b=0$. So the only solution is the trivial solution, and these functions are linearly independent in $\mathscr{F}$.

11. Since $\sin^2 x+\cos^2 x=1$, the functions are linearly dependent.

12. Suppose that $ae^x+be^{-x}=0$ for all x. Setting $x=0$ gives $ae^0+be^0=a+b=0$, while setting $x=\ln 2$ gives $2a+\frac{1}{2}b=0$. Since the coefficient matrix,

$$\begin{bmatrix}1 & 1\\ 2 & \frac{1}{2}\end{bmatrix},$$

has nonzero determinant, the only solution to the system is the trivial solution, so that these functions are linearly independent in $\mathscr{F}$.

13. Using properties of the logarithm, we have $\ln(2x)=\ln 2+\ln x$ and $\ln(x^2)=2\ln x$, so we want to see if $\{1,\ln 2+\ln x,2\ln x\}$ are linearly independent over $\mathscr{F}$. But clearly they are not, since by inspection

$$(-2\ln 2)\cdot 1+2\cdot(\ln 2+\ln x)-2\ln x=0.$$

So these functions are linearly dependent in $\mathscr{F}$.

14. We wish to find solutions to $a\sin x+b\sin 2x+c\sin 3x=0$ that hold for all x. Setting

$$\begin{aligned}x=\frac{\pi}{2}\quad&\Rightarrow\quad a-c=0\\ x=\frac{\pi}{6}\quad&\Rightarrow\quad \frac{1}{2}a+\frac{\sqrt{3}}{2}b=0\\ x=\frac{\pi}{3}\quad&\Rightarrow\quad \frac{\sqrt{3}}{2}a-\frac{\sqrt{3}}{2}b=0.\end{aligned}$$

Adding the second and third equations gives $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)a=0$, so that $a=0$; then the first and third equations force $b=c=0$. So the only solution is the trivial solution, and thus these functions are linearly independent in $\mathscr{F}$.

15. We want to prove that if the Wronskian is not identically zero, then f and g are linearly independent. We do this by proving the contrapositive: assume f and g are linearly dependent. If either f or g is zero, then clearly the Wronskian is zero as well. So assume that neither f nor g is zero; since they are linearly dependent, each is a nonzero multiple of the other. Say $f(x) = ag(x)$, so that $f'(x) = ag'(x)$. Now, the Wronskian is

$$\begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x) = ag(x)g'(x) - g(x)(ag'(x)) = ag(x)g'(x) - ag(x)g'(x) = 0.$$

Thus the Wronskian is identically zero, proving the result.

16. Exercise 10: $W(x) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} = -\cos^2 x - \sin^2 x = -1 \neq 0$. Since the Wronskian is not zero, the functions are linearly independent.

Exercise 11:

$$\begin{aligned} W(x) &= \begin{vmatrix} 1 & \sin^2 x & \cos^2 x \\ 0 & 2\sin x\cos x & -2\sin x\cos x \\ 0 & 2(\cos^2 x - \sin^2 x) & 2(\sin^2 x - \cos^2 x) \end{vmatrix} \\ &= \begin{vmatrix} 1 & \sin^2 x & \cos^2 x \\ 0 & \sin 2x & -\sin 2x \\ 0 & 2\cos 2x & -2\cos 2x \end{vmatrix} \\ &= \begin{vmatrix} \sin 2x & -\sin 2x \\ 2\cos 2x & -2\cos 2x \end{vmatrix} \\ &= 0. \end{aligned}$$

Since the Wronskian is zero, the functions are linearly dependent.

Exercise 12: $W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2 \neq 0$. Since the Wronskian is not zero, the functions are linearly independent.

Exercise 13:

$$W(x) = \begin{vmatrix} 1 & \ln(2x) & \ln(x^2) \\ 0 & \frac{1}{x} & \frac{2}{x} \\ 0 & -\frac{1}{x^2} & -\frac{2}{x^2} \end{vmatrix} = \begin{vmatrix} \frac{1}{x} & \frac{2}{x} \\ -\frac{1}{x^2} & -\frac{2}{x^2} \end{vmatrix} = 0.$$

Since the Wronskian is zero, the functions are linearly dependent.

Exercise 14: $W(x) = \begin{vmatrix} \sin x & \sin 2x & \sin 3x \\ \cos x & 2\cos 2x & 3\cos 3x \\ -\sin x & -4\sin 2x & -9\sin 3x \end{vmatrix}$. Rather than computing the determinant and showing it is not identically zero, it suffices to find some value of x such that $W(x) \neq 0$. Let $x = \frac{\pi}{3}$; then

$$\begin{aligned} W\left(\frac{\pi}{3}\right) &= \begin{vmatrix} \sin\frac{\pi}{3} & \sin\frac{2\pi}{3} & \sin\pi \\ \cos\frac{\pi}{3} & 2\cos\frac{2\pi}{3} & 3\cos\pi \\ -\sin\frac{\pi}{3} & -4\sin\frac{2\pi}{3} & -9\sin\pi \end{vmatrix} \\ &= \begin{vmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & -1 & -3 \\ -\frac{\sqrt{3}}{2} & -2\sqrt{3} & 0 \end{vmatrix} \\ &= 3\begin{vmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -2\sqrt{3} \end{vmatrix} \\ &= 3\left(-3 + \frac{3}{4}\right) = -\frac{27}{4} \neq 0. \end{aligned}$$

So the Wronskian is not identically zero and therefore the functions are linearly independent.

17. **(a)** Yes, it is. Suppose

$$a(\mathbf{u}+\mathbf{v})+b(\mathbf{v}+\mathbf{w})+c(\mathbf{u}+\mathbf{w})=(a+c)\mathbf{u}+(a+b)\mathbf{v}+(b+c)\mathbf{w}=\mathbf{0}.$$

Since $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ are linearly independent, it follows that $a+c=0$, $a+b=0$, and $b+c=0$. Subtracting the latter two from each other gives $a-c=0$; combining this with the first equation shows that $a=c=0$, and then $b=0$. So the only solution to the system is the trivial one, so these three vectors are linearly independent.

(b) No, they are not, since

$$(\mathbf{u}-\mathbf{v})+(\mathbf{v}-\mathbf{w})-(\mathbf{u}-\mathbf{w})=\mathbf{0}.$$

18. Since $\dim M_{22}=4$ but $\mathcal{B}$ has only three elements, it cannot be a basis.

19. If

$$a\begin{bmatrix}1&0\\0&1\end{bmatrix}+b\begin{bmatrix}0&-1\\1&0\end{bmatrix}+c\begin{bmatrix}1&1\\1&1\end{bmatrix}+d\begin{bmatrix}1&1\\1&-1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix},$$

then

$$\begin{aligned}a\qquad+c+d&=0\\-b+c+d&=0\\b+c+d&=0\\a\qquad+c-d&=0.\end{aligned}$$

Subtracting the second and third equations shows that $b=0$; subtracting the first and fourth shows that $d=0$. Then from the second equation, $c=0$ as well, which forces $a=0$. So the only solution is the trivial solution, and these matrices are linearly independent. Since there are four of them, and $\dim M_{22}=4$, it follows that they form a basis for M_{22}.

20. Since $\dim M_{22}=4$, if these vectors are linearly independent then they span and thus form a basis. To see if they are linearly independent, we want to solve

$$c_1\begin{bmatrix}1&0\\0&1\end{bmatrix}+c_2\begin{bmatrix}0&1\\1&0\end{bmatrix}+c_3\begin{bmatrix}1&1\\0&1\end{bmatrix}+c_4\begin{bmatrix}1&0\\1&1\end{bmatrix}=0.$$

Solving this linear system gives $c1=-2c_4$, $c_2=-c_4$, and $c_3=c_4$. For example, letting $c_4=1$ gives

$$-2\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}0&1\\1&0\end{bmatrix}+\begin{bmatrix}1&1\\0&1\end{bmatrix}+\begin{bmatrix}1&0\\1&1\end{bmatrix}=0.$$

Thus the matrices are linearly dependent so do not form a basis (see Theorem 6.10).

21. Since $\dim M_{22}=4$ but $\mathcal{B}$ has five elements, it cannot be a basis.

22. Since $\dim \mathscr{P}_2=3$, if these vectors are linearly independent then they span and thus form a basis. To see if they are linearly independent, we want to solve

$$c_1x+c_2(1+x)+c_3(x-x^2)=c_2+(c_1+c_2+c_3)x-c_3x^2=0.$$

Rather than setting up the augmented matrix and row-reducing it, simply note that the first and third terms force $c_2=c_3=0$; then the second term forces $c_1=0$. So the only solution is the trivial solution, and these three polynomials are linearly independent, so they form a basis for $\mathscr{P}_2$.

23. Since $\dim \mathscr{P}_2 = 3$, if these vectors are linearly independent then they span and thus form a basis. To see if they are linearly independent, we want to solve

$$c_1(1-x) + c_2(1-x^2) + c_3(x-x^2) = (c_1+c_2) + (-c_1+c_3)x + (-c_2-c_3)x^2 = 0.$$

To solve the resulting linear system, we row-reduce its augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This has general solution $c_1 = t$, $c_2 = -t$, $c_3 = t$, so setting $t = 1$ we get

$$(1-x) - (1-x^2) + (x-x^2) = 0.$$

Therefore the vectors are not linearly independent, so they cannot form a basis for $\mathscr{P}_2$, which has dimension 3.

24. Since $\dim \mathscr{P}_2 = 3$ but there are only two vectors in $\mathcal{B}$, they cannot form a basis.

25. Since $\dim \mathscr{P}_2 = 3$ but there are four vectors in $\mathcal{B}$, they must be linearly dependent, so cannot form a basis.

26. We want to find constants c_1, c_2, c_3, and c_4 such that

$$c_1E_{22} + c_2E_{21} + c_3E_{12} + c_4E_{11} = A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Setting up the equivalent linear system gives

$$\begin{aligned} c_1 &= 4 \\ c_2 &= 3 \\ c_3 &= 2 \\ c_4 &= 1. \end{aligned}$$

This system obviously has the solution $c_1 = 4$, $c_2 = 3$, $c_3 = 2$, $c_4 = 1$, so the coordinate vector is

$$[A]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

27. We want to find constants c_1, c_2, c_3, and c_4 such that

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Setting up the equivalent linear system gives

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1 \\ c_2 + c_3 + c_4 &= 2 \\ c_3 + c_4 &= 3 \\ c_4 &= 4. \end{aligned}$$

By backwards substitution, the solution is $c_4 = 4$, $c_3 = -1$, $c_2 = -1$, $c_1 = -1$, so the coordinate vector is

$$[A]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}.$$

28. We want to find constants c_1, c_2, and c_3 such that

$$p(x) = 1 + 2x + 3x^2 = c_1(1 + x) + c_2(1 - x) + c_3(x^2) = (c_1 + c_2) + (c_1 - c_2)x + c_3x^2.$$

The corresponding linear system is

$$\begin{aligned} c_1 + c_2 \quad &= 1 \\ c_1 - c_2 \quad &= 2 \\ c_3 &= 3. \end{aligned}$$

Row-reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 3 \end{array}\right],$$

so that $c_1 = \frac{3}{2}$, $c_2 = -\frac{1}{2}$, and $c_3 = 3$. Thus the coordinate vector is

$$[p]_{\mathcal{B}} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 3 \end{bmatrix}.$$

29. We want to find constants c_1, c_2, and c_3 such that

$$p(x) = 2 - x + 3x^2 = c_1(1) + c_2(1 + x) + c_3(-1 + x^2) = (c_1 + c_2 - c_3) + c_2x + c_3x^2.$$

Equating coefficients shows that $c_3 = 3$ and $c_2 = -1$; substituting into the equation $c_1 + c_2 - c_3 = 2$ gives $c_1 = 6$, so that the coordinate vector is

$$[p]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix}.$$

30. To show $\mathcal{B}$ is a basis, we must show that it spans V and that the elements of $\mathcal{B}$ are linearly independent. $\mathcal{B}$ spans V, since if $\mathbf{v} \in V$, then it can be written as a linear combination of vectors in $\mathcal{B}$ by hypothesis. To show they are linearly independent, note that $\mathbf{0} = \sum 0\mathbf{u}_i$; since the representation of $\mathbf{0}$ is unique by assumption, it follows that the only solution to $\mathbf{0} = \sum c_i\mathbf{u}_i$ is the trivial solution, so that $\mathcal{B} = \{\mathbf{u}_i\}$ is a linearly independent set. Thus $\mathcal{B}$ is a basis for V.

31. $[c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k]_{\mathcal{B}} = [c_1\mathbf{u}_1]_{\mathcal{B}} + \cdots + [c_k\mathbf{u}_k]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_k[\mathbf{u}_k]_{\mathcal{B}}$.

32. Suppose $c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$. Then clearly

$$\mathbf{0} = [c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n]_{\mathcal{B}}.$$

But by Exercise 31, the latter expression is equal to

$$c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_n[\mathbf{u}_n]_{\mathcal{B}};$$

since $[\mathbf{u}_i]_{\mathcal{B}}$ are linearly independent in $\mathbb{R}^n$, we get $c_i = 0$ for all i, so that the only solution to the original equation is the trivial solution. Thus the $\mathbf{u}_i$ are linearly independent in V.

33. First suppose that $\operatorname{span}(\mathbf{u}_1, \ldots, \mathbf{u}_m) = V$. Choose

$$\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

Then $\sum c_i\mathbf{u}_i \in V$ since $V = \text{span}(\mathbf{u}_i)$. Therefore by Exercise 31

$$\mathbf{x} = [c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_m[\mathbf{u}_m]_{\mathcal{B}},$$

which shows that $\mathbf{x} \in \text{span}(S)$.

For the converse, suppose that $\text{span}(S) = \mathbb{R}^n$. Choose $\mathbf{v} \in V$; then $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$, and using Exercise 31 again,

$$[\mathbf{v}]_{\mathcal{B}} = \sum c_i[\mathbf{u}_i]_{\mathcal{B}} = \left[\sum c_i\mathbf{u}_i\right]_{\mathcal{B}}.$$

But since $\mathcal{B}$ is a basis, and the two elements $\mathbf{v}$ and $\sum c_i\mathbf{u}_i$ of V have the same coordinate vector, they must be the same element. Thus $\mathbf{v} = \sum c_i\mathbf{u}_i$, so that $\{\mathbf{u}_i\}$ spans V.

34. Using the standard basis $\{x^2, x, 1\}$ for $\mathscr{P}_2$, if $p(x) = ax^2 + bx + c \in V$, then $p(0) = c = 0$, so that $c = 0$ and therefore $p(x) = ax^2 + bx$. Since x^2 and x are linearly independent and span V, $\{x, x^2\}$ is a basis, so that $\dim V = 2$.

35. Two polynomials in V are $p(x) = 1 - x$ and $q(x) = 1 - x^2$, since $p(1) = q(1) = 0$. If we show these polynomials are linearly independent and span V, we are done. They are linearly independent, since if

$$a(1 - x) + b(1 - x^2) = (a + b) - ax - bx^2 = 0,$$

then equating coefficients shows that $a = b = 0$. To see that they span, suppose $r(x) = c + bx + ax^2 \in V$. Then $r(1) = a + b + c = 0$, so that $c = -b - a$ and therefore

$$r(x) = -(b + a) + bx + ax^2 = -b(1 - x) - a(1 - x^2),$$

so that r is a linear combination of p and q. Thus $\{p(x), q(x)\}$ forms a basis for V, which has dimension 2.

36. Suppose $p(x) = c + bx + ax^2$; then $xp'(x) = bx + 2ax^2$. So $p(x) = xp'(x)$ means that $c = 0$ and $a = 2a$, so that $a = 0$. Thus $p(x) = bx$. Then clearly $V = \text{span}(x)$, so that $\{x\}$ forms a basis, and $\dim V = 1$.

37. Upper triangular matrices in M_{22} have the form

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = aE_{11} + bE_{12} + dE_{22}.$$

Since E_{11}, E_{12}, and E_{22} span V, and we know they are linearly independent, it follows that they form a basis for V, which therefore has dimension 3.

38. Skew-symmetric matrices in M_{22} have the form

$$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = bA.$$

Since A spans V, and $\{A\}$ is a linearly independent set, we see that $\{A\}$ is a basis for V, which therefore has dimension 1.

39. Suppose that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} a & a + b \\ c & c + d \end{bmatrix}, \qquad BA = \begin{bmatrix} a + c & b + d \\ c & d \end{bmatrix}.$$

Then $AB = BA$ means that $a = a + c$, so that $c = 0$, and also $a + b = b + d$, so that $a = d$. So the matrix A has the form

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + bE_{12} = aI + bE_{12}.$$

I and E_{12} are linearly independent since $I = E_{11} + E_{22}$, and $\{E_{11}, E_{22}, E_{12}\}$ is a linearly independent set. Since they span, we see that $\{I, E_{12}\}$ forms a basis for V, which therefore has dimension 2.

40. $\dim M_{nn} = n^2$. For symmetric matrices, we have $a_{ij} = a_{ji}$. Therefore specifying the entries on or above the diagonal determines the matrix, and these entries may take any value. Since there are

$$n + (n-1) + \cdots + 2 + 1 = \frac{n(n+1)}{2}$$

entries on or above the diagonal, this is the dimension of the space of symmetric matrices.

41. $\dim M_{nn} = n^2$. For skew-symmetric matrices, we have $a_{ij} = -a_{ji}$. Therefore all the diagonal entries are zero; further, specifying the entries above the diagonal determines the matrix, and these entries may take any value. Since there are

$$(n-1) + \cdots + 2 + 1 = \frac{n(n-1)}{2}$$

entries above the diagonal, this is the dimension of the space of symmetric matrices.

42. Following the hint, Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for $U \cap W$. By Theorem 6.10(e), this can be extended to a basis $\mathcal{C}$ for U and to a basis $\mathcal{D}$ for W. Consider the set $\mathcal{C} \cup \mathcal{D}$. Clearly it spans $U + W$, since an element of $U + W$ is $\mathbf{u} + \mathbf{w}$ for $\mathbf{u} \in U$, $\mathbf{w} \in W$; since $\mathcal{C}$ and $\mathcal{D}$ are bases for U and W, the result follows. Next we must show that this set is linearly independent. Suppose $\dim U = m$ and $\dim V = n$; then

$$\mathcal{C} = \{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\} \cup \mathcal{B}, \qquad \mathcal{D} = \{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\} \cup \mathcal{B}.$$

Suppose that a linear combination of $\mathcal{C} \cup \mathcal{D}$ is the zero vector. This means that

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + c_{k+1}\mathbf{u}_{k+1} + \cdots + c_m\mathbf{u}_m + d_{k+1}\mathbf{w}_{k+1} + d_n\mathbf{w}_n = A + C + D = \mathbf{0}.$$

This means that $C = -A - D$. Since $A \in U \cap W$ and $D \in W$, we see that $-A - D \in W$ so that $C \in W$. But also $C \in U$, since it is a combination of basis vectors for U. Thus $C \in U \cap W$, which is impossible unless all of the c_i are zero, since the $\mathbf{u}_{k+1}$ are elements of U not in the subspace $U \cap W$. We are left with $A + D = 0$; this is a linear combination of basis elements of W, so that all the a_i and all the d_j must be zero as well. Therefore $\mathcal{C} \cup \mathcal{D}$ is linearly independent.

It follows that $\mathcal{C} \cup \mathcal{D}$ forms a basis for $U + W$, so it remains to count the elements in this set. Clearly $\mathcal{C} \cap \mathcal{D} = \mathcal{B}$, for any basis element in common between $\mathcal{C}$ and $\mathcal{D}$ must be in $U \cap W$, so it must be in $\mathcal{B}$. Write $\#X$ for the number of elements in the set X. Then

$$\dim(U + W) = \#(\mathcal{C} \cup \mathcal{D}) = \#\mathcal{C} + \#\mathcal{D} - \#\mathcal{B} = n + m - k = \dim U + \dim W - \dim(U \cap W).$$

43. Let $\dim U = m$, with basis $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$, and $\dim V = n$, with basis $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

(a) Claim that a basis for $U \times V$ is $\mathcal{D} = \{(\mathbf{u}_i, \mathbf{0}), (\mathbf{0}, \mathbf{v}_j)\}$. This will also show that $\dim(U \times V) = mn$, by counting the basis vectors. First we want to show that $\mathcal{D}$ spans. Let $\mathbf{x} = (\mathbf{u}, \mathbf{v}) \in U \times V$. Then

$$\mathbf{x} = (\mathbf{u}, \mathbf{v}) = \left(\sum_{i=1}^{m} c_i\mathbf{u}_i,\ \sum_{j=1}^{n} d_j\mathbf{v}_j\right) = \sum_{i=1}^{m} c_i(\mathbf{u}_i, \mathbf{0}) + \sum_{j=1}^{n} d_j(\mathbf{0}, \mathbf{v}_j),$$

proving that $\mathcal{D}$ spans $U \times V$. To see that $\mathcal{D}$ is linearly independent, suppose that $\mathbf{0}$ is a linear combination of the elements of $\mathcal{D}$. This means that

$$\mathbf{0} = (\mathbf{0}, \mathbf{0}) = \sum_{i=1}^{m} c_i(\mathbf{u}_i, \mathbf{0}) + \sum_{j=1}^{n} d_j(\mathbf{0}, \mathbf{v}_j) = \left(\sum_{i=1}^{m} c_i\mathbf{u}_i, \sum_{j=1}^{n} d_j\mathbf{v}_j\right).$$

But the $\mathbf{u}_i$ are linearly independent, so that $\sum_{i=1}^{m} c_i\mathbf{u}_i = \mathbf{0}$ means that all of the c_i are zero; similarly the $\mathbf{v}_j$ are linearly independent, so that $\sum_{j=1}^{n} d_j\mathbf{v}_j = \mathbf{0}$ means that all of the d_j are zero. Thus $\mathcal{D}$ is linearly independent.

So $\mathcal{D}$ is a basis for $U \times W$, and therefore $\dim(U \times W) = mn$.

(b) Let $\dim W = k$, with basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$. Claim that $\mathcal{D} = \{(\mathbf{w}_i, \mathbf{w}_i)\}$ is a basis for Δ; proving this claim also shows, by counting the basis vectors, that $\dim \Delta = \dim W$. To see that $\mathcal{D}$ spans Δ, choose $(\mathbf{w}, \mathbf{w}) \in \Delta$; then $\mathbf{w} = \sum c_i \mathbf{w}_i$, so that

$$(\mathbf{w}, \mathbf{w}) = \left(\sum c_i \mathbf{w}_i, \sum c_i \mathbf{w}_i\right) = \sum (c_i \mathbf{w}_i, c_i \mathbf{w}_i) = \sum c_i (\mathbf{w}_i, \mathbf{w}_i).$$

Thus $\mathcal{D}$ spans Δ. To see that $\mathcal{D}$ is linearly independent, suppose that $\sum c_i (\mathbf{w}_i, \mathbf{w}_i) = \mathbf{0}$. Then

$$\mathbf{0} = \sum c_i (\mathbf{w}_i, \mathbf{w}_i) = \sum (c_i \mathbf{w}_i, c_i \mathbf{w}_i) = \left(\sum c_i \mathbf{w}_i, \sum c_i \mathbf{w}_i\right).$$

This means that $\sum c_i \mathbf{w}_i = \mathbf{0}$; since the $\mathbf{w}_i$ are linearly independent, we see that all the c_i are zero. So $\mathcal{D}$ is linearly independent, showing that it is a basis for Δ and thus that $\dim \Delta = \dim W$.

44. Suppose that $\mathcal{P}$ is finite dimensional, and that $\{p_1(x), \ldots, p_n(x)\}$ is a basis. Let m be the largest power of x that appears in any of the basis elements, and consider $q(x) = x^{m+1}$. Since any linear combination of the p_i has powers of x at most equal to m, it follows that $q \notin \operatorname{span}\{p_i\}$, so that $\mathcal{P}$ cannot have a finite basis.

45. Start by adding all three standard basis vectors for $\mathcal{P}_2$ to the set, yielding $\{1+x, 1+x+x^2, 1, x, x^2\}$. Since $(1+x)+x^2 = 1+x+x^2$, we see that $\{1+x, 1+x+x^2, x^2\}$ is linearly dependent, so discard x^2, leaving $\{1+x, 1+x+x^2, 1, x\}$. Next, since $(1+x) = 1+x$, the set $\{1+x, 1, x\}$ is linearly dependent, so discard x, leaving $\mathcal{B} = \{1+x, 1+x+x^2, 1\}$. This is a set of three vectors, so to prove that it is a basis, it suffices to show that it spans. But by construction, the standard basis vectors not in $\mathcal{B}$ are linear combinations of elements of $\mathcal{B}$. Hence $\mathcal{B}$ is a basis for $\mathcal{P}_2$.

46. Start by adding the standard basis vectors E_{ij} to the set, giving

$$\mathcal{B} = \left\{C = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E_{11}, E_{12}, E_{21}, E_{22}\right\}.$$

Since $E_{11} = D - C$ is the difference of the two given matrices, we discard it from $\mathcal{B}$, leaving us with five elements. Next, $C = E_{12} + E_{22}$, so discard E_{22}, leaving us with $\mathcal{B} = \{C, D, E_{12}, E_{21}\}$. This has four elements, and $\dim M_{22} = 4$, so to prove that this is a basis, it suffices to prove that it spans. But by construction, the standard basis vectors not in $\mathcal{B}$ are linear combinations of elements of $\mathcal{B}$. So $\mathcal{B}$ spans and therefore forms a basis for M_{22}.

47. Start by adding the standard basis vectors E_{ij} to the set, giving

$$\mathcal{B} = \left\{I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, E_{11}, E_{12}, E_{21}, E_{22}\right\}.$$

Since $E_{21} = \frac{1}{2}(C+D)$, we discard it from $\mathcal{B}$. Next, $I = E_{11} + E_{22}$, so discard E_{22} from $\mathcal{B}$. Finally, $E_{12} = \frac{1}{2}(C-D)$, so we discard it from $\mathcal{B}$. This leaves us with $\mathcal{B} = \{I, C, D, E_{11}\}$. This has four elements, and $\dim M_{22} = 4$, so to prove that this is a basis, it suffices to prove that it spans. But by construction, the standard basis vectors not in $\mathcal{B}$ are linear combinations of elements of $\mathcal{B}$. So $\mathcal{B}$ spans and therefore forms a basis for M_{22}.

48. Start by adding the standard basis vectors E_{ij} to the set, giving

$$\mathcal{B} = \left\{I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_{11}, E_{12}, E_{21}, E_{22}\right\}.$$

Since $I + C = E_{11} + E_{12} + E_{21} + E_{22}$, these are a linearly dependent set; discard E_{22} from $\mathcal{B}$. Next, $C = E_{12} + E_{21}$, so discard E_{21} from $\mathcal{B}$. This leaves us with $\mathcal{B} = \{I, C, E_{11}, E_{12}\}$. This has four elements, and $\dim M_{22} = 4$, so to prove that this is a basis, it suffices to prove that it spans. But by construction, the standard basis vectors not in $\mathcal{B}$ are linear combinations of elements of $\mathcal{B}$. So $\mathcal{B}$ spans and therefore forms a basis for M_{22}.

49. The standard basis for $\mathcal{P}_1$ is $\{1, x\}$. If we show that each of these vectors is in the span of the given vectors, then those vectors span $\mathcal{P}_1$. But $1 \in \operatorname{span}(1, 1+x, 2x)$, and $x = (1+x) - 1 \in \operatorname{span}(1, 1+x, 2x)$. Thus

$$\operatorname{span}(1, 1+x, 2x) \supseteq \operatorname{span}(1, x) = \mathcal{P}_1,$$

so that a basis for the span is $\{1, x\}$.

50. Since $(1-2x) + (2x - x^2) = 1 - x^2$, these three elements are linearly independent, so we can discard $1 - x^2$ from the set without changing the span, leaving $\{1 - 2x, 2x - x^2, 1 + x^2\}$. Claim these are linearly independent. Suppose that

$$a(1-2x) + b(2x - x^2) + c(1 + x^2) = (a+c) + (-2a + 2b)x + (-b + c)x^2 = 0.$$

Then $a = -c$ and $a = b$ from the constant term and the coefficient of x, so that $b = -c$. But then $b = -c$ together with $-b + c = 0$ shows that $b = c = 0$, so that $a = 0$. Hence these three polynomials are linearly independent, so they form a basis for the span of the original set.

51. Since $(1-x) + (x - x^2) = 1 - x^2$ and $(1-x) - (x - x^2) = 1 - 2x + x^2$, we can remove both $1 - x^2$ and $1 - 2x + x^2$ from the set without changing the span. This leaves us with $\{1 - x, x - x^2\}$. Claim these are linearly independent. Suppose that

$$a(1-x) + b(x - x^2) = a + (-a + b)x - bx^2 = 0.$$

Then clearly $a = b = 0$. Hence these two polynomials are linearly independent, so they form a basis for the span of the original set.

52. Let

$$V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad V_4 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Then $V_3 + V_4 = 0$, so we can discard V_4 from the set without changing the span. Next, $V_3 = V_2 - V_1$, so we can discard V_3 from the set without changing the span. So we are left with $\{V_1, V_2\}$; these are linearly independent since they are not scalar multiples of one another. So this set forms a basis for the span of the given set.

53. Since $\cos 2x = \cos^2 x - \sin^2 x$, we can discard $\cos 2x$ from the set without changing the span, leaving us with $\{\sin^2 x, \cos^2 x\}$. Claim that this set is linearly independent. For suppose that $a \sin^2 x + b \cos^2 x = 0$ holds for all x. Setting $x = 0$ gives $b = 0$, while setting $x = \frac{\pi}{2}$ gives $a = 0$. Thus $a = b = 0$ and these functions are linearly independent. So $\{\sin^2 x, \cos^2 x\}$ forms a basis for the span of the original set.

54. Suppose that $a\mathbf{v} + c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$, so that $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = -a\mathbf{v}$. If $a \neq 0$, then dividing through by $-a$ shows that $\mathbf{v} \in \operatorname{span}(S)$, a contradiction. Therefore we must have $a = 0$, and we are left with

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0,$$

which implies (since the $\mathbf{v}_i$ are linearly independent) that all of the c_i are zero. Therefore all coefficients must be zero, so that S' is linearly independent.

55. If $\mathbf{v}_n \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$, then there are constants such that

$$\mathbf{v}_n = c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1}.$$

Now choose $\mathbf{x} \in V = \operatorname{span}(S)$. Then

$$\begin{aligned} \mathbf{x} &= a_1\mathbf{v}_1 + \cdots + a_{n-1}\mathbf{v}_{n-1} + a_n\mathbf{v}_n \\ &= a_1\mathbf{v}_1 + \cdots + a_{n-1}\mathbf{v}_{n-1} + a_n(c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1}) \\ &= (a_1 + a_nc_1)\mathbf{v}_1 + \cdots + (a_{n-1} + a_nc_{n-1})\mathbf{v}_{n-1}, \end{aligned}$$

showing that $\mathbf{x} \in \operatorname{span}(S')$. Then $V \subseteq \operatorname{span}(S') \subseteq \operatorname{span}(S) = V$, so that $\operatorname{span}(S') = V$.

56. Use induction on n. If $\mathcal{S} = \{\mathbf{v}_1\}$, then $\mathcal{S}$ is linearly independent so is a basis. Next, suppose the statement is true for all sets that span V and have no more than $n-1$ elements. Let $\mathcal{S} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. If $\mathcal{S}$ is linearly independent, then it is a basis and we are done. Otherwise it is a linearly dependent set, so we can write some element of S (say, $\mathbf{v}_n$ after a possible reordering) as a linear combination of the other, so that $\mathbf{v}_n \in \operatorname{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}\}$. Then by Exercise 55, we can remove $\mathbf{v}_n$ from $\mathcal{S}$, giving $\mathcal{S}' = \{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ that also spans V. By the inductive assumption, $\mathcal{S}'$ can be made into a basis for V by removing some (possibly no) elements, so $\mathcal{S}$ can.

57. Suppose $\mathbf{x} \in V$. Since $\mathcal{S} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, there are scalars $a_1, \ldots, a_n$ such that

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Since the c_i are nonzero, we also have

$$\mathbf{x} = \frac{a_1}{c_1}(c_1\mathbf{v}_1) + \frac{a_2}{c_2}\mathbf{v}_2 + \cdots + \frac{a_n}{c_n}(c_n\mathbf{v}_n),$$

so that $\mathcal{T} = \{c_1\mathbf{v}_1, \ldots, c_n\mathbf{v}_n\}$ spans V. To show $\mathcal{T}$ is linearly independent, the argument is much the same. Suppose that

$$b_1(c_1\mathbf{v}_1) + b_2(c_2\mathbf{v}_2) + \cdots + b_n(c_n\mathbf{v}_n) = (b_1c_1)\mathbf{v}_1 + (b_2c_2)\mathbf{v}_2 + \cdots + (b_nc_n)\mathbf{v}_n = \mathbf{v}_0.$$

Since $\mathcal{S}$ is linearly independent, it follows that $b_ic_i = 0$ for all i. Since all of the c_i are nonzero, all of the b_i must be zero, so that $\mathcal{T}$ is linearly independent. Therefore $\mathcal{T}$ is also a basis for V.

58. Let

$$\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}, \qquad \mathcal{T} = \{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \ldots, \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n\}.$$

By Exercise 21 in Section 2.3, to show that $\operatorname{span}(\mathcal{T}) = \operatorname{span}(\mathcal{S}) = V$ it suffices to show that every element of $\mathcal{S}$ is in $\operatorname{span}(\mathcal{T})$ and vice versa. But

$$\begin{aligned}\mathbf{v}_i &= (\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_{i-1} + \mathbf{v}_i) - (\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_{i-1}) \in \operatorname{span}(\mathcal{T}) \text{ and} \\ \mathbf{v}_1 + \cdots + \mathbf{v}_i &= (\mathbf{v}_1) + (\mathbf{v}_2) + \cdots + (\mathbf{v}_i) \in \operatorname{span}(\mathcal{S}).\end{aligned}$$

Thus the two spans are equal. Since $\mathcal{S}$ is a basis, and $\mathcal{T}$ has the same number of elements as $\mathcal{S}$, it must also be a basis, by Theorem 6.10(d).

59. **(a)**

$$\begin{aligned}
p_0(x) &= \frac{(x-a_1)(x-a_2)}{(a_0-a_1)(a_0-a_2)} = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{x^2-5x+6}{2} = \frac{1}{2}x^2 - \frac{5}{2}x + 3 \\
p_1(x) &= \frac{(x-a_0)(x-a_2)}{(a_1-a_0)(a_1-a_2)} = \frac{(x-1)(x-3)}{(2-1)(2-3)} = \frac{x^2-4x+3}{-1} = -x^2 + 4x - 3 \\
p_2(x) &= \frac{(x-a_0)(x-a_1)}{(a_2-a_0)(a_2-a_1)} = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{x^2-3x+2}{2} = \frac{1}{2}x^2 - \frac{3}{2}x + 1.
\end{aligned}$$

(b) If $i = j$, then substituting a_j for x in the definition of p_i gives a numerator equal to the denominator, so that $p_i(a_j) = 1$ for $i = j$. If $i \neq j$, however, then substituting a_j for x in the definition of p_i gives zero, since the term $x - a_j$ in the numerator of p_i becomes zero. Thus

$$p_i(a_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

60. **(a)** Note first that the degree of each p_i is n, since there are n factors in the numerator, so that $p_i \in \mathscr{P}_n$ for all i. Suppose that there are constants c_i such that

$$c_0p_0(x) + c_1p_1(x) + \cdots + c_np_n(x) = 0.$$

Then this equation must hold for all x. Substitute a_i for x. Then by Exercise 59(b), all $p_j(a_i)$ for $j \neq i$ vanish, leaving only $c_ip_i(a_i) = c_i$. Therefore $c_i = 0$. Since this holds for every i, we see that all of the c_i are zero, so that the p_i are linearly independent.

(b) Since $\dim \mathscr{P}_n = n+1$ and there are $n+1$ polynomials $p_0, p_1, \ldots, p_n$ that span, they must form a basis, by Theorem 6.10(d).

61. (a) This is similar to part (a) of Exercise 60. Suppose that $q(x) \in \mathscr{P}$, and that

$$q(x) = c_0p_0(x) + c_1p_1(x) + \cdots + c_np_n(x).$$

Then substituting a_i for x gives $q(a_i) = c_ip_i(a_i) = c_i$, since $p_j(a_i) = 0$ for $j \neq i$ by Exercise 59(b). Since the p_i form a basis for $\mathscr{P}$, any element of $\mathscr{P}$ has a unique representation as a linear combination of the p_i, so that unique representation must be the one we just found:

$$q(x) = q(a_0)p_0(x) + q(a_1)p_1(x) + \cdots + q(a_n)p_n(x).$$

(b) $q(x)$ passes through the given $n+1$ points if and only if $q(a_i) = c_1$. But this is exactly what we proved in part (a). Since the representation is unique in part (a), this is the only such degree n polynomial.

(c) (i) Here $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, so we use the Lagrange polynomials determined in Exercise 59(a) and therefore

$$\begin{aligned} q(x) &= c_0p_0(x) + c_1p_1(x) + c_2p_2(x) \\ &= 6\left(\frac{1}{2}x^2 - \frac{5}{2}x + 3\right) - \left(-x^2 + 4x - 3\right) - 2\left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right) \\ &= 3x^2 - 16x + 19. \end{aligned}$$

(ii) Here $a_0 = -1$, $a_1 = 0$, $a_2 = 3$. We first compute the Lagrange polynomials:

$$\begin{aligned} p_0(x) &= \frac{(x-a_1)(x-a_2)}{(a_0-a_1)(a_0-a_2)} = \frac{(x-0)(x-3)}{(-1-0)(-1-3)} = \frac{x^2-3x}{4} = \frac{1}{4}x^2 - \frac{3}{4}x \\ p_1(x) &= \frac{(x-a_0)(x-a_2)}{(a_1-a_0)(a_1-a_2)} = \frac{(x-(-1))(x-3)}{(0-(-1))(0-3)} = \frac{x^2-2x-3}{-3} = -\frac{1}{3}x^2 + \frac{2}{3}x + 1 \\ p_2(x) &= \frac{(x-a_0)(x-a_1)}{(a_2-a_0)(a_2-a_1)} = \frac{(x-(-1))(x-0)}{(3-(-1))(3-0)} = \frac{x^2+x}{12} = \frac{1}{12}x^2 + \frac{1}{12}x. \end{aligned}$$

Using these polynomials, we get

$$\begin{aligned} q(x) &= c_0p_0(x) + c_1p_1(x) + c_2p_2(x) \\ &= 10\left(\frac{1}{4}x^2 - \frac{3}{4}x\right) + 5\left(-\frac{1}{3}x^2 + \frac{2}{3}x + 1\right) + 2\left(\frac{1}{12}x^2 + \frac{1}{12}x\right) \\ &= \left(\frac{5}{2} - \frac{5}{3} + \frac{1}{6}\right)x^2 + \left(-\frac{15}{2} + \frac{10}{3} + \frac{1}{6}\right)x + 5 \\ &= x^2 - 4x + 5. \end{aligned}$$

62. Suppose that $(a_0, 0)$, $(a_1, 0), \ldots, (a_n, 0)$ be the $n+1$ distinct zeros. Then by Exercise 61(a), the only polynomial passing through these $n+1$ points is

$$q(x) = 0p_0(x) + 0p_1(x) + \cdots + 0p_n(x) = 0,$$

so it is the zero polynomial.

63. An $n \times n$ matrix is invertible if and only if it row-reduces to the identity matrix, so that its columns are linearly independent. Since n linearly independent columns in $\mathbb{Z}_p^n$ form a basis for $\mathbb{Z}_p^n$, finding the number of invertible matrices in $M_{nn}(\mathbb{Z}_p)$ is the same as finding the number of ways to construct a basis for $\mathbb{Z}_p^n$.

To construct a basis, we can start by choosing any nonzero vector $\mathbf{v}_1 \in \mathbb{Z}_p^n$. There are $p^n - 1$ ways to choose this vector (since $\mathbb{Z}_p^n$ has p^n elements, and we are excluding only one). For the second basis element $\mathbf{v}_2$, we can choose any vector not in $\text{span}(\mathbf{v}_1) = \{a_1\mathbf{v}_1 : a_1 \in \mathbb{Z}_p\}$. There are therefore $p^n - p$ choices for $\mathbf{v}_2$. Likewise, suppose we have chosen $\mathbf{v}_1, \dots, \mathbf{v}_{k-1}$. Then we can choose for $\mathbf{v}_k$ any vector not in the span of the previous vectors, so any vector not in

$$\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{k-1}) = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{k-1}\mathbf{v}_{k-1} : a_1, a_2, \dots, a_{k-1} \in \mathbb{Z}_p\}.$$

There are $p^n - p^{k-1}$ ways of choosing $\mathbf{v}_k$.

Thus the total number of bases is $(p^n - 1)(p^n - p)(p^n - p^2)(\cdots)(p^n - p^{n-1})$.

Exploration: Magic Squares

1. A classical magic square M contains each of the numbers $1, 2, \dots, n^2$ exactly once. Since $\text{wt}(M)$ is the common sum of any row (or column), and there are n rows, we have, using the comments preceding Exercise 51 in Section 2.4)

$$\text{wt}(M) = \frac{1}{n}(1 + 2 + \cdots + n^2) = \frac{1}{n} \cdot \frac{n^2(n^2+1)}{2} = \frac{n(n^2+1)}{2}.$$

2. A classical 3×3 magic square will have common sum $\frac{3(3^2+1)}{2} = 15$. Therefore that no two of 7, 8, and 9 can be in the same row, column, or diagonal, since the sum of any two of these numbers is at least 15. The central square cannot be 1, since then $1 + 2 + k = 15$ is impossible. Similarly, it cannot be any of 2 through 4, since then some row, column, or diagonal contains the numbers 1 and 2, which is impossible. The central square cannot be 7, 8, or 9, since then some line will contain two of those numbers. Finally, it cannot be 6, since then some line will contain both 6 and 9. So the central square must be 5. Also, 7 and 1 cannot be in the same line, since the third number would have to be 7. Finally, 9 cannot be in a corner, since it would then participate in three lines adding to 15. But there are only two possible sets of numbers, one of which is 9, adding to 15: 9, 5, 1 and 9, 4, and 2. Thus 9 must be on a side. Keeping these constraints in mind and trying a few configurations gives for example

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{bmatrix}$$

These examples are essentially the same — the second one is a rotated and reflected version of the first.

3. Since the magic squares in Exercise 2 have a weight of 15, subtracting $\frac{14}{3}$ from each entry will subtract 14 from each row and column while maintaining common sums, so the result will have weight 1. For example, using the second magic square above, we get

$$\begin{bmatrix} -\frac{8}{3} & \frac{7}{3} & \frac{4}{3} \\ \frac{13}{3} & \frac{1}{3} & -\frac{11}{3} \\ -\frac{2}{3} & -\frac{5}{3} & \frac{10}{3} \end{bmatrix}.$$

In general, starting from any 3×3 classical magic square, to construct a magic square of weight w, add $\frac{w-15}{3}$ to each entry in the magic square. This clearly preserves the common sum property, and the new sum of each row is $15 + 3\frac{w-15}{3} = w$.

4. **(a)** We must show that if A and B are magic squares, then so is $A + B$, and if c is any scalar, then cA is also a magic square. Suppose $\text{wt}(A) = a$ and $\text{wt}(B) = b$. Then the sum of the elements in any row of $A+B$ is the sum of the elements in that row of A plus the sum of the elements in that row of B, which is therefore $a+b$. The same argument shows that the sum of the elements in any column or either diagonal of $A+B$ is also $a+b$, so that $A+B \in Mag_3$ with weight $a+b$. In the same way, the sum of the elements in any row or column, or either diagonal, of cA is c times the sum of the elements in that line of A, so it is equal to ca. Thus $cA \in Mag_3$ with weight ca.

(b) If $A, B \in Mag_3^0$, then $\text{wt}(A) = \text{wt}(B) = 0$. From part (a), $\text{wt}(A + B) = 0$ as well, and also $\text{wt}(cA) = c\,\text{wt}(A) = 0$. Thus $A + B, cA \in Mag_3^0$, so by Theorem 6.2, Mag_3^0 is a subspace of Mag_3.

5. Let M be any 3×3 magic square, with $\text{wt}(M) = w$. Using Exercise 3, we define M_0 to be the magic square of weight 0 derived from M by adding $\frac{-w}{3}$ to every element of M. This is the same as adding the matrix

$$\begin{bmatrix} -\frac{w}{3} & -\frac{w}{3} & -\frac{w}{3} \\ -\frac{w}{3} & -\frac{w}{3} & -\frac{w}{3} \\ -\frac{w}{3} & -\frac{w}{3} & -\frac{w}{3} \end{bmatrix} = -\frac{w}{3}J$$

to M. Thus

$$M_0 = M - \frac{w}{3}J, \text{ so that } M = M_0 + \frac{w}{3}J.$$

Thus $k = \frac{w}{3}$.

6. The linear equations express the fact that every row and column sum each diagonal sum is zero:

$$\begin{aligned} a + b + c \qquad\qquad\qquad\qquad &= 0 \\ d + e + f \qquad\qquad &= 0 \\ g + h + i &= 0 \\ a \qquad + d \qquad\quad + g \qquad &= 0 \\ b \qquad + e \qquad\quad + h \quad &= 0 \\ c \qquad + f \qquad\quad + i &= 0 \\ a \qquad\quad + e \qquad\qquad + i &= 0 \\ c \quad + e \quad + g \qquad\qquad &= 0 \end{aligned}$$

Row-reducing the associated matrix gives

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So there are two free variables in the general solution, which gives the magic square

$$\begin{bmatrix} -s & -t & s+t \\ t+2s & 0 & -t-2s \\ -s-t & t & s \end{bmatrix}.$$

7. From Exercise 6, a basis for Mag_3^0 is given by

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix},$$

so that Mag_3^0 has dimension 2. Note that these matrices are linearly independent since they are not scalar multiples of each other. (The matrix in Exercise 6 is not the same as that given in the hint. To show that the two are just different forms of one another, make the substitutions $u = -s$, $v = s + t$; then $s = -v$ and $t = v - s = u + v$, so the matrix becomes

$$\begin{bmatrix} u & -u-v & v \\ -u+v & 0 & u-v \\ -v & u+v & -u \end{bmatrix},$$

which is the matrix given in the hint.)

8. From Exercise 5, an arbitrary element $M \in Mag_3$ can be written

$$M = M_0 + kJ, \quad M_0 \in Mag_3, \quad J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Since M_0 has basis $\{A, B\}$ (see Exercise 7), it follows that M_3 is spanned by $\mathcal{S} = \{A, B, J\}$. To see that this is a basis, we must prove that these matrices are linearly independent. Now, we have seen that A and B are linearly independent, so if $\mathcal{S}$ is linearly dependent, then $\operatorname{span}(\mathcal{S}) = \operatorname{span}(A, B) = Mag_3^0$. But this is impossible, since $\mathcal{S}$ spans Mag_3, which contains matrices not in Mag_3^0. Thus $\mathcal{S}$ is linearly independent, so it forms a basis for Mag_3, which therefore has dimension 3.

9. Using the matrix M from exercise 5, adding both diagonals and the middle column gives

$$(a + e + i) + (c + e + g) + (b + e + h) = (a + b + c) + (g + h + i) + 3e,$$

which is the sum of the first and third rows, plus 3 times the central entry. Since each row, column, and diagonal sum is w, we get $w + w + w = w + w + 3e$, so that $e = \frac{w}{3}$.

10. The sum of the squares in the entries of M are

$$\begin{aligned} s^2 &+ (-s-t)^2 + t^2 + (-s+t)^2 + 0^2 + (s-t)^2 + (-t)^2 + (s+t)^2 + (-s)^2 \\ &= s^2 + (s^2 + 2st + t^2) + t^2 + (s^2 - 2st + t^2) + (s^2 - 2st + t^2) + t^2 + (s^2 + 2st + t^2) + s^2 \\ &= 6s^2 + 6t^2. \end{aligned}$$

We have another way of computing this sum, however. Since M was obtained from a classical 3×3 magic square by subtracting 5 from each entry (see Exercise 5), the sum of the squares of the entries of M is also

$$(1-5)^2 + (2-5)^2 + \cdots + (9-5)^2 = 60.$$

So the equation is $6s^2 + 6t^2 = 60$, or $s^2 + t^2 = 10$, which is a circle of radius $\sqrt{10}$ centered at the origin. The only way to write 10 as a sum of two integral squares is $(\pm 1)^2 + (\pm 3)^2$, so this equation has the eight integral solutions

$$(s, t) = (-3, -1),\ (-3, 1),\ (-1, -3),\ (-1, 3),\ (3, -1),\ (3, 1),\ (1, -3),\ (1, 3).$$

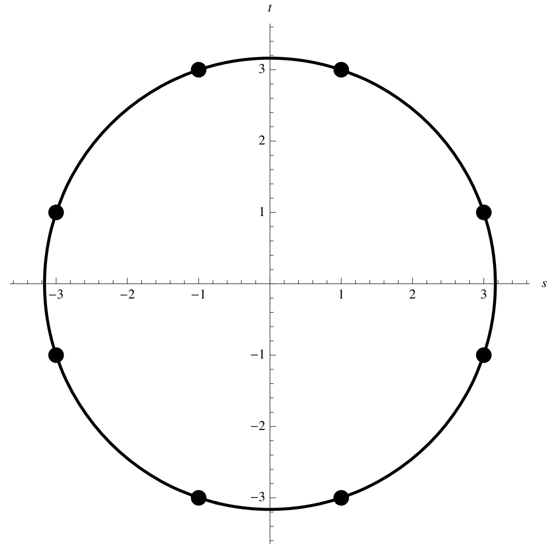
Substituting those values of s and t into

$$M + 5J = \begin{bmatrix} s+5 & -s-t+5 & t+5 \\ -s+t+5 & 5 & s-t+5 \\ -t+5 & s+t+5 & -s+5 \end{bmatrix}$$

gives the eight classical magic squares

$$\begin{bmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{bmatrix}, \quad \begin{bmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{bmatrix}, \quad \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}, \quad \begin{bmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{bmatrix},$$
$$\begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix}, \quad \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, \quad \begin{bmatrix} 6 & 7 & 2 \\ 1 & 5 & 9 \\ 8 & 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{bmatrix}.$$

These matrices can all be obtained from one another by some combination of row and column interchanges and matrix transposition. A plot of the circle, with these eight points marked, is



6.3 Change of Basis

1. **(a)**

$$\begin{bmatrix}2\\3\end{bmatrix} = b_1\begin{bmatrix}1\\0\end{bmatrix} + b_2\begin{bmatrix}0\\1\end{bmatrix} \quad\Rightarrow\quad \begin{aligned} b_1 &= 2\\ b_2 &= 3\end{aligned} \quad\Rightarrow\quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}2\\3\end{bmatrix}$$

$$\begin{bmatrix}2\\3\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} \quad\Rightarrow\quad \begin{aligned} c_1 &= \tfrac{5}{2}\\ c_2 &= -\tfrac{1}{2}\end{aligned} \quad\Rightarrow\quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}\frac{5}{2}\\-\frac{1}{2}\end{bmatrix}$$

(b) By Theorem 6.13, $[\ \mathcal{C} \mid \mathcal{B}\ ] \to [\ I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$:

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 1 & -1 & 0 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & \frac{1}{2} & -\frac{1}{2}\end{array}\right] \quad\Rightarrow\quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}$$

(c) $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}\frac{5}{2}\\-\frac{1}{2}\end{bmatrix}$, which is the same as the answer from part (a).

(d) By Theorem 6.13, $[\ \mathcal{B} \mid \mathcal{C}\ ] \to [\ I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{cc|cc}1 & 0 & 1 & 1\\ 0 & 1 & 1 & -1\end{array}\right] \quad\Rightarrow\quad P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

(e) $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{5}{2}\\-\frac{1}{2}\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$, which is the same as the answer from part (a).

2. **(a)**

$$\begin{bmatrix}4\\-1\end{bmatrix} = b_1\begin{bmatrix}1\\0\end{bmatrix} + b_2\begin{bmatrix}1\\1\end{bmatrix} \quad\Rightarrow\quad \begin{aligned} b_1 &= 5\\ b_2 &= -1\end{aligned} \quad\Rightarrow\quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}5\\-1\end{bmatrix}$$

$$\begin{bmatrix}4\\-1\end{bmatrix} = c_1\begin{bmatrix}0\\1\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} \quad\Rightarrow\quad \begin{aligned} c_1 &= -7\\ c_2 &= 2\end{aligned} \quad\Rightarrow\quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}-7\\2\end{bmatrix}$$

(b) By Theorem 6.13, $[\ \mathcal{C} \mid \mathcal{B}\ ] \to [\ I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$:

$$\left[\begin{array}{cc|cc}0 & 2 & 1 & 1\\ 1 & 3 & 0 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 0 & -\frac{3}{2} & -\frac{1}{2}\\ 0 & 1 & \frac{1}{2} & \frac{1}{2}\end{array}\right] \quad\Rightarrow\quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}-\frac{3}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

(c) $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$, which is the same as the answer from part (a).

(d) By Theorem 6.13, $[\ \mathcal{B} \mid \mathcal{C}\ ] \to [\ I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 3 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$$

(e) $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -7 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, which is the same as the answer from part (a).

3. **(a)**

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} b_1 &= 1 \\ b_2 &= 0 \\ b_3 &= -1 \end{aligned} \quad \Rightarrow \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \\ c_3 &= -1 \end{aligned} \quad \Rightarrow \quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

(b) By Theorem 6.13, $[\ \mathcal{C} \mid \mathcal{B}\ ] \to [\ I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

(c)

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

which is the same as the answer from part (a).

(d) By Theorem 6.13, $[\ \mathcal{B} \mid \mathcal{C}\ ] \to [\ I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(e)

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

which is the same as the answer from part (a).

4. **(a)**

$$\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = b_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} b_1 &= 1 \\ b_2 &= 5 \\ b_3 &= 3 \end{aligned} \quad \Rightarrow \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} c_1 &= -\tfrac{1}{2} \\ c_2 &= \tfrac{3}{2} \\ c_3 &= \tfrac{7}{2} \end{aligned} \quad \Rightarrow \quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{7}{2} \end{bmatrix}.$$

(b) By Theorem 6.13, $[\ \mathcal{C} \mid \mathcal{B}\ ] \to [\ I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(c)

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{7}{2} \end{bmatrix},$$

which is the same as the answer from part (a).

(d) By Theorem 6.13, $[\ \mathcal{B} \mid \mathcal{C}\ ] \to [\ I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(e)

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ \frac{7}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

which is the same as the answer from part (a).

5. Note that in terms of the standard basis,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(a)

$$2 - x = b_1(1) + b_2(x) \quad \Rightarrow \quad \begin{aligned} b_1 &= 2 \\ b_2 &= -1 \end{aligned} \quad \Rightarrow \quad [2-x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$2 - x = c_1(x) + c_2(1 + x) \quad \Rightarrow \quad \begin{aligned} c_1 &= -3 \\ c_2 &= 2 \end{aligned} \quad \Rightarrow \quad [2-x]_{\mathcal{C}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

(b) By Theorem 6.13, $[\ \mathcal{C} \mid \mathcal{B}\ ] \to [\ I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$:

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

(c) $[2-x]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[2-x]_{\mathcal{B}} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, which agrees with the answer in part (a).

(d) By Theorem 6.13, $[\ \mathcal{B} \mid \mathcal{C}\ ] \to [\ I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

(e) $[2-x]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[2-x]_{\mathcal{C}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, which agrees with the answer in part (a).

6. Note that in terms of the standard basis,

$$\mathcal{B}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\ \begin{bmatrix}1\\-1\end{bmatrix}\right\},\qquad \mathcal{C}=\left\{\begin{bmatrix}0\\2\end{bmatrix},\ \begin{bmatrix}4\\0\end{bmatrix}\right\}.$$

(a)

$$1+3x=b_1(1+x)+b_2(1-x)\quad\Rightarrow\quad \begin{array}{c}b_1=2\\ b_2=-1\end{array}\quad\Rightarrow\quad [1+3x]_{\mathcal{B}}=\begin{bmatrix}2\\-1\end{bmatrix}$$

$$1+3x=c_1(2x)+c_2(4)\quad\Rightarrow\quad \begin{array}{c}c_1=\frac{3}{2}\\ c_2=\frac{1}{4}\end{array}\quad\Rightarrow\quad [1+3x]_{\mathcal{C}}=\begin{bmatrix}\frac{3}{2}\\ \frac{1}{4}\end{bmatrix}$$

(b) By Theorem 6.13, $[\ \mathcal{C}\mid\mathcal{B}\ ]\to[\ I\mid P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$:

$$\left[\begin{array}{cc|cc}0&4&1&1\\2&0&1&-1\end{array}\right]\to\left[\begin{array}{cc|cc}1&0&\frac{1}{2}&-\frac{1}{2}\\0&1&\frac{1}{4}&\frac{1}{4}\end{array}\right]\quad\Rightarrow\quad P_{\mathcal{C}\leftarrow\mathcal{B}}=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{4}&\frac{1}{4}\end{bmatrix}.$$

(c) $[1+3x]_{\mathcal{C}}=P_{\mathcal{C}\leftarrow\mathcal{B}}[1+3x]_{\mathcal{B}}=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{4}&\frac{1}{4}\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}\frac{3}{2}\\ \frac{1}{4}\end{bmatrix}$, which agrees with the answer in part (a).

(d) By Theorem 6.13, $[\ \mathcal{B}\mid\mathcal{C}\ ]\to[\ I\mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{cc|cc}1&1&0&4\\1&-1&2&0\end{array}\right]\to\left[\begin{array}{cc|cc}1&0&1&2\\0&1&-1&2\end{array}\right]\quad\Rightarrow\quad P_{\mathcal{C}\leftarrow\mathcal{B}}=\begin{bmatrix}1&2\\-1&2\end{bmatrix}$$

(e) $[1+3x]_{\mathcal{B}}=P_{\mathcal{B}\leftarrow\mathcal{C}}[1+3x]_{\mathcal{C}}=\begin{bmatrix}1&2\\-1&2\end{bmatrix}\begin{bmatrix}\frac{3}{2}\\ \frac{1}{4}\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$, which agrees with the answer in part (a).

7. Note that in terms of the standard basis,

$$\mathcal{B}=\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\ \begin{bmatrix}0\\1\\1\end{bmatrix},\ \begin{bmatrix}0\\0\\1\end{bmatrix}\right\},\qquad \mathcal{C}=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\ \begin{bmatrix}0\\1\\0\end{bmatrix},\ \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}.$$

(a)

$$1+x^2=b_1(1+x+x^2)+b_2(x+x^2)+b_3(x^2)\quad\Rightarrow\quad \begin{array}{rcr}b_1&=&1\\ b_2&=&-1\\ b_3&=&1\end{array}\quad\Rightarrow\quad [1+x^2]_{\mathcal{B}}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$1+x^2=c_1(1)+c_2(x)+c_3(x^2)\quad\Rightarrow\quad \begin{array}{rcr}c_1&=&1\\ c_2&=&0\\ c_3&=&1\end{array}\quad\Rightarrow\quad [1+x^2]_{\mathcal{C}}=\begin{bmatrix}1\\0\\1\end{bmatrix}.$$

(b) By Theorem 6.13, $[\ \mathcal{C}\mid\mathcal{B}\ ]\to[\ I\mid P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$:

$$\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&0&1&1&1&1\end{array}\right]\quad\Rightarrow\quad P_{\mathcal{C}\leftarrow\mathcal{B}}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}.$$

(c) $[1+x^2]_{\mathcal{C}}=P_{\mathcal{C}\leftarrow\mathcal{B}}[1+x^2]_{\mathcal{B}}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}$, which agrees with the answer in part (a).

(d) By Theorem 6.13, $[\ \mathcal{B} \mid \mathcal{C}\ ] \to [\ I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(e) $[1+x^2]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[1+x^2]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, which agrees with the answer in part (a).

8. Note that in terms of the standard basis,

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a)

$$4-2x-x^2 = b_1(x) + b_2(1+x^2) + b_3(x+x^2) \quad \Rightarrow \quad \begin{array}{c} b_1 = 3 \\ b_2 = 4 \\ b_3 = -5 \end{array} \quad \Rightarrow \quad [4-2x-x^2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$$

$$4-2x-x^2 = c_1(1) + c_2(1+x) + c_3(x^2) \quad \Rightarrow \quad \begin{array}{c} c_1 = 6 \\ c_2 = -2 \\ c_3 = -1 \end{array} \quad \Rightarrow \quad [4-2x-x^2]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \\ -1 \end{bmatrix}$$

(b) By Theorem 6.13, $[\ \mathcal{C} \mid \mathcal{B}\ ] \to [\ I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(c) $[4-2x-x^2]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[4-2x-x^2]_{\mathcal{B}} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -1 \end{bmatrix}$, which agrees with the answer in part (a).

(d) By Theorem 6.13, $[\ \mathcal{B} \mid \mathcal{C}\ ] \to [\ I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

(e) $[4-2x-x^2]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[4-2x-x^2]_{\mathcal{C}} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$, which agrees with the answer in part (a).

9. **(a)**

$$\begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix} = b_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcr} b_1 & = & 4 \\ b_2 & = & 2 \\ b_3 & = & 0 \\ b_4 & = & -1 \end{array} \quad \Rightarrow \quad [A]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix} = b_1\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + b_2\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + b_3\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b_4\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcr} b_1 & = & \frac{5}{2} \\ b_2 & = & 0 \\ b_3 & = & -3 \\ b_4 & = & \frac{9}{2} \end{array} \quad \Rightarrow \quad [A]_{\mathcal{C}} = \begin{bmatrix} \frac{5}{2} \\ 0 \\ -3 \\ \frac{9}{2} \end{bmatrix}.$$

(b) By Theorem 6.13, $[\ \mathcal{C} \mid \mathcal{B}\ ] \to [\ I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$:

$$\left[\begin{array}{rrrr|rrrr} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{2} & -1 & -2 & -\frac{1}{2} \end{array}\right]$$

$$\Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ \frac{3}{2} & -1 & -2 & -\frac{1}{2} \end{bmatrix}$$

(c)

$$[A]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ \frac{3}{2} & -1 & -2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 0 \\ -3 \\ \frac{9}{2} \end{bmatrix},$$

which agrees with the answer in part (a).

(d) By Theorem 6.13, $[\ \mathcal{B} \mid \mathcal{C}\ ] \to [\ I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}.$$

(e)

$$[A]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[A]_{\mathcal{C}} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{2} \\ 0 \\ -3 \\ \frac{9}{2} \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix},$$

which agrees with the result in part (a).

10. (a)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b_4 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} b_1 &= 1 \\ b_2 &= 1 \\ b_3 &= 0 \\ b_4 &= 0 \end{aligned} \quad \Rightarrow \quad [A]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b_4 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} b_1 &= \tfrac{1}{3} \\ b_2 &= \tfrac{1}{3} \\ b_3 &= \tfrac{1}{3} \\ b_4 &= \tfrac{1}{3} \end{aligned} \quad \Rightarrow \quad [A]_{\mathcal{C}} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

(b) By Theorem 6.13, $[\ \mathcal{C} \mid \mathcal{B}\ ] \to [\ I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$:

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$\Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$$

(c)

$$[A]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

which agrees with the answer in part (a).

(d) By Theorem 6.13, $[\ \mathcal{B} \mid \mathcal{C}\ ] \to [\ I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\Rightarrow \quad P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

(e)

$$[A]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[A]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

which agrees with the result in part (a).

11. Note that in terms of the standard basis $\{\sin x, \cos x\}$,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

(a)

$$2\sin x - 3\cos x = b_1(\sin x + \cos x) + b_2(\cos x) \quad \Rightarrow \quad \begin{aligned} b_1 &= 2 \\ b_2 &= -5 \end{aligned}$$

$$\Rightarrow \quad [2\sin x - 3\cos x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$2\sin x - 3\cos x = c_1(\sin x + \cos x) + c_2(\sin x - \cos x) \quad \Rightarrow \quad \begin{aligned} c_1 &= -\tfrac{1}{2} \\ c_2 &= \tfrac{5}{2} \end{aligned}$$

$$\Rightarrow \quad [2\sin x - 3\cos x]_{\mathcal{C}} = \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}.$$

(b) By Theorem 6.13, $[\,\mathcal{C} \mid \mathcal{B}\,] \to [\,I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}\,]$:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

(c) $[2\sin x - 3\cos x]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[2\sin x - 3\cos x]_{\mathcal{B}} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$, which agrees with the answer in part (a).

(d) By Theorem 6.13, $[\,\mathcal{B} \mid \mathcal{C}\,] \to [\,I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\,]$:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

(e) $[2\sin x - 3\cos x]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[2\sin x - 3\cos x]_{\mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, which agrees with the answer in part (a).

12. Note that in terms of the standard basis $\{\sin x, \cos x\}$,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(a)

$$\sin x = b_1(\sin x + \cos x) + b_2(\cos x) \quad \Rightarrow \quad \begin{aligned} b_1 &= 1 \\ b_2 &= -1 \end{aligned} \quad \Rightarrow \quad [\sin x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\sin x = c_1(\cos x - \sin x) + c_2(\sin x + \cos x) \quad \Rightarrow \quad \begin{aligned} c_1 &= -\tfrac{1}{2} \\ c_2 &= \tfrac{1}{2} \end{aligned} \quad \Rightarrow \quad [\sin x]_{\mathcal{C}} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

(b) By Theorem 6.13, $[\,\mathcal{C} \mid \mathcal{B}\,] \to [\,I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}\,]$:

$$\left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{2} \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}.$$

(c) $[\sin x]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\sin x]_{\mathcal{B}} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, which agrees with the answer in part (a).

(d) By Theorem 6.13, $[\ \mathcal{B} \mid \mathcal{C}\ ] \to [\ I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\ ]$:

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \quad \Rightarrow \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$$

(e) $[\sin x]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\sin x]_{\mathcal{C}} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which agrees with the answer in part (a).

