Stage 2 Specialists Mathematics Investigation

MATHEMATICAL MODELS OF MISSILE FLIGHT PATHS

Introduction:

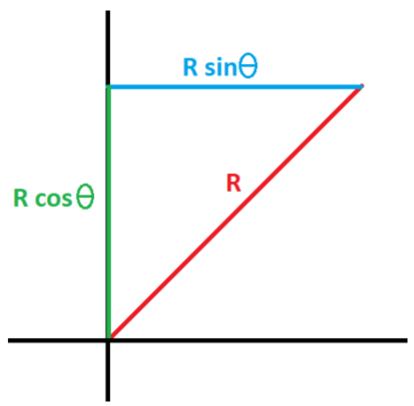
Vector calculus and parametric equations are a powerful mathematical tool used to describe the motion and behaviour of objects in various fields of science and engineering. Unlike traditional Cartesian equations, which express relationships between variables directly in terms of x and y coordinates, parametric equations define a set of equations using one or more auxiliary parameters, typically denoted as t. These parameters represent time or another independent variable, allowing us to describe complex curves, shapes, and motions with great precision. Parametric equations and vector calculus are especially useful when dealing with dynamic systems, such as the motion of projectiles, the trajectory of planets, or the path of particles in fluid dynamics.

This investigation aims to utilise vector calculus and parametric equations to model the flight path of Intercontinental Ballistic Missiles. In case 1, the basic parabolic trajectory of a missile will be analysed, where the only force acting on the missile is gravity. Case 2 explores the flight path of a missile that burns fuel to produce thrust, resulting in variable mass, consequently increasing the complexity of its trajectory. Case 3 utilises complex integration techniques and substitutions to determine the flight path of a missile when linear air drag, variable mass, and thrust are present. Case 4 introduces the concept of quadratic drag, non-explicitly solvable differential equations, and the Runge-Kutta method, an algorithm that numerically solves differential equations. Case 5 delves into the realistic flight path of one of the most common Intercontinental Ballistic Missiles, the "Minute Man", and includes quadratic air drag, variable mass, variable air density, variable gravity, and thrust in the parametric equations. Case 6 expands on the flight path of a missile and investigates the parametric equations of intercepting a missile given its trajectory.

Case 1: Basic Parabolic Trajectory of a Launched Object:

This section will analyse the basic parabolic motion of a projectile in ideal conditions. For simplicity, the trajectory of a ball being launched in a vacuum will be explored. A ball launched with initial velocity vector $\tilde{v}$ at an angle θ from the horizontal has a simple parabolic trajectory, which can be represented by a quadratic parametric equation of time t. Two dimensions of the projectile motion can be analysed, its horizontal motion $x(t)$, and its vertical motion $y(t)$, where $x(t)$ and $y(t)$ are functions of t. The initial velocity vector can be written as a two-dimensional vector, such that $\tilde{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$, where v_x, v_y are the horizontal and vertical components respectively. Using trigonometry, v_x, v_y can be represented in terms of $\tilde{v}$ and θ, where $v_x = |\tilde{v}|\cos\theta$ and $v_y = |\tilde{v}|\sin\theta$, as illustrated in Figure 1. Additionally, the initial launch position has position vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Figure 1: Diagram of vector components (How to write the horizontal and vertical components of a vector? | Homework.Study.com 2023)



Derivation of Vertical Parametric Equations:

Since $y(t)$ represents the vertical position of the ball, or the displacement of the ball from the origin at time t, the vertical velocity of the ball can be represented by $y'(t)$, and the vertical acceleration can be represented by $y''(t)$. In ideal conditions, the only force acting on the ball is the gravitational attraction to earth which provides constant acceleration to the ball, $g = -9.81\ ms^{-2}$. Since gravity is the only force affecting the ball, it can be deduced that $y''(t) = g$. The velocity function $y'(t)$ can be determined using the initial vertical velocity and the integral of $y''(t)$.

$$y'(t) = \int y''(t)\, dt + c$$
$$y'(t) = gt + c$$
When $t = 0$:
$$y'(0) = c$$
$$c = |\tilde{v}|\sin\theta$$
$$\therefore y'(t) = |\tilde{v}|\sin\theta + gt$$

Furthermore, the displacement function $y(t)$ can be determined using the initial vertical position, $y(0) = 0$, and the integral of $y'(t)$.

$$y(t) = \int y'(t)\, dt + c$$
$$y(t) = \frac{1}{2}gt^2 + |\tilde{v}|\sin\theta\, t + c$$
When $t = 0$: $y(0) = c, \therefore c = 0$
$$\therefore y(t) = -\frac{1}{2}gt^2 + |\tilde{v}|\sin\theta\, t$$

Hence, the three parametric equations that represent that vertical motion of the ball t seconds after it is launched is:

Acceleration	Velocity	Displacement
$\boldsymbol{y''(t) = g}$	$y'(t) = \|\tilde{v}\|\sin\theta + gt$	$y(t) = -\frac{1}{2}gt^2 + \|\tilde{v}\|\sin\theta\, t$

Derivation of Horizontal Parametric Equations:

Similarly, the horizontal velocity and acceleration of the ball can be represented as $y'(t)$ and $y''(t)$ respectively. Since it is known that there are no horizontal forces acting on the ball, and the initial velocity of the ball is known, the velocity function $x'(t)$ can be determined.

Acceleration	Velocity	Displacement
$\boldsymbol{x''(t) = 0}$	$x'(t) = \int x''(t)\, dt + c = c$ When $t = 0$: $x'(0) = c = \|\tilde{v}\|\cos\theta$ $\therefore x'(t) = \|\tilde{v}\|\cos\theta$	$x(t) = \int x'(t)\, dt + c = \|\tilde{v}\|\cos\theta\, t + c$ When $t = 0$: $x(0) = c = 0$ $\therefore x(t) = \|\tilde{v}\|\cos\theta\, t$

From this, trajectory of a ball being launched in a vacuum where gravity is the only active force is given by the parametric equation:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} |\tilde{v}|\cos\theta\, t \\ \frac{1}{2}gt^2 + |\tilde{v}|\sin\theta\, t \end{pmatrix} = \begin{pmatrix} |\tilde{v}|\cos\theta\, t \\ -4.91t^2 + |\tilde{v}|\sin\theta\, t \end{pmatrix}$$

From this, the trajectory is a ballistic missile can be displayed by the parametric equations derived. Neglecting the Earth's curvature and other external forces aside from gravity, the optimal launch angle and velocity of an ICBM fired from Moscow to New York can be determined. It is known that the distance between Moscow and New York is 7506 km, or 7506000 m, hence, the initial position of the ICBM is at $x(0) = 0$ and $x(t_f) = 7506$, where t_f is the time when the ICBM reaches New York. It is also known that the altitude of the ICBM should be 0 km at launch and at its destination, therefore, $y(0) = 0$ and $y(t_f) = 0$. The optimal launch angle is the angle that maximises the horizontal distance the ICMB can cover. The horizontal distance covered is limited by the airtime of the ICBM, which can be found by solving $y(t_f) = 0$, where the ICBM has an altitude of 0 km. This gives $t_f = \frac{|\tilde{v}|\sin\theta}{4.91}$. Substituting $t_f = \frac{|\tilde{v}|\sin\theta}{4.91}$ into $x(t)$, the horizontal distance covered from the ICBM's launch to its destination is given by $x\left(\frac{|\tilde{v}|\sin\theta}{4.91}\right) = |\tilde{v}|\cos\theta \frac{|\tilde{v}|\sin\theta}{4.91} = \frac{|\tilde{v}|^2\sin\theta\cos\theta}{4.91}$. The launch angle θ that maximises $x\left(\frac{|\tilde{v}|\sin\theta}{4.91}\right)$ is when its derivative, $\frac{d}{d\theta}\frac{|\tilde{v}|^2\sin\theta\cos\theta}{4.91} = 0$. This gives $\theta = \frac{\pi}{4} = 45°$. Since $x(t_f) = 7506$, $x\left(\frac{|\tilde{v}|\sin\frac{\pi}{4}}{4.91}\right) = 7506$, which gives $|\tilde{v}| = 8580$. This means that the optimal launch velocity of an ICBM from Moscow to New York is 8580 ms^{-1}.

Flight Duration	Optimal Launch Angle	Optimal Launch Velocity
$\boldsymbol{y(t) = 0}$ $\boldsymbol{-4.91t^2 + \lvert\tilde{v}\rvert\sin\theta\, t = 0}$ $\boldsymbol{\therefore t = \frac{\lvert\tilde{v}\rvert\sin\theta}{4.91}, (t > 0)}$	$\frac{d}{d\theta}\frac{\lvert\tilde{v}\rvert^2\sin\theta\cos\theta}{4.91} = 0$ $\frac{\lvert\tilde{v}\rvert^2}{4.91}(-\sin^2\theta + \cos^2\theta) = 0$ $\therefore \theta = \frac{\pi}{4}\ or\ 45°, (0 \le \theta \le \frac{\pi}{2})$	$x\left(\frac{\lvert\tilde{v}\rvert\sin\frac{\pi}{4}}{4.91}\right) = 7506000$ $\frac{\lvert\tilde{v}\rvert^2}{9.81} = 7506000$ $\lvert\tilde{v}\rvert = 8580$

Hence, the trajectory vector of the ICBM is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 8580\cos\frac{\pi}{4}t \\ -4.91t^2 + 8580\sin\frac{\pi}{4}t \end{pmatrix} = \begin{pmatrix} 6070t \\ -4.91t^2 + 6070t \end{pmatrix}$. Inversely, the destination of an ICBM can be predicted given the launch vector.

Derivatives of Displacement

Acceleration is the change in velocity per unit time, which can be denoted as $\frac{\Delta v}{\Delta t}$. The instantaneous acceleration is therefore the ratio of the change in velocity to change in time as the change in time approaches zero. This infinitely small change in velocity and time can be denoted dv and dt, such that the acceleration is $a = \frac{dv}{dt}$, which is also the derivative of the velocity function v with respect to time t. Furthermore, velocity is the change in displacement per unit time, written as $\frac{\Delta s}{\Delta t}$, where s is the displacement. The instantaneous velocity is the the change in displacement, Δs, divided by the time elapsed, Δt. When the time elapsed is reduced to an infinitely short time interval, the change in displacement and time is infinitely small and can be denoted ds and dt, such that the instantaneous velocity is the ratio of the two, $\frac{ds}{dt}$, which is also the derivative of the displacement function s with respect to time t. Substituting this derivative into $a = \frac{dv}{dt}$, the expression $a = \frac{d\left(\frac{ds}{dt}\right)}{dt}$ is obtained, which further simplifies to $a = \frac{d^2s}{dt^2}$. This shows that acceleration is the second derivative of displacement with respect to time.

The displacement of the missile from its original launch position can be represented as a two-dimensional position vector with horizontal and vertical components. Similarly, the velocity and acceleration vectors can be represented with horizontal and vertical components. Table 2 summarises the relationship between the acceleration, velocity, and displacement of the missile. Table 2:

Acceleration	Velocity	Displacement
$\boldsymbol{a = \begin{pmatrix} \frac{dv_x}{dt} \\ \frac{dv_y}{dt} \end{pmatrix} = \begin{pmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \end{pmatrix} = \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix}}$	$v = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$	$s = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

Case 2: Trajectory of Missile with Thrust and Varying Mass

When a missile produces thrust, it burns and ejects fuel. If the thrust is constant, it can be assumed that the rate at which fuel is burned and ejected is relatively constant, and can be expressed as a function of time t, so that the mass is represented by the function $M(t) = m_0 - ft$, where m_0 is the initial mass, and f is the fuel flow rate (mass of fuel burned per unit of time).

In this scenario, a missile is launched at an angle of 45° from the horizontal with a constant thrust of $500000N$. The missile has an initial mass of $10000kg$, a fuel flow rate of $50kgs^{-1}$, and a boost phase of $100s$. From this, the function for the missile's mass if given by: $M(t) = 10000 - 50t$.

