LaTeX Tutorial Project: Typesetting Mathematics

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Required packages: `amsmath`

Instructions: Typeset the statements depicted in each the following figures. Instead of copying each theorem word-for-word, you can try to summarize the points from the hypothesis and conclusion without sacrificing the practice in both *text style* and *display style* typesetting. For added practice in technical writing, you can then try to restate each theorem in paragraph form using (your own) formal language.

Exercise 1. Fundamental existence and uniqueness of solutions to first-order ordinary differential equations [5]:

Theorem (The Fundamental Existence-Uniqueness Theorem). *Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$ and assume that $\mathbf{f} \in C^1(E)$. Then there exists an $a > 0$ such that the initial value problem*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

has a unique solution $\mathbf{x}(t)$ on the interval $[-a, a]$.

Exercise 2. The Hahn-Banach theorem [3]:

4.3-2 Hahn-Banach Theorem (Normed spaces). *Let f be a bounded linear functional on a subspace Z of a normed space X. Then there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f to X and has the same norm,*

$$\|\tilde{f}\|_X = \|f\|_Z \tag{8}$$

where

$$\|\tilde{f}\|_X = \sup_{\substack{x \in X \\ \|x\|=1}} |\tilde{f}(x)|, \qquad \|f\|_Z = \sup_{\substack{x \in Z \\ \|x\|=1}} |f(x)|$$

(and $\|f\|_Z = 0$ in the trivial case $Z = \{0\}$).

Exercise 3. The dominated convergence theorem [6]:

> **Theorem 19 (General Lebesgue Dominated Convergence Theorem)** *Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g and dominates $\{f_n\}$ on E in the sense that*
>
> $$|f_n| \leq g_n \text{ on } E \text{ for all } n.$$
>
> $$\text{If } \lim_{n\to\infty} \int_E g_n = \int_E g < \infty, \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Exercise 4. The Moore-Penrose (pseudo)inverse [4]:

> **Theorem 7.21.** *Let $A \in \mathbb{R}^{p\times q}$ be given with SVD $A = U\Sigma V^T$ where $\Sigma \in \mathbb{R}^{p\times q}$ is defined by*
>
> $$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \dots & 0 \\ 0 & \dots & 0 & \sigma_k & 0 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}$$
>
> *and k is the number of nonzero singular values of A. Then, the Moore-Penrose pseudoinverse of A is*
>
> $$A^+ = V\Sigma^+ U^T$$
>
> *where $\Sigma^+ \in \mathbb{R}^{q\times p}$ is defined by*
>
> $$\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{\sigma_2} & \dots & \dots & 0 \\ \vdots & 0 & \ddots & \dots & 0 \\ 0 & \dots & 0 & \frac{1}{\sigma_k} & 0 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}.$$

Exercise 5. The central limit theorem [1]:

> **Theorem 3.4.1.** *Let $X_1, X_2, \dots$ be i.i.d. with $EX_i = \mu$, $var(X_i) = \sigma^2 \in (0,\infty)$. If $S_n = X_1 + \cdots + X_n$ then*
>
> $$(S_n - n\mu)/\sigma n^{1/2} \Rightarrow \chi$$
>
> *where χ has the standard normal distribution.*
>
> This notation is non-standard but convenient. To see the logic note that the square of a normal has a chi-squared distribution.

Exercise 6. Sturm-Liouville Theorems [2]: Here, first define a **Sturn-Liouville problem** as the following partial differential equation with the given boundary conditions. Then, comprise the list of theorems regarding such problems.

$$\boxed{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0} \qquad a < x < b, \qquad \boxed{\begin{aligned} \beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) &= 0 \\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) &= 0, \end{aligned}}$$

1. **All the eigenvalues λ are real.**
2. **There exist an infinite number of eigenvalues:**
$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$
   a. **There is a smallest eigenvalue, usually denoted λ_1.**
   b. **There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.**
3. **Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.**
4. **The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:**
$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$
Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).
5. **Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,**
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$
6. **Any eigenvalue can be related to its eigenfunction by the Rayleigh quotient:**
$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2\sigma\,dx},$$
where the boundary conditions may somewhat simplify this expression.

References

[1] R. Durrett. *Probability: Theory and Examples.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 5 edition, 2010.

[2] R. Haberman. *Applied Partial Differential Equations: With Fourier Series and Boundary Value Problems.* Pearson Prentice Hall, 2004.

[3] E. Kreyszig. *Introductory Functional Analysis with Applications.* Wiley classics library. Wiley, 1978.

[4] S. Pankavich. Matrix decomposition notes. Self-published, 2018.

[5] L. Perko. *Differential Equations and Dynamical Systems.* Springer-Verlag, Berlin, Heidelberg, 1991.

[6] H. L. Royden and P. Fitzpatrick. *Real Analysis.* Prentice Hall, 2010.