13. From Section 3.6, Example 3.58, if $\mathcal{B}$ is the usual basis and $\mathcal{C}$ is the basis whose basis vectors are the rotations of the usual basis vectors through $60°$, the matrix of the rotation is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} \cos 60° & \sin 60° \\ -\sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Then

(a)

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \sqrt{3} \\ 1 - \frac{3}{2}\sqrt{3} \end{bmatrix}.$$

(b) Since $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is an orthogonal matrix, we have

$$[\mathbf{x}]_{\mathcal{B}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}\,[\mathbf{x}]_{\mathcal{C}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{T}\,[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} + 2 \\ 2\sqrt{3} - 2 \end{bmatrix}.$$

14. From Section 3.6, Example 3.58, if $\mathcal{B}$ is the usual basis and $\mathcal{C}$ is the basis whose basis vectors are the rotations of the usual basis vectors through $135°$, the matrix of the rotation is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} \cos 135° & \sin 135° \\ -\sin 135° & \cos 135° \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

Then

(a)

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{5\sqrt{2}}{2} \end{bmatrix}.$$

(b) Since $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is an orthogonal matrix, we have

$$[\mathbf{x}]_{\mathcal{B}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}\,[\mathbf{x}]_{\mathcal{C}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{T}\,[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4\sqrt{2} \end{bmatrix}.$$

15. By definition, the columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the coordinates of $\mathcal{B}$ in the basis $\mathcal{C}$, so we have

$$\begin{aligned} [\mathbf{u}_1]_{\mathcal{C}} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{ so that } \mathbf{u}_1 = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ [\mathbf{u}_2]_{\mathcal{C}} &= \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \text{ so that } \mathbf{u}_2 = -1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}. \end{aligned}$$

Therefore

$$\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}.$$

16. Let $\mathcal{C} = \{p_1(x), p_2(x), p_3(x)\}$. By definition, the columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the coordinates of $\mathcal{B}$ in the basis $\mathcal{C}$, so that

$$[x]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \Rightarrow \quad x = p_1(x) - p_3(x),$$

$$[1+x]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad 1 + x = 2p_2(x) + p_3(x),$$

$$[1-x+x^2]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad 1 - x + x^2 = p_2(x) + p_3(x).$$

Subtract the third equation from the second, giving $p_2(x) = 2x - x^2$; substituting back into the third equation gives $p_3(x) = 1 - 3x + 2x^2$; then from the first equation, $p_1(x) = 1 - 2x + 2x^2$. So

$$\mathcal{C} = \{p_1(x), p_2(x), p_3(x)\} = \{1 - 2x + 2x^2, 2x - x^2, 1 - 3x + 2x^2\}.$$

17. Since $a = 1$ in this case and we are considering a quadratic, we have for the Taylor basis of $\mathscr{P}_2$

$$\mathcal{B} = \{1, x - 1, (x-1)^2\} = \{1, x - 1, x^2 - 2x + 1\}.$$

Let $\mathcal{C} = \{1, x, x^2\}$ be the standard basis for $\mathscr{P}_2$. Then

$$[p(x)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}.$$

To find the change-of-basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$, we reduce $\left[\mathcal{B} \mid \mathcal{C}\right] \to \left[I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\right]$:

$$\left[\mathcal{B} \mid \mathcal{C}\right] = \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Therefore

$$[p(x)]_{\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \\ -5 \end{bmatrix},$$

so that the Taylor polynomial centered at 1 is

$$-2 - 8(x-1) - 5(x-1)^2.$$

18. Since $a = -2$ in this case and we are considering a quadratic, we have for the Taylor basis of $\mathscr{P}_2$

$$\mathcal{B} = \{1, x + 2, (x+2)^2\} = \{1, x + 2, x^2 + 4x + 4\}.$$

Let $\mathcal{C} = \{1, x, x^2\}$ be the standard basis for $\mathscr{P}_2$. Then

$$[p(x)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}.$$

To find the change-of-basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$, we reduce $\left[\mathcal{B} \mid \mathcal{C}\right] \to \left[I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\right]$:

$$\left[\mathcal{B} \mid \mathcal{C}\right] = \left[\begin{array}{rrr|rrr} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Therefore

$$[p(x)]_{\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -23 \\ 22 \\ -5 \end{bmatrix},$$

so that the Taylor polynomial centered at -2 is

$$-23 + 22(x+2) - 5(x+2)^2.$$

19. Since $a = -1$ in this case and we are considering a cubic, we have for the Taylor basis of $\mathscr{P}_3$

$$\mathcal{B} = \{1, x+1, (x+1)^2, (x+1)^3\} = \{1, x+1, x^2+2x+1, x^3+3x^2+3x+1\}.$$

Let $\mathcal{C} = \{1, x, x^2, x^3\}$ be the standard basis for $\mathscr{P}_2$. Then

$$[p(x)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

To find the change-of-basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$, we reduce $\left[\mathcal{B} \mid \mathcal{C}\right] \to \left[I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\right]$:

$$\left[\mathcal{B} \mid \mathcal{C}\right] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

Therefore

$$[p(x)]_{\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix},$$

so that the Taylor polynomial centered at -1 is

$$-1 + 3(x+1) - 3(x+1)^2 + (x+1)^3.$$

20. Since $a = \frac{1}{2}$ in this case and we are considering a cubic, we have for the Taylor basis of $\mathscr{P}_3$

$$\mathcal{B} = \left\{1, x - \frac{1}{2}, \left(x - \frac{1}{2}\right)^2, \left(x - \frac{1}{2}\right)^3\right\} = \left\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{4}, x^3 - \frac{3}{2}x^2 + \frac{3}{4}x - \frac{1}{8}\right\}.$$

Let $\mathcal{C} = \{1, x, x^2, x^3\}$ be the standard basis for $\mathscr{P}_2$. Then

$$[p(x)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

To find the change-of-basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$, we reduce $\left[\mathcal{B} \mid \mathcal{C}\right] \to \left[I \mid P_{\mathcal{B}\leftarrow\mathcal{C}}\right]$:

$$\left[\mathcal{B} \mid \mathcal{C}\right] = \left[\begin{array}{cccc|cccc} 1 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & \frac{3}{4} & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & \frac{3}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

Therefore

$$[p(x)]_{\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & 1 & 1 & \frac{3}{4} \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ \frac{3}{4} \\ \frac{3}{2} \\ 1 \end{bmatrix},$$

so that the Taylor polynomial centered at $\frac{1}{2}$ is

$$\frac{1}{8} + \frac{3}{4}\left(x - \frac{1}{2}\right) + \frac{3}{2}\left(x - \frac{1}{2}\right)^2 + \left(x - \frac{1}{2}\right)^3.$$

21. Since $[x]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{C}}[x]_{\mathcal{C}}$ and $[x]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[x]_{\mathcal{B}}$, we get

$$[x]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{C}}[x]_{\mathcal{C}} = P_{\mathcal{D}\leftarrow\mathcal{C}}(P_{\mathcal{C}\leftarrow\mathcal{B}}[x]_{\mathcal{B}}) = (P_{\mathcal{D}\leftarrow\mathcal{C}}P_{\mathcal{C}\leftarrow\mathcal{B}})[x]_{\mathcal{B}}.$$

Since the change-of-basis matrix is unique (its columns are determined by the two bases), it follows that

$$P_{\mathcal{D}\leftarrow\mathcal{C}}P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{D}\leftarrow\mathcal{B}}.$$

22. By Theorem 6.10 (c), to show that $\mathcal{C}$ is a basis it suffices to show that it is linearly independent, since $\mathcal{C}$ has n elements and $\dim V = n$. Since P is invertible, the columns of P are linearly independent. But the given equation says that $[\mathbf{u}_i]_{\mathcal{B}} = \mathbf{p}_i$, since the elements of that column are the coordinates of $\mathbf{u}_i$ with respect to the basis $\mathcal{B}$. Since the columns of P are linearly independent, so is $\{[\mathbf{u}_1]_{\mathcal{B}}, [\mathbf{u}_2]_{\mathcal{B}}, \ldots, [\mathbf{u}_n]_{\mathcal{B}}\}$. But then by Theorem 6.7 in Section 6.2, it follows that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is linearly independent, so that $\mathcal{C}$ is a basis for V.

To show that $P = P_{\mathcal{B}\leftarrow\mathcal{C}}$, recall that the columns of $P_{\mathcal{B}\leftarrow\mathcal{C}}$ are the coordinate vectors of $\mathcal{C}$ with respect to the basis $\mathcal{B}$. But looking at the given equation, we see that this is exactly what the columns of P are. Thus $P = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

6.4 Linear Transformations

1. T is a linear transformation:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = T\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} = \begin{bmatrix} a+a'+b+b' \\ 0 \\ 0 \\ c+c'+d+d' \end{bmatrix}$$

$$= \begin{bmatrix} a+b \\ 0 \\ 0 \\ c+d \end{bmatrix} + \begin{bmatrix} a'+b' \\ 0 \\ 0 \\ c'+d' \end{bmatrix} = T\begin{bmatrix} a & b \\ c & d \end{bmatrix} + T\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

and

$$T\left(\alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}\right) = \begin{bmatrix} \alpha a + \alpha b & 0 \\ 0 & \alpha c + \alpha d \end{bmatrix} = \alpha\begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix} = \alpha T\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

2. T is not a linear transformation:

$$T\left(\begin{bmatrix} w & x \\ y & z \end{bmatrix} + \begin{bmatrix} w' & x' \\ y' & z' \end{bmatrix}\right) = T\left(\begin{bmatrix} w+w' & x+x' \\ y+y' & z+z' \end{bmatrix}\right) = \begin{bmatrix} 1 & w+w'-z-z' \\ x+x'-y-y' & 1 \end{bmatrix}$$

while

$$T\left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}\right) + T\left(\begin{bmatrix} w' & x' \\ y' & z' \end{bmatrix}\right) = \begin{bmatrix} 1 & w-z \\ x-y & 1 \end{bmatrix} + \begin{bmatrix} 1 & w'-z' \\ x'-y' & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & w+w'-z-z' \\ x+x'-y-y' & 2 \end{bmatrix}$$

Since the two are not equal, T is not a linear transformation.

3. T is a linear transformation. If $A, C \in M_{nn}$ and α is a scalar, then

$$\begin{aligned} T(A+C) &= (A+C)B = AB + CB = T(A) + T(C) \\ T(\alpha A) &= (\alpha A)B = \alpha(AB) = \alpha T(A). \end{aligned}$$

4. T is a linear transformation. If $A, C \in M_{nn}$ and α is a scalar, then

$$\begin{aligned} T(A+C) &= (A+C)B - B(A+C) = AB + CB - BA - BC = (AB - BA) + (CB - BC) \\ &= T(A) + T(C) \\ T(\alpha A) &= (\alpha A)B - B(\alpha A) = \alpha(AB) - \alpha(BA) = \alpha(AB - BA) = \alpha T(A). \end{aligned}$$

5. T is a linear transformation, since by Exercise 44 in Section 3.2, $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$ and $\operatorname{tr}(\alpha A) = \alpha \operatorname{tr}(A)$.

6. T is not a linear transformation. For example, in M_{22}, let $A = I$; then $T(2A) = T(2I) = 2 \cdot 2 = 4$, while $2T(A) = 2T(I) = 2(1 \cdot 1) = 2$.

7. T is not a linear transformation. For example, let A be any invertible matrix in M_{nn}; then $-A$ is invertible as well, so that $\operatorname{rank}(A) = \operatorname{rank}(-A) = n$. Then

$$T(A) + T(-A) = \operatorname{rank}(A) + \operatorname{rank}(-A) = 2n \text{ while } T(A + (-A)) = T(O) = \operatorname{rank}(O) = 0.$$

8. Since $T(\mathbf{0}) = 1 + x + x^2 \neq \mathbf{0}$, T is not a linear transformation by Theorem 6.14(a).

9. T is a linear transformation. Let $p(x) = a + bx + cx^2$ and $p'(x) = a' + b'x + c'x^2$ be elements of $\mathscr{P}_2$, and let α be a scalar. Then

$$\begin{aligned} (T(p+p'))(d) &= (T((a+a') + (b+b')x + (c+c')x^2))(d) \\ &= (a+a') + (b+b')(d+1) + (b+b')(d+1)^2 \\ &= \left(a + b(d+1) + b(d+1)^2\right) + \left(a' + b'(d+1) + b'(d+1)^2\right) \\ &= (T(p))(d) + (T(p'))(d), \end{aligned}$$

and

$$\begin{aligned} T(\alpha p)(d) = (T(\alpha a + \alpha bx + \alpha cx^2))(d) &= \alpha a + \alpha b(d+1) + \alpha b(d+1)^2 \\ &= \alpha\left(a + b(d+1) + b(d+1)^2\right) = \alpha(T(p))(d). \end{aligned}$$

10. T is a linear transformation. Let $f, g \in \mathscr{F}$ and α be a scalar. Then

$$\begin{aligned} (T(f+g))(x) &= (f+g)(x^2) = f(x^2) + g(x^2) = (T(f))(x) + (T(g))(x) \\ (T(\alpha f))(x) &= (\alpha f)(x^2) = \alpha f(x^2) = \alpha(T(f))(x). \end{aligned}$$

11. T is not a linear transformation. For example, let f be any function in $\mathscr{F}$ that takes on a value other than 0 or 1. Then

$$(T(2f))(x) = ((2f)(x))^2 = (2f(x))^2 = 4f(x)^2 \neq 2f(x)^2 = 2(T(f))(x),$$

where the inequality holds because of the conditions on f.

12. This is similar to Exercise 10; T is a linear transformation. Let $f, g \in \mathscr{F}$ and α be a scalar. Then

$$\begin{aligned} (T(f+g))(c) &= (f+g)(c) = f(c) + g(c) = (T(f))(c) + (T(g))(c) \\ (T(\alpha f))(c) &= (\alpha f)(c) = \alpha f(c) = \alpha(T(f))(c) \end{aligned}$$

13. For T, we have (with α a scalar)

$$\begin{aligned}T\left(\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}\right)&=T\begin{bmatrix}a+c\\b+d\end{bmatrix}=(a+c)+(a+c+b+d)x=(a+(a+b)x)+(c+(c+d)x)\\&=T\begin{bmatrix}a\\b\end{bmatrix}+T\begin{bmatrix}c\\d\end{bmatrix}\\T\left(\alpha\begin{bmatrix}a\\b\end{bmatrix}\right)&=T\begin{bmatrix}\alpha a\\\alpha b\end{bmatrix}=\alpha a+(\alpha a+\alpha b)x=\alpha(a+(a+b)x)=\alpha T\begin{bmatrix}a\\b\end{bmatrix}.\end{aligned}$$

For S, suppose that $p(x), q(x)\in\mathscr{P}_1$ and α is a scalar; then

$$\begin{aligned}T(p(x)+q(x))&=x(p(x)+q(x))=xp(x)+xq(x)=T(p(x))+T(q(x))\\T(\alpha p(x))&=x(\alpha p(x))=\alpha(xp(x))=\alpha T(p(x)).\end{aligned}$$

14. Using the comment following the definition of a linear transformation, we have

$$T\begin{bmatrix}5\\2\end{bmatrix}=T\left(5\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}\right)=5T\begin{bmatrix}1\\0\end{bmatrix}+2T\begin{bmatrix}0\\1\end{bmatrix}=5\begin{bmatrix}1\\2\\-1\end{bmatrix}+2\begin{bmatrix}3\\0\\4\end{bmatrix}=\begin{bmatrix}5\\10\\-5\end{bmatrix}+\begin{bmatrix}6\\0\\8\end{bmatrix}=\begin{bmatrix}11\\10\\3\end{bmatrix}$$

In a similar way,

$$T\begin{bmatrix}a\\b\end{bmatrix}=T\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)=aT\begin{bmatrix}1\\0\end{bmatrix}+bT\begin{bmatrix}0\\1\end{bmatrix}=a\begin{bmatrix}1\\2\\-1\end{bmatrix}+b\begin{bmatrix}3\\0\\4\end{bmatrix}=\begin{bmatrix}a\\2a\\-a\end{bmatrix}+\begin{bmatrix}3b\\0\\4b\end{bmatrix}=\begin{bmatrix}a+3b\\2a\\-a+4b\end{bmatrix}$$

15. We must first find the coordinates of $\begin{bmatrix}-7\\9\end{bmatrix}$ in the basis $\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}3\\-1\end{bmatrix}\right\}$. So we want to find c, d such that

$$\begin{bmatrix}-7\\9\end{bmatrix}=c\begin{bmatrix}1\\1\end{bmatrix}+d\begin{bmatrix}3\\-1\end{bmatrix}\quad\Rightarrow\quad\begin{aligned}c+3d&=-7\\c-d&=9\end{aligned}\quad\Rightarrow\quad\begin{aligned}c&=5\\d&=-4.\end{aligned}$$

Then using the comment following the definition of a linear transformation, we have

$$T\begin{bmatrix}-7\\9\end{bmatrix}=T\left(5\begin{bmatrix}1\\1\end{bmatrix}-4\begin{bmatrix}3\\-1\end{bmatrix}\right)=5T\begin{bmatrix}1\\1\end{bmatrix}-4T\begin{bmatrix}3\\-1\end{bmatrix}=5(1-2x)-4(x+2x^2)=-8x^2-14x+5.$$

The second part is much the same, except that we need to find c, d such that

$$\begin{bmatrix}a\\b\end{bmatrix}=c\begin{bmatrix}1\\1\end{bmatrix}+d\begin{bmatrix}3\\-1\end{bmatrix}\quad\Rightarrow\quad\begin{aligned}c+3d&=a\\c-d&=b\end{aligned}\quad\Rightarrow\quad\begin{aligned}c&=\tfrac{a+3b}{4}\\d&=\tfrac{a-b}{4}.\end{aligned}$$

Then using the comment following the definition of a linear transformation, we have

$$\begin{aligned}T\begin{bmatrix}a\\b\end{bmatrix}&=T\left(\frac{a+3b}{4}\begin{bmatrix}1\\1\end{bmatrix}+\frac{a-b}{4}\begin{bmatrix}3\\-1\end{bmatrix}\right)\\&=\frac{a+3b}{4}T\begin{bmatrix}1\\1\end{bmatrix}+\frac{a-b}{4}T\begin{bmatrix}3\\-1\end{bmatrix}\\&=\frac{a+3b}{4}(1-2x)+\frac{a-b}{4}(x+2x^2)\\&=\frac{a-b}{2}x^2-\frac{a+7b}{4}x+\frac{a+3b}{4}.\end{aligned}$$

16. Since T is linear,

$$\begin{aligned}T(6+x-4x^2)&=T(6)+T(x)+T(-4x^2)=6T(1)+T(x)-4T(x^2)\\&=6(3-2x)+(4x-x^2)-4(2+2x^2)=10-8x-9x^2.\end{aligned}$$

Similarly,

$$\begin{aligned}T(a+bx+cx^2) &= T(a)+T(bx)+T(cx^2) = aT(1)+bT(x)+cT(x^2)\\ &= a(3-2x)+b(4x-x^2)+c(2+2x^2) = (3a+2c)+(4b-2a)x+(2c-b)x^2.\end{aligned}$$

17. We do the second part first, then substitute for a, b, and c to get the first part. We must find the coordinates of $a+bx+cx^2$ in the basis $\left\{1+x, x+x^2, 1+x^2\right\}$. So we want to find r, s, t such that

$$a+bx+cx^2 = r(1+x)+s(x+x^2)+t(1+x^2) = (r+t)+(r+s)x+(s+t)x^2 \quad\Rightarrow$$

$$\begin{array}{rcrcrcl} r & & & + & t & = & a\\ r & + & s & & & = & b\\ & & s & + & t & = & c\end{array} \quad\Rightarrow\quad \begin{array}{rcl} r & = & \frac{a+b-c}{2}\\ s & = & \frac{-a+b+c}{2}\\ t & = & \frac{a-b+c}{2}.\end{array}$$

Then using the comment following the definition of a linear transformation, we have

$$\begin{aligned}T(a+bx+cx^2) &= T\left(\frac{a+b-c}{2}(1+x)+\frac{-a+b+c}{2}(x+x^2)+\frac{a-b+c}{2}(1+x^2)\right)\\ &= \frac{a+b-c}{2}T(1+x)+\frac{-a+b+c}{2}T(x+x^2)+\frac{a-b+c}{2}T(1+x^2)\\ &= \frac{a+b-c}{2}(1+x^2)+\frac{-a+b+c}{2}(x-x^2)+\frac{a-b+c}{2}(1+x+x^2)\\ &= a+cx+\frac{3a-b-c}{2}x^2\end{aligned}$$

For the second part, substitute $a=4$, $b=-1$, and $c=3$ to get

$$T(4-x+3x^2) = 4+3x+\frac{12+1-3}{2}x^2 = 4+3x+5x^2.$$

18. We do the second part first, then substitute for a, b, c, and d to get the first part. We must find the coordinates of

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix} \text{ in the basis } \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

So we want to find r, s, t, u such that

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix} = r\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+s\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}+t\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}+u\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \quad\Rightarrow\quad \begin{array}{rcrcrcrcl} r & + & s & + & t & + & u & = & a\\ & & s & + & t & + & u & = & b\\ & & & & t & + & u & = & c\\ & & & & & & u & = & d\end{array} \quad\Rightarrow\quad \begin{array}{rcl} r & = & a-b\\ s & = & b-c\\ t & = & c-d\\ u & = & d.\end{array}$$

Then using the comment following the definition of a linear transformation, we have

$$\begin{aligned}T\begin{bmatrix}a & b\\ c & d\end{bmatrix} &= T\left((a-b)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+(b-c)\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}+(c-d)\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}++d\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right)\\ &= (a-b)T\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+(b-c)T\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}+(c-d)T\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}++dT\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\\ &= (a-b)\cdot 1+(b-c)\cdot 2+(c-d)\cdot 3+d\cdot 4\\ &= a+b+c+d.\end{aligned}$$

For the second part, substitute $a=1$, $b=3$, $c=4$, $d=2$ to get

$$T\begin{bmatrix}1 & 3\\ 4 & 2\end{bmatrix} = 1+3+4+2 = 10.$$

19. Let

$$T(E_{11}) = a, \quad T(E_{12}) = b, \quad T(E_{21}) = c, \quad T(E_{22}) = d.$$

Then

$$T\begin{bmatrix} w & x \\ y & z \end{bmatrix} = t\left(wE_{11} + xE_{12} + yE_{21} + zE_{22}\right) = wTE_{11} + xTE_{12} + yTE_{21} + zTE_{22} = aw + bx + cy + dz.$$

20. Writing

$$\begin{bmatrix} 0 \\ 6 \\ -8 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

means that we must have $b = -4$, and then $a = 6$, so that if T were a linear transformation, then

$$T\begin{bmatrix} 0 \\ 6 \\ -8 \end{bmatrix} = 6T\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = 6(1+x) - 4(2 - x + x^2) = -2 + 10x - 4x^2 \neq -2 + 2x^2.$$

This is a contradiction, so T cannot be linear.

21. Choose $\mathbf{v} \in V$. Then $-\mathbf{v} = \mathbf{0} - \mathbf{v}$ by definition, so using part (a) of Theorem 6.14 and the definition of a linear transformation,

$$T(-\mathbf{v}) = T(\mathbf{0} - \mathbf{v}) = T(\mathbf{0}) - T(\mathbf{v}) = \mathbf{0} - T(\mathbf{v}) = -T(\mathbf{v}).$$

22. Choose $\mathbf{v} \in V$. Then there are scalars $c_1, \ldots, c_n$ such that $\mathbf{v} = \sum_{i=1}^n c_i\mathbf{v}_i$ since the $\mathbf{v}_i$ are a basis. Since T is linear, using the comment following the definition of a linear transformation we get

$$T(\mathbf{v}) = T\left(\sum_{i=1}^n c_i\mathbf{v}_i\right) = \sum_{i=1}^n c_iT(\mathbf{v}_i) = \sum_{i=1}^n c_i\mathbf{v}_i = \mathbf{v}.$$

Since $T(\mathbf{v}) = \mathbf{v}$ for every vector $\mathbf{v} \in V$, T is the identity transformation.

23. We must show that if $p(x) \in \mathscr{P}_n$, then $Tp(x) = p'(x)$. Suppose that $p(x) = \sum_{k=0}^n a_kx^k$. Then since T is linear, using the comment following the definition of a linear transformation we get

$$Tp(x) = T\left(\sum_{k=0}^n a_kx^k\right) = \sum_{k=0}^n a_kT(x^k) = a_0T(1) + \sum_{k=1}^n a_kT(x^k) = \sum_{k=1}^n ka_kx^{k-1} = p'(x).$$

24. **(a)** We prove the contrapositive: suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent in V; then we have

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0},$$

where not all of the c_i are zero. Then

$$T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n) = T(\mathbf{0}) = \mathbf{0},$$

showing that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly dependent in W. Therefore if $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly independent in W, we must also have that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent in V.

(b) For example, the zero transformation from V to W, which takes every element of V to $\mathbf{0}$, shows that the converse is false. There are many other examples as well, however. For instance, let

$$T : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Then $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent in V, but

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

are linearly dependent.

25. We have

$$\begin{aligned}(S\circ T)\begin{bmatrix}2\\1\end{bmatrix} &= S\left(T\begin{bmatrix}2\\1\end{bmatrix}\right) = S\begin{bmatrix}5\\-1\end{bmatrix} = \begin{bmatrix}4 & -1\\0 & 6\end{bmatrix}\\ (S\circ T)\begin{bmatrix}x\\y\end{bmatrix} &= S\left(T\begin{bmatrix}x\\y\end{bmatrix}\right) = S\begin{bmatrix}2x+y\\-y\end{bmatrix} = \begin{bmatrix}2x & -y\\0 & 2x+2y\end{bmatrix}.\end{aligned}$$

Since the domain of T is $\mathbb{R}^2$ but the codomain of S is M_{22}, $T\circ S$ does not make sense and we cannot compute it.

26. We have

$$\begin{aligned}(S\circ T)(3+2x-x^2) &= S(T(3+2x-x^2)) = S(2+2\cdot(-1)x) = S(2-2x)\\ &= 2+(2-2)x+2\cdot(-2)x^2 = 2-4x^2\\ (S\circ T)(a+bx+cx^2) &= S(T(a+bx+cx^2)) = S(b+2cx) = b+(b+2c)x+2(2c)x^2\\ &= b+(b+2c)x+4cx^2.\end{aligned}$$

Since the domain of S is $\mathscr{P}_1$, which equals the codomain of T, we can compute $T\circ S$:

$$(T\circ S)(a+bx) = T(S(a+bx)) = T(a+(a+b)x+2bx^2) = (a+b)+2\cdot 2bx = (a+b)+4bx.$$

27. The Chain Rule says that $(f\circ g)'(x) = g'(x)f'(g(x))$. Let $g(x)=x+1$ and $f(x)=p(x)$. Then

$$\begin{aligned}&(S\circ T)(p(x)) = S(T(p(x))) = S(p'(x)) = p'(x+1)\\ &(T\circ S)(p(x)) = T(S(p(x))) = T(p(x+1)) = T((p\circ g)(x)) = (p\circ g)'(x) = g'(x)p'(g(x)) = p'(x+1)\end{aligned}$$

since the derivative of $g(x)$ is 1.

28. Use the Chain Rule from the previous exercise, and let $g(x)=x+1$:

$$\begin{aligned}&(S\circ T)(p(x)) = S(T(p(x))) = S(xp'(x)) = (x+1)p'(x+1)\\ &(T\circ S)(p(x)) = T(S(p(x))) = T(p(x+1)) = T((p\circ g)(x)) = x(p\circ g)'(x) = xg'(x)p'(g(x)) = xp'(x+1)\end{aligned}$$

since the derivative of $g(x)$ is 1.

29. We must verify that $S\circ T = I_{\mathbb{R}^2}$ and $T\circ S = I_{\mathbb{R}^2}$.

$$\begin{aligned}(S\circ T)\begin{bmatrix}x\\y\end{bmatrix} &= S\left(T\begin{bmatrix}x\\y\end{bmatrix}\right) = S\begin{bmatrix}x-y\\-3x+4y\end{bmatrix} = \begin{bmatrix}4(x-y)+(-3x+4y)\\3(x-y)+(-3x+4y)\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}\\ (T\circ S)\begin{bmatrix}x\\y\end{bmatrix} &= T\left(S\begin{bmatrix}x\\y\end{bmatrix}\right) = T\begin{bmatrix}4x+y\\3x+y\end{bmatrix} = \begin{bmatrix}(4x+y)-(3x+y)\\-3(4x+y)+4(3x+y)\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}.\end{aligned}$$

Thus $(S\circ T)\begin{bmatrix}x\\y\end{bmatrix} = (T\circ S)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$ for any vector $\begin{bmatrix}x\\y\end{bmatrix}$, so both compositions are the identity and we are done.

30. We must verify that $S\circ T = I_{\mathscr{P}_1}$ and $T\circ S = I_{\mathscr{P}_1}$.

$$\begin{aligned}(S\circ T)(a+bx) &= S(T(a+bx)) = S\left(\frac{b}{2}+(a+2b)x\right) = \left(-4\cdot\frac{b}{2}+(a+2b)\right)+2\cdot\frac{b}{2}x = a+bx\\ (T\circ S)(a+bx) &= T(S(a+bx)) = T((-4a+b)+2ax) = \frac{2a}{2}+((-4a+b)+2\cdot 2a)x = a+bx.\end{aligned}$$

Thus $(S\circ T)(a+bx) = (T\circ S)(a+bx) = a+bx$ for any $a+bx\in\mathscr{P}_1$, so both compositions are the identity and we are done.

31. Suppose T' and T'' are such that $T' \circ T = T'' \circ T = I_V$ and $T \circ T' = T \circ T'' = I_W$. To show that $T' = T''$, it suffices to show that $T'(\mathbf{w}) = T''(\mathbf{w})$ for any vector $\mathbf{w} \in W$. But

$$T'(\mathbf{w}) = I_V \circ T'(\mathbf{w}) = T'' \circ T \circ T'(\mathbf{w}) = T'' \circ ((T \circ T')\mathbf{w}) = T'' \circ I_W(\mathbf{w}) = T''(\mathbf{w}).$$

32. (a) First, if $T(\mathbf{v}) = \pm\mathbf{v}$, then clearly $\{\mathbf{v}, T\mathbf{v}\}$ is linearly dependent. For the converse, if $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly dependent, then for some c, d not both zero $c\mathbf{v} + dT\mathbf{v} = \mathbf{0}$. Then

$$T(c\mathbf{v} + dT\mathbf{v}) = cT\mathbf{v} + d(T \circ T)\mathbf{v} = cT\mathbf{v} + d\mathbf{v} = T\mathbf{0} = \mathbf{0}$$

since $T \circ T = I_V$. Thus

$$\begin{aligned} c\mathbf{v} + dT\mathbf{v} &= \mathbf{0} \\ cT\mathbf{v} + \quad d\mathbf{v} &= \mathbf{0}. \end{aligned}$$

Subtracting these two equations gives $c(T\mathbf{v} - \mathbf{v}) - d(T\mathbf{v} - \mathbf{v}) = (c - d)(T\mathbf{v} - \mathbf{v}) = \mathbf{0}$. But this means that either $c = d \neq 0$ or that $T\mathbf{v} = \mathbf{v}$. If $c = d \neq 0$, then divide through by c in the equation $c\mathbf{v} + dT\mathbf{v} = 0$ to get $\mathbf{v} + T\mathbf{v} = 0$, so that $T\mathbf{v} = -\mathbf{v}$. Hence $T\mathbf{v} = \pm\mathbf{v}$.

(b) Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$. Then

$$(T \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = T\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} -(-x) \\ -(-y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

so that $T \circ T = I_V$, and $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} -x \\ -y \end{bmatrix}$ are linearly dependent.

33. (a) First, if $T(\mathbf{v}) = \mathbf{v}$, then clearly $\{\mathbf{v}, T\mathbf{v}\} = \{\mathbf{v}, \mathbf{v}\}$ is linearly dependent, and if $T(\mathbf{v}) = \mathbf{0}$, then $\{\mathbf{v}, T\mathbf{v}\} = \{\mathbf{v}, \mathbf{0}\}$ is linearly dependent. For the converse, suppose that $\{\mathbf{v}, T\mathbf{v}\}$ is linearly dependent. Then there exist c, d, not both zero, such that

$$c\mathbf{v} + dT\mathbf{v} = \mathbf{0}.$$

As in Exercise 32, we get

$$T(c\mathbf{v} + dT\mathbf{v}) = cT\mathbf{v} + d(T \circ T)\mathbf{v} = cT\mathbf{v} + dT\mathbf{v} = T\mathbf{0} = \mathbf{0}$$

since $T \circ T = T$. Thus

$$\begin{aligned} (c + d)T\mathbf{v} &= \mathbf{0} \\ c\mathbf{v} + \qquad dT\mathbf{v} &= \mathbf{0}. \end{aligned}$$

Subtracting the two equations gives $c(\mathbf{v} - T\mathbf{v}) = \mathbf{0}$, so that either $T\mathbf{v} = \mathbf{v}$ or $c = 0$. But if $c = 0$, then $c\mathbf{v} + dT\mathbf{v} = dT\mathbf{v} = \mathbf{0}$, so that since c and d were not both zero, d cannot be zero, so that $T\mathbf{v} = \mathbf{0}$. Thus either $T\mathbf{v} = \mathbf{v}$ or $T\mathbf{v} = \mathbf{0}$.

(b) For example, let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$. This is a projection operator, so we already know that $T \circ T = T$; to show this via computation,

$$(T \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = T\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}.$$

Then for example $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ while $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{0}$.

34. To show that $S+T$ is a linear transformation, we must show it obeys the rules of a linear transformation. Using the definition of $S+T$ and of cT, as well as the fact that both S and T are linear, we have

$$\begin{aligned}(S+T)(\mathbf{u}+\mathbf{v}) &= S(\mathbf{u}+\mathbf{v})+T(\mathbf{u}+\mathbf{v}) = S\mathbf{u}+S\mathbf{v}+T\mathbf{u}+T\mathbf{v} = (S\mathbf{u}+T\mathbf{u})+(S\mathbf{v}+T\mathbf{v})\\ &= (S+T)\mathbf{u}+(S+T)\mathbf{v}\\ (c(S+T))\mathbf{v} &= (cS+cT)\mathbf{v} = (cS)\mathbf{v}+(cT)\mathbf{v} = c(S\mathbf{v})+c(T\mathbf{v}) = c(S\mathbf{v}+T\mathbf{v}) = c(S+T)\mathbf{v}.\end{aligned}$$

35. We show that $\mathscr{L} = \mathscr{L}(V, W)$ satisfies the ten axioms of a vector space. We use freely below the fact that V and W are both vector spaces. Also, note that $S = T$ in $\mathscr{L}(V, W)$ simply means that $S\mathbf{v} = T\mathbf{v}$ for all $\mathbf{v} \in V$.

1. From Exercise 34, $S, T \in \mathscr{L}$ implies that $S+T \in \mathscr{L}$.

2. $(S+T)\mathbf{v} = S\mathbf{v}+T\mathbf{v} = T\mathbf{v}+S\mathbf{v} = (T+S)\mathbf{v}$.

3. $((R+S)+T)\mathbf{v} = (R+S)\mathbf{v}+T\mathbf{v} = R\mathbf{v}+S\mathbf{v}+T\mathbf{v} = R\mathbf{v}+(S+T)\mathbf{v} = (R+(S+T))\mathbf{v}$.

4. Let $Z : V \to W$ be the map $Z(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in \mathbf{V}$. Then Z is the zero map, so it is linear and thus in $\mathscr{L}$, and
$$(S+Z)\mathbf{v} = S\mathbf{v}+Z\mathbf{v} = S\mathbf{v}+\mathbf{0} = S\mathbf{v}.$$

5. If $S \in \mathscr{L}$, let $-S : V \to W$ be the map $(-S)\mathbf{v} = -S\mathbf{v}$. By Exercise 34, $-S \in \mathscr{L}$, and
$$(S+(-S))\mathbf{v} = S\mathbf{v}+(-S)\mathbf{v} = S\mathbf{v}-S\mathbf{v} = \mathbf{0}.$$

6. By Exercise 34, if $S \in \mathscr{L}$, then $cS \in \mathscr{L}$.

7. $(c(S+T))\mathbf{v} = c(S+T)\mathbf{v} = c(S\mathbf{v}+T\mathbf{v}) = cS\mathbf{v}+cT\mathbf{v} = (cS)\mathbf{v}+(cT)\mathbf{v} = ((cS)+(cT))\mathbf{v}$.

8. $((c+d)S)\mathbf{v} = (c+d)S\mathbf{v} = cS\mathbf{v}+dS\mathbf{v} = (cS)\mathbf{v}+(cT)\mathbf{v} = ((cS)+(cT))\mathbf{v}$.

9. $(c(dS))\mathbf{v} = c(dS)\mathbf{v} = c(dS\mathbf{v}) = (cd)S\mathbf{v} = ((cd)S)\mathbf{v}$.

10. By Exercise 34, $(1S)\mathbf{v} = 1S\mathbf{v} = S\mathbf{v}$.

36. To show that these linear transformations are equal, it suffices to show that they have the same value on every element of V.

(a) $(R\circ(S+T))\mathbf{v} = R((S+T)\mathbf{v}) = R(S\mathbf{v}+T\mathbf{v}) = R(S\mathbf{v})+R(T\mathbf{v}) = (R\circ S)\mathbf{v}+(R\circ T)\mathbf{v}$.

(b)
$$\begin{aligned}(c(R\circ S))\mathbf{v} &= c(R\circ S)\mathbf{v} = cR(S\mathbf{v}) = (cR)(S\mathbf{v}) = ((cR)\circ S)\mathbf{v}\\ (c(R\circ S))\mathbf{v} &= c(R\circ S)\mathbf{v} = cR(S\mathbf{v}) = (cR)(S\mathbf{v}) = R((cS)\mathbf{v}) = (R\circ(cS))\mathbf{v}.\end{aligned}$$

6.5 The Kernel and Range of a Linear Transformation

1. **(a)** Only (ii) is in $\ker(T)$, since

(i) $T\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix} \neq O$,

(ii) $T\begin{bmatrix}0 & 4\\ 2 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = O$,

(iii) $T\begin{bmatrix}3 & 0\\ 0 & -3\end{bmatrix} = \begin{bmatrix}3 & 0\\ 0 & -3\end{bmatrix} \neq O$.

(b) From the definition of T, a matrix is in range(T) if and only if its two off-diagonal elements are zero. Thus only (iii) is in range(T).

2. **(a)** T is explicitly defined by
$$T\begin{bmatrix}a & b\\ c & d\end{bmatrix} = a+d$$

Thus

(i) $T\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = 1 + 3 = 4$

(ii) $T\begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix} = 0 + 0 = 0$

(iii) $T\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = 1 + (-1) = 0,$

and therefore (ii) and (iii) are in $\ker(T)$ but (i) is not.

(b) All three of these are in range(T). For example,

(i) $T\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 + 0 = 0,$

(ii) $T\begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix} = 2 + 0 = 2,$

(iii) $T\begin{bmatrix} \frac{\sqrt{2}}{2} & \pi \\ 12 & 0 \end{bmatrix} = \frac{\sqrt{2}}{2} + 0 = \frac{\sqrt{2}}{2}.$

(c) $\ker(T)$ is the set of all matrices in M_{22} whose diagonal elements are negatives of each other. range(T) is all real numbers.

3. (a) We have

(i) $T(1+x) = T(1 + 1x + 0x^2) = \begin{bmatrix} 1-1 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0},$

(ii) $T(x - x^2) = T(0 + 1x - 1x^2) = \begin{bmatrix} 0-1 \\ 1+(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \neq \mathbf{0},$

(iii) $T(1 + x - x^2) = T(1 + 1x - 1x^2) = \begin{bmatrix} 1-1 \\ 1+(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0},$

so only (iii) is in range(T).

(b) All of them are. For example,

(i) $T(1 + x - x^2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, from part (a),

(ii) $T(1) = \begin{bmatrix} 1-0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$

(iii) $T(x^2) = \begin{bmatrix} 0-0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

(c) $\ker(T)$ consists of all polynomials $a + bx + cx^2$ such that $a = b$ and $b = -c$; parametrizing via $a = t$ gives $\ker(T) = \{a + ax - ax^2\}$, so that $\ker(T)$ consists of all multiples of $1 + x - x^2$. range(T) is all of $\mathbb{R}^2$, since it is a subspace of $\mathbb{R}^2$ containing both $\mathbf{e}_1$ and $\mathbf{e}_2$ (see items (ii) and (iii) of part (b)).

4. (a) Since $xp'(x) = 0$ if and only if $p'(x) = 0$, we see that only (i) is in $\ker(T)$. For the other two, we have $T(x) = xx' = x$ and $T(x^2) = x(x^2)' = 2x^2$.

(b) A polynomial in range(T) must be divisible by x; that is, it must have no constant term. Thus (i) is not in range(T). However, $x = T(x)$ from part (a), and also from part (a), we see that $T\left(\frac{1}{2}x^2\right) = x^2$. Thus (ii) and (iii) are in range(T).

(c) $\ker(T)$ consists of all polynomials whose derivative vanishes, so it consists of constant polynomials. range(T) consists of all polynomials with zero constant terms, since given any such polynomial, we may write it as $xg(x)$ for some polynomial $g(x)$; then clearly $T\left(\int g(x)\,dx\right) = xg(x)$.

5. Since $\ker(T) = \left\{\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right\}$, a basis for $\ker(T)$ is

$$\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}.$$

Similarly, since $\text{range}(T) = \left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right\}$, a basis for $\text{range}(T)$ is

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

Thus $\dim\ker(T) = 2$ and $\dim\text{range}(T) = 2$. The Rank Theorem says that

$$4 = \dim M_{22} = \text{nullity}(T) + \text{rank}(T) = \dim\ker(T) + \dim\text{range}(T) = 2 + 2,$$

which is true.

6. Since

$$\ker(T) = \left\{\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right\},$$

a basis for $\ker(T)$ is

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}.$$

From Exercise 2, $\text{range}(T) = \mathbb{R}$. Thus $\dim\ker(T) = 3$ and $\dim\text{range}(T) = 1$. The Rank Theorem says that

$$4 = \dim M_{22} = \text{nullity}(T) + \text{rank}(T) = \dim\ker(T) + \dim\text{range}(T) = 3 + 1,$$

which is true.

7. Since $\ker(T) = \left\{a + ax - ax^2\right\}$, a basis for $\ker(T)$ is $\{1 + x - x^2\}$. From Exercise 3, $\text{range}(T) = \mathbb{R}^2$. Thus $\dim\ker(T) = 1$ and $\dim\text{range}(T) = 2$. The Rank Theorem says that

$$3 = \dim \mathscr{P}_2 = \text{nullity}(T) + \text{rank}(T) = \dim\ker(T) + \dim\text{range}(T) = 1 + 2,$$

which is true.

8. Since $\ker(T)$ is the constant polynomials, it has a basis $\{1\}$. From Exercise 4, $\text{range}(T)$ is the set of polynomials with nonzero constant terms, or $\{bx + cx^2\}$, so it has a basis $\{x, x^2\}$. Thus $\dim\ker(T) = 1$ and $\dim\text{range}(T) = 2$. The Rank Theorem says that

$$3 = \dim \mathscr{P}_2 = \text{nullity}(T) + \text{rank}(T) = \dim\ker(T) + \dim\text{range}(T) = 1 + 2,$$

which is true.

9. Since

$$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0-0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

it follows that $\text{range}(T) = \mathbb{R}^2$, since it is a subspace of $\mathbb{R}^2$ containing both $\mathbf{e}_1$ and $\mathbf{e}_2$. Thus $\text{rank}(T) = \dim\text{range}(T) = 2$, so that by the Rank Theorem,

$$\text{nullity}(T) = \dim M_{22} - \text{rank}(T) = 4 - 2 = 2.$$

10. Since

$$T(1-x) = \begin{bmatrix} 1-0 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad T(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

it follows that $\text{range}(T) = \mathbb{R}^2$, since it is a subspace of $\mathbb{R}^2$ containing both $\mathbf{e}_1$ and $\mathbf{e}_2$. Thus $\text{rank}(T) = \dim\text{range}(T) = 2$, so that by the Rank Theorem,

$$\text{nullity}(T) = \dim \mathscr{P}_2 - \text{rank}(T) = 3 - 2 = 1.$$

11. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$T(A) = AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a-b & -a+b \\ c-d & -c+d \end{bmatrix} = \begin{bmatrix} a-b & -(a-b) \\ c-d & -(c-d) \end{bmatrix}.$$

Thus $T(A) = O$ if and only if $a = b$ and $c = d$, so that

$$\ker(T) = \left\{ \begin{bmatrix} a & a \\ c & c \end{bmatrix} \right\},$$

so that a basis for $\ker(T)$ is

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

Hence $\operatorname{nullity}(T) = \dim \ker(T) = 2$, so that by the Rank Theorem

$$\operatorname{rank}(T) = \dim M_{22} - \operatorname{nullity}(T) = 4 - 2 = 2.$$

12. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$T(A) = AB - BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} - \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} -c & a-d \\ 0 & c \end{bmatrix}.$$

So $A \in \ker(T)$ if and only if $c = 0$ and $a = d$, so that

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

These two matrices are linearly independent since they are not multiples of one another, so that $\operatorname{nullity}(T) = 2$. Since $\dim M_{22} = 4$, we see that by the Rank Theorem

$$\operatorname{rank}(T) = \dim M_{22} - \operatorname{nullity}(T) = 4 - 2 = 2.$$

13. The derivative of $p(x) = a + bx + cx^2$ is $b + 2cx$; then $p'(0) = b$, so this is zero if $b = 0$. Thus $\ker(T) = \{a + cx^2\}$, so that a basis for $\ker(T)$ is $\{1, x^2\}$. Then $\operatorname{nullity}(T) = \dim \ker(T) = 2$, so that by the Rank Theorem,

$$\operatorname{rank}(T) = \dim \mathscr{P}_2 - \operatorname{nullity}(T) = 3 - 2 = 1.$$

14. We have

$$T(A) = A - A^T = \begin{bmatrix} 0 & a_{12} - a_{21} & a_{13} - a_{31} \\ a_{12} - a_{21} & 0 & a_{23} - a_{32} \\ a_{31} - a_{13} & a_{32} - a_{23} & 0 \end{bmatrix},$$

so that $A \in \ker(T)$ if and only if $a_{12} = a_{21}$, $a_{13} = a_{31}$, and $a_{23} = a_{32}$; that is, if and only if A is symmetric. By Exercise 40 in Section 6.2, $\operatorname{nullity}(T) = \dim \ker(T) = \frac{3 \cdot 4}{2} = 6$. Then by the Rank Theorem,

$$\operatorname{rank}(T) = \dim M_{33} - \operatorname{nullity}(T) = 9 - 6 = 3.$$

15. **(a)** Yes, T is one-to-one. If

$$T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a - b \\ a + 2b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then $2a - b = 0$ and $a + 2b = 0$. The only solution to this pair of equations is $a = b = 0$, so that the only element of the kernel is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and thus T is one-to-one.

(b) Since T is one-to-one and the dimension of the domain and codomain are both 2, T must be onto by Theorem 6.21.

16. (a) Yes, T is one-to-one. If

$$T\begin{bmatrix} a \\ b \end{bmatrix} = (a - 2b) + (3a + b)x + (a + b)x^2 = 0,$$

then $a + b = 0$ from the third term. Since $3a + b = 0$ as well, subtracting gives $2a = 0$, so that $a = 0$. The third term then forces $b = 0$. Thus $\begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{0}$ and T is one-to-one.

(b) T cannot be onto. $\mathbb{R}^2$ has dimension 2, so by the rank theorem $2 = \dim \mathbb{R}^2 = \operatorname{rank}(T) + \operatorname{nullity}(T) \geq \operatorname{rank}(T)$. It follows that the dimension of the range of T is at most 2 (in fact, since from part (a) $\operatorname{nullity}(T) = 0$, the dimension of the range is equal to 2). But $\dim \mathscr{P}_2 = 3$, so T is not onto.

17. (a) If $T(a + bx + cx^2) = \mathbf{0}$, then

$$\begin{aligned} 2a - b \phantom{{}- 3c} &= 0 \\ a + b - 3c &= 0 \\ -a \phantom{{}+b} + c &= 0. \end{aligned}$$

Row-reducing the augmented matrix gives

$$\left[\begin{array}{rrr|r} 2 & -1 & 0 & 0 \\ 1 & 1 & -3 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This has nontrivial solutions $a = t$, $b = 2t$, $c = t$, so that

$$\ker(T) = \{t + 2tx + tx^2\} = \{t(1 + 2x + x^2)\}.$$

So T is not one-to-one.

(b) Since $\dim \mathscr{P}_2 = \dim \mathbb{R}^3 = 3$, the fact that T is not one-to-one implies, by Theorem 6.21, that T is not onto.

18. (a) By Exercise 10, $\operatorname{nullity}(T) = 1$, so that $\ker(T) \neq \{\mathbf{0}\}$, so that T is not one-to-one.

(b) By Exercise 10, $\operatorname{range}(T) = \mathbb{R}^2$, so that T is onto by definition.

19. (a) If $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = O$, then $a - b = a + b = 0$ and $b - c = b + c = 0$; these equations give $a = b = c = 0$ as the only solution. Thus T is one-to-one.

(b) By the Rank Theorem,

$$\dim \operatorname{range}(T) = \dim \mathbb{R}^3 - \dim \ker(T) = 3 - 0 = 3.$$

However, $\dim M_{22} = 4 > 3 = \dim \operatorname{range}(T)$, so that T cannot be onto.

20. (a) If $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = O$, then $a + b + c = 0$, $b - 2c = 0$, and $a - c = 0$. The latter two equations give $a = c$ and $b = 2c$, so that from the first equation $4c = 0$ and thus $c = 0$. It follows that $a = b = 0$. Thus T is one-to-one.

(b) By Exercise 40 in Section 6.2, $\dim W = \frac{2 \cdot 3}{2} = 3 = \dim \mathbb{R}^3$. Since the dimensions of the domain and codomain are equal and T is one-to-one, Theorem 6.21 implies that T is also onto.

21. These vector spaces are isomorphic. Bases for the two vector spaces are

$$D_3: \ \{E_{11},\ E_{22},\ E_{33}\}, \qquad \mathbb{R}^3: \{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}.$$

Since both have dimension 3, we know that $D_3 \cong \mathbb{R}^3$. Define

$$T(aE_{11} + bE_{22} + cE_{33}) = a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3.$$

Then T is a linear transformation from the way we have defined it. Since the $\mathbf{e}_i$ are linearly independent, it follows that $a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3 = \mathbf{0}$ if and only if $a = b = c = 0$. Thus $\ker(T) = \{0\}$ so that T is one-to-one. Since $\dim D_3 = \dim \mathbb{R}^3$, we know that T is onto as well, so it is an isomorphism.

22. By Exercise 40 in Section 6.2, $\dim S_3 = \frac{3\cdot 4}{2} = 6$. Since an upper triangular matrix has three entries that are forced to be zero (those below the diagonal), and the other six may take any value, we see that $\dim U_3 = 6$ as well. Thus $S_3 \cong U_3$. Define

$$T: S_3 \to D_3: \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

Then any matrix in $\ker(T)$ must have $a_{11} = a_{12} = a_{13} = a_{22} = a_{23} = a_{33} = 0$, so that it must itself be the zero matrix. Thus $\ker(T) = \{\mathbf{0}\}$ so that T is one-to-one. Since T is one-to-one and $\dim S_3 = \dim U_3 = 6$, it follows that T is onto and is therefore an isomorphism.

23. By Exercises 40 and 41 in Section 6.2, $\dim S_3 = \frac{3\cdot 4}{2} = 6$ while $\dim S_3' = \frac{2\cdot 3}{2} = 3$. Since their dimensions are unequal, $S_3 \ncong S_3'$.

24. Since a basis for $\mathscr{P}_2$ is $\{1, x, x^2\}$, we have $\dim \mathscr{P}_2 = 3$. If $p(x) \in W$, say $p(x) = a + bx + cx^2 + dx^3$, then $p(0) = 0$ implies that $a = 0$, so that $p(x) = bx + cx^2 + dx^3$; a basis for this subspace is $\{x, x^2, x^3\}$ so that $\dim W = 3$ as well. Therefore $\mathscr{P}_2 \cong W$. Define

$$T: \mathscr{P}_2 \to W: a + bx + cx^2 \mapsto ax + bx^2 + cx^3.$$

Then $T(a + bx + cx^2) = ax + bx^2 + cx^3 = 0$ implies that $a = b = c = 0$. Thus T is one-to-one; since the dimensions of domain and codomain are equal, T is also onto, so it is an isomorphism.

25. A basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$ is $\{1, i\}$, so that $\dim \mathbb{C} = 2$. A basis for $\mathbb{R}^2$ is $\{\mathbf{e}_1, \mathbf{e}_2\}$, so that $\dim \mathbb{R}^2 = 2$. Thus $\mathbb{C} \cong \mathbb{R}^2$. Define

$$T: \mathbb{C} \to \mathbb{R}^2: a + bi \mapsto \begin{bmatrix} a \\ b \end{bmatrix}.$$

Then $T(a + bi) = \mathbf{0}$ means that $a = b = 0$. Thus T is one-to-one; since the dimensions of domain and codomain are equal, T is also onto, so it is an isomorphism.

26. V consists of all 2×2 matrices whose upper-left and lower-right entries are negatives of each other, so that

$$V = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right\}.$$

Thus a basis for V is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\},$$

and therefore $\dim V = 3$. Since $\dim W = \dim \mathbb{R}^2 = 2$, these vector spaces are not isomorphic.

27. Since the dimensions of the domain and codomain are the same, it suffices to show that T is one-to-one. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$. Then

$$\begin{aligned} p(x) + p'(x) &= (a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n) + (a_1 + 2a_2 x + \cdots + n a_n x^{n-1}) \\ &= (a_0 + a_1) + (a_1 + 2a_2)x + \cdots + (a_{n-1} + n a_n)x^{n-1} + a_n x^n. \end{aligned}$$

If $p(x) + p'(x) = 0$, then the coefficient of x^n is zero, so that $a_n = 0$. But then the coefficient of x^{n-1} is $0 = a_{n-1} + na_n = a_{n-1}$, so that $a_{n-1} = 0$. We continue this process, determining that $a_i = 0$ for all i. Thus $p(x) = 0$, so that T is one-to-one and is thus an isomorphism.

28. Since the dimensions of the domain and codomain are the same, it suffices to show that T is one-to-one. Now, $T(x^k) = (x-2)^k$. Suppose that

$$T(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1(x-2) + \cdots + a_n(x-2)^n = 0.$$

If we show that $\{1, x-2, (x-2)^2, \ldots, (x-2)^n\}$ is a basis for $\mathscr{P}_n$, then it follows that all the a_i are zero, so that $\ker(T) = \{\mathbf{0}\}$ and T is one-to-one. To prove they form a basis, it suffices to show they are linearly independent, since there are $n+1$ elements in the set and $\dim \mathscr{P}_n = n+1$. We do this by induction. For $n = 0$, this is clear. For $n = 1$, suppose that $a + b(x-2) = (a-2b) + bx = 0$. Then $b = 0$ and thus $a = 0$, so that linear independence for $n = 1$ is established. Now suppose that $\{1, x-2, \ldots, (x-2)^k\}$ is a linearly independent set, and suppose that

$$a_0 + a_1(x-2) + \cdots + a_{k-1}(x-2)^{k-1} + a_k(x-2)^k = 0.$$

If this polynomial were expanded, the only term involving x^k comes from the last term of the sum, and it is a_kx^k. Therefor $a_k = 0$, and then the other $a_i = 0$ by induction. Thus this set forms a basis for $\mathscr{P}_n$, so that T is one-to-one and thus is an isomorphism.

29. Since the dimensions of the domain and codomain are the same, it suffices to show that T is one-to-one. That is, we want to show that $x^np\left(\frac{1}{x}\right) = 0$ implies that $p(x) = 0$. Suppose

$$p(x) = a_0 + a_1x + \cdots + a_nx^n.$$

Then

$$x^np\left(\frac{1}{x}\right) = x^n\left(a_0 + a_1\left(\frac{1}{x}\right) + \cdots + a_n\left(\frac{1}{x^n}\right)\right) = a_0x^n + a_1x^{n-1} + \cdots + a_n.$$

Therefore if $x^np\left(\frac{1}{x}\right) = 0$, all of the a_i are zero so that $p(x) = 0$. Thus T is one-to-one and is an isomorphism.

30. **(a)** As the hint suggests, define a map $T : \mathscr{C}[0,1] \to \mathscr{C}[2,3]$ by $(T(f))(x) = f(x-2)$ for $f \in \mathscr{C}[0,1]$. Thus for example $(T(f))(2.5) = f(2.5-2) = f(0.5)$. T is linear, since

$$\begin{aligned}(T(f+g))(x) &= (f+g)(x-2) = f(x-2) + g(x-2) = (T(f))(x) + (T(g))(x)\\ (T(cf))(x) &= (cf)(x-2) = cf(x-2) = c(T(f))(x).\end{aligned}$$

We must show that T is one-to-one and onto. First suppose that $T(f) = 0$; that is, suppose that $f \in \mathscr{C}[0,1]$ and that $f(x-2) = 0$ for all $x \in [2,3]$. Let $y = x-2$; then $x \in [2,3]$ means $y \in [0,1]$, and we get $f(y) = 0$ for all $y \in [0,1]$. But this means that $f = 0$ in $\mathscr{C}[0,1]$. Thus T is one-to-one. To see that it is onto, choose $f \in \mathscr{C}[2,3]$, and let $g(x) = f(x+2)$. Then

$$(T(g))(x) = g(x-2) = f(x+2-2) = f(x),$$

so that $T(g) = f$ and thus T is onto. Since T is one-to-one and onto, it is an isomorphism.

(b) This is similar to part (a). Define a map $T : \mathscr{C}[0,1] \to \mathscr{C}[a,a+1]$ by $(T(f))(x) = f(x-a)$ for $f \in \mathscr{C}[0,1]$. T is proved linear in a similar fashion to part (a). We must show that T is one-to-one and onto. First suppose that $T(f) = 0$; that is, suppose that $f \in \mathscr{C}[0,1]$ and that $f(x-a) = 0$ for all $x \in [a,a+1]$. Let $y = x-a$; then $x \in [a,a+1]$ means $y \in [0,1]$, and we get $f(y) = 0$ for all $y \in [0,1]$. But this means that $f = 0$ in $\mathscr{C}[0,1]$. Thus T is one-to-one. To see that it is onto, choose $f \in \mathscr{C}[a,a+1]$, and let $g(x) = f(x+a)$. Then

$$(T(g))(x) = g(x-a) = f(x+a-a) = f(x),$$

so that $T(g) = f$ and thus T is onto. Since T is one-to-one and onto, it is an isomorphism.

31. This is similar to Exercise 30. Define a map $T : \mathscr{C}[0,1] \to \mathscr{C}[0,2]$ by $(T(f))(x) = f\left(\frac{1}{2}x\right)$ for $f \in \mathscr{C}[0,1]$. Thus for example $(T(f))(1) = f\left(\frac{1}{2} \cdot 1\right) = f\left(\frac{1}{2}\right)$. T is proved linear in a similar fashion to part (a) of Exercise 30. We must show that T is one-to-one and onto. First suppose that $T(f) = 0$; that is, suppose that $f \in \mathscr{C}[0,1]$ and that $f\left(\frac{1}{2}x\right) = 0$ for all $x \in [0,2]$. Let $y = \frac{1}{2}x$; then $x \in [0,2]$ means $y \in [0,1]$, and we get $f(y) = 0$ for all $y \in [0,1]$. But this means that $f = 0$ in $\mathscr{C}[0,1]$. Thus T is one-to-one. To see that it is onto, choose $f \in \mathscr{C}[0,2]$, and let $g(x) = f(2x)$. Then

$$(T(g))(x) = g\left(\frac{1}{2}x\right) = f\left(2 \cdot \frac{1}{2}x\right) = f(x),$$

so that $T(g) = f$ and thus T is onto. Since T is one-to-one and onto, it is an isomorphism.

32. This is again similar to Exercises 30 and 31, and is the most general form. Define a map $T : \mathscr{C}[a,b] \to \mathscr{C}[c,d]$ by $(T(f))(x) = f\left(\frac{b-a}{d-c}(x-c) + a\right)$ for $f \in \mathscr{C}[a,b]$. Then $(T(f))(c) = f(a)$ and $(T(f))(d) = b - a + a = b$. T is proved linear in a similar fashion to part (a) of Exercise 30. We must show that T is one-to-one and onto. First suppose that $T(f) = 0$; that is, suppose that $f \in \mathscr{C}[a,b]$ and that $f\left(\frac{b-a}{d-c}(x-c) + a\right) = 0$ for all $x \in [c,d]$. Let $y = \frac{b-a}{d-c}(x-c) + a$; then $x \in [c,d]$ means $y \in [a,b]$, and we get $f(y) = 0$ for all $y \in [a,b]$. But this means that $f = 0$ in $\mathscr{C}[a,b]$. Thus T is one-to-one. To see that it is onto, choose $f \in \mathscr{C}[c,d]$, and let $g(x) = f\left(\frac{d-c}{b-a}(x-a) + c\right)$. Then

$$(T(g))(x) = g\left(\frac{b-a}{d-c}(x-c) + a\right) = f\left(\frac{d-c}{b-a}\left(\left(\frac{b-a}{d-c}(x-c) + a\right) - a\right) + c\right) = f(x),$$

so that $T(g) = f$ and thus T is onto. Since T is one-to-one and onto, it is an isomorphism.

33. **(a)** Suppose that $(S \circ T)(\mathbf{u}) = \mathbf{0}$. Then $(S \circ T)(\mathbf{u}) = S(T\mathbf{u}) = \mathbf{0}$. Since S is one-to-one, we must have $T\mathbf{u} = \mathbf{0}$. But then since T is one-to-one, it follows that $\mathbf{u} = 0$. Thus $S \circ T$ is one-to-one.

(b) Choose $\mathbf{w} \in W$. Since S is onto, we can find some $\mathbf{v} \in V$ such that $S\mathbf{v} = \mathbf{w}$. But T is onto as well, so there is some $\mathbf{u} \in U$ such that $T\mathbf{u} = \mathbf{v}$. Then

$$(S \circ T)(\mathbf{u}) = S(T\mathbf{u}) = S\mathbf{v} = \mathbf{w},$$

so that $S \circ T$ is onto.

34. **(a)** Suppose that $T\mathbf{u} = \mathbf{0}$. Then since $S\mathbf{0} = \mathbf{0}$, we have

$$(S \circ T)\mathbf{u} = S(T\mathbf{u}) = S\mathbf{0} = \mathbf{0}.$$

But $S \circ T$ is one-to-one, so we must have $\mathbf{u} = \mathbf{0}$, so that T is one-to-one as well. Intuitively, if T takes two elements of U to the same element of V, then certainly $S \circ T$ takes those two elements of U to the same element of W.

(b) Choose $\mathbf{w} \in W$. Since $S \circ T$ is onto, there is some $\mathbf{u} \in U$ such that $(S \circ T)\mathbf{u} = \mathbf{w}$. Let $\mathbf{v} = T\mathbf{u}$. Then

$$S\mathbf{v} = S(T\mathbf{u}) = (S \circ T)\mathbf{u} = \mathbf{w},$$

so that S is onto. Intuitively, if $S \circ T$ "hits" every element of W, then surely S must.

35. **(a)** Suppose that $T : V \to W$ and that $\dim V < \dim W$. Then range(T) is a subspace of W, and the Rank Theorem asserts that

$$\dim \operatorname{range}(T) = \dim V - \operatorname{nullity}(T) \le \dim V < \dim W.$$

Theorem 6.11(b) in Section 6.2 states that if U is a subspace of W, then $\dim U = \dim W$ if and only if $U = W$. But here $\dim \operatorname{range}(T) \ne \dim W$, and range$(T)$ is a subspace of W, so that range$(T) \ne W$ and T cannot be onto.

(b) We want to show that $\ker(T) \neq \{0\}$. Since $\operatorname{range}(T)$ is a subspace of W, then $\dim W \geq \dim \operatorname{range}(T) = \operatorname{rank}(T)$. Then the Rank Theorem asserts that

$$\dim W + \operatorname{nullity}(T) \geq \operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V,$$

so that $\dim\ker(T) = \operatorname{nullity}(T) \geq \dim V - \dim W > 0$. Thus $\ker(T) \neq \{\mathbf{0}\}$, so that T is not one-to-one.

36. Since $\dim \mathscr{P}_n = \dim \mathbb{R}^{n+1} = n+1$, we need only show that T is a one-to-one linear transformation. It is linear since

$$(T(p+q))(x) = \begin{bmatrix} (p+q)(a_0) \\ (p+q)(a_1) \\ \vdots \\ (p+q)(a_n) \end{bmatrix} = \begin{bmatrix} p(a_0)+q(a_0) \\ p(a_1)+q(a_1) \\ \vdots \\ p(a_n)+q(a_n) \end{bmatrix} = \begin{bmatrix} p(a_0) \\ p(a_1) \\ \vdots \\ p(a_n) \end{bmatrix} + \begin{bmatrix} q(a_0) \\ q(a_1) \\ \vdots \\ q(a_n) \end{bmatrix} = (T(p))(x) + (T(q))(x)$$

$$(T(cp))(x) = \begin{bmatrix} (cp)(a_0) \\ (cp)(a_1) \\ \vdots \\ (cp)(a_n) \end{bmatrix} = \begin{bmatrix} cp(a_0) \\ cp(a_1) \\ \vdots \\ cp(a_n) \end{bmatrix} = c\begin{bmatrix} p(a_0) \\ p(a_1) \\ \vdots \\ p(a_n) \end{bmatrix} = c(T(p))(x).$$

Now suppose $T(p) = 0$. That means that

$$p(a_0) = p(a_1) = \cdots = p(a_n) = 0,$$

so that p has $n+1$ zeroes since the a_i are distinct. But Exercise 62 in Section 6.2 implies that $p(x)$ must be the zero polynomial, so that $\ker(T) = \{\mathbf{0}\}$ and thus T is one-to-one.

37. By the Rank Theorem, since $T : V \to V$ and $T^2 : V \to V$, we have

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V = \operatorname{rank}(T^2) + \operatorname{nullity}(T^2).$$

Since $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, it follows that $\operatorname{nullity}(T) = \operatorname{nullity}(T^2)$, and thus that $\dim\ker(T) = \dim\ker(T^2)$. So if we show $\ker(T) \subseteq \ker(T^2)$, we can conclude that $\ker(T) = \ker(T^2)$. Choose $\mathbf{v} \in \ker(T)$; then $T^2\mathbf{v} = T(T\mathbf{v}) = T\mathbf{0} = \mathbf{0}$. Hence $\ker(T) = \ker(T^2)$.

Now choose $\mathbf{v} \in \operatorname{range}(T) \cap \ker(T)$; we want to show $\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \in \operatorname{range}(T)$, there is $\mathbf{w} \in V$ such that $\mathbf{v} = T\mathbf{w}$. Then since $\mathbf{v} \in \ker(T)$,

$$\mathbf{0} = T\mathbf{v} = T(T\mathbf{w}) = T^2\mathbf{w},$$

so that $\mathbf{w} \in \ker(T^2) = \ker(T)$. So $\mathbf{w}$ is also in $\ker(T)$. As a result, $\mathbf{v} = T\mathbf{w} = \mathbf{0}$, as we were to show.

38. (a) We have

$$\begin{aligned} T\left((\mathbf{u}_1, \mathbf{w}_1) + (\mathbf{u}_2, \mathbf{w}_2)\right) &= T(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{w}_1 + \mathbf{w}_2) = \mathbf{u}_1 + \mathbf{u}_2 - \mathbf{w}_1 - \mathbf{w}_2 \\ &= (\mathbf{u}_1 - \mathbf{w}_1) + (\mathbf{u}_2 - \mathbf{w}_2) = T(\mathbf{u}_1, \mathbf{w}_1) + T(\mathbf{u}_2, \mathbf{w}_2) \\ T\left(c(\mathbf{u}, \mathbf{w})\right) &= T(c\mathbf{u}, c\mathbf{w}) = c\mathbf{u} - c\mathbf{w} = c(\mathbf{u} - \mathbf{w}) = cT(\mathbf{u}, \mathbf{w}). \end{aligned}$$

(b) By definition, $U + W = \{\mathbf{u} + \mathbf{w} \;:\; \mathbf{u} \in U,\ \mathbf{w} \in W\}$. So if $\mathbf{u} + \mathbf{w} \in U + W$, we have

$$T(\mathbf{u}, -\mathbf{w}) = \mathbf{u} + \mathbf{w},$$

so that $U + W \subset \operatorname{range}(T)$. But since $T(\mathbf{u}, \mathbf{w}) = \mathbf{u} - \mathbf{w} \in U + W$, we also have $\operatorname{range}(T) \subset U + W$. Thus $\operatorname{range}(T) = U + W$.

(c) Suppose $\mathbf{v} = (\mathbf{u}, \mathbf{w}) \in \ker(T)$. Then $\mathbf{u} - \mathbf{w} = \mathbf{0}$, so that $\mathbf{u} = \mathbf{w}$ and thus $\mathbf{u} \in U \cap W$, and $\mathbf{v} = (\mathbf{u}, \mathbf{u})$. So

$$\ker(T) = \{(\mathbf{u}, \mathbf{u}) : \mathbf{u} \in U\} \subset U \times W$$

is a subspace of $U \times W$. Then

$$S : U \cap W \to \ker(T) : \mathbf{u} \mapsto (\mathbf{u}, \mathbf{u})$$

is clearly a linear transformation that is one-to-one and onto. Thus S is an isomorphism between $U \cap W$ and $\ker T$.

(d) From Exercise 43(b) in Section 6.2, $\dim(U \times W) = \dim U + \dim W$. Then the Rank Theorem gives

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(U \times W) = \dim U + \dim W.$$

But from parts (a) and (b), we have

$$\operatorname{rank}(T) = \dim \operatorname{range}(T) = \dim(U + W), \qquad \operatorname{nullity}(T) = \dim \ker(T) = \dim(U \cap W).$$

Thus

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(U + W) + \dim(U \cap W) = \dim U + \dim W.$$

Rearranging this equation gives $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.