As mentioned in the previous section, the acceleration of the missile is $a = \frac{F}{m}$. However, the mass of the missile is a function, and therefore the acceleration is $a = \frac{F}{M(t)}$. The motion of the missile can be broken down into horizontal and vertical components and can be mathematically analysed using calculus. In the boost phase, the forces acting vertically on the missile are its thrust and gravity. Thus, $acceleration = y''(t) = \frac{T}{M(t)} - g$. By integrating with respect to time, the vertical velocity and displacement can be found:

Velocity	Displacement
$\boldsymbol{y'(t) = \int y''(t)\,dt}$ $\boldsymbol{y'(t) = \int \frac{T\sin\theta}{m_0 - ft} - g\,dt}$ $\boldsymbol{y'(t) = -\frac{T\sin\theta}{f}\ln\lvert m_0 - ft\rvert - gt + c}$ Since at $\boldsymbol{t = 0, y'(t) = 0, \therefore c = \frac{T\sin\theta}{f}\ln m_0}$ $\boldsymbol{y'(t) = -\frac{T\sin\theta}{f}\ln\lvert m_0 - ft\rvert - gt + \frac{T\sin\theta}{f}\ln m_0}$	$y(t) = \int y'(t)\,dt$ $y(t) = \int -\frac{T\sin\theta}{f}\ln\lvert m_0 - ft\rvert - gt + \frac{T\sin\theta}{f}\ln m_0\,dt$ $y(t) = \frac{T\sin\theta}{f^2}[(m_0 - ft)\ln(m_0 - ft) + ft] - \frac{gt^2}{2} + \frac{T\sin\theta}{f}\ln m_0\,t + c$ Since at $t = 0,\ y(t) = 0, \therefore c = -\frac{T\sin\theta}{f^2}m_0\ln m_0$ $y(t) = \frac{T\sin\theta}{f^2}[(m_0 - ft)\ln(m_0 - ft) + ft] + \frac{T\sin\theta}{f}\ln m_0\,t - \frac{gt^2}{2} - \frac{T\sin\theta}{f^2}m_0\ln m_0$

The same process can be completed to determine the functions for the missile's horizontal velocity and displacement:

Velocity	Displacement
$\boldsymbol{x'(t) = \int x''(t)\,dt}$ $\boldsymbol{x'(t) = \int \frac{T\cos\theta}{m_0 - ft}dt}$ $\boldsymbol{x'(t) = -\frac{T\cos\theta}{f}\ln\lvert m_0 - ft\rvert + c}$ Since at $\boldsymbol{t = 0, x'(t) = 0, \therefore c = \frac{T\cos\theta}{f}\ln m_0}$ $\boldsymbol{x'(t) = -\frac{T\cos\theta}{f}\ln\lvert m_0 - ft\rvert + \frac{T\cos\theta}{f}\ln m_0}$	$x(t) = \int x'(t)\,dt$ $x(t) = \int -\frac{T\cos\theta}{f}\ln\lvert m_0 - ft\rvert + \frac{T\cos\theta}{f}\ln m_0\,dt$ $x(t) = \frac{T\cos\theta}{f^2}[(m_0 - ft)\ln(m_0 - ft) + ft] + \frac{T\cos\theta}{f}\ln m_0\,t + c$ Since at $t = 0,\ x(t) = 0, \therefore c = -\frac{T\cos\theta}{f^2}m_0\ln m_0$ $x(t) = \frac{T\cos\theta}{f^2}[(m_0 - ft)\ln(m_0 - ft) + ft] + \frac{T\cos\theta}{f}\ln m_0\,t - \frac{T\cos\theta}{f^2}m_0\ln m_0$

Therefore, the displacement of the missile in the first 100 seconds, where it is ejecting mass and being accelerated, is given by the parametric equations:

$$\boxed{\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} \frac{T\cos\theta}{f^2}[(m_0 - ft)\ln(m_0 - ft) + ft] + \frac{T\cos\theta}{f}\ln m_0\,t - \frac{T\cos\theta}{f^2}m_0\ln m_0 \\ \frac{T\sin\theta}{f^2}[(m_0 - ft)\ln(m_0 - ft) + ft] + \frac{T\sin\theta}{f}\ln m_0\,t - \frac{T\sin\theta}{f^2}m_0\ln m_0 - \frac{gt^2}{2} \end{pmatrix} \\ &\approx \begin{pmatrix} (1414000 - 7071t)\ln(10000 - 50t) + 72200t - 13030000 \\ (1414000 - 7071t)\ln(10000 - 50t) + 72200t - 13030000 - 4.905t^2 \end{pmatrix}\end{aligned}}$$

The velocity of the missile in the first 100 seconds is given by the parametric equations:

$$\boxed{\begin{aligned}\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} &= \begin{pmatrix} -\frac{T\cos\theta}{f}\ln\lvert m_0 - ft\rvert + \frac{T\cos\theta}{f}\ln m_0 \\ -\frac{T\sin\theta}{f}\ln\lvert m_0 - ft\rvert + \frac{T\sin\theta}{f}\ln m_0 - gt \end{pmatrix} \\ &\approx \begin{pmatrix} -7071\ln(10000 - 50t) + 65130 \\ -7071\ln(10000 - 50t) + 65130 - 9.81t \end{pmatrix}\end{aligned}}$$

At the end of the boost phase ($t = 100$), $\begin{pmatrix} x(100) \\ y(100) \end{pmatrix} = \begin{pmatrix} 210800 \\ 161800 \end{pmatrix}$, ad $\begin{pmatrix} x'(100) \\ y'(100) \end{pmatrix} = \begin{pmatrix} 4905 \\ 3924 \end{pmatrix}$. Afterwards, the missile enters free fall, and its motion follows that described by the parametric equations of free fall. When the conditions at $t = 100$ are substituted:

$$\begin{pmatrix} x(t-100) \\ y(t-100) \end{pmatrix} = \begin{pmatrix} \lvert\tilde{v}\rvert\cos\theta\, t \\ -4.905t^2 + \lvert\tilde{v}\rvert\sin\theta\, t \end{pmatrix} + \begin{pmatrix} 210800 \\ 161800 \end{pmatrix} \approx \begin{pmatrix} 4905t + 210800 \\ -4.905t^2 + 3924t + 161800 \end{pmatrix}$$

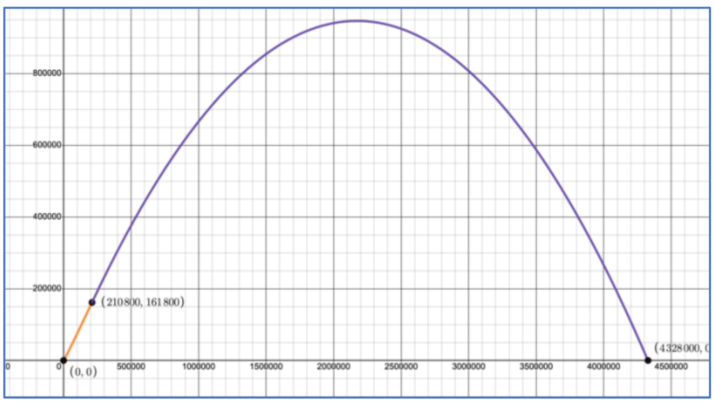


Figure 2 (Flight Path of Missile)

The destination of the missile can be determined by solving for $y(t) = 0$, where the missile reaches the ground:

$$-4.905(t - 100)^2 + 3924(t - 100) + 161800 = 0$$
$$\therefore t = 939.3\ (t > 0)$$

This can be substituted into $x(t - 100)$ so that $x(939.3) = 4328000$. From this, it can be determined that the destination of the missile is 4328 km away from the initial launch position, in the direction that the missile was launched. Figure 2 shows the complete trajectory of the missile from $t = 0$ to $t = 939.3$. It is evident that the missile follows a pseudo-parabolic trajectory where the boost phase is linear. It is also evident that when the missile is losing mass, it attains higher velocities and thus covers greater distances.'

Case 3: Trajectory of Missile with Linear Drag, Thrust, and Variable Mass

When a missile is in the air and moving, it must push through the atmosphere, and the surrounding air particles create a drag force on the missile, opposite and parallel to its motion. The drag force acting on an object can be roughly approximated using linear drag, where the drag is proportional to the velocity of the object, so that $d(v)=kv$, where k is a constant, and v is the velocity of the object. The drag vector has horizontal and vertical components, such that the horizontal component is kv_x and the vertical component is kv_y, where v_x, v_y are the horizontal and vertical components of the missile's velocity vector. During the boost phase, the forces present are thrust, gravity, and drag, hence, the horizontal and vertical forces acting on the missile can be shown by a vector, so that $F=\begin{pmatrix} T\cos\theta - D \\ T\sin\theta - D - M(t)g \end{pmatrix}$. From this, the acceleration vector can be obtained, where $a=\begin{pmatrix} \frac{T\cos\theta - kv}{m_0-ft} \\ \frac{T\cos\theta-kv}{m_0-ft}-g \end{pmatrix}$.

The acceleration vector can be integrated with respect to time to determine the velocity vector and can be further integrated with respect to time to obtain the displacement vector which can be used to find the trajectory of the missile during the boost phase $0\le t\le t_f$, where t_f is the time elapsed during the boost phase. The horizontal and vertical components of displacement vector can be subscripted with b, so that the displacement of the missile in the boost phase is x_b and y_b.

Horizontal Velocity

As mentioned in Table 2, acceleration is the derivative of velocity with respect to time.

$$x_b''(t)=\frac{dv_x}{dt_b}=\frac{T\cos\theta-kv_{x_b}}{m_0-ft}$$

Let $T_x=T\cos\theta$:

This is a separable differential equation, and can be solved by isolating v_x and t:

$$\frac{1}{T_x-kv_x}dv_x=\frac{1}{(m_0-ft_b)}dt$$

$$\int\frac{1}{T_x-kv_x}dv_x=\int\frac{1}{(m_0-ft_b)}dt$$

$$-\frac{\ln(T_x-kv_x)}{k}=-\frac{\ln(m_0-ft_b)}{f}+c$$

At $t=0, v_x=0 \therefore c=-\frac{\ln(T_x)}{k}+\frac{\ln m_0}{f}=\ln\frac{(m_0)^{\frac{1}{f}}}{(T_x)^{\frac{1}{k}}}$

$$\ln(T_x-kv_x)^{-\frac{1}{k}}=\ln(m_0-ft_b)^{-\frac{1}{f}}+\ln\frac{(m_0)^{\frac{1}{f}}}{(T_x)^{\frac{1}{k}}}$$

$$\ln(T_x-kv_x)^{-\frac{1}{k}}=\ln(m_0-ft_b)^{-\frac{1}{f}}\frac{(m_0)^{\frac{1}{f}}}{(T_x)^{\frac{1}{k}}}$$

$$(T_x-kv_x)^{-\frac{1}{k}}=(m_0-ft_b)^{-\frac{1}{f}}\frac{(m_0)^{\frac{1}{f}}}{(T_x)^{\frac{1}{k}}}$$

$$v_x=(m_0)^{\frac{-k}{f}}(T_x)\frac{(m_0-ft_b)^{\frac{k}{f}}}{-k}+\frac{T_x}{k}$$

$$v_x=\frac{T_x-(m_0-ft_b)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}(T_x)}{k}$$

$$\therefore x_b'(t)=\frac{T_x\left(1-(m_0-ft)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}\right)}{k}$$

Vertical Velocity

$$y_b''(t)=\frac{dv_y}{dt_b}=\frac{T\cos\theta-kv_y}{m_0-ft}-g$$

$$\frac{dv_y}{dt_b}+\frac{kv_y}{m_0-ft}=\frac{T\cos\theta}{m_0-ft}-g$$

Let $T sin\,\theta=T_y, \mu(t)=e^{\int\frac{k}{m_0-ft}dt}=(m_0-ft)^{-\frac{k}{f}}$, and multiply $\mu(t)$ to both sides:

$$(m_0-ft)^{-\frac{k}{f}}\frac{dv_y}{dt_b}-\frac{\left(k(m_0-ft)^{-\frac{k}{f}}\right)v_y}{m_0-ft}=-(g-\frac{T_y}{m_0-ft})(m_0-ft)^{-\frac{k}{f}}$$

Substitute $-\frac{k(m_0-ft)^{-\frac{k}{f}}}{m_0-ft}=\frac{d}{dt}(m_0-ft)^{-\frac{k}{f}}$:

$$\frac{dv_y}{dt}\left((m_0-ft)^{-\frac{k}{f}}\right)+\frac{d}{dt}\left((m_0-ft)^{-\frac{k}{f}}\right)v_y=-(g-\frac{T_y}{m_0-ft})(m_0-ft)^{-\frac{k}{f}}$$

Apply the reverse product rule, $\frac{dg}{dt}f+\frac{df}{dt}g=\frac{d}{dt}fg$:

$$\frac{d}{dt_b}\left((m_0-ft)^{-\frac{k}{f}}v_y\right)=-(g-\frac{T_y}{m_0-ft})(m_0-ft)^{-\frac{k}{f}}$$

Integrate both sides of the equation with respect to t:

$$\int\frac{d}{dt}\left((m_0-ft)^{-\frac{k}{f}}v_y\right)dt=\int-(g-\frac{T_y}{m_0-ft})(m_0-ft)^{-\frac{k}{f}}dt$$

$$(m_0-ft)^{-\frac{k}{f}}v_y=\frac{fgkt-gkm_0+T_yk-T_yf}{k(k-f)(m_0-ft)^{\frac{k}{f}}}+c$$

$$v_y=\frac{gkm_0-fgkt-T_yk+T_yf}{k(f-k)}+c(m_0-ft)^{\frac{k}{f}}$$

$At\ t=0, v_y=0,$

$$\therefore c=-\left(\frac{gkm_0-T_yk+T_yf}{k(f-k)}\right)(m_0)^{-\frac{k}{f}}=\frac{-kgm_0-(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}$$

$$v_y=\frac{T_y}{k}+\frac{g(m_0-ft_b)}{f-k}+\frac{-kgm_0-(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}(m_0-ft_b)^{\frac{k}{f}}$$

$$\therefore y_b'(t)=\frac{T_y}{k}+\frac{g(m_0-ft)}{f-k}+\frac{-kgm_0-(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}(m_0-ft)^{\frac{k}{f}}$$

These can be further integrated to find the horizontal and vertical displacement functions:

Horizontal Displacement

$$x_b(t)=\int x_b'(t)\,dt=\int(m_0)^{\frac{-k}{f}}(T_x)\frac{(m_0-ft)^{\frac{k}{f}}}{-k}+\frac{T_x}{k}dt$$

$$x_b(t)=\frac{(m_0)^{\frac{-k}{f}}(T_x)}{-k}\int(m_0-ft)^{\frac{k}{f}}dt+\frac{T_x}{k}t+c$$

$$x_b(t)=\frac{(m_0)^{\frac{-k}{f}}(T_x)}{k}\left(\frac{(m_0-ft)^{\frac{k}{f}+1}}{f(\frac{k}{f}+1)}\right)+\frac{T_x}{k}t+c$$

$At\ t=0, x(0)=0$

$$\therefore c=-\frac{(m_0)^{\frac{-k}{f}}(T_x)}{k}\left(\frac{(m_0)^{\frac{k}{f}+1}}{f(\frac{k}{f}+1)}\right)=-\frac{T_xm_0}{fk(\frac{k}{f}+1)}$$

$$\therefore x_b(t)=\left(\frac{(m_0)^{\frac{-k}{f}}(T_x)(m_0-ft)^{\frac{k}{f}+1}-T_xm_0}{fk(\frac{k}{f}+1)}\right)+\frac{T_x}{k}t$$