6.6 The Matrix of a Linear Transformation

1. Computing $T(\mathbf{v})$ directly gives

$$T(\mathbf{v}) = T(4 + 2x) = 2 - 4x.$$

To compute the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$, we compute the value of T on each basis element of $\mathcal{B}$:

$$[T(1)]_{\mathcal{C}} = [0 - x]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad [T(x)]_{\mathcal{C}} = [1 - 0x]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow$$

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\ [T(1)]_{\mathcal{C}} \quad [T(x)]_{\mathcal{C}} \right] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[4 + 2x]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} = [2 - 4x]_{\mathcal{C}} = [T(4 + 2x)]_{\mathcal{C}}.$$

2. Computing $T(\mathbf{v})$ directly gives (as in Exercise 1)

$$T(\mathbf{v}) = T(4 + 2x) = 2 - 4x.$$

In terms of the basis $\mathcal{B}$, we have $4 + 2x = 3(1 + x) + (1 - x)$, so that

$$[4 + 2x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Now,

$$[T(1 + x)]_{\mathcal{C}} = [1 - x]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad [T(1 - x)]_{\mathcal{C}} = [-1 - x]_{\mathcal{C}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \Rightarrow$$

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\ [T(1 + x)]_{\mathcal{C}} \quad [T(1 - x)]_{\mathcal{C}} \right] = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[4 + 2x]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} = [2 - 4x]_{\mathcal{C}} = [T(4 + 2x)]_{\mathcal{C}}.$$

3. Computing $T(\mathbf{v})$ directly gives

$$T(\mathbf{v}) = T(a + bx + cx^2) = a + b(x+2) + c(x+2)^2.$$

In terms of the standard basis $\mathcal{B}$, we have

$$[a + bx + cx^2]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Now,

$$[T(1)]_{\mathcal{C}} = [1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_{\mathcal{C}} = [x+2]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [T(x^2)]_{\mathcal{C}} = [(x+2)^2]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\; [T(1)]_{\mathcal{C}} \quad [T(x)]_{\mathcal{C}} \quad [T(x^2)]_{\mathcal{C}} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[a + bx + cx^2]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [a + b(x+2) + c(x+2)^2]_{\mathcal{C}} = [T(a + bx + cx^2)]_{\mathcal{C}}.$$

4. Computing $T(\mathbf{v})$ directly gives

$$T(\mathbf{v}) = T(a + bx + cx^2) = a + b(x+2) + c(x+2)^2 = (a + 2b + 4c) + (b + 4c)x + cx^2.$$

To compute $[a + bx + cx^2]_{\mathcal{B}}$, we want to solve

$$a + bx + cx^2 = r + s(x+2) + t(x+2)^2 = (r + 2s + 4t) + (s + 4t)x + tx^2$$

for r, s, and t. The x^2 terms give $t = c$; then from the linear term $b = s + 4c$, so that $s = b - 4c$; finally, $a = r + 2(b - 4c) + 4c = r + 2b - 4c$, so that $r = a - 2b + 4c$. Hence

$$[a + bx + cx^2]_{\mathcal{B}} = \begin{bmatrix} a - 2b + 4c \\ b - 4c \\ c \end{bmatrix}.$$

Now,

$$[T(1)]_{\mathcal{C}} = [1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$[T(x+2)]_{\mathcal{C}} = [x+4]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix},$$

$$[T((x+2)^2)]_{\mathcal{C}} = [(x+4)^2]_{\mathcal{C}} = [x^2 + 8x + 16]_{\mathcal{C}} = \begin{bmatrix} 16 \\ 8 \\ 1 \end{bmatrix},$$

so that

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\; [T(1)]_{\mathcal{C}} \quad [T(x+2)]_{\mathcal{C}} \quad [T((x+2)^2)]_{\mathcal{C}} \right] = \begin{bmatrix} 1 & 4 & 16 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned}
[T]_{\mathcal{C}\leftarrow\mathcal{B}}[a+bx+cx^2]_{\mathcal{B}} &= \begin{bmatrix} 1 & 4 & 16 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a-2b+4c \\ b-4c \\ c \end{bmatrix} \\
&= \begin{bmatrix} (a-2b+4c)+4(b-4c)+16c \\ (b-4c)+8c \\ c \end{bmatrix} \\
&= \begin{bmatrix} a+2b+4c \\ b+4c \\ c \end{bmatrix} \\
&= [(a+2b+4c)+(b+4c)x+cx^2]_{\mathcal{C}} = [T(a+bx+cx^2)]_{\mathcal{C}}.
\end{aligned}$$

5. Directly,

$$T(a+bx+cx^2) = \begin{bmatrix} a \\ a+b+c \end{bmatrix}.$$

Using the standard basis $\mathcal{B}$, we have

$$[a+bx+cx^2]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Now,

$$[T(1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad [T(x)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad [T(x^2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so that

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [T(1)]_{\mathcal{C}} & [T(x)]_{\mathcal{C}} & [T(x^2)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[a+bx+cx^2]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix}_{\mathcal{C}} = [T(a+bx+cx^2)]_{\mathcal{C}}.$$

6. Directly, as in Exercise 5,

$$T(a+bx+cx^2) = \begin{bmatrix} a \\ a+b+c \end{bmatrix}.$$

Using the basis $\mathcal{B}$, we have

$$[a+bx+cx^2]_{\mathcal{B}} = \begin{bmatrix} c \\ b \\ a \end{bmatrix}.$$

Now,

$$[T(x^2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad [T(x)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad [T(1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so that

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [T(x^2)]_{\mathcal{C}} & [T(x)]_{\mathcal{C}} & [T(1)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[a+bx+cx^2]_{\mathcal{B}} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} -(b+c) \\ a+b+c \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix}_{\mathcal{C}} = [T(a+bx+cx^2)]_{\mathcal{C}}.$$

7. Computing directly gives

$$T\mathbf{v} = T\begin{bmatrix}-7\\7\end{bmatrix} = \begin{bmatrix}-7+2\cdot 7\\-(-7)\\7\end{bmatrix} = \begin{bmatrix}7\\7\\7\end{bmatrix}.$$

We first compute the coordinates of $\begin{bmatrix}-7\\7\end{bmatrix}$ in $\mathcal{B}$. We want to solve

$$\begin{bmatrix}-7\\7\end{bmatrix} = r\begin{bmatrix}1\\2\end{bmatrix} + s\begin{bmatrix}3\\-1\end{bmatrix}$$

for r and s; that is, we want r and s such that $-7 = r + 3s$ and $7 = 2r - s$. Solving this pair of equations gives $r = 2$ and $s = -3$. Thus

$$\begin{bmatrix}-7\\7\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}2\\-3\end{bmatrix}.$$

Now,

$$\left[T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix}5\\-1\\2\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}6\\-3\\2\end{bmatrix}, \quad \left[T\left(\begin{bmatrix}3\\-1\end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix}1\\-3\\-1\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}4\\-2\\-1\end{bmatrix},$$

so that

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\ \left[T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right]_{\mathcal{C}} \quad \left[T\left(\begin{bmatrix}3\\-1\end{bmatrix}\right)\right]_{\mathcal{C}}\right] = \begin{bmatrix}6 & 4\\-3 & -2\\2 & -1\end{bmatrix}.$$

Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}\begin{bmatrix}-7\\7\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}6 & 4\\-3 & -2\\2 & -1\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}0\\0\\7\end{bmatrix} = \begin{bmatrix}7\\7\\7\end{bmatrix}_{\mathcal{C}} = \left[T\begin{bmatrix}-7\\7\end{bmatrix}\right]_{\mathcal{C}}.$$

8. Computing directly gives (as in the statement of Exercise 7)

$$T\mathbf{v} = T\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a+2b\\-a\\b\end{bmatrix}.$$

We first compute the coordinates of $\begin{bmatrix}a\\b\end{bmatrix}$ in $\mathcal{B}$. We want to solve

$$\begin{bmatrix}a\\b\end{bmatrix} = r\begin{bmatrix}1\\2\end{bmatrix} + s\begin{bmatrix}3\\-1\end{bmatrix}$$

for r and s; that is, we want r and s such that $a = r + 3s$ and $b = 2r - s$. Solving this pair of equations gives $r = \frac{1}{7}(a + 3b)$ and $s = \frac{1}{7}(2a - b)$. Thus

$$\begin{bmatrix}a\\b\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}\frac{1}{7}(a+3b)\\\frac{1}{7}(2a-b)\end{bmatrix}.$$

Now,

$$\left[T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix}5\\-1\\2\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}6\\-3\\2\end{bmatrix}, \quad \left[T\left(\begin{bmatrix}3\\-1\end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix}1\\-3\\-1\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}4\\-2\\-1\end{bmatrix},$$

so that

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\ \left[T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right]_{\mathcal{C}} \quad \left[T\left(\begin{bmatrix}3\\-1\end{bmatrix}\right)\right]_{\mathcal{C}}\right] = \begin{bmatrix}6 & 4\\-3 & -2\\2 & -1\end{bmatrix}.$$

Then

$$\begin{aligned}
[T]_{\mathcal{C}\leftarrow\mathcal{B}}\begin{bmatrix}a\\b\end{bmatrix}_{\mathcal{B}} &= \begin{bmatrix}6 & 4\\-3 & -2\\2 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{7}(a+3b)\\\frac{1}{7}(2a-b)\end{bmatrix}\\
&= \begin{bmatrix}\frac{6}{7}(a+3b)+\frac{4}{7}(2a-b)\\-\frac{3}{7}(a+3b)-\frac{2}{7}(2a-b)\\\frac{2}{7}(a+3b)-\frac{1}{7}(2a-b)\end{bmatrix}\\
&= \begin{bmatrix}2a+2b\\-a-b\\b\end{bmatrix} = \begin{bmatrix}a+2b\\-a\\b\end{bmatrix}_{\mathcal{C}} = \left[T\begin{bmatrix}a\\b\end{bmatrix}\right]_{\mathcal{C}}.
\end{aligned}$$

9. Computing directly gives

$$T\mathbf{v} = TA = A^T = \begin{bmatrix}a & c\\b & d\end{bmatrix}.$$

Since $A = aE_{11} + bE_{12} + cE_{21} + dE_{22}$, the coordinate vector of A in $\mathcal{B}$ is

$$[A]_{\mathcal{B}} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}.$$

Now,

$$\begin{aligned}
[TE_{11}]_{\mathcal{C}} &= [E_{11}^T]_{\mathcal{C}} = [E_{11}]_{\mathcal{C}} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix},\\
[TE_{12}]_{\mathcal{C}} &= [E_{12}^T]_{\mathcal{C}} = [E_{21}]_{\mathcal{C}} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix},\\
[TE_{21}]_{\mathcal{C}} &= [E_{21}^T]_{\mathcal{C}} = [E_{12}]_{\mathcal{C}} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix},\\
[TE_{22}]_{\mathcal{C}} &= [E_{22}^T]_{\mathcal{C}} = [E_{22}]_{\mathcal{C}} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.
\end{aligned}$$

Therefore

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}[TE_{11}]_{\mathcal{C}} & [TE_{12}]_{\mathcal{C}} & [TE_{21}]_{\mathcal{C}} & [TE_{22}]_{\mathcal{C}}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}.$$

Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}a\\c\\b\\d\end{bmatrix} = \begin{bmatrix}a & c\\b & d\end{bmatrix}_{\mathcal{C}} = [TA]_{\mathcal{C}}.$$

10. Computing directly gives

$$T\mathbf{v} = TA = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Since $A = aE_{11} + bE_{12} + cE_{21} + dE_{22}$, the coordinate vector of A in $\mathcal{B}$ is

$$[A]_{\mathcal{B}} = \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}.$$

Now,

$$[TE_{22}]_{\mathcal{C}} = [E_{22}^T]_{\mathcal{C}} = [E_{22}]_{\mathcal{C}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$[TE_{21}]_{\mathcal{C}} = [E_{21}^T]_{\mathcal{C}} = [E_{12}]_{\mathcal{C}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$[TE_{12}]_{\mathcal{C}} = [E_{12}^T]_{\mathcal{C}} = [E_{21}]_{\mathcal{C}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$[TE_{11}]_{\mathcal{C}} = [E_{11}^T]_{\mathcal{C}} = [E_{11}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\, [TE_{22}]_{\mathcal{C}} \quad [TE_{21}]_{\mathcal{C}} \quad [TE_{12}]_{\mathcal{C}} \quad [TE_{11}]_{\mathcal{C}} \right] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = \begin{bmatrix} c \\ b \\ d \\ a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}_{\mathcal{C}} = [TA]_{\mathcal{C}}.$$

11. Computing directly gives

$$\begin{aligned} T\mathbf{v} = TA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \\ = \begin{bmatrix} a-b & b-a \\ c-d & d-c \end{bmatrix} - \begin{bmatrix} a-c & b-d \\ c-a & d-b \end{bmatrix} = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix}. \end{aligned}$$

Since $A = aE_{11} + bE_{12} + cE_{21} + dE_{22}$, the coordinate vector of A in $\mathcal{B}$ is

$$[A]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Now,

$$[TE_{11}]_{\mathcal{C}} = [E_{11}B - BE_{11}]_{\mathcal{C}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix},$$

$$[TE_{12}]_{\mathcal{C}} = [E_{12}B - BE_{12}]_{\mathcal{C}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$[TE_{21}]_{\mathcal{C}} = [E_{21}B - BE_{21}]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$[TE_{22}]_{\mathcal{C}} = [E_{22}B - BE_{22}]_{\mathcal{C}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

Therefore

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\, [TE_{11}]_{\mathcal{C}} \quad [TE_{12}]_{\mathcal{C}} \quad [TE_{21}]_{\mathcal{C}} \quad [TE_{22}]_{\mathcal{C}} \right] = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} c-b \\ d-a \\ a-d \\ b-c \end{bmatrix} = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix}_{\mathcal{C}} = [TA]_{\mathcal{C}}.$$

12. Computing directly gives

$$T\mathbf{v} = TA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}.$$

Using the given basis $\mathcal{B}$ for M_{22}, the coordinate vector of A is

$$[A]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Now,

$$[TE_{11}]_{\mathcal{C}} = [E_{11} - E_{11}^T]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [TE_{12}]_{\mathcal{C}} = [E_{12} - E_{12}^T]_{\mathcal{C}} = [E_{12} - E_{21}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

$$[TE_{21}]_{\mathcal{C}} = [E_{21} - E_{21}^T]_{\mathcal{C}} = [E_{21} - E_{12}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad [TE_{22}]_{\mathcal{C}} = [E_{22} - E_{22}^T]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [TE_{22}]_{\mathcal{C}} & [TE_{21}]_{\mathcal{C}} & [TE_{12}]_{\mathcal{C}} & [TE_{11}]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ b-c \\ c-b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}_{\mathcal{C}} = [TA]_{\mathcal{C}}.$$

13. (a) If $a\sin x + b\cos x \in W$, then

$$D(a\sin x + b\cos x) = D(a\sin x) + D(b\cos x) = aD\sin x + bD\cos x = a\cos x - b\sin x \in W.$$

(b) Since

$$[\sin x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad [\cos x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then

$$[D\sin x]_{\mathcal{B}} = [\cos x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad [D\cos x]_{\mathcal{B}} = [-\sin x]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \Rightarrow \qquad [D]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(c) $[f(x)]_{\mathcal{B}} = [3\sin x - 5\cos x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, so

$$[D(f(x))]_{\mathcal{B}} = [D]_{\mathcal{B}}[f(x)]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix},$$

so that $D(f(x)) = f'(x) = 5\sin x + 3\cos x$. Directly, since $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, we also get $f'(x) = 3\cos x - 5(-\sin x) = 5\sin x + 3\cos x$.

14. (a) If $ae^{2x} + be^{-2x} \in W$, then

$$D(ae^{2x} + be^{-2x}) = D(ae^{2x}) + D(be^{-2x}) = aDe^{2x} + bDe^{-2x} = (2a)e^{2x} + (-2b)e^{-2x} \in W.$$

(b) Since

$$[e^{2x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad [e^{-2x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then

$$[De^{2x}]_{\mathcal{B}} = [2e^{2x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad [De^{-2x}]_{\mathcal{B}} = [-2e^{-2x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \qquad \Rightarrow \qquad [D]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

(c) $[f(x)]_{\mathcal{B}} = [e^{2x} - 3e^{-2x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, so

$$[D(f(x))]_{\mathcal{B}} = [D]_{\mathcal{B}}[f(x)]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix},$$

so that $D(f(x)) = f'(x) = 2e^{2x} + 6e^{-3x}$. Directly, $f'(x) = e^{2x}\cdot 2 - 3e^{-2x}\cdot(-2) = 2e^{2x} + 6e^{-3x}$.

15. (a) Since

$$[e^{2x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [e^{2x}\cos x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad [e^{2x}\sin x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

then

$$[De^{2x}]_{\mathcal{B}} = [2e^{2x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix},$$

$$[D(e^{2x}\cos x)]_{\mathcal{B}} = [2e^{2x}\cos x - e^{2x}\sin x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix},$$

$$[D(e^{2x}\sin x)]_{\mathcal{B}} = [2e^{2x}\sin x + e^{2x}\cos x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Thus

$$[D]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}.$$

(b) $[f(x)]_{\mathcal{B}} = [3e^{2x} - e^{2x}\cos x + 2e^{2x}\sin x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, so

$$[D(f(x))]_{\mathcal{B}} = [D]_{\mathcal{B}}[f(x)]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix},$$

so that $D(f(x)) = f'(x) = 6e^{2x} + 5e^{2x}\sin x$. Directly,

$$f'(x) = 3e^{2x}\cdot 2 - (2e^{2x}\cos x - e^{2x}\sin x) + (4e^{2x}\sin x + 2e^{2x}\cos x) = 6e^{2x} + 5e^{2x}\sin x.$$

16. (a) Since

$$[\cos x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad [\sin x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [x\cos x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad [x\sin x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

then

$$[D\cos x]_{\mathcal{B}} = [-\sin x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad [D\sin x]_{\mathcal{B}} = [\cos x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$[D(x\cos x)]_{\mathcal{B}} = [\cos x - x\sin x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad [D(x\sin x)]_{\mathcal{B}} = [\sin x + x\cos x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Thus

$$[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

(b) $[f(x)]_{\mathcal{B}} = [\cos x + 2x\cos x]_{\mathcal{B}} = \begin{bmatrix}1\\0\\2\\0\end{bmatrix}$, so

$$[D(f(x))]_{\mathcal{B}} = [D]_{\mathcal{B}}[f(x)]_{\mathcal{B}} = \begin{bmatrix}0 & 1 & 1 & 0\\ -1 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}1\\0\\2\\0\end{bmatrix} = \begin{bmatrix}2\\-1\\0\\-2\end{bmatrix},$$

so that $D(f(x)) = f'(x) = 2\cos x - \sin x - 2x\sin x$. Directly,

$$f'(x) = -\sin x + 2\cos x - 2x\sin x = 2\cos x - \sin x - 2x\sin x.$$

17. **(a)** Let $p(x) = a + bx$; then

$$(S\circ T)p(x) = S(Tp(x)) = S\left(\begin{bmatrix}p(0)\\p(1)\end{bmatrix}\right) = S\begin{bmatrix}a\\a+b\end{bmatrix} = \begin{bmatrix}a-2(a+b)\\2a-(a+b)\end{bmatrix} = \begin{bmatrix}-a-2b\\a-b\end{bmatrix}.$$

Then

$$[(S\circ T)(1)]_{\mathcal{D}} = \begin{bmatrix}-1\\1\end{bmatrix}_{\mathcal{D}} = \begin{bmatrix}-1\\1\end{bmatrix},\quad [(S\circ T)(x)]_{\mathcal{D}} = \begin{bmatrix}-2\\-1\end{bmatrix}_{\mathcal{D}} = \begin{bmatrix}-2\\-1\end{bmatrix},$$

so that

$$[S\circ T]_{\mathcal{D}\leftarrow\mathcal{B}} = \begin{bmatrix}-1 & -2\\ 1 & -1\end{bmatrix}.$$

(b) We have

$$[T(1)]_{\mathcal{C}} = \begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}1\\1\end{bmatrix},\qquad [T(x)]_{\mathcal{C}} = \begin{bmatrix}0\\1\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}0\\1\end{bmatrix},$$

so that

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}.$$

Next,

$$\left[S\begin{bmatrix}1\\0\end{bmatrix}\right]_{\mathcal{D}} = \begin{bmatrix}1\\2\end{bmatrix}_{\mathcal{D}} = \begin{bmatrix}1\\2\end{bmatrix},\qquad \left[S\begin{bmatrix}0\\1\end{bmatrix}\right]_{\mathcal{D}} = \begin{bmatrix}-2\\-1\end{bmatrix}_{\mathcal{D}} = \begin{bmatrix}-2\\-1\end{bmatrix},$$

so that

$$[S]_{\mathcal{D}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & -2\\ 2 & -1\end{bmatrix}.$$

Then

$$[S\circ T]_{\mathcal{D}\leftarrow\mathcal{B}} = [S]_{\mathcal{D}\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & -2\\ 2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}-1 & -2\\ 1 & -1\end{bmatrix},$$

which matches the result from part (a).

18. **(a)** Let $p(x) = a + bx$; then

$$(S\circ T)p(x) = S(Tp(x)) = S(a + b(x+1)) = S((a+b) + bx) = (a+b) + b(x+1) = (a+2b) + bx.$$

Then

$$[(S\circ T)(1)]_{\mathcal{D}} = \begin{bmatrix}1\\0\\0\end{bmatrix}_{\mathcal{D}} = \begin{bmatrix}1\\0\\0\end{bmatrix},\quad [(S\circ T)(x)]_{\mathcal{D}} = \begin{bmatrix}2\\1\\0\end{bmatrix}_{\mathcal{D}} = \begin{bmatrix}2\\1\\0\end{bmatrix},$$

so that

$$[S\circ T]_{\mathcal{D}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 2\\ 0 & 1\\ 0 & 0\end{bmatrix}.$$

(b) We have

$$[T(1)]_{\mathcal{C}} = [1]_{\mathcal{C}} = \begin{bmatrix}1\\0\\0\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad [T(x)]_{\mathcal{C}} = [x+1]_{\mathcal{C}} = \begin{bmatrix}1\\1\\0\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}1\\1\\0\end{bmatrix},$$

so that

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix}.$$

Next,

$$[S(1)]_{\mathcal{D}} = [1]_{\mathcal{D}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad [S(x)]_{\mathcal{D}} = [x+1]_{\mathcal{D}} = \begin{bmatrix}1\\1\\0\end{bmatrix},$$

$$[S(x^2)]_{\mathcal{D}} = [(x+1)^2]_{\mathcal{D}} = [x^2+2x+1]_{\mathcal{D}} = \begin{bmatrix}1\\2\\1\end{bmatrix},$$

so that

$$[S]_{\mathcal{D}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}.$$

Then

$$[S\circ T]_{\mathcal{D}\leftarrow\mathcal{B}} = [S]_{\mathcal{D}\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 1\\0 & 0\end{bmatrix},$$

which matches the result from part (a).

19. In Exercise 1, the basis for $\mathscr{P}_1$ in both the domain and codomain was already the standard basis $\mathcal{E}$, so we can use the matrix computed there:

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}.$$

Since this matrix has determinant 1, it is invertible, so that T is invertible as well. Then

$$[T^{-1}(a+bx)]_{\mathcal{E}\leftarrow\mathcal{E}} = ([T]_{\mathcal{E}\leftarrow\mathcal{E}})^{-1}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}^{-1}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}-b\\a\end{bmatrix}.$$

So $T^{-1}(a+bx) = -b+ax$.

20. From Exercise 5,

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\end{bmatrix}.$$

Since this matrix is not square, it is not invertible, so that T is not invertible.

21. We first compute $[T]_{\mathcal{E}\leftarrow\mathcal{E}}$, where $\mathcal{E} = \{1, x, x^2\}$ is the standard basis:

$$[T(1)]_{\mathcal{E}} = [1]_{\mathcal{E}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad [T(x)]_{\mathcal{E}} = [x+2]_{\mathcal{E}} = \begin{bmatrix}2\\1\\0\end{bmatrix}, \quad [T(x^2)]_{\mathcal{E}} = [(x+2)^2]_{\mathcal{E}} = [x^2+4x+4]_{\mathcal{E}} = \begin{bmatrix}4\\4\\1\end{bmatrix},$$

so that

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix}1 & 2 & 4\\0 & 1 & 4\\0 & 0 & 1\end{bmatrix}.$$

Since this is an upper triangular matrix with nonzero diagonal entries, it is invertible, so that T is invertible as well. To find T^{-1}, we have

$$\begin{aligned}[T^{-1}(a+bx+cx^2)]_{\mathcal{E}\leftarrow\mathcal{E}} = ([T]_{\mathcal{E}\leftarrow\mathcal{E}})^{-1}\begin{bmatrix}a\\b\\c\end{bmatrix} &= \begin{bmatrix}1&2&4\\0&1&4\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}a\\b\\c\end{bmatrix}\\ &= \begin{bmatrix}1&-2&4\\0&1&-4\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a-2b+4c\\b-4c\\c\end{bmatrix}.\end{aligned}$$

So $T^{-1}(a+bx) = (a-2b+4c)+(b-4c)x+cx^2 = a+b(x-2)+c(x-2)^2 = p(x-2)$.

22. We first compute $[T]_{\mathcal{E}\leftarrow\mathcal{E}}$, where $\mathcal{E}=\{1,x,x^2\}$ is the standard basis:

$$[T(1)]_{\mathcal{E}} = [0]_{\mathcal{E}} = \begin{bmatrix}0\\0\\0\end{bmatrix},\quad [T(x)]_{\mathcal{E}} = [1]_{\mathcal{E}} = \begin{bmatrix}1\\0\\0\end{bmatrix},\quad [T(x^2)]_{\mathcal{E}} = [2x]_{\mathcal{E}} = \begin{bmatrix}0\\2\\0\end{bmatrix},$$

so that

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}.$$

Since this is an upper triangular matrix with a zero entry on the diagonal, it has determinant zero, so is not invertible; therefore, T is not invertible either.

23. We first compute $[T]_{\mathcal{E}\leftarrow\mathcal{E}}$, where $\mathcal{E}=\{1,x,x^2\}$ is the standard basis:

$$[T(1)]_{\mathcal{E}} = [1+0]_{\mathcal{E}} = [1]_{\mathcal{E}} = \begin{bmatrix}1\\0\\0\end{bmatrix},\quad [T(x)]_{\mathcal{E}} = [x+1]_{\mathcal{E}} = \begin{bmatrix}1\\1\\0\end{bmatrix},\quad [T(x^2)]_{\mathcal{E}} = [x^2+2x]_{\mathcal{E}} = \begin{bmatrix}0\\2\\1\end{bmatrix},$$

so that

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix}1&1&0\\0&1&2\\0&0&1\end{bmatrix}.$$

Since this is an upper triangular matrix with nonzero diagonal entries, it is invertible, so that T is invertible as well. To find T^{-1}, we have

$$\begin{aligned}[T^{-1}(a+bx+cx^2)]_{\mathcal{E}\leftarrow\mathcal{E}} = ([T]_{\mathcal{E}\leftarrow\mathcal{E}})^{-1}\begin{bmatrix}a\\b\\c\end{bmatrix} &= \begin{bmatrix}1&1&0\\0&1&2\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}a\\b\\c\end{bmatrix}\\ &= \begin{bmatrix}1&-1&2\\0&1&-2\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a-b+2c\\b-2c\\c\end{bmatrix}.\end{aligned}$$

So $T^{-1}(a+bx+cx^2) = (a-b+2c)+(b-2c)x+cx^2$.

24. We first compute $[T]_{\mathcal{E}\leftarrow\mathcal{E}}$, where $\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is the standard basis:

$$[TE_{11}]_{\mathcal{E}} = \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}3 & 2\\2 & 1\end{bmatrix}\right)_{\mathcal{E}} = \begin{bmatrix}3 & 2\\0 & 0\end{bmatrix}_{\mathcal{E}} = \begin{bmatrix}3\\2\\0\\0\end{bmatrix},$$

$$[TE_{12}]_{\mathcal{E}} = \left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}3 & 2\\2 & 1\end{bmatrix}\right)_{\mathcal{E}} = \begin{bmatrix}2 & 1\\0 & 0\end{bmatrix}_{\mathcal{E}} = \begin{bmatrix}2\\1\\0\\0\end{bmatrix},$$

$$[TE_{21}]_{\mathcal{E}} = \left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}3 & 2\\2 & 1\end{bmatrix}\right)_{\mathcal{E}} = \begin{bmatrix}0 & 0\\3 & 2\end{bmatrix}_{\mathcal{E}} = \begin{bmatrix}0\\0\\3\\2\end{bmatrix},$$

$$[TE_{22}]_{\mathcal{E}} = \left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\2 & 1\end{bmatrix}\right)_{\mathcal{E}} = \begin{bmatrix}0 & 0\\2 & 1\end{bmatrix}_{\mathcal{E}} = \begin{bmatrix}0\\0\\2\\1\end{bmatrix},$$

so that

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix}3 & 2 & 0 & 0\\2 & 1 & 0 & 0\\0 & 0 & 3 & 2\\0 & 0 & 2 & 1\end{bmatrix}.$$

This matrix is block triangular, and each diagonal block is invertible, so it is invertible and therefore T is invertible as well. To find T^{-1}, we have

$$\left[T^{-1}\begin{bmatrix}a & b\\c & d\end{bmatrix}\right]_{\mathcal{E}\leftarrow\mathcal{E}} = ([T]_{\mathcal{E}\leftarrow\mathcal{E}})^{-1}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}3 & 2 & 0 & 0\\2 & 1 & 0 & 0\\0 & 0 & 3 & 2\\0 & 0 & 2 & 1\end{bmatrix}^{-1}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$$
$$= \begin{bmatrix}-1 & 2 & 0 & 0\\2 & -3 & 0 & 0\\0 & 0 & -1 & 2\\0 & 0 & 2 & -3\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}-a+2b\\2a-3b\\-c+2d\\2c-3d\end{bmatrix}.$$

So $T^{-1}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}-a+2b & 2a-3b\\-c+2d & 2c-3d\end{bmatrix}$.

25. From Exercise 11, since the bases given there were both the standard basis,

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix}0 & -1 & 1 & 0\\-1 & 0 & 0 & 1\\1 & 0 & 0 & -1\\0 & 1 & -1 & 0\end{bmatrix}.$$

This matrix has determinant zero (for example, the second and third rows are multiples of one another), so that T is not invertible.

26. From Exercise 12, since the bases given there were both the standard basis,

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix}0 & 0 & 0 & 0\\0 & 1 & -1 & 0\\0 & -1 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}.$$

This matrix has determinant zero, so that T is not invertible.

27. In Exercise 13, the basis is $\mathcal{B} = \{\sin x, \cos x\}$. Using the method of Example 6.83,

$$\left[\int (a\sin x + b\cos x)\,dx\right]_{\mathcal{B}} = [D]^{-1}_{\mathcal{B}\leftarrow\mathcal{B}} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Using $[D]_{\mathcal{B}\leftarrow\mathcal{B}}$ from Exercise 13, we get

$$\left[\int (\sin x - 3\cos x)\,dx\right]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}.$$

So $\int(\sin x - 3\cos x)\,dx = -3\sin x - \cos x + C$. This is the same answer as integrating directly.

28. In Exercise 14, the basis is $\mathcal{B} = \{e^{2x}, e^{-2x}\}$. Using the method of Example 6.83,

$$\left[\int \left(ae^{2x} + be^{-2x}\right)\,dx\right]_{\mathcal{B}} = [D]^{-1}_{\mathcal{B}\leftarrow\mathcal{B}} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Using $[D]_{\mathcal{B}\leftarrow\mathcal{B}}$ from Exercise 14, we get

$$\left[\int 5e^{-2x}\,dx\right]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{5}{2} \end{bmatrix}.$$

So $\int 5e^{-2x}\,dx = -\frac{5}{2}e^{-2x} + C$. This is the same answer as integrating directly.

29. In Exercise 15, the basis is $\mathcal{B} = \{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\}$. Using the method of Example 6.83,

$$\left[\int \left(ae^{2x} + be^{-2x}\right)\,dx\right]_{\mathcal{B}} = [D]^{-1}_{\mathcal{B}\leftarrow\mathcal{B}} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Using $[D]_{\mathcal{B}\leftarrow\mathcal{B}}$ from Exercise 15, we get

$$\left[\int \left(e^{2x}\cos x - 2e^{2x}\sin x\right)\,dx\right]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}.$$

So $\int \left(e^{2x}\cos x - 2e^{2x}\sin x\right)\,dx = \frac{4}{5}e^{2x}\cos x - \frac{3}{5}e^{2x}\sin x + C$. Checking via differentiation gives

$$\begin{aligned}\left(\frac{4}{5}e^{2x}\cos x - \frac{3}{5}e^{2x}\sin x + C\right)' &= \frac{4}{5}\left(2e^{2x}\cos x - e^{2x}\sin x\right) - \frac{3}{5}\left(2e^{2x}\sin x + e^{2x}\cos x\right) \\ &= e^{2x}\cos x - 2e^{2x}\sin x.\end{aligned}$$

30. In Exercise 16, the basis is $\mathcal{B} = \{\cos x, \sin x, x\cos x, x\sin x\}$. Using the method of Example 6.83,

$$\left[\int \left(a\cos x + b\sin x + cx\cos x + dx\sin x\right)\,dx\right]_{\mathcal{B}} = [D]^{-1}_{\mathcal{B}\leftarrow\mathcal{B}} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Using $[D]_{\mathcal{B}\leftarrow\mathcal{B}}$ from Exercise 16, we get

$$\left[\int \left(x\cos x + x\sin x\right)\,dx\right]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

So $\int \left(x\cos x + x\sin x\right)\,dx = \cos x + \sin x - x\cos x + x\sin x + C$. Checking via differentiation gives

$$\begin{aligned}(\cos x + \sin x - x\cos x + x\sin x + C)' &= -\sin x + \cos x - \cos x + x\sin x + \sin x + x\cos x \\ &= x\cos x + x\sin x.\end{aligned}$$

31. With $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, we have

$$[T\mathbf{e}_1]_{\mathcal{E}} = \begin{bmatrix} -4\cdot 0 \\ 1+5\cdot 0 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad [T\mathbf{e}_2]_{\mathcal{E}} = \begin{bmatrix} -4\cdot 1 \\ 0+5\cdot 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}.$$

Therefore

$$[T]_{\mathcal{E}} = \begin{bmatrix} 0 & -4 \\ 1 & 5 \end{bmatrix}.$$

Using the method of Example 6.86(b), the eigenvectors of $[T]_{\mathcal{E}}$ will give us a diagonalizing basis $\mathcal{C}$. Using methods from Chapter 4, we find the eigenvalues and corresponding eigenvectors to be

$$\lambda_1 = 4,\ \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 1,\ \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}.$$

Thus

$$\mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}, \text{ and } [T]_{\mathcal{C}} = \begin{bmatrix} -1 & -4 \\ 1 & 1 \end{bmatrix}^{-1} [T]_{\mathcal{E}} \begin{bmatrix} -1 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

32. With $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, we have

$$[T\mathbf{e}_1]_{\mathcal{E}} = \begin{bmatrix} 1-0 \\ 1+0 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad [T\mathbf{e}_2]_{\mathcal{E}} = \begin{bmatrix} 0-1 \\ 0+1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Using the method of Example 6.86(b), the eigenvectors of $[T]_{\mathcal{E}}$ will give us a diagonalizing basis $\mathcal{C}$. Using methods from Chapter 4, we find the eigenvalues and corresponding eigenvectors to be

$$\lambda_1 = 1+i,\ \mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \qquad \lambda_2 = 1-i,\ \mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Since the eigenvectors are not real, T is not diagonalizable.

33. With $\mathcal{E} = \{1, x\}$, we have

$$[T(1)]_{\mathcal{E}} = [4+x]_{\mathcal{E}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad [T(x)]_{\mathcal{E}} = [2+3x]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Therefore

$$[T]_{\mathcal{E}} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}.$$

Using the method of Example 6.86(b), the eigenvectors of $[T]_{\mathcal{E}}$ will give us a diagonalizing basis $\mathcal{C}$. Using methods from Chapter 4, we find the eigenvalues and corresponding eigenvectors to be

$$\lambda_1 = 5,\ \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 2,\ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Thus

$$\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}, \text{ and } [T]_{\mathcal{C}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} [T]_{\mathcal{E}} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}.$$

34. With $\mathcal{E} = \{1, x, x^2\}$, we have

$$[T(1)]_{\mathcal{E}} = [1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_{\mathcal{E}} = [x+1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$[T(x^2)]_{\mathcal{E}} = [(x+1)^2]_{\mathcal{E}} = [x^2+2x+1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the method of Example 6.86(b), the eigenvectors of $[T]_{\mathcal{E}}$ will give us a diagonalizing basis $\mathcal{C}$. Since this is an upper triangular matrix with 1s on the diagonal, the only eigenvalue is 1; using methods from Chapter 4 we find the eigenspace to be one-dimensional with basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Thus T is not diagonalizable.

35. With $\mathcal{E} = \{1, x\}$, we have

$$[T(1)]_{\mathcal{E}} = [1 + x \cdot 0]_{\mathcal{E}} = [1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad [T(x)]_{\mathcal{E}} = [x + x \cdot 1]_{\mathcal{E}} = [2x]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Therefore

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Thus T is already diagonal with respect to the standard basis $\mathcal{E}$.

36. With $\mathcal{E} = \{1, x, x^2\}$, we have

$$\begin{aligned} [T(1)]_{\mathcal{E}} &= [1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ [T(x)]_{\mathcal{E}} &= [3x+2]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \\ [T(x^2)]_{\mathcal{E}} &= [(3x+2)^2]_{\mathcal{E}} = [9x^2 + 12x + 4]_{\mathcal{E}} = \begin{bmatrix} 4 \\ 12 \\ 9 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 9 \\ 12 \\ 4 \end{bmatrix}. \end{aligned}$$

Therefore

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 9 \end{bmatrix}.$$

Using the method of Example 6.86(b), the eigenvectors of $[T]_{\mathcal{E}}$ will give us a diagonalizing basis $\mathcal{C}$. Since this is an upper triangular matrix with 1s on the diagonal, the eigenvalues are 1, 3, and 4; using methods from Chapter 4 we find the corresponding eigenvectors to be

$$\lambda_1 = 1,\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 3,\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \lambda_3 = 9,\ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Thus

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}, \text{ and } [T]_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} [T]_{\mathcal{E}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

37. Let ℓ have direction vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, and let T be the reflection in ℓ. Let $\mathbf{d}' = \begin{bmatrix} -d_2 \\ d_1 \end{bmatrix}$. As in Example 6.85, we may as well assume for simplicity that $\mathbf{d}$ is a unit vector; then $\mathbf{d}'$ is as well. Continuing to follow Example 6.85, $\mathcal{D} = \{\mathbf{d}, \mathbf{d}'\}$ is a basis for $\mathbb{R}^2$. Since $\mathbf{d}'$ is orthogonal to $\mathbf{d}$, it follows that $T\mathbf{d}' = -\mathbf{d}'$. Thus

$$[T(\mathbf{d})]_{\mathcal{D}} = [\mathbf{d}]_{\mathcal{D}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad [T(\mathbf{d}')]_{\mathcal{D}} = [-\mathbf{d}']_{\mathcal{D}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Thus

$$[T]_{\mathcal{D}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now, $[T]_{\mathcal{E}} = P_{\mathcal{E}\leftarrow\mathcal{D}}[T]_{\mathcal{D}}P_{\mathcal{D}\leftarrow\mathcal{E}}$, so we must find $P_{\mathcal{E}\leftarrow\mathcal{D}}$. The columns of this matrix are the expression of the basis vectors of $\mathcal{D}$ in the standard basis $\mathcal{E}$, so we have

$$P_{\mathcal{E}\leftarrow\mathcal{D}} = \begin{bmatrix} d_1 & d_2 \\ d_2 & -d_1 \end{bmatrix}.$$

Since we assumed $\mathbf{d}$ was a unit vector, this is an orthogonal matrix, so that

$$\begin{aligned} [T]_{\mathcal{E}} &= P_{\mathcal{E}\leftarrow\mathcal{D}}[T]_{\mathcal{D}}P_{\mathcal{D}\leftarrow\mathcal{E}} = P_{\mathcal{E}\leftarrow\mathcal{D}}[T]_{\mathcal{D}}P_{\mathcal{E}\leftarrow\mathcal{D}}^{-1} = P_{\mathcal{E}\leftarrow\mathcal{D}}[T]_{\mathcal{D}}P_{\mathcal{E}\leftarrow\mathcal{D}}^{T} \\ &= \begin{bmatrix} d_1 & d_2 \\ d_2 & -d_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} d_1 & d_2 \\ d_2 & -d_1 \end{bmatrix} = \begin{bmatrix} d_1^2 - d_2^2 & 2d_1d_2 \\ 2d_1d_2 & d_2^2 - d_1^2 \end{bmatrix}. \end{aligned}$$

Note that this is the same as the answer from Exercise 26 in Section 3.6, with the assumption that $\mathbf{d}$ is a unit vector.

38. Let T be the projection onto W. From Example 5.11, we have an orthogonal basis for $\mathbb{R}^2$,

$$\left\{\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right\},$$

where $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for W and $\{\mathbf{u}_3\}$ is a basis for $W^\perp$. Then

$$[T(\mathbf{u}_1)]_{\mathcal{D}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(\mathbf{u}_2)]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [T(\mathbf{u}_3)]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{so that} \qquad [T]_{\mathcal{D}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Further, since the columns of $P_{\mathcal{E}\leftarrow\mathcal{D}}$ are the expression of the basis vectors of $\mathcal{D}$ in the standard basis $\mathcal{E}$,

$$P_{\mathcal{E}\leftarrow\mathcal{D}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \quad \Rightarrow \quad P_{\mathcal{E}\leftarrow\mathcal{D}}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{bmatrix}.$$

Thus

$$\begin{aligned} [T]_{\mathcal{E}} &= P_{\mathcal{E}\leftarrow\mathcal{D}}[T]_{\mathcal{D}}P_{\mathcal{D}\leftarrow\mathcal{E}} = P_{\mathcal{E}\leftarrow\mathcal{D}}[T]_{\mathcal{D}}P_{\mathcal{E}\leftarrow\mathcal{D}}^{-1} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}. \end{aligned}$$

If $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, then

$$\operatorname{proj}_W \mathbf{v} = [T]_{\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix},$$

which is the same as was found in Example 11 in Section 5.2.

39. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, and suppose that A is such that $A[\mathbf{v}]_{\mathcal{B}} = [T\mathbf{v}]_{\mathcal{C}}$ for all $\mathbf{v} \in V$. Then $A[\mathbf{v}_i]_{\mathcal{B}} = [T\mathbf{v}_i]_{\mathcal{C}}$. But $[\mathbf{v}_i]_{\mathcal{B}} = \mathbf{e}_i$ since the $\mathbf{v}_i$ are the basis vectors of $\mathcal{B}$, so this equation says that $A\mathbf{e}_i = [T\mathbf{v}_i]_{\mathcal{C}}$. The left-hand side of this equation is the i^{th} column of A, while the right-hand side is the i^{th} column of $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$. Thus those i^{th} columns are equal; since this holds for all i, we get $A = [T]_{\mathcal{C}\leftarrow\mathcal{B}}$.

40. Suppose that $[\mathbf{v}]_{\mathcal{B}} \in \text{null}([T]_{\mathcal{C}\leftarrow\mathcal{B}})$. That means that

$$\mathbf{0} = [T]_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T\mathbf{v}]_{\mathcal{C}},$$

so that $[T\mathbf{v}]_{\mathcal{C}} = \mathbf{0}$ and thus $\mathbf{v} \in \ker(T)$. Similarly, if $\mathbf{v} \in \ker(T)$, then

$$\mathbf{0} = [T\mathbf{v}]_{\mathcal{C}} = [T]_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}},$$

so that $[\mathbf{v}]_{\mathcal{B}} \in \text{null}([T]_{\mathcal{C}\leftarrow\mathcal{B}})$. Then the map $V \to \mathbb{R}^n : \mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is an isomorphism mapping $\ker(T)$ to $\text{null}([T]_{\mathcal{C}\leftarrow\mathcal{B}})$, so the two subspaces are isomorphic and thus have the same dimension. So $\text{nullity}(T) = \text{nullity}([T]_{\mathcal{C}\leftarrow\mathcal{B}})$.

41. Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{b}_i]_{\mathcal{B}} = [T(\mathbf{b}_i)]_{\mathcal{C}}$$

by definition of the matrix of T with respect to these bases, and denote by $\mathbf{a}_i$ the columns of $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$. Then $\text{rank}([T]_{\mathcal{C}\leftarrow\mathcal{B}}) = \dim \text{col}[T]_{\mathcal{C}\leftarrow\mathcal{B}}$, which is the number of linearly independent $\mathbf{a}_i$, and $\text{rank}(T) = \dim \text{range}(T)$, which is the number of linearly independent $T(\mathbf{v}_i)$. Now, since

$$\mathbf{a}_i = [T]_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{b}_i]_{\mathcal{B}} = [T(\mathbf{b}_i)]_{\mathcal{C}}$$

we see that $\{\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, \ldots, \mathbf{a}_{i_m}\}$ are linearly independent if and only if $\{T(\mathbf{b}_{i_1}), \ldots, T(\mathbf{b})_{i_m}\}$ are linearly independent, so $\text{col}(A) = \text{span}(\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, \ldots, \mathbf{a}_{i_m})$ if and only if $\text{range}(T) = \text{span}(T(\mathbf{b}_{i_1}), \ldots, T(\mathbf{b})_{i_m})$ and therefore $\text{rank}(A) = \text{rank}(T)$.

42. T is diagonalizable if and only if there is some basis $\mathcal{C}$ and matrix P such that $[T]_{\mathcal{C}} = P^{-1}AP$ is diagonal. Thus if T is diagonalizable, so is A. For the reverse, if A is diagonalizable, then the basis $\mathcal{C}$ whose elements are the columns of P is a basis in which $[T]_{\mathcal{C}}$ is diagonal.

43. Let $\dim V = n$ and $\dim W = m$. Then $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ is an $m \times n$ matrix, and by Theorem 3.26 in Section 3.5, $\text{rank}([T]_{\mathcal{C}\leftarrow\mathcal{B}}) + \text{nullity}([T]_{\mathcal{C}\leftarrow\mathcal{B}}) = n$. But by Exercises 40 and 41, $\text{rank}([T]_{\mathcal{C}\leftarrow\mathcal{B}}) = \text{rank}(T)$ and $\text{nullity}([T]_{\mathcal{C}\leftarrow\mathcal{B}}) = \text{nullity}(T)$; since $\dim V = n$, we get

$$\dim V = \text{rank}(T) + \text{nullity}(T).$$

44. Claim that $[T]_{\mathcal{C}'\leftarrow\mathcal{B}'} = P_{\mathcal{C}'\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{B}'}$. To prove this, it is enough to show that for all $\mathbf{x} \in V$,

$$[\mathbf{x}]_{\mathcal{C}'} = P_{\mathcal{C}'\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{B}'}[\mathbf{x}]_{\mathcal{B}'}$$

since the matrix of T with respect to $\mathcal{B}'$ and $\mathcal{C}'$ is unique by Exercise 39. And indeed

$$P_{\mathcal{C}'\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{B}'}[\mathbf{x}]_{\mathcal{B}'} = P_{\mathcal{C}'\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{B}}\left(P_{\mathcal{B}\leftarrow\mathcal{B}'}[\mathbf{x}]_{\mathcal{B}'}\right) = P_{\mathcal{C}'\leftarrow\mathcal{C}}\left([T]_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}\right) = P_{\mathcal{C}'\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{C}'}.$$

45. For $T \in \mathcal{L}(V, W)$, define $\varphi(T) = [T]_{\mathcal{C}\leftarrow\mathcal{B}} \in M_{mn}$. Claim that φ is an isomorphism. We must first show it is linear:

$$\begin{aligned}
\varphi(S+T) &= [(S+T)]_{\mathcal{C}\leftarrow\mathcal{B}} \\
&= \left[\ [(S+T)\mathbf{b}_1]_{\mathcal{C}} \quad \cdots \quad [(S+T)\mathbf{b}_n]_{\mathcal{C}} \right] \\
&= \left[\ [S\mathbf{b}_1]_{\mathcal{C}} + [T\mathbf{b}_1]_{\mathcal{C}} \quad \cdots \quad [S\mathbf{b}_n]_{\mathcal{C}} + [T\mathbf{b}_n]_{\mathcal{C}} \right] \\
&= \left[\ [S\mathbf{b}_1]_{\mathcal{C}} \quad \cdots \quad [S\mathbf{b}_n]_{\mathcal{C}} \right] + \left[\ [T\mathbf{b}_1]_{\mathcal{C}} \quad \cdots \quad [T\mathbf{b}_n]_{\mathcal{C}} \right] \\
&= \varphi(S) + \varphi(T).
\end{aligned}$$

Similarly

$$\begin{aligned}\varphi(cS) &= [(cS)]_{\mathcal{C}\leftarrow\mathcal{B}} \\ &= \begin{bmatrix} [(cS)\mathbf{b}_1]_{\mathcal{C}} & \cdots & [(cS)\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \\ &= \begin{bmatrix} c[S\mathbf{b}_1]_{\mathcal{C}} & \cdots & c[S\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \\ &= c\begin{bmatrix} [S\mathbf{b}_1]_{\mathcal{C}} & \cdots & [S\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \\ &= c\varphi(S).\end{aligned}$$

Next we show that $\ker\varphi = \mathbf{0}$, so that φ is one-to-one. Suppose that $\varphi(T) = \mathbf{0}$. That means that $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = [0]_{mn}$, so that $\operatorname{nullity}([T]_{\mathcal{C}\leftarrow\mathcal{B}}) = n$. By Exercise 40, this is equal to $\operatorname{nullity}(T)$. Thus $\operatorname{nullity}(T) = n = \dim V$, so that T is the zero linear transformation, showing that φ is one-to-one.

Finally, since $\dim M_{mn} = mn$, we will have shown that φ is an isomorphism if we prove that $\dim\mathcal{L}(V,W) = mn$. But if $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ and $\{\mathbf{w}_1,\ldots,\mathbf{w}_m\}$ are bases for V and W, then a linear map from V to W sends each $\mathbf{v}_i$ to some linear combination of the $\mathbf{w}_j$, so it is a sum of maps that send $\mathbf{v}_i$ to $\mathbf{w}_k$ and the other $\mathbf{v}_i$ to zero. There are mn of these maps, which are clearly linearly independent, so that $\dim\mathcal{L}(V,W) = mn$.

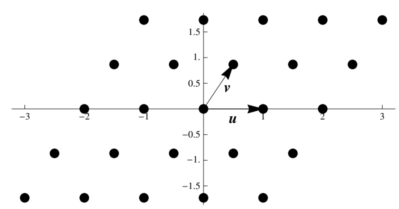
46. From Exercise 45 with $W = \mathbb{R}$, we have $\dim W = \dim\mathbb{R} = 1$, so that $M_{1n} \cong V$. Thus

$$V^* = \mathcal{L}(V,W) = \mathcal{L}(V,\mathbb{R}) \cong M_{1n} \cong V.$$

Exploration: Tiles, Lattices, and the Crystallographic Restriction

1. Let P be the pattern in Figure 6.15. We are given, from the figure, that $P+\mathbf{u} = P$ and that $P+\mathbf{v} = P$. Therefore if a is a positive integer, then $P + a\mathbf{u} = (P+\mathbf{u}) + (a-1)\mathbf{u} = P + (a-1)\mathbf{u}$, and by induction this is just P. Similarly if b is a positive integer then $P + b\mathbf{v} = (P+\mathbf{v}) + (b-1)\mathbf{v} = P + (b-1)\mathbf{v}$, which again by induction is equal to P. If a is a negative integer, then since $P + (-a)\mathbf{u} = P$, it follows that $P = P - (-a)\mathbf{u} = P + a\mathbf{u}$, and similarly for $b < 0$ and $\mathbf{v}$.

2. The lattice points correspond to centers of the trefoils in Figure 6.15.



3. Let θ be the smallest positive angle through which the lattice exhibits rotational symmetry. Let R_O^θ denote the operation of rotation through an angle $\theta > 0$ around a center of rotation O, and let P be a pattern or lattice. Now we have

$$R_O^{n\theta}(P) = \left(R_O^\theta\right)^n(P) = P,\ n\in\mathbb{Z},\ n > 0,$$

since rotation through $n\theta$ is the same as applying a rotation through θ n times. If $n < 0$, then rotating through $n\theta$ gives a figure which, if rotated back through $(-n)\theta$, is P, so it must be P as well. This shows that P is rotationally symmetric under any integer multiple of θ.

Next, let m be the smallest positive integer such that $m\theta \ge 360^\circ$, and suppose that $m\theta = 360^\circ + \varphi$. Note that $R_O^{360^\circ}(P) = P$, so that $R_O^{360^\circ+\varphi}(P) = R_O^\varphi(P)$. We want to show that $\frac{360^\circ}{\theta}$ is an integer, so it suffices to show that $\varphi = 0$. Now, since m is minimal, we know that

$$m\theta - \varphi = 360^\circ > (m-1)\theta,$$

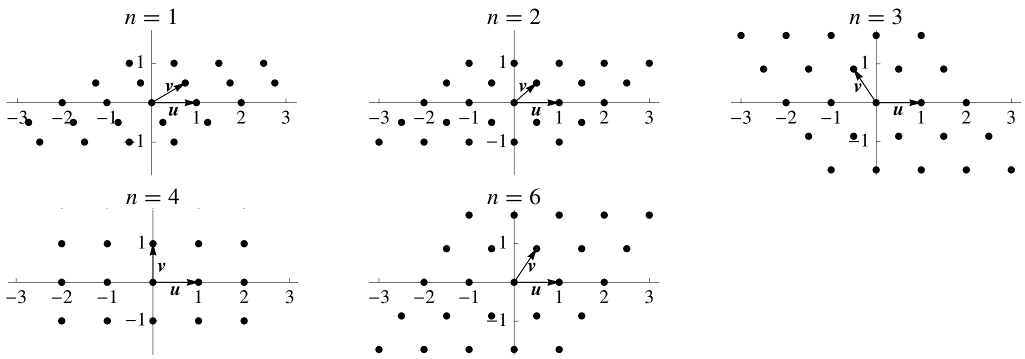
so that $m\theta - \varphi > m\theta - \theta$ and thus $-\varphi > -\theta$, so that $\varphi < \theta$. But

$$P = R_O^{m\theta}(P) = R_O^{360^\circ+\varphi}(P) = R_O^\varphi(P),$$

and we see that the lattice is rotationally symmetric under a rotation of $\psi < \theta'$. Since $\varphi \ge 0$ and θ is the smallest positive such angle, it follows that $\varphi = 0$. Thus $\theta = \frac{360^\circ}{m}$.

4. The smallest positive angle of rotational symmetry in the lattice of Exercise 2 is $60°$; this rotational symmetry is also present in the original figure, in Figure 6.14 or 6.15.

5. Some examples are below:



It is impossible to draw a lattice with 8-fold rotational symmetry, as the rest of this section will show.

6. **(a)** With $\theta = 60°$, we have

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix},$$

and

$$[R_\theta]_{\mathcal{E}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

(b) From the discussion in part (a),

$$R_\theta(\mathbf{u}) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \mathbf{v},$$
$$R_\theta(\mathbf{v}) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = -\mathbf{u} + \mathbf{v}.$$

With $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$, we have

$$[R_\theta(\mathbf{u})]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad [R_\theta(\mathbf{v})]_{\mathcal{B}} = [-\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad [R_\theta]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

7. Since the lattice is invariant under R_θ, $\mathbf{u}$ and $\mathbf{v}$ are mapped to lattice points under this rotation. But each lattice point is by definition $a\mathbf{u} + b\mathbf{v}$ where a and b are integers. Thus

$$[R_\theta(\mathbf{u})]_{\mathcal{B}} = [a\mathbf{u} + c\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad [R_\theta(\mathbf{v})]_{\mathcal{B}} = [b\mathbf{u} + d\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} b \\ d \end{bmatrix} \qquad \Rightarrow \qquad [R_\theta]_{\mathcal{B}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and the matrix entries are integers.

8. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$. Then by Theorem 6.29 in Section 6.6, $[R_\theta]_{\mathcal{E}} = P^{-1}[R_\theta]_{\mathcal{B}}P$. But this means that $[R_\theta]_{\mathcal{E}} \sim [R_\theta]_{\mathcal{B}}$ (that is, the two matrices are similar). By Exercise 35 in Section 4.4, this implies that

$$2\cos\theta = \operatorname{tr}[R_\theta]_{\mathcal{E}} = \operatorname{tr}[R_\theta]_{\mathcal{B}} = a + d,$$

which is an integer.

9. If $2\cos\theta = m$ is an integer, then m must be 0, ± 1, or ± 2, so the angles are angles whose cosines are 0, $\pm\frac{1}{2}$, or ± 1; these angles are

$$\theta = 60^\circ,\ 90^\circ, 120^\circ,\ 180^\circ,\ 240^\circ,\ 270^\circ,\ 300^\circ,\ 360^\circ,$$

corresponding to

$$n = 1,\ 2,\ 3,\ 4,\ \text{or } 6.$$

These are the only possible n-fold rotational symmetries.

10. Left to the reader.

6.7 Applications

1. Here $a = -3$, so by Theorem 6.32, $\{e^{3t}\}$ is a basis for solutions, which therefore have the form $y(t) = ce^{3t}$. Since $y(1) = ce^3 = 2$, we have $c = 2e^{-3}$, so that $y(t) = 2e^{-3}e^{3t} = 2e^{3t-3}$.

2. Here $a = 1$, so by Theorem 6.32, $\{e^{-t}\}$ is a basis for solutions, which therefore have the form $x(t) = ce^{-t}$. Since $x(1) = ce^{-1} = 1$, we have $c = e$, so that $x(t) = ee^{-t} = e^{1-t}$.

3. $y'' - 7y' + 12y = 0$ corresponds to the characteristic equation $\lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4) = 0$. Then $\lambda_1 = 3$ and $\lambda_2 = 4$, so Theorem 6.33 tells us that the general solution has the form $y(t) = c_1e^{3t} + c_2e^{4t}$. Then

$$\begin{array}{c} y(0) = 1 \\ y(1) = 1 \end{array} \quad\Rightarrow\quad \begin{array}{rcrcl} c_1 & + & c_2 & = & 1 \\ e^3c_1 & + & e^4c_2 & = & 1 \end{array} \quad\Rightarrow\quad \begin{array}{l} c_1 = \dfrac{1 - e^4}{e^3 - e^4} \\ c_2 = \dfrac{e^3 - 1}{e^3 - e^4} \end{array}$$

so that

$$y(t) = \frac{1}{e^3 - e^4}\left(\left(1 - e^4\right)e^{3t} + \left(e^3 - 1\right)e^{4t}\right).$$

4. $x'' + x' - 12x = 0$ corresponds to the characteristic equation $\lambda^2 + \lambda - 12 = (\lambda - 3)(\lambda + 4) = 0$. Then $\lambda_1 = 3$ and $\lambda_2 = -4$, so Theorem 6.33 tells us that the general solution has the form $x(t) = c_1e^{3t} + c_2e^{-4t}$. Then

$$\begin{array}{c} x(0) = 0 \\ x'(0) = 1 \end{array} \quad\Rightarrow\quad \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ 3c_1 & - & 4c_2 & = & 1 \end{array} \quad\Rightarrow\quad \begin{array}{l} c_1 = \dfrac{1}{7} \\ c_2 = -\dfrac{1}{7} \end{array}$$

so that

$$y(t) = \frac{1}{7}\left(e^{3t} - e^{-4t}\right).$$

5. $f'' - f' - f = 0$ corresponds to the characteristic equation $\lambda^2 - \lambda - 1 = 0$, so that $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Therefore by Theorem 6.33, the general solution is $f(t) = c_1e^{(1+\sqrt{5})t/2} + c_2e^{(1-\sqrt{5})t/2}$. So

$$\begin{array}{c} f(0) = 0 \\ f(1) = 1 \end{array} \quad\Rightarrow\quad \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ e^{(1+\sqrt{5})/2}c_1 & + & e^{(1-\sqrt{5})/2}c_2 & = & 1 \end{array} \quad\Rightarrow\quad \begin{array}{l} c_1 = \dfrac{e^{(\sqrt{5}-1)/2}}{e^{\sqrt{5}} - 1} \\ c_2 = -\dfrac{e^{(\sqrt{5}-1)/2}}{e^{\sqrt{5}} - 1} \end{array}$$

so that

$$f(t) = \frac{e^{(\sqrt{5}-1)/2}}{e^{\sqrt{5}} - 1}\left(e^{(1+\sqrt{5})t/2} - e^{(1-\sqrt{5})t/2}\right).$$

6. $g'' - 2g = 0$ corresponds to the characteristic equation $\lambda^2 - 2 = 0$, so that $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{2}$. Therefore by Theorem 6.33, the general solution is $g(t) = c_1 e^{t\sqrt{2}} + c_2 e^{-t\sqrt{2}}$. So

$$\begin{aligned} g(0) &= 1 \\ g(1) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 + \phantom{e^{-\sqrt{2}}} c_2 &= 1 \\ e^{\sqrt{2}} c_1 + e^{-\sqrt{2}} c_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= -\frac{1}{e^{2\sqrt{2}} - 1} \\ c_2 &= \frac{e^{2\sqrt{2}}}{e^{2\sqrt{2}} - 1} \end{aligned}$$

so that

$$g(t) = \frac{1}{e^{2\sqrt{2}} - 1} \left(-e^{t\sqrt{2}} + e^{2\sqrt{2} - t\sqrt{2}} \right).$$

7. $y'' - 2y' + y = 0$ corresponds to the characteristic equation $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$, so that $\lambda_1 = \lambda_2 = 1$. Therefore by Theorem 6.33, the general solution is $y(t) = c_1 e^t + c_2 t e^t$. So

$$\begin{aligned} y(0) &= 1 \\ y(1) &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ ec_1 + ec_2 &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= \frac{1 - e}{e} \end{aligned}$$

so that

$$y(t) = e^t + \frac{1 - e}{e} t e^t = e^t + (1 - e) t e^{t-1}.$$

8. $x'' + 4x' + 4x = 0$ corresponds to the characteristic equation $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$, so that $\lambda_1 = \lambda_2 = -2$. Therefore by Theorem 6.33, the general solution is $x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$. So

$$\begin{aligned} x(0) &= 1 \\ x'(0) &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ -2c_1 + c_2 &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= 3 \end{aligned}$$

so that

$$y(t) = e^{-2t} + 3te^{-2t}.$$

9. $y'' - k^2 y = 0$ corresponds to the characteristic equation $\lambda^2 - k^2 = 0$, so that $\lambda_1 = k$ and $\lambda_2 = -k$. Therefore by Theorem 6.33, the general solution is $y(t) = c_1 e^{kt} + c_2 e^{-kt}$. So

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 + c_2 &= 1 \\ kc_1 - kc_2 &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= \frac{k + 1}{2k} \\ c_2 &= \frac{k - 1}{2k} \end{aligned}$$

so that

$$y(t) = \frac{1}{2k} \left((k + 1) e^{kt} + (k - 1) e^{-kt} \right).$$

10. $y'' - 2ky' + k^2 y = 0$ corresponds to the characteristic equation $\lambda^2 - 2k\lambda + k^2 = (\lambda - k)^2 = 0$, so that $\lambda_1 = \lambda_2 = k$. Therefore by Theorem 6.33, the general solution is $y(t) = c_1 e^{kt} + c_2 t e^{kt}$. So

$$\begin{aligned} y(0) &= 1 \\ y(1) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_1 + c_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

so that

$$y(t) = e^{kt} - te^{-kt}.$$

11. $f'' - 2f' + 5f = 0$ corresponds to the characteristic equation $\lambda^2 - 2\lambda + 5 = 0$, so that $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Therefore by Theorem 6.33, the general solution is $f(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t$. Then

$$\begin{aligned} f(0) &= 1 \\ f\left(\frac{\pi}{4}\right) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 \phantom{+ e^{\pi/4} c_2} &= 1 \\ e^{\pi/4} c_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= 0 \end{aligned}$$

so that

$$y(t) = e^t \cos 2t.$$

12. $h'' - 4h' + 5h = 0$ corresponds to the characteristic equation $\lambda^2 - 4\lambda + 5 = 0$, so that $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. Therefore by Theorem 6.33, the general solution is $h(t) = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t$. Then $h'(t) = c_1(2e^{2t}\cos t - e^{2t}\sin t) + c_2(2e^{2t}\sin t + e^{2t}\cos t)$, and

$$\begin{aligned} h(0) &= 0 \\ h'(0) &= -1 \end{aligned} \quad \Rightarrow \quad \begin{array}{rcl} c_1 \phantom{{}+c_2} & = & 0 \\ 2c_1 + c_2 & = & -1 \end{array} \quad \Rightarrow \quad \begin{aligned} c_1 &= 0 \\ c_2 &= -1 \end{aligned}$$

so that

$$y(t) = -e^{2t}\sin t.$$

13. **(a)** We start with $p(t) = ce^{kt}$; then $p(0) = ce^{0k} = c = 100$, so that $c = 100$. Next, $p(3) = 100e^{3k} = 1600$, so that $e^{3k} = 16$ and $k = \frac{\ln 16}{3}$. Hence

$$p(t) = 100e^{(\ln 16/3)t} = 100\left(e^{\ln 16}\right)^{t/3} = 100 \cdot 16^{t/3}.$$

(b) The population doubles when $p(t) = 100e^{(\ln 16/3)t} = 200$. Simplifying gives

$$\frac{\ln 16}{3}t = \ln 2, \text{ so that } t = \frac{3\ln 2}{\ln 16} = \frac{3\ln 2}{4\ln 2} = \frac{3}{4}.$$

The population doubles in $\frac{3}{4}$ of an hour, or 45 minutes.

(c) We want to find t such that $p(t) = 100e^{(\ln 16/3)t} = 1000000$. Simplifying gives

$$\frac{\ln 16}{3}t = \ln 10000, \text{ so that } t = \frac{3\ln 10000}{\ln 16} = \frac{12\ln 10}{4\ln 2} = \frac{3\ln 10}{\ln 2} \approx 9.966 \text{ hours.}$$

14. **(a)** Start with $p(t) = ce^{kt}$; then $p(0) = ce^{0k} = c = 76$. Next, $p(1) = 76e^{1k} = 92$, so that $k = \ln\frac{92}{76}$, and therefore $p(t) = 76e^{t\ln(92/76)}$. For the year 2000, we get

$$p(10) = 76e^{10\ln(92/76)} \approx 513.5.$$

This is almost twice the actual population in 2000.

(b) Start with $p(t) = ce^{kt}$; then $p(0) = ce^{0k} = c = 203$. Next, $p(1) = 203e^{1k} = 227$, so that $k = \ln\frac{227}{203}$, and therefore $p(t) = 203e^{t\ln(227/203)}$. For the year 2000, we get

$$p(3) = 203e^{3\ln(227/203)} \approx 283.8.$$

This is a pretty good approximation.

(c) Since the first estimate is very high but the second estimate is close, we can conclude that population growth was lower than exponential in the early 20$^{\text{th}}$ century, but has been close to exponential since 1970. This could be attributed for example to three major wars during the period from 1900 to 1970.

15. **(a)** Let $m(t) = ae^{-ct}$. Then $m(0) = ae^{-c\cdot 0} = a = 50$, so that $m(t) = 50e^{-ct}$. Since the half-life is 1590 years, we know that 25 mg will remain after that time, so that

$$m(1590) = 50e^{-1590c} = 25.$$

Simplifying gives $-1590c = \ln\frac{1}{2} = -\ln 2$, so that $c = \frac{\ln 2}{1590}$. So finally, $m(t) = 50e^{-t\ln 2/1590}$. After 1000 years, the amount remaining is

$$m(1000) = 50e^{-1000\ln 2/1590} \approx 32.33 \text{ mg.}$$

(b) Only 10 mg will remain when $m(t) = 10$. Solving $50e^{-t\ln 2/1590} = 10$ gives

$$-t\frac{\ln 2}{1590} = \ln\frac{1}{5} = -\ln 5, \text{ so that } t = \frac{1590\ln 5}{\ln 2} \approx 3692 \text{ years.}$$

16. With $m(t) = ae^{-ct}$, a half-life of 5730 years means that $m(5730) = ae^{-5730c} = \frac{a}{2}$, so that $e^{-5730c} = \frac{1}{2}$ and thus $c = -\frac{\ln(1/2)}{5730} = \frac{\ln 2}{5730}$. To find out how old the posts are, we want to find t such that

$$m(t) = ae^{-t\ln 2/5730} = 0.45a, \text{ which gives } t = -\frac{5730\ln 0.45}{\ln 2} \approx 6600 \text{ years.}$$

17. Following Example 6.92, the general equation of the spring motion is $x(t) = c_1\cos\sqrt{K}\,t + c_2\sin\sqrt{K}\,t$. Then $x(0) = 10 = c_1\cos 0 + c_2\sin 0 = c_1$, so that $c_1 = 10$. Then

$$x(10) = 5 = 10\cos 10\sqrt{K} + c_2\sin 10\sqrt{K}, \text{ so that } c_2 = \frac{5 - 10\cos 10\sqrt{K}}{\sin 10\sqrt{K}}.$$

Therefore the equation of the spring, giving its length at time t, is

$$c_2 = 10\cos\sqrt{K}\,t + \frac{5 - 10\cos 10\sqrt{K}}{\sin 10\sqrt{K}}\sin\sqrt{K}\,t.$$

18. From the analysis in Example 6.92, since $K = \frac{k}{m}$ and $m = 50$, we have $K = \frac{k}{50}$, so that

$$x(t) = c_1\cos\sqrt{K}\,t + c_2\sin\sqrt{K}\,t = c_1\cos\sqrt{\frac{k}{50}}\,t + c_2\sin\sqrt{\frac{k}{50}}\,t.$$

Since the period of $\sin bt$ and $\cos bt$ are each $\frac{2\pi}{b}$, the fact that the period is 10 means that $\frac{2\pi}{\sqrt{k/50}} = 10$, so that $\sqrt{\frac{k}{50}} = \frac{\pi}{5}$ and thus $k = 2\pi^2$.

19. The differential equation for the pendulum motion is the same as the equation of spring motion, so we apply the analysis from Example 6.92. Recall that the period of $\sin bt$ and $\cos bt$ are each $\frac{2\pi}{b}$.

(a) Since $\theta'' + \frac{g}{L}\theta = 0$, we have $K = \frac{g}{L}$ and

$$\theta(t) = c_1\cos\sqrt{K}\,t + c_2\sin\sqrt{K}\,t = c_1\cos\sqrt{\frac{g}{L}}\,t + c_2\sin\sqrt{\frac{g}{L}}\,t.$$

Recall that the period of $\sin bt$ and $\cos bt$ are each $\frac{2\pi}{b}$. Then the period is $P = \frac{2\pi}{\sqrt{g\mathrm{L}}} = 2\pi\sqrt{\frac{L}{g}}$. In this case, since $L = 1$, we get $P = 2\pi\sqrt{\frac{1}{g}} = \frac{2\pi}{\sqrt{g}}$.

(b) Since θ_1 does not appear in the formula, the period does not depend on θ_1. It depends only on the gravitational constant g and the length of the pendulum L. (Note that it also does not depend on the weight of the pendulum).

20. We want to show that if y_1 and y_2 are two solutions, and c is a constant, then $y_1 + y_2$ and cy_1 are also solutions. But this follows directly from properties of the derivative:

$$(y_1 + y_2)'' + a(y_1 + y_2)' + b(y_1 + y_2) = (y_1'' + ay_1' + by_1) + (y_2'' + ay_2' + by_2) = 0 + 0 = 0$$
$$(cy_1)'' + a(cy_1)' + b(cy_1) = cy_1'' + acy_1' + bcy_1 = c(y_1'' + ay_1' + by_1) = c \cdot 0 = 0.$$

Thus the solution set is a subspace of $\mathscr{F}$.