Vertical Displacement

$$y_b(t)=\int v_y(t_b)\,dt=\int\frac{T_y}{k}+\frac{g(m_0-ft)}{f-k}+\frac{-kgm_0-(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}(m_0-ft)^{\frac{k}{f}}dt$$

$$y_b(t)=\frac{T_y}{k}t+\frac{g}{f-k}\int m_0-ft\,dt+\frac{-kgm_0-(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\int(m_0-ft)^{\frac{k}{f}}dt+c$$

$$y_b(t)=\frac{T_y}{k}t+\frac{gm_0}{f-k}t-\frac{gf}{2(f-k)}t^2+\frac{kgm_0+(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\left(\frac{(m_0-ft)^{\frac{k}{f}+1}}{f\left(\frac{k}{f}+1\right)}\right)+c$$

$$At\ t=0, y_b(0)=0, \therefore c=-\frac{kgm_0+(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\left(\frac{(m_0)^{\frac{k}{f}+1}}{f\left(\frac{k}{f}+1\right)}\right)$$

$$\therefore y_b(t)=\frac{T_y}{k}t+\frac{gm}{f-k}t-\frac{gf}{2(f-k)}t^2+\frac{kgm_0+(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\left(\frac{(m_0-ft)^{\frac{k}{f}+1}-(m_0)^{\frac{k}{f}+1}}{f\left(\frac{k}{f}+1\right)}\right)$$

After the boost phase $t > t_f$, the forces acting on the missile is the drag force and gravity. The mass of the missile also does not change during the free fall stage and remains the same at $M(t_f) = m_0 - f t_f$, which can be denoted m_f. The displacement of the missile during this phase can be denoted by x_f and y_f. Hence, the acceleration vector of the missile during this phase is $\begin{pmatrix} x_f''(t) \\ y_f''(t) \end{pmatrix} = \begin{pmatrix} -\frac{kv_x}{m_f} \\ -\frac{kv_y}{m_f} - g \end{pmatrix}$. This can be integrated with respect to time to determine the velocity and displacement vectors of the missile during this phase.

Horizontal Acceleration

$$x_f''(t) = \frac{dv_x}{dt} = -\frac{kv}{m_f}$$
$$\int \frac{1}{kv_x} dv_x = -\int \frac{1}{m_f} dt_u$$
$$\frac{\ln(kv_x)}{k} = -\frac{t}{m_f} + c$$
$$\ln(kv_x) = -\frac{kt}{m_f} + kc$$
$$v_x = Ae^{-\frac{kt}{m_f}}, \{A \in \mathbb{R}\}$$

At $t = 0$ of the free fall stage, $v_x = x_b'(t_f) = \frac{T_x\left(1-(m_f)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}\right)}{k}$:

$$A = \frac{T_x\left(1-(m_f)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}\right)}{k}$$
$$\therefore x_f'(t) = \frac{T_x\left(1-(m_f)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}\right)}{k} e^{-\frac{kt}{m_f}}$$

Vertical Acceleration

$$y_f''(t) = \frac{dv_y}{dt} = -\frac{kv_y + m_f g}{m_f}$$
$$\int \frac{1}{kv_y + m_f g} d_y = -\int \frac{1}{m_f} dt$$
$$\frac{\ln(kv_y + m_f g)}{k} = -\frac{t}{m_f} + c$$
$$\ln(kv_y + m_f g) = -\frac{kt}{m_f} + c$$
$$kv_y + m_f g = e^{-\frac{kt}{m_f}} e^c$$
$$v_y = \frac{e^{-\frac{kt}{m_f}} e^c - m_f g}{k} = Be^{-\frac{kt}{m_f}} - \frac{m_f g}{k}, \{B \in \mathbb{R}\}$$

$At\ t = 0$ of the free fall stage, $v_y = \frac{T_y}{k} + \frac{g(m_f)}{f-k} + \frac{-kgm_0-(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}}$:

$$B = \frac{T_y}{k} + \frac{g(m_f)}{f-k} + \frac{-kgm_0 - (f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + \frac{m_f g}{k}$$
$$\therefore y_f'(t) = e^{-\frac{kt}{m_f}}\left(\frac{T_y}{k} + \frac{g(m_f)}{f-k} + \frac{-kgm_0 - (f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + \frac{m_f g}{k}\right) - \frac{m_f g}{k}$$

Horizontal Displacement

$$x_f(t) = \int x_f'(t)\, dt$$
$$x_f(t) = \int \frac{T_x\left(1-(m_f)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}\right)}{k} e^{-\frac{kt}{m_f}} dt$$
$$x_f(t) = \frac{-T_x m_f\left(1-(m_f)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}\right)}{k^2} e^{-\frac{kt}{m_f}} + c$$

At $t = 0$ of the free fall stage, and the end of the boost phase, the horizontal displacement of the missile is $x_b(t_f) = \left(\frac{(m_0)^{\frac{-k}{f}}(T_x)(m_f)^{\frac{k}{f}+1} - T_x m_0}{fk(\frac{k}{f}+1)}\right) + t_f \frac{T_x}{k}$.

Hence:

$$x_f(0) = -\left(T_x - (m_f)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}(T_x)\right)\frac{m_f}{k^2} + c$$
$$= \left(\frac{(m_0)^{\frac{-k}{f}}(T_x)(m_f)^{\frac{k}{f}+1} - T_x m_0}{fk(\frac{k}{f}+1)}\right) + t_f \frac{T_x}{k}$$
$$c = \left(\frac{(m_0)^{\frac{-k}{f}}(T_x)(m_f)^{\frac{k}{f}+1} - T_x m_0}{fk(\frac{k}{f}+1)}\right) + t_f \frac{T_x}{k} + \left(T_x - (m_f)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}(T_x)\right)\frac{m_f}{k^2}$$
$$\therefore x_f(t) = -\left(T_x - (m_f)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}(T_x)\right)\frac{m_f}{k^2}\left(e^{-\frac{kt_u}{m_f}} - 1\right) + \left(\frac{(m_0)^{\frac{-k}{f}}(T_x)(m_f)^{\frac{k}{f}+1} - T_x m_0}{fk\left(\frac{k}{f}+1\right)}\right) + t_f \frac{T_x}{k}$$

Vertical Displacement

$$y_f(t) = \int e^{-\frac{kt_u}{m_f}} \frac{\left(T_y + \frac{kg(m_f)}{f-k} + \frac{-kgm_0 - (f-k)T_y}{(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + m_f g\right)}{k} - \frac{m_f g}{k} dt$$
$$y_f(t) = e^{-\frac{kt}{m_f}} \frac{-m_f\left(T_y + \frac{kg(m_f)}{f-k} + \frac{-kgm_0 - (f-k)T_y}{(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + m_f g\right)}{k^2} - \frac{m_f g}{k} t + c$$

At the start of the free fall stage and the end of the boost phase, the vertical displacement of the missile is $y_b(t_f) = t_f\left(\frac{T_y}{k} + \frac{gm_0}{f-k} - \frac{t_f}{2}\frac{gf}{(f-k)}\right) + \frac{kgm_0+(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\left(\frac{(m_f)^{\frac{k}{f}+1} - (m_0)^{\frac{k}{f}+1}}{f\left(\frac{k}{f}+1\right)}\right)$. Hence:

$$y_f(0) = \frac{-m_f\left(T_y + \frac{kg(m_f)}{f-k} + \frac{-kgm_0 - (f-k)T_y}{(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + m_f g\right)}{k^2} + c = y_b(t_f)$$
$$= t_f\left(\frac{T_y}{k} + \frac{gm}{f-k} - \frac{t_f}{2}\frac{gf}{(f-k)}\right) + \frac{kgm_0 + (f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\left(\frac{(m_f)^{\frac{k}{f}+1} - (m_0)^{\frac{k}{f}+1}}{f\left(\frac{k}{f}+1\right)}\right)$$
$$c = t_f\left(\frac{T_y}{k} + \frac{gm}{f-k} - \frac{t_f}{2}\frac{gf}{(f-k)}\right) + \frac{kgm_0 + (f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\left(\frac{(m_f)^{\frac{k}{f}+1} - (m_0)^{\frac{k}{f}+1}}{f\left(\frac{k}{f}+1\right)}\right) + \frac{m_f\left(T_y + \frac{kg(m_f)}{f-k} + \frac{-kgm_0 - (f-k)T_y}{(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + \frac{m_f g}{k}\right)}{k^2}$$
$$\therefore y_f(t) = \frac{m_f\left(T_y + \frac{kg(m_f)}{f-k} + \frac{-kgm_0-(f-k)T_y}{(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + m_f g\right)}{k^2}\left(1 - e^{-\frac{kt}{m_f}}\right) - \frac{m_f g}{k} t + t_f\left(\frac{T_y}{k} + \frac{gm_0}{f-k} - \frac{t_f}{2}\frac{gf}{(f-k)}\right) + \frac{kgm_0+(f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\left(\frac{(m_f)^{\frac{k}{f}+1} - (m_0)^{\frac{k}{f}+1}}{f\left(\frac{k}{f}+1\right)}\right).$$

It is known that at time t_{max} when the missile is at its maximum altitude, the vertical velocity of the missile is zero. From this, the t_{max} can be determined:

$$y_f'(t_{max}) = e^{-\frac{kt}{m_f}}\left(\frac{T_y}{k} + \frac{g(m_f)}{f-k} + \frac{-kgm_0 - (f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + \frac{m_f g}{k}\right) - \frac{m_f g}{k} = 0$$

$$t_{max} = -\frac{m_f}{k}\ln\left(\frac{m_f g}{\left(T_y + \frac{kg(m_f)}{f-k} + \frac{-kgm_0 - (f-k)T_y}{(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + m_f g\right)}\right)$$

Furthermore, the time at which the missile reaches its destination can be determined by solving $y_f(t) = 0$:

$$y_f(t) = \frac{m_f\left(T_y + \frac{kg(m_f)}{f-k} + \frac{-kgm_0 - (f-k)T_y}{(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + m_f g\right)}{k^2}\left(1 - e^{-\frac{kt}{m_f}}\right) - \frac{m_f g}{k}t + t_f\left(\frac{T_y}{k} + \frac{gm_0}{f-k} - \frac{t_f}{2}\frac{gf}{(f-k)}\right) + \frac{kgm_0 + (f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\left(\frac{(m_f)^{\frac{k}{f}+1} - (m_0)^{\frac{k}{f}+1}}{f\left(\frac{k}{f}+1\right)}\right) = 0$$

However, there is no explicit symbolic solution to the expression above. In this scenario, a missile is launched at an angle of $45°$ from the horizontal with a constant thrust of $500000N$. The missile has an initial mass of $10000kg$, a fuel flow rate of $50kgs^{-1}$, and a boost phase of $100s$. The drag coefficient in this scenario is $k = 20$.

The parametric equations of the missile's displacement during its boost phase, $0 < t < 100$, are shown below:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \left(\frac{(m_0)^{\frac{-k}{f}}(T_x)(m_0 - ft)^{\frac{k}{f}+1} - T_x m_0}{fk\left(\frac{k}{f}+1\right)}\right) + \frac{T_x}{k}t \\ \frac{T_y}{k}t + \frac{gm_0}{f-k}t - \frac{gf}{2(f-k)}t^2 + \frac{kgm_0 + (f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\left(\frac{(m_0 - ft)^{\frac{k}{f}+1} - (m_0)^{\frac{k}{f}+1}}{f\left(\frac{k}{f}+1\right)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 6.3435(10000 - 50t)^{1.4} - 2525400 + 17678 \\ 21011t - 8.3333t^2 + 7.5396(10000 - 50t)^{1.4} - 3001600 \end{pmatrix}$$

After the boost phase, and during the free fall stage, the parametric equations of the missile's displacement are:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -\left(T_x - (m_f)^{\frac{k}{f}}(m_0)^{\frac{-k}{f}}(T_x)\right)\frac{m_f}{k^2}\left(e^{-\frac{kt}{m_f}} - 1\right) + \left(\frac{(m_0)^{\frac{-k}{f}}(T_x)(m_f)^{\frac{k}{f}+1} - T_x m_0}{fk\left(\frac{k}{f}+1\right)}\right) + t_f\frac{T_x}{k} \\ \frac{m_f\left(T_y + \frac{kg(m_f)}{f-k} + \frac{-kgm_0 - (f-k)T_y}{(f-k)(m_0)^{\frac{k}{f}}}(m_f)^{\frac{k}{f}} + m_f g\right)}{k^2}\left(1 - e^{-\frac{kt}{m_f}}\right) - \frac{m_f g}{k}t + t_f\left(\frac{T_y}{k} + \frac{gm_0}{f-k} - \frac{t_f}{2}\frac{gf}{(f-k)}\right) + \frac{kgm_0 + (f-k)T_y}{k(f-k)(m_0)^{\frac{k}{f}}}\left(\frac{(m_f)^{\frac{k}{f}+1} - (m_0)^{\frac{k}{f}+1}}{f\left(\frac{k}{f}+1\right)}\right) \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -1070100e^{-0.004t} + 1269500 \\ -1480200e^{-0.004} - 2500t + 1633820 \end{pmatrix}$$

The position of the missile at the end of the boost phase, at its maximum altitude, and when it reaches its destination can be determined as well:

Position at end of boost phase	Position at maximum altitude	Destination
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 199330 \\ 153580 \end{pmatrix}$	$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 817610 \\ 469940 \end{pmatrix}$	$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1172300 \\ 0 \end{pmatrix}$

This confirms that the maximum altitude of the missile is $469.94\ km$ and that the target of the missile is $1172.3\ km$ away from the launch site, in the direction the missile was launched. Figure 3 summarises the trajectory of the missile given by the parametric equations:

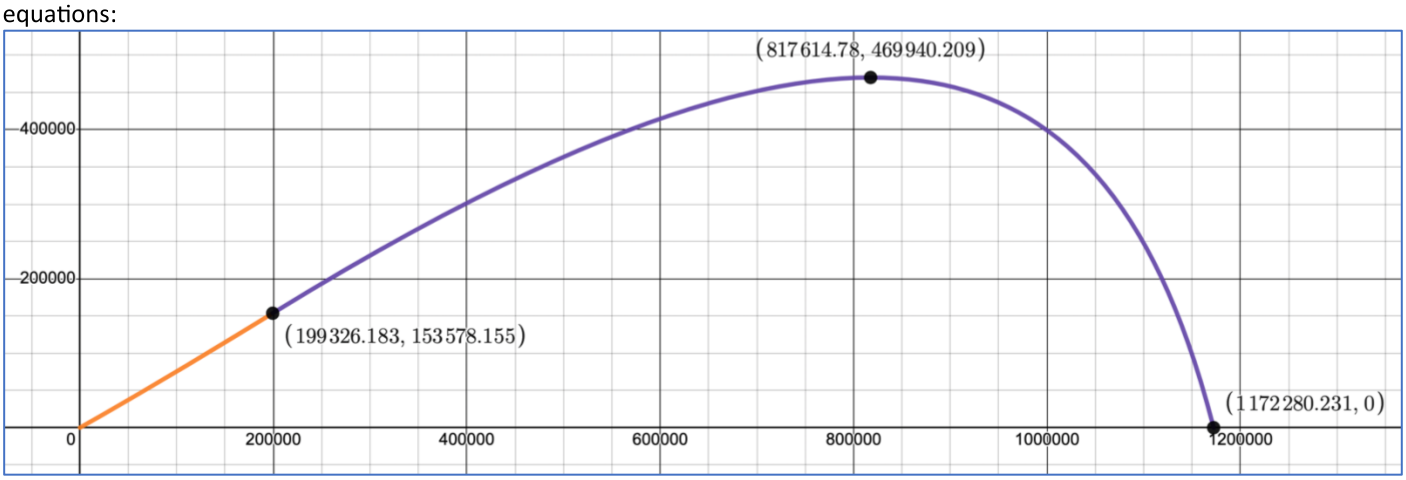
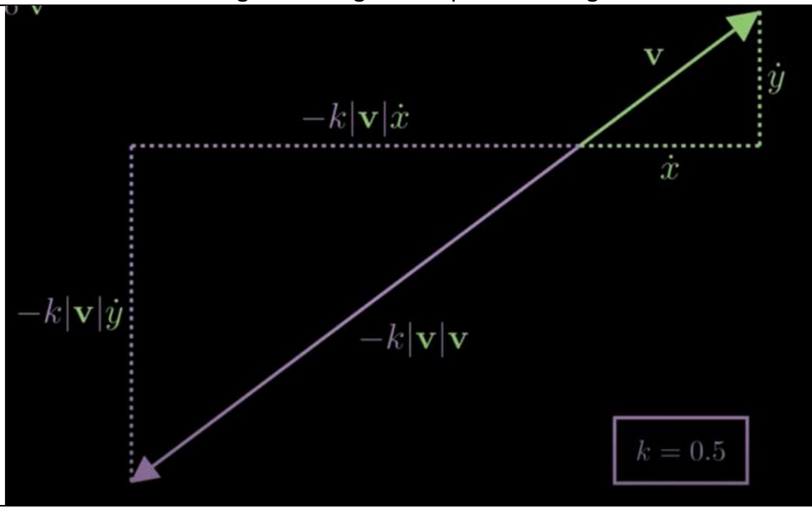


Figure 3: Flight Path of Missile with Linear Drag

Case 4: Trajectory of Missile with Quadratic Drag, Thrust, and Variable Mass

At high velocities, the drag force acting on an object is quadratically proportional to the object's velocity, and is given by the equation $D(v) = \frac{1}{2}\rho C A v^2$, where ρ is the air density, C is the missile's drag coefficient (This is approximated by the 3D geometry of the missile), A is the cross-sectional area, and v is the velocity of the missile. The horizontal and vertical components of drag can be broken down using simple geometry of similar triangles. It is known that the drag vector is always opposite and parallel to the velocity vector, and that it has a magnitude of the velocity vector multiplied by itself and some constant. Hence, the ratio between the velocity vector and the drag vector is some constant k multiplied by the magnitude of the velocity vector $|\vec{v}|$, that is the ratio $k|v|$. Similar triangles make it evident that the vertical component of the drag vector is $D_y = k|\vec{v}|v_y$, and the horizontal component is $D_x = k|\vec{v}|v_x$. This is shown in Figure 4. Additionally, by Pythagorean Theorem, $|\vec{v}| = \sqrt{(v_x)^2 + (v_y)^2}$.

As mentioned in Table 2, acceleration is the derivative of velocity with respect to time. During the boost phase, thrust is acting on the missile, and after the boost phase, in the free fall phase, only gravity and drag is acting on the missile, thus:

Acceleration Vector	Figure 4: Diagram of quadratic drag
During Boost Phase: $\begin{pmatrix} x_b''(t) \\ y_b''(t) \end{pmatrix} = \begin{pmatrix} \dfrac{T_x - k\sqrt{(v_x)^2 + (v_y)^2}\,v_x}{(m_0 - ft)} \\ \dfrac{T_y - k\sqrt{(v_x)^2 + (v_y)^2}\,v_y}{(m_0 - ft)} - g \end{pmatrix} = \begin{pmatrix} \dfrac{dv_x}{dt} \\ \dfrac{dv_y}{dt} \end{pmatrix}$ During Free Fall Stage: $\begin{pmatrix} x_f''(t) \\ y_f''(t) \end{pmatrix} = \begin{pmatrix} \dfrac{-k\sqrt{(v_x)^2 + (v_y)^2}\,v_x}{(m_0 - ft)} \\ \dfrac{-k\sqrt{(v_x)^2 + (v_y)^2}\,v_y}{(m_0 - ft)} - g \end{pmatrix}$	

Since these differential equations are non-linear first-order differential equations, integrating the velocity vector is too complex, and no explicit expression for $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ exists. Instead, the differential equation can be solved numerically by technological means. The Runge-Kutta method will be utilised to numerically solve the differential equations, and Python will be the programming language used.

The Runge-Kutta method is like the Euler method. Consider a differential equation $\frac{dy}{dt} = f(t, y)$. In the Euler method, small steps are taken in the t values, and are used to calculate the y values, such that: $y_{n+1} = y_n + h \times f(t_n, y_n)$, where h is the small step in the t value, and y_n is the y value corresponding to each t_n value for each step. Euler's method produces a good approximation of most functions, much suffers when the differential equation is complex or volatile, where Euler's method lags unless very small steps are used. The Runge-Kutta method is an extension of Euler's method, such that: $y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, and $k_1 = f(t_n, y_n), k_2 = f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right), k_3 = f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right), k_4 = f(t_n + h, y_n + hk_3)$. It is evident that the Runge-Kutta method utilises the weighted average of the estimated y values. It can be interpreted that k_1 is the slope at the beginning of the step using Euler's method. k_2 is the slope at the midpoint of the step that utilises k_1. k_3 is another estimate of the slope at the midpoint of the step but uses k_2 to estimate. k_4 is the slope at the end of the step and utilises k_3. More weight is given to the k_2, k_3 as they represent the slope at the midpoint of the step. Thus, the Runge-Kutta method can make more accurate estimations of a function than Euler's method. Since the Runge-Kutta method utilises the average of 4 slopes that are based off of previously calculated slopes, it is often referred to as the fourth order Runge-Kutta method, abbreviated RK4. In its expanded form, the RK4:

$$y_{n+1} = y_n + \frac{h}{6}\left(f(t_n, y_n) + 2f\left(t_n + \frac{h}{2}, y_n + h\frac{f(t_n, y_n)}{2}\right) + 2f\left(t_n + \frac{h}{2}, y_n + h\frac{f\left(t_n + \frac{h}{2}, y_n + h\frac{f(t_n, y_n)}{2}\right)}{2}\right) + f\left(t_n + h, y_n + hf\left(t_n + \frac{h}{2}, y_n + h\frac{f\left(t_n + \frac{h}{2}, y_n + h\frac{f(t_n, y_n)}{2}\right)}{2}\right)\right)\right)$$

```
def RK4_1(x, y, dx, dydx):

    # Calculate slopes
    k1 = dx*dydx(x, y)
    k2 = dx*dydx(x+dx/2., y+k1/2.)
    k3 = dx*dydx(x+dx/2., y+k2/2.)
    k4 = dx*dydx(x+dx, y+k3)
```

As shown, RK4 involves fourth degree nested derivatives. From this, the error produced by RK4 is also in the order of four, that is if the step is multiplied by $\frac{1}{2}$, the error is multiplied by $\frac{1}{16}$, which is a significant improvement from the Euler method, where the error produced is linear, such that halving the step only halves the error.

The Python code for RK4 is shown on the left:

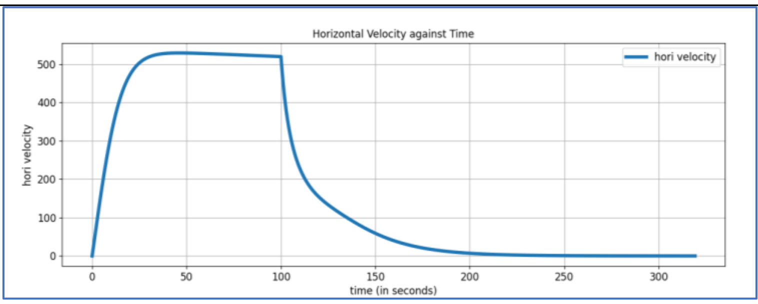
The numerical values of the scenario must be substituted into the equation to numerically solve the differential equations. In this scenario, a missile is launched at an angle of $45°$ from the horizontal with a constant thrust of $500000N$. The missile has an initial mass of $10000kg$, a fuel flow rate of $50kgs^{-1}$, and a boost phase of $100s$. The drag coefficient in this scenario is $k = 1$. Hence, the system of differential equations that describe the trajectory of the missile is:

$$\begin{pmatrix} x_b''(t) \\ y_b''(t) \end{pmatrix} = \begin{pmatrix} \dfrac{500000 \cos\frac{\pi}{4} - \sqrt{(v_x)^2 + (v_y)^2}\, v_x}{(10000 - 50t)} \\ \dfrac{500000 \sin\frac{\pi}{4} - \sqrt{(v_x)^2 + (v_y)^2}\, v_y}{(10000 - 50t)} - 9.81 \end{pmatrix}, \begin{pmatrix} x_f''(t) \\ y_f''(t) \end{pmatrix} = \begin{pmatrix} \dfrac{-\sqrt{(v_x)^2 + (v_y)^2}\, v_x}{(10000 - 50t)} \\ \dfrac{-\sqrt{(v_x)^2 + (v_y)^2}\, v_y}{(10000 - 50t)} - 9.81 \end{pmatrix}$$

Where $\begin{pmatrix} x_b''(t) \\ y_b''(t) \end{pmatrix}$ is the acceleration vector of the missile during the boost phase $(0 < t < 100)$, and , $\begin{pmatrix} x_f''(t) \\ y_f''(t) \end{pmatrix}$ is the acceleration vector of the missile during the free fall stage $(t \geq 100)$.

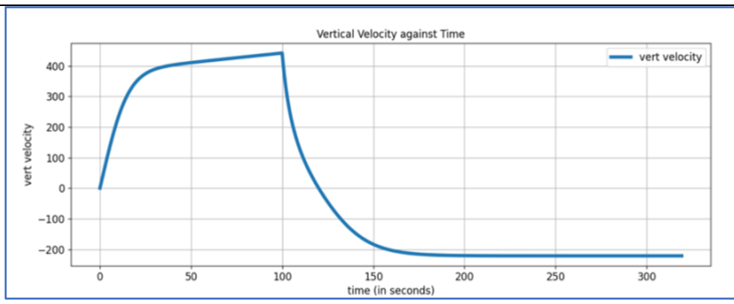
RK4 can find $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ from $\begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix}$, and can be reiterated to find $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ from $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$.

Horizontal Velocity



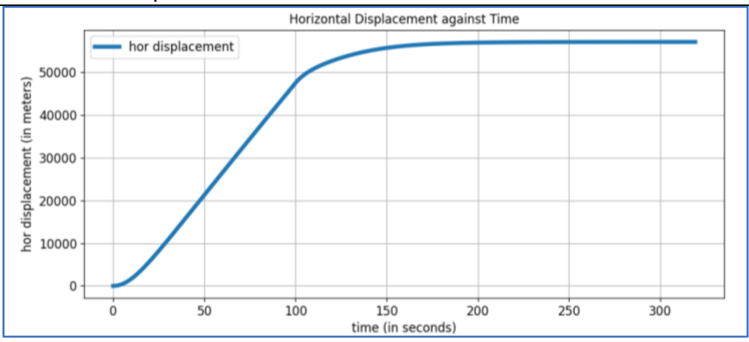
The horizontal velocity increases at a decreasing rate in the first 30 seconds of flight, but plateaus since the drag force opposes the thrust and results in zero acceleration. This is the terminal velocity of the missile in its boost phase. 100 seconds into the missile's flight, it stops producing thrust, and drag slows down the missile. As expected, the missile loses all its horizontal velocity due to drag, and plateaus at zero.