21. Since we know that $\dim S = 2$, it suffices to show that $e^{\lambda t}$ and $te^{\lambda t}$ are solutions (are in S) and that they are linearly independent. Note that λ is a solution of $\lambda^2 + a\lambda + b = 0$. Then

$$\left(e^{\lambda t}\right)'' + a\left(e^{\lambda t}\right)' + be^{\lambda t} = \lambda^2 e^{\lambda t} + a\lambda e^{\lambda t} + be^{\lambda t} = e^{\lambda t}(\lambda^2 + a\lambda + b) = 0.$$

For $te^{\lambda t}$, note that

$$\left(te^{\lambda t}\right)' = e^{\lambda t} + \lambda te^{\lambda t},$$
$$\left(te^{\lambda t}\right)'' = \left(e^{\lambda t} + \lambda te^{\lambda t}\right)' = \lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 te^{\lambda t} = 2\lambda e^{\lambda t} + \lambda^2 te^{\lambda t}.$$

Then

$$\begin{aligned}\left(te^{\lambda t}\right)'' + a\left(te^{\lambda t}\right)' + bte^{\lambda t} &= 2\lambda e^{\lambda t} + \lambda^2 te^{\lambda t} + a\left(e^{\lambda t} + \lambda te^{\lambda t}\right) + bte^{\lambda t}\\ &= (2\lambda + a)e^{\lambda t} + te^{\lambda t}(\lambda^2 + a\lambda + b)\\ &= (2\lambda + a)e^{\lambda t}.\end{aligned}$$

But the solutions of $\lambda^2 + a\lambda + b$ are $\lambda = \frac{-a\pm\sqrt{a^2-4b}}{2}$, so if the two roots are equal, we must have $a^2 - 4b = 0$, so that $\lambda = -\frac{a}{2}$, so that $2\lambda = -a$ and then $2\lambda + a = 0$. So the remaining term vanishes, and $te^{\lambda t}$ is a solution.

To see that the two are linearly independent, suppose $c_1e^{\lambda t} + c_2te^{\lambda t}$ is zero for all values of t. Setting $t = 0$ gives $c_1 = 0$, so that $c_2te^{\lambda t}$ is zero. Then set $t = 1$ to get $c_2 = 0$. Thus the two are linearly independent, so they form a basis for the solution set S.

22. Suppose that $c_1e^{pt}\cos qt + c_2e^{pt}\sin qt$ is identically zero. Setting $t = 0$ gives $c_1 = 0$, so that $c_2e^{pt}\sin qt$ is identically zero, which clearly forces $c_2 = 0$, proving linear independence.

Chapter Review

1. **(a)** False. All we can say is that $\dim V \le n$, so that any spanning set for V contains *at most* n vectors. For example, let $V = \mathbb{R}$, $\mathbf{v}_1 = \mathbf{e}_1$ and $\mathbf{v}_2 = 2\mathbf{e}_1$. Then $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$, but V has spanning sets consisting of only one vector.

(b) True. See Exercise 17(a) in Section 6.2.

(c) True. For example,

$$I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \quad J = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \quad L = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \quad R = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}.$$

Then $R - I = E_{11}$, $L - I = E_{12}$, $J + I - L = E_{21}$, $J + I - R = E_{22}$. Therefore this set of four invertible matrices spans; since $\dim M_{22} = 4$, it is a basis.

(d) False. By Exercise 2 in Section 6.5, the kernel of the map $\text{tr} : M_{22} \to \mathbb{R}$ has dimension 3, so that matrices with trace 0 span a subspace of M_{22} of dimension 3.

(e) False. In general, $\|\mathbf{x} + \mathbf{y}\| \ne \|\mathbf{x}\| + \|\mathbf{y}\|$. For example, take $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{y} = \mathbf{e}_2$ in $\mathbb{R}^2$.

(f) True. If T is one-to-one and $\{\mathbf{u}_i\}$ is a basis for V, then $\{T\mathbf{u}_i\}$ is a linearly independent set by Theorem 6.22. It spans range(T), so it is a basis for range(T). Thus if T is onto, range$(T) = W$, so that $\{T\mathbf{u}_i\}$ is a basis for W and therefore $\dim V = \dim W$.

(g) False. This is only true if T is onto. For example, let $T : \mathbb{R} \to \mathbb{R}$ be $T(x) = 0$. Then $\ker T = \mathbb{R}$ but the codomain is not $\{\mathbf{0}\}$ (although the range is).

(h) True. If nullity$(T) = 4$, then $\text{rank}\, T = \dim M_{33} - \text{nullity}(T) = 9 - 4 = 5$. So range$(T)$ is a five-dimensional subspace of $\mathscr{P}_4$, so it is the entire space.

(i) True. Since $\dim \mathscr{P}_3 = 4$, it suffices to show that $\dim V = 4$. But polynomials in V are of the form $a + bx + cx^2 + dx^3 + ex^4$ where $a + b + c + d + e = 0$, so they are of the form $(-b - c - d - e) + bx + cx^2 + dx^3 + ex^4$, where b, c, d, and e are arbitrary. Thus a basis for V is $\{x, x^2, x^3, x^4\}$, so $\dim V = 4$.

(j) False. See Example 6.80 in Section 6.6.

2. This is the subspace $\{\mathbf{0}\}$, since the only vector $\begin{bmatrix}x\\y\end{bmatrix}$ satisfying $x^2 + 3y^2 = 0$ is $\begin{bmatrix}0\\0\end{bmatrix}$.

3. The conditions $a + b = a + c$ and $a + c = c + d$ imply that $b = c$ and $a = d$, so that

$$W = \left\{\begin{bmatrix}a & b\\b & a\end{bmatrix}\right\}.$$

(Note that matrices of this form satisfy all of the original equalities.) To show that this is a subspace, we must show that it is closed under sums and scalar multiplication. Thus suppose

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad \begin{bmatrix} a' & b' \\ b' & a' \end{bmatrix} \in W, \qquad c \text{ a scalar.}$$

Then

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} + \begin{bmatrix} a' & b' \\ b' & a' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ b+b' & a+a' \end{bmatrix} \in W$$
$$c\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} ca & cb \\ cb & ca \end{bmatrix} \in W.$$

4. To show that this is a subspace, we must show that it is closed under sums and scalar multiplication. Suppose $p(x), q(x) \in W$. Then

$$x^3(p+q)\left(\frac{1}{x}\right) = x^3\left(p\left(\frac{1}{x}\right) + q\left(\frac{1}{x}\right)\right) = x^3 p\left(\frac{1}{x}\right) + x^3 q\left(\frac{1}{x}\right) = p(x) + q(x) = (p+q)(x),$$

so that $p+q \in W$. Next, if c is a scalar, then

$$x^3(cp)\left(\frac{1}{x}\right) = cx^3 p\left(\frac{1}{x}\right) = cp(x) = (cp)(x),$$

so that $cp \in W$. Thus W is a subspace.

5. To show that this is a subspace, we must show that it is closed under sums and scalar multiplication. Suppose $f(x), g(x) \in W$. Then

$$(f+g)(x+\pi) = f(x+\pi) + g(x+\pi) = f(x) + g(x) = (f+g)(x),$$

so that $f+g \in W$. Next, if c is a scalar, then

$$(cf)(x+\pi) = cf(x+\pi) = cf(x) = (cf)(x),$$

so that $cf \in W$. Thus W is a subspace.

6. They are linearly dependent. By the double angle formula, $\cos 2x = 1 - 2\sin^2 x$, so that

$$1 - \cos 2x - 2\sin^2 x = 1 - \cos 2x - \frac{2}{3}\left(3\sin^2 x\right) = 0.$$

7. Suppose that $cA + dB = O$. Taking the transpose of both sides gives $cA^T + dB^T = O$; since A is symmetric and B is skew-symmetric, this is just $cA - dB = O$. Adding these two equations gives $2cA = O$. Since A is nonzero, we must have $c = 0$, so that $dB = O$. Then since B is nonzero, we must have $d = 0$. Thus A and B are linearly independent.

8. The condition $a + d = b + c$ means that $a = b + c - d$, so that

$$W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = b + c\right\} = \left\{\begin{bmatrix} b+c-d & b \\ c & d \end{bmatrix}\right\}.$$

Then define $\mathcal{B}$ by finding three matrices in W in each of which exactly one of b, c, and d is 1 while the other are zero:

$$\mathcal{B} = \left\{B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\ D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

Consider the expression

$$bB + cC + dD = \begin{bmatrix} b+c-d & b \\ c & d \end{bmatrix}.$$

If this is O, then clearly $b = c = d = 0$, so that B, C, and D are linearly independent. But also, $bB + cC + dD$ is a general matrix in W, so that B, C, and D span W. Thus $\{B, C, D\}$ is a basis for W.

9. Suppose $p(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5 \in \mathscr{P}_5$ with $p(-x) = p(x)$. Then

$$\begin{aligned}
p(1) = p(-1) &\Rightarrow a + b + c + d + e + f = a - b + c - d + e - f \Rightarrow b + d + f = 0 \\
p(2) = p(-2) &\Rightarrow a + 2b + 4c + 8d + 16e + 32f = a - 2b + 4c - 8d + 16e - 32f \Rightarrow b + 4d + 16f = 0 \\
p(3) = p(-3) &\Rightarrow a + 3b + 9c + 27d + 81e + 243f = a - 3b + 9c - 27d + 81e - 243f \\
&\qquad\qquad \Rightarrow b + 9d + 81f = 0.
\end{aligned}$$

These three equations in the unknowns b, d, f have only the trivial solution, so that $b = d = f = 0$, and $p(x) = a + cx^2 + dx^4$. Any polynomial of this form clearly has $p(x) = p(-x)$, since $(-x)^2 = x^2$ and $(-x)^4 = x^4$. So $\{1, x^2, x^4\}$ is a basis for W.

10. Let $\mathcal{B} = \{1, 1 + x, 1 + x + x^2\} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{1 + x, x + x^2, 1 + x^2\} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$. Then

$$[\mathbf{c}_1]_\mathcal{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad [\mathbf{c}_2]_\mathcal{B} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad [\mathbf{c}_3]_\mathcal{B} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

so that by Theorem 6.13

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{C}}^{-1} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

11. To show that T is linear, we must show that $T(\mathbf{x} + \mathbf{z}) = T\mathbf{x} + T\mathbf{z}$ and that $T(c\mathbf{x}) = cT\mathbf{x}$, where c is a scalar.

$$\begin{aligned}
T(\mathbf{x} + \mathbf{z}) &= \mathbf{y}(\mathbf{x} + \mathbf{z})^T\mathbf{y} = \mathbf{y}\left(\mathbf{x}^T + \mathbf{z}^T\right)\mathbf{y} = \mathbf{y}\mathbf{x}^T\mathbf{y} + \mathbf{y}\mathbf{z}^T\mathbf{y} = T\mathbf{x} + T\mathbf{z} \\
T(c\mathbf{x}) &= \mathbf{y}(c\mathbf{x})^T\mathbf{y} = c\mathbf{y}\mathbf{x}^T\mathbf{y} = cT\mathbf{x}.
\end{aligned}$$

Thus T is linear.

12. T is not a linear transformation. Let $A \in M_{nn}$ and c be a scalar other than 0 or 1. Then

$$T(cA) = (cA)^T(cA) = c^2A^TA = c^2T(A).$$

Since $c \neq 0, 1$, it follows that $c^2T(A) \neq cT(A)$, so that T is not linear.

13. To show that T is linear, we must show that $(T(p + q))(x) = Tp(x) + Tq(x)$ and that $(T(cp))(x) = cTp(x)$, where c is a scalar.

$$\begin{aligned}
(T(p + q))(x) &= (p + q)(2x - 1) = p(2x - 1) + q(2x - 1) = Tp(x) + Tq(x) \\
(T(cp))(x) &= (cp)(2x - 1) = cp(2x - 1) = Tp(x).
\end{aligned}$$

Thus T is linear.

14. We first write $5 - 3x + 2x^2$ in terms of 1, $1 + x$, and $1 + x + x^2$:

$$5 - 3x + 2x^2 = a(1) + b(1 + x) + c(1 + x + x^2) = (a + b + c) + (b + c)x + cx^2.$$

Then $c = 2$, so that (from the x term) $b = -5$, and then $a = 8$. Thus

$$\begin{aligned}
T(5 - 3x + 2x^2) &= 8T(1) - 5T(1 + x) + 2T(1 + x + x^2) \\
&= 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 2 & 3 \end{bmatrix}.
\end{aligned}$$

15. Since $T : M_{nn} \to \mathbb{R}$ is onto, we know from the rank theorem that

$$n^2 = \dim M_{nn} = \operatorname{rank}(T) + \operatorname{nullity}(T) = 1 + \operatorname{nullity}(T),$$

so that $\operatorname{nullity}(T) = n^2 - 1$.

16. **(a)** We want a map $T : M_{22} \to M_{22}$ such that $T(A) = O$ if and only if A is upper triangular. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

an obvious map to try is

$$T(A) = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}.$$

Clearly the matrices sent to O by T are exactly the upper triangular matrices (together with O), which is the subspace W. To see that T is linear, note that we can rewrite T as $T(A) = a_{21}E_{21}$. Then

$$\begin{aligned} T(A+B) &= (A+B)_{21}E_{21} = (a_{21}+b_{21})E_{21} = a_{21}E_{21} + b_{21}E_{21} = T(A) + T(B) \\ T(cA) &= (cA)_{21}E_{21} = ca_{21}E_{21} = cT(A). \end{aligned}$$

(b) For this map, we want the image of any matrix A to be upper triangular. With A as above, an obvious map to try is the one that sets the entry below the main diagonal to zero:

$$T(A) = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$

It is obvious that $\operatorname{range} T = W$, since any upper triangular matrix is the image of itself under T, and every matrix of the form $T(A)$ is upper triangular by construction. To see that T is linear, write $T(A) = A - a_{21}E_{21}$; then

$$\begin{aligned} T(A+B) &= (A+B) - (A+B)_{21}E_{21} = (A+B) - (a_{21}+b_{21})E_{21} \\ &= (A - a_{21}E_{21}) + (B - b_{21}E_{21}) = T(A) + T(B) \\ T(cA) &= (cA) - (cA)_{21}E_{21} = cA - ca_{21}E_{21} = c(A - a_{21}E_{21}) = cT(A). \end{aligned}$$

17. We have

$$\begin{aligned} T(1) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \cdot E_{11} + 0 \cdot E_{12} + 0 \cdot E_{21} + 1 \cdot E_{22} \quad \Rightarrow \quad [T(1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ T(x) &= T((x+1)-1) = T(x+1) - T(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= 0 \cdot E_{11} + 1 \cdot E_{12} + 0 \cdot E_{21} + 0 \cdot E_{22} \quad \Rightarrow \quad [T(x)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ T(x^2) &= T((1+x+x^2) - (1+x)) = T(1+x+x^2) - T(1+x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix} \\ &= -1 \cdot E_{11} - 2 \cdot E_{12} + 1 \cdot E_{21} - 1 \cdot E_{22} \quad \Rightarrow \quad [T(1)]_{\mathcal{C}} = \begin{bmatrix} -1 \\ -2 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Since these vectors are the columns of $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$, we have

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

18. We must show that S spans V and that S is linearly independent. First, since every vector can be written as a linear combination of elements of S in one way, S spans V. Next, suppose

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Since we know that we can write $\mathbf{0}$ as the linear combination of the elements of S with zero coefficients, and since the representation of $\mathbf{0}$ is unique by hypothesis, it follows that $c_i = 0$ for all i, so that S is linearly independent. Thus S is a basis for V.

19. If $\mathbf{u} \in U$, then $T\mathbf{u} \in \ker(S)$, so that $(S \circ T)\mathbf{u} = S(T\mathbf{u}) = \mathbf{0}$. Thus $S \circ T$ is the zero map.

20. Since $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is a basis for V, it follows that $\operatorname{range}(T) = V$ since any element of V can be expressed as a linear combination of elements in $\operatorname{range}(T)$. By Theorem 6.30 (p) and (t), this means that T is invertible.

Chapter 7

Distance and Approximation

7.1 Inner Product Spaces

1. With $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$, we get

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = 2 \cdot 1 \cdot 4 + 3 \cdot (-2) \cdot 3 = -10$.

(b) $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{2 \cdot 1 \cdot 1 + 3 \cdot (-2) \cdot (-2)} = \sqrt{14}$.

(c) $\mathrm{d}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left\| \begin{bmatrix} -3 \\ -5 \end{bmatrix} \right\| = \sqrt{2 \cdot (-3) \cdot (-3) + 3 \cdot (-5) \cdot (-5)} = \sqrt{93}$.

2. With $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$, we get

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v} = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = -4$.

(b) $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}} = \sqrt{10}$.

(c) $\mathrm{d}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left\| \begin{bmatrix} -3 \\ -5 \end{bmatrix} \right\| = \sqrt{\begin{bmatrix} -3 & -5 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ -5 \end{bmatrix}} = \sqrt{189}$.

3. We want a vector $\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that

$$\langle \mathbf{u}, \mathbf{w} \rangle = 2 \cdot 1 \cdot a + 3 \cdot (-2) \cdot b = 2a - 6b = 0.$$

Any vector of the form $\begin{bmatrix} 3t \\ t \end{bmatrix}$ is such a vector; for example, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

4. We want a vector $\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that

$$\langle \mathbf{u}, \mathbf{w} \rangle = \mathbf{u}^T a \mathbf{w} = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 2a - 4b = 0.$$

There are many such vectors, for example $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

5.

(a) $\langle p(x), q(x) \rangle = \left\langle 3 - 2x, 1 + x + x^2 \right\rangle = 3 \cdot 1 + (-2) \cdot 1 + 0 \cdot 1 = 1$.

(b) $\|p(x)\| = \sqrt{\langle p(x), p(x) \rangle} = \sqrt{3 \cdot 3 + (-2) \cdot (-2) + 0 \cdot 0} = \sqrt{13}$.

(c) $\mathrm{d}(p(x), q(x)) = \|p(x) - q(x)\| = \left\| 2 - 3x - x^2 \right\| = \sqrt{2^2 + (-3)^2 + (-1)^2} = \sqrt{14}$.

6.

(a) $$\begin{aligned}\langle p(x), q(x)\rangle &= \int_0^1 (3-2x)(1+x+x^2)\,dx = \int_0^1 (-2x^3+x^2+x+3)\,dx \\ &= \left(-\frac{1}{2}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+3x\right)\Big|_0^1 = \frac{10}{3}.\end{aligned}$$

(b) $$\|p(x)\| = \sqrt{\langle p(x), p(x)\rangle} = \sqrt{\int_0^1 (3-2x)^2\,dx} = \sqrt{\left(9x-6x^2+\frac{4}{3}x^3\right)\Big|_0^1} = \sqrt{\frac{13}{3}}.$$

(c) $$\begin{aligned}\mathrm{d}(p(x), q(x)) &= \|p(x)-q(x)\| = \left\|2-3x-x^2\right\| = \sqrt{\int_0^1 (2-3x-x^2)^2\,dx} \\ &= \sqrt{\left(\frac{1}{5}x^5+\frac{3}{2}x^4+\frac{5}{3}x^3-6x^2+4x\right)\Big|_0^1} = \sqrt{\frac{41}{30}}.\end{aligned}$$

7. We want a polynomial $r(x) = a+bx+cx^2$ such that

$$\langle p(x), r(x)\rangle = 3\cdot a - 2\cdot b + 0\cdot c = 3a-2b = 0.$$

There are many such polynomials; two examples are $2-3x$ and $4-6x+7x^2$.

8. We want a polynomial $r(x) = a+bx+cx^2$ such that

$$\begin{aligned}\langle p(x), r(x)\rangle &= \int_0^1 (3-2x)(a+bx+cx^2)\,dx \\ &= \int_0^1 (3a+(3b-2a)x+(3c-2b)x^2-2cx^3)\,dx \\ &= \left(3ax+\frac{3b-2a}{2}x^2+\frac{3c-2b}{3}x^2-\frac{c}{2}x^4\right)\Big|_0^1 \\ &= 2a+\frac{5}{6}b+\frac{1}{2}c = 0.\end{aligned}$$

There are many such polynomials. Three examples are $-1+4x^2$, $5-12x$, and $12x-20x^2$.

9. Recall the double-angle formulas $\sin^2 x = \frac{1}{2}-\frac{1}{2}\cos 2x$ and $\cos^2 x = \frac{1}{2}+\frac{1}{2}\cos 2x$. Then

(a)

$$\begin{aligned}\langle f, g\rangle &= \int_0^{2\pi} f(x)g(x)\,dx = \int_0^{2\pi} \left(\sin^2 x + \sin x\cos x\right)\,dx \\ &= \int_0^{2\pi} \left(\frac{1}{2}-\frac{1}{2}\cos 2x + \sin x\cos x\right)\,dx = \left[\frac{1}{2}x-\frac{1}{4}\sin 2x+\frac{1}{2}\sin^2 x\right]_0^{2\pi} = \pi.\end{aligned}$$

(b) $\|f\| = \int_0^{2\pi} f(x)^2\,dx = \int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} \left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)\,dx = \left[\frac{1}{2}x-\frac{1}{4}\sin 2x\right]_0^{2\pi} = \pi.$

(c)

$$\begin{aligned}\mathrm{d}(f, g) = \|f-g\| &= \int_0^{2\pi} (-\cos x)^2\,dx = \int_0^{2\pi} \cos^2 x\,dx = \int_0^{2\pi} \left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)\,dx \\ &= \left[\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{2\pi} = \pi.\end{aligned}$$

10. We want to find a function g such that $\int_0^{2\pi} \sin x \cdot g(x)\,dx = 0$. Note from the previous exercise that $\left[\frac{1}{2}\sin^2 x\right]_0^{2\pi} = 0$, so choose $g(x) = \cos x$.

11. We must show it satisfies all four properties of an inner product. First,

$$\langle p(x), q(x)\rangle = p(a)q(a) + p(b)q(b) + p(c)q(c) = q(a)p(a) + q(b)p(b) + q(c)p(c) = \langle q(x), p(x)\rangle .$$

Next,

$$\begin{aligned}\langle p(x), q(x) + r(x)\rangle &= p(a)(q(a) + r(a)) + p(b)(q(b) + r(b)) + p(c)(q(c) + r(c))\\ &= p(a)q(a) + p(a)r(a) + p(b)q(b) + p(b)r(b) + p(c)q(c) + p(c)r(c)\\ &= p(a)q(a) + p(b)q(b) + p(c)q(c) + p(a)r(a) + p(b)r(b) + p(c)r(c)\\ &= \langle p(x), q(x)\rangle + \langle p(x), r(x)\rangle .\end{aligned}$$

Third,

$$\langle dp(x), q(x)\rangle = dp(a)q(a) + dp(b)q(b) + dp(c)q(c) = d(p(a)q(a) + p(b)q(b) + p(c)q(c)) = d\,\langle p(x), q(x)\rangle .$$

Finally,

$$\langle p(x), p(x)\rangle = p(a)p(a) + p(b)q(b) + p(c)q(c) = p(a)^2 + p(b)^2 + p(c)^2 \geq 0,$$

and if equality holds, then $p(a) = p(b) = p(c) = 0$ since squares are always nonnegative. But $p \in \mathscr{P}_2$, so if it is nonzero, it has at most two roots. Since a, b, and c are distinct, it follows that $p(x) = 0$.

12. We must show it satisfies all four properties of an inner product. First,

$$\langle p(x), q(x)\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2) = q(0)p(0) + q(1)p(1) + q(2)p(2) = \langle q(x), p(x)\rangle .$$

Next,

$$\begin{aligned}\langle p(x), q(x) + r(x)\rangle &= p(0)(q(0) + r(0)) + p(1)(q(1) + r(1)) + p(2)(q(2) + r(2))\\ &= p(0)q(0) + p(0)r(0) + p(1)q(1) + p(1)r(1) + p(2)q(2) + p(2)r(2)\\ &= p(0)q(0) + p(1)q(1) + p(2)q(2) + p(0)r(0) + p(1)r(1) + p(2)r(2)\\ &= \langle p(x), q(x)\rangle + \langle p(x), r(x)\rangle .\end{aligned}$$

Third,

$$\langle cp(x), q(x)\rangle = cp(0)q(0) + cp(1)q(1) + cp(2)q(2) = c(p(0)q(0) + p(1)q(1) + p(2)q(2)) = c\,\langle p(x), q(x)\rangle .$$

Finally,

$$\langle p(x), p(x)\rangle = p(0)p(0) + p(1)q(1) + p(2)q(2) = p(0)^2 + p(1)^2 + p(2)^2 \geq 0,$$

and if equality holds, then 0, 1, and 2 are all roots of p since squares are always nonnegative. But $p \in \mathscr{P}_2$, so if it is nonzero, it has at most two roots. It follows that $p(x) = 0$.

13. The fourth axiom does not hold. For example, let $\mathbf{u} = \begin{bmatrix}0\\1\end{bmatrix}$. Then $\langle \mathbf{u}, \mathbf{u}\rangle = 0$ but $\mathbf{u} \neq \mathbf{0}$.

14. The fourth axiom does not hold. For example, let $\mathbf{u} = \begin{bmatrix}1\\1\end{bmatrix}$. Then $\langle \mathbf{u}, \mathbf{u}\rangle = 1\cdot 1 - 1\cdot 1 = 0$, but $\mathbf{u} \neq \mathbf{0}$.

Alternatively, if $\mathbf{u} = \begin{bmatrix}1\\2\end{bmatrix}$, then $\mathbf{u}\cdot\mathbf{u} = -3 < 0$, so the fourth axiom again fails.

15. The fourth axiom does not hold. For example, let $\mathbf{u} = \begin{bmatrix}1\\0\end{bmatrix}$. Then $\langle \mathbf{u}, \mathbf{u}\rangle = 0\cdot 1 + 1\cdot 0 = 0$, but $\mathbf{u} \neq \mathbf{0}$.

Alternatively, if $\mathbf{u} = \begin{bmatrix}1\\-2\end{bmatrix}$, then $\mathbf{u}\cdot\mathbf{u} = -4 < 0$, so the fourth axiom again fails.

16. The fourth axiom does not hold. For example, if p is any nonzero polynomial with zero constant term, then $\langle p(x), p(x)\rangle = 0$ but $p(x)$ is not the zero polynomial.

17. The fourth axiom does not hold. For example, let $p(x) = 1 - x$; then $p(1) = 0$ so that $\langle p(x), p(x)\rangle = 0$, but $p(x)$ is not the zero polynomial.

18. The fourth axiom does not hold. For example $E_{12}E_{22} = O$, so that $\langle E_{12}, E_{22}\rangle = \det O = 0$, but $E_{11} \neq O$. Alternatively, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; then $A \neq O$ but $\langle A, A\rangle = 0$.

19. Examining Example 7.3, the matrix should be $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$; and indeed

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4u_1 + u_2 & u_1 + 4u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4u_1v_1 + u_2v_1 + u_1v_2 + 4u_2v_2,$$

as desired.

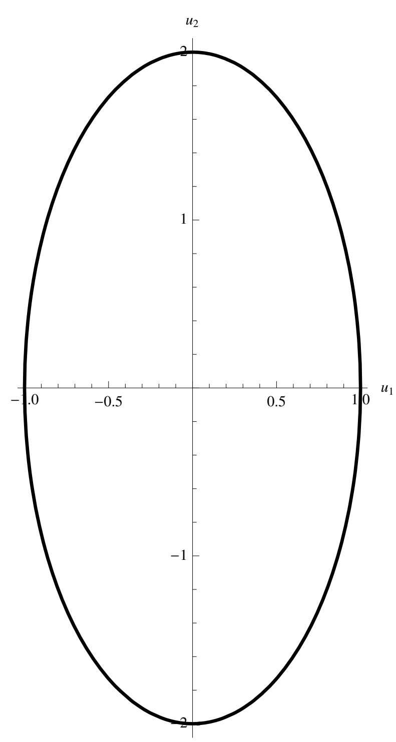
20. Examining Example 7.3, the matrix should be $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$; and indeed

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + 2u_2 & 2u_1 + 5u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1v_1 + 2u_2v_1 + 2u_1v_2 + 5u_2v_2,$$

as desired.

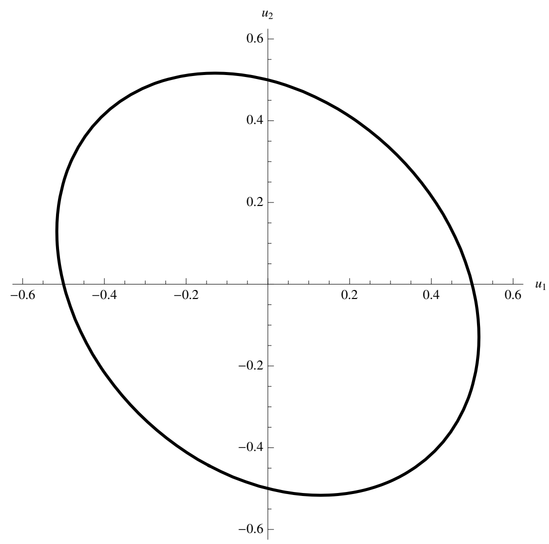
21. The unit circle is

$$\left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \ : \ u_1^2 + \frac{1}{4}u_2^2 = 1 \right\}.$$



22. The unit circle is

$$\left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \ : \ 4u_1^2 + 2u_1u_2 + 4u_2^2 = 1 \right\}.$$



23. We have, using properties of the inner product,

$$\begin{aligned} \langle \mathbf{u}, c\mathbf{v} \rangle &= \langle c\mathbf{v}, \mathbf{u} \rangle && \text{(Property 1)} \\ &= c\,\langle \mathbf{v}, \mathbf{u} \rangle && \text{(Property 3)} \\ &= c\,\langle \mathbf{u}, \mathbf{v} \rangle && \text{(Property 1).} \end{aligned}$$

24. Let $\mathbf{w}$ be any vector. Then $\langle \mathbf{u}, \mathbf{0} \rangle = \langle \mathbf{u}, 0\mathbf{w} \rangle$ which, by Theorem 7.1(b), is equal to $0\,\langle \mathbf{u}, \mathbf{w} \rangle = 0$. Similarly, $\langle \mathbf{0}, \mathbf{v} \rangle = \langle 0\mathbf{w}, \mathbf{v} \rangle$ which, by property 3 of inner products, is equal to $0\,\langle \mathbf{w}, \mathbf{v} \rangle = 0$.

25. Using properties of the inner product and Theorem 7.1 as needed,

$$\begin{aligned} \langle \mathbf{u} + \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle &= \langle \mathbf{u}, \mathbf{v} - \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, -\mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, -\mathbf{w} \rangle \\ &= \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{w}, \mathbf{w} \rangle \\ &= 1 - 5 + 0 - \|\mathbf{w}\|^2 \\ &= -4 - 4 = -8. \end{aligned}$$

26. Using properties of the inner product and Theorem 7.1 as needed,

$$\begin{aligned} \langle 2\mathbf{v} - \mathbf{w}, 3\mathbf{u} + 2\mathbf{w} \rangle &= \langle 2\mathbf{v}, 3\mathbf{u} + 2\mathbf{w} \rangle + \langle -\mathbf{w}, 3\mathbf{u} + 2\mathbf{w} \rangle \\ &= \langle 2\mathbf{v}, 3\mathbf{u} \rangle + \langle 2\mathbf{v}, 2\mathbf{w} \rangle + \langle -\mathbf{w}, 3\mathbf{u} \rangle + \langle -\mathbf{w}, 2\mathbf{w} \rangle \\ &= 6\,\langle \mathbf{v}, \mathbf{u} \rangle + 4\,\langle \mathbf{v}, \mathbf{w} \rangle - 3\,\langle \mathbf{w}, \mathbf{u} \rangle - 2\,\langle \mathbf{w}, \mathbf{w} \rangle \\ &= 6\,\langle \mathbf{u}, \mathbf{v} \rangle + 4\,\langle \mathbf{v}, \mathbf{w} \rangle - 3\,\langle \mathbf{u}, \mathbf{w} \rangle - 2\,\|\mathbf{w}\|^2 \\ &= 6 \cdot 1 + 4 \cdot 0 - 3 \cdot 5 - 2 \cdot 2^2 = -17. \end{aligned}$$

27. Using properties of the inner product and Theorem 7.1 as needed,

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\| &= \sqrt{\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle} \\ &= \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle} \\ &= \sqrt{\|\mathbf{u}\|^2 + 2\,\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2} \\ &= \sqrt{1 + 2 \cdot 1 + 3} = \sqrt{6}. \end{aligned}$$

28. Using properties of the inner product and Theorem 7.1 as needed,

$$\begin{aligned}
\|2\mathbf{u}-3\mathbf{v}+\mathbf{w}\| &= \sqrt{\langle 2\mathbf{u}-3\mathbf{v}+\mathbf{w}, 2\mathbf{u}-3\mathbf{v}+\mathbf{w}\rangle} \\
&= \sqrt{4\langle \mathbf{u},\mathbf{u}\rangle - 12\langle \mathbf{u},\mathbf{v}\rangle - 6\langle \mathbf{v},\mathbf{w}\rangle + 4\langle \mathbf{u},\mathbf{w}\rangle + 9\langle \mathbf{v},\mathbf{v}\rangle + \langle \mathbf{w},\mathbf{w}\rangle} \\
&= \sqrt{4\|\mathbf{u}\|^2 - 12\langle \mathbf{u},\mathbf{v}\rangle - 6\langle \mathbf{v},\mathbf{w}\rangle + 4\langle \mathbf{u},\mathbf{w}\rangle + 9\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2} \\
&= \sqrt{4\cdot 1^2 - 12\cdot 1 - 6\cdot 0 + 4\cdot 5 + 9\cdot\left(\sqrt{3}\right)^2 + 2^2} \\
&= \sqrt{43}.
\end{aligned}$$

29. $\mathbf{u}+\mathbf{v}=\mathbf{w}$ means that $\mathbf{u}+\mathbf{v}-\mathbf{w}=\mathbf{0}$; this is true if and only if $\langle \mathbf{u}+\mathbf{v}-\mathbf{w}, \mathbf{u}+\mathbf{v}-\mathbf{w}\rangle = 0$. Computing this inner product gives

$$\begin{aligned}
\langle \mathbf{u}+\mathbf{v}-\mathbf{w}, \mathbf{u}+\mathbf{v}-\mathbf{w}\rangle &= \langle \mathbf{u},\mathbf{u}\rangle + 2\langle \mathbf{u},\mathbf{v}\rangle - 2\langle \mathbf{u},\mathbf{w}\rangle - 2\langle \mathbf{v},\mathbf{w}\rangle + \langle \mathbf{v},\mathbf{v}\rangle + \langle -\mathbf{w},-\mathbf{w}\rangle \\
&= \|\mathbf{u}\|^2 + 2\cdot 1 - 2\cdot 5 - 2\cdot 0 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \\
&= 1 + 2 - 10 + 3 + 2 = 0.
\end{aligned}$$

30. Since $\|\mathbf{u}\| = 1$, we also have $1 = \|\mathbf{u}\|^2 = \langle \mathbf{u},\mathbf{u}\rangle$. Similarly, $1 = \langle \mathbf{v},\mathbf{v}\rangle$. But then

$$0 \le \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = \langle \mathbf{u},\mathbf{u}\rangle + 2\langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{v}\rangle = 2 + 2\langle \mathbf{u},\mathbf{v}\rangle.$$

Solving this inequality gives $2\langle \mathbf{u},\mathbf{v}\rangle \ge -2$, so that $\langle \mathbf{u},\mathbf{v}\rangle \ge -1$.

31. We have

$$\langle \mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle = \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{u},\mathbf{v}\rangle - \langle \mathbf{v},\mathbf{u}\rangle - \langle \mathbf{v},\mathbf{v}\rangle.$$

Since inner products are commutative, the middle two terms cancel, leaving us with

$$\langle \mathbf{u},\mathbf{u}\rangle - \langle \mathbf{v},\mathbf{v}\rangle = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2.$$

32. We have

$$\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{u}\rangle + \langle \mathbf{v},\mathbf{v}\rangle.$$

Since inner products are commutative, the middle two terms are the same, and we get

$$\langle \mathbf{u},\mathbf{u}\rangle + 2\langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{v}\rangle = \|\mathbf{u}\|^2 + 2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{v}\|^2.$$

33. From Exercise 32, we have

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{v}\|^2. \tag{1}$$

Substituting $-\mathbf{v}$ for $\mathbf{v}$ gives another identity,

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\langle \mathbf{u},-\mathbf{v}\rangle + \|-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{v}\|^2. \tag{2}$$

Adding these two gives

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2,$$

and dividing through by 2 gives the required identity.

34. Solve (1) and (2) in the previous exercise for $\langle \mathbf{u},\mathbf{v}\rangle$, giving

$$\begin{aligned}
\langle \mathbf{u},\mathbf{v}\rangle &= \frac{1}{2}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2\right) \\
\langle \mathbf{u},\mathbf{v}\rangle &= \frac{1}{2}\left(-\|\mathbf{u}-\mathbf{v}\|^2 + \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right).
\end{aligned}$$

Add these two equations, giving

$$2\langle \mathbf{u}, \mathbf{v}\rangle = \frac{1}{2}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2\right);$$

dividing through by 2 gives

$$\langle \mathbf{u}, \mathbf{v}\rangle = \frac{1}{4}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2\right) = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2.$$

35. Use Exercise 34. First, if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$, then Exercise 34 implies that $\langle \mathbf{u}, \mathbf{v}\rangle = 0$. Second, suppose that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Then again from Exercise 34,

$$0 = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 = \frac{1}{4}\left(\|\mathbf{u}+\mathbf{v}\| + \|\mathbf{u}-\mathbf{v}\|\right)\left(\|\mathbf{u}+\mathbf{v}\| - \|\mathbf{u}-\mathbf{v}\|\right),$$

so that one of the two factors must be zero. If the first factor is zero, then both norms are zero, so they are equal. If the second factor is zero, then we have $\|\mathbf{u}+\mathbf{v}\| - \|\mathbf{u}-\mathbf{v}\| = 0$ and again the norms are equal.

36. From (2) in Exercise 33, we have

$$\mathrm{d}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2}.$$

This is equal to $\sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2}$ if and only if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$, which is to say when $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

37. The inner product we are working with is $\langle \mathbf{u}, \mathbf{v}\rangle = 2u_1v_1 + 3u_2v_2$, so that

$$\begin{aligned}
\mathbf{v}_1 &= \begin{bmatrix}1\\0\end{bmatrix}\\
\mathbf{v}_2 &= \begin{bmatrix}1\\1\end{bmatrix} - \frac{2\cdot 1\cdot 1 + 3\cdot 1\cdot 0}{2\cdot 1\cdot 1 + 3\cdot 0\cdot 0}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}.
\end{aligned}$$

The orthogonal basis is $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$.

38. The inner product we are working with is

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T A\mathbf{v} = \mathbf{u}^T\begin{bmatrix}4 & -2\\-2 & 7\end{bmatrix} = 4u_1v_1 - 2u_1v_2 - 2u_2v_1 + 7u_2v_2.$$

Then

$$\begin{aligned}
\mathbf{v}_1 &= \begin{bmatrix}1\\0\end{bmatrix}\\
\mathbf{v}_2 &= \begin{bmatrix}1\\1\end{bmatrix} - \frac{4\cdot 1\cdot 1 - 2\cdot 1\cdot 1 - 2\cdot 1\cdot 0 + 7\cdot 0\cdot 1}{4\cdot 1\cdot 1 - 2\cdot 1\cdot 0 - 2\cdot 1\cdot 0 + 7\cdot 0\cdot 0}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}.
\end{aligned}$$

39. The inner product we are working with is $\left\langle a_0 + a_1x + a_2x^2, b_0 + b_1x + b_2x^2\right\rangle = a_0b_0 + a_1b_1 + a_2b_2$. Then

$$\begin{aligned}
\mathbf{v}_1 &= 1\\
\mathbf{v}_2 &= (1+x) - \frac{1\cdot(1+x)}{1\cdot 1}1 = (1+x) - 1 = x\\
\mathbf{v}_3 &= (1+x+x^2) - \frac{1\cdot(1+x+x^2)}{1\cdot 1}1 - \frac{x\cdot(1+x+x^2)}{x\cdot x}x = (1+x+x^2) - 1 - x = x^2.
\end{aligned}$$

The orthogonal basis is $\{1, x, x^2\}$.

40. The inner product we are working with is $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx$. Then

$$\begin{aligned}
\mathbf{v}_1 &= 1\\
\mathbf{v}_2 &= (1+x) - \frac{1\cdot(1+x)}{1\cdot 1}1 = (1+x) - \frac{\int_0^1 (1+x)\,dx}{\int_0^1 1\,dx}1 = (1+x) - \frac{3}{2}1 = -\frac{1}{2} + x\\
\mathbf{v}_3 &= (1+x+x^2) - \frac{1\cdot(1+x+x^2)}{1\cdot 1}1 - \frac{\left(-\frac{1}{2}+x\right)\cdot(1+x+x^2)}{\left(-\frac{1}{2}+x\right)\cdot\left(-\frac{1}{2}+x\right)}\left(-\frac{1}{2}+x\right)\\
&= (1+x+x^2) - \frac{\int_0^1 (1+x+x^2)\,dx}{\int_0^1 1\,dx}1 - \frac{\int_0^1 \left(-\frac{1}{2}+x\right)(1+x+x^2)\,dx}{\int_0^1 \left(-\frac{1}{2}+x\right)\cdot\left(-\frac{1}{2}+x\right)}\left(-\frac{1}{2}+x\right)\\
&= (1+x+x^2) - \frac{11}{6}1 - \frac{1/6}{1/12}\left(-\frac{1}{2}+x\right)\\
&= (1+x+x^2) - \frac{11}{6}1 - 2\left(-\frac{1}{2}+x\right)\\
&= \frac{1}{6} - x + x^2.
\end{aligned}$$

41. The inner product here is $\langle f, g\rangle = \int_{-1}^1 f(x)g(x)\,dx$.

(a) To normalize the first three Legendre polynomials, we find the norm of each:

$$\begin{aligned}
\|1\| &= \sqrt{\int_{-1}^1 1\cdot 1\,dx} = \sqrt{\int_{-1}^1 1\,dx} = \sqrt{2}\\
\|x\| &= \sqrt{\int_{-1}^1 x\cdot x\,dx} = \sqrt{\int_{-1}^1 x^2\,dx} = \sqrt{\left[\frac{1}{3}x^3\right]_{-1}^1} = \sqrt{\frac{2}{3}}\\
\left\|x^2 - \frac{1}{3}\right\| &= \sqrt{\int_{-1}^1 \left(x^2-\frac{1}{3}\right)\left(x^2-\frac{1}{3}\right)\,dx} = \sqrt{\int_{-1}^1 \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)\,dx}\\
&= \sqrt{\left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right]_{-1}^1} = \frac{2\sqrt{2}}{3\sqrt{5}}.
\end{aligned}$$

So the normalized Legendre polynomials are

$$\frac{1}{\|1\|} = \frac{1}{\sqrt{2}}, \qquad \frac{x}{\|x\|} = \frac{\sqrt{6}}{2}x, \qquad \frac{x^2-\frac{1}{3}}{\left\|x^2-\frac{1}{3}\right\|} = \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2-\frac{1}{3}\right) = \frac{\sqrt{5}}{2\sqrt{2}}(3x^2-1).$$

(b) For the computation, we will use the unnormalized Legendre polynomials given in Example 3.8, 1, x, and $x^2 - \frac{1}{3}$, and normalize the fourth one at the end. With $\mathbf{x}_4 = x^3$, we have

$$\begin{aligned}
\mathbf{v}_4 &= \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 \\
&= x^3 - \frac{\int_{-1}^1 x^3 \cdot 1\,dx}{\int_{-1}^1 1 \cdot 1\,dx}1 - \frac{\int_{-1}^1 x^3 \cdot x\,dx}{\int_{-1}^1 x \cdot x\,dx}x - \frac{\int_{-1}^1 x^3\left(x^2 - \frac{1}{3}\right)\,dx}{\int_{-1}^1 \left(x^2 - \frac{1}{3}\right)\left(x^2 - \frac{1}{3}\right)\,dx}\left(x^2 - \frac{1}{3}\right) \\
&= x^3 - \frac{\int_{-1}^1 x^3\,dx}{\int_{-1}^1 1\,dx}1 - \frac{\int_{-1}^1 x^4\,dx}{\int_{-1}^1 x^2\,dx}x - \frac{\int_{-1}^1 \left(x^5 - \frac{1}{3}x^3\right)\,dx}{\int_{-1}^1 \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)\,dx}\left(x^2 - \frac{1}{3}\right) \\
&= x^3 - \frac{\left[\frac{1}{4}x^4\right]_{-1}^1}{2}1 - \frac{\left[\frac{1}{5}x^5\right]_{-1}^1}{\left[\frac{1}{3}x^3\right]_{-1}^1}x - \frac{\left[\frac{1}{6}x^6 - \frac{1}{12}x^4\right]_{-1}^1}{\left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right]_{-1}^1}\left(x^2 - \frac{1}{3}\right) \\
&= x^3 - 0 - \frac{2/5}{2/3}x - 0 \\
&= x^3 - \frac{3}{5}x.
\end{aligned}$$

The norm of this polynomial is

$$\begin{aligned}
\|\mathbf{v}_4\| = \sqrt{\int_{-1}^1 \left(x^3 - \frac{3}{5}x\right)\left(x^3 - \frac{3}{5}x\right)\,dx} &= \sqrt{\int_{-1}^1 \left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right)\,dx} \\
&= \sqrt{\left[\frac{1}{7}x^7 - \frac{6}{25}x^5 + \frac{3}{25}x^3\right]_{-1}^1} = \frac{2\sqrt{14}}{35}.
\end{aligned}$$

Thus the normalized Legendre polynomial is

$$\frac{35}{2\sqrt{14}}\left(x^3 - \frac{3}{5}x\right) = \frac{\sqrt{14}}{4}(5x^3 - 3x).$$

42. **(a)** Starting with $\ell_0(x) = 1$, $\ell_1(x) = x$, $\ell_2(x) = x^2 - \frac{1}{3}$, $\ell_3(x) = x^3 - \frac{3}{5}x$, we have

$$\begin{aligned}
\ell_0(1) &= 1 &&\Rightarrow\quad L_0(x) = 1 \\
\ell_1(1) &= 1 &&\Rightarrow\quad L_1(x) = x \\
\ell_2(1) &= 1 - \frac{1}{3} = \frac{2}{3} &&\Rightarrow\quad L_2(x) = \frac{3}{2}\left(x^2 - \frac{1}{3}\right) = \frac{3}{2}x^2 - \frac{1}{2} \\
\ell_3(1) &= 1 - \frac{3}{5} - \frac{2}{5} &&\Rightarrow\quad L_3(x) = \frac{5}{2}\left(x^3 - \frac{3}{5}x\right) = \frac{5}{2}x^3 - \frac{3}{2}x.
\end{aligned}$$

(b) For L_2 and L_3, the given recurrence says

$$\begin{aligned}
L_2(x) &= \frac{2 \cdot 2 - 1}{2}xL_1(x) - \frac{2 - 1}{2}L_0(x) = \frac{3}{2}x^2 - \frac{1}{2} \\
L_3(x) &= \frac{2 \cdot 3 - 1}{3}xL_2(x) - \frac{3 - 1}{3}L_1(x) = \frac{5}{3}x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{2}{3}x = \frac{5}{2}x^3 - \frac{3}{2}x,
\end{aligned}$$

so that the recurrence indeed works for $n = 2$ and $n = 3$. Then

$$\begin{aligned}
L_4(x) &= \frac{2 \cdot 4 - 1}{4}xL_3(x) - \frac{4 - 1}{4}L_2(x) = \frac{7}{4}x\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) - \frac{3}{4}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) \\
&= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} \\
L_5(x) &= \frac{2 \cdot 5 - 1}{5}xL_4(x) - \frac{5 - 1}{5}L_3(x) = \frac{9}{5}x\left(\frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}\right) - \frac{4}{5}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) \\
&= \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x.
\end{aligned}$$

43. Let $\mathcal{B} = \{\mathbf{u}_i\}$ be an orthogonal basis for W. Then

$$\langle \operatorname{proj}_W(\mathbf{v}), \mathbf{u}_j \rangle = \left\langle \sum \frac{\langle \mathbf{u}_i, \mathbf{v} \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i, \mathbf{u}_j \right\rangle = \left\langle \frac{\langle \mathbf{u}_j, \mathbf{v} \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \mathbf{u}_j, \mathbf{u}_j \right\rangle = \frac{\langle \mathbf{u}_j, \mathbf{v} \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \langle \mathbf{u}_j, \mathbf{u}_j \rangle = \langle \mathbf{u}_j, \mathbf{v} \rangle .$$

Then

$$\langle \operatorname{perp}_W(\mathbf{v}), \mathbf{u}_j \rangle = \langle \mathbf{v} - \operatorname{proj}_W(\mathbf{v}), \mathbf{u}_j \rangle = \langle \mathbf{v}, \mathbf{u}_j \rangle - \langle \operatorname{proj}_W(\mathbf{v}), \mathbf{u}_j \rangle = \langle \mathbf{v}, \mathbf{u}_j \rangle - \langle \mathbf{u}_j, \mathbf{v} \rangle = 0.$$

Thus $\operatorname{perp}_W(\mathbf{v})$ is orthogonal to every vector in $\mathcal{B}$, so that $\operatorname{perp}_W(\mathbf{v})$ is orthogonal to every $\mathbf{w} \in W$.

44. (a) We have

$$\langle t\mathbf{u} + \mathbf{v}, t\mathbf{u} + \mathbf{v} \rangle = t^2 \langle \mathbf{u}, \mathbf{u} \rangle + 2t \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{u}\|^2 t^2 + 2 \langle \mathbf{u}, \mathbf{v} \rangle t + \|\mathbf{v}\|^2 \geq 0.$$

This is a quadratic $at^2 + bt + c$ in t, where $a = \|\mathbf{u}\|^2$, $b = 2 \langle \mathbf{u}, \mathbf{v} \rangle$, and $c = \|\mathbf{v}\|^2$.

(b) Since $a = \|\mathbf{u}\|^2 > 0$, this is an upward-opening parabola, so its vertex is its lowest point. That vertex is at $-\frac{b}{2a}$, so the lowest point, which must be nonnegative, is

$$a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c = \frac{b^2 - 2b^2 + 4ac}{4a} = \frac{4ac - b^2}{4a} \geq 0.$$

Since $4a > 0$, we must have $4ac \geq b^2$, so that $\sqrt{ac} \geq \frac{1}{2} |b|$.

(c) Substituting back in the values of a, b, and c from part (a) gives

$$\sqrt{ac} = \sqrt{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2} = \|\mathbf{u}\| \|\mathbf{v}\| \quad \geq \quad \frac{1}{2} |b| = \frac{1}{2} |2 \langle \mathbf{u}, \mathbf{v} \rangle| = |\langle \mathbf{u}, \mathbf{v} \rangle| .$$

Thus $\|\mathbf{u}\| \|\mathbf{v}\| \geq |\langle \mathbf{u}, \mathbf{v} \rangle|$, which is the Cauchy-Schwarz inequality.

Exploration: Vectors and Matrices with Complex Entries

1. Recall that if $z \in \mathbb{C}$, then $\overline{z}z = |z|^2$. Then for $\mathbf{v}$ as given, we have

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\overline{v}_1 v_1 + \overline{v}_2 v_2 + \cdots + \overline{v}_n v_n} = \sqrt{|v_1|^2 + \cdots + |v_n|^2}.$$

Finally, we must show that $|z|^2 = \left|z^2\right|$. But

$$|z|^2 = (\overline{z}z)^2 = \overline{z}\overline{z}z^2 = \overline{z^2}z^2 = \left|z^2\right| ,$$

so the above is in fact equal to $\sqrt{|v_1^2| + \cdots + |v_n^2|}$.

2. (a) $\mathbf{u} \cdot \mathbf{v} = \overline{i} \cdot (2 - 3i) + \overline{1} \cdot (1 + 5i) = -i(2 - 3i) + (1 + 5i) = -2 + 3i.$

(b) By Exercise 1, $\|\mathbf{u}\| = \sqrt{|i^2| + |1^2|} = \sqrt{|-1| + |1|} = \sqrt{2}.$

(c) By Exercise 1,

$$\|\mathbf{v}\| = \sqrt{|(2 - 3i)^2| + |(1 + 5i)^2|} = \sqrt{(2 + 3i)(2 - 3i) + (1 - 5i)(1 + 5i)} = \sqrt{13 + 26} = \sqrt{39}.$$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left\| \begin{bmatrix} -2 + 4i \\ -5i \end{bmatrix} \right\| = \sqrt{|-2 + 4i|^2 + |-5i|^2} = \sqrt{(-2 - 4i)(-2 + 4i) + (5i)(-5i)} =$
$3\sqrt{5}.$

(e) If $\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ is orthogonal to $\mathbf{u}$, then

$$\mathbf{u} \cdot \mathbf{w} = \bar{i} \cdot a + \bar{1} \cdot b = -ai + b = 0,$$

so that $b = ai$. Thus for example setting $a = 1$, we get

$$\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

(f) If $\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ is orthogonal to $\mathbf{v}$, then

$$\mathbf{u} \cdot \mathbf{w} = \overline{2-3i} \cdot a + \overline{1+5i} \cdot b = (2+3i)a + (1-5i)b = 0.$$

So $b = -\frac{2+3i}{1-5i}a$; setting for example $a = 1-5i$, we get $b = -2-3i$, and one possible vector is

$$\begin{bmatrix} 1-5i \\ -2-3i \end{bmatrix}$$

3. (a) $\overline{\mathbf{v} \cdot \mathbf{u}} = \overline{\overline{v}_1 u_1 + \cdots + \overline{v}_n u_n} = \overline{\overline{v_1}} \overline{u_1} + \cdots + \overline{\overline{v_n}} \overline{u_n} = v_1 \overline{u}_1 + \cdots + v_n \overline{u}_n = \overline{u}_1 v_1 + \cdots + \overline{u}_n v_n = \mathbf{u} \cdot \mathbf{v}$.

(b)

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \overline{u}_1(v_1 + w_1) + \overline{u}_2(v_2 + w_2) + \cdots \overline{u}_n(v_n + w_n) \\ &= \overline{u}_1 v_1 + \overline{u}_1 w_1 + \overline{u}_2 v_2 + \overline{u}_2 w_2 + \cdots + \overline{u}_n v_n + \overline{u}_n w_n \\ &= (\overline{u}_1 v_1 + \overline{u}_2 v_2 + \cdots + \overline{u}_n v_n) + (\overline{u}_1 w_1 + \overline{u}_2 w_2 + \cdots + \overline{u}_n w_n) \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}. \end{aligned}$$

(c) First,

$$(c\mathbf{u}) \cdot \mathbf{v} = \overline{cu_1} v_1 + \cdots + \overline{cu_n} v_n = \overline{c}\overline{u}_1 v_1 + \cdots + \overline{c}\overline{u}_n v_n = \overline{c}(\overline{u}_1 v_1 + \cdots + \overline{n}_n v_n) = \overline{c}\mathbf{u} \cdot \mathbf{v}.$$

For the second equality, we have, using part (a),

$$\mathbf{u} \cdot (c\mathbf{v}) = \overline{(c\mathbf{u}) \cdot \mathbf{v}} = \overline{\overline{c}\mathbf{u} \cdot \mathbf{v}} = c\overline{\mathbf{u} \cdot \mathbf{v}} = c\mathbf{v} \cdot \mathbf{u}.$$

(d) By Exercise 1, $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 = \left|u_1^2\right| + \cdot + \left|u_n^2\right| \geq 0$. Further, equality holds if and only if each $\left|u_i^2\right| = |u_i|^2 = 0$, so all the u_i must be zero and therefore $\mathbf{u} = \mathbf{0}$.

4. Just apply the definition: conjugate A and take its transpose:

(a) $A^* = \overline{A}^T = \begin{bmatrix} -i & i \\ -2i & 3 \end{bmatrix}$.

(b) $A^* = \overline{A}^T = \begin{bmatrix} 2 & 5-2i \\ 5+2i & -1 \end{bmatrix}$.

(c) $A^* = \overline{A}^T = \begin{bmatrix} 2+i & 4 \\ 1-3i & 0 \\ -2 & 3+4i \end{bmatrix}$.

(d) $A^* = \overline{A}^T = \begin{bmatrix} -3i & 1+i & 1-i \\ 0 & 4 & 0 \\ 1-i & -i & i \end{bmatrix}$.

5. These facts follow directly from standard properties of complex numbers together with the fact that matrix conjugation is element by element:

(a) $\overline{\overline{A}} = \left[\overline{\overline{a_{ij}}}\right] = [a_{ij}] = A$.

(b) $\overline{A+B} = \left[\overline{a_{ij}+b_{ij}}\right] = \left[\overline{a_{ij}}+\overline{b_{ij}}\right] = \left[\overline{a_{ij}}\right] + \left[\overline{b_{ij}}\right] = \overline{A}+\overline{B}$.

(c) $\overline{cA} = \left[\overline{ca_{ij}}\right] = \left[\overline{c}\cdot\overline{a_{ij}}\right] = \overline{c}\left[\overline{a_{ij}}\right] = \overline{c}\overline{A}$.

(d) Let $C = AB$; then $c_{ij} = \sum_k a_{ik}b_{kj}$, so that

$$\overline{AB}_{ij} = \overline{C}_{ij} = \left[\overline{\sum_k a_{ik}b_{kj}}\right]_{ij} = \left[\sum_k \overline{a_{ik}}\overline{b_{ik}}\right]_{ij} = \left[\overline{A}\cdot\overline{B}\right]_{ij},$$

so that $\overline{AB} = \overline{A}\cdot\overline{B}$.

(e) The ij entry of $\left(\overline{A}\right)^T$ is $\overline{a_{ji}}$. The ij entry of A^T is a_{ji}, so its conjugate, which is the ij entry of $\overline{A^T}$, is $\overline{a_{ji}}$. So the two matrices have the same entries and thus are equal.

6. These facts follow directly from the facts in Exercise 5.

(a) $(A^*)^* = \left(\overline{A^T}\right)^* = \overline{\left(\overline{A^T}\right)}^T = \left(\overline{\overline{A}}^T\right)^T = \left(A^T\right)^T = A$.

(b) $(A+B)^* = \overline{(A+B)^T} = \overline{A^T} + \overline{B^T} = A^* + B^*$.

(c) $(cA)^* = \overline{(cA)^T} = \overline{cA^T} = \overline{c}\overline{A^T} = \overline{c}A^*$.

(d) $(AB)^* = \overline{(AB)^T} = \overline{B^TA^T} = \overline{B^T}\cdot\overline{A^T} = B^*A^*$.

7. We have $\mathbf{u}\cdot\mathbf{v} = \sum \overline{u_i}v_i = \begin{bmatrix}\overline{u_1} & \overline{u_2} & \cdots & \overline{u_n}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix} = \overline{\mathbf{u}^T}\mathbf{v} = \mathbf{u}^*\mathbf{v}$.

8. If A is Hermitian, then $\overline{a_{ij}} = a_{ji}$ for all i and j. In particular, when $i = j$, this gives $\overline{a_{ii}} = a_{ii}$, so that a_{ii} is equal to its own conjugate and therefore must be real.

9. **(a)** A is not Hermitian since it does not have real diagonal entries.

(b) A is not Hermitian since $a_{12} = 2-3i$ and $\overline{a_{21}} = \overline{2-3i} = 2+3i$; the two are not equal.

(c) A is not Hermitian since $a_{12} = 1-5i$ and $\overline{a_{21}} = -1-5i$; the two are not equal.

(d) A is Hermitian, since every entry is the conjugate of the same entry in its transpose.

(e) A is not Hermitian. Since it is a real matrix, it is equal to its conjugate, but $A^T \neq A$ since A is not symmetric.

(f) A is Hermitian. Since it is a real matrix, it is equal to its conjugate, and it is symmetric, so that $\overline{A}^T = A^T = A$.

10. Suppose that λ is an eigenvalue of a Hermitian matrix A, with corresponding eigenvector $\mathbf{v}$. Now,

$$\mathbf{v}^*A = \mathbf{v}^*A^* = (A\mathbf{v})^* = (\lambda\mathbf{v})^*$$

since A is Hermitian. But by Exercise 6(c), $(\lambda\mathbf{v})^* = \overline{\lambda}\mathbf{v}^*$. Then

$$\lambda(\mathbf{v}^*\mathbf{v}) = \mathbf{v}^*(\lambda\mathbf{v}) = \mathbf{v}^*(A\mathbf{v}) = (\mathbf{v}^*A)\mathbf{v} = \left(\overline{\lambda}\mathbf{v}^*\right)\mathbf{v} = \overline{\lambda}(\mathbf{v}^*\mathbf{v}).$$

Then since $\mathbf{v} \neq \mathbf{0}$, we have by Exercise 7 $\mathbf{v}^*\mathbf{v} = \mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2 \neq 0$. So we can divide the above equation through by $\mathbf{v}^*\mathbf{v}$ to get $\lambda = \overline{\lambda}$, so that λ is real.

11. Following the hint, adapt the proof of Theorem 5.19. Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of a Hermitian matrix A corresponding to eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Then

$$\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) = (\lambda_1\mathbf{v}_1) \cdot \mathbf{v}_2 = (A\mathbf{v}_1) \cdot \mathbf{v}_2 = (A\mathbf{v}_1)^*\mathbf{v}_2 = (\mathbf{v}_1^* A^*)\mathbf{v}_2.$$

But A is Hermitian, so $A^* = A$ and we continue:

$$(\mathbf{v}_1^* A^*)\mathbf{v}_2 = (\mathbf{v}_1^* A)\mathbf{v}_2 = \mathbf{v}_1^*(A\mathbf{v}_2) = \mathbf{v}_1^*(\lambda_2\mathbf{v}_2) = \lambda_2(\mathbf{v}_1^*\mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2).$$

Thus $\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2)$, so that $(\lambda_1 - \lambda_2)(\mathbf{v}_1 \cdot \mathbf{v}_2) = 0$. But $\lambda_1 \neq \lambda_2$ by assumption, so we must have $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

12. **(a)** Multiplying, we get $AA^* = I$, so that A is unitary and $A^{-1} = A^* = \begin{bmatrix} -\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}$.

(b) Multiplying, we get $AA^* = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq I$, so that A is not unitary.

(c) Multiplying, we get $AA^* = I$, so that A is unitary and $A^{-1} = A^* = \begin{bmatrix} \frac{3}{5} & -\frac{4i}{5} \\ -\frac{4}{5} & -\frac{3i}{5} \end{bmatrix}$.

(d) Multiplying, we get $AA^* = I$, so that A is unitary and $A^{-1} = A^* = \begin{bmatrix} \frac{1-i}{\sqrt{6}} & 0 & \frac{-1+i}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$.

13. First show (a) $\Leftrightarrow$ (b). Let $\mathbf{u}_i$ denote the i^{th} column of a matrix U. Then the i^{th} row of U^* is $\overline{\mathbf{u}_i}$. Then by the definition of matrix multiplication,

$$(U^*U)_{ij} = \mathbf{u}_i \cdot \mathbf{u}_j.$$

Then the columns of U form an orthonormal set if and only if

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases},$$

which holds if and only if

$$(U^*U)_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases},$$

which is to say that $U^*U = I$. Thus the columns of U are orthogonal if and only if U is unitary.

Now apply this to the matrix U^*. We get that U^* is unitary if and only if the columns of U^* are orthogonal. But U^* is unitary if and only if U is, since

$$(U^*)^*U^* = UU^*,$$

and the columns of U^* are just the conjugates of the rows of U, so that (letting U_i be the i^{th} row of U)

$$\overline{U_i} \cdot \overline{U_j} = \overline{U_i}^* \overline{U_j} = (U_i)^T \overline{U_j} = U_j \cdot U_i.$$

So the columns of U^* are orthogonal if and only if the rows of U are orthogonal. This proves (a) $\Leftrightarrow$ (c).

Next, we show (a) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a). To prove (a) $\Rightarrow$ (e), assume that U is unitary. Then $U^*U = I$, so that

$$U\mathbf{x} \cdot U\mathbf{y} = (U\mathbf{x})^* U\mathbf{y} = \mathbf{x}^*U^*U\mathbf{y} = \mathbf{x}^*\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

Next, for (e) $\Rightarrow$ (d), assume that (e) holds. Then $Q\mathbf{x} \cdot Q\mathbf{x} = \mathbf{x} \cdot \mathbf{x}$, so that $\|Q\mathbf{x}\| = \sqrt{Q\mathbf{x} \cdot Q\mathbf{x}} = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \|\mathbf{x}\|$. Next, we show (d) $\Rightarrow$ (e). Let $\mathbf{u}_i$ denote the i^{th} column of U. Then

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{y} &= \frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right) \\
&= \frac{1}{4}\left(\|U(\mathbf{x}+\mathbf{y})\|^2 - \|U(\mathbf{x}-\mathbf{y})\|^2\right) \\
&= \frac{1}{4}\left(\|U\mathbf{x}+U\mathbf{y}\|^2 - \|U\mathbf{x}-U\mathbf{y}\|^2\right) \\
&= U\mathbf{x} \cdot U\mathbf{y}.
\end{aligned}$$

Thus (d) $\Rightarrow$ (e). Finally, to see that (e) $\Rightarrow$ (a), let $\mathbf{e}_i$ be the i^{th} standard basis vector. Then $\mathbf{u}_i = U\mathbf{e}_i$, so that since (e) holds,

$$\mathbf{u}_i \cdot \mathbf{u}_j = U\mathbf{e}_i \cdot U\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

This shows that the columns of U form an orthonormal set, so that U is unitary.

14. (a) Since

$$\begin{aligned}
\begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} &= -\frac{i}{\sqrt{2}} \cdot \left(-\frac{i}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} = 0 \\
\begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} &= -\frac{i}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} - \frac{i}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} = 1 \\
\begin{bmatrix} -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} &= \frac{i}{\sqrt{2}} \cdot \left(-\frac{i}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}} \cdot \frac{i}{\sqrt{2}} = 1,
\end{aligned}$$

the columns of the matrix are orthonormal, so the matrix is unitary.

(b) The dot product of the first column with itself is

$$\overline{1+i}(1+i) + \overline{1-i}(1-i) = (1-i)(1+i) + (1+i)(1-i) = 4 \neq 1,$$

so that the matrix is not unitary.

(c) Since

$$\begin{aligned}
\begin{bmatrix} \frac{3}{5} \\ \frac{4i}{5} \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{5} \\ \frac{3i}{5} \end{bmatrix} &= \frac{3}{5} \cdot \left(-\frac{4}{5}\right) - \frac{4i}{5} \cdot \frac{3i}{5} = 0 \\
\begin{bmatrix} \frac{3}{5} \\ \frac{4i}{5} \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{5} \\ \frac{4i}{5} \end{bmatrix} &= \frac{3}{5} \cdot \frac{3}{5} - \frac{4i}{5} \cdot \frac{4i}{5} = 1 \\
\begin{bmatrix} -\frac{4}{5} \\ \frac{3i}{5} \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{5} \\ \frac{3i}{5} \end{bmatrix} &= -\frac{4}{5} \cdot \left(-\frac{4}{5}\right) - \frac{3i}{5} \cdot \frac{3i}{5} = 1,
\end{aligned}$$

the columns of the matrix are orthonormal, so the matrix is unitary.

(d) Clearly column 2 is a unit vector, and is orthogonal to each of the other two columns, so we need only show the first and third columns are unit vectors and are orthogonal. But

$$\begin{bmatrix}\frac{1+i}{\sqrt{6}}\\0\\\frac{-1-i}{\sqrt{3}}\end{bmatrix}\cdot\begin{bmatrix}\frac{2}{\sqrt{6}}\\0\\\frac{1}{\sqrt{3}}\end{bmatrix}=\frac{1-i}{\sqrt{6}}\cdot\frac{2}{\sqrt{6}}+0+\frac{-1+i}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}=\frac{2-2i}{6}+\frac{-1+i}{3}=0$$

$$\begin{bmatrix}\frac{1+i}{\sqrt{6}}\\0\\\frac{-1-i}{\sqrt{3}}\end{bmatrix}\cdot\begin{bmatrix}\frac{1+i}{\sqrt{6}}\\0\\\frac{-1-i}{\sqrt{3}}\end{bmatrix}=\frac{1-i}{\sqrt{6}}\cdot\frac{1+i}{\sqrt{6}}+0+\frac{-1+i}{\sqrt{3}}\cdot\frac{-1-i}{\sqrt{3}}=\frac{2}{6}+\frac{2}{3}=1$$

$$\begin{bmatrix}\frac{2}{\sqrt{6}}\\0\\\frac{1}{\sqrt{3}}\end{bmatrix}\cdot\begin{bmatrix}\frac{2}{\sqrt{6}}\\0\\\frac{1}{\sqrt{3}}\end{bmatrix}=\frac{2}{\sqrt{6}}\cdot\frac{2}{\sqrt{6}}+0+\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}=\frac{4}{6}+\frac{1}{3}=1,$$

and we are done.