Vertical Velocity



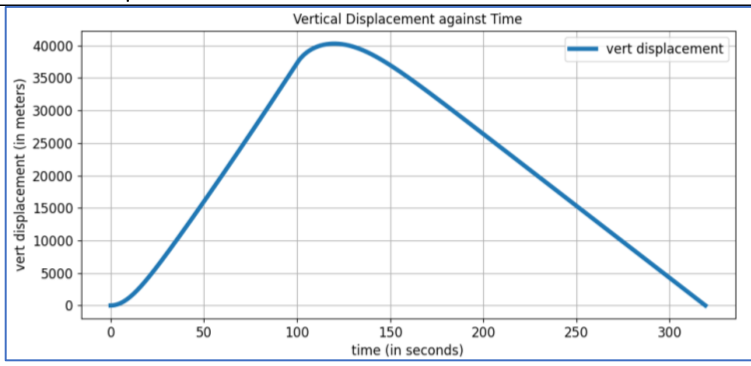
The vertical velocity increases at a decreasing rate during the first 100 seconds, which is attributed to drag, which is decreasing the acceleration of the missile. After 100 seconds, the missiles stop producing thrust, and drag slows down the missile. Since gravity is affecting the missile, it falls downwards and is opposed by drag until it reaches its terminal velocity at about $-210\ ms^{-1}$.

Horizontal Displacement



The horizontal displacement increases at an increasing rate in the first 30 seconds of flight, since its velocity is increasing during this time. When the missile's velocity plateaus, its displacement increases approximately linearly due to constant velocity. From 100 to 200 seconds, the missile's velocity decreases but remains positive, and thus the displacement begins to increase at a decreasing rate. After 200 seconds, the velocity of the missile is approximately 0, and thus, the displacement stays constant. This horizontal asymptote is the maximum horizontal displacement of the missile

Vertical Displacement



The vertical displacement increases at an increasing rate in the first 100 seconds of flight. This is due to the increasing velocity during the boost phase. From 100 to 120 seconds, the vertical displacement of the missiles increases at a decreasing rate. This is due to the decreasing velocity of the missile. From 120 seconds to about 150 seconds, the displacement of the missile decreases at an increasing rate. This is due to the velocity of the missile being negative and decreasing. After 150 seconds, the velocity of the missile is approximately constant, so the displacement decreases approximately linearly.

The time when the missile attains maximum altitude can be determined by: $y'(t) = 0$. This can be solved using numerical methods where the computer program stores the value of t when $y'(t - \delta t) > 0$ and $y'(t) \leq 0$, where δt is the time step. The program calculates this to be $t_{max} = 120.04$, where t_{max} is the time of maximum altitude. This can be substituted to find the maximum altitude, where $y_{max} = y(t_{max})$. The program calculates this to be $y_{max} = 40284$. The flight time of the missile can be found as well, by solving $y(t_f) = 0$, where t_f is the time when the missile reaches the ground again and has vertical displacement of zero. Using the condition $y(t - \delta t) > 0$ and $y(t) \leq 0$, the program calculates this to be $t_f = 319.27$. This can be substituted to find the range of the missile, so that $x(t_f) = 57140$. This means that that missile's destination is about $57.140\ km$ from the launch site, in the direction it was launched. Figure 4 shows the solution to the system of differential equations that describe the trajectory of the missile:

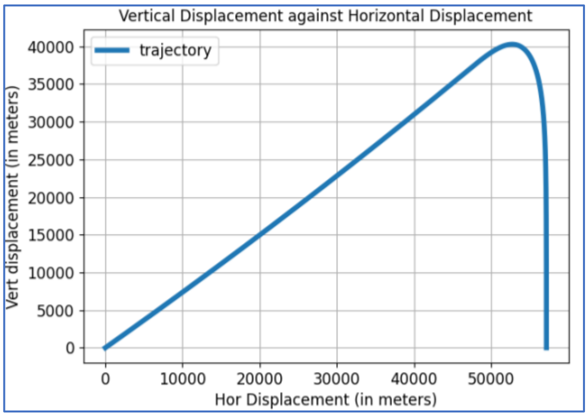


Figure 5: Flight Path with Quadratic Drag

Breakdown of ICBM Flight Specifications and Relevant Equations

All ballistic missiles are propelled by thrusters to gain enough velocity to enter sub-orbital altitudes. The forces provided by the thrusters are non-constant, and thus, the ballistic missile undergoes non-uniform acceleration. The acceleration curves of generic ICBMs are shown in Figure 2. Note, these are the acceleration curves in Figure 2 show the acceleration of the actual missile in non-ideal conditions (Drag and other factors affect it). This section of the investigation focuses on a 3-boost phase 180-sec ICBM, such as the American 'Minute Man' and its acceleration profile. 'Minute Man' has a flight sequence that is depicted in Figure 3, where from $t = 0$ to $t = 60$, the missile is powered by the first stage thruster. The second stage thruster begins ignition at $t = 60$ until $t = 120$, and the third stage thruster ignites from $t = 120$ to $t = 180$, in seconds. After each boost stage, the missile ejects the used fuel tanks and thrusters, which decreases its mass.

Figure 6: Acceleration curve of ICBM (Figure A5: Acceleration profiles for ICBMs. 2023)

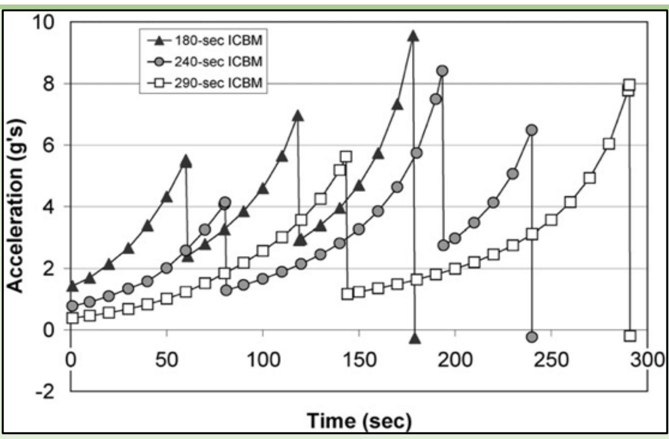
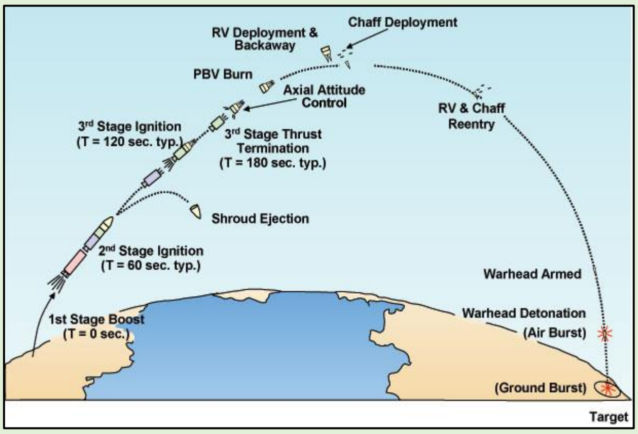


Figure 7: Minuteman Flight Sequence (Minuteman Flight Sequence 2023)



It is evident that the acceleration profile of an ICBM is highly complex and involves multiple factors such as thrust, drag, and gravity. These factors will be analysed and implemented into the parametric equations of the ICBM's flight trajectory step by step. To construct an accurate trajectory of 'Minute Man' its physical properties need to be obtained to produce accurate physics equations that describe the forces acting on the missile during its flight. All data collected is rounded to 5 significant figures (*Minuteman 3* 2018). Table 1:

	1st Boost Stage	2nd Boost Stage	3rd Boost Stage	Post-Boost Stage
Initial Mass (kg)	35300	12223	5191.0	1591.0
Thrust (kN)	933.0	267.70	152.00	0
Fuel flow rate (kgs^{-1})	346.42	103.95	53.333	0
Missile Radius (m)	0.83500	0.66500	0.66000	0.44000

The fuel being exhausted form the missile generates thrust and leads to a linear change of the missile's mass. The equation for the mass of the missile with respect to time can be denoted as $M(t) = m_0 - ft$, where m_0 is the initial mass, and f is the fuel flow rate (mass of fuel burned per unit of time).

The drag force acting on the missile due to air resistance can be represented as a function of the missile's velocity, which can be denoted as $D(v) = \frac{1}{2}\rho CAv^2$, where ρ is the air density, C is the missile's drag coefficient (This is approximated by the 3D geometry of the missile), A is the cross-sectional area, and v is the velocity of the missile. Since the ICBM reaches high altitudes, the air density is variable, and the changing air density significantly affects the trajectory of the ICBM. The air density at sea level is $1.2250\ kgm^{-3}$, and exponentially decays with altitude. Hence, the air density at any altitude y metres can be given by the equation $\rho(y) = 1.225e^{-\frac{y}{10400}}$ (Pilot Institute 2022). This is a sufficiently accurate approximation of Earth's air density derived from the ideal gas law. Hence, the drag force on the missile can be represented by $D(v) = \frac{1}{2}\rho(y)CAv^2$. Since the cross-sectional area, drag coefficient of the missile, and 0.6125 are constants in the equation, the product of the three constant terms can be denoted as a constant k, where $k = 0.6125CA$. Therefore, $D(v) = k\rho(y)v^2 = ke^{-\frac{y}{10400}}v^2$. This shows that the drag force is exponentially proportional on the inverse of the missile's altitude, and quadratically proportional to its velocity, and thus, if the vertical velocity of the missile is greater than zero, the drag force ultimately decreases due to the more significant effect of lower air density than high velocity. Hence, missiles are designed to gain altitude as quickly as possible to push through the dense atmosphere at lower altitudes.

Another force acting on the missile is gravity. The varying gravitational acceleration at high altitude leads to inconsistent gravitational acceleration experienced by the missile. The Earth's gravitational acceleration at sea level is approximately $g_0 = 9.8067ms^{-2}$. The gravitational acceleration experienced by an object on Earth is given by $g = \frac{GM}{r^2}$, where GM is the product of Earth's mass and the gravitational constant, and r is the distance between the object and Earth's centre. Hence, the gravitational acceleration at sea level is $g_0 = \frac{GM}{(6378100)^2}$, as the mean radius of Earth is 6378100 m. Therefore, the gravitational acceleration of object y meters above sea level is: $g(y) = \frac{GM}{(6378100+y)^2} = \frac{GM}{(6378100)^2}\left(\frac{63781000}{6378100+y}\right)^2 = g_0\left(\frac{63781000}{6378100+y}\right)^2$.

The sum of forces acting on the missile during its flight can be represented as a 2D vector, such that $\sum F = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$, where F_x and F_y are the vertical and horizontal components respectively.

Vector Equations of Missile Flight During Boost Stages

The forces acting on the missile during its flight are gravity, drag, and thrust. When analysed in a two-dimensional frame, each of these forces can be represented as a 2D vector, such that $g = \begin{pmatrix} g_x \\ g_y \end{pmatrix}$, $D = \begin{pmatrix} D_x \\ D_y \end{pmatrix}$, $T = \begin{pmatrix} T_x \\ T_y \end{pmatrix}$, where the vertical and horizontal components of an arbitrary vector $\hat{\imath}$ is $\hat{\imath} \sin\theta$ and $\hat{\imath}\cos\theta$ respectively, and θ is the angle of the vector from the horizontal.
The sum of forces acting on the missile can be represented as: $\sum F = T - D - mg$, where T is the thrust of the missile, D is the drag force, and mg is the product of the missile's mass and the gravitational acceleration (weight). Since $F = ma$ where a is the missile's acceleration, the equation can be rearranged to isolate a such that: $a = \frac{T-D}{m} - g$. In two-dimensional vector form, the acceleration of the missile is: $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} \frac{T_x - D_x}{m} - g \\ \frac{T_y - D_y}{m} - g \end{pmatrix}$. Gravity only acts in the downward direction, so the horizontal component of the gravity vector is zero, hence, $\vec{a} = \begin{pmatrix} \frac{T_x - D_x}{m} \\ \frac{T_y - D_y}{m} - g \end{pmatrix}$. The horizontal and vertical components of drag can be broken down using simple geometry of similar triangles. Previously explained, the drag force vector is: $\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \begin{pmatrix} ke^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right) \\ ke^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right) \end{pmatrix}$

The expression for the missile's acceleration can now be expanded into terms of x and y so that:

$$\vec{\boldsymbol{a}} = \begin{pmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \end{pmatrix} = \begin{pmatrix} \dfrac{T\cos\theta - ke^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{m_0 - ft} \\ \dfrac{T\sin\theta - ke^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{m_0 - ft} - g_0\left(\dfrac{6378100}{6378100 + y}\right)^2 \end{pmatrix}$$

Stage-1 Boost Phase Vector Equations

The trajectory of the missile in stage one of the boost phase, i.e., $0 < t \leq 60$, can be determined by solving the system of equations of the acceleration vector in Table 3. Since the equations involve second-order derivatives of the missile's displacement vector, the system of equations can be referred to as a system of Second-Order Differential Equations, also abbreviated SODEs. SODEs are highly complex and require immense computational power and time to solve. Because of the computationally intensive nature of solving systems of SODEs, Python and the RK4 method previously explained will be utilised to graph, solve, and evaluate various equations in this investigation.
As shown in Figure 3, "Minute Man" has a three-stage boost phase. In the first boost phase, the missile is completely intact, and has an initial mass of 35300 kg at $t = 0$. The missile's thrusters burn and eject fuel at a rate of 346.42 kgs^{-1}, which generates about 933.00 kN of thrust. The parameter k in the vector equation determines the intensity of the drag force. As established previously, $k = 0.6125CA$, where C is the drag coefficient of the missile and A is its cross-sectional area. The typical drag coefficient of a rocket-shaped object is about 0.75 (*Velocity During Recovery* 2023). The cross-sectional radius of "Minute Man" in stage one is about $0.835m$, hence, the cross-sectional area is $A = \pi 0.835^2 = 2.1904m^2$. By substituting these calculations into k , $k = (0.6125)(0.75)(2.1904) = 1.0062$. In this scenario, the missile is launched at 45° from the horizontal. It is also assumed that the thrusters maintain an angle of 45° from the horizontal during its boost phase, thus producing thrust at a 45° from the horizontal. This is a sufficiently accurate approximation as "Minute Man" is already equipped with fins and stabilisers that prevent it from tilting during its boost phase.
These specifications can be substituted into the equation for the acceleration vector in Table 3, so that:

$$\begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \end{pmatrix} = \begin{pmatrix} \dfrac{933000\cos\theta - 1.0062e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{35300 - 346.42t} \\ \dfrac{933000\sin\theta - 1.0062e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{35300 - 346.42t} - 9.8067\left(\dfrac{6378100}{6378100 + y}\right)^2 \end{pmatrix}$$

As shown, the only unknown parameter in this equation is the launch angle. When the launch angle is given, the equation can be solved to accurately represent the trajectory of "Minute Man" as a function of time t. In this situation, the launch angle is 45°.