15. (a) With A as given, the characteristic polynomial is $(\lambda-2)^2-i\cdot(-i)=\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)$. Thus the eigenvalues are $\lambda_1=1$ and $\lambda_2=3$. Corresponding eigenvectors are

$$\lambda=1:\ \begin{bmatrix}1\\i\end{bmatrix},\qquad \lambda=3:\ \begin{bmatrix}1+i\\1-i\end{bmatrix}.$$

Since these eigenvectors correspond to different eigenvalues, they are already orthogonal. Their norms are

$$\left\|\begin{bmatrix}1\\i\end{bmatrix}\right\|=\sqrt{1\cdot 1+(-i)\cdot i}=\sqrt{2},\qquad \left\|\begin{bmatrix}1+i\\1-i\end{bmatrix}\right\|=\sqrt{|1+i|^2+|1-i|^2}=2.$$

So defining a matrix U whose columns are the normalized eigenvectors gives

$$U=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1+i}{2}\\ \frac{i}{\sqrt{2}} & \frac{1-i}{2}\end{bmatrix},$$

which is unitary. Then

$$U^*AU=\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}.$$

(b) With A as given, the characteristic polynomial is λ^2+1. Thus the eigenvalues are $\lambda_1=i$ and $\lambda_2=-i$. Corresponding eigenvectors are

$$\lambda=1:\ \begin{bmatrix}i\\1\end{bmatrix},\qquad \lambda=3:\ \begin{bmatrix}-i\\1\end{bmatrix}.$$

Since these eigenvectors correspond to different eigenvalues, they are already orthogonal. Their norms are

$$\left\|\begin{bmatrix}i\\1\end{bmatrix}\right\|=\sqrt{|i|^2+|1|^2}=\sqrt{2},\qquad \left\|\begin{bmatrix}-i\\1\end{bmatrix}\right\|=\sqrt{|-i|^2+|1|^2}=\sqrt{2}.$$

So defining a matrix U whose columns are the normalized eigenvectors gives

$$U=\begin{bmatrix}\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix},$$

which is unitary. Then

$$U^*AU=\begin{bmatrix}i & 0\\0 & -i\end{bmatrix}.$$

(c) With A as given, the characteristic polynomial is $(\lambda+1)\lambda-(1+i)(1-i) = \lambda^2+\lambda-2 = (\lambda+2)(\lambda-1)$. Thus the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$. Corresponding eigenvectors are

$$\lambda = 1: \begin{bmatrix} 1+i \\ 2 \end{bmatrix}, \qquad \lambda = 3: \begin{bmatrix} 1+i \\ -1 \end{bmatrix}.$$

Since these eigenvectors correspond to different eigenvalues, they are already orthogonal. Their norms are

$$\left\| \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \right\| = \sqrt{|1+i|^2 + |2|^2} = \sqrt{6}, \qquad \left\| \begin{bmatrix} 1+i \\ -1 \end{bmatrix} \right\| = \sqrt{|1+i|^2 + |1|^2} = \sqrt{3}.$$

So defining a matrix U whose columns are the normalized eigenvectors gives

$$U = \begin{bmatrix} \frac{1+i}{\sqrt{6}} & \frac{1+i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix},$$

which is unitary. Then

$$U^*AU = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

(d) With A as given, the characteristic polynomial is

$$(\lambda-1)\left((\lambda-2)(\lambda-3) - (1-i)(1+i)\right) = (\lambda-1)\left(\lambda^2 - 5\lambda + 4\right) = (\lambda-1)^2(\lambda-4)$$

Thus the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$. Corresponding eigenvectors are

$$\lambda = 1: \begin{bmatrix} 0 \\ -1+i \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \lambda = 4: \begin{bmatrix} 0 \\ 1-i \\ 2 \end{bmatrix}.$$

The eigenvectors corresponding to different eigenvalues are already orthogonal, and the two eigenvectors corresponding to $\lambda = 1$ are also orthogonal, so these vectors form an orthogonal set. Their norms are

$$\left\| \begin{bmatrix} 0 \\ -1+i \\ 1 \end{bmatrix} \right\| = \sqrt{|0|^2 + |-1+i|^2 + |1|^2} = \sqrt{3}, \qquad \left\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\| = 1,$$

$$\left\| \begin{bmatrix} 0 \\ 1-i \\ 2 \end{bmatrix} \right\| = \sqrt{|0|^2 + |1-i|^2 + |2|^2} = \sqrt{6}.$$

So defining a matrix U whose columns are the normalized eigenvectors gives

$$U = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1+i}{\sqrt{3}} & 0 & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix},$$

which is unitary. Then

$$U^*AU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

16. If A is Hermitian, then $A^* = A$, so that $A^*A = AA = AA^*$. Next, if U is unitary, then $UU^* = I$. But if U is unitary, then its rows are orthonormal. Since the columns of U^* are the conjugates of the rows of U, they are orthonormal as well and therefore U^* is also unitary. Hence $UU^* = U^*U = I$ and therefore U is normal. Finally, if A is skew-Hermitian, then $A^* = -A$; then $A^*A = -AA = A(-A) = AA^*$. So all three of these types of matrices are normal.

17. By the result quoted just before Exercise 16, if A is unitarily diagonalizable, then $A^*A = AA^*$; this is the definition of A being normal.

Exploration: Geometric Inequalities and Optimization Problems

1. We have

$$|\mathbf{u}\cdot\mathbf{v}| = \sqrt{x}\cdot\sqrt{y} + \sqrt{y}\cdot\sqrt{x} = 2\sqrt{x}\sqrt{y}$$
$$\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2} = \sqrt{x+y},$$

so that $2\sqrt{x}\sqrt{y} \le \sqrt{x+y}\cdot\sqrt{x+y} = x+y$, and thus

$$\sqrt{xy} \le \frac{x+y}{2}.$$

2. (a) Since $\left(\sqrt{x}-\sqrt{y}\right)^2$ is a square, it is nonnegative. Expanding then gives $x - 2\sqrt{x}\sqrt{y} + y \ge 0$, so that $2\sqrt{xy} \le x+y$ and thus

$$\sqrt{xy} \le \frac{x+y}{2}.$$

(b) Let $AD = a$ and $BD = b$. Then by the Pythagorean Theorem,

$$CD^2 = a^2 - x^2 = b^2 - y^2,$$

so that $2CD^2 = (a^2+b^2) - x^2 - y^2 = (x+y)^2 - (x^2+y^2) = 2xy$, so that $CD = \sqrt{xy}$. But CD is clearly at most equal to a radius of the circle, since it extends from a diameter to the circle. Since the radius is $\frac{x+y}{2}$, we again get

$$\sqrt{xy} \le \frac{x+y}{2}.$$

3. Since the area is $xy = 100$, we have $10 = \sqrt{xy} \le \frac{x+y}{2}$, so that $40 \le 2(x+y)$, which is the perimeter. By the AMGM, equality holds when $x = y$; that is, when $x = y = 10$, so the square of side 10 is the rectangle with smallest perimeter.

4. Let $\frac{1}{x}$ play the role of y in the AMGM. Then

$$\sqrt{x\cdot\frac{1}{x}} \le \frac{x+\frac{1}{x}}{2}, \text{ or } 2\sqrt{1} = 2 \le x + \frac{1}{x}.$$

Equality holds if and only if $x = \frac{1}{x}$, i.e., when $x = 1$. Thus $1 + \frac{1}{1} = 2$ is the minimum value for $f(x)$ for $x > 0$.

5. If the dimensions of the cut-out square are $x \times x$, then the base of the box will be $(10-2x)\times(10-2x)$, so its volume will be $x(10-2x)^2$. Then

$$\sqrt[3]{V} \le \frac{x + (10-2x) + (10-2x)}{3} = \frac{20-3x}{3},$$

with equality holding if $x = 10 - 2x$. Thus $x = \frac{10}{3}$, so that $10 - 2x = \frac{10}{3}$ as well. Then the largest possible volume is $\frac{10}{3}\times\frac{10}{3}\times\frac{10}{3}$.

6. We have

$$\sqrt[3]{f} = \frac{(x+y)+(y+z)+(z+x)}{3} = \frac{2}{3}(x+y+z),$$

with equality holding if $x+y = y+z = z+x$. But this implies that $x = y = z$; since $xyz = 1$, we must have $x = y = z = 1$, so that $f(x,y,z) = (1+1)(1+1)(1+1) = 8$. This is the minimum value of f subject to the constraint $xyz = 1$.

7. Use the substitution $u = x - y$. Then the original expression becomes

$$u + y + \frac{8}{uy}.$$

Using the AMGM for three variables, we get

$$\sqrt[3]{u \cdot y \cdot \frac{8}{uy}} \le \frac{u + y + \frac{8}{uy}}{3}, \text{ or } 3\sqrt[3]{8} = 6 \le u + y + \frac{8}{uy},$$

with equality holding if and only if $u = y = \frac{8}{uy}$; this implies that $u = y = 2$, so that $x = u + y = 4$. So the minimum values is $4 + \frac{8}{2 \cdot 2} = 6$, which occurs when $x = 4$ and $y = 2$.

8. Follow the method of Example 7.11. $x^2 + 2y^2 + z^2$ is the square of the norm of $\mathbf{v} = \begin{bmatrix} x \\ \sqrt{2}\,y \\ z \end{bmatrix}$, so that $x + 2y + 4z$ is the dot product of that vector with $\mathbf{u} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 4 \end{bmatrix}$. Then the component-wise form of the Cauchy-Schwarz Inequality gives

$$(1 \cdot x + \sqrt{2} \cdot \sqrt{2}\,y + 4 \cdot z)^2 = (x + 2y + 4z)^2 \le \left(1^2 + \left(\sqrt{2}\right)^2 + 4^2\right)(x^2 + 2y^2 + z^2) = 19(x^2 + 2y^2 + z^2) = 19.$$

So $x + 2y + 4z \le \sqrt{19}$, and the maximum value is $\sqrt{19}$.

9. Follow the method of Example 7.11. Here $f(x, y, z)$ is the square of the norm of $\mathbf{v} = \begin{bmatrix} x \\ y \\ \frac{z}{\sqrt{2}} \end{bmatrix}$, so set $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$. Then the component-wise form of the Cauchy-Schwarz Inequality gives

$$\left(\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ \frac{z}{\sqrt{2}} \end{bmatrix}\right)^2 = (x + y + z)^2 \le \left(1^2 + 1^2 + \left(\sqrt{2}\right)^2\right)\left(x^2 + y^2 + \frac{1}{2}z^2\right).$$

Since $x + y + z = 10$, we get

$$100le4\left(x^2 + y^2 + \frac{1}{2}z^2\right), \text{ or } 25 \le x^2 + y^2 + \frac{1}{2}z^2.$$

Thus the minimum value of the function is 25.

10. Follow the method of Example 7.11 with $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$. (This works because $\|\mathbf{v}\|^2 = \sin^2\theta + \cos^2\theta = 1$.) Then the component-wise form of the Cauchy-Schwarz Inequality gives

$$(\mathbf{u} \cdot \mathbf{v})^2 = (\sin\theta + \cos\theta)^2 \le (1^2 + 1^2)(\sin^2\theta + \cos^2\theta) = 2.$$

Therefore $\sin\theta + \cos\theta \le \sqrt{2}$, so the maximum value of the expression is $\sqrt{2}$.

11. We want to minimize $x^2 + y^2$ subject to the constraint $x + 2y = 5$. Thus set $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then the component-wise form of the Cauchy-Schwarz Inequality gives

$$(x + 2y)^2 \le (1^2 + 2^2)(x^2 + y^2).$$

Since $x+2y=5$, we get

$$25 \le 5(x^2+y^2), \text{ or } 5 \le x^2+y^2.$$

Thus the minimum value is 5, which occurs at equality. To find the point where equality occurs, substitution $5-2y$ for x to get

$$5 = (5-2y)^2+y^2 = 25-20y+5y^2, \text{ so that } y^2-4y+4=0.$$

This has the solution $y=2$; then $x=5-2y=1$. So the point on $x+2y=5$ closest to the origin is $(1,2)$.

12. To show that $\sqrt{xy} \ge \frac{2}{1/x+1/y}$, apply the AMGM to $x' = \frac{1}{x}$ and $y' = \frac{1}{y}$, giving

$$\sqrt{x'y'} \le \frac{x'+y'}{2}, \text{ or } \sqrt{\frac{1}{x'y'}} \ge \frac{2}{x'+y'}, \text{ or } \sqrt{xy} \ge \frac{2}{1/x+1/y}.$$

Since this inequality results from the AMGM, we have equality when $x'=y'$, so when $x=y$. For the first inequality, note first that $(x-y)^2 = x^2-2xy+y^2 \ge 0$, so that $x^2+y^2 \ge 2xy$. Then

$$\frac{x^2+y^2}{2} = \frac{x^2+y^2+(x^2+y^2)}{4} \ge \frac{x^2+2xy+y^2}{4} = \left(\frac{x+y}{2}\right)^2, \text{ so that } \sqrt{\frac{x^2+y^2}{2}} \ge \frac{x+y}{2}.$$

Equality occurs when $x^2+y^2=2xy$, which is the same as $(x-y)^2=0$, so that $x=y$.

13. The area of the rectangle shown is $2xy$, where $x^2+y^2=r^2$. From Exercise 12, we have

$$\sqrt{\frac{x^2+y^2}{2}} \ge \sqrt{xy} \Rightarrow \sqrt{x^2+y^2} \ge \sqrt{2xy} \Rightarrow \sqrt{r^2} = r \ge \sqrt{2xy} = \sqrt{A} \Rightarrow A \le r^2.$$

The maximum occurs at equality, and the maximum area is $A=r^2$. By Exercise 12, this happens when $x=y$.

14. Following the hint, we have

$$\frac{(x+y)^2}{xy} = (x+y)\left(\frac{1}{x}+\frac{1}{y}\right).$$

So from Exercise 12, we have

$$\frac{x+y}{2} \ge \frac{2}{1/x+1/y} \Rightarrow (x+y)\left(\frac{1}{x}+\frac{1}{y}\right) \ge 4.$$

Thus the minimum value is 4; this minimum occurs when $x=y$, so when

$$(x+x)\left(\frac{1}{x}+\frac{1}{x}\right) = \frac{4x}{x} = 4,$$

or $x=y=1$.

15. Squaring the inequalities in Exercise 12 gives

$$\frac{x^2+y^2}{2} \ge \left(\frac{x+y}{2}\right)^2 \ge \left(\frac{2}{1/x+1/y}\right)^2.$$

Substituting $x+\frac{1}{x}$ for x and $y+\frac{1}{y}$ for y gives from the first inequality

$$\frac{\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2}{2} \ge \left(\frac{\left(x+\frac{1}{x}\right)+\left(y+\frac{1}{y}\right)}{2}\right)^2.$$

Since the constraint we are working with is $x+y=1$, we get

$$\left(\frac{\left(x+\frac{1}{x}\right)+\left(y+\frac{1}{y}\right)}{2}\right)^2=\left(\frac{(x+y)+\left(\frac{1}{x}+\frac{1}{y}\right)}{2}\right)^2=\left(\frac{1}{2}+\frac{\frac{1}{x}+\frac{1}{y}}{2}\right)^2\geq\left(\frac{1}{2}+\frac{2}{x+y}\right),$$

using the second and fourth terms in Exercise 12 for the final inequality. But now again since $x+y=1$, we get

$$\left(\frac{1}{2}+\frac{2}{x+y}\right)^2=\left(\frac{5}{2}\right)^2=\frac{25}{4},$$

so that $f(x,y)=2\cdot\frac{25}{4}=\frac{25}{2}$ is the minimum when $x+y=1$. This equality holds when $x=y$, so that $x=y=\frac{1}{2}$.

7.2 Norms and Distance Functions

1. We have

$$\begin{aligned}\|\mathbf{u}\|_E&=\sqrt{(-1)^2+4^2+(-5)^2}=\sqrt{42}\\ \|\mathbf{u}\|_s&=|-1|+|4|+|-5|=10\\ \|\mathbf{u}\|_m&=\max\left(|-1|,|4|,|-5|\right)=5.\end{aligned}$$

2. We have

$$\begin{aligned}\|\mathbf{v}\|_E&=\sqrt{2^2+(-2)^2+0^2}=2\sqrt{2}\\ \|\mathbf{v}\|_s&=|2|+|-2|+|0|=4\\ \|\mathbf{v}\|_m&=\max\left(|2|,|-2|,|0|\right)=2.\end{aligned}$$

3. $\mathbf{u}-\mathbf{v}=\begin{bmatrix}-3\\6\\-5\end{bmatrix}$, so

$$\begin{aligned}d_E(\mathbf{u},\mathbf{v})&=\sqrt{(-3)^2+6^2+(-5)^2}=\sqrt{70}\\ d_s(\mathbf{u},\mathbf{v})&=|-3|+|6|+|-5|=14\\ d_m(\mathbf{u},\mathbf{v})&=\max\left(|-3|,|6|,|-5|\right)=6.\end{aligned}$$

4. **(a)** $d_s(\mathbf{u},\mathbf{v})$ measures the total difference in magnitude between corresponding components.

(b) $d_m(\mathbf{u},\mathbf{v})$ measure the greatest difference between any pair of corresponding components.

5. $\|\mathbf{u}\|_H=w(\mathbf{u})$, which is the number of 1s in $\mathbf{u}$, or 4. $\|\mathbf{v}\|_H=w(\mathbf{v})$, which is the number of 1s in $\mathbf{v}$, or 5.

6. $d_H(\mathbf{u},\mathbf{v})=w(\mathbf{u}-\mathbf{v})=w(\mathbf{u})+w(\mathbf{v})$ less twice the overlap, or $4+5-2\cdot 2=5$; this is the number of 1s in $\mathbf{u}-\mathbf{v}$.

7. **(a)** If $\|\mathbf{v}\|_E=\sqrt{\sum v_i^2}=\max\{|v_i|\}=\|\mathbf{v}\|_m$, then squaring both sides gives $\sum v_i^2=\max\{|v_i|^2\}=\max\{v_i^2\}$. Since $v_i^2\geq 0$ for all i, equality holds if and only if exactly one of the v_i is nonzero.

(b) Suppose $\|\mathbf{v}\|_s=\sum|v_i|=\max\{|v_i|\}=\|\mathbf{v}\|_m$. Since $|v_i|\geq 0$ for all i, equality holds if and only if exactly one of the v_i is nonzero.

(c) By parts (a) and (b), $\|\mathbf{v}\|_E=\|\mathbf{v}\|_m=\|\mathbf{v}\|_s$ if and only if exactly one of the components of $\mathbf{v}$ is nonzero.

8. **(a)** Since

$$\|\mathbf{u}+\mathbf{v}\|_E = \sum (u_i+v_i)^2 = \sum u_i^2 + \sum v_i^2 + 2\sum u_iv_i = \|\mathbf{u}\|_E + \|\mathbf{v}\|_E + 2\mathbf{u}\cdot\mathbf{v},$$

we see that $\|\mathbf{u}+\mathbf{v}\|_E = \|\mathbf{u}\|_E + \|\mathbf{v}\|_E$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$; that is, if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) We have

$$\|\mathbf{u}+\mathbf{v}\|_s = \sum |u_i+v_i|, \qquad \|\mathbf{u}\|_s + \|\mathbf{v}\|_s = \sum (|u_i| + |v_i|).$$

For a given i, if the signs of u_i and v_i are the same, then $|u_i+v_i| = |u_i| + |v_i|$, while if they are different, then $|u_i+v_i| < |u_i| + |v_i|$. Thus the two norms are equal exactly when u_i and v_i have the same sign for each i, which is to say that $u_iv_i \geq 0$ for all i.

(c) We have

$$\|\mathbf{u}+\mathbf{v}\|_s = \max\{|u_i+v_i|\}, \qquad \|\mathbf{u}\|_s + \|\mathbf{v}\|_s = \max\{|u_i|\} + \max\{|v_i|\}.$$

Suppose that $|u_r| = \max\{|u_i|\}$ and $|v_t| = \max\{|v_i|\}$. Then for any j, we know that

$$|u_j+v_j| \leq |u_j| + |v_j| \leq |u_r| + |v_t|,$$

where the first inequality is the triangle inequality and the second comes from the condition on r and t. Choose j such that $|u_j+v_j| = \|\mathbf{u}+\mathbf{v}\|_s$. If the two norms are equal, then

$$|u_j+v_j| = \|\mathbf{u}+\mathbf{v}\|_s = \|\mathbf{u}\|_s + \|\mathbf{v}\|_s = |u_r| + |v_t|.$$

Thus both of the inequalities above must be equalities. The first is an equality if u_j and v_j have the same sign. For the second, since $|u_j| \leq |u_r|$ and $|v_j| \leq |v_t|$, equality means that $|u_j| = |u_r|$ and $|v_j| = |v_t|$. Putting this together gives the following (somewhat complicated) condition for equality to hold in the triangle inequality for the max norm: equality holds if there is some component j such that u_j and v_j have the same sign and such that $|u_j|$ is a component of maximal magnitude of $\mathbf{u}$ and $|v_j|$ is a component of maximal magnitude of $\mathbf{v}$.

9. Let v_k be a component of $\mathbf{v}$ with maximum magnitude. Then the result follows from the string of equalities and inequalities

$$\|\mathbf{v}\|_m = |v_k| = \sqrt{v_k^2} \leq \sqrt{\sum v_i^2} = \|\mathbf{v}\|_E.$$

10. We have $\|\mathbf{v}\|_E^2 = \sum v_i^2 \leq \left(\sum |v_i|\right)^2 \leq \|\mathbf{v}\|_s^2$. Taking square roots gives the desired result since norms are always nonnegative.

11. We have $\|\mathbf{v}\|_s = \sum |v_i| \leq \sum \max\{|v_i|\} = n\max\{|v_i|\} = n\|\mathbf{v}\|_m$.

12. We have $\|\mathbf{v}\|_E = \sqrt{\sum v_i^2} \leq \sqrt{\sum (\max\{v_i\})^2} = \sqrt{n\max\{|v_i|^2\}} = \sqrt{n}\max\{|v_i|\} = \sqrt{n}\,\|\mathbf{v}\|_m$.

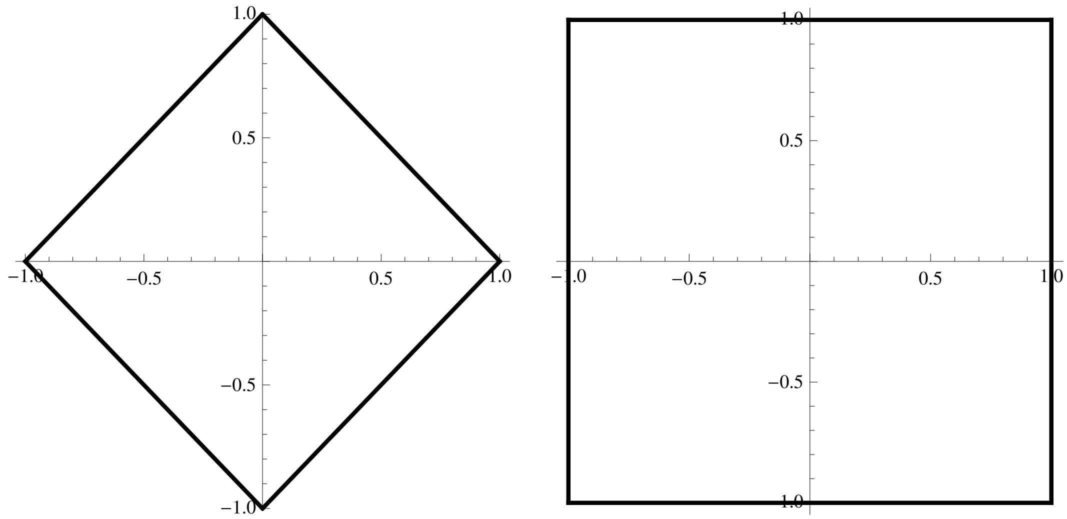
13. The unit circle in $\mathbb{R}^2$ relative to the sum norm is

$$\{(x,y) \;:\; |x| + |y| = 1\} = \{(x,y) \;:\; x+y=1 \text{ or } x-y=1 \text{ or } -x+y=1 \text{ or } -x-y=1\}.$$

The unit circle in $\mathbb{R}^2$ relative to the max norm is

$$\{(x,y) \;:\; \max\{|x|, |y|\} = 1\}.$$

These unit circles are shown below:



14. For example, in $\mathbb{R}^2$, let $\mathbf{u} = \langle 2, 1\rangle$ and $\mathbf{v} = \langle 1, 2\rangle$. Then $\|\mathbf{u}\| = \|\mathbf{v}\| = 2 + 1 = 3$. Also, $\|\mathbf{u} + \mathbf{v}\| = \|\langle 3, 3\rangle\| = 6$ and $\|\mathbf{u} - \mathbf{v}\| = \|\langle 1, -1\rangle\| = 2$. But then

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = 3^2 + 3^2 = 18 \neq \frac{1}{2}\|\mathbf{u} + \mathbf{v}\|^2 + \frac{1}{2}\|\mathbf{u} - \mathbf{v}\|^2 = 18 + 2 = 20.$$

15. Clearly $\left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\| \geq 0$, and since $\max\{|2a|, |3b|\} = 0$ if and only if both a and b are zero, we see that the norm is zero only for the zero vector. Thus property (1) is satisfied. For property 2,

$$\left\|c\begin{bmatrix} a \\ b \end{bmatrix}\right\| = \left\|\begin{bmatrix} ca \\ cb \end{bmatrix}\right\| = \max\{|2ac|, |3bc|\} = |c|\max\{|2a|, |3b|\}.$$

Finally,

$$\left\|\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right\| = \left\|\begin{bmatrix} a + c \\ b + d \end{bmatrix}\right\| = \max\{|2(a + c)|, |3(b + d)|\}.$$

Now use the triangle inequality:

$$\max\{|2(a + c)|, |3(b + d)|\} \leq \max\{|2a| + |2c|, |3b| + |3d|\} \leq \max\{|2a|, |3b|\} + \max\{|2c|, |3d|\}.$$

But the last expression is just $\left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\| + \left\|\begin{bmatrix} c \\ d \end{bmatrix}\right\|$, and we are done.

16. Clearly $\max_{i,j}\{|a_{ij}|\} \geq 0$, and is zero only when all of the a_{ij} are zero, so when $A = O$. Thus property (1) is satisfied. For property (2),

$$\|cA\| = \max_{i,j}\{|ca_{ij}|\} = \max_{i,j}\{|c|\,|a_{ij}|\} = |c|\max_{i,j}\{|a_{ij}|\} = |c|\,\|A\|.$$

For property (3), using the triangle inequality for absolute values gives

$$\|A + B\| = \max_{i,j}\{|a_{ij} + b_{ij}|\} \leq \max_{i,j}\{|a_{ij}| + |b_{ij}|\} \leq \max_{i,j}\{|a_{ij}|\} + \max_{i,j}\{|b_{ij}|\} = \|A\| + \|B\|.$$

17. Clearly $\|f\| \geq 0$, since $|f| \geq 0$. Since f is continuous, $|f|$ is also continuous, so its integral is zero if and only if it is identically zero. So property (1) holds. For property (2), use standard properties of integrals:

$$\|cf\| = \int_0^1 |cf(x)|\ dx = \int_0^1 |c| \cdot |f(x)|\ dx = |c| \int_0^1 |f(x)|\ dx = |c|\,\|f\|\,.$$

Finally, property (3) relies on the triangle inequality for absolute value:

$$\|f+g\| = \int_0^1 |f(x)+g(x)|\ dx \leq \int_0^1 (|f(x)|+|g(x)|)\ dx = \int_0^1 |f(x)|\ dx + \int_0^1 |g(x)|\ dx = \|f\| + \|g\|\,.$$

18. Since the norm is the maximum of a set of nonnegative numbers, it is always nonnegative, and is zero exactly when all of the numbers are zero, which means that f is the zero function on $[0,1]$. So property (1) holds. For property (2), we have

$$\|cf\| = \max_{0\leq x\leq 1}\{|cf(x)|\} = \max_{0\leq x\leq 1}\{|c|\,|f(x)|\} = |c| \max_{0\leq x\leq 1}\{|f(x)|\} = |c|\,\|f\|\,.$$

Finally, property (3) relies on the triangle inequality for absolute values:

$$\|f+g\| = \max_{0\leq x\leq 1}\{|f(x)+g(x)|\} \leq \max_{0\leq x\leq 1}\{|f(x)|+|g(x)|\} \leq \max_{0\leq x\leq 1}\{|f(x)|\}+\max_{0\leq x\leq 1}\{|g(x)|\} = \|f\|+\|g\|\,.$$

19. By property (2) of the definition of a norm,

$$\mathrm{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = |-1|\,\|\mathbf{u}-\mathbf{v}\| = \|(-1)(\mathbf{u}-\mathbf{v})\| = \|\mathbf{v}-\mathbf{u}\| = \mathrm{d}(\mathbf{v},\mathbf{u}).$$

20. The Frobenius norm is $\|A\|_F = \sqrt{2^2+3^2+4^2+1^2} = \sqrt{30}$. From Theorem 7.7 and the comment following, $\|A\|_1$ is the largest column sum of $|A|$, which is $2+4=6$, while $\|A\|_\infty$ is the largest row sum of $|A|$, which is $2+3=4+1=5$.

21. The Frobenius norm is $\|A\|_F = \sqrt{0^2+(-1)^2+(-3)^2+3^2} = \sqrt{19}$. From Theorem 7.7 and the comment following, $\|A\|_1$ is the largest column sum of $|A|$, which is $|-1|+3=4$, while $\|A\|_\infty$ is the largest row sum, which is $|-3|+3=6$.

22. The Frobenius norm is $\|A\|_F = \sqrt{1^2+5^2+(-2)^2+(-1)^2} = \sqrt{31}$. From Theorem 7.7 and the comment following, $\|A\|_1$ is the largest column sum of $|A|$, which is $5+|-1|=6$, while $\|A\|_\infty$ is the largest row sum, which is $1+5=6$.

23. The Frobenius norm is $\|A\|_F = \sqrt{2^2+1^2+1^2+1^2+3^2+2^2+1^2+1^2+3^2} = \sqrt{31}$. From Theorem 7.7 and the comment following, $\|A\|_1$ is the largest column sum of $|A|$, which is $1+2+3=6$, while $\|A\|_\infty$ is the largest row sum, which is $1+3+2=6$.

24. The Frobenius norm is $\|A\|_F = \sqrt{0^2+(-5)^2+2^2+3^2+1^2+(-3)^2+(-4)^2+(-4)^2+3^2} = \sqrt{89}$. From Theorem 7.7 and the comment following, $\|A\|_1$ is the largest column sum of $|A|$, which is $|-5|+1+|-4|=10$, while $\|A\|_\infty$ is the largest row sum, which is $|-4|+|-4|+3=11$.

25. The Frobenius norm is $\|A\|_F = \sqrt{4^2+(-2)^2+(-1)^2+0^2+(-1)^2+2^2+3^2+(-3)^2+0^2} = \sqrt{44} = 2\sqrt{11}$. From Theorem 7.7 and the comment following, $\|A\|_1$ is the largest column sum of $|A|$, which is $4+0+3=7$, while $\|A\|_\infty$ is the largest row sum, which is $4+|-2|+|-1|=7$.

26. Since $\|A\|_1 = 6$, which is the sum of the first column, such a vector $\mathbf{x}$ is $\mathbf{e}_1$, since

$$\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}2\\ 4\end{bmatrix}, \text{ and } \left\|\begin{bmatrix}2\\ 4\end{bmatrix}\right\|_s = 6.$$

Since $\|A\|_\infty = 5$, which is the sum of (say) the first row, let $\mathbf{y} = \begin{bmatrix}1\\ 1\end{bmatrix}$; then

$$\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}5\\ 5\end{bmatrix}, \text{ and } \left\|\begin{bmatrix}5\\ 5\end{bmatrix}\right\|_m = 5.$$

27. Since $\|A\|_1 = 4$, which is the sum of the magnitudes of the second column, such a vector $\mathbf{x}$ is $\mathbf{e}_2$, since

$$\begin{bmatrix} 0 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \text{ and } \left\| \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\|_s = |-1| + 3 = 4.$$

Since $\|A\|_\infty = 6$, which is the sum of the magnitudes of the second row, let $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ since the first column in that row is negative; then

$$\begin{bmatrix} 0 & -1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}, \text{ and } \left\| \begin{bmatrix} -1 \\ 6 \end{bmatrix} \right\|_m = 6.$$

28. Since $\|A\|_1 = 6$, which is the sum of the magnitudes of the second column, such a vector $\mathbf{x}$ is $\mathbf{e}_2$, since

$$\begin{bmatrix} 1 & 5 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \text{ and } \left\| \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right\|_s = 5 + |-1| = 6.$$

Since $\|A\|_\infty = 6$, which is the sum of the first row, let $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; then

$$\begin{bmatrix} 1 & 5 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}, \text{ and } \left\| \begin{bmatrix} 6 \\ -3 \end{bmatrix} \right\|_m = 6.$$

29. Since $\|A\|_1 = 6$, which is the sum of the magnitudes of the third column, such a vector $\mathbf{x}$ is $\mathbf{e}_3$, since

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{ and } \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|_s = 6.$$

Since $\|A\|_\infty = 6$, which is the sum of the second row, let $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; then

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix}, \text{ and } \left\| \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix} \right\|_m = 6.$$

30. Since $\|A\|_1 = 10$, which is the sum of the magnitudes of the second column, such a vector $\mathbf{x}$ is $\mathbf{e}_2$, since

$$\begin{bmatrix} 0 & -5 & 2 \\ 3 & 1 & -3 \\ -4 & -4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}, \text{ and } \left\| \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix} \right\|_s = 10.$$

Since $\|A\|_\infty = 11$, which is the sum of the magnitudes of the third row, let $\mathbf{y} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ since the entries in the first two columns are negative; then

$$\begin{bmatrix} 0 & -5 & 2 \\ 3 & 1 & -3 \\ -4 & -4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ 11 \end{bmatrix}, \text{ and } \left\| \begin{bmatrix} 7 \\ -7 \\ 11 \end{bmatrix} \right\|_m = 11.$$

31. Since $\|A\|_1 = 7$, which is the sum of the magnitudes of the first column, such a vector $\mathbf{x}$ is $\mathbf{e}_1$, since

$$\begin{bmatrix} 4 & -2 & -1 \\ 0 & -1 & 2 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \text{ and } \left\| \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \right\|_s = 7.$$

Since $\|A\|_\infty = 7$, which is the sum of the magnitudes of the first row, let $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ since the entries in the last two columns are negative; then

$$\begin{bmatrix} 4 & -2 & -1 \\ 0 & -1 & 2 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 6 \end{bmatrix}, \text{ and } \left\| \begin{bmatrix} 7 \\ -1 \\ 6 \end{bmatrix} \right\|_m = 7.$$

32. Let $\mathbf{A}_i$ be the i^{th} row of A, let $M = \max_{j=1,2,\ldots,n} \{\|\mathbf{A}_j\|_s\}$ be the maximum absolute row sum, and let $\mathbf{x}$ be any vector with $\|\mathbf{x}\|_m = 1$ (which means that $|x_i| \le 1$ for all i). Then

$$\begin{aligned} \|A\mathbf{x}\|_m &= \max_i \{|\mathbf{A}_i \mathbf{x}|\} \\ &= \max_i \{|a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n|\} \\ &\le \max_i \{|a_{i1}|\,|x_1| + |a_{i2}|\,|x_2| + \cdots + |a_{i1}|\,|x_n|\} \\ &\le \max_i \{|a_{i1}| + |a_{i2}| + \cdots + |a_{in}|\} \\ &= \max_i \{\|\mathbf{A}_i\|_s\} = M. \end{aligned}$$

Thus $\|A\mathbf{x}\|_m$ is at most equal to the maximum absolute row sum. Next, we find a specific unit vector $\mathbf{x}$ so that it is actually equal to M. Let row k be the row of A that gives the maximum sum norm; that is, such that $\|\mathbf{A}_k\|_s = M$. Let

$$y_j = \begin{cases} 1 & A_{kj} \ge 0 \\ -1 & A_{kj} < 0 \end{cases}, \; j = 1, 2, \ldots, n.$$

Then with $\mathbf{y} = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^T$, we have $\|\mathbf{y}\|_m = 1$, and we have corrected any negative signs in row k, so that $a_{kj}y_j \ge 0$ for all j and therefore the k^{th} entry of $A\mathbf{y}$ is

$$\sum_j a_{kj}y_j = \sum_j |a_{kj}y_j| = \sum_j |a_{kj}|\,|y_j| = \sum_j |a_{kj}| = \|\mathbf{A}_k\|_s = M.$$

So we have shown that the ∞ norm of A is bounded by the maximum row sum, and then found a vector $\mathbf{x}$ for which $\|A\mathbf{x}\| \le \|A\|_\infty$ is equal to the maximum row sum. Thus $\|A\|_\infty$ is equal to the maximum row sum, as desired.

33. **(a)** If $\|\cdot\|$ is an operator norm, then $\|I\| = \max_{\|\hat{\mathbf{x}}\|=1} \|I\hat{\mathbf{x}}\| = \max_{\|\hat{\mathbf{x}}\|=1} \|\hat{\mathbf{x}}\| = 1$.

(b) No, there is not, for $n > 1$, since in the Frobenius norm, $\|I\|_F = \sqrt{\sum 1^2} = \sqrt{n} \ne 1$. When $n = 1$, the Frobenius norm is the same as the absolute value norm on the real number line.

34. Let λ be an eigenvalue, and $\mathbf{v}$ a corresponding unit vector. Normalize $\mathbf{v}$ to $\hat{\mathbf{v}}$. Then $A\hat{\mathbf{v}} = \lambda\hat{\mathbf{v}}$, so that

$$\|A\| = \max_{\|\hat{\mathbf{x}}\|=1} \|A\hat{\mathbf{x}}\| \ge \|A\hat{\mathbf{v}}\| = \|\lambda\hat{\mathbf{v}}\| = |\lambda|\,\|\hat{\mathbf{v}}\| = |\lambda|\,.$$

35. Computing the inverse gives

$$A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}.$$

Then

$$\begin{aligned} \operatorname{cond}_1(A) &= \|A\|_1 \left\|A^{-1}\right\|_1 = (3+4)\,(1+|-2|) = 21 \\ \operatorname{cond}_\infty(A) &= \|A\|_\infty \left\|A^{-1}\right\|_\infty = (4+2)\left(|-2| + \frac{3}{2}\right) = 21. \end{aligned}$$

The matrix is well-conditioned.

36. Since $\det A = 0$, by definition $\operatorname{cond}_1(A) = \operatorname{cond}_\infty(A) = \infty$, so this matrix is ill-conditioned.

37. Computing the inverse gives

$$A^{-1} = \begin{bmatrix} 100 & -99 \\ -100 & 100 \end{bmatrix}.$$

Then

$$\begin{aligned} \operatorname{cond}_1(A) &= \|A\|_1 \left\|A^{-1}\right\|_1 = (1+1)(100+|-100|) = 400 \\ \operatorname{cond}_\infty(A) &= \|A\|_\infty \left\|A^{-1}\right\|_\infty = (1+1)(100+|-100|) = 400. \end{aligned}$$

The matrix is ill-conditioned.

38. Computing the inverse gives

$$A^{-1} = \begin{bmatrix} \frac{2001}{50} & -2 \\ -\frac{3001}{100} & \frac{3}{2} \end{bmatrix}.$$

Then

$$\begin{aligned} \operatorname{cond}_1(A) &= \|A\|_1 \left\|A^{-1}\right\|_1 = (200+4002)\left(\frac{2001}{50} + \frac{3001}{100}\right) = 294266 \\ \operatorname{cond}_\infty(A) &= \|A\|_\infty \left\|A^{-1}\right\|_\infty = (3001+4002)\left(\frac{2001}{50} + 2\right) = 294266. \end{aligned}$$

The matrix is ill-conditioned.

39. Computing the inverse gives

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 6 & -1 & -1 \\ -5 & 1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \operatorname{cond}_1(A) &= \|A\|_1 \left\|A^{-1}\right\|_1 = (1+5+1)(0+6+5) = 77 \\ \operatorname{cond}_\infty(A) &= \|A\|_\infty \left\|A^{-1}\right\|_\infty = (5+5+6)(6+1+1) = 128. \end{aligned}$$

The matrix is moderately ill-conditioned.

40. Computing the inverse gives

$$A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}.$$

Then

$$\begin{aligned} \operatorname{cond}_1(A) &= \|A\|_1 \left\|A^{-1}\right\|_1 = \left(1 + \frac{1}{2} + \frac{1}{3}\right)(36+192+180) = 748 \\ \operatorname{cond}_\infty(A) &= \|A\|_\infty \left\|A^{-1}\right\|_\infty = \left(1 + \frac{1}{2} + \frac{1}{3}\right)(36+192+180) = 748. \end{aligned}$$

The matrix is ill-conditioned.

41. **(a)** Computing the inverse gives

$$A^{-1} = \frac{1}{1-k}\begin{bmatrix} 1 & -k \\ -1 & 1 \end{bmatrix}.$$

Then

$$\operatorname{cond}_\infty(A) = \|A\|_\infty \left\|A^{-1}\right\|_\infty = \max\{|k|+1, 2\} \max\left\{\left|\frac{k}{k-1}\right| + \left|\frac{1}{k-1}\right|, \left|\frac{2}{k-1}\right|\right\}.$$

(b) As $k \to 1$, since the second factor goes to ∞, $\operatorname{cond}_\infty(A) \to \infty$. This makes sense since when $k = 1$, A is not invertible.

42. Since the matrix norm is compatible, we have from the definition that $\|A\mathbf{x}\| \le \|A\|\,\|\mathbf{x}\|$ for any $\mathbf{x}$, so that

$$\|\mathbf{b}\| = \left\|AA^{-1}\mathbf{b}\right\| \le \|A\|\left\|A^{-1}\mathbf{b}\right\|, \text{ so that } \frac{1}{\|A^{-1}\mathbf{b}\|} \le \frac{\|A\|}{\|\mathbf{b}\|}.$$

Then

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\left\|A^{-1}\Delta\mathbf{b}\right\|}{\|A^{-1}\mathbf{b}\|} \le \frac{\left\|A^{-1}\right\|\|\Delta\mathbf{b}\|}{\|A^{-1}\mathbf{b}\|} \le \frac{\|A\|}{\|\mathbf{b}\|}\left\|A^{-1}\right\|\|\Delta\mathbf{b}\| = \operatorname{cond}(A)\frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}.$$

43. **(a)** Since

$$A^{-1} = \begin{bmatrix} -\frac{9}{10} & 1 \\ 1 & -1 \end{bmatrix},$$

we have $\operatorname{cond}_\infty(A) = (10 + 10)(1 + 1) = 40$.

(b) Here

$$\Delta A = \begin{bmatrix} 10 & 10 \\ 10 & 11 \end{bmatrix} - \begin{bmatrix} 10 & 10 \\ 10 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \|\Delta A\| = 2.$$

Therefore

$$\frac{\|\Delta x\|_m}{\|x'\|_m} \le \operatorname{cond}_\infty(A)\frac{\|\Delta A\|_\infty}{\|A\|_\infty} = 40 \cdot \frac{2}{20},$$

so that this can produce at most a change of a factor of 4, or a 400% relative change.

(c) Row-reducing gives

$$\begin{aligned}
\left[A \mid \mathbf{b}\right] &= \left[\begin{array}{cc|c} 10 & 10 & 100 \\ 10 & 9 & 99 \end{array}\right] &\to& \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 1 \end{array}\right] \\
\left[A' \mid \mathbf{b}\right] &= \left[\begin{array}{cc|c} 10 & 10 & 100 \\ 10 & 11 & 99 \end{array}\right] &\to& \left[\begin{array}{cc|c} 1 & 0 & 11 \\ 0 & 1 & -1 \end{array}\right].
\end{aligned}$$

Thus

$$\mathbf{x} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix} 11 \\ -1 \end{bmatrix}, \text{ so that } \Delta\mathbf{x} = \mathbf{x}' - \mathbf{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

and $\frac{\|\Delta\mathbf{x}\|_m}{\|\mathbf{x}\|_m} = \frac{2}{9}$, for approximately a 22% relative error.

(d) Using Exercise 42,

$$\Delta\mathbf{b} = \begin{bmatrix} 100 \\ 101 \end{bmatrix} - \begin{bmatrix} 100 \\ 99 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

so that $\|\Delta\mathbf{b}\|_m = 2$ and

$$\frac{\|\Delta\mathbf{x}\|_m}{\|\mathbf{x}\|_m} \le \operatorname{cond}_\infty(A)\frac{\|\Delta\mathbf{b}\|_m}{\|\mathbf{b}\|_m} = 40 \cdot \frac{2}{100} = 0.8,$$

so that at most an 80% change can result.

(e) Row-reducing $\left[A \mid \mathbf{b}\right]$ gives the same result as above, while

$$\left[A \mid \mathbf{b}'\right] = \begin{bmatrix} 10 & 10 & 100 \\ 10 & 9 & 101 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -1 \end{bmatrix}.$$

Thus

$$\Delta\mathbf{x} = \begin{bmatrix} 11 \\ -1 \end{bmatrix} - \begin{bmatrix} 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

and again $\frac{\|\Delta\mathbf{x}\|_m}{\|\mathbf{x}\|_m} = \frac{2}{9}$, for about a 22% actual relative error.

44. **(a)** Since

$$A^{-1} = \begin{bmatrix} -10 & 3 & 5 \\ 4 & -1 & -2 \\ 7 & -2 & -3 \end{bmatrix},$$

we have $\operatorname{cond}_1(A) = (1 + 5 + |-1|)(|-10| + 4 + 7) = 147$.

(b) Here

$$\Delta A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 0 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \|\Delta A\|_1 = 1.$$

Therefore

$$\frac{\|\Delta x\|_s}{\|x'\|_s} \le \operatorname{cond}_1(A)\frac{\|\Delta A\|_1}{\|A\|_1} = 147 \cdot \frac{1}{7} = 21,$$

so that this can produce at most a change of a factor of 21, or a 2100% relative change.

(c) Row-reducing gives

$$\left[A \mid \mathbf{b}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 5 & 0 & 2 \\ 1 & -1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

$$\left[A' \mid \mathbf{b}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 5 & 0 & 2 \\ 1 & -1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{11}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 5 \end{array}\right].$$

Thus

$$\mathbf{x} = \begin{bmatrix} 11 \\ -4 \\ -6 \end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix} -\frac{11}{2} \\ \frac{3}{2} \\ 5 \end{bmatrix}, \text{ so that } \Delta\mathbf{x} = \mathbf{x}' - \mathbf{x} = \begin{bmatrix} -\frac{33}{2} \\ \frac{11}{2} \\ 11 \end{bmatrix},$$

and $\frac{\|\Delta\mathbf{x}\|_s}{\|\mathbf{x}\|_s} = \frac{33}{21} \approx 1.57$, for approximately a 157% actual relative error.

(d) Using Exercise 42,

$$\Delta\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},$$

so that $\|\Delta\mathbf{b}\|_s = 1$ and

$$\frac{\|\Delta\mathbf{x}\|_s}{\|\mathbf{x}\|_s} \le \operatorname{cond}_1(A)\frac{\|\Delta\mathbf{b}\|_s}{\|\mathbf{b}\|_s} = 147 \cdot \frac{1}{6} = 24.5,$$

so that at most a 2450% relative change can result.

(e) Row-reducing $\left[A \mid \mathbf{b}\right]$ gives the same result as above, while

$$\left[A \mid \mathbf{b}'\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 5 & 0 & 1 \\ 1 & -1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

Thus

$$\Delta\mathbf{x} = \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 11 \\ -4 \\ -6 \end{bmatrix} = \begin{bmatrix} -15 \\ 5 \\ 10 \end{bmatrix}$$

and $\frac{\|\Delta\mathbf{x}\|_s}{\|\mathbf{x}\|_s} = \frac{30}{21} \approx 1.43$, for about a 143% actual relative error.

45. From Exercise 33(a), $1 = \|I\| = \left\|A^{-1}A\right\| \le \left\|A^{-1}\right\| \|A\| = \operatorname{cond}(A)$.

46. From Exercise 33(a),

$$\operatorname{cond}(AB) = \left\|(AB)^{-1}\right\| \|AB\| = \left\|B^{-1}A^{-1}\right\| \|AB\| \le \left\|A^{-1}\right\| \|A\| \left\|B^{-1}\right\| \|B\| = \operatorname{cond}(A)\operatorname{cond}(B).$$

47. From Theorem 4.18 (b) in Chapter 4, if λ_n is an eigenvalue of A, then $\frac{1}{\lambda_n}$ is an eigenvalue of A^{-1}. By Exercise 34, $\|A\| \ge |\lambda_1|$ and $\left\|A^{-1}\right\| \ge \left|\frac{1}{\lambda_n}\right|$, so that

$$\operatorname{cond}(A) = \left\|A^{-1}\right\| \|A\| \ge \frac{|\lambda_1|}{|\lambda_n|}.$$

48. With

$$A = \begin{bmatrix} 7 & -1 \\ 1 & -5 \end{bmatrix},$$

we have

$$M = -D^{-1}(L+U) = -\begin{bmatrix} 7 & 0 \\ 0 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{1}{5} & 0 \end{bmatrix},$$

so that $\|M\|_\infty = \frac{1}{5}$. Then

$$\mathbf{b} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} \quad\Rightarrow\quad \mathbf{c} = D^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{6}{7} \\ \frac{4}{5} \end{bmatrix} \quad\Rightarrow\quad \|\mathbf{c}\|_m = \frac{6}{7}.$$

Note that $\mathbf{x}_0 = 0$, so that $\mathbf{x}_1 = \mathbf{c}$ and then $\|M\|_\infty^k \|\mathbf{x}_1 - \mathbf{x}_0\|_m = \|M\|_\infty^k \|\mathbf{c}\|_m < 0.0005$ means that $k > \frac{4+\log_{10}\left(\|\mathbf{c}\|_m/5\right)}{\log_{10}\left(1/\|M\|_\infty\right)}$. In this case, with $\|M\|_\infty = \frac{1}{5}$ and $\|\mathbf{c}\|_m = \frac{6}{7}$, we get

$$k > \frac{4+\log_{10}(6/35)}{\log_{10} 5} \approx 4.6, \text{ so that } k \ge 5.$$

Computing $\mathbf{x}_5$ using $\mathbf{x}_{k+1} = M\mathbf{x}_k + \mathbf{c}$ gives $\mathbf{x}_5 = \begin{bmatrix} 0.999 \\ 0.999 \end{bmatrix}$, as compared with an actual solution of $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

49. With

$$A = \begin{bmatrix} 4.5 & -0.5 \\ 1 & -3.5 \end{bmatrix},$$

we have

$$M = -D^{-1}(L+U) = -\begin{bmatrix} 4.5 & 0 \\ 0 & -3.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} & 0 \\ 0 & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{9} \\ \frac{2}{7} & 0 \end{bmatrix},$$

so that $\|M\|_\infty = \frac{2}{7}$. Then

$$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad\Rightarrow\quad \mathbf{c} = D^{-1}\mathbf{b} = \begin{bmatrix} \frac{2}{9} & 0 \\ 0 & -\frac{2}{7} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} \\ \frac{2}{7} \end{bmatrix} \quad\Rightarrow\quad \|\mathbf{c}\|_m = \frac{2}{7}.$$

Note that $\mathbf{x}_0 = 0$, so that $\mathbf{x}_1 = \mathbf{c}$ and then $\|M\|_\infty^k \|\mathbf{x}_1 - \mathbf{x}_0\|_m = \|M\|_\infty^k \|\mathbf{c}\|_m < 0.0005$ means that $k > \frac{4+\log_{10}\left(\|\mathbf{c}\|_m/5\right)}{\log_{10}\left(1/\|M\|_\infty\right)}$. In this case, with $\|M\|_\infty = \frac{2}{7}$ and $\|\mathbf{c}\|_m = \frac{2}{7}$, we get

$$k > \frac{4+\log_{10}(2/35)}{\log_{10}(7/2)} \approx 5.06, \text{ so that } k \ge 6.$$

Computing $\mathbf{x}_6$ using $\mathbf{x}_{k+1} = M\mathbf{x}_k + \mathbf{c}$ gives $\mathbf{x}_6 = \begin{bmatrix} 0.2622 \\ 0.3606 \end{bmatrix}$, as compared with an actual solution of $\mathbf{x} = \begin{bmatrix} 0.2623 \\ 0.3606 \end{bmatrix}$.

50. With

$$A = \begin{bmatrix} 20 & 1 & -1 \\ 1 & -10 & 1 \\ -1 & 1 & 10 \end{bmatrix},$$

we have

$$M = -D^{-1}(L+U) = -\begin{bmatrix} 20 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{20} & \frac{1}{20} \\ \frac{1}{10} & 0 & \frac{1}{10} \\ \frac{1}{10} & -\frac{1}{10} & 0 \end{bmatrix}$$

so that $\|M\|_\infty = \frac{2}{10} = \frac{1}{5}$. Then

$$\mathbf{b} = \begin{bmatrix} 17 \\ 13 \\ 18 \end{bmatrix} \quad \Rightarrow \quad \mathbf{c} = D^{-1}\mathbf{b} = \begin{bmatrix} 0 & \frac{1}{20} & -\frac{1}{20} \\ -\frac{1}{10} & 0 & -\frac{1}{10} \\ -\frac{1}{10} & \frac{1}{10} & 0 \end{bmatrix} \begin{bmatrix} 17 \\ 13 \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{17}{20} \\ -\frac{13}{20} \\ \frac{9}{5} \end{bmatrix} \quad \Rightarrow \quad \|\mathbf{c}\|_m = \frac{9}{5}.$$

Note that $\mathbf{x}_0 = 0$, so that $\mathbf{x}_1 = \mathbf{c}$ and then $\|M\|_\infty^k \|\mathbf{x}_1 - \mathbf{x}_0\|_m = \|M\|_\infty^k \|\mathbf{c}\|_m < 0.0005$ means that $k > \frac{4+\log_{10}\left(\|\mathbf{c}\|_m/5\right)}{\log_{10}\left(1/\|M\|_\infty\right)}$. In this case, with $\|M\|_\infty = \frac{1}{5}$ and $\|\mathbf{c}\|_m = \frac{9}{5}$, we get

$$k > \frac{4 + \log_{10}(9/25)}{\log_{10} 5} \approx 5.08, \text{ so that } k \ge 6.$$

Computing $\mathbf{x}_6$ using $\mathbf{x}_{k+1} = M\mathbf{x}_k + \mathbf{c}$ gives $\mathbf{x}_6 = \begin{bmatrix} 0.999 \\ -1.000 \\ 1.999 \end{bmatrix}$, as compared with an actual solution of $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

51. With

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

we have

$$M = -D^{-1}(L+U) = -\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{3} & 0 \end{bmatrix}$$

so that $\|M\|_\infty = \frac{2}{4} = \frac{1}{2}$. Then

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{c} = D^{-1}\mathbf{b} = \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{3} \end{bmatrix} \quad \Rightarrow \quad \|\mathbf{c}\|_m = \frac{1}{3}.$$

Note that $\mathbf{x}_0 = 0$, so that $\mathbf{x}_1 = \mathbf{c}$ and then $\|M\|_\infty^k \|\mathbf{x}_1 - \mathbf{x}_0\|_m = \|M\|_\infty^k \|\mathbf{c}\|_m < 0.0005$ means that $k > \frac{4+\log_{10}\left(\|\mathbf{c}\|_m/5\right)}{\log_{10}\left(1/\|M\|_\infty\right)}$. In this case, with $\|M\|_\infty = \frac{1}{2}$ and $\|\mathbf{c}\|_m = \frac{1}{3}$, we get

$$k > \frac{4 + \log_{10}(1/15)}{\log_{10} 2} \approx 9.3, \text{ so that } k \ge 10.$$

Computing $\mathbf{x}_{10}$ using $\mathbf{x}_{k+1} = M\mathbf{x}_k + \mathbf{c}$ gives $\mathbf{x}_{10} = \begin{bmatrix} 0.299 \\ 0.099 \\ 0.299 \end{bmatrix}$, as compared with an actual solution of $\mathbf{x} = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.3 \end{bmatrix}$.

52. The sum and max norms are operator norms induced from the sum and max norms on $\mathbb{R}^n$, so they are matrix norms.

(a) If C and D are any two matrices and $\|\cdot\|$ is either the sum or max norm, then $\|CD\| \le \|C\| \, \|D\|$. Now assume A is such that $\|A\| = \delta < 1$. Then

$$\|A^n\| \le \|A\|^n = \delta^n.$$

Thus $\lim_{n\to\infty} \|A^n\| = \lim_{n\to\infty} \delta^n = 0$. So by property (1) in the definition of matrix norms, $A^n \to O$.

(b) A computation shows that

$$(I - A)(I + A + \cdots + A^n) = I - A^{n+1},$$

since all the other terms cancel in pairs. Thus

$$\begin{aligned} (I - A)(I + A + A^2 + A^3 + \cdots) &= (I - A) \lim_{n\to\infty} (I + A + \cdots + A^n) \\ &= \lim_{n\to\infty} ((I - A)(I + A + \cdots + A^n)) \\ &= \lim_{n\to\infty} \left(I - A^{n+1}\right) \\ &= I. \end{aligned}$$

It follows that $I - A$ is invertible and that its inverse is $I + A + A^2 + A^3 + \cdots$.

(c) Recall that a consumption matrix C is a matrix such that all entries are nonnegative and the sum of the entries in each column does not exceed 1. A consumption matrix C is productive if $I - C$ is invertible and $(I - C)^{-1} \ge O$; that is, if it has only nonnegative entries.

To prove Corollary 3.35, use $\|\cdot\|_\infty$, the max norm. Note that since $C \ge O$, the sum of the absolute values of the entries of a row is the same as the sum of the entries of the row. If the sum of each row of C is less than 1, then $\|C\|_\infty < 1$, so by part (b), $I - C$ is invertible. Since $C \ge O$, it is clear that each entry of $(I - C)^{-1} = I + C + C^2 + C^3 + \cdots$ is nonnegative, so that $(I - C)^{-1} \ge O$. Thus C is productive.

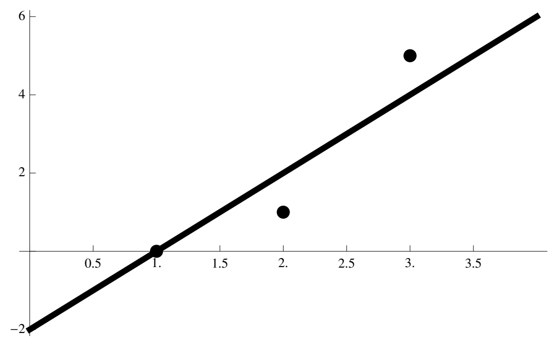
The proof of Corollary 3.36 is similar, but it uses $\|\cdot\|_1$, the sum norm. If the sum of each column of C is less than 1, then $\|C\|_1 < 1$. The rest of the argument is identical to that in the preceding paragraph.

7.3 Least Squares Approximation

1. The corresponding data points predicted by the line $y = -2 + 2x$ are $(1, 0)$, $(2, 2)$, and $(3, 4)$, so the least squares error is

$$\sqrt{(0-0)^2 + (2-1)^2 + (4-5)^2} = \sqrt{2}.$$

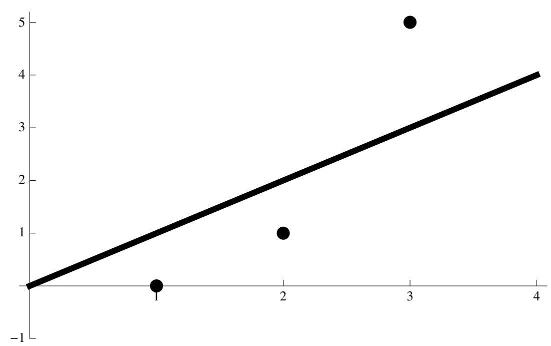
A plot of the points and the line on the same set of axes is



2. The corresponding data points predicted by the line $y = x$ are $(1, 1)$, $(2, 2)$, and $(3, 3)$, so the least squares error is

$$\sqrt{(1-0)^2 + (2-1)^2 + (3-5)^2} = \sqrt{6}.$$

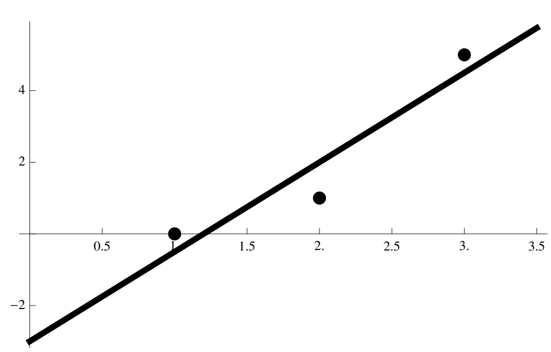
A plot of the points and the line on the same set of axes is



3. The corresponding data points predicted by the line $y = -3 + \frac{5}{2}x$ are $\left(1, -\frac{1}{2}\right)$, $(2, 2)$, and $\left(3, \frac{9}{2}\right)$, so the least squares error is

$$\sqrt{\left(-\frac{1}{2} - 0\right)^2 + (2-1)^2 + \left(\frac{9}{2} - 5\right)^2} = \frac{\sqrt{6}}{2}.$$

A plot of the points and the line on the same set of axes is



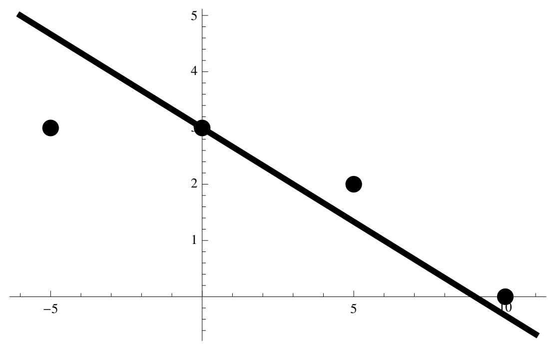
4. The corresponding data points predicted by the line $y = 3 - \frac{1}{3}x$ are

$$\left(-5, 3 - \frac{1}{3}(-5)\right) = \left(-5, \frac{14}{3}\right), \qquad \left(0, 3 - \frac{1}{3} \cdot 0\right) = (0, 3),$$
$$\left(5, 3 - \frac{1}{3} \cdot 5\right) = \left(5, \frac{4}{3}\right), \qquad \left(10, 3 - \frac{1}{3} \cdot 10\right) = \left(10, -\frac{1}{3}\right),$$

so the least squares error is

$$\sqrt{\left(3 - \frac{14}{3}\right)^2 + (3-3)^2 + \left(2 - \frac{4}{3}\right)^2 + \left(0 - \left(-\frac{1}{3}\right)\right)^2} = \sqrt{\frac{25}{9} + 0 + \frac{4}{9} + \frac{1}{9}} = \frac{\sqrt{30}}{3}.$$

A plot of the points and the line on the same set of axes is



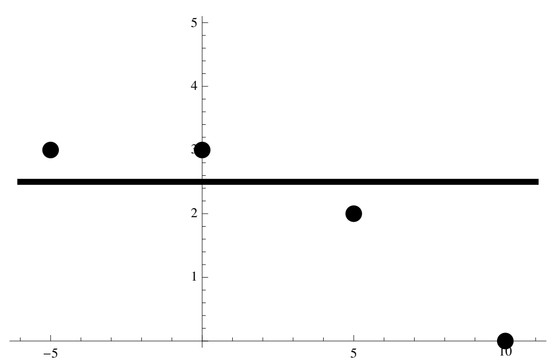
5. The corresponding data points predicted by the line $y = 3 - \frac{1}{3}x$ are

$$\left(-5, \frac{5}{2}\right), \quad \left(0, \frac{5}{2}\right), \quad \left(5, \frac{5}{2}\right), \quad \left(10, \frac{5}{2}\right),$$

so the least squares error is

$$\sqrt{\left(3 - \frac{5}{2}\right)^2 + \left(3 - \frac{5}{2}\right)^2 + \left(2 - \frac{5}{2}\right)^2 + \left(0 - \left(\frac{5}{2}\right)\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{25}{4}} = \sqrt{7}.$$

A plot of the points and the line on the same set of axes is



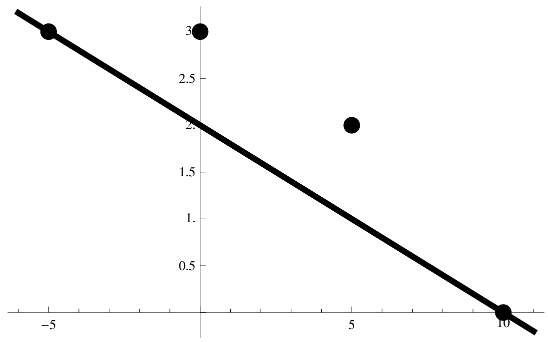
6. The corresponding data points predicted by the line $y = 2 - \frac{1}{5}x$ are

$$(-5, 3), \quad (0, 2), \quad (5, 1), \quad (10, 0),$$

so the least squares error is

$$\sqrt{(3-3)^2 + (3-2)^2 + (2-1)^2 + (0-0)\,6^2} = \sqrt{2}.$$

A plot of the points and the line on the same set of axes is



7. We are looking for a line $y = a + bx$ that best fits the given points. Using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}.$$

Since

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \Rightarrow (A^T A)^{-1} = \begin{bmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{5}{2} \end{bmatrix}.$$

Thus the least squares approximation is $y = -3 + \frac{5}{2}x$. To find the least squares error, compute

$$\mathbf{e} = \mathbf{b} - A\overline{\mathbf{x}} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}.$$

Then $\|\mathbf{e}\| = \sqrt{\left(\frac{1}{2}\right)^2 + (-1)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{6}}{2}$.

8. We are looking for a line $y = a + bx$ that best fits the given points. Using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}.$$

Since

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \Rightarrow (A^T A)^{-1} = \begin{bmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{25}{3} \\ -\frac{5}{2} \end{bmatrix}.$$

Thus the least squares approximation is $y = \frac{25}{3} - \frac{5}{2}x$. To find the least squares error, compute

$$\mathbf{e} = \mathbf{b} - A\overline{\mathbf{x}} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{25}{3} \\ -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \end{bmatrix}.$$

Then $\|\mathbf{e}\| = \sqrt{\left(\frac{1}{6}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)^2} = \frac{\sqrt{6}}{6}$.

9. We are looking for a line $y = a+bx$ that best fits the given points. Using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}.$$

Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{3} \\ -2 \end{bmatrix}.$$

Thus the least squares approximation is $y = \frac{11}{3} - 2x$. To find the least squares error, compute

$$\mathbf{e} = \mathbf{b} - A\overline{\mathbf{x}} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{11}{3} \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Then $\|\mathbf{e}\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{6}}{3}$.

10. We are looking for a line $y = a+bx$ that best fits the given points. Using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}.$$

Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ 1 \end{bmatrix}.$$

Thus the least squares approximation is $y = \frac{8}{3} + x$. To find the least squares error, compute

$$\mathbf{e} = \mathbf{b} - A\overline{\mathbf{x}} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{8}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Then $\|\mathbf{e}\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{6}}{3}$.

11. We are looking for a line $y = a+bx$ that best fits the given points. Using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix}.$$

Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 150 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{50} \\ -\frac{1}{50} & \frac{1}{125} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{50} \\ -\frac{1}{50} & \frac{1}{125} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{7}{10} \\ \frac{8}{25} \end{bmatrix}.$$

Thus the least squares approximation is $y = \frac{7}{10} + \frac{8}{25}x$. To find the least squares error, compute

$$\mathbf{e} = \mathbf{b} - A\overline{\mathbf{x}} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} \frac{7}{10} \\ \frac{8}{25} \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} \\ \frac{3}{10} \\ -\frac{3}{10} \\ \frac{1}{10} \end{bmatrix}.$$

Then $\|\mathbf{e}\| = \sqrt{\left(-\frac{1}{10}\right)^2 + \left(\frac{3}{10}\right)^2 + \left(-\frac{3}{10}\right)^2 + \left(\frac{1}{10}\right)^2} = \frac{\sqrt{5}}{5}$.

12. We are looking for a line $y = a+bx$ that best fits the given points. Using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 150 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{50} \\ -\frac{1}{50} & \frac{1}{125} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{50} \\ -\frac{1}{50} & \frac{1}{125} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{5} \end{bmatrix}.$$

Thus the least squares approximation is $y = \frac{5}{2} - \frac{1}{5}x$. To find the least squares error, compute

$$\mathbf{e} = \mathbf{b} - A\overline{\mathbf{x}} = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

Then $\|\mathbf{e}\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$.

13. We are looking for a line $y = a+bx$ that best fits the given points. Using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}.$$

Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{11}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{11}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{7}{5} \end{bmatrix}.$$

Thus the least squares approximation is $y = -\frac{1}{5} + \frac{7}{5}x$. To find the least squares error, compute

$$\mathbf{e} = \mathbf{b} - A\overline{\mathbf{x}} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} \\ \frac{7}{5} \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \\ 0 \\ -\frac{2}{5} \\ \frac{1}{5} \end{bmatrix}.$$

Then $\|\mathbf{e}\| = \sqrt{\left(-\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + 0^2 + \left(-\frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2} = \frac{\sqrt{10}}{5}$.

14. We are looking for a line $y = a+bx$ that best fits the given points. Using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 5 \\ 3 \\ 0 \end{bmatrix}.$$

Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{11}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{11}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 5 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{127}{10} \\ -\frac{5}{2} \end{bmatrix}.$$

Thus the least squares approximation is $y = \frac{127}{10} - \frac{5}{2}x$. To find the least squares error, compute

$$\mathbf{e} = \mathbf{b} - A\overline{\mathbf{x}} = \begin{bmatrix} 10 \\ 8 \\ 5 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \frac{127}{10} \\ -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{3}{10} \\ -\frac{1}{5} \\ \frac{3}{10} \\ -\frac{1}{5} \end{bmatrix}.$$

Then $\|\mathbf{e}\| = \sqrt{\left(-\frac{1}{5}\right)^2 + \left(\frac{3}{10}\right)^2 + \left(-\frac{1}{5}\right)^2 + \left(\frac{3}{10}\right)^2 + \left(-\frac{1}{5}\right)^2} = \frac{\sqrt{30}}{10}$.

15. Following the method of Example 7.28, we substitute the given values of x into the quadratic equation $a + bx + cx^2 = y$ to get

$$\begin{aligned} a + b + c &= 1 \\ a + 2b + 4c &= -2 \\ a + 3b + 9c &= 3 \\ a + 4b + 16c &= 4, \end{aligned}$$

which in matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}.$$

Thus we want the least squares approximation to $A\mathbf{x} = \mathbf{b}$, where A is the 4×3 matrix on the left and $\mathbf{b}$ is the matrix on the right in the above equation. Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{31}{4} & -\frac{27}{4} & \frac{5}{4} \\ -\frac{27}{4} & \frac{129}{20} & -\frac{5}{4} \\ \frac{5}{4} & -\frac{5}{4} & \frac{1}{4} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{31}{4} & -\frac{27}{4} & \frac{5}{4} \\ -\frac{27}{4} & \frac{129}{20} & -\frac{5}{4} \\ \frac{5}{4} & -\frac{5}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{18}{5} \\ 1 \end{bmatrix}.$$

Thus the least squares approximation is $y = 3 - \frac{18}{5}x + x^2$.

16. Following the method of Example 7.28, we substitute the given values of x into the quadratic equation $a + bx + cx^2 = y$ to get

$$\begin{aligned} a + b + c &= 6 \\ a + 2b + 4c &= 0 \\ a + 3b + 9c &= 0 \\ a + 4b + 16c &= 2, \end{aligned}$$

which in matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

Thus we want the least squares approximation to $A\mathbf{x} = \mathbf{b}$, where A is the 4×3 matrix on the left and $\mathbf{b}$ is the matrix on the right in the above equation. Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{31}{4} & -\frac{27}{4} & \frac{5}{4} \\ -\frac{27}{4} & \frac{129}{20} & -\frac{5}{4} \\ \frac{5}{4} & -\frac{5}{4} & \frac{1}{4} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{31}{4} & -\frac{27}{4} & \frac{5}{4} \\ -\frac{27}{4} & \frac{129}{20} & -\frac{5}{4} \\ \frac{5}{4} & -\frac{5}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ -\frac{56}{5} \\ 2 \end{bmatrix}.$$

Thus the least squares approximation is $y = 15 - \frac{56}{5}x + 2x^2$.

17. Following the method of Example 7.28, we substitute the given values of x into the quadratic equation $a + bx + cx^2 = y$ to get

$$\begin{aligned} a - 2b + 4c &= 4 \\ a - b + c &= 7 \\ a \phantom{{}- 2b + 4c} &= 3 \\ a + b + c &= 0 \\ a + 2b + 4c &= -1, \end{aligned}$$

which in matrix form is

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 3 \\ 0 \\ -1 \end{bmatrix}.$$

Thus we want the least squares approximation to $A\mathbf{x} = \mathbf{b}$, where A is the 5×3 matrix on the left and $\mathbf{b}$ is the matrix on the right in the above equation. Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{17}{35} & 0 & -\frac{1}{7} \\ 0 & \frac{1}{10} & 0 \\ -\frac{1}{7} & 0 & \frac{1}{14} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{17}{35} & 0 & -\frac{1}{7} \\ 0 & \frac{1}{10} & 0 \\ -\frac{1}{7} & 0 & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{18}{5} \\ -\frac{17}{10} \\ -\frac{1}{2} \end{bmatrix}.$$

Thus the least squares approximation is $y = \frac{18}{5} - \frac{17}{10}x - \frac{1}{2}x^2$.

18. Following the method of Example 7.28, we substitute the given values of x into the quadratic equation $a + bx + cx^2 = y$ to get

$$\begin{aligned} a - 2b + 4c &= 0 \\ a - b + c &= -11 \\ a &= -10 \\ a + b + c &= -9 \\ a + 2b + 4c &= 8, \end{aligned}$$

which in matrix form is

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -11 \\ -10 \\ -9 \\ 8 \end{bmatrix}.$$

Thus we want the least squares approximation to $A\mathbf{x} = \mathbf{b}$, where A is the 5×3 matrix on the left and $\mathbf{b}$ is the matrix on the right in the above equation. Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{17}{35} & 0 & -\frac{1}{7} \\ 0 & \frac{1}{10} & 0 \\ -\frac{1}{7} & 0 & \frac{1}{14} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{17}{35} & 0 & -\frac{1}{7} \\ 0 & \frac{1}{10} & 0 \\ -\frac{1}{7} & 0 & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -11 \\ -10 \\ -9 \\ 8 \end{bmatrix} = \begin{bmatrix} -\frac{62}{5} \\ \frac{9}{5} \\ 4 \end{bmatrix}.$$

Thus the least squares approximation is $y = -\frac{62}{5} + \frac{9}{5}x + 4x^2$.

19. The normal equation is $A^TA\mathbf{x} = A^T\mathbf{b}$. We have

$$A^TA = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix},$$

so that the normal form of this system is

$$\begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Thus

$$\mathbf{x} = \begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{11}{30} \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{7}{15} \end{bmatrix}$$

is the least squares solution.

20. The normal equation is $A^TA\mathbf{x} = A^T\mathbf{b}$. We have

$$A^TA = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -6 \\ -6 & 9 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix},$$

so that the normal form of this system is

$$\begin{bmatrix} 14 & -6 \\ -6 & 9 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}.$$

Thus

$$\mathbf{x} = \begin{bmatrix} 14 & -6 \\ -6 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{15} \\ \frac{1}{15} & \frac{7}{45} \end{bmatrix} \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{2}{15} \\ -\frac{1}{15} \end{bmatrix}$$

is the least squares solution.

21. The normal equation is $A^TA\mathbf{x} = A^T\mathbf{b}$. We have

$$A^TA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 8 & 38 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ -21 \end{bmatrix},$$

so that the normal form of this system is

$$\begin{bmatrix} 14 & 8 \\ 8 & 38 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 12 \\ -21 \end{bmatrix}.$$

Thus

$$\mathbf{x} = \begin{bmatrix} 14 & 8 \\ 8 & 38 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ -21 \end{bmatrix} = \begin{bmatrix} \frac{19}{234} & -\frac{2}{117} \\ -\frac{2}{117} & \frac{7}{234} \end{bmatrix} \begin{bmatrix} 12 \\ -21 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{5}{6} \end{bmatrix}$$

is the least squares solution.

22. The normal equation is $A^TA\mathbf{x} = A^T\mathbf{b}$. We have

$$A^TA = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \end{bmatrix},$$

so that the normal form of this system is

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 12 \\ -2 \end{bmatrix}.$$

Thus

$$\mathbf{x} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix} \begin{bmatrix} 12 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{22}{9} \\ \frac{8}{9} \end{bmatrix}$$

is the least squares solution.

23. Instead of trying to invert A^TA, we instead use an alternative method, row-reducing $\left[A^TA \mid A^T\mathbf{b}\right]$. We will see that A^TA does not reduce to the identity matrix, so that the system does not have a unique solution. We get

$$A^TA = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 0 & 3 & -2 & -1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 2 & 2 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \\ -1 \end{bmatrix}.$$

Row-reducing, we have

$$\left[\begin{array}{cccc|c} 3 & 0 & 2 & 1 & 2 \\ 0 & 3 & -2 & -1 & -5 \\ 2 & -2 & 3 & 2 & 3 \\ 1 & -1 & 2 & 2 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So A^TA is not invertible, and therefore there is not a unique solution. The general solution is

$$\begin{bmatrix} 4+t \\ -5-t \\ -5-2t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ -5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \end{bmatrix}.$$

24. Instead of trying to invert A^TA, we instead use an alternative method, row-reducing $\left[A^TA \mid A^T\mathbf{b}\right]$. We will see that A^TA does not reduce to the identity matrix, so that the system does not have a unique solution. We get

$$A^TA = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 & 0 \\ 0 & 3 & 1 & 2 \\ 3 & 1 & 4 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 8 \\ -2 \end{bmatrix}.$$

Row-reducing, we have

$$\left[\begin{array}{cccc|c} 3 & 0 & 3 & 0 & 3 \\ 0 & 3 & 1 & 2 & 3 \\ 3 & 1 & 4 & 0 & 8 \\ 0 & 2 & 0 & 2 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -5 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So A^TA is not invertible, and therefore there is not a unique solution. The general solution is

$$\begin{bmatrix} -5-t \\ -1-t \\ 6+t \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 6 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

25. Solving these equations corresponds to solving $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 3 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 6 \\ 11 \\ 0 \end{bmatrix}.$$

To solve, we row-reduce $\left[A^TA \mid A^T\mathbf{b}\right]$:

$$A^TA = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 1 & -1 & 2 & 0 \\ -1 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 3 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 7 & -5 \\ 7 & 6 & -5 \\ -5 & -5 & 7 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 1 & -1 & 2 & 0 \\ -1 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 11 \\ 0 \end{bmatrix} = \begin{bmatrix} 35 \\ 18 \\ -1 \end{bmatrix}.$$

Row-reducing, we have

$$\left[\begin{array}{ccc|c} 11 & 7 & -5 & 35 \\ 7 & 6 & -5 & 18 \\ -5 & -5 & 7 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{42}{11} \\ 0 & 1 & 0 & \frac{19}{11} \\ 0 & 0 & 1 & \frac{42}{11} \end{array}\right].$$

Thus the best approximation is

$$\overline{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{42}{11} \\ \frac{19}{11} \\ \frac{42}{11} \end{bmatrix}.$$

26. Solving these equations corresponds to solving $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 21 \\ 7 \\ 14 \\ 0 \end{bmatrix}.$$

To solve, we row-reduce $\left[A^TA \mid A^T\mathbf{b}\right]$:

$$A^TA = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 4 \\ 6 & 15 & 5 \\ 4 & 5 & 4 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 7 \\ 14 \\ 0 \end{bmatrix} = \begin{bmatrix} 35 \\ 84 \\ 14 \end{bmatrix}.$$

Row-reducing, we have

$$\left[\begin{array}{ccc|c} 6 & 6 & 4 & 35 \\ 6 & 15 & 5 & 84 \\ 4 & 5 & 4 & 14 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{469}{66} \\ 0 & 1 & 0 & \frac{224}{33} \\ 0 & 0 & 1 & -\frac{133}{11} \end{array}\right].$$

Thus the best approximation is

$$\overline{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{469}{66} \\ \frac{224}{33} \\ -\frac{133}{11} \end{bmatrix}.$$

27. With $A\mathbf{x} = QR\mathbf{x} = \mathbf{b}$, we get

$$R\mathbf{x} = Q^T\mathbf{b}, \quad \text{or} \quad \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Back-substitution gives the solution

$$\mathbf{x} = \begin{bmatrix} \frac{5}{3} \\ -2 \end{bmatrix}.$$

28. With $A\mathbf{x} = QR\mathbf{x} = \mathbf{b}$, we get

$$R\mathbf{x} = Q^T\mathbf{b}, \quad \text{or} \quad \begin{bmatrix} \sqrt{6} & -\frac{\sqrt{6}}{2} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \sqrt{2} \end{bmatrix}.$$

Back-substitution gives the solution

$$\mathbf{x} = \begin{bmatrix} \frac{4}{3} \\ 2 \end{bmatrix}.$$

29. We are looking for a line $y = a+bx$ that best fits the given points. Using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & 20 \\ 1 & 40 \\ 1 & 48 \\ 1 & 60 \\ 1 & 80 \\ 1 & 100 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 14.5 \\ 31 \\ 36 \\ 45.5 \\ 59 \\ 73.5 \end{bmatrix}.$$

Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 20 & 40 & 48 & 60 & 80 & 100 \end{bmatrix} \begin{bmatrix} 1 & 20 \\ 1 & 40 \\ 1 & 48 \\ 1 & 60 \\ 1 & 80 \\ 1 & 100 \end{bmatrix} = \begin{bmatrix} 6 & 348 \\ 348 & 24304 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{1519}{1545} & -\frac{29}{2060} \\ -\frac{29}{2060} & \frac{1}{4120} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{1519}{1545} & -\frac{29}{2060} \\ -\frac{29}{2060} & \frac{1}{4120} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 20 & 40 & 48 & 60 & 80 & 100 \end{bmatrix} \begin{bmatrix} 14.5 \\ 31 \\ 36 \\ 45.5 \\ 59 \\ 73.5 \end{bmatrix} = \begin{bmatrix} 0.918 \\ 0.730 \end{bmatrix}.$$

Thus the least squares approximation is $y = 0.918 + 0.730x$.

30. **(a)** We are looking for a line $L = a + bF$ that best fits the given points. Using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 6 \\ 1 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 7.4 \\ 9.6 \\ 11.5 \\ 13.6 \end{bmatrix}.$$

Since

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 6 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 20 \\ 20 & 120 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{20} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{20} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} 7.4 \\ 9.6 \\ 11.5 \\ 13.6 \end{bmatrix} = \begin{bmatrix} 5.4 \\ 1.025 \end{bmatrix}.$$

Thus the least squares approximation is $L = 5.4 + 1.025F$. $a = 5.4$ represents the length of the spring when no force is applied ($F = 0$).

(b) When a weight of 5 ounces is attached, the spring length will be about $5.4 + 1.025 \cdot 5 = 10.525$ inches.

31. **(a)** We are looking for a line $y = a + bx$ that best fits the given points. Letting $x = 0$ correspond to 1920 and using the least squares approximation theorem, we want a solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 10 \\ 1 & 20 \\ 1 & 30 \\ 1 & 40 \\ 1 & 50 \\ 1 & 60 \\ 1 & 70 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 54.1 \\ 59.7 \\ 62.9 \\ 68.2 \\ 69.7 \\ 70.8 \\ 73.7 \\ 75.4 \end{bmatrix}.$$

Since

$$A^TA = \begin{bmatrix} 8 & 280 \\ 280 & 14000 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{5}{12} & -\frac{1}{120} \\ -\frac{1}{120} & \frac{1}{4200} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{5}{12} & -\frac{1}{120} \\ -\frac{1}{120} & \frac{1}{4200} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 10 & 20 & 30 & 40 & 50 & 60 & 70 \end{bmatrix} \begin{bmatrix} 54.1 \\ 59.7 \\ 62.9 \\ 68.2 \\ 69.7 \\ 70.8 \\ 73.7 \\ 75.4 \end{bmatrix} = \begin{bmatrix} 56.63 \\ 0.291 \end{bmatrix}.$$

Thus the least squares approximation is $y = 56.63 + 0.291x$. The model predicts that the life expectancy of someone born in 2000 is $y(80) = 56.63 + 0.291 \cdot 80 = 79.91$ years.

(b) To see how good the model is, compute the least squares error:

$$\mathbf{e}=\mathbf{b}-A\overline{\mathbf{x}}=\begin{bmatrix}54.1\\59.7\\62.9\\68.2\\69.7\\70.8\\73.7\\75.4\end{bmatrix}-\begin{bmatrix}1&0\\1&10\\1&20\\1&30\\1&40\\1&50\\1&60\\1&70\end{bmatrix}\begin{bmatrix}56.63\\0.291\end{bmatrix}=\begin{bmatrix}-2.53\\0.16\\0.45\\2.84\\1.43\\-0.38\\-0.39\\-1.6\end{bmatrix},\text{ so }\|\mathbf{e}\|\approx 4.43.$$

The model is not bad, with an error of under 10% of the initial value.

32. **(a)** Following the method of Example 7.28, we substitute the given values of x into the equation $a+bx+cx^2=y$ to get

$$\begin{aligned}a+0.5b+0.25c&=11\\a+\quad b+\quad c&=17\\a+1.5b+2.25c&=21\\a+\ 2b+\ 4c&=23\\a+\ 3b+\ 9c&=18,\end{aligned}$$

which in matrix form is

$$\begin{bmatrix}1&0.5&0.25\\1&1&1\\1&1.5&2.25\\1&2&4\\1&3&9\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}11\\17\\21\\23\\18\end{bmatrix}.$$

Thus we want the least squares approximation to $A\mathbf{x}=\mathbf{b}$, where A is the 5×3 matrix on the left and $\mathbf{b}$ is the matrix on the right in the above equation. Since

$$A^TA=\begin{bmatrix}1&1&1&1&1\\0.5&1&1.5&2&3\\0.25&1&2.25&4&9\end{bmatrix}\begin{bmatrix}1&0.5&0.25\\1&1&1\\1&1.5&2.25\\1&2&4\\1&3&9\end{bmatrix}=\begin{bmatrix}5&8&16.5\\8&16.5&39.5\\16.5&39.5&103.125\end{bmatrix}\Rightarrow$$

$$(A^TA)^{-1}=\begin{bmatrix}3.330&-4.082&1.031\\-4.082&5.735&-1.543\\1.031&-1.543&0.436\end{bmatrix},$$

we get

$$\overline{\mathbf{x}}=(A^TA)^{-1}A^T\mathbf{b}=\begin{bmatrix}3.330&-4.082&1.031\\-4.082&5.735&-1.543\\1.031&-1.543&0.436\end{bmatrix}\begin{bmatrix}1&1&1&1&1\\0.5&1&1.5&2&3\\0.25&1&2.25&4&9\end{bmatrix}\begin{bmatrix}11\\17\\21\\23\\18\end{bmatrix}=\begin{bmatrix}1.918\\20.306\\-4.972\end{bmatrix}.$$

Thus the least squares approximation is $y=1.918+20.306t-4.972t^2$.

(b) The height at which the object was released is $y(0)\approx 1.918$ m. Its initial velocity was $y'(0)\approx 20.306$ m/sec, and its acceleration due to gravity is $y''(t)\approx -9.944$ m/s^2.

(c) The object hits the ground when $y(t)=0$. Solving the quadratic using the quadratic formula gives $y\approx -0.092$ and $y\approx 4.176$ seconds. So the object hits the ground after about 4.176 seconds.

33. **(a)** If $p(t) = ce^{kt}$, then $p(0) = 150$ means that $c = 150$, so that $p(t) = 150e^{kt}$. Taking logs gives $\ln p(t) = \ln 150 + kt \approx 5.011 + kt$. We wish to find the value of k which best fits the data. Let $t = 0$ correspond to 1950 and $t = 1$ to 1960. Then $kt = \ln p(t) - \ln 150 = \ln \frac{p(t)}{150}$, so

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} k \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \ln \frac{179}{150} \\ \ln \frac{203}{150} \\ \ln \frac{227}{150} \\ \ln \frac{250}{150} \\ \ln \frac{281}{150} \end{bmatrix} \approx \begin{bmatrix} 0.177 \\ 0.303 \\ 0.414 \\ 0.511 \\ 0.628 \end{bmatrix}.$$

Since

$$A^TA = \begin{bmatrix} 55 \end{bmatrix} \text{ and thus } (A^TA)^{-1} = \begin{bmatrix} \frac{1}{55} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{1}{55} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.177 \\ 0.303 \\ 0.414 \\ 0.511 \\ 0.628 \end{bmatrix} = 0.131.$$

Thus the least squares approximation is $p(t) = 150e^{0.131t}$.

(b) 2010 corresponds to $t = 6$, so the predicted population is $p(6) = 150e^{0.131 \cdot 6} \approx 329.19$ million.

34. Let $t = 0$ correspond to 1970.

(a) For a quadratic approximation, follow the method of Example 7.28, and substitute the given values of x into the equation $a + bx + cx^2 = y$ to get

$$\begin{aligned} a \qquad\qquad\quad &= 29.3 \\ a + 5b + 25c &= 44.7 \\ a + 10b + 100c &= 143.8 \\ a + 15b + 225c &= 371.6 \\ a + 20b + 400c &= 597.5 \\ a + 25b + 625c &= 1110.8 \\ a + 30b + 900c &= 1895.6 \\ a + 35b + 1225c &= 2476.6, \end{aligned}$$

which in matrix form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 25 \\ 1 & 10 & 100 \\ 1 & 15 & 225 \\ 1 & 20 & 400 \\ 1 & 25 & 625 \\ 1 & 30 & 900 \\ 1 & 35 & 1225 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 29.3 \\ 44.7 \\ 143.8 \\ 371.6 \\ 597.5 \\ 1110.8 \\ 1895.6 \\ 2476.6 \end{bmatrix}.$$

Thus we want the least squares approximation to $A\mathbf{x} = \mathbf{b}$, where A is the 8×3 matrix on the left and $\mathbf{b}$ is the matrix on the right in the above equation. Since

$$A^TA = \begin{bmatrix} 8 & 140 & 3500 \\ 140 & 3500 & 98000 \\ 3500 & 98000 & 2922500 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{17}{24} & -\frac{3}{40} & \frac{1}{600} \\ -\frac{3}{40} & \frac{53}{4200} & -\frac{1}{3000} \\ \frac{1}{600} & -\frac{1}{3000} & \frac{1}{105000} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{17}{24} & -\frac{3}{40} & \frac{1}{600} \\ -\frac{3}{40} & \frac{53}{4200} & -\frac{1}{3000} \\ \frac{1}{600} & -\frac{1}{3000} & \frac{1}{105000} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 25 \\ 1 & 10 & 100 \\ 1 & 15 & 225 \\ 1 & 20 & 400 \\ 1 & 25 & 625 \\ 1 & 30 & 900 \\ 1 & 35 & 1225 \end{bmatrix}^T \begin{bmatrix} 29.3 \\ 44.7 \\ 143.8 \\ 371.6 \\ 597.5 \\ 1110.8 \\ 1895.6 \\ 2476.6 \end{bmatrix} = \begin{bmatrix} 57.063 \\ -20.335 \\ 2.589 \end{bmatrix}.$$

Thus the least squares approximation is $y = 57.063 - 20.335t + 2.589t^2$.

(b) With $y(t) = ce^{kt}$, since $y(0) = 29.3$ we have $c = 29.3$ so that $p(t) = 29.3e^{kt}$. Taking logs gives $\ln p(t) = \ln 29.3 + kt$, or $kt = \ln \frac{p(t)}{29.3}$. Then

$$A = \begin{bmatrix} 0 \\ 5 \\ 10 \\ 15 \\ 20 \\ 25 \\ 30 \\ 35 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \ln \frac{44.7}{29.3} \\ \ln \frac{143.8}{29.3} \\ \ln \frac{371.6}{29.3} \\ \ln \frac{597.5}{29.3} \\ \ln \frac{1110.8}{29.3} \\ \ln \frac{1895.6}{29.3} \\ \ln \frac{2476.6}{29.3} \end{bmatrix} \approx \begin{bmatrix} 0.422 \\ 1.591 \\ 2.540 \\ 3.015 \\ 3.635 \\ 4.170 \\ 4.437 \end{bmatrix}.$$

Then $A^TA = \begin{bmatrix} 3500 \end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix} 487.696 \end{bmatrix}$, so that $\overline{x} = \begin{bmatrix} \frac{487.696}{3500} \end{bmatrix} \approx \begin{bmatrix} 0.1393 \end{bmatrix}$. Thus $y(t) = 29.3e^{0.1393t}$.

(c) The models produce the following predictions:

Year	Actual	Quadratic	Exponential
1970	29.3	57.1	29.3
1975	44.7	20.1	58.8
1980	143.8	112.6	118.0
1985	371.6	334.6	236.8
1990	597.5	686.0	475.1
1995	1110.8	1166.8	953.5
2000	1895.6	1777.1	1913.3
2005	2476.6	2516.9	3839.4

The quadratic appears to be a better fit overall, even though it does a poor job in the first 15 years or so. The exponential model simply grows too fast.

(d) Using the quadratic approximation, we get for 2010 and 2015

$$y(40) = 57.063 - 20.335 \cdot 40 + 2.589 \cdot 40^2 = 3386.1$$
$$y(45) = 57.063 - 20.335 \cdot 45 + 2.589 \cdot 45^2 = 4384.7.$$

35. Using Example 7.29, taking logs gives

$$A = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \\ 90 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} kt_1 \\ kt_2 \\ kt_2 \end{bmatrix} = \begin{bmatrix} \ln \frac{172}{200} \\ \ln \frac{148}{200} \\ \ln \frac{128}{200} \end{bmatrix}.$$

Then $A^TA = \begin{bmatrix}12600\end{bmatrix}$ and $A^T\mathbf{b} \approx \begin{bmatrix}-62.8\end{bmatrix}$, so that $\overline{\mathbf{x}} \approx \begin{bmatrix}-\frac{62.8}{12600}\end{bmatrix} \approx -0.005$. Thus $k = -0.005$ and $p(t) = ce^{-0.005t}$. To compute the half-life, we want to solve $\frac{1}{2}c = ce^{-0.005t}$, so that $-0.005t = \ln\frac{1}{2} = -\ln 2$; then $t = \frac{\ln 2}{0.005} \approx 139$ days.

36. We want to find the best fit $z = a + bx + cy$ to the given data; the data give the equations

$$\begin{aligned} a \qquad - 4c &= 0 \\ a + 5b \qquad &= 0 \\ a + 4b - c &= 1 \\ a + b - 3c &= 1 \\ a - b - 5c &= -2, \end{aligned}$$

which in matrix form is

$$\begin{bmatrix} 1 & 0 & -4 \\ 1 & 5 & 0 \\ 1 & 4 & -1 \\ 1 & 1 & -3 \\ 1 & -1 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}.$$

Thus we want the least squares approximation to $A\mathbf{x} = \mathbf{b}$, where A is the 5×3 matrix on the left and $\mathbf{b}$ is the matrix on the right in the above equation. Since

$$A^TA = \begin{bmatrix} 5 & 9 & -13 \\ 9 & 43 & -2 \\ -13 & -2 & 51 \end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix} \frac{2189}{15} & -\frac{433}{15} & \frac{541}{15} \\ -\frac{433}{15} & \frac{86}{15} & -\frac{107}{15} \\ \frac{541}{15} & -\frac{107}{15} & \frac{134}{15} \end{bmatrix},$$

we get

$$\overline{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} \frac{2189}{15} & -\frac{433}{15} & \frac{541}{15} \\ -\frac{433}{15} & \frac{86}{15} & -\frac{107}{15} \\ \frac{541}{15} & -\frac{107}{15} & \frac{134}{15} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 4 & 1 & -1 \\ -4 & 0 & -1 & -3 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{43}{3} \\ -\frac{8}{3} \\ \frac{11}{3} \end{bmatrix}.$$

Thus the least squares approximation is $y = \frac{1}{3}(43 - 8x + 11y)$.

37. We have

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \Rightarrow \quad A^TA = \begin{bmatrix} 2 \end{bmatrix},\ (A^TA)^{-1} = \begin{bmatrix} \frac{1}{2} \end{bmatrix}.$$

So the standard matrix of the orthogonal projection onto W is

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then

$$\operatorname{proj}_W(\mathbf{v}) = P\mathbf{v} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ \frac{7}{2} \end{bmatrix}.$$

38. We have

$$A = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & -2 \end{bmatrix} \quad \Rightarrow \quad A^TA = \begin{bmatrix} 5 \end{bmatrix},\ (A^TA)^{-1} = \begin{bmatrix} \frac{1}{5} \end{bmatrix}.$$

So the standard matrix of the orthogonal projection onto W is

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}.$$

Then

$$\text{proj}_W(\mathbf{v}) = P\mathbf{v} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{bmatrix}.$$

39. We have

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad A^TA = \begin{bmatrix} 3 \end{bmatrix},\ (A^TA)^{-1} = \begin{bmatrix} \frac{1}{3} \end{bmatrix}.$$

So the standard matrix of the orthogonal projection onto W is

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Then

$$\text{proj}_W(\mathbf{v}) = P\mathbf{v} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

40. We have

$$A = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \quad \Rightarrow \quad A^TA = \begin{bmatrix} 9 \end{bmatrix},\ (A^TA)^{-1} = \begin{bmatrix} \frac{1}{9} \end{bmatrix}.$$

So the standard matrix of the orthogonal projection onto W is

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{9} \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{4}{9} & -\frac{2}{9} \\ \frac{4}{9} & \frac{4}{9} & -\frac{2}{9} \\ -\frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix}.$$

Then

$$\text{proj}_W(\mathbf{v}) = P\mathbf{v} = \begin{bmatrix} \frac{4}{9} & \frac{4}{9} & -\frac{2}{9} \\ \frac{4}{9} & \frac{4}{9} & -\frac{2}{9} \\ -\frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{bmatrix}.$$

41. We have

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},\ (A^TA)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

So the standard matrix of the orthogonal projection onto W is

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix}.$$

Then

$$\text{proj}_W(\mathbf{v}) = P\mathbf{v} = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}.$$

42. We have

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ 1 & -1 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \Rightarrow \quad A^T A = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}, \ (A^T A)^{-1} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

So the standard matrix of the orthogonal projection onto W is

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Then

$$\operatorname{proj}_W(\mathbf{v}) = P\mathbf{v} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

43. With the new basis, we have

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \quad \Rightarrow \quad A^T A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix}, \ (A^T A)^{-1} = \begin{bmatrix} \frac{11}{6} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

So the standard matrix of the orthogonal projection onto W is

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{11}{6} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

This agrees with the matrix found in Example 7.31.

44. **(a)** To see that P is symmetric, compute its transpose:

$$\begin{aligned} P^T = \left(A\left(A^T A\right)^{-1} A^T\right)^T = \left(A^T\right)^T \left(\left(A^T A\right)^{-1}\right)^T A^T &= A\left(\left(A^T A\right)^T\right)^{-1} A^T \\ &= A\left(A^T A\right)^{-1} A^T = P, \end{aligned}$$

recalling that $\left(B^T\right)^{-1} = \left(B^{-1}\right)^T$ for any invertible matrix B.

(b) To show that P is idempotent, we must show that $P^2 = PP = P$:

$$\begin{aligned} PP = \left(A\left(A^T A\right)^{-1} A^T\right)\left(A\left(A^T A\right)^{-1} A^T\right) &= A\left(\left(A^T A\right)^{-1}\left(A^T A\right)\right)\left(A^T A\right)^{-1} A^T \\ &= A\left(A^T A\right)^{-1} A^T = P. \end{aligned}$$

45. We have

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \Rightarrow \quad A^T A = \begin{bmatrix} 5 \end{bmatrix}, \ (A^T A)^{-1} = \begin{bmatrix} \frac{1}{5} \end{bmatrix}.$$

So the pseudoinverse of A is

$$A^+ = (A^T A)^{-1} A^T = \begin{bmatrix} \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

46. We have

$$A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \quad \Rightarrow \quad A^T A = \begin{bmatrix} 6 \end{bmatrix}, \ (A^T A)^{-1} = \begin{bmatrix} \frac{1}{6} \end{bmatrix}.$$

So the pseudoinverse of A is

$$A^+ = (A^T A)^{-1} A^T = \begin{bmatrix} \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{bmatrix}.$$

47. We have

$$A=\begin{bmatrix}1&3\\-1&1\\0&2\end{bmatrix}\quad\Rightarrow\quad A^T=\begin{bmatrix}1&-1&0\\3&1&2\end{bmatrix}\quad\Rightarrow\quad A^TA=\begin{bmatrix}2&2\\2&14\end{bmatrix},\ (A^TA)^{-1}=\begin{bmatrix}\frac{7}{12}&-\frac{1}{12}\\-\frac{1}{12}&\frac{1}{12}\end{bmatrix}.$$

So the pseudoinverse of A is

$$A^+=(A^TA)^{-1}A^T=\begin{bmatrix}\frac{7}{12}&-\frac{1}{12}\\-\frac{1}{12}&\frac{1}{12}\end{bmatrix}\begin{bmatrix}1&-1&0\\3&1&2\end{bmatrix}=\begin{bmatrix}\frac{1}{3}&-\frac{2}{3}&-\frac{1}{6}\\\frac{1}{6}&\frac{1}{6}&\frac{1}{6}\end{bmatrix}.$$

48. We have

$$A=\begin{bmatrix}1&3\\3&1\\2&2\end{bmatrix}\quad\Rightarrow\quad A^T=\begin{bmatrix}1&3&2\\3&1&2\end{bmatrix}\quad\Rightarrow\quad A^TA=\begin{bmatrix}14&10\\10&14\end{bmatrix},\ (A^TA)^{-1}=\begin{bmatrix}\frac{7}{48}&-\frac{5}{48}\\-\frac{5}{48}&\frac{7}{48}\end{bmatrix}.$$

So the pseudoinverse of A is

$$A^+=(A^TA)^{-1}A^T=\begin{bmatrix}\frac{7}{48}&-\frac{5}{48}\\-\frac{5}{48}&\frac{7}{48}\end{bmatrix}\begin{bmatrix}1&3&2\\3&1&2\end{bmatrix}=\begin{bmatrix}-\frac{1}{6}&\frac{1}{3}&\frac{1}{12}\\\frac{1}{3}&-\frac{1}{6}&\frac{1}{12}\end{bmatrix}.$$

49. We have

$$A=\begin{bmatrix}1&1\\0&1\end{bmatrix}\quad\Rightarrow\quad A^T=\begin{bmatrix}1&0\\1&1\end{bmatrix}\quad\Rightarrow\quad A^TA=\begin{bmatrix}1&1\\1&2\end{bmatrix},\ (A^TA)^{-1}=\begin{bmatrix}2&-1\\-1&1\end{bmatrix}.$$

So the pseudoinverse of A is

$$A^+=(A^TA)^{-1}A^T=\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix}=\begin{bmatrix}1&-1\\0&1\end{bmatrix}.$$

Note that this is the inverse A^{-1} of A.

50. We have

$$A=\begin{bmatrix}1&2\\3&4\end{bmatrix}\quad\Rightarrow\quad A^T=\begin{bmatrix}1&3\\2&4\end{bmatrix}\quad\Rightarrow\quad A^TA=\begin{bmatrix}10&14\\14&20\end{bmatrix},\ (A^TA)^{-1}=\begin{bmatrix}5&-\frac{7}{2}\\-\frac{7}{2}&\frac{5}{2}\end{bmatrix}.$$

So the pseudoinverse of A is

$$A^+=(A^TA)^{-1}A^T=\begin{bmatrix}5&-\frac{7}{2}\\-\frac{7}{2}&\frac{5}{2}\end{bmatrix}\begin{bmatrix}1&3\\2&4\end{bmatrix}=\begin{bmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{bmatrix}.$$

Note that this is the inverse A^{-1} of A.

51. We have

$$A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&1\\1&1&1\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}1&1&0&1\\0&0&1&1\\0&1&1&1\end{bmatrix}\Rightarrow A^TA=\begin{bmatrix}3&1&2\\1&2&2\\2&2&3\end{bmatrix},\ (A^TA)^{-1}=\begin{bmatrix}\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\\frac{1}{3}&\frac{5}{3}&-\frac{4}{3}\\-\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\end{bmatrix}.$$

So the pseudoinverse of A is

$$A^+=(A^TA)^{-1}A^T=\begin{bmatrix}\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\\frac{1}{3}&\frac{5}{3}&-\frac{4}{3}\\-\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\end{bmatrix}\begin{bmatrix}1&1&0&1\\0&0&1&1\\0&1&1&1\end{bmatrix}=\begin{bmatrix}\frac{2}{3}&0&-\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&-1&\frac{1}{3}&\frac{2}{3}\\-\frac{2}{3}&1&\frac{1}{3}&-\frac{1}{3}\end{bmatrix}.$$

52. We have

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 2 \end{bmatrix},$$

so that

$$A^TA = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 6 & -3 \\ -2 & -3 & 9 \end{bmatrix}, \ (A^TA)^{-1} = \begin{bmatrix} \frac{15}{7} & -1 & \frac{1}{7} \\ -1 & \frac{2}{3} & 0 \\ \frac{1}{7} & 0 & \frac{1}{7} \end{bmatrix}.$$

So the pseudoinverse of A is

$$A^+ = (A^TA)^{-1}A^T = \begin{bmatrix} \frac{15}{7} & -1 & \frac{1}{7} \\ -1 & \frac{2}{3} & 0 \\ \frac{1}{7} & 0 & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & -\frac{8}{7} & \frac{6}{7} & \frac{2}{7} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & \frac{2}{7} \end{bmatrix}.$$

53. **(a)** Since A is square with linearly independent columns, it is invertible, so that

$$A^+ = \left(A^TA\right)^{-1} A^T = A^{-1}\left(A^T\right)^{-1} A^T = A^{-1}.$$

(b) Let the columns of A be $\mathbf{q}_i$; then $\mathbf{q}_i \cdot \mathbf{q}_j = 0$ if $i \neq j$ and 1 if $i = j$. Then

$$A^TA = \begin{bmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{bmatrix} = I_n,$$

so that $A^+ = \left(A^TA\right)^{-1} A^T = I_nA^T = A^T$. (Note that this makes sense, for if A were square, it would be an orthogonal matrix and then part (a) would hold and $A^T = A^{-1}$.)

54. $A^+AA^+ = \left(\left(A^TA\right)^{-1} A^T\right) AA^+ = \left(A^TA\right)^{-1}\left(A^TA\right) A^+ = A^+$.

55. We have $A^+A = \left(A^TA\right)^{-1} A^TA = \left(A^TA\right)^{-1}\left(A^TA\right) = I$, which is certainly a symmetric matrix.

56. **(a)** $(cA)^+ = \left(cA^TcA\right)^{-1} cA^T = \frac{1}{c^2}\left(A^TA\right)^{-1} cA^T = \frac{1}{c}\left(A^TA\right)^{-1} A^T = \frac{1}{c}A^T$.

(b) By Exercise 53(a), if A is square, then $A^+ = A^{-1}$. Also A^{-1} is square with linearly independent columns, so that $\left(A^{-1}\right)^+ = \left(A^{-1}\right)^{-1} = A$. Then$(A^+)^+ = \left(A^{-1}\right)^+ = A$.

(c) Since A is square, Exercise 53(a) applies to both A and A^T, so that

$$\left(A^T\right)^+ = \left(A^T\right)^{-1} = \left(A^{-1}\right)^T = \left(A^+\right)^T.$$

57. The fact that not all the points lie on the same vertical line means that there is at least one pair i, j such that $x_i \neq x_j$. Constructing the matrix A gives

$$A = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_i \\ \vdots & \vdots \\ 1 & x_j \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}.$$

Since $x_i \neq x_j$, the columns are not multiples of one another, so must be linearly independent. Since the columns of A are linearly independent, the points have a unique least squares approximating line, by part (b) of the Least Squares Theorem.

58. Constructing the matrix A gives

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{bmatrix}.$$

The solution(s) to $A\mathbf{x} = \mathbf{b}$ are the least squares approximating polynomials of degree k; by the Least Squares Theorem, there is a unique solution when the columns of A are linearly independent. Now suppose that

$$a_0\mathbf{a}_1 + a_1\mathbf{a}_2 + \cdots + a_k\mathbf{a}_{k+1} = 0.$$

This means that $a_0 + a_1x_i + \cdots + a_kx_i^k = 0$ for all i. But since at least $k+1$ of the x_i are distinct, the polynomial $a_0 + a_1x + \cdots + a_kx^k$ has at least $k+1$ distinct roots, so it must be the zero polynomial by Exercise 62 in Section 6.2. Thus all of the a_i are zero, so the columns of A are linearly independent.

7.4 The Singular Value Decomposition

1. We have

$$A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

Then the singular values of A are the square roots of the eigenvalues of this matrix which, since the matrix is diagonal, are 4 and 9. Thus the singular values of A are $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{4} = 2$.

2. We have

$$A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

Then the singular values of A are the square roots of the eigenvalues of this matrix.

$$\det(A^TA - \lambda I) = \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 16 = \lambda^2 - 10\lambda + 9 = (\lambda - 9)(\lambda - 1),$$

so that the eigenvalues are 9 and 1. Thus the singular values of A are $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{1} = 1$.

3. We have

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then the singular values of A are the square roots of the eigenvalues of this matrix.

$$\det(A^TA - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2),$$

so that the singular values of A are $\sigma_1 = \sqrt{2}$ and $\sigma_2 = \sqrt{0} = 0$.

4. We have

$$A^TA = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}.$$

Then the singular values of A are the square roots of the eigenvalues of this matrix.

$$\det(A^TA - \lambda I) = \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1),$$

so that the singular values of A are $\sigma_1 = \sqrt{4} = 2$ and $\sigma_2 = \sqrt{1} = 1$.

5. We have

$$A^TA = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 25 \end{bmatrix}.$$

Then the singular values of A are the square roots of the eigenvalues of this matrix.

$$\det(A^TA - \lambda I) = \begin{vmatrix} 25 - \lambda \end{vmatrix} = 25 - \lambda,$$

so that the singular value of A is $\sigma_1 = \sqrt{25} = 5$.

6. We have

$$A^TA = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}.$$

Then the singular values of A are the square roots of the eigenvalues of this matrix.

$$\det(A^TA - \lambda I) = \begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = (9-\lambda)(16-\lambda) - 144 = \lambda^2 - 25\lambda = \lambda(\lambda - 25),$$

so that the singular values of A are $\sigma_1 = \sqrt{25} = 5$ and $\sigma_2 = \sqrt{0} = 0$.

7. We have

$$A^TA = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

Then the singular values of A are the square roots of the eigenvalues of this matrix. Since it is diagonal, its eigenvalues are 9 and 4, so that the singular values of A are $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{4} = 2$.

8. We have

$$A^TA = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Then the singular values of A are the square roots of the eigenvalues of this matrix.

$$\det(A^TA - \lambda I) = \begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 16 = \lambda^2 - 10\lambda + 9 = (\lambda - 9)(\lambda - 1),$$

so that the eigenvalues are 9 and 1. Thus the singular values of A are $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{1} = 1$.

9. We have

$$A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Then the singular values of A are the square roots of the eigenvalues of this matrix.

$$\det(A^TA - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 2 \\ 0 & 4-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} = (4-\lambda)\left((4-\lambda)(1-\lambda) - 4\right) = -\lambda(\lambda - 5)(\lambda - 4),$$

so that the eigenvalues are 5, 4, and 0. Thus the singular values of A are $\sigma_1 = \sqrt{5}$, $\sigma_2 = \sqrt{4} = 2$, and $\sigma_3 = \sqrt{0} = 0$.

10. We have

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 9 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Then the singular values of A are the square roots of the eigenvalues of this matrix.

$$\det(A^TA-\lambda I)=\begin{vmatrix}2-\lambda & 0 & 2\\ 0 & 9-\lambda & 0\\ 2 & 0 & 2-\lambda\end{vmatrix}=(9-\lambda)\left((2-\lambda)^2-4\right)=-(\lambda-9)(\lambda-4)\lambda$$

so that the eigenvalues are 9, 4, and 0. Thus the singular values of A are $\sigma_1=\sqrt{9}=3$, $\sigma_2=\sqrt{4}=2$, and $\sigma_3=\sqrt{0}=0$.

11. From Exercise 3,

$$A^TA=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},$$

with eigenvalues 2 and 0, so that the singular values of A are $\sigma_1=\sqrt{2}$ and $\sigma_2=\sqrt{0}=0$. To find the corresponding eigenspaces, row-reduce:

$$\begin{aligned}\left[A^TA-2I\mid 0\right]&=\left[\begin{array}{rr|r}-1 & 1 & 0\\ 1 & -1 & 0\end{array}\right]\to\left[\begin{array}{rr|r}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]\Rightarrow \mathbf{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix}\\ \left[A^TA-0I\mid 0\right]&=\left[\begin{array}{rr|r}1 & 1 & 0\\ 1 & 1 & 0\end{array}\right]\to\left[\begin{array}{rr|r}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]\Rightarrow \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}.\end{aligned}$$

These vectors are already orthogonal since they correspond to distinct eigenvalues. Normalizing each of them gives

$$\mathbf{v}_1=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix},\qquad \mathbf{v}_2=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix},\quad\text{so that}\quad V=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}.$$

The first column of the matrix U is

$$\mathbf{u}_1=\frac{1}{\sigma_1}A\mathbf{v}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}.$$

We do not use $\mathbf{v}_2$ since it corresponds to an eigenvalue of zero. We must extend this to an orthonormal basis of $\mathbb{R}^2$ by adjoining a second vector, $\mathbf{u}_2$, to get an orthogonal matrix. Clearly $\mathbf{e}_2$ is such a vector. Thus

$$A=U\Sigma V^T=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

is a singular value decomposition of A.

12. We have

$$A^TA=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}.$$

The characteristic polynomial of A^TA is

$$\det(A^TA-\lambda I)=\begin{vmatrix}2-\lambda & 2\\ 2 & 2-\lambda\end{vmatrix}=(2-\lambda)^2-4=\lambda^2-4\lambda=\lambda(\lambda-4).$$

Thus the eigenvalues are $\lambda_1=4$ and $\lambda_2=0$ (and so the singular values are $\sigma_1=2$ and $\sigma_2=0$). To find the corresponding eigenspaces, row-reduce:

$$\begin{aligned}\left[A^TA-4I\mid 0\right]&=\left[\begin{array}{rr|r}-2 & 2 & 0\\ 2 & -2 & 0\end{array}\right]\to\left[\begin{array}{rr|r}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]\Rightarrow \mathbf{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix}\\ \left[A^TA-0I\mid 0\right]&=\left[\begin{array}{rr|r}2 & 2 & 0\\ 2 & 2 & 0\end{array}\right]\to\left[\begin{array}{rr|r}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]\Rightarrow \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}.\end{aligned}$$

To find the matrix V, we must normalize each of these vectors (they are already orthogonal since they correspond to distinct eigenvalues). This gives

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \text{so that} \quad V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

The first column of the matrix U is

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

We do not use $\mathbf{v}_2$ since it corresponds to an eigenvalue of zero. We must extend this to an orthonormal basis of $\mathbb{R}^2$ by adjoining a second vector, $\mathbf{u}_2$, to get an orthogonal matrix. By looking at the matrix V, we see that such a vector is $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. Thus

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

is a singular value decomposition of A.

13. We have

$$A^T A = \begin{bmatrix} 0 & -3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}.$$

This matrix is diagonal, so its eigenvalues are its diagonal entries, which are $\lambda_1 = 9$ and $\lambda_2 = 4$. Therefore the singular values of A are $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{4} = 2$. To find the corresponding eigenspaces, row-reduce:

$$\begin{aligned} \begin{bmatrix} A^T A - 9I \mid 0 \end{bmatrix} &= \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} A^T A - 4I \mid 0 \end{bmatrix} &= \left[\begin{array}{cc|c} 4 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

These vectors are already orthonormal, so we set

$$\mathbf{v}_1 = \mathbf{x}_1, \quad \mathbf{v}_2 = \mathbf{x}_2, \quad \text{so that} \quad V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then U is found by

$$\begin{aligned} \mathbf{u}_1 &= \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{3} \begin{bmatrix} 0 & -2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ \mathbf{u}_2 &= \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{2} \begin{bmatrix} 0 & -2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \end{aligned}$$

Thus

$$A = U\Sigma V^T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a singular value decomposition of A.

14. We have

$$A^TA = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

This matrix is diagonal, so its eigenvalues are its diagonal entries. So the only eigenvalue is 2 and the singular values of A are $\sigma_1 = \sigma_2 = \sqrt{2}$. To find the corresponding eigenspace, row-reduce:

$$\left[A^TA - 2I \mid 0\right] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

These vectors are already orthonormal, so we set

$$\mathbf{v}_1 = \mathbf{x}_1, \quad \mathbf{v}_2 = \mathbf{x}_2, \quad \text{so that} \quad V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then U is found by

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a singular value decomposition of A.

15. From Exercise 5,

$$A^TA = \begin{bmatrix} 25 \end{bmatrix},$$

so that the only eigenvalue is 25 and therefore the only singular value is $\sigma_1 = \sqrt{25} = 5$. To find the corresponding eigenspace, row-reduce:

$$\left[A^TA - 25I \mid 0\right] = \left[0 \mid 0\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \end{bmatrix}.$$

This single vector forms an orthonormal set; let $\mathbf{v}_1 = \mathbf{x}_1$, so that

$$V = \begin{bmatrix} \mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}.$$

Then U is found by

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}.$$

We extend this to an orthonormal basis for $\mathbb{R}^2$ by adjoining the orthogonal unit vector $\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$. Since Σ should be 2×1 and there is only one singular value, we adjoin a zero to the singular value. Thus

$$A = U\Sigma V^T = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

is a singular value decomposition of A.

16. From Exercise 6,

$$A^TA = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix},$$

with eigenvalues 25 and 0, so that the singular values of A are $\sigma_1 = \sqrt{25} = 5$ and $\sigma_2 = \sqrt{0} = 0$. To find the corresponding eigenspaces, row-reduce:

$$\begin{aligned}
\begin{bmatrix} A^TA - 25I \mid 0 \end{bmatrix} &= \left[\begin{array}{cc|c} -16 & 12 & 0 \\ 12 & -9 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\
\begin{bmatrix} A^TA - 0I \mid 0 \end{bmatrix} &= \left[\begin{array}{cc|c} 9 & 12 & 0 \\ 12 & 16 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{4}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.
\end{aligned}$$

To find the matrix V, we must normalize each of these vectors (they are already orthogonal since they correspond to distinct eigenvalues). This gives

$$\mathbf{v}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}, \quad \text{so that} \quad V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

The first column of the matrix U is

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{5}\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}.$$

Then U is the 1×1 matrix containing 1 as its entry. Since Σ should be 1×2 and there is only one nonzero singular value, we adjoin a zero to the singular value, so that

$$A = U\Sigma V^T = \begin{bmatrix} 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

is a singular value decomposition of A.

17. From Exercise 7,

$$A^TA = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix},$$

with eigenvalues 9 and 4, so with singular values 3 and 2. To find the corresponding eigenspaces, row-reduce:

$$\begin{aligned}
\begin{bmatrix} A^TA - 9I \mid 0 \end{bmatrix} &= \left[\begin{array}{cc|c} -5 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\begin{bmatrix} A^TA - 4I \mid 0 \end{bmatrix} &= \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\end{aligned}$$

These vectors are already orthonormal, so we set

$$\mathbf{v}_1 = \mathbf{x}_1, \quad \mathbf{v}_2 = \mathbf{x}_2, \quad \text{so that} \quad V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then U is found by

$$\begin{aligned}
\mathbf{u}_1 &= \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
\mathbf{u}_2 &= \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.
\end{aligned}$$

We must extend $\mathbf{u}_1$ and $\mathbf{u}_2$ to an orthonormal basis for $\mathbb{R}^3$. Clearly $\mathbf{u}_3 = \mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ works. Then U is the matrix whose columns are the $\mathbf{u}_i$. Now, Σ should be 3×2, so we add a row of zeros at the bottom, giving

$$A = U\Sigma V^T = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

for a singular value decomposition of A.

18. From Exercise 8,

$$A^TA = \begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix},$$

with eigenvalues 9 and 1 and singular values $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{1} = 1$. To find the corresponding eigenspaces, row-reduce:

$$\begin{aligned}\left[A^TA - 9I \mid 0\right] &= \left[\begin{array}{cc|c}-4 & -4 & 0\\ -4 & -4 & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix}1\\-1\end{bmatrix}\\ \left[A^TA - I \mid 0\right] &= \left[\begin{array}{cc|c}4 & -4 & 0\\ -4 & 4 & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix}1\\1\end{bmatrix}.\end{aligned}$$

These vectors are already orthogonal, but we must normalize them:

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}, \quad \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}, \quad \text{so that} \quad V = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

Then U is found by

$$\begin{aligned}\mathbf{u}_1 &= \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix}0 & 1\\ 1 & 0\\ 2 & -2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}-\frac{1}{3\sqrt{2}}\\ \frac{1}{3\sqrt{2}}\\ \frac{4}{3\sqrt{2}}\end{bmatrix}\\ \mathbf{u}_2 &= \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{1}\begin{bmatrix}0 & 1\\ 1 & 0\\ 2 & -2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{bmatrix}.\end{aligned}$$

We must extend $\mathbf{u}_1$ and $\mathbf{u}_2$ to an orthonormal basis for $\mathbb{R}^3$. To keep computations simpler, we multiply $\mathbf{u}_1$ by $3\sqrt{2}$ and $\mathbf{v}_1$ by $\sqrt{2}$. Then adjoin $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ to these vectors and row-reduce:

$$\begin{bmatrix}-1 & 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 1 & 0\\ 4 & 0 & 0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0 & \frac{1}{4}\\ 0 & 1 & 0 & 1 & -\frac{1}{4}\\ 0 & 0 & 1 & -1 & \frac{1}{2}\end{bmatrix}.$$

Then a basis for the column space, which is $\mathbb{R}^3$, is given by the columns of the original matrix corresponding to leading 1's, which is $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{x}_3 = \mathbf{e}_1$. Finally, we must turn this into an orthonormal basis:

$$\mathbf{x}_3' = \mathbf{x}_3 - \frac{-1\cdot 1 + 1\cdot 0 + 4\cdot 0}{-1\cdot(-1) + 1\cdot 1 + 4\cdot 4}\begin{bmatrix}-1\\1\\4\end{bmatrix} - \frac{1\cdot 1 + 1\cdot 0 + 0\cdot 0}{1\cdot 1 + 1\cdot 1 + 0\cdot 0}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{1}{18}\begin{bmatrix}-1\\1\\4\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{4}{9}\\ -\frac{4}{9}\\ \frac{2}{9}\end{bmatrix}.$$

Normalizing this vector gives

$$\mathbf{u}_3 = \frac{\mathbf{x}_3'}{\|\mathbf{x}_3\|} = \begin{bmatrix}\frac{2}{3}\\ -\frac{2}{3}\\ \frac{1}{3}\end{bmatrix}.$$

Then U is the matrix whose columns are the $\mathbf{u}_i$. Now, Σ should be 3×2, so we add a row of zeros at the bottom, giving

$$A = U\Sigma V^T = \begin{bmatrix} -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ \frac{4}{3\sqrt{2}} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

for a singular value decomposition of A.

19. From Exercise 9,

$$A^TA = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

with eigenvalues 5, 4, and 0, so the singular values are $\sigma_1 = \sqrt{5}$, $\sigma_2 = 2$, and $\sigma_3 = 0$. To find the corresponding eigenspaces, row-reduce:

$$\left[A^TA - 5I \mid 0\right] = \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[A^TA - 4I \mid 0\right] = \left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[A^TA - 0I \mid 0\right] = \left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

These vectors are already orthogonal, but we must normalize them:

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \quad \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix},$$

so that

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Then U is found by

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{2} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We do not use $\mathbf{v}_3$ since it corresponds to a zero singular value. Then $\mathbf{u}_1$ and $\mathbf{u}_2$ already form a basis for $\mathbb{R}^2$. Now, Σ should be 3×2, so we add a column of zeros on the right, giving

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}$$

for a singular value decomposition of A.

20. We have

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Then the singular values of A are the square roots of the eigenvalues of this matrix.

$$\det(A^TA - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = -\lambda^2(\lambda - 6),$$

so that the eigenvalues are 6 and 0. Thus the singular values of A are $\sigma_1 = \sqrt{6}$ and $\sigma_2 = \sqrt{0} = 0$. To find the corresponding eigenspaces, row-reduce:

$$\left[A^TA - 6I \mid 0\right] = \left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[A^TA - 0I \mid 0\right] = \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

We must orthogonalize $\mathbf{x}_2$ and $\mathbf{x}_3$. Use Gram-Schmidt:

$$\mathbf{x}_3' = \mathbf{x}_3 - \frac{\mathbf{x}_2 \cdot \mathbf{x}_3}{\mathbf{x}_2 \cdot \mathbf{x}_2}\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

Multiply through by 2 to clear fractions, giving

$$\mathbf{x}_3'' = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

Finally, we normalize the vectors:

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{v}_3 = \frac{\mathbf{x}_3''}{\|\mathbf{x}_3''\|} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix} \quad \Rightarrow$$

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{6}} \end{bmatrix}.$$

Then U is found by

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

We do not use $\mathbf{v}_2$ or $\mathbf{v}_3$ since they correspond to a zero singular value. Then we extend $\mathbf{u}_1$ to an orthonormal basis of $\mathbb{R}^2$ by adding the orthogonal unit vector $\mathbf{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. Finally, Σ must be 2×3, so we adjoin zeros as required:

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

for a singular value decomposition of A.

21. Using the singular values and the matrices U and V with columns $\mathbf{u}_i$ and $\mathbf{v}_i$ from Exercise 11, we get

$$A = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T = \sqrt{2}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}.$$

22. Using the singular values and the matrices U and V with columns $\mathbf{u}_i$ and $\mathbf{v}_i$ from Exercise 14, we get

$$A = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T = \sqrt{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \sqrt{2}\begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}.$$

23. Using the singular values and the matrices U and V with columns $\mathbf{u}_i$ and $\mathbf{v}_i$ from Exercise 17, we get

$$A = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T = 3\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} + 2\begin{bmatrix}0\\0\\-1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}.$$

24. Using the singular values and the matrices U and V with columns $\mathbf{u}_i$ and $\mathbf{v}_i$ from Exercise 19, we get

$$A = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T = \sqrt{5}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}}\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix}.$$

25. For example, in Exercise 11, we extended $\mathbf{u}_1$ to a basis for $\mathbb{R}^2$ by adjoining $\mathbf{e}_2$; we could instead have adjoined $-\mathbf{e}_2$, giving a different SVD:

$$A = U\Sigma V^T = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}.$$

In Exercise 14, we could have used $\mathbf{x}_1 = \begin{bmatrix}-1\\0\end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix}0\\-1\end{bmatrix}$ as eigenvectors, which would change both U and V and give a different SVD:

$$A = U\Sigma V^T = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}.$$

26. **(a)** If A is symmetric, then $A^TA = AA = A^2$. If λ is any eigenvalue of A, then λ^2 is an eigenvalue of $A^2 = A^TA$ and therefore $\sqrt{\lambda^2} = |\lambda|$ is a singular value of A.

(b) If A is positive definite, then A is symmetric with positive eigenvalues, so that in part (a), $|\lambda| = \lambda$. Thus the singular values of A are just the eigenvalues of A.

27. **(a)** We want to show that $A = U\Sigma V^T$ is an orthogonal diagonalization of A. Since A is positive definite, its singular values are its eigenvalues, by Exercise 26(b), so that Σ is a diagonal matrix D whose entries are the eigenvalues of A. It remains to show that U and V^T are orthogonal and that $U = V = Q$ where Q is orthogonal, for then $A = QDQ^T$, so that we indeed have an orthogonal diagonalization of A. Now,

$$\mathbf{u}_i = \frac{1}{\sigma_i}A\mathbf{v}_i = \frac{1}{\lambda_i}A\mathbf{v}_i.$$

But $\mathbf{v}_i$ is an eigenvector corresponding to λ_i, so continuing,

$$\frac{1}{\lambda_i}A\mathbf{v}_i = \frac{1}{\lambda_i}\lambda_i\mathbf{v}_i = \mathbf{v}_i.$$

Thus the columns of U and V are the same, so that $U = V$. Further, V is an orthogonal matrix by construction, since its entries are orthonormal eigenvectors for A.

(b) From part (a), $U = V = Q$, an orthogonal matrix, and by Exercise 26, the singular values of A are the eigenvalues of A. Then the outer product form of the SVD is

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_n \mathbf{u}_n \mathbf{v}_n^T = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \cdots + \lambda_n \mathbf{q}_n \mathbf{q}_n^T,$$

which is the spectral decomposition of A since the $\mathbf{q}_i$ are a set of orthonormal eigenvectors corresponding to the λ_i.

28. Since U and V are both orthogonal, we know that $U^{-1} = U^T$ and $V^{-1} = V^T$. Then $A = U\Sigma V^T$ means that

$$U^T A V = U^T U \Sigma V^T V = U^{-1} U \Sigma V^{-1} V = \Sigma.$$

Since U^T, A, and V are all invertible, their product, Σ, is also invertible. Then since all the matrices are invertible,

$$A^{-1} = \left(U\Sigma V^T\right)^{-1} = \left(V^T\right)^{-1} \Sigma^{-1} U^{-1} = \left(V^T\right)^T \Sigma^{-1} U^T = V\Sigma^{-1} U^T,$$

showing that $V\Sigma^{-1}U^T$ is an SVD of A^{-1} since Σ^{-1} is also a diagonal matrix and V and U^T are orthogonal matrices.

29. We have $AA^T = \left(U\Sigma V^T\right)\left(U\Sigma V^T\right)^T = U\Sigma(V^T V)\Sigma U^T = U\Sigma^2 U^T$, so that (since U and U^T are orthogonal matrices) $U^{-1}\left(AA^T\right)U = U^T\left(AA^T\right)U = \Sigma^2$, which is a diagonal matrix. Therefore U diagonalizes AA^T, so by Theorem 4.23 in Section 4.4, it follows that the columns of U are eigenvectors of AA^T.

30. To show that A and A^T have the same singular values, we must show that $A^T A$ and AA^T have the same eigenvalues. But from Exercise 29, $AA^T = U\Sigma^2 U^T$; a similar string of equalities to that in Exercise 29 gives $A^T A = V\Sigma^2 V^T$. Since the diagonal matrix in both cases gives the eigenvalues, and those diagonal matrices are the same, AA^T and $A^T A$ have the same eigenvalues.

31. To show that A and QA have the same singular values, it suffices to show that $A^T A$ and $(QA)^T QA$ have the same eigenvalues. But

$$(QA)^T QA = A^T Q^T QA = A^T A$$

since Q is orthogonal. So in fact the two matrices are equal so they must have the same eigenvalues. Thus A and QA have the same singular values.

32. We know that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthonormal set, so it is linearly independent. Next, since $A = U\Sigma V^T$, we get $A^T = \left(U\Sigma V^T\right)^T = V\Sigma^T U^T$, so that $V\Sigma^T = A^T U$. But this means that $\mathbf{v}_i\sigma_i = A^T\mathbf{u}_i$, so that $\mathbf{v}_i = \frac{1}{\sigma_i}A^T\mathbf{u}_i$, so that $\mathbf{v}_i \in \operatorname{row}(A)$. It remains only to show that $\dim\operatorname{row}(A) = r$, for then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an r-element linearly independent set in $\operatorname{row}(A)$, so it must form a basis. But by part (e), $\{v_{r+1}, \ldots, v_n\}$ forms a basis for (A), which therefore has dimension $n - r$. By Theorem 5.10, $(\operatorname{row}(A))^\perp = (A)$, so that $\dim\operatorname{row}(A) = n - \dim(A) = n - (n - r) = r$.

33. Since $\operatorname{rank}(A) = 1$ but A is a map into $\mathbb{R}^2$, the image of the unit circle is a solid ellipsoid. The only nonzero singular value of A is $\sqrt{2}$, so the equation of the ellipsoid is

$$\left(\frac{y_1}{\sqrt{2}}\right)^2 \le 1, \quad \text{or} \quad y_1^2 \le 2.$$

This is the interval $[-\sqrt{2}, \sqrt{2}]$ along the y_1-axis.

34. Since $\operatorname{rank}(A) = 2$ but A is a map into $\mathbb{R}^3$, the image of the unit circle is a solid ellipsoid. The nonzero singular values of A are $\sigma_1 = 3$ and $\sigma_2 = 2$, so the equation of the ellipsoid is

$$\left(\frac{y_1}{\sigma_1}\right)^2 + \left(\frac{y_2}{\sigma_2}\right)^2 \le 1, \quad \text{or} \quad \frac{y_1^2}{9} + \frac{y_2^2}{4} \le 1.$$

This is an ellipse in $\mathbb{R}^2$ with semimajor axis 3 and semiminor axis 2 together with its interior.

35. Since $\operatorname{rank}(A) = 2$ and A is a map into $\mathbb{R}^2$, the image of the unit sphere is an ellipsoid. The nonzero singular values of A are $\sigma_1 = \sqrt{5}$ and $\sigma_2 = 2$, so the equation of the ellipsoid is

$$\left(\frac{y_1}{\sigma_1}\right)^2 + \left(\frac{y_2}{\sigma_2}\right)^2 = 1, \quad \text{or} \quad \frac{y_1^2}{5} + \frac{y_2^2}{4} = 1.$$

This is an ellipse in $\mathbb{R}^2$ with semimajor axis $\sqrt{5}$ and semiminor axis 2.

36. Since $\operatorname{rank}(A) = 2$ but A is a map into $\mathbb{R}^3$, the image of the unit sphere is a solid ellipsoid. The nonzero singular values of A are $\sigma_1 = 3$ and $\sigma_2 = 2$, so the equation of the ellipsoid is

$$\left(\frac{y_1}{\sigma_1}\right)^2 + \left(\frac{y_2}{\sigma_2}\right)^2 \le 1, \quad \text{or} \quad \frac{y_1^2}{9} + \frac{y_2^2}{4} \le 1.$$

This is an ellipse in $\mathbb{R}^2$ with semimajor axis 3 and semiminor axis 2 lying in the y_1y_2-plane, together with its interior in that plane.

37. **(a)** From example 7.37, $\|A\|_2$ is the largest singular value, $\sqrt{2}$. Thus $\|A\|_2 = \sqrt{2}$.

(b) Since A is not invertible, $\operatorname{cond}_2(A) = \infty$.

38. **(a)** From example 7.37, $\|A\|_2$ is the largest singular value, 3. Thus $\|A\|_2 = 3$.

(b) Since A is not invertible, $\operatorname{cond}_2(A) = \infty$.

39. We first compute the singular values of A:

$$A^TA = \begin{bmatrix} 1 & 1 \\ 0.9 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.9 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1.9 \\ 1.9 & 1.81 \end{bmatrix}.$$

Then the characteristic polynomial for A^TA is

$$\begin{vmatrix} 2-\lambda & 1.9 \\ 1.9 & 1.81-\lambda \end{vmatrix} = (2-\lambda)(1.81-\lambda) - 3.61 = \lambda^2 - 3.81x + 0.01.$$

Solving using a CAS gives $\lambda_1 \approx 3.807$ and $\lambda_2 \approx 0.00263$, so that the singular values are $\sigma_1 = \sqrt{3.807} \approx 1.951$ and $\sigma_2 = \sqrt{0.00263} \approx 0.0512$.

(a) From Example 7.37, $\|A\|_2 = \sigma_1 \approx 1.95$.

(b) $\operatorname{cond}_2(A) = \frac{\sigma_1}{\sigma_2} \approx 38.1$.

40. We first compute the singular values of A:

$$A^TA = \begin{bmatrix} 10 & 100 \\ 10 & 100 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 10 & 0 \\ 100 & 100 & 1 \end{bmatrix} = \begin{bmatrix} 10100 & 10100 & 100 \\ 10100 & 10100 & 100 \\ 100 & 100 & 1 \end{bmatrix}.$$

Using technology, the characteristic polynomial is $-\lambda^3 + 20201\lambda^2 - 200\lambda$, with roots $\lambda_1 \approx 20201$, $\lambda_2 \approx 0.0099$, and $\lambda_3 = 0$, so that the singular values are $\sigma_1 \approx 142.13$, $\sigma_2 \approx 0.0995$, and $\sigma_3 = 0$.

(a) From Example 7.37, $\|A\|_2 = \sigma_1 \approx 142.13$.

(b) Since A is not invertible, $\operatorname{cond}_2(A) = \infty$.