Stage-2 Boost Phase Vector Equations

$$\begin{pmatrix}\frac{d^2x}{dt^2}\\ \frac{d^2y}{dt^2}\end{pmatrix} = \begin{pmatrix} \dfrac{267700\cos\theta - 0.63821e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{12223-103.95t} \\ \dfrac{267700\sin\theta - 0.63281e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{12223-103.95t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 \end{pmatrix}$$

In the second boost stage, “Minute Man” has an initial mass of 12223 kg. The missile’s thrusters burn and eject fuel at a rate of $103.95kgs^{-1}$, which generates about 267.70 kN of thrust. The parameter k in the vector equation determines the intensity of the drag force. The cross-sectional radius of “Minute Man” in stage two is about $0.665m$, hence, the cross-sectional area is $A = \pi 0.665^2 = 1.3893m^2$. By substituting these calculations into k , $k = 0.63821$.

Stage-3 Boost Phase Vector Equations

$$\begin{pmatrix}\frac{d^2x}{dt^2}\\ \frac{d^2y}{dt^2}\end{pmatrix} = \begin{pmatrix} \dfrac{152000\cos\theta - 0.62864e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{5191-53.333t} \\ \dfrac{152000\sin\theta - 0.62864e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{5191-53.333t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 \end{pmatrix}$$

In the third boost stage, “Minute Man” has an initial mass of 5191 kg. The missile’s thrusters burn and eject fuel at a rate of $53.333kgs^{-1}$, which generates about 152 kN of thrust. The parameter k in the vector equation determines the intensity of the drag force. The cross-sectional radius of “Minute Man” in stage two is about $0.66m$, hence, the cross-sectional area is $A = \pi 0.66^2 = 1.3685m^2$. By substituting these calculations into k , $k = 0.62864$.

Free Fall Phase Vector Equations

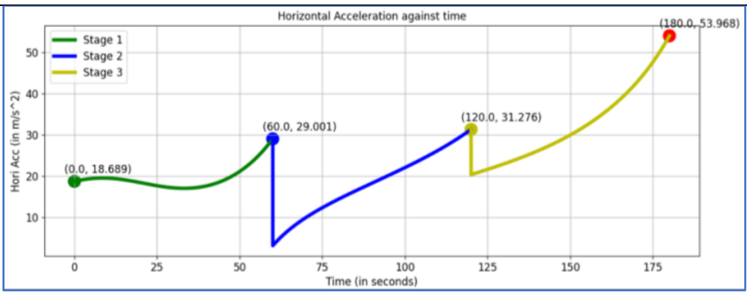
$$\begin{pmatrix}\frac{d^2x}{dt^2}\\ \frac{d^2y}{dt^2}\end{pmatrix} = \begin{pmatrix} \dfrac{-0.63821e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{12223-103.95t} \\ \dfrac{-0.63281e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{12223-103.95t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 \end{pmatrix}$$

In the third boost stage, “Minute Man” ejects all its fuel tanks and thrusters, such that its mass is 1591.0 kg. It stops producing thrust, and stops burning fuel, thus its mass remains constant, and $Thrust = 0$. The radius of the missile becomes 0.44 m, and thus its cross-sectional area is given by $A = \pi 0.44^2 = 0.60821$. By substituting these values into k , $k = 0.27940$.

Numerically Solving the Differential Equations of Boost Phase

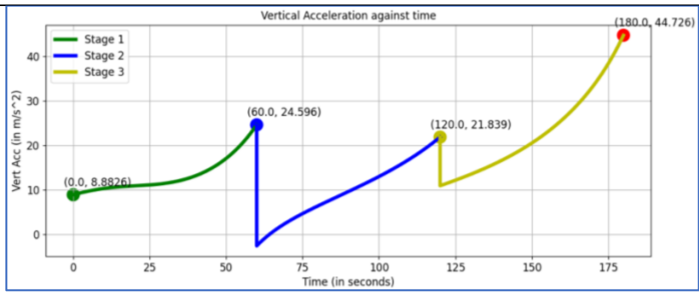
Now that the differential equations that describe of the missile’s flight are known, the system of SODEs can be numerically solved using the RK4 method. The initial conditions were coded into the program, and the initial velocity and displacement for each stage are the final conditions of the previous stage.

Horizontal Acceleration



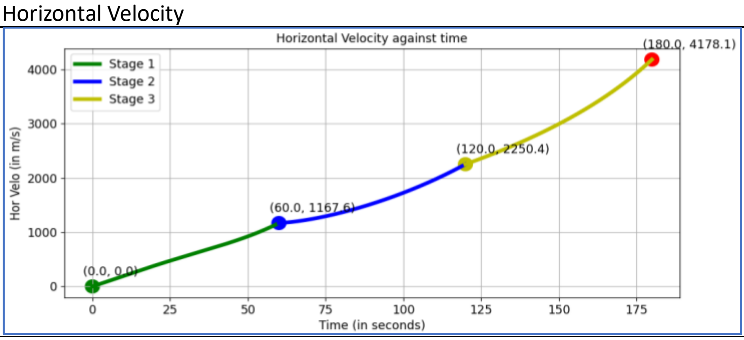
This graph shows how the horizontal acceleration of the missile changes with time. In the first 60 seconds, it is evident that the acceleration of the missile increases. The acceleration increases and then decreases, and then increases again. This is due to the drag force increasing as the missile gains velocity, but also decreasing as the missile gains altitude, and the decreasing mass of the missile, ultimately leading to an increase in acceleration. In the second boost phase, $60 \leq t < 120$, the missile steadily gains acceleration due to its reduction in mass, and the decrease in air density which decreases drag. From $120 \leq t < 180$, a similar trend is evident, where the acceleration of the missile increases. However, the acceleration is increasing at a greater rate. This is because of the weaker drag force present due to the missile’s altitude. Between boost phases, the acceleration of the missile decreases sharply. This is due to the reduction of thrust, which decreases the thrust to mass ratio of the missile, and consequently decreases its acceleration.

Vertical Acceleration

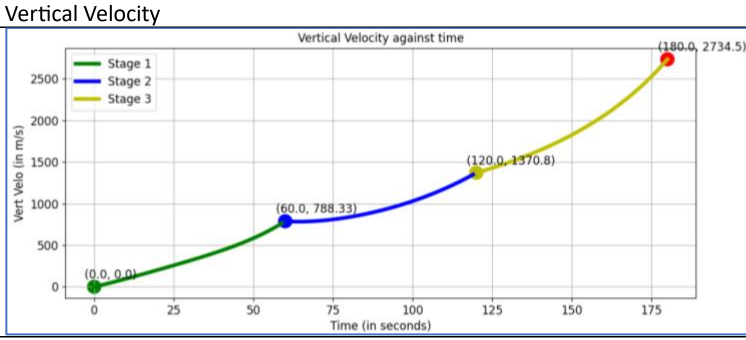


This graph shows how the vertical acceleration of the missile changes with time. In the first 60 seconds, it is evident that the acceleration of the missile increases. The acceleration increases at a decreasing rate, then an increasing rate. This is due to the drag force increasing as the missile gains velocity, but also decreasing as the missile gains altitude, and the decreasing mass of the missile, ultimately leading to an increase in acceleration. In the second boost phase, $60 \leq t < 120$, the missile steadily gains acceleration due to its reduction in mass, and the decrease in air density which decreases drag. From $120 \leq t < 180$, a similar trend is evident, where the acceleration of the missile increases. However, the acceleration is increasing at a greater rate. It is evident that the vertical acceleration is always less than the horizontal acceleration. This is because gravity is a vertical force and does not affect the horizontal component of the acceleration vector. Due to gravity, the vertical acceleration is reduced, and so is its velocity and displacement.

Horizontal Velocity



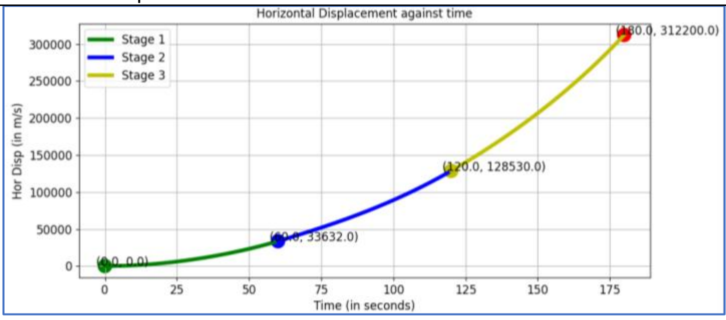
Vertical Velocity



The horizontal velocity of the missile steadily increases during the boost phase. This is because acceleration is the derivative of velocity, the slope of the acceleration-time graph illustrates the rate of change of the velocity with respect to time, and the acceleration of the missile does not fluctuate a lot and stays around $\sim 20\, ms^{-2}$. The velocity-time graph is achieved by integrating the acceleration-time graph using RK4, such that $x'(t) = \int x''(t)\, dt + v_0$, where v_0 is the initial velocity of the missile during any stage, which is also the final velocity of the previous missile stage.

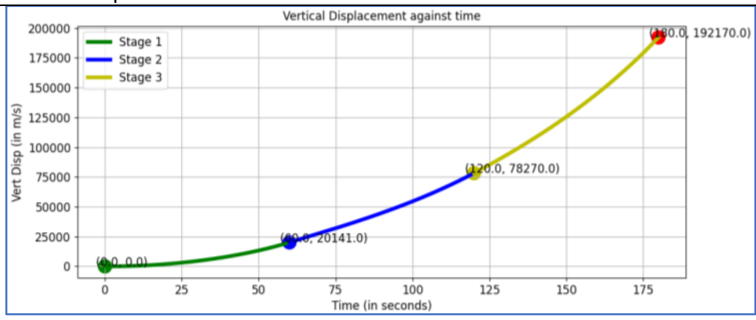
The vertical velocity curve of the missile is less smooth than the horizontal curve, since the vertical acceleration undergoes greater sudden changes than that of the horizontal acceleration. It is also evident that the vertical velocity is less than the horizontal velocity, which due to the effects of gravity. The velocity-time graph is achieved by integrating the acceleration-time graph using RK4, such that $y'(t) = \int y''(t)\, dt + v_0$, where v_0 is the initial velocity of the missile during any stage, which is also the final velocity of the previous missile stage.

Horizontal Displacement



The horizontal displacement of the missile increases at an increasing rate, and achieves a horizontal displacement of 312200 meters at the end of the boost phase. This is determined by integrating the velocity function using RK4, so that $x'(t) = \int x''(t)\, dt + x_0$, where x_0 is the initial displacement at that stage, or the final displacement of the previous stage.

Vertical Displacement



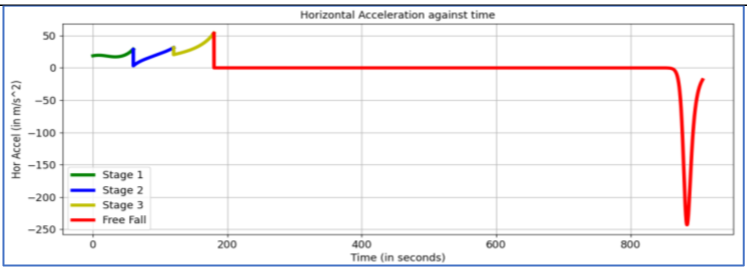
The vertical displacement increases at an increasing rate. However, it achieves less vertical height than horizontal distance due to gravity, which decreases velocity, and thus decreases vertical displacement. At the end of the boost phase, the altitude of the missile is 192170 meters. This is determined by integrating the velocity function using RK4, so that $y'(t) = \int y''(t)\, dt + y_0$, where y_0 is the initial displacement at that stage, or the final displacement of the previous stage.

Numerically Solving the Differential Equations of Free Fall Phase

The final conditions of the boost phase can be substituted into the differential equations of the free fall phase such that:

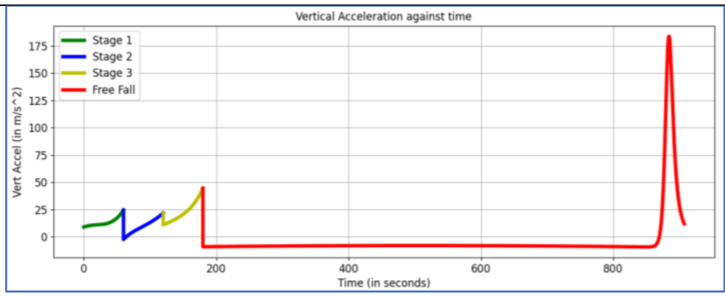
$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \int x''(t)\, dt \\ \int y''(t)\, dt \end{pmatrix} + \begin{pmatrix} v_{x0} \\ v_{y0} \end{pmatrix}$, where $\begin{pmatrix} v_{x0} \\ v_{y0} \end{pmatrix} = \begin{pmatrix} 4178.1 \\ 2734.5 \end{pmatrix}$, and $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \int x'(t)\, dt \\ \int y'(t)\, dt \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, where $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 312200 \\ 192170 \end{pmatrix}$.