41. From Exercise 11,

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

is a singular value decomposition of the matrix A from Exercise 3. So

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

42. From Exercise 18,

$$A = U\Sigma V^T = \begin{bmatrix} -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{4}{3\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

is a singular value decomposition of the matrix A from Exercise 8. So

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{5}{9} & \frac{2}{9} \\ \frac{5}{9} & \frac{4}{9} & -\frac{2}{9} \end{bmatrix}.$$

43. From Exercise 19,

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}$$

for a singular value decomposition of the matrix A from Exercise 9. So

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & \frac{1}{2} \\ \frac{1}{5} & 0 \end{bmatrix}.$$

44. We have

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 9 & 0 \\ 2 & 0 & 2 \end{bmatrix}.$$

Then the characteristic polynomial of A^TA is

$$\begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 9-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = (9-\lambda)\left((2-\lambda)^2-4\right) = (9-\lambda)(\lambda^2-4\lambda) = -\lambda(\lambda-9)(\lambda-4).$$

Thus the eigenvalues are $\lambda_1 = 9$, $\lambda_2 = 4$, and $\lambda_3 = 0$, so that the singular values are $\sigma_1 = 3$, $\sigma_2 = 2$, and $\sigma_3 = 0$. To find the corresponding eigenspaces, row-reduce:

$$\begin{bmatrix} A^TA - 9I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} -7 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A^TA - 4I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 5 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A^TA - 0I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 9 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

These vectors are already orthogonal, but we must normalize them:

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{v}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix},$$

so that

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then U is found by

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

We must extend $\mathbf{u}_1$ and $\mathbf{u}_2$ to an orthonormal basis for $\mathbb{R}^3$; clearly

$$\mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

works. Then U is the matrix whose columns are the $\mathbf{u}_i$. Now, Σ should be 3×3, so we add zeros as appropriate, giving

$$A = U\Sigma V^T = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

for a singular value decomposition of A. Then

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & -\frac{1}{3} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}.$$

45. We have

$$A^TA = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}.$$

Then the characteristic polynomial of A^TA is

$$\begin{vmatrix} 5-\lambda & 10 \\ 10 & 20-\lambda \end{vmatrix} = (5-\lambda)(20-\lambda) - 100 = \lambda^2 - 25\lambda = \lambda(\lambda - 25).$$

Thus the eigenvalues are $\lambda_1 = 25$ and $\lambda_2 = 0$, so that the singular values are $\sigma_1 = 5$ and $\sigma_2 = 0$. To find the corresponding eigenspaces, row-reduce:

$$\begin{bmatrix} A^TA - 25I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} -20 & 10 & 0 \\ 10 & -5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} A^TA - 0I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} 5 & 10 & 0 \\ 10 & 20 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

These vectors are already orthogonal, but we must normalize them:

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \quad \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \quad \text{so that} \quad V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

Then U is found by

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

We must extend $\mathbf{u}_1$ to an orthonormal basis for $\mathbb{R}^3$; clearly

$$\mathbf{u}_2 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

works. Then U is the matrix whose columns are the $\mathbf{u}_i$. Now, Σ should be 2×2, so we add zeros as appropriate, giving

$$U = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

for a singular value decomposition of A. Then

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{25} & \frac{2}{25} \\ \frac{2}{25} & \frac{4}{25} \end{bmatrix},$$

so that

$$\overline{\mathbf{x}} = A^+\mathbf{b} = \begin{bmatrix} \frac{1}{25} & \frac{2}{25} \\ \frac{2}{25} & \frac{4}{25} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{13}{25} \\ \frac{26}{25} \end{bmatrix}.$$

46. We have

$$A^TA = \begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Since this matrix is diagonal, we can read off its eigenvalues, which are $\lambda_1 = 9$, $\lambda_2 = 4$, and $\lambda_3 = 0$, so that the singular values are $\sigma_1 = 3$, $\sigma_2 = 2$ and $\sigma_3 = 0$. To find the corresponding eigenspaces, row-reduce:

$$\begin{bmatrix} A^TA - 9I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A^TA - 4I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A^TA - 0I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

These vectors are already orthonormal, so setting $\mathbf{v}_i = \mathbf{x}_i$, we get

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then U is found by

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{2}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

These vectors form an orthonormal basis for $\mathbb{R}^2$, so that U is the matrix whose columns are the $\mathbf{u}_i$. Now, Σ should be 2×3, so we add a column of zeros on the right, giving

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

for a singular value decomposition of A. Then

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

so that

$$\overline{\mathbf{x}} = A^+\mathbf{b} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

47. We have

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}.$$

The characteristic polynomial of A^TA is

$$\begin{vmatrix} 3-\lambda & 3 \\ 3 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 9 = \lambda(\lambda - 6),$$

so that its eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = 0$. Therefore the singular values of A are $\sigma_1 = \sqrt{6}$ and $\sigma_2 = 0$. To find the corresponding eigenspaces, row-reduce:

$$\begin{aligned} \left[A^TA - 6I \mid 0\right] &= \left[\begin{array}{rr|r} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \left[A^TA - 0I \mid 0\right] &= \left[\begin{array}{rr|r} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

These vectors are already orthogonal, but we must normalize them:

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix},$$

so that

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then U is found by

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

We must extend this to an orthonormal basis for $\mathbb{R}^3$. Clearly adjoining $\mathbf{e}_1$ and $\mathbf{e}_2$ forms a basis; we must orthonormalize it. Multiplying $\mathbf{u}_1$ through by $\sqrt{3}$, work instead with $\mathbf{u}_1' = \begin{bmatrix}1\\1\\1\end{bmatrix}$ to simplify computations. Then

$$\mathbf{u}_2' = \mathbf{e}_1 - \frac{\mathbf{e}_1\cdot\mathbf{u}_1'}{\mathbf{u}_1'\cdot\mathbf{u}_1'}\mathbf{u}_1' = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\-\frac{1}{3}\end{bmatrix}$$

$$\mathbf{u}_3' = \mathbf{e}_2 - \frac{\mathbf{e}_2\cdot\mathbf{u}_1'}{\mathbf{u}_1'\cdot\mathbf{u}_1'}\mathbf{u}_1' - \frac{\mathbf{e}_2\cdot\mathbf{u}_2'}{\mathbf{u}_2'\cdot\mathbf{u}_2'}\mathbf{u}_2' = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1/3}{2/3}\begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix} + \begin{bmatrix}\frac{1}{3}\\-\frac{1}{6}\\-\frac{1}{6}\end{bmatrix} = \begin{bmatrix}0\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}.$$

Normalizing these vectors gives

$$\mathbf{u}_2 = \frac{\mathbf{u}_2'}{\|\mathbf{u}_2'\|} = \begin{bmatrix}\frac{2}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\end{bmatrix}, \qquad \mathbf{u}_3 = \frac{\mathbf{u}_3'}{\|\mathbf{u}_3'\|} = \begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}.$$

These vectors form an orthonormal basis for $\mathbb{R}^2$, so that U is the matrix whose columns are the $\mathbf{u}_i$. Now, Σ should be 3×2, so we add two rows of zeros on the bottom, giving

$$U = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}}\end{bmatrix}, \quad \Sigma = \begin{bmatrix}\sqrt{6} & 0\\0&0\\0&0\end{bmatrix}, \quad V = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

for a singular value decomposition of A. Then

$$A^+ = V\Sigma^+U^T = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}} & 0 & 0\\ 0&0&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{6} & \frac{1}{6} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6}\end{bmatrix},$$

so that

$$\overline{\mathbf{x}} = A^+\mathbf{b} = \begin{bmatrix}\frac{1}{6} & \frac{1}{6} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6}\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}.$$

48. We have

$$A^TA = \begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix} = \begin{bmatrix}2&0&2\\0&1&0\\2&0&2\end{bmatrix}.$$

The characteristic polynomial of A^TA is

$$\begin{vmatrix}2-\lambda & 0 & 2\\ 0 & 1-\lambda & 0\\ 2 & 0 & 2-\lambda\end{vmatrix} = (1-\lambda)\left((2-\lambda)^2-4\right) = -\lambda(\lambda-1)(\lambda-4),$$

so that its eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 1$, and $\lambda_3 = 0$. Therefore the singular values of A are $\sigma_1 = 2$,

$\sigma_2 = 1$ and $\sigma_3 = 0$. To find the corresponding eigenspaces, row-reduce:

$$\left[A^TA - 4I \mid 0\right] = \left[\begin{array}{rrr|r} -2 & 0 & 2 & 0 \\ 0 & -3 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[A^TA - 1I \mid 0\right] = \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[A^TA - 0I \mid 0\right] = \left[\begin{array}{rrr|r} 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

These vectors are already orthogonal, but we must normalize them:

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix},$$

so that

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then U is found by

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We must extend this to an orthonormal basis for $\mathbb{R}^3$; clearly adjoining

$$\mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

does the trick. Then U is the matrix whose columns are the $\mathbf{u}_i$. Now, Σ should be 3×3, so we add zeros as appropriate, giving

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

for a singular value decomposition of A. Then

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix},$$

so that

$$\overline{\mathbf{x}} = A^+\mathbf{b} = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}.$$

49. **(a)** The normal equation is $A^TA\overline{\mathbf{x}} = A^T\mathbf{b}$. Now,

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$
$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Row-reducing, we get

$$\left[A^TA \mid A^T\mathbf{b}\right] = \left[\begin{array}{cc|c} 2 & 2 & 1 \\ 2 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right].$$

Therefore $\overline{\mathbf{x}} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$, which satisfies $x + y = \frac{1}{2}$.

(b) Letting $y = t$, the general solution is

$$\overline{\mathbf{x}} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - t \\ t \end{bmatrix}.$$

The length of this solution vector is

$$\sqrt{\left(\frac{1}{2} - t\right)^2 + t^2} = \sqrt{\frac{1}{4} - t + 2t^2}.$$

(c) To minimize this length, it suffices to minimize $\frac{1}{4} - t + 2t^2$. This is an upward-opening parabola, so its minimum occurs at $t = -\frac{b}{2a} = \frac{1}{4}$. So the $\overline{\mathbf{x}}$ of minimum length is

$$\begin{bmatrix} \frac{1}{2} - \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix},$$

as found in Example 7.40.

50. The definition of pseudoinverse in Section 7.3 is $A^+ = \left(A^TA\right)^{-1}A^T$; we must show that this is the same as $V\Sigma^+U^T$, where $U\Sigma V^T$ is an SVD for A. Now, $A^T = \left(U\Sigma V^T\right)^T = V\Sigma^TU^T = V\Sigma U^T$, and

$$A^TA = \left(V\Sigma U^T\right)\left(U\Sigma V^T\right) = V\Sigma^2V^T,$$

since Σ is diagonal, thus symmetric, and U is orthogonal. Since A has linearly independent columns, the rank of A is the number of columns, so by Theorem 7.15, A has r nonzero singular values, so A^TA has only nonzero eigenvalues. Thus A^TA is invertible, and

$$\left(A^TA\right)^{-1} = \left(V\Sigma^2V^T\right)^{-1} = \left(V^T\right)^{-1}\left(\Sigma^2\right)^{-1}V^{-1} = V\left(\Sigma^{-1}\right)^2V^T.$$

So finally,

$$\left(A^TA\right)^{-1}A^T = \left(V\left(\Sigma^{-1}\right)^2V^T\right)\left(V\Sigma U^T\right) = V\left(\Sigma^{-1}\right)^2\left(V^TV\right)\Sigma U^T = V\Sigma^{-1}U^T = V\Sigma^+U^T,$$

which is the definition of A^+ in this section. Note that since Σ is invertible, Exercise 28 tells us that $\Sigma^{-1} = \Sigma^+$.

51. Let $A = U\Sigma V^T$ be an SVD for A. Suppose that

$$\Sigma = \begin{bmatrix} D_{r\times r} & O \\ O & O \end{bmatrix}$$

where D is an invertible diagonal matrix. Then

$$\Sigma^+\Sigma = \begin{bmatrix} I_{r\times r} & O \\ O & O \end{bmatrix}$$

since the nonzero entries of Σ are the inverses of the nonzero entries of Σ. Therefore

$$\Sigma\Sigma^+\Sigma = \Sigma\left(\Sigma^+\Sigma\right) = \begin{bmatrix} D_{r\times r} & O \\ O & O \end{bmatrix}\begin{bmatrix} I_{r\times r} & O \\ O & O \end{bmatrix} = \Sigma.$$

Similarly

$$\Sigma^+\Sigma\Sigma^+ = \left(\Sigma^+\Sigma\right)\Sigma^+ = \begin{bmatrix} I_{r\times r} & O \\ O & O \end{bmatrix}\begin{bmatrix} D_{r\times r}^{-1} & O \\ O & O \end{bmatrix} = \Sigma^+.$$

(a) Just compute the product:

$$\begin{aligned} AA^+A = \left(U\Sigma V^T\right)\left(V\Sigma^+U^T\right)\left(U\Sigma V^T\right) = U\Sigma\left(V^TV\right)\Sigma^+\left(U^TU\right)\Sigma V^T &= U\Sigma\Sigma^+\Sigma V^T \\ &= U\Sigma V^T = A. \end{aligned}$$

(b) Just compute the product:

$$\begin{aligned} A^+AA^+ = \left(V\Sigma^+U^T\right)\left(U\Sigma V^T\right)\left(V\Sigma^+U^T\right) &= V\Sigma^+\left(U^TU\right)\Sigma\left(V^TV\right)\Sigma^+U^T \\ &= V\Sigma^+\Sigma\Sigma^+U^T = V\Sigma^+U^T = A^+. \end{aligned}$$

(c) Compute the transpose of AA^+:

$$\begin{aligned} \left(AA^+\right)^T = \left(AV\Sigma^+U^T\right)^T = U\left(\Sigma^+\right)^T V^TA^T &= U\left(\Sigma^+\right)^T\left(V^TV\right)\Sigma^TU^T \\ &= U\left(\Sigma^+\right)^T\Sigma^TU^T = U\left(\Sigma\Sigma^+\right)^TU^T. \end{aligned}$$

But $\Sigma\Sigma^+$ is a block matrix with the identity in the upper left and zeros elsewhere, so it is symmetric, and this expression becomes

$$U\left(\Sigma\Sigma^+\right)^TU^T = U\Sigma\Sigma^+U^T = \left(U\Sigma V^T\right)\left(V\Sigma^+U^T\right) = AA^+,$$

so that AA^+ is symmetric. A similar argument shows that A^+A is symmetric.

52. As suggested in the hint, suppose that A' is a matrix satisfying the Penrose conditions for A. Then using the fact that A^+ and A' each satisfy the Penrose conditions for A, we have

$$\begin{aligned} A' &= A'AA' && \text{(Condition (b) for } A'\text{)} \\ &= A'\left(AA^+A\right)A' && \text{(Condition (a) for } A^+\text{)} \\ &= A'\left(AA^+\right)\left(AA'\right) \\ &= A'\left(A^+\right)^TA^T\left(A'\right)^TA^T && \text{(Condition (c) for } A' \text{ and } A^+\text{)} \\ &= A'\left(A^+\right)^T\left(AA'A\right)^T \\ &= A'\left(A^+\right)^TA^T && \text{(Condition (a) for } A'\text{)} \\ &= A'AA^+ && \text{(Condition (c) for } A^+\text{).} \end{aligned}$$

Similarly,

$$\begin{aligned} A^+ &= A^+AA^+ && \text{(Condition (b) for } A^+\text{)} \\ &= A^+\left(AA'A\right)A^+ && \text{(Condition (a) for } A'\text{)} \\ &= \left(A^+A\right)\left(A'A\right)A^+ \\ &= A^T\left(A^+\right)^TA^T\left(A'\right)^TA^+ && \text{(Condition (c) for } A' \text{ and } A^+\text{)} \\ &= \left(AA^+A\right)^T\left(A'\right)^TA^+ \\ &= A^T\left(A'\right)^TA^+ && \text{(Condition (a) for } A^+\text{)} \\ &= A'A^TA^+ && \text{(Condition (c) for } A'\text{).} \end{aligned}$$

Since A' and A^+ are equal to the same thing, they are equal to each other.

53. Since $A^+AA^+ = A^+$, the matrix A satisfies Penrose condition (a) for A^+. Similarly, since $AA^+A = A$, the matrix A satisfies Penrose condition (b) for A^+. Finally, since $(A^+A)^T = A^+A$, the matrix A satisfies the first of the Penrose conditions in (c) for A^+; similarly, since $(AA^+)^T = AA^+$, the matrix A satisfies the second of the Penrose conditions in (c) for A^+. Since A satisfies all of the Penrose conditions for A^+, and since by Exercise 52 $(A^+)^+$ is the only matrix satisfying those conditions for A^+, we must have $A = (A^+)^+$.

54. As in Exercise 53, we show that $(A^+)^T$ satisfies the Penrose conditions for A^T; then uniqueness tells us that $(A^+)^T = \left(A^T\right)^+$.

(a) $A^T(A^+)^T A^T = (AA^+A)^T = A^T$.

(b) $(A^+)^T A^T (A^+)^T = (A^+AA^+)^T = (A^+)^T$.

(c) We must show that both $A^T(A^+)^T$ and $(A^+)^T A^T$ are symmetric:

$$\begin{aligned}\left(A^T(A^+)^T\right)^T &= \left((A^+)^T\right)^T\left(A^T\right)^T = A^+A = \left(A^+A\right)^T = A^T\left(A^+\right)^T\\ \left((A^+)^T A^T\right)^T &= \left(A^T\right)^T\left((A^+)^T\right)^T = AA^+ = \left(AA^+\right)^T = \left(A^+\right)^T A^T.\end{aligned}$$

So $(A^+)^T = \left(A^T\right)^+$.

55. We show that A satisfies the Penrose conditions for A; then uniqueness tells us that $A^+ = A$.

(a), (b) $AAA = A^2A = AA = A^2 = A$ since A is idempotent.

(c) We must show that AA is symmetric. But $AA = A^2 = A$, which is symmetric.

So A satisfies all three Penrose conditions and thus $A^+ = A$.

56. It suffices to show that A^+Q^T satisfies the Penrose conditions for QA. Using the fact that Q is orthogonal,

(a) $(QA)\left(A^+Q^T\right)(QA) = Q\left(AA^+\right)\left(Q^TQ\right)A = Q\left(AA^+A\right) = QA$.

(b) $\left(A^+Q^T\right)(QA)\left(A^+Q^T\right) = A^+\left(Q^TQ\right)AA^+Q^T = \left(A^+AA^+\right)Q^T = A^+Q^T$.

(c) We must show that both $(QA)\left(A^+Q^T\right)$ and $\left(A^+Q^T\right)(QA)$ are symmetric:

$$\begin{aligned}\left((QA)\left(A^+Q^T\right)\right)^T &= \left(Q^T\right)^T\left(A^+\right)^T A^TQ^T = Q\left(AA^+\right)^T Q^T = Q\left(AA^+\right)Q^T = (QA)\left(A^+Q^T\right)\\ \left(\left(A^+Q^T\right)(QA)\right)^T &= A^TQ^T\left(Q^T\right)^T\left(A^+\right)^T = A^T\left(A^+\right)^T = \left(A^+A\right)^T\\ &= A^+A = A^+\left(Q^TQ\right)A = \left(A^+Q^T\right)(QA).\end{aligned}$$

So $A^+Q^T = (QA)^+$.

57. Since A is positive definite, it is symmetric by definition. Then by Exercise 27(a), the SVD for A is its orthogonal diagonalization, and $U = V$. The computations are identical to those in Exercise 27(a).

58. Recall that

$$\|A\|_1 = \max\left\{\sum_{i=1}^n |a_{ij}|\right\}, \qquad \|A\|_\infty = \max\left\{\sum_{j=1}^n |a_{ij}|\right\}.$$

If A is diagonal, then the sum of any row or column is just the diagonal entry in that row or column, so that

$$\|A\|_1 = \max\left\{\sum_{i=1}^n |a_{ij}|\right\} = \max\left\{|a_{jj}|\right\}, \qquad \|A\|_\infty = \max\left\{\sum_{j=1}^n |a_{ij}|\right\} = \max\left\{|a_{ii}|\right\},$$

and the two are clearly equal. For $\|A\|_2$, by Exercise 26, since A is symmetric, being diagonal, it follows that the singular values of A are the absolute values of its eigenvalues, which are the absolute values of its diagonal entries. Further, from the material in the text following Example 7.37, we know that $\|A\|_2$ is the largest singular value, which is the maximum magnitude of the diagonal entries. But this is just the value of $\|A\|_1$ and $\|A\|_\infty$.

59. Let σ be the largest singular value of A. Then $\|A\|_2^2 = \sigma^2 = |\lambda|$, where λ is the eigenvalue of A^TA of largest magnitude. Next, since $\left\|A^T\right\|_1 = \|A\|_\infty$, we have $\left\|A^TA\right\|_1 = \left\|A^T\right\|_1 \|A\|_1 = \|A\|_1 \|A\|_\infty$. But by Exercise 34 in Section 7.2, $\left\|A^TA\right\|_1 \ge |\lambda|$. Thus

$$\|A\|_2^2 = |\lambda| \le \left\|A^TA\right\|_1 = \|A\|_1 \|A\|_\infty .$$

60. Let $A = U\Sigma V^T$ be an SVD for a square matrix A. Since U is orthogonal, we have

$$A = U\Sigma V^T = U\Sigma\left(U^TU\right)V^T = \left(U\Sigma U^T\right)\left(UV^T\right) = RQ.$$

The eigenvalues of R are the same as the eigenvalues of Σ, since $R \sim \Sigma$; thus the eigenvalues of R are all nonnegative. Now, since A is square, Σ is an $n \times n$ symmetric matrix, since it is a diagonal matrix, so that

$$R^T = \left(U\Sigma U^T\right)^T = U\Sigma^TU^T = U\Sigma U^T.$$

Hence R is a symmetric matrix with nonnegative eigenvalues, so it is positive semidefinite. Q is orthogonal since both U and V are, so that both U and V^T are; see Theorem 5.8(d) in Section 5.1.

61. From Exercise 11,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

is a singular value decomposition of the matrix A from Exercise 3. Then

$$R = U\Sigma U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$Q = UV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

so that a polar decomposition is

$$A = RQ = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

62. From Exercise 14,

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a singular value decomposition of A. Then

$$R = U\Sigma U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$Q = UV^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

so that a polar decomposition is

$$A = RQ = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

63. We have

$$A^T A = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix}.$$

Then the characteristic polynomial of $A^T A$ is

$$\begin{vmatrix} 10-\lambda & 5 \\ 5 & 5-\lambda \end{vmatrix} = \lambda^2 - 15\lambda + 25.$$

This has roots $\lambda_1,\ \lambda_2 = \frac{15\pm5\sqrt{5}}{2}$, so that these are the eigenvalues of $A^T A$. Thus the singular values of A are $\sigma_1 = \sqrt{\lambda_1}$ and $\sigma_2 = \sqrt{\lambda_2}$. Using a CAS to row-reduce $A - \lambda_1 I$ and $A - \lambda_2 I$ gives eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

These vectors are already orthogonal, but we must normalize them. Using a CAS, we get

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \\ \frac{2}{\sqrt{10+2\sqrt{5}}} \end{bmatrix}, \quad \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}} \\ \frac{2}{\sqrt{10-2\sqrt{5}}} \end{bmatrix},$$

so that

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}} \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{2}{\sqrt{10-2\sqrt{5}}} \end{bmatrix}.$$

Using technology again, we get

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} \frac{2}{\sqrt{10+2\sqrt{5}}} \\ -\frac{5+3\sqrt{5}}{2\sqrt{25+10\sqrt{5}}} \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} \frac{5-\sqrt{5}}{2\sqrt{25-10\sqrt{5}}} \\ \frac{-5+3\sqrt{5}}{2\sqrt{25-10\sqrt{5}}} \end{bmatrix}.$$

Then U is the matrix whose columns are the $\mathbf{u}_i$. Then

$$U = \begin{bmatrix} \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{5-\sqrt{5}}{2\sqrt{25-10\sqrt{5}}} \\ -\frac{5+3\sqrt{5}}{2\sqrt{25+10\sqrt{5}}} & \frac{-5+3\sqrt{5}}{2\sqrt{25-10\sqrt{5}}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{\frac{15+5\sqrt{5}}{2}} & 0 \\ 0 & \sqrt{\frac{15-5\sqrt{5}}{2}} \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}} \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{2}{\sqrt{10-2\sqrt{5}}} \end{bmatrix}.$$

gives a singular value decomposition of A. Then using technology,

$$R = U\Sigma U^T = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \qquad Q = UV^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and $A = RQ$. Whew.

64. We have

$$A^T A = \begin{bmatrix} 4 & -2 & 4 \\ 2 & 2 & -1 \\ -3 & 6 & 6 \end{bmatrix} \begin{bmatrix} 4 & 2 & -3 \\ -2 & 2 & 6 \\ 4 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 81 \end{bmatrix}.$$

This is a diagonal matrix, so its eigenvalues are $\lambda_1 = 81$, $\lambda_2 = 36$, and $\lambda_3 = 9$, so that the singular values of A are $\sigma_1 = 9$, $\sigma_2 = 6$, and $\sigma_3 = 3$. Row-reduce to find the corresponding eigenspaces:

$$\begin{aligned}
\left[A^TA - 81I \mid 0\right] &= \left[\begin{array}{rrr|r} -45 & 0 & 0 & 0 \\ 0 & -72 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\left[A^TA - 36I \mid 0\right] &= \left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ 0 & -27 & 0 & 0 \\ 0 & 0 & 45 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\left[A^TA - 9I \mid 0\right] &= \left[\begin{array}{rrr|r} 27 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 72 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

These vectors are already orthonormal, so V is the matrix

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Next,

$$\begin{aligned}
\mathbf{u}_1 &= \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{9}\begin{bmatrix} 4 & 2 & -3 \\ -2 & 2 & 6 \\ 4 & -1 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \\
\mathbf{u}_2 &= \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{6}\begin{bmatrix} 4 & 2 & -3 \\ -2 & 2 & 6 \\ 4 & -1 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \\
\mathbf{u}_3 &= \frac{1}{\sigma_3} A\mathbf{v}_3 = \frac{1}{3}\begin{bmatrix} 4 & 2 & -3 \\ -2 & 2 & 6 \\ 4 & -1 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}.
\end{aligned}$$

Then U is the matrix whose columns are the $\mathbf{u}_i$, so that

$$U = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

gives a singular value decomposition of A. So

$$\begin{aligned}
R = U\Sigma U^T &= \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix} \\
Q = UV^T &= \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}
\end{aligned}$$

and $A = RQ$.

7.5 Applications

1. As in Example 7.41, let $W = \mathscr{P}[-1,1]$ be the subspace of linear polynomials in $\mathscr{C}[-1,1]$. Then $\{1, x\}$ is an orthogonal basis for W, and we want to find

$$g(x) = \operatorname{proj}_W\left(x^2\right) = \frac{\left\langle 1, x^2\right\rangle}{\langle 1, 1\rangle}1 + \frac{\left\langle x, x^2\right\rangle}{\langle x, x\rangle}x.$$

Computing the various inner products, we get

$$\begin{aligned} \left\langle 1, x^2\right\rangle &= \int_{-1}^{1} x^2\,dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3} & \left\langle x, x^2\right\rangle &= \int_{-1}^{1} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^{1} = 0 \\ \langle 1, 1\rangle &= \int_{-1}^{1} 1\,dx = [x]_{-1}^{1} = 2 & \langle x, x\rangle &= \int_{-1}^{1} x^2\,dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3}. \end{aligned}$$

Therefore the best linear approximation is

$$g(x) = \frac{2/3}{2}1 + \frac{0}{2/3}x = \frac{1}{3}.$$

2. As in Example 7.41, let $W = \mathscr{P}[-1,1]$ be the subspace of linear polynomials in $\mathscr{C}[-1,1]$. Then $\{1, x\}$ is an orthogonal basis for W, and we want to find

$$g(x) = \operatorname{proj}_W\left(x^2 + 2x\right) = \frac{\left\langle 1, x^2 + 2x\right\rangle}{\langle 1, 1\rangle}1 + \frac{\left\langle x, x^2 + 2x\right\rangle}{\langle x, x\rangle}x.$$

Computing the various inner products, we get

$$\begin{aligned} \left\langle 1, x^2 + 2x\right\rangle &= \int_{-1}^{1} \left(x^2 + 2x\right)\,dx = \left[\frac{1}{3}x^3 + x^2\right]_{-1}^{1} = \frac{2}{3} \\ \left\langle x, x^2 + 2x\right\rangle &= \int_{-1}^{1} \left(x^3 + 2x^2\right)\,dx = \left[\frac{1}{4}x^4 + \frac{2}{3}x^3\right]_{-1}^{1} = \frac{4}{3}. \end{aligned}$$

$\langle 1, 1\rangle$ and $\langle x, x\rangle$ were computed in Exercise 1. Therefore the best linear approximation is

$$g(x) = \frac{2/3}{2}1 + \frac{4/3}{2/3}x = \frac{1}{3} + 2x.$$

3. As in Example 7.41, let $W = \mathscr{P}[-1,1]$ be the subspace of linear polynomials in $\mathscr{C}[-1,1]$. Then $\{1, x\}$ is an orthogonal basis for W, and we want to find

$$g(x) = \operatorname{proj}_W\left(x^3\right) = \frac{\left\langle 1, x^3\right\rangle}{\langle 1, 1\rangle}1 + \frac{\left\langle x, x^3\right\rangle}{\langle x, x\rangle}x.$$

Computing the various inner products, we get

$$\left\langle 1, x^3\right\rangle = \int_{-1}^{1} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^{1} = 0 \qquad \left\langle x, x^3\right\rangle = \int_{-1}^{1} x^4\,dx = \left[\frac{1}{5}x^5\right]_{-1}^{1} = \frac{2}{5}.$$

$\langle 1, 1\rangle$ and $\langle x, x\rangle$ were computed in Exercise 1. Therefore the best linear approximation is

$$g(x) = \frac{0}{2}1 + \frac{2/5}{2/3}x = \frac{3}{5}x.$$

4. As in Example 7.41, let $W = \mathscr{P}[-1,1]$ be the subspace of linear polynomials in $\mathscr{C}[-1,1]$. Then $\{1, x\}$ is an orthogonal basis for W, and we want to find

$$g(x) = \operatorname{proj}_W\left(x^3\right) = \frac{\left\langle 1, \sin\frac{\pi x}{2}\right\rangle}{\langle 1, 1\rangle} 1 + \frac{\left\langle x, \sin\frac{\pi x}{2}\right\rangle}{\langle x, x\rangle} x.$$

Computing the various inner products, we get

$$\left\langle 1, \sin\frac{\pi x}{2}\right\rangle = \int_{-1}^{1} \sin\frac{\pi x}{2}\, dx = \left[-\cos\frac{\pi x}{2}\right]_{-1}^{1} = 0$$
$$\left\langle x, \sin\frac{\pi x}{2}\right\rangle = \int_{-1}^{1} x \sin\frac{\pi x}{2}\, dx = \left[-\frac{2}{\pi} x \cos\frac{\pi x}{2}\right]_{-1}^{1} + \frac{2}{\pi}\int_{-1}^{1} \cos\frac{\pi x}{2}\, dx = \frac{2}{\pi}\left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_{-1}^{1} = \frac{8}{\pi^2}.$$

$\langle 1, 1\rangle$ and $\langle x, x\rangle$ were computed in the previous exercises. Therefore the best linear approximation is

$$g(x) = \frac{0}{2} 1 + \frac{8/\pi^2}{2/3} x = \frac{12}{\pi^2} x.$$

5. As in the text example, let $W = \mathscr{P}[-1,1]$ be the subspace of quadratic polynomials in $\mathscr{C}[-1,1]$. Then $\{1, x, x^2 - \frac{1}{3}\}$ is an orthogonal basis for W, and we want to find

$$g(x) = \operatorname{proj}_W(|x|) = \frac{\langle 1, |x|\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle x, |x|\rangle}{\langle x, x\rangle} x + \frac{\left\langle x^2 - \frac{1}{3}, |x|\right\rangle}{\left\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\right\rangle}.$$

Computing the various inner products, we get

$$\begin{aligned}
\langle 1, |x|\rangle &= \int_{-1}^{1} |x|\, dx = \int_{-1}^{0} (-x)\, dx + \int_{0}^{1} x\, dx = \left[-\frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{1} = 1 \\
\langle x, |x|\rangle &= \int_{-1}^{1} x\,|x|\, dx = \int_{-1}^{0} (-x^2)\, dx + \int_{0}^{1} x^2\, dx = \left[-\frac{1}{3}x^3\right]_{-1}^{0} + \left[\frac{1}{3}x^3\right]_{0}^{1} = 0 \\
\left\langle x^2 - \frac{1}{3}, |x|\right\rangle &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right) |x|\, dx \\
&= -\int_{-1}^{0} \left(x^3 - \frac{1}{3}x\right) dx + \int_{0}^{1} \left(x^3 - \frac{1}{3}x\right) dx \\
&= -\left[\frac{1}{4}x^4 - \frac{1}{6}x^2\right]_{-1}^{0} + \left[\frac{1}{4}x^4 - \frac{1}{6}x^2\right]_{0}^{1} \\
&= \frac{1}{6} \\
\langle 1, 1\rangle &= \int_{-1}^{1} 1\, dx = [x]_{-1}^{1} = 2 \\
\langle x, x\rangle &= \int_{-1}^{1} x^2\, dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3} \\
\left\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\right\rangle &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx = \left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right]_{-1}^{1} = \frac{8}{45}.
\end{aligned}$$

Therefore the best quadratic approximation is

$$g(x) = \frac{1}{2} 1 + \frac{0}{2/3} x + \frac{1/6}{8/45}\left(x^2 - \frac{1}{3}\right) = \frac{1}{2} + \frac{15}{16}x^2 - \frac{5}{16}.$$

6. As in the text example, let $W = \mathscr{P}[-1,1]$ be the subspace of quadratic polynomials in $\mathscr{C}[-1,1]$. Then $\{1, x, x^2 - \frac{1}{3}\}$ is an orthogonal basis for W, and we want to find

$$g(x) = \operatorname{proj}_W(|x|) = \frac{\left\langle 1, \cos\frac{\pi x}{2}\right\rangle}{\langle 1,1\rangle}1 + \frac{\left\langle x, \cos\frac{\pi x}{2}\right\rangle}{\langle x,x\rangle}x + \frac{\left\langle x^2-\frac{1}{3}, \cos\frac{\pi x}{2}\right\rangle}{\left\langle x^2-\frac{1}{3}, x^2-\frac{1}{3}\right\rangle}.$$

Computing the various inner products, we get

$$\begin{aligned}
\left\langle 1, \cos\frac{\pi x}{2}\right\rangle &= \int_{-1}^{1} \cos\frac{\pi x}{2}\,dx = \left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_{-1}^{1} = \frac{4}{\pi} \\
\left\langle x, \cos\frac{\pi x}{2}\right\rangle &= \int_{-1}^{1} x\cos\frac{\pi x}{2}\,dx = \left[\frac{2}{\pi}x\sin\frac{\pi x}{2}\right]_{-1}^{1} - \int_{-1}^{1}\sin\frac{\pi x}{2}\,dx = 0 - \left[-\frac{2}{\pi}\cos\frac{\pi x}{2}\right]_{-1}^{1} = 0 \\
\left\langle x^2 - \frac{1}{3}, \cos\frac{\pi x}{2}\right\rangle &= \left\langle x^2, \cos\frac{\pi x}{2}\right\rangle - \frac{1}{3}\left\langle 1, \cos\frac{\pi x}{2}\right\rangle \\
&= \int_{-1}^{1} x^2\cos\frac{\pi x}{2}\,dx - \frac{1}{3}\cdot\frac{4}{\pi} \\
&= \left[\frac{8x}{\pi^2}\cos\frac{\pi x}{2} + \frac{2}{\pi^3}(\pi^2x^2-8)\sin\frac{\pi x}{2}\right]_{-1}^{1} - \frac{4}{3\pi} \\
&= \left(\frac{4}{\pi} - \frac{32}{\pi^3}\right) - \frac{4}{3\pi} \\
&= \frac{8\pi^2 - 96}{3\pi^3}.
\end{aligned}$$

$\langle 1,1\rangle$, $\langle x,x\rangle$, and $\left\langle x^2-\frac{1}{3}, x^2-\frac{1}{3}\right\rangle$ were computed in Exercise 5. Therefore the best quadratic approximation is

$$\begin{aligned}
g(x) &= \frac{4/\pi}{2}1 + \frac{0}{2/3}x + \frac{(8\pi^2-96)/(3\pi^3)}{8/45}\left(x^2 - \frac{1}{3}\right) \\
&= \frac{2}{\pi} + \frac{15\pi^2 - 180}{\pi^3}x^2 - \frac{5\pi^2-60}{\pi^3} \\
&= \frac{60 - 3\pi^2}{\pi^3} + \frac{15\pi^2-180}{\pi^3}x^2.
\end{aligned}$$

7. Set $\mathbf{v}_1 = \mathbf{x}_1 = 1$, and $\mathbf{x}_2 = x$; then

$$\langle 1,1\rangle = \int_0^1 1\,dx = [x]_0^1 = 1, \qquad \langle 1,x\rangle = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2},$$

so that

$$\mathbf{v}_2 = x - \frac{\langle 1,x\rangle}{\langle 1,1\rangle} = x - \frac{1/2}{1}1 = x - \frac{1}{2}.$$

So an orthogonal basis for $\mathscr{P}_1[0,1]$ is $\left\{1, x - \frac{1}{2}\right\}$.

8. From Exercise 7, we know that we can take $\mathbf{v}_1 = 1$ and $\mathbf{v}_2 = x - \frac{1}{2}$. Then let $\mathbf{x}_3 = x^2$. Now,

$$\begin{aligned}
\left\langle 1, x^2\right\rangle &= \int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}, \\
\left\langle x - \frac{1}{2}, x^2\right\rangle &= \left\langle x, x^2\right\rangle - \frac{1}{2}\left\langle 1, x^2\right\rangle = \int_0^1 x^3\,dx - \frac{1}{2}\cdot\frac{1}{3} = \left[\frac{1}{4}x^4\right]_0^1 - \frac{1}{6} = \frac{1}{12} \\
\left\langle x - \frac{1}{2}, x - \frac{1}{2}\right\rangle &= \langle x,x\rangle - \langle 1,x\rangle + \frac{1}{4}\langle 1,1\rangle = \frac{1}{3} - \frac{1}{2} + \frac{1}{4}\cdot 1 = \frac{1}{12},
\end{aligned}$$

so that, using the other inner products we computed in Exercise 7,

$$\mathbf{v}_3 = x^2 - \frac{\langle 1, x^2\rangle}{\langle 1, 1\rangle}1 - \frac{\langle x-\frac{1}{2}, x^2\rangle}{\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle}\left(x-\frac{1}{2}\right) = x^2 - \frac{1}{3}\cdot 1 - \frac{1/12}{1/12}\left(x-\frac{1}{2}\right) = x^2 - x + \frac{1}{6}.$$

Thus an orthogonal basis for $\mathscr{P}_2[0,1]$ is $\left\{1, x-\frac{1}{2}, x^2-x+\frac{1}{6}\right\}$.

9. Using the basis from Exercise 7 together with the various inner products computed in Exercises 7 and 8, the best linear approximation is

$$g(x) = \frac{\langle 1, x^2\rangle}{\langle 1, 1\rangle}1 + \frac{\langle x-\frac{1}{2}, x^2\rangle}{\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle}\left(x-\frac{1}{2}\right) = \frac{1/3}{1}1 + \frac{1/12}{1/12}\left(x-\frac{1}{2}\right) = x - \frac{1}{6}.$$

10. Use the basis from Exercise 7 together with the various inner products computed in Exercises 7 and 8. Additionally,

$$\begin{aligned}\langle 1, \sqrt{x}\rangle &= \int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}\\ \left\langle x-\frac{1}{2}, \sqrt{x}\right\rangle &= \int_0^1 \left(x-\frac{1}{2}\right)x^{1/2}\,dx = \int_0^1 \left(x^{3/2}-\frac{1}{2}x^{1/2}\right)\,dx = \left[\frac{2}{5}x^{5/2}-\frac{1}{3}x^{3/2}\right]_0^1 = \frac{1}{15}.\end{aligned}$$

Then the best linear approximation is

$$g(x) = \frac{\langle 1, \sqrt{x}\rangle}{\langle 1, 1\rangle}1 + \frac{\langle x-\frac{1}{2}, \sqrt{x}\rangle}{\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle}\left(x-\frac{1}{2}\right) = \frac{2/3}{1}1 + \frac{1/15}{1/12}\left(x-\frac{1}{2}\right) = \frac{2}{3} + \frac{4}{5}x - \frac{2}{5} = \frac{4}{15} + \frac{4}{5}x.$$

11. Use the basis from Exercise 7 together with the various inner products computed in Exercises 7 and 8. Additionally,

$$\begin{aligned}\langle 1, e^x\rangle &= \int_0^1 e^x\,dx = [e^x]_0^1 = e-1\\ \left\langle x-\frac{1}{2}, e^x\right\rangle &= \int_0^1 \left(x-\frac{1}{2}\right)e^x\,dx = \int_0^1 \left(xe^x-\frac{1}{2}e^x\right)\,dx = \left[xe^x - e^x - \frac{1}{2}e^x\right]_0^1 = \frac{3}{2} - \frac{1}{2}e.\end{aligned}$$

Then the best linear approximation is

$$\begin{aligned}g(x) &= \frac{\langle 1, e^x\rangle}{\langle 1, 1\rangle}1 + \frac{\langle x-\frac{1}{2}, e^x\rangle}{\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle}\left(x-\frac{1}{2}\right)\\ &= \frac{e-1}{1}1 + \frac{\frac{3}{2}-\frac{1}{2}e}{1/12}\left(x-\frac{1}{2}\right)\\ &= e-1+(18-6e)x-(9-3e)\\ &= (18-6e)x+4e-10.\end{aligned}$$

12. Use the basis from Exercise 7 together with the various inner products computed in Exercises 7 and 8. Additionally,

$$\begin{aligned}\left\langle 1, \sin\frac{\pi x}{2}\right\rangle &= \int_0^1 \sin\frac{\pi x}{2}\,dx = \left[-\frac{2}{\pi}\cos\frac{\pi x}{2}\right]_0^1 = \frac{2}{\pi}\\ \left\langle x-\frac{1}{2}, \sin\frac{\pi x}{2}\right\rangle &= \int_0^1 \left(x-\frac{1}{2}\right)\sin\frac{\pi x}{2}\,dx\\ &= \left[\frac{4}{\pi^2}\sin\frac{\pi x}{2} - \frac{2}{\pi}x\cos\frac{\pi x}{2} + \frac{1}{\pi}\cos\frac{\pi x}{2}\right]_0^1\\ &= \left(\frac{4}{\pi^2}-0+0\right) - \left(0-0+\frac{1}{\pi}\right)\\ &= \frac{4-\pi}{\pi^2}.\end{aligned}$$

Then the best linear approximation is

$$\begin{aligned}
g(x) &= \frac{\left\langle 1, \sin\frac{\pi x}{2}\right\rangle}{\langle 1,1\rangle}1 + \frac{\left\langle x-\frac{1}{2}, \sin\frac{\pi x}{2}\right\rangle}{\left\langle x-\frac{1}{2}, x-\frac{1}{2}\right\rangle}\left(x-\frac{1}{2}\right) \\
&= \frac{2/\pi}{1}1 + \frac{(4-\pi)/\pi^2}{1/12}\left(x-\frac{1}{2}\right) \\
&= \frac{2}{\pi} + \frac{48-12\pi}{\pi^2}x - \frac{24-6\pi}{\pi^2} \\
&= \frac{8\pi-24}{\pi^2} + \frac{48-12\pi}{\pi^2}x.
\end{aligned}$$

13. Use the basis from Exercise 8 together with the various inner products computed in Exercises 7 and 8. Additionally,

$$\begin{aligned}
\left\langle x^2-x+\frac{1}{6}, x^2-x+\frac{1}{6}\right\rangle &= \int_0^1 \left(x^2-x+\frac{1}{6}\right)^2 dx \\
&= \int_0^1 \left(x^4-2x^3+\frac{4}{3}x^2-\frac{1}{3}x+\frac{1}{36}\right) dx \\
&= \left[\frac{1}{5}x^5-\frac{1}{2}x^4+\frac{4}{9}x^3-\frac{1}{6}x^2+\frac{1}{36}x\right]_0^1 \\
&= \frac{1}{180} \\
\langle 1, x^3\rangle &= \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4} \\
\left\langle x-\frac{1}{2}, x^3\right\rangle &= \int_0^1 \left(x-\frac{1}{2}\right)x^3\,dx = \int_0^1 \left(x^4-\frac{1}{2}x^3\right) dx = \left[\frac{1}{5}x^5-\frac{1}{8}x^4\right]_0^1 = \frac{3}{40} \\
\left\langle x^2-x+\frac{1}{6}, x^3\right\rangle &= \int_0^1 \left(x^2-x+\frac{1}{6}\right)x^3\,dx \\
&= \int_0^1 \left(x^5-x^4+\frac{1}{6}x^3\right) dx \\
&= \left[\frac{1}{6}x^6-\frac{1}{5}x^5+\frac{1}{24}x^4\right]_0^1 \\
&= \frac{1}{120}.
\end{aligned}$$

Then the best quadratic approximation is

$$\begin{aligned}
g(x) &= \frac{\langle 1, x^3\rangle}{\langle 1,1\rangle}1 + \frac{\left\langle x-\frac{1}{2}, x^3\right\rangle}{\left\langle x-\frac{1}{2}, x-\frac{1}{2}\right\rangle}\left(x-\frac{1}{2}\right) + \frac{\left\langle x^2-x+\frac{1}{6}, x^3\right\rangle}{\left\langle x^2-x+\frac{1}{6}, x^2-x+\frac{1}{6}\right\rangle}\left(x^2-x+\frac{1}{6}\right) \\
&= \frac{1/4}{1}1 + \frac{3/40}{1/12}\left(x-\frac{1}{2}\right) + \frac{1/120}{1/180}\left(x^2-x+\frac{1}{6}\right) \\
&= \frac{1}{4} + \frac{9}{10}x - \frac{9}{20} + \frac{3}{2}x^2 - \frac{3}{2}x + \frac{1}{4} \\
&= \frac{1}{20} - \frac{3}{5}x + \frac{3}{2}x^2.
\end{aligned}$$

14. Use the basis from Exercise 8 together with the various inner products computed in Exercises 7, 8, and 10. Additionally,

$$\begin{aligned}\left\langle x^2 - x + \frac{1}{6}, \sqrt{x}\right\rangle &= \langle x^2, \sqrt{x}\rangle - \langle x, \sqrt{x}\rangle + \frac{1}{6}\langle 1, \sqrt{x}\rangle \\ &= \int_0^1 x^{5/2}\,dx - \frac{2}{5} + \frac{1}{6}\cdot\frac{2}{3} \\ &= \left[\frac{2}{7}x^{7/2}\right]_0^1 - \frac{2}{5} + \frac{1}{9} \\ &= -\frac{1}{315}.\end{aligned}$$

Then the best quadratic approximation is

$$\begin{aligned}g(x) &= \frac{\langle 1, \sqrt{x}\rangle}{\langle 1, 1\rangle}1 + \frac{\left\langle x - \frac{1}{2}, \sqrt{x}\right\rangle}{\left\langle x - \frac{1}{2}, x - \frac{1}{2}\right\rangle}\left(x - \frac{1}{2}\right) + \frac{\left\langle x^2 - x + \frac{1}{6}, \sqrt{x}\right\rangle}{\left\langle x^2 - x + \frac{1}{6}, x^2 - x + \frac{1}{6}\right\rangle}\left(x^2 - x + \frac{1}{6}\right) \\ &= \frac{2/3}{1}1 + \frac{1/15}{1/12}\left(x - \frac{1}{2}\right) + \frac{-1/315}{1/180}\left(x^2 - x + \frac{1}{6}\right) \\ &= \frac{2}{3} + \frac{4}{5}x - \frac{2}{5} - \frac{4}{7}x^2 + \frac{4}{7}x - \frac{2}{21} \\ &= \frac{6}{35} + \frac{48}{35}x - \frac{4}{7}x^2.\end{aligned}$$

15. Use the basis from Exercise 8 together with the various inner products computed in previous exercises. Additionally,

$$\begin{aligned}\left\langle x^2 - x + \frac{1}{6}, e^x\right\rangle &= \langle x^2, e^x\rangle - \langle x, e^x\rangle + \frac{1}{6}\langle 1, e^x\rangle \\ &= \int_0^1 x^2 e^x\, x - 1 + \frac{1}{6}(e - 1) \\ &= \left[x^2e^x - 2xe^x + 2e^x\right]_0^1 + \frac{1}{6}e - \frac{7}{6} \\ &= e - 2 + \frac{1}{6}e - \frac{7}{6} = \frac{7}{6}e - \frac{19}{6}.\end{aligned}$$

Then the best quadratic approximation is

$$\begin{aligned}g(x) &= \frac{\langle 1, e^x\rangle}{\langle 1, 1\rangle}1 + \frac{\left\langle x - \frac{1}{2}, e^x\right\rangle}{\left\langle x - \frac{1}{2}, x - \frac{1}{2}\right\rangle}\left(x - \frac{1}{2}\right) + \frac{\left\langle x^2 - x + \frac{1}{6}, e^x\right\rangle}{\left\langle x^2 - x + \frac{1}{6}, x^2 - x + \frac{1}{6}\right\rangle}\left(x^2 - x + \frac{1}{6}\right) \\ &= \frac{e-1}{1}1 + \frac{\frac{3}{2} - \frac{1}{2}e}{1/12}\left(x - \frac{1}{2}\right) + \frac{\frac{7}{6}e - \frac{19}{6}}{1/180}\left(x^2 - x + \frac{1}{6}\right) \\ &= e - 1 + (18 - 6e)\left(x - \frac{1}{2}\right) + (210e - 570)\left(x^2 - x + \frac{1}{6}\right) \\ &= (39e - 105) + (588 - 216e)x + (210e - 570)x^2.\end{aligned}$$

16. Use the basis from Exercise 8 together with the various inner products computed in previous exercises. Additionally,

$$\begin{aligned}\left\langle x^2 - x + \frac{1}{6}, \sin\frac{\pi x}{2}\right\rangle &= \left\langle x^2, \sin\frac{\pi x}{2}\right\rangle - \left\langle x, \sin\frac{\pi x}{2}\right\rangle + \frac{1}{6}\left\langle 1, \sin\frac{\pi x}{2}\right\rangle \\ &= \int_0^1 x^2 \sin\frac{\pi x}{2}\,dx - \frac{4}{\pi^2} + \frac{1}{6}\cdot\frac{2}{\pi} \\ &= \left[\frac{8}{\pi^2}x\sin\frac{\pi x}{2} + \frac{16-2\pi^2x^2}{\pi^3}\cos\frac{\pi x}{2}\right]_0^1 - \frac{4}{\pi^2} + \frac{1}{3\pi} \\ &= \frac{8}{\pi^2} - \frac{16}{\pi^3} - \frac{4}{\pi^2} + \frac{1}{3\pi} \\ &= \frac{\pi^2 + 12\pi - 48}{3\pi^3}.\end{aligned}$$

Then the best quadratic approximation is

$$\begin{aligned} g(x) &= \frac{\left\langle 1, \sin\frac{\pi x}{2}\right\rangle}{\langle 1, 1\rangle}1 + \frac{\left\langle x - \frac{1}{2}, \sin\frac{\pi x}{2}\right\rangle}{\left\langle x-\frac{1}{2}, x-\frac{1}{2}\right\rangle}\left(x - \frac{1}{2}\right) + \frac{\left\langle x^2 - x + \frac{1}{6}, \sin\frac{\pi x}{2}\right\rangle}{\left\langle x^2 - x + \frac{1}{6}, x^2 - x + \frac{1}{6}\right\rangle}\left(x^2 - x + \frac{1}{6}\right) \\ &= \frac{2/\pi}{1}1 + \frac{(4-\pi)/\pi^2}{1/12}\left(x - \frac{1}{2}\right) + \frac{(\pi^2+12\pi-48)/(3\pi^3)}{1/180}\left(x^2 - x + \frac{1}{6}\right) \\ &= \frac{2}{\pi} + \frac{48-12\pi}{\pi^2}x - \frac{24-6\pi}{\pi^2} + \frac{60(\pi^2+12\pi-48)}{\pi^3}x^2 - \frac{60(\pi^2+12\pi-48)}{\pi^3}x \\ &\qquad + \frac{10(\pi^2+12\pi-48)}{\pi^3} \\ &= \frac{18\pi^2 + 96\pi - 480}{\pi^3} + \frac{-72\pi^2 - 672\pi + 2880}{\pi^3}x + \frac{60\pi^2 + 720\pi - 2880}{\pi^3}x^2.\end{aligned}$$

17. Assuming k is an integer, recall that cosine is an even function (that is, $\cos(-x) = \cos x$), so that

$$\begin{aligned}\langle 1, \cos kx\rangle &= \int_{-\pi}^{\pi}\cos kx\,dx = \left[\frac{1}{k}\sin kx\right]_{-\pi}^{\pi} = \frac{1}{k}(\sin k\pi - \sin(-k\pi)) = 0 \\ \langle 1, \sin kx\rangle &= \int_{-\pi}^{\pi}\sin kx\,dx = \left[-\frac{1}{k}\cos kx\right]_{-\pi}^{\pi} = \frac{1}{k}(\cos k\pi - \cos(-k\pi)) = \frac{1}{k}(\cos k\pi - \cos k\pi) = 0.\end{aligned}$$

18. Assuming j and k are integers, use the identity

$$\cos(j+k)x + \cos(j-k)x = 2\cos jx\cos kx;$$

this is easily derived by expanding the left-hand side using the formula for $\cos(\alpha+\beta)$. Then if $j \neq k$

$$\begin{aligned}\langle \cos jx, \cos kx\rangle &= \int_{-\pi}^{\pi}\cos jx\cos kx\,dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\left(\cos(j+k)x + \cos(j-k)x\right)\,dx \\ &= \frac{1}{2}\left[\frac{1}{j+k}\sin(j+k)x + \frac{1}{j-k}\sin(j-k)x\right]_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

Note that this computation fails if $j = k$, since then $\cos(j-k)x = 1$ and the integral evaluates differently.

19. Assuming j and k are integers, use the identity

$$\cos(j-k)x - \cos(j+k)x = 2\sin jx \sin kx;$$

this is easily derived by expanding the left-hand side using the formula for $\cos(\alpha+\beta)$. Then if $j \neq k$

$$\begin{aligned}
\langle \sin jx, \sin kx\rangle &= \int_{-\pi}^{\pi} \sin jx \sin kx \, dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos(j-k)x - \cos(j+k)x\right)\, dx \\
&= \frac{1}{2}\left[\frac{1}{j-k}\sin(j-k)x - \frac{1}{j+k}\sin(j+k)x\right]_{-\pi}^{\pi} \\
&= 0.
\end{aligned}$$

Note that this computation fails if $j = k$, since then $\cos(j-k)x = 1$ and the integral evaluates differently.

20. $\|1\|^2 = \langle 1, 1\rangle = \int_{-\pi}^{\pi} 1^2\, dx = [x]_{-\pi}^{\pi} = 2\pi$. Also,

$$\begin{aligned}
\|\cos kx\|^2 &= \langle \cos kx, \cos kx\rangle \\
&= \int_{-\pi}^{\pi} \cos^2 kx \, dx \\
&= \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2kx\right) dx \\
&= \left[\frac{1}{2}x + \frac{1}{4k}\sin 2kx\right]_{-\pi}^{\pi} \\
&= \pi.
\end{aligned}$$

21. We have

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| \, dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} (-x)\, dx + \int_{0}^{\pi} x\, dx\right) = \frac{1}{2\pi}\left(\left[-\frac{1}{2}x^2\right]_{-\pi}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{\pi}\right) = \frac{\pi}{2} \\
a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos x\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-x\cos x)\, dx + \int_{0}^{\pi} x\cos x\, dx\right) \\
&= \frac{1}{\pi}\left([-x\sin x - \cos x]_{-\pi}^{0} + [x\sin x + \cos x]_0^{\pi}\right) \\
&= -\frac{4}{\pi} \\
a_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos 2x\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-x\cos 2x)\, dx + \int_{0}^{\pi} x\cos 2x\, dx\right) \\
&= \frac{1}{\pi}\left(-\left[\frac{1}{4}\cos 2x + \frac{1}{2}x\sin 2x\right]_{-\pi}^{0} + \left[\frac{1}{4}\cos 2x + \frac{1}{2}x\sin 2x\right]_{0}^{\pi}\right) \\
&= 0 \\
a_3 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos 3x\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-x\cos 3x)\, dx + \int_{0}^{\pi} x\cos 3x\, dx\right) \\
&= \frac{1}{\pi}\left(-\left[\frac{1}{9}\cos 3x + \frac{1}{3}x\sin 3x\right]_{-\pi}^{0} + \left[\frac{1}{9}\cos 3x + \frac{1}{3}x\sin 3x\right]_{0}^{\pi}\right) \\
&= -\frac{4}{9\pi}.
\end{aligned}$$

For the b's, we have

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin kx \, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-x \sin kx)\, dx + \int_{0}^{\pi} x \sin kx \, dx\right).$$

In the first integral, substitute $u = -x$; then $du = -dx$ and the integral becomes

$$\int_{-\pi}^{0} (-x \sin kx)\, dx = \int_{\pi}^{0} u \sin(-ku) \cdot (-1)\, du = \int_{\pi}^{0} u \sin u \, du.$$

This is just the negative of the second integral, so their sum vanishes and $b_k = 0$ for all k. Therefore the third-order Fourier approximation of $f(x) = |x|$ on $[-\pi, \pi]$ is

$$a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x.$$

22. We have

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{2\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi} = \frac{\pi^2}{3} \\
a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos x \, dx = \frac{1}{\pi}\left[2x\cos x + (x^2 - 2)\sin x\right]_{-\pi}^{\pi} = -4 \\
a_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos 2x \, dx = \frac{1}{\pi}\left[\frac{1}{2}x\cos 2x + \frac{1}{4}(2x^2 - 1)\sin 2x\right]_{-\pi}^{\pi} = 1 \\
a_3 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos 3x \, dx = \frac{1}{\pi}\left[\frac{2}{9}x\cos 3x + \frac{1}{27}(9x^2 - 2)\sin 3x\right]_{-\pi}^{\pi} = -\frac{4}{9}.
\end{aligned}$$

For the b's, we have

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin kx \, dx \\
&= \frac{1}{\pi}\left[\frac{2}{k^2}x \sin kx + \frac{2 - k^2x^2}{k^3}\cos kx\right]_{-\pi}^{\pi} \\
&= \frac{2 - k^2\pi^2}{k^3}\cos k\pi - \frac{2 - k^2(-\pi)^2}{k^3}\cos(-k\pi) \\
&= 0.
\end{aligned}$$

Thus $b_k = 0$ for all k. Therefore the third-order Fourier approximation of $f(x) = x^2$ on $[-\pi, \pi]$ is

$$a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x.$$

23. We have

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} 1\, dx\right) = \frac{1}{2} \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx \, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} \cos kx \, dx\right) = \frac{1}{\pi}\left[\frac{1}{k}\sin kx\right]_{0}^{\pi} = 0 \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx \, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} \sin kx \, dx\right) = \frac{1}{\pi}\left[-\frac{1}{k}\cos kx\right]_{0}^{\pi} = \frac{1 - (-1)^k}{\pi k}.
\end{aligned}$$

24. We have

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-1)\,dx + \int_0^{\pi} 1\,dx\right) = 0\\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx\\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-\cos kx)\,dx + \int_0^{\pi}\cos kx\,dx\right)\\
&= \frac{1}{\pi}\left(\left[-\frac{1}{k}\sin kx\right]_{-\pi}^{0} + \left[\frac{1}{k}\sin kx\right]_0^{\pi}\right)\\
&= 0\\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx\\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-\sin x)\,dx + \int_0^{\pi}\sin kx\,dx\right)\\
&= \frac{1}{\pi}\left(\left[\frac{1}{k}\cos kx\right]_{-\pi}^{0} + \left[-\frac{1}{k}\cos kx\right]_0^{\pi}\right)\\
&= \frac{1}{\pi}\left(\frac{1-(-1)^k}{k} + \frac{1-(-1)^k}{k}\right)\\
&= \frac{2(1-(-1)^k)}{\pi k}.
\end{aligned}$$

25. We have

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - x)\,dx = \frac{1}{2\pi}\left[\pi x - \frac{1}{2}x^2\right]_{-\pi}^{\pi} = \pi\\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)\cos kx\,dx = \frac{1}{\pi}\left[\frac{\pi}{k}\sin kx - \frac{1}{k^2}\cos kx - \frac{1}{k}x\sin kx\right]_{-\pi}^{\pi}\\
&= 0\\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)\sin kx\,dx = \frac{1}{\pi}\left[-\frac{\pi}{k}\cos kx - \frac{1}{k^2}\sin kx + \frac{1}{k}x\cos kx\right]_{-\pi}^{\pi}\\
&= \frac{2(-1)^k}{k}.
\end{aligned}$$

26. We have

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-x)\,dx + \int_0^{\pi} x\,dx\right) = \frac{1}{2\pi}\left(\left[-\frac{1}{2}x^2\right]_{-\pi}^{0} + \left[\frac{1}{2}x^2\right]_0^{\pi}\right) = \frac{\pi}{2}\\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos kx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x\cos kx)\,dx + \int_0^{\pi} x\cos kx\,dx\right)\\
&= \frac{1}{\pi}\left(\left[-\frac{1}{k}x\sin kx - \frac{1}{k^2}\cos kx\right]_{-\pi}^{0} + \left[\frac{1}{k}x\sin kx + \frac{1}{k^2}\cos kx\right]_0^{\pi}\right)\\
&= \frac{1}{\pi}\left(\left(0 - \frac{1}{k^2}\right) - \left(\frac{\pi}{k}\sin(-k\pi) - \frac{1}{k^2}\cos(-k\pi)\right) + \left(\frac{\pi}{k}\sin k\pi + \frac{1}{k^2}\cos k\pi\right) - \left(0 + \frac{1}{k^2}\right)\right)\\
&= \frac{1}{\pi}\left(-\frac{2}{k^2} + \frac{2}{k^2}\cos k\pi\right)\\
&= \frac{2((-1)^k - 1)}{\pi k^2}
\end{aligned}$$

For the b's, we have

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin kx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-x\sin kx)\,dx + \int_0^{\pi} x\sin kx\,dx\right).$$

In the first integral on the right, substitute $u = -x$; then $du = -dx$ and the integral becomes

$$\begin{aligned}\int_{-\pi}^{0} (-x\sin kx)\,dx + \int_0^{\pi} x\sin kx\,dx &= \int_{\pi}^{0} u\sin(-ku)\cdot(-1\,du) + \int_0^{\pi} x\sin kx\,dx\\ &= \int_{\pi}^{0} u\sin ku\,du + \int_0^{\pi} x\sin kx\,dx\\ &= -\int_0^{\pi} u\sin ku\,du + \int_0^{\pi} x\sin kx\,dx\\ &= 0.\end{aligned}$$

Therefore $b_k = 0$ for all k.

27. **(a)** For any function $f(x)$, we have

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx.$$

In the first integral on the right, make the substitution $u = -x$; then $du = -dx$, and $x = -\pi$ corresponds to $u = \pi$ while $x = 0$ corresponds to $u = 0$. Then if f is odd, the sum above becomes

$$\begin{aligned}\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx &= \int_{\pi}^{0} f(-u)\cdot(-\,du) + \int_0^{\pi} f(x)\,dx\\ &= \int_{\pi}^{0} (-f(u))\cdot(-\,du) + \int_0^{\pi} f(x)\,dx\\ &= \int_{\pi}^{0} f(u)\,du + \int_0^{\pi} f(x)\,dx\\ &= -\int_0^{\pi} f(u)\,du + \int_0^{\pi} f(x)\,dx\\ &= 0.\end{aligned}$$

(b) Note that $\cos kx$ is even, since $\cos(k(-x)) = \cos(-kx) = \cos kx$. If $f(x)$ is odd, then

$$f(-x)\cos(k(-x)) = -f(x)\cos kx,$$

so that $f(x)\cos kx$ is an odd function. From part (a), we see that

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0.$$

28. **(a)** For any function $f(x)$, we have

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx.$$

In the first integral on the right, make the substitution $u = -x$; then $du = -dx$, and $x = -\pi$

corresponds to $u = \pi$ while $x = 0$ corresponds to $u = 0$. Then if f is even, the sum above becomes

$$\begin{aligned}\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx &= \int_{\pi}^{0} f(-u)\cdot(-\,du) + \int_0^{\pi} f(x)\,dx \\ &= \int_{\pi}^{0} f(u)\cdot(-\,du) + \int_0^{\pi} f(x)\,dx \\ &= -\int_{\pi}^{0} f(u)\,du + \int_0^{\pi} f(x)\,dx \\ &= \int_0^{\pi} f(u)\,du + \int_0^{\pi} f(x)\,dx \\ &= 2\int_0^{\pi} f(x)\,dx.\end{aligned}$$

(b) Note that $\sin kx$ is odd, since $\sin(k(-x)) = \sin(-kx) = -\sin kx$. If $f(x)$ is even, then

$$f(-x)\sin(k(-x)) = f(x)\cdot(-\sin kx) = -f(x)\sin kx,$$

so that $f(x)\sin kx$ is an odd function. From part (a) of Exercise 27, we see that

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = 0.$$

Chapter Review

1. **(a)** True. Since 1 and π are both positive scalars, this is a weighted dot product; see Section 7.1.

(b) True. This corresponds to the dot product $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T A\mathbf{v}$, since $A = \begin{bmatrix} 4 & -2 \\ -2 & 4\end{bmatrix}$ is a positive definite matrix (its eigenvalues are 2 and 6).

(c) False. Any matrix A of trace zero will have $\langle A, A\rangle = 0$; for example, take $A = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$.

(d) True. This is a calculation:

$$\|\mathbf{u}+\mathbf{v}\| = \sqrt{\langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle} = \sqrt{\|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2} = \sqrt{4^2 + 2\cdot 2 + \left(\sqrt{5}\right)^2} = \sqrt{25} = 5.$$

(e) True. An element of $\mathbb{R}^1$ is a vector $\begin{bmatrix} v_1\end{bmatrix}$; then the sum norm is $\sum |v_i| = |v_1|$, the max norm is $\max\{|v_i|\} = |v_1|$, and the Euclidean norm is $\sqrt{\sum v_i^2} = \sqrt{v_1^2} = |v_1|$.

(f) False. The condition number satisfies

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}'\|} \le \operatorname{cond}(A)\frac{\|\Delta A\|}{\|A\|},$$

where $\|A\|$ is some matrix norm. The left-hand side measures the maximum percentage change, but even if that is small, $\operatorname{cond}(A)$ could be large.

(g) True. From the inequality in the previous part, if $\operatorname{cond}(A)$ is small, so is the left-hand side, so the matrix is well-conditioned.

(h) False. It is not necessarily *unique*. See Theorems 7.9 and 7.10 in Section 7.3.

(i) True. By Theorem 7.11 in Section 7.3, the standard matrix of P is $P = A\left(A^TA\right)^{-1}A^T$. But since A has orthonormal columns, it follows that $A^TA = I$, so that $P = AA^T$.

(j) False. By Exercise 26 in Section 7.4, the singular values of A are the absolute value of the eigenvalues of A. If A is positive definite, then the eigenvalues are positive, so this statement is true.

2. It is not. For example, $\langle x, x\rangle = 0\cdot 1 + 1\cdot 0 = 0$, but $x \neq 0$.

3. This is an inner product. We show that it satisfies all four properties for an inner product.

1. $\langle A, B\rangle = \operatorname{tr}\left(A^TB\right) = \operatorname{tr}\left(A^TB\right)^T = \operatorname{tr}\left(B^TA\right) = \langle B, A\rangle$.

2. $\langle A, B+C\rangle = \operatorname{tr}\left(A^T(B+C)\right) = \operatorname{tr}\left(A^TB\right) + \operatorname{tr}\left(A^TC\right) = \langle A, B\rangle + \langle A, C\rangle$.

3. $\langle cA, B\rangle = \operatorname{tr}\left(cA^TB\right) = c\operatorname{tr}\left(A^TB\right) = c\,\langle A, B\rangle$.

4. Let $A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$; then

$$A^TA = \begin{bmatrix} a & c \\ b & d\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2\end{bmatrix}.$$

Thus $\langle A, A\rangle = \operatorname{tr}\left(A^TA\right) = a^2+b^2+c^2+d^2 \geq 0$, and $\langle A, A\rangle = 0$ if and only if $a=b=c=d=0$; that is, if and only if $A = O$.

4. False. For example, let $f(x) = x$, $g(x) = x$, and $h(x) = -x$. Then $g(x) + h(x) = 0$, and

$$\begin{aligned}\langle f, g+h\rangle &= \left(\max_{0\leq x\leq 1} f(x)\right)\left(\max_{0\leq x\leq 1}(g(x)+h(x))\right) = 1\cdot 0 = 0\\ \langle f, g\rangle + \langle f, h\rangle &= \left(\max_{0\leq x\leq 1} f(x)\right)\left(\max_{0\leq x\leq 1} g(x)\right) + \left(\max_{0\leq x\leq 1} f(x)\right)\left(\max_{0\leq x\leq 1} h(x)\right) = 1\cdot 1 + 1\cdot 0 = 1.\end{aligned}$$

The two are not equal, so property (2) is not satisfied.

5. We have

$$\left\|1+x+x^2\right\| = \sqrt{\langle 1+x+x^2, 1+x+x^2\rangle} = \sqrt{1\cdot 1 + 1\cdot 1 + 1\cdot 1} = \sqrt{3}.$$

6. We have

$$\begin{aligned}\mathrm{d}(x, x^2) = \left\|x - x^2\right\| = \sqrt{\langle x-x^2, x-x^2\rangle} &= \sqrt{\int_0^1 \left(x-x^2\right)^2\,dx}\\ &= \sqrt{\int_0^1 \left(x^2 - 2x^3 + x^4\right)\,dx} = \sqrt{\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1} = \sqrt{\frac{1}{30}}.\end{aligned}$$

7. Start with

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2\end{bmatrix}.$$

Then

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1\end{bmatrix}, \qquad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{v}_1, \mathbf{x}_2\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1.$$

Computing the two inner products, we get

$$\begin{aligned}\langle \mathbf{v}_1, \mathbf{x}_2\rangle &= \mathbf{v}_1^TA\mathbf{x}_2 = \begin{bmatrix} 1 & 1\end{bmatrix}\begin{bmatrix} 6 & 4 \\ 4 & 6\end{bmatrix}\begin{bmatrix} 1 \\ 2\end{bmatrix} = 30\\ \langle \mathbf{v}_1, \mathbf{v}_1\rangle &= \mathbf{v}_1^TA\mathbf{v}_2 = \begin{bmatrix} 1 & 1\end{bmatrix}\begin{bmatrix} 6 & 4 \\ 4 & 6\end{bmatrix}\begin{bmatrix} 1 \\ 2\end{bmatrix} = 20,\end{aligned}$$

so that

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2\end{bmatrix} - \frac{30}{20}\begin{bmatrix} 1 \\ 1\end{bmatrix} = \begin{bmatrix} 1 \\ 2\end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2}\end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2}\end{bmatrix}.$$

8. Start with $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, $\mathbf{x}_3 = x^2$. Then

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 = 1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{v}_1, \mathbf{x}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = x - \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx}1 = x - \frac{1}{2} \cdot 1 = x - \frac{1}{2} \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{v}_1, \mathbf{x}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{x}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 \\
&= x^2 - \frac{\int_0^1 x^2\,dx}{\int_0^1 1\,dx}1 - \frac{\int_0^1 \left(x - \frac{1}{2}\right) x^2\,dx}{\int_0^1 \left(x - \frac{1}{2}\right)^2\,dx}\left(x - \frac{1}{2}\right) \\
&= x^2 - \frac{1}{3} \cdot 1 - \frac{\int_0^1 \left(x^3 - \frac{1}{2}x^2\right)\,dx}{\int_0^1 \left(x - \frac{1}{2}\right)^2\,dx}\left(x - \frac{1}{2}\right) \\
&= x^2 - \frac{1}{3} - \frac{\left[\frac{1}{4}x^4 - \frac{1}{6}x^3\right]_0^1}{\left[\frac{1}{2}\left(x - \frac{1}{2}\right)^2\right]_0^1}\left(x - \frac{1}{2}\right) \\
&= x^2 - \frac{1}{3} - \frac{1/12}{1/12}\left(x - \frac{1}{2}\right) \\
&= x^2 - \frac{1}{3} - x + \frac{1}{2} \\
&= x^2 - x + \frac{1}{6}.
\end{aligned}$$

9. This is not a norm since it fails to satisfy property (2) of a norm (see Section 7.2):

$$\|c\mathbf{v}\| = c\mathbf{v}^T c\mathbf{v} = c^2 \mathbf{v}^T \mathbf{v} = c^2 \|\mathbf{v}\| \neq c \|\mathbf{v}\|$$

if c is a scalar other than 0 or 1.

10. This is a norm. Let $p(x) = a + bx$; then $p(0) = a$ and $p(1) = a + b$. We show it satisfies all three properties:

1. $\|p(x)\| = |p(0)| + |p(1) - p(0)| = |a| + |a + b - a| = |a| + |b| \geq 0$. Also, $\|p(x)\| = 0$ if and only if $|a| + |b| = 0$; since $|a|$ and $|b|$ are both nonnegative, their sum is zero if and only if they are both zero. So $\|p(x)\| = 0$ if and only if $p(x) = 0$.