Horizontal Acceleration



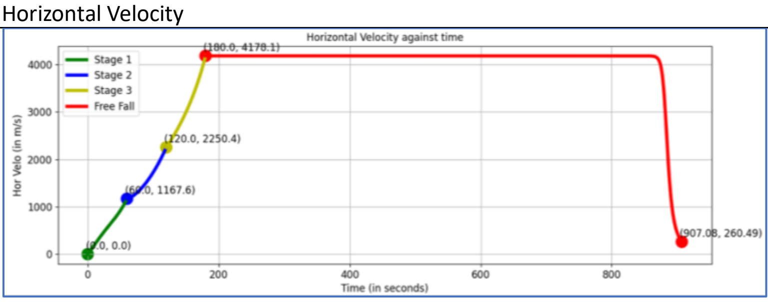
The graph of the acceleration against time shows that the horizontal acceleration of the missile drops to zero after the boost stage. At the altitude the missile is, air density is near zero and thus the drag force is negligible. The missile is not producing thrust, and there is nearly no drag, thus, there is negligible force acting on the missile, and thus it is not accelerating. As the missile drops closer to Earth's surface, the atmosphere becomes denser and the drag force increases, which is reflected by the sudden spike in deceleration at around 900 seconds into the missile's flight. As the missile loses velocity, the drag force decreases, and thus the deceleration decreases.

Vertical Acceleration



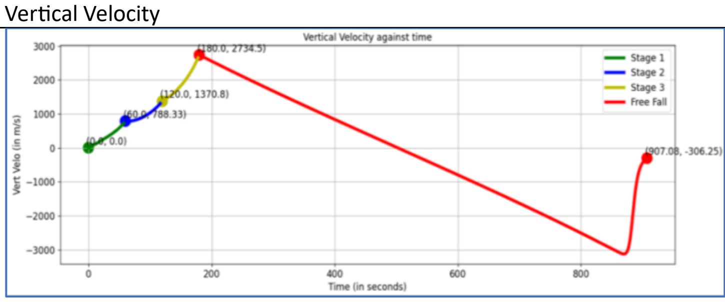
The vertical acceleration follows a similar trend to the horizontal acceleration, where drag is negligible and only gravity is present during the free fall stage. At about 900 seconds, the missile renters the Earth's atmosphere at high downwards velocity. This is counteracted by drag pushing up, thus generating positive acceleration on the missile. When the missile loses most of its velocity to drag, the drag force decreases, consequently decreasing the upwards acceleration on the missile.

Horizontal Velocity



The horizontal velocity stays constant for most of the free fall, due to near zero acceleration. As it re-enters the atmosphere, it loses most of its velocity to drag.

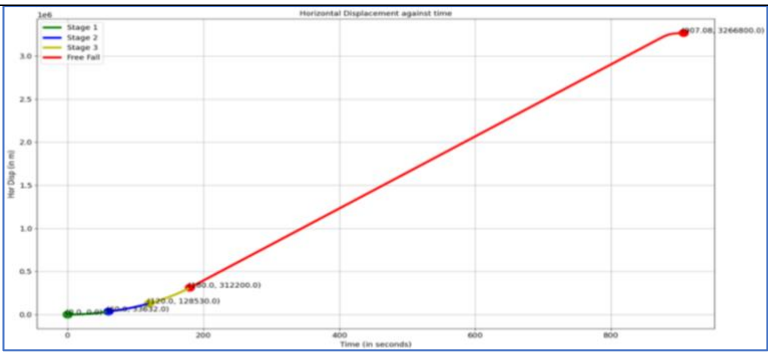
Vertical Velocity



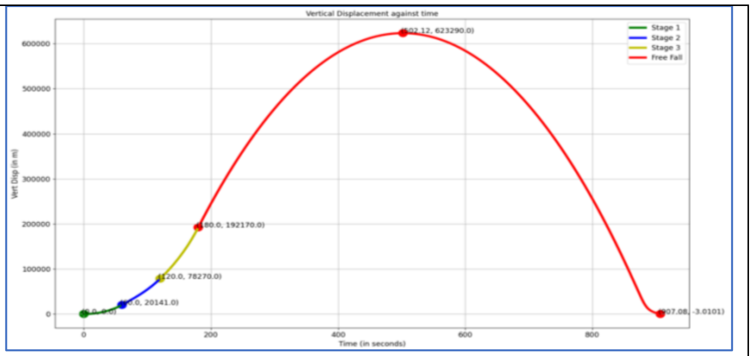
The vertical velocity steadily decreases due during free fall due to gravity. As it re-enters the atmosphere, its downwards velocity is opposed by drag, which slows down the missile. However, there is not enough time for the missile to reach terminal velocity before it reaches its destination.

Horizontal Displacement

Vertical Displacement



The horizontal displacement of the missile increases linearly after the boost phase due to the near constant velocity. When the missile renter's dense atmosphere, the decrease in velocity causes the rate of increase of the displacement to decrease. At the end of its flight, the missile achieves a horizontal range of 3266800 meters (3266.8 km)



The vertical displacement of the missile resembles parabolic motion, due to the linear decrease in its vertical velocity. It achieves a maximum of altitude of 623290 meters at 502.12 seconds into its flight. The missile begins to fall afterward and makes landfall at 907.08 seconds into its flight.

Summary of Parametric Equations of 'Minute Man' Trajectory

First Boost Stage Velocity

$$\begin{pmatrix} \displaystyle\int \frac{933000\cos\theta - 1.0062e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{35300-346.42t}\,dt \\ \displaystyle\int \frac{933000\sin\theta - 1.0062e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{35300-346.42t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 dt \end{pmatrix}$$

First Boost Stage Displacement

$$\begin{pmatrix} \displaystyle\iint \frac{933000\cos\theta - 1.0062e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{35300-346.42t}\,dt\,dt \\ \displaystyle\iint \frac{933000\sin\theta - 1.0062e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{35300-346.42t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 dt\,dt \end{pmatrix}$$

Second Boost Stage Velocity

$$\begin{pmatrix} \displaystyle\int \frac{267700\cos\theta - 0.63821e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{12223-103.95t}\,dt + 1167.6 \\ \displaystyle\int \frac{267700\cos\theta - 0.63821e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{12223-103.95t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 dt + 788.33 \end{pmatrix}$$

Second Boost Stage Displacement

$$\begin{pmatrix} \displaystyle\iint \frac{267700\cos\theta - 0.63821e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{12223-103.95t}\,dt\,dt + 33632 \\ \displaystyle\iint \frac{267700\cos\theta - 0.63821e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{12223-103.95t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 dt\,dt + 20141 \end{pmatrix}$$

Third Boost Stage Velocity

$$\begin{pmatrix} \displaystyle\int \frac{152000\cos\theta - 0.62864e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{5191-53.333t}\,dt + 2250.4 \\ \displaystyle\int \frac{152000\cos\theta - 0.62864e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{5191-53.333t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 dt + 1370.8 \end{pmatrix}$$

Third Boost Stage Displacement

$$\begin{pmatrix} \displaystyle\iint \frac{152000\cos\theta - 0.62864e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{5191-53.333t}\,dt\,dt + 128530 \\ \displaystyle\iint \frac{152000\cos\theta - 0.62864e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{5191-53.333t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 dt\,dt + 78270 \end{pmatrix}$$

Free Fall Stage Velocity

$$\begin{pmatrix} \displaystyle\int \frac{-0.63821e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{12223-103.95t}\,dt + 4178.1 \\ \displaystyle\int \frac{-0.63821e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{12223-103.95t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 dt + 2734.5 \end{pmatrix}$$

Free Fall Stage Displacement

$$\begin{pmatrix} \displaystyle\iint \frac{-0.63821e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dx}{dt}\right)}{12223-103.95t}\,dt\,dt + 312200 \\ \displaystyle\iint \frac{-0.63821e^{-\frac{y}{10400}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\left(\frac{dy}{dt}\right)}{12223-103.95t} - 9.8067\left(\frac{6378100}{6378100+y}\right)^2 dt\,dt + 192170 \end{pmatrix}$$

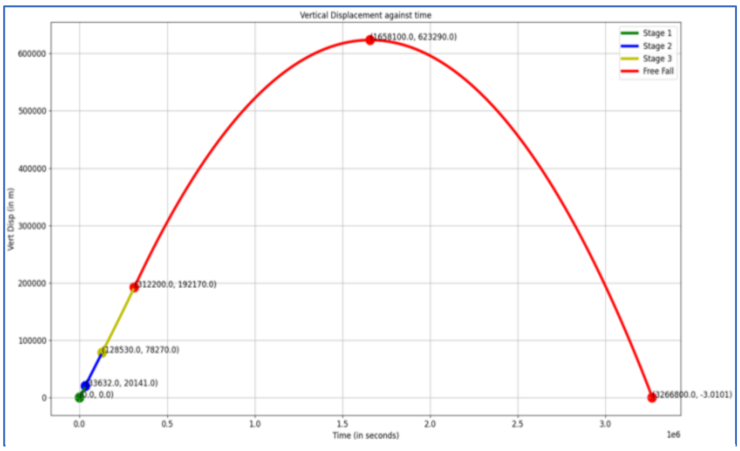


Figure 8: Flight Path of Minute Man

The trajectory formed by this system of parametric equations describes the flight path of 'Minute Man" when it is launched at an angle of 45° from the horizontal. The path resembles that of parabolic projectile motion, which is largely due to the absence of drag during the free fall stage, rendering the flight path nearly identical to parabolic motion. However, as the missile is thrusted away from Earth, and as it re-enters Earth's atmosphere, the missile's trajectory becomes highly unpredictable without the aid of parametric equations. From this, the target of the ICBM can be predicted, which is 3266.8 km away from the ICBM launch site.

The values of the ICBM's range, maximum altitude, flight duration, and flight trajectory are consistent with that found in experiments and practical records, which supports that the system of differential equations are an accurate model of the ICBM's flight path.

Additionally, the acceleration curve suggests that the missile's motion is most predictable during the free fall stage. Counter-ICBM measures can take advantage of this and intercept ICBMs during this stage.

Case 6: 3-Dimensional Extension of ICBM Flight and Interception Opportunities

Furthermore, the trajectory of an ICBM can be easily extended into a 3-Dimensional environment, by adding a third component to the acceleration, velocity, and displacement vectors, such that the $x(t)$ gives the horizontal displacement, $y(t)$ component gives the vertical displacement, and $z(t)$ gives the depth displacement. It can be assumed that there are no forces acting on the missile in the z direction, so that there is no change in velocity nor displacement in the z direction.

In this scenario, the country targeted by the ICBM has an advanced Low-Earth-Orbit Anti-Ballistic Missile system (LEOABM) that utilises satellites equipped with high-intensity lasers to trigger the ICBM's explosive wiring and causes it to explode mid-flight. To minimise the damage caused by the explosion of the ICBM, the country decides to intercept the ICBM at its maximum altitude, at $t = 502.12$, so that the position of the missile with respect to its launch site is: $\begin{pmatrix} x(502.12) \\ y(502.12) \\ z(502.12) \end{pmatrix} = \begin{pmatrix} 1658100 \\ 623290 \\ 0 \end{pmatrix}$. Most LEOABM orbit the Earth at an altitude of $2000\ km$, therefore, the position of the LEOABM at any time t is $\begin{pmatrix} x_l \\ 2000000 \\ z_l \end{pmatrix}$, where x_l and z_l are the x and z positions of the LEOABM with respect to the launch site at any time. Since lasers travel at the speed of light, are unaffected by drag, and negligibly affected by gravity, a beam of light travelling through space from the LEOABM to the ICBM can be approximated to be a straight line that passes through the positions of the LEOABM and the ICBM. Therefore, the direction vector of the beam of light travelling from the LEOABM to the ICBM is: $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_l \\ 2000000 \\ z_l \end{pmatrix} - \begin{pmatrix} 1658100 \\ 623290 \\ 0 \end{pmatrix} = \begin{pmatrix} x_l - 1658100 \\ 1376710 \\ z_l \end{pmatrix}$. From this, the equation of the line that passes through the LEOABM and the ICBM is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_l \\ 2000000 \\ z_l \end{pmatrix} - \lambda \begin{pmatrix} x_l - 1658100 \\ 1376710 \\ z_l \end{pmatrix}$, where λ is an auxillary parameter of the function. This gives the path of the laser beam.

Furthermore, the rotation needed to aim the LEOABM to the OCBM can be determined by analysing the angle of the direction vector of the laser's path to the $\hat{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \hat{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ unit vectors. The angle θ between any 2 vectors $\tilde{a}$ and $\tilde{b}$ is given by the formula: $\theta = \arccos \frac{\tilde{a} \cdot \tilde{b}}{|\tilde{a}||\tilde{b}|}$, where $\tilde{a} \cdot \tilde{b}$ is the scalar product of $\tilde{a}$ and $\tilde{b}$. This is visually depicted in Figure 9.