2. We have

$$\begin{aligned}
\|cp(x)\| &= |cp(0)| + |cp(1) - cp(0)| = |c|\,|p(0)| + |c(p(1) - p(0))| \\
&= |c|\,|p(0)| + |c|\,|p(1) - p(0)| = |c|\,(|p(0)| + |p(1) - p(0)|) = |c|\,\|p(x)\|\,.
\end{aligned}$$

3. Using the triangle inequality for the usual absolute value, we have

$$\begin{aligned}
\|(p + q)(x)\| &= |(p + q)(0)| + |(p + q)(1) - (p + q)(0)| \\
&= |p(0) + q(0)| + |p(1) - p(0) + q(1) - q(0)| \\
&\leq |p(0)| + |q(0)| + |p(1) - p(0)| + |q(1) - q(0)| \\
&= (|p(0)| + |p(1) - p(0)|) + (|q(0)| + |q(1) - q(0)|) \\
&= \|p(x)\| + \|q(x)\|\,.
\end{aligned}$$

11. The condition number is defined by $\operatorname{cond}(A) = \left\|A^{-1}\right\| \|A\|$, where $\|\cdot\|$ is a matrix norm. Computing A^{-1} gives

$$A^{-1} = \begin{bmatrix} 1 & -11 & 10 \\ -11 & -990 & 1000 \\ 10 & 1000 & -1000 \end{bmatrix}.$$

Then using $\|\cdot\|_1$ gives

$$\left\|A^{-1}\right\|_1 \|A\|_1 = 2010 \cdot 1.21 = 2432.1$$

and $\|\cdot\|_\infty$ gives

$$\left\|A^{-1}\right\|_\infty \|A\|_\infty = 2010 \cdot 1.21 = 2432.1.$$

By either measure, A is ill-conditioned.

12. If $Q = \begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n\end{bmatrix}$ is orthogonal, then each $\mathbf{q}_i$ is a unit vector. This means that

$$\|\mathbf{q}_i\| = \sqrt{\sum_{j=1}^n q_{ij}^2} = \sqrt{1} = 1.$$

Then

$$\|Q\|_F = \sqrt{\sum_{i,j=1}^n q_{ij}^2} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n q_{ij}^2} = \sqrt{\sum_{i=1}^n 1} = \sqrt{n}.$$

13. Using the method of Example 7.26, we have

$$A = \begin{bmatrix}1 & 1\\1 & 2\\1 & 3\\1 & 4\end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix}2\\3\\5\\7\end{bmatrix},$$

so that

$$A^TA = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\\1 & 3\\1 & 4\end{bmatrix} = \begin{bmatrix}4 & 10\\10 & 30\end{bmatrix}.$$

This matrix is invertible, and

$$\left(A^TA\right)^{-1} = \begin{bmatrix}\frac{3}{2} & -\frac{1}{2}\\-\frac{1}{2} & \frac{1}{5}\end{bmatrix}.$$

Then

$$\overline{\mathbf{x}} = \left(A^TA\right)^{-1}A^T\mathbf{b} = \begin{bmatrix}\frac{3}{2} & -\frac{1}{2}\\-\frac{1}{2} & \frac{1}{5}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\end{bmatrix}\begin{bmatrix}2\\3\\5\\7\end{bmatrix} = \begin{bmatrix}0\\\frac{17}{10}\end{bmatrix}.$$

So the least squares solution is $y = 0 + \frac{17}{10}x = \frac{17}{10}x$.

14. Using the method of Example 7.25, we have

$$A = \begin{bmatrix}1 & 2\\1 & 0\\2 & -1\\0 & 5\end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix}1\\0\\-1\\3\end{bmatrix},$$

so that

$$A^TA = \begin{bmatrix}1 & 1 & 2 & 0\\2 & 0 & -1 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 0\\2 & -1\\0 & 5\end{bmatrix} = \begin{bmatrix}6 & 0\\0 & 30\end{bmatrix}.$$

This matrix is invertible, and

$$\left(A^TA\right)^{-1} = \begin{bmatrix}\frac{1}{6} & 0\\0 & \frac{1}{30}\end{bmatrix}.$$

Then

$$\overline{\mathbf{x}} = \left(A^TA\right)^{-1} A^T\mathbf{b} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{30} \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ \frac{3}{5} \end{bmatrix}.$$

15. See Theorem 7.11. With

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

we have

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

so that

$$\left(A^TA\right)^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Then the standard matrix of the projection onto the column space W of A is

$$P = A\left(A^TA\right)^{-1} A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix},$$

so that the projection of $\mathbf{x}$ onto W is

$$P\mathbf{x} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{2}{3} \\ \frac{5}{3} \end{bmatrix}.$$

16. Note that $\mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}^T$. So we should try $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$. Then the column space of A is $\operatorname{span}(\mathbf{u}, \mathbf{v})$, so $P = A\left(A^TA\right)^{-1} A^T$ is the projection matrix onto this span. We now compute P. First, since $\mathbf{u}$ and $\mathbf{v}$ are orthonormal,

$$\mathbf{u}^T\mathbf{u} = \mathbf{u}\cdot\mathbf{u} = 1, \qquad \mathbf{v}^T\mathbf{v} = \mathbf{v}\cdot\mathbf{v} = 1, \qquad \mathbf{u}^T\mathbf{v} = \mathbf{u}\cdot\mathbf{v} = 0, \qquad \mathbf{v}^T\mathbf{u} = \mathbf{v}\cdot\mathbf{u} = 0.$$

Then

$$A^TA = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \end{bmatrix} \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^T\mathbf{u} & \mathbf{u}^T\mathbf{v} \\ \mathbf{v}^T\mathbf{u} & \mathbf{v}^T\mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad\Rightarrow\quad \left(A^TA\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the standard matrix of the projection onto the column space of A (which is the space spanned by $\mathbf{u}$ and $\mathbf{v}$) is

$$P = A\left(A^TA\right)^{-1} A^T = AA^T = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \end{bmatrix} = \mathbf{u}\mathbf{u}^T + \mathbf{v}\mathbf{v}^T,$$

as required.

17. **(a)** We have

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

This diagonal matrix has eigenvalues $\lambda_1 = \lambda_2 = 2$, so the singular values of A are $\sigma_1 = \sigma_2 = \sqrt{2}$.

(b) Compute the corresponding eigenvectors of A^TA by row-reduction:

$$\begin{bmatrix} A^TA - 2I \mid 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and then

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To compute U, we have

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

We must extend this to an orthonormal basis of $\mathbb{R}^3$; clearly adjoining the vector $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is sufficient. Finally, since Σ must be a 2×3 matrix, we append a row of zeroes to the bottom. Then

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

give an SVD for A, and $A = U\Sigma V^T$.

(c) With Σ as above, we compute Σ^+ by transposing Σ and inverting all the nonzero entries:

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

Then

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

18. (a) We have

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}.$$

The characteristic polynomial is $-\lambda^3 + 6\lambda^2 = -\lambda^2(\lambda - 6)$, so $\lambda_1 = 6$ and $\lambda_2 = 0$ are the eigenvalues of A^TA. Therefore the singular values of A are $\sigma_1 = \sqrt{6}$ and $\sigma_2 = 0$.

(b) Compute the corresponding eigenvectors of A^TA by row-reduction:

$$\begin{bmatrix} A^TA - 6I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} -4 & 2 & -2 & 0 \\ 2 & -4 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A^TA - 0I \mid 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ 2 & 2 & -2 & 0 \\ -2 & -2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

We must orthogonalize $\{\mathbf{x}_2, \mathbf{x}_3\}$:

$$\mathbf{x}_3' = \mathbf{x}_3 - \frac{\langle \mathbf{x}_2, \mathbf{x}_3\rangle}{\langle \mathbf{x}_2, \mathbf{x}_2\rangle}\mathbf{x}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{-1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\1\end{bmatrix}.$$

Multiply through by 2 to clear fractions, giving $\mathbf{x}_3'' = \begin{bmatrix}1\\1\\2\end{bmatrix}$. Finally, normalize all three of these vectors:

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix}-\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}, \qquad \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}, \qquad \mathbf{v}_3 = \frac{\mathbf{x}_3''}{\|\mathbf{x}_3''\|} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\end{bmatrix}.$$

Then

$$V = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{bmatrix}.$$

To compute U, we have

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix}1 & 1 & -1\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}.$$

We must extend this to an orthonormal basis of $\mathbb{R}^2$; clearly adjoining the vector

$$\mathbf{u}_2 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}$$

is sufficient. Finally, since Σ must be a 2×3 matrix, we append two columns of zeroes to the right. Then

$$U = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}, \quad \Sigma = \begin{bmatrix}\sqrt{6} & 0 & 0\\0 & 0 & 0\end{bmatrix}, \quad V = \begin{bmatrix}-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{bmatrix}$$

give an SVD for A, and $A = U\Sigma V^T$.

(c) With Σ as above, we compute Σ^+ by transposing Σ and inverting all the nonzero entries:

$$\Sigma^+ = \begin{bmatrix}\frac{1}{\sqrt{6}} & 0\\0 & 0\\0 & 0\end{bmatrix}.$$

Then

$$A^+ = V\Sigma^+U^T = \begin{bmatrix}-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}} & 0\\0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{6} & \frac{1}{6}\\\frac{1}{6} & \frac{1}{6}\\-\frac{1}{6} & -\frac{1}{6}\end{bmatrix}.$$

19. It suffices to show that the eigenvalues of A^TA and those of $(PAQ)^T(PAQ)$ are the same. Now,

$$(PAQ)^T(PAQ) = Q^TA^TP^TPAQ = Q^T\left(A^TA\right)Q$$

since P is an orthogonal matrix. Thus A^TA and $(PAQ)^T(PAQ)$ are similar via the matrix Q. By Theorem 4.22 in Section 4.4, the eigenvalues of A^TA are the same as those of $(PAQ)^T(PAQ)$, so that the singular values of A and PAQ are the same.

20. By the Penrose conditions, we have

$$\begin{aligned}
\left(A^+\right)^2 &= \left(A^+AA^+\right)\left(A^+AA^+\right) \\
&= A^+\left(AA^+\right)\left(A^+A\right)A^+ \\
&= A^+\left(AA^+\right)^T\left(A^+A\right)^TA^+ \\
&= A^+\left(A^+\right)^T\left(A^TA^T\right)\left(A^+\right)^TA^+ \\
&= A^+\left(A^+\right)^T\left(A^2\right)^T\left(A^+\right)^TA^+ \\
&= A^+\left(A^+\right)^TO^T\left(A^+\right)^TA^+ \\
&= O.
\end{aligned}$$

Chapter 8

Codes

8.1 Code Vectors

1. We must have $\mathbf{1}\cdot\mathbf{u} = [1, 1, 1, 1, 1]\cdot[1, 0, 1, 1, d] = 0$ in $\mathbb{Z}_2$. Since this dot product is

$$1\cdot 1 + 1\cdot 0 + 1\cdot 1 + 1\cdot 1 + 1\cdot d = 3 + d = 1 + d \text{ in } \mathbb{Z}_2,$$

we must have $d = 1$, so that the parity check code vector is $\mathbf{v} = [1, 0, 1, 1, 1]$.

2. We must have $\mathbf{1}\cdot\mathbf{u} = [1, 1, 1, 1, 1, 1]\cdot[1, 1, 0, 1, 1, d] = 0$ in $\mathbb{Z}_2$. Since this dot product is

$$1\cdot 1 + 1\cdot 1 + 1\cdot 0 + 1\cdot 1 + 1\cdot 1 + 1\cdot d = 4 + d = d \text{ in } \mathbb{Z}_2,$$

we must have $d = 0$, so that the parity check code vector is $\mathbf{v} = [1, 1, 0, 1, 1, 0]$.

3. Compute $\mathbf{1}\cdot\mathbf{v}$:

$$\mathbf{1}\cdot\mathbf{v} = 1\cdot 1 + 1\cdot 0 + 1\cdot 1 + 1\cdot 0 = 2 = 0 \text{ in } \mathbb{Z}_2.$$

Since the sum is zero, a single error could not have occurred.

4. Compute $\mathbf{1}\cdot\mathbf{v}$:

$$\mathbf{1}\cdot\mathbf{v} = 1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 1\cdot 0 + 1\cdot 1 + 1\cdot 1 = 5 = 1 \text{ in } \mathbb{Z}_2.$$

Since the sum is not zero, a single error may have occurred.

5. Compute $\mathbf{1}\cdot\mathbf{v}$:

$$\mathbf{1}\cdot\mathbf{v} = 1\cdot 0 + 1\cdot 1 + 1\cdot 0 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 = 4 = 0 \text{ in } \mathbb{Z}_2.$$

Since the sum is zero, a single error could not have occurred.

6. Compute $\mathbf{1}\cdot\mathbf{v}$:

$$\mathbf{1}\cdot\mathbf{v} = 1\cdot 1 + 1\cdot 1 + 1\cdot 0 + 1\cdot 1 + 1\cdot 0 + 1\cdot 1 + 1\cdot 1 + 1\cdot 1 = 6 = 0 \text{ in } \mathbb{Z}_2.$$

Since the sum is zero, a single error could not have occurred.

7. We want $[1, 1, 1, 1, 1]\cdot[1, 2, 2, 2, d] = 0$ in $\mathbb{Z}_3$. The dot product is

$$1\cdot 1 + 1\cdot 2 + 1\cdot 2 + 1\cdot 2 + 1\cdot d = 7 + d = 1 + d \text{ in } \mathbb{Z}_3,$$

so the check digit must be $d = 2$, since $1 + 2 = 0$ in $\mathbb{Z}_3$.

8. We want $[1, 1, 1, 1, 1]\cdot[3, 4, 2, 3, d] = 0$ in $\mathbb{Z}_5$. The dot product is

$$1\cdot 3 + 1\cdot 4 + 1\cdot 2 + 1\cdot 3 + 1\cdot d = 12 + d = 2 + d \text{ in } \mathbb{Z}_5,$$

so the check digit must be $d = 3$, since $2 + 3 = 0$ in $\mathbb{Z}_5$.

9. We want $[1, 1, 1, 1, 1, 1] \cdot [1, 5, 6, 4, 5, d] = 0$ in $\mathbb{Z}_7$. The dot product is

$$1 \cdot 1 + 1 \cdot 5 + 1 \cdot 6 + 1 \cdot 4 + 1 \cdot 5 + 1 \cdot d = 21 + d = d \text{ in } \mathbb{Z}_7,$$

so the check digit must be $d = 0$.

10. We want $[1, 1, 1, 1, 1, 1, 1] \cdot [3, 0, 7, 5, 6, 8, d] = 0$ in $\mathbb{Z}_9$. The dot product is

$$1 \cdot 3 + 1 \cdot 0 + 1 \cdot 7 + 1 \cdot 5 + 1 \cdot 6 + 1 \cdot 8 + 1 \cdot d = 29 + d = 2 + d \text{ in } \mathbb{Z}_9,$$

so the check digit must be $d = 7$, since $2 + 7 = 0$ in $\mathbb{Z}_9$.

11. Suppose $\mathbf{u} = [u_1, u_2, \ldots, u_n]$, and suppose that $\mathbf{v}$ differs from $\mathbf{u}$ in exactly one component, say the j^{th} component. Then $\mathbf{v} = [u_1, u_2, \ldots, v_j, \ldots, u_n]$ where $u_j \neq v_j$. So

$$\begin{aligned}
\mathbf{c} \cdot (\mathbf{u} - \mathbf{v}) &= [1, 1, \ldots, 1] \cdot ([u_1, u_2, \ldots, u_j, \ldots, u_n] - [u_1, u_2, \ldots, v_j, \ldots, u_n]) \\
&= [1, 1, \ldots, 1] \cdot [0, 0, \ldots, 0, u_j - v_j, 0, \ldots, 0] \\
&= u_j - v_j \neq 0.
\end{aligned}$$

Thus $\mathbf{c} \cdot (\mathbf{u} - \mathbf{v}) \neq 0$ so that $\mathbf{c} \cdot \mathbf{u} \neq \mathbf{c} \cdot \mathbf{v}$.

12. With $\mathbf{c} = [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1]$, we want $\mathbf{c} \cdot \mathbf{u} = 0$ in $\mathbb{Z}_{10}$:

$$\begin{aligned}
\mathbf{c} \cdot \mathbf{u} &= [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1] \cdot [0, 5, 9, 4, 6, 4, 7, 0, 0, 2, 7, d] \\
&= 3 \cdot 0 + 1 \cdot 5 + 3 \cdot 9 + 1 \cdot 4 + 3 \cdot 6 + 1 \cdot 4 + 3 \cdot 7 + 1 \cdot 0 + 3 \cdot 0 + 1 \cdot 2 + 3 \cdot 7 + 1 \cdot d \\
&= 102 + d = 2 + d \text{ in } \mathbb{Z}_{10}.
\end{aligned}$$

So the check digit must be $d = 8$.

13. With $\mathbf{c} = [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1]$, we want $\mathbf{c} \cdot \mathbf{u} = 0$ in $\mathbb{Z}_{10}$:

$$\begin{aligned}
\mathbf{c} \cdot \mathbf{u} &= [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1] \cdot [0, 1, 4, 0, 1, 4, 1, 8, 4, 1, 2, d] \\
&= 3 \cdot 0 + 1 \cdot 1 + 3 \cdot 4 + 1 \cdot 0 + 3 \cdot 1 + 1 \cdot 4 + 3 \cdot 1 + 1 \cdot 8 + 3 \cdot 4 + 1 \cdot 1 + 3 \cdot 2 + 1 \cdot d \\
&= 50 + d = d \text{ in } \mathbb{Z}_{10}.
\end{aligned}$$

So the check digit must be $d = 0$.

14. Let $\mathbf{u} = [0, 4, 6, 9, 5, 6, 1, 8, 2, 0, 1, 5]$ and $\mathbf{c} = [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1]$. Then

(a) We check if $\mathbf{c} \cdot \mathbf{u} = 0$:

$$\begin{aligned}
\mathbf{c} \cdot \mathbf{u} &= [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1] \cdot [0, 4, 6, 9, 5, 6, 1, 8, 2, 0, 1, 5] \\
&= 3 \cdot 0 + 1 \cdot 4 + 3 \cdot 6 + 1 \cdot 9 + 3 \cdot 5 + 1 \cdot 6 + 3 \cdot 1 + 1 \cdot 8 + 3 \cdot 2 + 1 \cdot 0 + 3 \cdot 1 + 1 \cdot 5 \\
&= 77 = 7 \text{ in } \mathbb{Z}_{10}.
\end{aligned}$$

Since the dot product is nonzero, the UPC cannot be correct.

(b) Substituting an unknown digit d for the 6 in the third position, we get

$$\begin{aligned}
\mathbf{c} \cdot \mathbf{u} &= [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1] \cdot [0, 4, d, 9, 5, 6, 1, 8, 2, 0, 1, 5] \\
&= 3 \cdot 0 + 1 \cdot 4 + 3 \cdot d + 1 \cdot 9 + 3 \cdot 5 + 1 \cdot 6 + 3 \cdot 1 + 1 \cdot 8 + 3 \cdot 2 + 1 \cdot 0 + 3 \cdot 1 + 1 \cdot 5 \\
&= 59 + 3d = 9 + 3d \text{ in } \mathbb{Z}_{10}.
\end{aligned}$$

Solving $9 + 3d = 0$ in $\mathbb{Z}_{10}$ gives $d = 7$, so the correct third digit is 7.

15. Suppose $\mathbf{u} = [u_1, u_2, \ldots, u_{12}]$ is a correct UPC, and suppose that $\mathbf{v}$ differs from $\mathbf{u}$ in exactly one component, say the j^{th} component. Then $\mathbf{v} = [u_1, u_2, \ldots, v_j, \ldots, u_{12}]$ where $u_j \neq v_j$. So

$$\begin{aligned}\mathbf{c}\cdot(\mathbf{u}-\mathbf{v}) &= [3, 1, \ldots, 1]\cdot([u_1, u_2, \ldots, u_j, \ldots, u_{12}] - [u_1, u_2, \ldots, v_j, \ldots, u_{12}])\\ &= [1, 1, \ldots, 1]\cdot[0, 0, \ldots, 0, u_j - v_j, 0, \ldots, 0]\\ &= 1(u_j - v_j) \text{ or } 3(u_j - v_j).\end{aligned}$$

If $1(u_j - v_j) = 0 \in \mathbb{Z}_{10}$, then $u_j = v_j \in \mathbb{Z}_{10}$. Since both u_j and v_j are between 0 and 9, they must be equal. Similarly, if $3(u_j - v_j) = 0 \in \mathbb{Z}_{10}$, then since $3x = 0 \in \mathbb{Z}_{10}$ implies that $x = 0$, we see again that $u_j = v_j \in \mathbb{Z}_{10}$. So neither of these can be zero, since $u_j \neq v_j$. Thus $\mathbf{c}\cdot(\mathbf{u}-\mathbf{v}) \neq 0$ so that $0 = \mathbf{c}\cdot\mathbf{u} \neq \mathbf{c}\cdot\mathbf{v}$. Since $\mathbf{c}\cdot\mathbf{v} \neq 0$, this error will be detected.

16. (a) Compute $\mathbf{c}\cdot\mathbf{u}'$, where $\mathbf{u}'$ is the given UPC with the second and third components transposed:

$$\begin{aligned}\mathbf{c}\cdot\mathbf{u}' &= [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1]\cdot[0, 4, 7, 9, 2, 7, 0, 2, 0, 9, 4, 6]\\ &= 3\cdot 0 + 1\cdot 4 + 3\cdot 7 + 1\cdot 9 + 3\cdot 2 + 1\cdot 7 + 3\cdot 0 + 1\cdot 2 + 3\cdot 0 + 1\cdot 9 + 3\cdot 4 + 1\cdot 6\\ &= 76 = 6 \text{ in } \mathbb{Z}_{10}.\end{aligned}$$

Since the dot product is nonzero, this error will be detected.

(b) Since $3\cdot 4 + 9 = 21$ while $3\cdot 9 + 4 = 31$, and $21 = 31 = 1$ in $\mathbb{Z}_{10}$, transposing the third and fourth entries in the UPC will not change the dot product. So this transposition error will not be detected.

(c) Suppose that transposing u_i and u_{i+1} does not change the dot product, so that if the UPC with the entries transposed is $\mathbf{u}'$, then $\mathbf{c}\cdot\mathbf{u} = \mathbf{c}\cdot\mathbf{u}' = 0$. Then

$$\mathbf{u}' = \mathbf{u} + \mathbf{e}_i(u_{i+1} - u_i) + \mathbf{e}_{i+1}(u_i - u_{i+1}),$$

and thus

$$\begin{aligned}0 = \mathbf{c}\cdot\mathbf{u}' &= \mathbf{c}\cdot(\mathbf{u} + \mathbf{e}_i(u_{i+1} - u_i) + \mathbf{e}_{i+1}(u_i - u_{i+1}))\\ &= \mathbf{c}\cdot\mathbf{u} + \mathbf{c}\cdot\mathbf{e}_i(u_{i+1} - u_i) + \mathbf{c}\cdot\mathbf{e}_{i+1}(u_i - u_{i+1}))\\ &= c_i(u_{i+1} - u_i) + c_{i+1}(u_i - u_{i+1})\\ &= u_i(c_{i+1} - c_i) + u_{i+1}(c_i - c_{i+1}).\end{aligned}$$

If i is even, then $c_i = 1$ and $c_{i+1} = 3$, and we get $2u_i - 2u_{i+1} = 2u_i + 8u_{i+1} = 0$ in $\mathbb{Z}_{10}$. If i is odd, then $c_i = 3$ and $c_{i+1} = 1$, so we get $-2u_i + 2u_{i+1} = 8u_i + 2u_{i+1} = 0$ in $\mathbb{Z}_{10}$. Note that in part (b) above, $i = 3$ is odd, and $8u_3 + 2u_4 = 8\cdot 4 + 2\cdot 9 = 32 + 18 = 50 = 0$ in $\mathbb{Z}_{10}$, as expected.

17. Here $\mathbf{c} = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]$ and $\mathbf{u} = [0, 3, 8, 7, 9, 7, 9, 9, 3, d]$, so that

$$\begin{aligned}\mathbf{c}\cdot\mathbf{u} &= [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]\cdot[0, 3, 8, 7, 9, 7, 9, 9, 3, d]\\ &= 10\cdot 0 + 9\cdot 3 + 8\cdot 8 + 7\cdot 7 + 6\cdot 9 + 5\cdot 7 + 4\cdot 9 + 3\cdot 9 + 2\cdot 3 + 1\cdot d\\ &= 298 + d = 1 + d \text{ in } \mathbb{Z}_{11}.\end{aligned}$$

So the check digit is $d = X$, since X represents 10, and $1 + 10 = 0$ in $\mathbb{Z}_{11}$.

18. Here $\mathbf{c} = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]$ and $\mathbf{u} = [0, 3, 9, 4, 7, 5, 6, 8, 2, d]$, so that

$$\begin{aligned}\mathbf{c}\cdot\mathbf{u} &= [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]\cdot[0, 3, 9, 4, 7, 5, 6, 8, 2, d]\\ &= 10\cdot 0 + 9\cdot 3 + 8\cdot 9 + 7\cdot 4 + 6\cdot 7 + 5\cdot 5 + 4\cdot 6 + 3\cdot 8 + 2\cdot 2 + 1\cdot d\\ &= 246 + d = 4 + d \text{ in } \mathbb{Z}_{11}.\end{aligned}$$

So the check digit is $d = 7$, since $4 + 7 = 0$ in $\mathbb{Z}_{11}$.

19. (a) Let $\mathbf{c}$ be the check vector as in the previous exercise, and let $\mathbf{u} = [0, 4, 4, 9, 5, 0, 8, 3, 5, 6]$. Then

$$\begin{aligned}\mathbf{c}\cdot\mathbf{u} &= [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]\cdot[0, 4, 4, 9, 5, 0, 8, 3, 5, 6]\\ &= 10\cdot 0 + 9\cdot 4 + 8\cdot 4 + 7\cdot 9 + 6\cdot 5 + 5\cdot 0 + 4\cdot 8 + 3\cdot 3 + 2\cdot 5 + 1\cdot 6\\ &= 218 = 9 \text{ in } \mathbb{Z}_{11}.\end{aligned}$$

Since the dot product is not zero, this ISBN-10 is not correct.

(b) Substitute an unknown digit d for the 5 in the fifth entry and recompute the dot product:

$$\begin{aligned}\mathbf{c}\cdot\mathbf{u} &= [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]\cdot[0, 4, 4, 9, d, 0, 8, 3, 5, 6]\\ &= 10\cdot 0 + 9\cdot 4 + 8\cdot 4 + 7\cdot 9 + 6\cdot d + 5\cdot 0 + 4\cdot 8 + 3\cdot 3 + 2\cdot 5 + 1\cdot 6\\ &= 188 + 6d = 1 + 6d \text{ in } \mathbb{Z}_{11}.\end{aligned}$$

Solving $1 + 6d = 0$ in $\mathbb{Z}_{11}$ gives $d = 9$. The correct digit was 9.

20. (a) Let $\mathbf{u}'$ be $\mathbf{u}$ with the fourth and fifth entries transposed: $\mathbf{u}' = [0, 6, 7, 7, 9, 6, 2, 9, 0, 6]$. Then

$$\mathbf{c}\cdot(\mathbf{u}-\mathbf{u}') = \mathbf{c}\cdot[0, 0, 0, 2, -2, 0, 0, 0, 0, 0] = 7\cdot 2 + 6\cdot(-2) = 2 \neq 0.$$

So this transposition error will be detected.

(b) See part (c).

(c) A correct ISBN-10 vector $\mathbf{u}$ satisfies the check constraint (in $\mathbb{Z}_{11}$)

$$\sum_{i=1}^{10}(11-i)u_i = 0.$$

Suppose that u_j and u_{j+1} are transposed, giving a new vector $\mathbf{u}'$. Then $u_i = u_i'$ except for $i = j$ or $i = j+1$, so that

$$\begin{aligned}\mathbf{c}\cdot\mathbf{u}' &= \sum_{i=1}^{10}(11-i)u_i'\\ &= \sum_{i=1}^{10}(11-i)u_i - (11-j)u_j + (11-j)u_j' - (11-(j+1))u_{j+1} + (11-(j+1))u_{j+1}'\\ &= 0 + (11-j)(u_j' - u_j) + (11-(j+1))(u_{j+1}' - u_{j+1})\\ &= (11-j)(u_{j+1} - u_j) + (10-j)(u_j - u_{j+1})\\ &= u_{j+1} - u_j.\end{aligned}$$

So unless u_j and u_{j+1} are equal, in which case the transposition does not change the vector, the resulting checksum will be nonzero, so the error will be detected.

21. (a) Let $\mathbf{c}$ be the ISBN-10 check vector, and let $\mathbf{u}' = [0, 8, 3, 7, 0, 9, 9, 0, 2, 6]$. Then

$$\begin{aligned}\mathbf{c}\cdot\mathbf{u}' &= [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]\cdot[0, 8, 3, 7, 0, 9, 9, 0, 2, 6]\\ &= 10\cdot 0 + 9\cdot 8 + 8\cdot 3 + 7\cdot 7 + 6\cdot 0 + 5\cdot 9 + 4\cdot 9 + 3\cdot 0 + 2\cdot 2 + 1\cdot 6\\ &= 236 = 5 \neq 0 \text{ in } \mathbb{Z}_{11}.\end{aligned}$$

Since the dot product is not zero, this ISBN-10 is not correct.

(b) From part (c) of Exercise 20, the dot product we got in part (a) is the difference (in $\mathbb{Z}_{11}$ of the two transposed entries in the correct vector $\mathbf{u}$. So we are looking for two entries $\mathbf{u}_j' = \mathbf{u}_{j+1}$ and $\mathbf{u}_{j+1}' = \mathbf{u}_j$ such that

$$\mathbf{u}_{j+1} - \mathbf{u}_j = \mathbf{u}_j' - \mathbf{u}_{j+1}' = 5 \text{ in } \mathbb{Z}_{11}.$$

The only such pair are the second and third entries, so the correct ISBN-10 must have been $[0, 3, 8, 7, 0, 9, 9, 0, 2, 6]$.

(c) We were able to correct the error in the ISBN-10 in part (b) because there was only one pair of entries satisfying the required difference constraint. But if an ISBN-10 has two such pairs, we have no way of knowing which pair was actually transposed. For example, consider the vector $\mathbf{u}' = [0, 8, 3, 7, 0, 9, 9, 6, 1, 1]$. Then computing $\mathbf{c} \cdot \mathbf{u}'$ again gives 5, but we do not know whether the transposition occurred between positions 2 and 3 or between positions 8 and 9. The correct ISBN-10 could have been either

$$[0, 3, 8, 7, 0, 9, 9, 6, 1, 1] \text{ or } [0, 8, 3, 7, 0, 9, 9, 1, 6, 1].$$

8.2 Error-Correcting Codes

1. For example, suppose [0, 0] is transmitted and an error occurs in the last component, so that [0, 0, 0, 1] is received. Since this is not a legal code vector, there must have been an error; however, the received vector differs from two legal code vectors, [0, 0, 0, 0] and [0, 1, 0, 1], in one position each, so we cannot tell which binary digit is in error.

2. Any single or double error resulting from a mistransmission of 0 will give a result vector $\mathbf{u}_0$ that has one or two 1's. Any single double error resulting from a mistransmission of 1 will give a result vector $\mathbf{u}_1$ that has one or two 0's, so that it has three or four 1's. So by counting the number of 1's, we can tell what vector was actually transmitted: if there are two or fewer, then a 0 was transmitted; otherwise, a 1 was transmitted.

3. With

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

we get

$$G\mathbf{x} = G\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

4. With G as in Example 8.7 and Exercise 3,

$$G\mathbf{x} = G\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

5. With G as in Example 8.7 and Exercise 3,

$$G\mathbf{x} = G\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\end{bmatrix}.$$

6. With

$$P = \begin{bmatrix}1 & 1 & 0 & 1 & 1 & 0 & 0\\1 & 0 & 1 & 1 & 0 & 1 & 0\\0 & 1 & 1 & 1 & 0 & 0 & 1\end{bmatrix},$$

we have

$$P\mathbf{c}' = P\begin{bmatrix}0\\1\\0\\0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Since the product is zero, no single-bit error occurred. The first four components of the received code are the original message vector, so the original $\mathbf{x} = [0,\ 1,\ 0,\ 0]^T$.

7. With

$$P = \begin{bmatrix}1 & 1 & 0 & 1 & 1 & 0 & 0\\1 & 0 & 1 & 1 & 0 & 1 & 0\\0 & 1 & 1 & 1 & 0 & 0 & 1\end{bmatrix},$$

we have

$$P\mathbf{c}' = P\begin{bmatrix}1\\1\\0\\0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}.$$

Since the product is nonzero, there was an error; since the result vector matches the second column of P, we conclude that the second binary digit was in error. Making that change, $\mathbf{x}$ is the corrected first four columns of the result, or $[1,\ 0,\ 0,\ 0]^T$.

8. With

$$P = \begin{bmatrix}1 & 1 & 0 & 1 & 1 & 0 & 0\\1 & 0 & 1 & 1 & 0 & 1 & 0\\0 & 1 & 1 & 1 & 0 & 0 & 1\end{bmatrix},$$

we have

$$P\mathbf{c}' = P\begin{bmatrix}0\\0\\1\\1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Since the product is nonzero, there was an error; since the result vector matches the sixth column of P, we conclude that the sixth binary digit was in error. The sixth digit is part of the check code, not of the message, so that the correct message is the first four digits of the result, or $\mathbf{x} = [0, 0, 1, 1]^T$.

9. (a) If $\mathbf{a} = [a_1, a_2, a_3, a_4, a_5, a_6] \in \mathbb{Z}_2^6$ is an arbitrary vector, the encoded vector has a seventh component a_7 such that $\sum_{i=1}^7 a_i = 0 \in \mathbb{Z}_2$. This can be encoded using the matrix $P = [1, 1, 1, 1, 1, 1, 1]$, since

$$P \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \sum_{i=1}^{7} a_i.$$

(b) Using the definitions, we have $B = [1, 1, 1, 1, 1, 1]$, $n = 7$, and $k = 6$. Then a standard generator matrix for this code is

$$\begin{bmatrix} I_6 \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

(c) By Theorem 8.1, this is an error-correcting code if and only if the columns of P are nonzero and distinct. But all the columns of P are the same, so this is not an error-correcting code.

10. (a) The code words are found by multiplying G by each possible input:

$$\begin{aligned}
[0, 0] &\quad\to\quad G\begin{bmatrix} 0 \\ 0 \end{bmatrix} = [0, 0, 0, 0, 0]^T \\
[0, 1] &\quad\to\quad G\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [0, 1, 0, 1, 1]^T \\
[1, 0] &\quad\to\quad G\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1, 0, 1, 0, 1]^T \\
[1, 1] &\quad\to\quad G\begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1, 1, 1, 1, 0]^T.
\end{aligned}$$

(b) Here $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, so that the parity check matrix is

$$P = \begin{bmatrix} B & I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Since the columns of P are nonzero and distinct, this is a single error-correcting code.

11. (a) The code words are found by multiplying G by each possible input:

$$\begin{array}{lcl}
[0,\,0,\,0] & \to & G\begin{bmatrix}0\\0\end{bmatrix} = [0,\,0,\,0,\,0,\,0,\,0]^T \\
[0,\,0,\,1] & \to & G\begin{bmatrix}0\\1\end{bmatrix} = [0,\,0,\,1,\,0,\,0,\,1]^T \\
[0,\,1,\,0] & \to & G\begin{bmatrix}1\\0\end{bmatrix} = [0,\,1,\,0,\,0,\,1,\,1]^T \\
[0,\,1,\,1] & \to & G\begin{bmatrix}1\\1\end{bmatrix} = [0,\,1,\,1,\,0,\,1,\,0]^T \\
[1,\,0,\,0] & \to & G\begin{bmatrix}0\\0\end{bmatrix} = [1,\,0,\,0,\,1,\,1,\,1]^T \\
[1,\,0,\,1] & \to & G\begin{bmatrix}0\\1\end{bmatrix} = [1,\,0,\,1,\,1,\,1,\,0]^T \\
[1,\,1,\,0] & \to & G\begin{bmatrix}1\\0\end{bmatrix} = [1,\,1,\,0,\,1,\,0,\,0]^T \\
[1,\,1,\,1] & \to & G\begin{bmatrix}1\\1\end{bmatrix} = [1,\,1,\,1,\,1,\,0,\,1]^T.
\end{array}$$

(b) Here $B = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix}$, so that the parity check matrix is

$$P = \begin{bmatrix}B & I_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1 & 0 & 0\\1 & 1 & 0 & 0 & 1 & 0\\1 & 1 & 1 & 0 & 0 & 1\end{bmatrix}.$$

The columns of P are nonzero, but the third and sixth columns are the same, so this is not a single error-correcting code.

12. The code in Example 8.6 has generating matrix

$$G = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

So we can identify $A = \begin{bmatrix}1\\1\end{bmatrix}$ and then $P = \begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\end{bmatrix}$, appending a copy of the identity matrix to A. From P we see that $k = 1$ and $n - k = 2$, so that $n = 3$. Therefore this is a $(3, 1)$ Hamming code.

13. With $n = 15$ and $k = 11$, there are $n - k = 4$ parity check equations, and P has 15 columns. The four parity check equations give rise to the last four columns of P as a 4×4 identity matrix; the other eleven columns must be the other 11 nonzero elements of $\mathbb{Z}_2^4$ in some order. One possible P is

$$P = \begin{bmatrix}1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0\\1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0\\1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1\end{bmatrix}.$$

Using Theorem 8.1, we get

$$A = B = \begin{bmatrix}1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0\\1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1\end{bmatrix}.$$

Therefore the generating matrix for the Hamming (15,11) code is

$$G = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1\\ 1&1&1&1&0&1&1&1&0&0&0\\ 1&1&1&0&1&1&0&0&1&1&0\\ 1&1&0&1&1&0&1&0&1&0&1\\ 1&0&1&1&1&0&0&1&0&1&1 \end{bmatrix}.$$

14. Suppose that $B = A$; then

$$G = \begin{bmatrix} I_k \\ A \end{bmatrix}, \qquad P = \begin{bmatrix} A & I_{n-k} \end{bmatrix}.$$

Note that A is $(n-k) \times k$. Choose $\mathbf{x} \in \mathbb{Z}_2^n$. Then

$$PG\mathbf{x} = \begin{bmatrix} A & I_{n-k} \end{bmatrix} \begin{bmatrix} I_k \\ A \end{bmatrix} \mathbf{x} = (AI_k + I_{n-k}A)\mathbf{x} = 2A\mathbf{x}.$$

However, in $\mathbb{Z}_2$, $2 = 0$, so that $2A\mathbf{x} = 0A\mathbf{x} = 0$.

15. Suppose

$$G = \begin{bmatrix} I_k \\ A \end{bmatrix}, \qquad P = \begin{bmatrix} A & I_{n-k} \end{bmatrix}$$

are standard generator and parity check matrices for the same code. Choose $\mathbf{x} \in \mathbb{Z}_2^k$, and let $\mathbf{c} = G\mathbf{x}$ be the encoding of $\mathbf{x}$. If there is a transmission error in the i^{th} column of $\mathbf{c}$, receiving $\mathbf{c}'$ instead, then $P\mathbf{c}' = \mathbf{p}_i$. Similarly, if there is a transmission error in the j^{th} column of $\mathbf{c}$, receiving $\mathbf{c}''$ instead, then $P\mathbf{c}'' = \mathbf{p}_j$. Thus if $\mathbf{p}_i = \mathbf{p}_j$, we cannot tell whether the received vector corresponds to an error in the i^{th} or j^{th} column.

8.3 Dual Codes

1. Add column 2 to column 1, giving

$$G' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which is in standard form. Since we did not use R1, this is the same code as C.

2. Perform the following set of column operations:

$$\begin{bmatrix} 1&0&1\\ 1&0&0\\ 0&1&1\\ 1&1&0\\ 0&0&1 \end{bmatrix} \xrightarrow{C_1 \leftrightarrow C_3} \begin{bmatrix} 1&0&1\\ 0&0&1\\ 1&1&0\\ 0&1&1\\ 1&0&0 \end{bmatrix} \xrightarrow{C_2 \leftrightarrow C_3} \begin{bmatrix} 1&1&0\\ 0&1&0\\ 1&0&1\\ 0&1&1\\ 1&0&0 \end{bmatrix}.$$

Continue, adding columns to each other:

$$\begin{bmatrix}1&1&0\\0&1&0\\1&0&1\\0&1&1\\1&0&0\end{bmatrix} \xrightarrow{C_1+C_2\to C_2} \begin{bmatrix}1&0&0\\0&1&0\\1&1&1\\0&1&1\\1&1&0\end{bmatrix} \begin{matrix}\xrightarrow{C_3+C_1\to C_1}\\ \\ \xrightarrow{C_3+C_2\to C_2}\end{matrix} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\1&0&1\\1&1&0\end{bmatrix} = G'.$$

This matrix G' is in standard form. Since we did not use R1, the code C' is the same as the code C.

3. Since the first row of G is zero, we will need to use the operation R1, so the resulting code C' will be equivalent to but not equal to C. First exchange the first and fourth rows, then apply the following column operations:

$$\begin{bmatrix}1&1&1\\1&0&1\\0&1&1\\0&0&0\end{bmatrix} \begin{matrix}\xrightarrow{C_1+C_2\to C_2}\\ \\ \xrightarrow{C_1+C_3\to C_3}\end{matrix} \begin{bmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&0\end{bmatrix} \xrightarrow{C_3+C_2\to C_2} \begin{bmatrix}1&0&0\\1&1&0\\0&0&1\\0&0&0\end{bmatrix} \xrightarrow{C_2+C_1\to C_1} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix} = G'.$$

This matrix G' is in standard form.

4. Since the first and second rows of G are identical, we will need to use the operation R1, so the resulting code C' will be equivalent to but not equal to C. First exchange the first and third rows, then apply the following column operation:

$$\begin{bmatrix}1&0\\1&1\\1&1\\0&0\\1&0\end{bmatrix} \xrightarrow{C_2+C_1\to C_1} \begin{bmatrix}1&0\\0&1\\0&1\\0&0\\1&0\end{bmatrix} = G'.$$

This matrix G' is in standard form.

5. Since there is only one row, and the last entry is not a 1 (the one-dimensional identity matrix), we must use C1. Interchange the second and third columns, giving

$$P' = \begin{bmatrix}1&0&1\end{bmatrix}.$$

This is in standard form; since we used C1, the corresponding code is equivalent to but not equal to C.

6. Perform the following row operations:

$$\begin{bmatrix}1&1&0&1\\1&1&1&1\end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix}1&1&1&1\\1&1&0&1\end{bmatrix} \xrightarrow{R_2+R_1\to R_1} \begin{bmatrix}0&0&1&0\\1&1&0&1\end{bmatrix} = P'.$$

This matrix P' is in standard form. Since we did not use C1, the corresponding code is equal to C.

7. Since the third and fourth columns of P are identical, we cannot get a 3×3 identity matrix in columns three through five without using C1. So the resulting code C' will be equivalent to but not equal to C. First exchange columns 1 and 4, and then perform the following row operations:

$$\begin{bmatrix}1&1&1&0&0\\0&1&0&1&1\\1&0&1&0&1\end{bmatrix} \xrightarrow{R_1+R_3\to R_3} \begin{bmatrix}1&1&1&0&0\\0&1&0&1&1\\0&1&0&0&1\end{bmatrix} \xrightarrow{R_3+R_2\to R_2} \begin{bmatrix}1&1&1&0&0\\0&0&0&1&0\\0&1&0&0&1\end{bmatrix} = P'.$$

This matrix P' is in standard form.

8. Since the third column of P is zero, we cannot get a 2×2 identity matrix in the last two columns without using C1. So the resulting code C' will be equivalent to but not equal to C. First exchange columns 2 and 3, and then perform the following row operation:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = P'.$$

This matrix P' is in standard form.

9. To find the dual code, we must find the set of vectors $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ such that $\mathbf{c} \cdot \mathbf{x} = 0$ for every $\mathbf{c} \in C$. This is the same as finding the null space of the matrix A whose rows are the codewords in C, so that

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $A\mathbf{x} = \mathbf{0}$ gives two equations, one of which is all zeros, so we can ignore it. The second equation is $x_2 = 0$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix},$$

and thus

$$C^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

10. To find the dual code, we must find the set of vectors $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ such that $\mathbf{c} \cdot \mathbf{x} = 0$ for every $\mathbf{c} \in C$. This is the same as finding the null space of the matrix A whose rows are the codewords in C, so that

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then $A\mathbf{x} = \mathbf{0}$ gives four equations, one of which is all zeros, so we can ignore it. The other equations give the linear system

$$\begin{aligned} x_1 + x_2 \phantom{{}+x_3} &= 0 \\ x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0. \end{aligned}$$

The second equation forces $x_3 = 0$, and then either $x_1 = x_2 = 0$ or $x_1 = x_2 = 1$. Thus

$$C^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

11. To find the dual code, we must find the set of vectors $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ such that $\mathbf{c} \cdot \mathbf{x} = 0$ for every $\mathbf{c} \in C$. This is the same as finding the null space of the matrix A whose rows are the codewords in C, so that

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $A\mathbf{x} = \mathbf{0}$ gives four equations, one of which is all zeros, so we can ignore it. The second equation is $x_2 = 0$, and the fourth is $x_4 = 0$. The third is $x_2 + x_4 = 0$, which is satisfied by $x_2 = x_4 = 0$. Finally, x_1 and x_3 are arbitrary. Thus

$$C^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

12. To find the dual code, we must find the set of vectors $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T$ such that $\mathbf{c} \cdot \mathbf{x} = 0$ for every $\mathbf{c} \in C$. This is the same as finding the null space of the matrix A whose rows are the codewords in C, so that

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then $A\mathbf{x} = \mathbf{0}$ gives four equations, one of which is all zeros, so we can ignore it. The remaining three equations give the linear system

$$\begin{aligned} x_2 + x_3 \phantom{{}+x_4} + x_5 &= 0 \\ x_1 \phantom{{}+x_2+x_3} + x_4 \phantom{{}+x_5} &= 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 0. \end{aligned}$$

Note that the third equation is the sum of the first two, so we need only consider the first two. The second equation gives $x_1 = x_4$, while the first tells us that x_2, x_3, and x_5 must have an even number of 1's. Thus

$$C^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

13. Let

$$P^\perp = G^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

$P^\perp$ is in standard form, with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus

$$G^\perp = \begin{bmatrix} I \\ A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

14. Since

$$G^T = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

is not in standard form, add the first row to the second; now it is in standard form and

$$P^\perp = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

so that

$$G^{\perp} = \begin{bmatrix} I \\ A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

15. P^T is not in standard form, but adding the second column of P^T to the first gives

$$G^{\perp} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},$$

which is in standard form with

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then

$$P^{\perp} = \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

16. P^T is not in standard form, but adding the third column of P^T to the first gives

$$G^{\perp} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

which is in standard form with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then

$$P^{\perp} = \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

17. By Theorem 8.2,

$$G^{\perp} = P^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P^{\perp} = G^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

18. **(a)** For E_3, the parity check is that the sum of the entries of the codeword is zero, so the parity check matrix must be $P = \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. Then $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$, so that

$$G = \begin{bmatrix} I \\ A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

For Rep_3, 0 maps to $\mathbf{0}$ and 1 maps to $\mathbf{1}$, so the generator matrix must be

$$G' = \begin{bmatrix} I \\ A' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ so that } A' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then

$$P' = \begin{bmatrix} A' & I \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(b) Note that $G' = P^T$ from part (a). While $P' \neq G^T$, it can be transformed into G^T by adding the second row to the first row and then exchanging the first and second rows, so these are parity check matrices for the same code.

19. The argument is much the same as in Exercise 18. For E_n, the parity check is that the sum of the entries of the codeword is zero, so the parity check matrix must be the $1 \times n$ matrix $P = \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$. Then A is the $1 \times (n-1)$ matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$, so that G is the $n \times (n-1)$ matrix

$$G = \begin{bmatrix} I_{n-1} \\ A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

For Rep_n, 0 maps to $\mathbf{0}$ and 1 maps to $\mathbf{1}$, so the generator matrix must be the $n \times 1$ matrix

$$G' = \begin{bmatrix} I \\ A' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

so that A' is the $(n-1) \times 1$ matrix

$$A' = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = A^T.$$

Then P' is the $(n-1) \times n$ matrix

$$P' = \begin{bmatrix} A' & I_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

We see that $G' = P^T$ and that P' can be transformed into G^T by adding the last row to each of the other rows and then moving the last row to the first row (this is the same as several row exchanges). Therefore E_n and Rep_n are dual codes.

20. We must show that if $\mathbf{x} \in D^\perp$, then $\mathbf{x} \in C^\perp$. But $\mathbf{x} \in D^\perp$ means that $\mathbf{d} \cdot \mathbf{x} = 0$ for all $\mathbf{d} \in D$. Since $C \subset D$, certainly also $\mathbf{c} \cdot \mathbf{x} = 0$ for all $\mathbf{c} \in C$, so that $\mathbf{x} \in C^\perp$.

21. It suffices to show that if G is a generator matrix for C, then it is also a generator matrix for $\left(C^\perp\right)^\perp$. If G is a generator matrix for C, then G^T is a parity check matrix for $C^\perp$ by Theorem 8.2(a). Then by Theorem 8.2(b), we see that $\left(G^T\right)^T = G$ is a generator matrix for $\left(C^\perp\right)^\perp$.

22. Let G and P be generator and parity check matrices for C. We want $G^\perp = G$. But $G^\perp = P^T$, so we want $G = P^T$. Since $n = 6$, it follows from $G = P^T$ that $n - k = k$, so that $k = 3$. So one such code is

$$G = \begin{bmatrix} I_3 \\ I_3 \end{bmatrix}, \qquad P = \begin{bmatrix} I_3 & I_3 \end{bmatrix}.$$

8.4 Linear Codes

1. Since $\mathbf{0} \notin C$, this is not a subspace, hence is not a linear code.

2. Since $\mathbf{0} \in C$ and $\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \mathbf{0} \in C$, this subset is closed under addition, so is a subspace and thus is a linear code.

3. Since $\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} \notin C$, this subset is not closed under addition, so is not a subspace and thus is not a linear code.

4. Since $\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} \notin C$, this subset is not closed under addition, so is not a subspace and thus is not a linear code.

5. Since $\mathbf{0} \in C$ and $\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} \in C$, and similarly for other pairs of vectors, this subset is closed under addition, so is a subspace and thus is a linear code.

6. Since

$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \notin C,$$

this subset is not closed under addition, so is not a subspace and thus is not a linear code.

7. If $w(\mathbf{x}) = m$ and $w(\mathbf{y}) = n$, then consider $\mathbf{x} + \mathbf{y}$. It has a 1 anywhere that either $\mathbf{x}$ or $\mathbf{y}$ does, but not where they both do. In the overlap, where they both have 1's, all of these 1's become 0's in the sum. So

$$w(\mathbf{x} + \mathbf{y}) = w(\mathbf{x}) + w(\mathbf{y}) - 2 \text{ overlap}.$$

So if $w(\mathbf{x})$ and $w(\mathbf{y})$ are both even, so is $w(\mathbf{x} + \mathbf{y})$. Therefore E_n is closed under addition since E_n consists of all vectors with even weight, so it is a linear code.

8. Since $w(\mathbf{0}) = 0$, we see that $\mathbf{0} \notin O_n$, so that O_n is not a subspace and thus not a linear code.

9. The other two bases are the other two pairs of nonzero vectors:

$$\left\{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\} \quad \left(\text{note that } \begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)$$

$$\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\} \quad \left(\text{note that } \begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right).$$

10. **(a)** Since G is $n \times k$, G is 9×4. Also, P is $(n-k) \times n = 5 \times 9$.

(b) G is $n \times k$; P is $(n-k) \times n$.

11. If $\mathbf{c} \in C$ and $\mathbf{c}' \in C^\perp$, then $\mathbf{c} \in \left(C^\perp\right)^\perp$ by definition of the dual code. Therefore $C \subset \left(C^\perp\right)^\perp$. But by the proof of Theorem 8.4(a), $\dim \left(C^\perp\right)^\perp = n - \dim C^\perp = n - (n - \dim C) = \dim C$. So the dimensions of C and $\left(C^\perp\right)^\perp$ are equal; since the first is a subset of the second, it must be the case that $C = \left(C^\perp\right)^\perp$.

12. By Theorem 8.4(b), C has 2^k vectors and $C^\perp$ has 2^{n-k} vectors. If $C = C^\perp$, then $2^k = 2^{n-k}$; multiply through by 2^k to get $2^{2k} = 2^n$, so that $n = 2k$ is even.

13. Starting with the basis for R_2 given in the text,

$$\left\{\mathbf{r}_{21} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{r}_{22} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \mathbf{r}_{23} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\},$$

a basis for R_3 is, by the definitions,

$$\left\{\begin{bmatrix}\mathbf{r}_{21}\\\mathbf{r}_{21}\end{bmatrix}, \begin{bmatrix}\mathbf{r}_{22}\\\mathbf{r}_{22}\end{bmatrix}, \begin{bmatrix}\mathbf{r}_{23}\\\mathbf{r}_{23}\end{bmatrix}, \begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right\} = \left\{\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\1\\1\\1\end{bmatrix}\right\}.$$

So the set of vectors in R_3 is the subspace generated by this basis, which will have 16 vectors (since there are 4 basis elements):

$$\left\{\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\\0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}\right\}.$$

14. **(a)** Since $G_0 = 1$, we get

$$G_1 = \begin{bmatrix}G_0 & \mathbf{0}\\G_0 & \mathbf{1}\end{bmatrix} = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$$

$$G_2 = \begin{bmatrix}G_1 & \mathbf{0}\\G_1 & \mathbf{1}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 0 & 1\\1 & 1 & 1\end{bmatrix}$$

$$G_3 = \begin{bmatrix}G_2 & \mathbf{0}\\G_2 & \mathbf{1}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\1 & 1 & 0 & 0\\1 & 0 & 1 & 0\\1 & 1 & 1 & 0\\1 & 0 & 0 & 1\\1 & 1 & 0 & 1\\1 & 0 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}.$$

(b) Use induction on n. For $n = 1$, since the columns of G_1 form a basis for R_1, we see that G_1 generates R_1. Now assume G_{n-1} generates R_{n-1}. Then we must show that G_n generates all vectors of the form

$$\begin{bmatrix}\mathbf{u}\\\mathbf{u}\end{bmatrix}, \begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix},$$

where $\mathbf{u}$ is a basis vector for R_{n-1}. Let $\mathbf{u}$ be a basis vector for R_{n-1}, and choose $\mathbf{x}$ such that $G_{n-1}\mathbf{x} = \mathbf{u}$; such an $\mathbf{x}$ exists since G_{n-1} generates R_{n-1}. Then

$$\begin{aligned} G_n \begin{bmatrix} \mathbf{x} \\ \mathbf{0} \end{bmatrix} &= \begin{bmatrix} G_{n-1} & 0 \\ G_{n-1} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} G_{n-1}\mathbf{x} \\ G_{n-1}\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{u} \\ \mathbf{u} \end{bmatrix} \\ G_n \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} &= \begin{bmatrix} G_{n-1} & 0 \\ G_{n-1} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} G_{n-1}\mathbf{0} + 0 \cdot \mathbf{1} \\ G_{n-1}\mathbf{0} + 1 \cdot \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}, \end{aligned}$$

as required.

15. G_2 can be put into standard form by the following sequence of column operations:

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{C_2+C_1 \to C_2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} \xrightarrow{C_2+C_1 \to C_1} \\ \xrightarrow{C_3+C_2 \to C_3} \end{matrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{C_3+C_1 \to C_1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A \\ I \end{bmatrix}.$$

This matrix is in standard form, and since we did not use the operation R1, it represents the same code, namely R_2. So $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, and thus the associated parity check matrix is

$$P = \begin{bmatrix} I & A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

16. G cannot be put into standard form without using a row operation, since the 4×4 matrix consisting of the bottom four rows of G has determinant zero. Instead, we use Theorem 8.2: compute a basis for $C^\perp$ and then a generator matrix $G^\perp$ for $C^\perp$; then $\left(G^\perp\right)^T$ is a parity check matrix for C, since $\left(C^\perp\right)^\perp = C$. Now, C is a four-dimensional subspace of $\mathbb{Z}_2^7$, so its complement is three-dimensional. Examining the basis of C given in Exercise 13, clearly a basis for $C^\perp$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

so that $G^\perp$ is the matrix whose columns are these vectors. (Alternatively, if desired one can construct the matrix whose rows are the entries of the basis for R_3 and determine its nullspace, as in the text.) Then

$$P = \left(G^\perp\right)^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

17. Let $E \subseteq C$ be the set of even vectors and $O \subseteq C$ the set of odd vectors. Suppose that O is nonempty, and choose $\mathbf{c}_0 \in O$. Consider the map

$$\psi : E \to C : \mathbf{e} \mapsto \mathbf{c}_0 + \mathbf{e},$$

and let $O' = \{\mathbf{c}_0 + \mathbf{e} : \mathbf{e} \in E\}$ be the image of this map. Note that ψ is one-to-one, since $\mathbf{c}_0 + \mathbf{e}_1 = \mathbf{c}_0 + \mathbf{e}_2$ implies that $\mathbf{e}_1 = \mathbf{e}_2$.

Claim $O' = O$. First, for any $\mathbf{e} \in E$,

$$w(\mathbf{c}_0 + \mathbf{e}) = w(\mathbf{c}_0) + w(\mathbf{e}) - 2 \text{ overlap},$$

from Exercise 7. But $w(\mathbf{c}_0)$ is odd and $w(\mathbf{e})$ is even, so that $w(\mathbf{c}_0+\mathbf{e})$ is odd and therefore $\mathbf{c}_0+\mathbf{e}\in O$. Thus $O'\subseteq O$. Next, choose $\mathbf{o}\in O$. Note that $\mathbf{c}_0+\mathbf{c}_0=\mathbf{0}$ since we are working in $\mathbb{Z}_2$, so that

$$\mathbf{o}=\mathbf{0}+\mathbf{o}=(\mathbf{c}_0+\mathbf{c}_0)+\mathbf{o}=\mathbf{c}_0+(\mathbf{c}_0+\mathbf{o}).$$

But $\mathbf{c}_0+\mathbf{o}\in E$, since its weight is $w(\mathbf{c}_0)+w(\mathbf{o})-2$ overlap, and all three terms are even. Thus $\mathbf{o}=\mathbf{c}_0+\mathbf{e}'$ for $\mathbf{e}'\in E$, so that $\mathbf{o}\in O'$. Hence $O\subset O'$, so that the $O=O'$.

Finally, we see that ψ is a one-to-one map from E onto O, so that the two sets have the same number of elements. Thus exactly have of the code vectors have odd weight.

8.5 The Minimum Distance of a Code

1. Since $x-y=x+y$ in $\mathbb{Z}_2$, we have

$$\mathrm{d}_H(x,y)=\|x-y\|_H=\|x+y\|_H=w(x+y).$$

So the distance between two vectors is just the weight of their sum. Denoting by $\mathbf{c}_0$, $\mathbf{c}_1$, and $\mathbf{c}_2$ the vectors in C, we have

$$w(\mathbf{c}_0+\mathbf{c}_1)=w\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=1,\qquad w(\mathbf{c}_0+\mathbf{c}_2)=w\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=2,\qquad w(\mathbf{c}_1+\mathbf{c}_2)=w\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=1.$$

So the minimum distance is $\mathrm{d}(C)=\min(1,2,1)=1$.

2. Since $x-y=x+y$ in $\mathbb{Z}_2$, we have

$$\mathrm{d}_H(x,y)=\|x-y\|_H=\|x+y\|_H=w(x+y).$$

So the distance between two vectors is just the weight of their sum. Denoting by $\mathbf{c}_0$, $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ the vectors in C, we have

$$\begin{gathered}w(\mathbf{c}_0+\mathbf{c}_1)=w\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)=4,\qquad w(\mathbf{c}_0+\mathbf{c}_2)=w\left(\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right)=2,\qquad w(\mathbf{c}_0+\mathbf{c}_3)=w\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right)=2,\\ w(\mathbf{c}_1+\mathbf{c}_2)=w\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right)=2,\qquad w(\mathbf{c}_1+\mathbf{c}_3)=w\left(\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right)=2,\qquad w(\mathbf{c}_2+\mathbf{c}_3)=w\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)=4.\end{gathered}$$

So the minimum distance is $\mathrm{d}(C)=\min(2,4)=2$.

3. Note that

$$\mathrm{d}_H\left(\begin{bmatrix}1\\1\\0\\\vdots\\0\end{bmatrix},\mathbf{0}\right)=2,$$

so that $\mathrm{d}(C)\leq 2$. Now choose $\mathbf{x},\mathbf{y}\in E_n$, neither of which is the zero vector, with $\mathbf{x}\neq\mathbf{y}$. Then since $\mathbf{x}+\mathbf{y}\in E_n$ and is nonzero, we know that $\mathbf{x}+\mathbf{y}$ has at least two ones, so its weight is at least 2. Therefore no pair of nonzero unequal vectors have a distance less than 2, so that $\mathrm{d}(C)=2$.

4. Since $\mathrm{Rep}_n=\{\mathbf{0},\mathbf{1}\}$, we see that $\mathrm{d}(C)=w(\mathbf{0}+\mathbf{1})=n$.

5. The columns of $G = \begin{bmatrix} A \\ I \end{bmatrix}$ form a basis for the code vectors of C. Since no two columns of A are identical, and the weight of the sum of any two columns of the identity matrix is 2, it follows that the weight of the sum of any two columns of G is at least 3. Thus $\mathrm{d}(C) \geq 3$. But $P = \begin{bmatrix} I & A \end{bmatrix}$, and $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$, so by Theorem 8.7, $\mathrm{d}(C) \leq 3$. Therefore $\mathrm{d}(C) = 3$.

6. It is easy to see that the rows of P are linearly independent (if they were not, the sum of two or three of them would be zero, and it is not). Therefore $\mathrm{rank}(P) = 3$, so that the smallest d for which P has d linearly dependent columns is $d = 4$, and it follows from Theorem 8.7 that $\mathrm{d}(C) = 4$.

7. Let $C = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$ respectively. Then

$$\mathrm{d}_H(\mathbf{c}_1, \mathbf{c}_2) = w\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right) = 3, \qquad \mathrm{d}_H(\mathbf{c}_1, \mathbf{c}_3) = w\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) = 4, \qquad \mathrm{d}_H(\mathbf{c}_1, \mathbf{c}_4) = w\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = 3$$

$$\mathrm{d}_H(\mathbf{c}_2, \mathbf{c}_3) = w\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = 3, \qquad \mathrm{d}_H(\mathbf{c}_2, \mathbf{c}_4) = w\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) = 4, \qquad \mathrm{d}_H(\mathbf{c}_3, \mathbf{c}_4) = w\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right) = 3.$$

Thus $\mathrm{d}(C) = 3$. Since

$$\mathrm{d}_H(\mathbf{c}_1, \mathbf{u}) = w\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = 2, \qquad \mathrm{d}_H(\mathbf{c}_2, \mathbf{u}) = w\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = 1,$$

$$\mathrm{d}_H(\mathbf{c}_3, \mathbf{u}) = w\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right) = 4, \qquad \mathrm{d}_H(\mathbf{c}_4, \mathbf{u}) = w\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right) = 3,$$

we conclude that $\mathbf{u}$ should decode as $\mathbf{c}_2$. Since

$$\mathrm{d}_H(\mathbf{c}_1, \mathbf{v}) = w\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = 3, \qquad \mathrm{d}_H(\mathbf{c}_2, \mathbf{v}) = w\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = 2,$$

$$\mathrm{d}_H(\mathbf{c}_3, \mathbf{v}) = w\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right) = 3, \qquad \mathrm{d}_H(\mathbf{c}_4, \mathbf{v}) = w\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right) = 2,$$

the identity of $\mathbf{v}$ is not determined. It could be decoded as either $\mathbf{c}_2$ or $\mathbf{c}_4$ using nearest neighbor decoding. Since

$$\mathrm{d}_H(\mathbf{c}_1, \mathbf{w}) = w\left(\begin{bmatrix}1\\0\\0\\1\\1\end{bmatrix}\right) = 3, \qquad \mathrm{d}_H(\mathbf{c}_2, \mathbf{w}) = w\left(\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}\right) = 2,$$

$$\mathrm{d}_H(\mathbf{c}_3, \mathbf{w}) = w\left(\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}\right) = 1, \qquad \mathrm{d}_H(\mathbf{c}_4, \mathbf{w}) = w\left(\begin{bmatrix}0\\1\\1\\1\\1\end{bmatrix}\right) = 4,$$

we conclude that $\mathbf{w}$ should decode as $\mathbf{c}_3$.

8. The code generated by (the columns of) G is

$$C = \left\{\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\\0\\0\\1\end{bmatrix}\right\}.$$

Letting the i^{th} element of C be $\mathbf{c}_i$, we have:

- $\min\{\mathrm{d}_H(\mathbf{c}_i, \mathbf{u})\} = 1$ only for $\mathbf{c}_5$, so that $\mathbf{u}$ decodes as $\mathbf{c}_5$.
- $\min\{\mathrm{d}_H(\mathbf{c}_i, \mathbf{v})\} = 1$ only for $\mathbf{c}_8$, so that $\mathbf{v}$ decodes as $\mathbf{c}_8$.
- $\min\{\mathrm{d}_H(\mathbf{c}_i, \mathbf{w})\} = 1$ only for $\mathbf{c}_4$, so that $\mathbf{w}$ decodes as $\mathbf{c}_4$.

9. From Exercise 4, $\mathrm{d}(\mathrm{Rep}_n) = 1$, so that $\mathrm{Rep}_8 = \{\mathbf{0}, \mathbf{1}\} \subset \mathbb{Z}_2^8$ is such a code.

10. If such a code exists, then its parity check matrix P is $(8-2) \times 8 = 6 \times 8$. The rank of such a matrix can be at most 6, so that the maximum number of linearly independent columns is 6. But that means that the minimum number of linearly dependent columns is at most 7. Thus we cannot have $d = 8$, and such a code cannot exist.

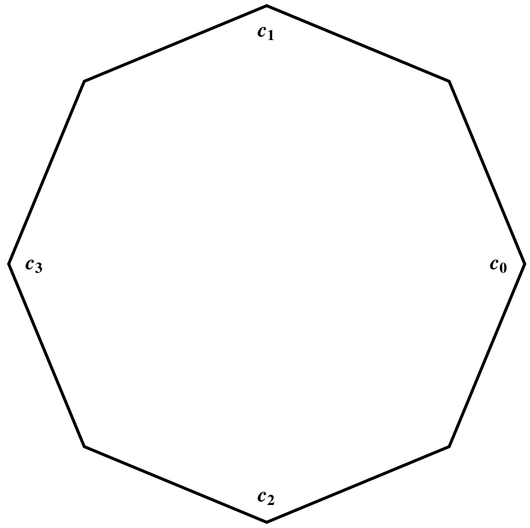
11. If such a code exists, then its parity check matrix P is $(8-5) \times 8 = 3 \times 8$. The rank of such a matrix can be at most 3, so that the maximum number of linearly independent columns is 3. But that means that the minimum number of linearly dependent columns is at most 4. Thus we cannot have $d = 5$, and such a code cannot exist.

12. By Theorem 8.5, R_3 is an $(8, 4)$ linear code, and from Example 8.16, it has minimum distance $d = 2^{3-1} = 4$, so it is an $(8, 4, 4)$ linear code.

13. Model the code vectors as four vertices of an octagon. With

$$\mathbf{c}_0 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \ \mathbf{c}_1 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \ \mathbf{c}_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix},$$

we have



14. If $\mathbf{c}, \mathbf{d} \in C$, then since we are working in $\mathbb{Z}_2$, we have $\mathbf{c} - \mathbf{d} = \mathbf{c} + \mathbf{d}$, and then

$$\mathrm{d}_H(\mathbf{c}, \mathbf{d}) = \|\mathbf{c} - \mathbf{d}\|_H = \|\mathbf{c} + \mathbf{d}\|_H = w(\mathbf{c} + \mathbf{d}).$$

But $\mathbf{c} + \mathbf{d} \in C$. Thus

$$\mathrm{d}_H(C) = \min\{\mathrm{d}_H(\mathbf{c}, \mathbf{d}) : \mathbf{c}, \mathbf{d} \in C\} = \min\{w(\mathbf{c} + \mathbf{d}) : \mathbf{c}, \mathbf{d} \in C\} = \min\{w(\mathbf{c}) : \mathbf{c} \in C,\ \mathbf{c} \neq \mathbf{0}\}.$$

15. A parity check matrix for any linear (n, k) code is $(n - k) \times n$, so that $\operatorname{rank}(P) \neq n - k$. Then the number of linearly independent columns of P is at most $n - k$, so that the minimum number of linearly dependent columns is at most $n - k + 1$. Thus $d \leq n - k + 1$, so that $d - 1 \leq n - k$.

16. From the previous exercise, we know that $d \leq n - k + 1$, so that $n - k \geq d - 1$. Now, if every set of $n - k$ columns of P are linearly independent, then $n - k < d$. Together, these two inequalities force $n - k = d - 1$, or $d = n - k + 1$. Conversely, if $d = n - k + 1$, then Theorem 8.7 says that $n - k + 1$ is the smallest integer for which P has that many linearly dependent columns. But if this is true, then it must be the case that every set of $n - k$ columns is linearly independent.