The angle α between r and $\hat{\imath}$ is:
$\alpha = \arccos \frac{x_l - 1658100}{\sqrt{((x_l - 1658100)^2 + 1376710^2 + z_l^2)}}$
The angle β between r and $\hat{\jmath}$ is:
$\beta = \arccos \frac{1376710}{\sqrt{((x_l - 1658100)^2 + 1376710^2 + z_l^2)}}$
The angle γ between r and $\hat{k}$ is:
$\gamma = \arccos \frac{z_1}{\sqrt{((x_l - 1658100)^2 + 1376710^2 + z_l^2)}}$

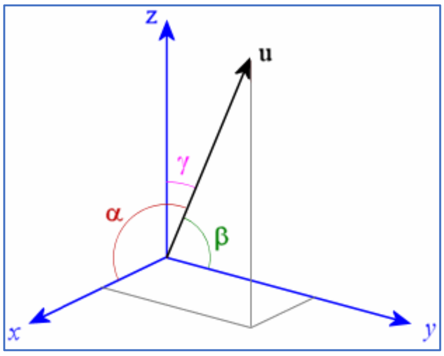


Figure 9: Angle between vector and axes (Bourne 2020)

The position and rotation of the LEOABM is always available to the defending country, and in this scenario, is $\begin{pmatrix} 100000 \\ 2000000 \\ -300000 \end{pmatrix}, \begin{pmatrix} 45° \\ 24° \\ 81° \end{pmatrix}$ respectively. Therefore:

$\alpha = \arccos \frac{100000 - 1658100}{\sqrt{((100000 - 1658100)^2 + 1376710^2 + 300000^2)}} = 137.88°$
$\beta = \arccos \frac{1376710}{\sqrt{((100000 - 1658100)^2 + 1376710^2 + 300000^2)}} = 49.054°$
$\gamma = \arccos \frac{-300000}{\sqrt{((100000 - 1658100)^2 + 1376710^2 + 300000^2)}} = 98.210°$

Hence, the rotation required to aim the LEOABM towards the ICBM is: $\begin{pmatrix} 45° - 137.88° \\ 24° - 49.054° \\ 81° - 98.210° \end{pmatrix} = \begin{pmatrix} -92.876° \\ -25.054° \\ -17.210° \end{pmatrix}$

Figure 10 shows the 3D trajectory of the ICBM, and the laser beam from the LEOABM towards the ICBM shown in black:

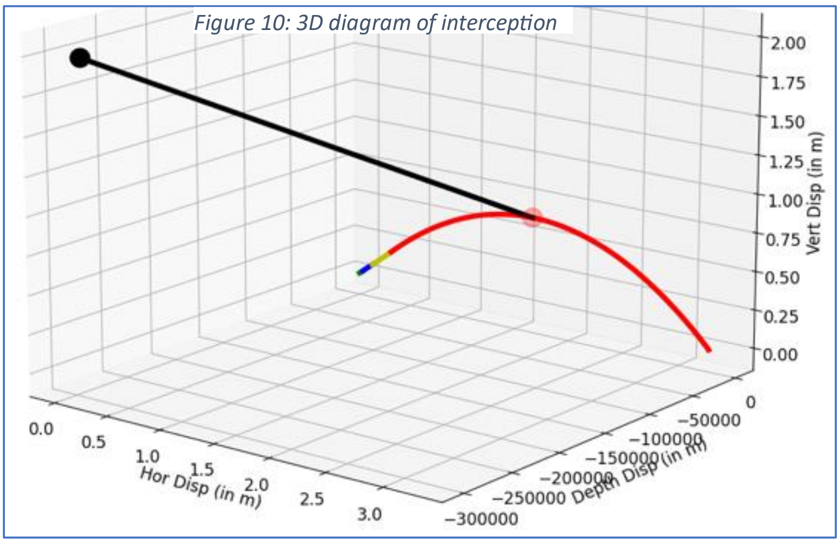


Figure 10: 3D diagram of interception

Limitations and Assumptions:

There were various assumptions made in this investigation regarding the motion of the missile, which consequently affected the validity and accuracy of the calculations and findings.

In Mathematical Models:

To keep the investigation within the scope of the course, assumptions and simplifications were made to reduce the complexities of equations. Case 1, 2, and 3 were investigated with the assumption that the missile was moving without any external forces aside from drag, gravity, and thrust, which is not an accurate representation of real missile flights. Case 4 and 5 attempted to consider factors affecting the missile's flight path with a more practical approach, by including complex differential equations that express the effects of quadratic drag, altitude and its impact on air density and gravity. However, assumptions such as the absence of wind, the effect of the rocket's spin, and its exact aerodynamics were made. It is impossible to perfectly model the aerodynamics of a missile without running computationally intensive simulations. Despite this, the effect of the missile's spin, wind, and aerodynamics are not significant, and would not have greatly changed the flight path of the missile. Another assumption made through all cases was that the thrust produced was produced at a constant angle, which is not representative of reality. This is mitigated since the wobble and the missile is insignificant and the average thrust angle stays mostly constant in practical cases.

In Solutions:

The symbolic explicit solutions to the integrals of the acceleration vectors were solved with the aid of online resources such as WolframAlpha and were cross-examined to guarantee the validity of the solutions. However, the substitution of numerical values into the equations reduced the accuracy of the parametric equations in modelling the missile's flight path. The initial conditions of the free fall stage equations were obtained from the final conditions of the boost phase of the missile's flight. In cases 1 to 3, the conditions substituted into the equations were rounded to four significant figures, and in cases 4 to 6, they were rounded to five significant figures. This would have hindered the numerical accuracy of the modelled flight path.

Furthermore, the usage of RK4 to numerically solve the system of differential equations limited the mathematical depth achieved within the investigation, as the algebraic solution to the system did not exist and could not be explored. The RK4 method also does not provide an exact solution to the system of differential equations, and only provided a high-precision approximation of the flight path of the missile. This inaccuracy was mitigated by setting the time-step size to 0.001 seconds, which meant there were 1000 calculations made for every second of the missile's flight. Although the numerical solution is not an exact representation of the missile's flight path, it was a sufficiently accurate solution.

In Graphs:

Desmos.com and Matplotlib were utilised to graph the solutions to the missile's flight paths. The calculations and renders from these programs may not have been exact, and computational glitches may have occurred. However, the models generated are consistent with those found in literature and practical experiments, thus, the models generated are sufficiently accurate.

Conclusion:

Findings:

This investigation aimed to model the flight path of a missile. Factors affecting the missile's flight path were methodologically explored. Case 1 investigated the effect of gravity on the missile's flight path and explained the parabolic trajectory of the missile with the use of vector calculus. Additionally, parametric equations were introduced, which represented the motion of the missile in 2 dimensions, namely horizontal and vertical directions. Case 2 explored the impact of decreasing mass and thrust. Integration concepts such as separable differential was introduced and solved to obtain the velocity and displacement functions of the missile's horizontal and vertical motion. Case 3 added linear drag into the equations. Linear drag significantly increased the complexity of the flight path, and the equations associated with it. More complex integration techniques and substitutions were used to solve for the velocity and displacement functions, which were further utilised to find the missile's maximum height, range, and ultimately, its flight path, which resembled a skewed parabolic path. Case 4 delved deeper and analysed the effects of quadratic drag. This section introduced the concept of first-order non-separable differential equations that did not have symbolic explicit solutions, and utilising numerical solving techniques to solve the equation. The fourth order Runge-Kutta method was analysed, explained, and implemented into a Python program, which was used to numerically integrate and graph the velocity and displacement parametric equations of the missile's flight. Case 5 included several more factors that affect the flight path of the "Minute Man" ICBM. Factors such as air density's variation with altitude, gravity's variation with altitude, and multi-stage thrust were implemented into the equations. This further complicated the flight path of the missile, which had to be separated into multiple flight stages. The flight path of each stage was numerically solved, and the final conditions of each stage were substituted as the initial conditions for the next stage. The acceleration, velocity, displacement, and trajectory graphs of the missile's flight were obtained, which were further utilised to determine an anti-ballistic missile interception strategy. In Case 6, the 2-dimensional flight path of missile flight was expanded into 3-Dimensions. The parametric equations of a laser beam's path were determined using vector mathematics, and a 3-dimensional diagram was constructed to demonstrate the process of intercepting an ICBM.

Discussion:

Overall, this investigation was successful in its modelling of missile trajectories and explored a variety of simulations to achieve this goal. Multiple cases were used to discuss the effects of external factors.

Some challenges were encountered, causing assumptions to be made in some cases to keep the investigation within the confinities of the page count and course outline. The equations utilised were physically accurate and adhered to the laws of physics. To improve the validity of findings, more research into literature and experiments could have been done into finding exact values for various constants, such as drag or lift coefficients, and the effects of wind or other complex external factors.

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Appendix:

Code for RK4 method in Python:

Fourth-order Runge-Kutta function:

```
def RK4_4(x, y, z, w, v, dx, dydx, dzdx, dwdx, dvdx):

    k1 = dx*dydx(x, y, z, w, v)
    h1 = dx*dzdx(x, y, z, w, v)
    n1 = dx*dwdx(x, y, z, w, v)
    l1 = dx*dvdx(x, y, z, w, v)

    k2 = dx*dydx(x+dx/2., y+k1/2., z+h1/2., w+n1/2., v+l1/2.)
    h2 = dx*dzdx(x+dx/2., y+k1/2., z+h1/2., w+n1/2., v+l1/2.)
    n2 = dx*dwdx(x+dx/2., y+k1/2., z+h1/2., w+n1/2., v+l1/2.)
    l2 = dx*dvdx(x+dx/2., y+k1/2., z+h1/2., w+n1/2., v+l1/2.)

    k3 = dx*dydx(x+dx/2., y+k2/2., z+h2/2., w+n2/2., v+l2/2.)
    h3 = dx*dzdx(x+dx/2., y+k2/2., z+h2/2., w+n2/2., v+l2/2.)
    n3 = dx*dwdx(x+dx/2., y+k2/2., z+h2/2., w+n2/2., v+l2/2.)
    l3 = dx*dvdx(x+dx/2., y+k2/2., z+h2/2., w+n2/2., v+l2/2.)

    k4 = dx*dydx(x+dx, y+k3, z+h3, w+n3, v+l3)
    h4 = dx*dzdx(x+dx, y+k3, z+h3, w+n3, v+l3)
    n4 = dx*dwdx(x+dx, y+k3, z+h3, w+n3, v+l3)
    l4 = dx*dvdx(x+dx, y+k3, z+h3, w+n3, v+l3)

    y = y + 1./6.*(k1+2*k2+2*k3+k4)
    z = z + 1./6.*(h1+2*h2+2*h3+h4)
    w = w + 1./6.*(n1+2*n2+2*n3+n4)
    v = v + 1./6.*(l1+2*l2+2*l3+l4)

    #return graph of derivative last
    derivative1 = k1 / dx
    derivative2 = h1 / dx
    return y, z, w, v, derivative1, derivative2
```

Main.py code for “Minute Man” flight:

```
import RungeKutta as rk
import matplotlib.pyplot as plt
import matplotlib as mpl
import numpy as np
import sigfig
from sigfig import round
from mpl_toolkits import mplot3d
from mpl_toolkits.mplot3d import Axes3D

plt.rcParams.update({'font.size': 12})

#Physics related variables
```

```
angle = np.pi / 4
thrust0 = 933000.
thrust1 = 267700.
thrust2 = 152000.
thrustf = 0.
m0 = 35300.
m1 = 12223.
m2 = 5191.0
mf = 1591.0
f0 = 346.42
f1 = 103.95
f2 = 53.333
ff = 0.
k0 = 1.0062
k1 = 0.63821
k2 = 0.62864
kf = 0.27940
g= 9.8067

#time related variables
t0 = 0.
vx0 = 0.
sx0 = 0.
vy0 = 0.
sy0 = 0.
ax0 = thrust0 / m0 * np.cos(angle)
ay0 = thrust0 / m0 * np.sin(angle) - g

dt = 0.01
t_end = 907.08
n_steps = int(round((t_end-t0)/dt))    # number of timesteps
boost1 = 60.
boost2 = 120.
boost3 = 180.

#initiating arrays
vx_arr = np.zeros(n_steps + 1)   # create an array of zeros for Y
sx_arr = np.zeros(n_steps + 1)
vy_arr = np.zeros(n_steps + 1)   # create an array of zeros for Y
sy_arr = np.zeros(n_steps + 1)   # create an array of zeros for P
ax_arr = np.zeros(n_steps + 1)   # create an array of zeros for P
ay_arr = np.zeros(n_steps + 1)   # create an array of zeros for P
z_arr = np.zeros(n_steps + 1)

t_arr = np.zeros(n_steps + 1)   # create an array of zeros for t
t_arr[0] = t0              # add starttime to array
```

```
vx_arr[0] = vx0             # add initial value of Y to array
sx_arr[0] = sx0
vy_arr[0] = vy0             # add initial value of Y to array
sy_arr[0] = sy0
ax_arr[0] = ax0
ay_arr[0] = ay0

#Differential equations
def dvxdt(t, vx, vy, sx, sy):
  #helper functions
  drag = 0
  mass = 0
  thrust = 0
  if(t < boost1 and t >= 0):
    mass = m0 - f0 * t
    thrust = thrust0
    drag = k0 * np.e**(-sy/10400) * np.sqrt((vx*vx)+(vy*vy)) * vx
  elif(t < boost2 and t >= boost1):
    mass = m1 - f1 * (t - boost1)
    thrust = thrust1
    drag = k1 * np.e**(-sy/10400) * np.sqrt((vx*vx)+(vy*vy)) * vx
  elif(t < boost3 and t >= boost2):
    mass = m2 - f2 * (t - boost2)
    thrust = thrust2
    drag = k2 * np.e**(-sy/10400) * np.sqrt((vx*vx)+(vy*vy)) * vx
  else:
    mass = mf
    thrust = thrustf
    drag = kf * np.e**(-sy/10400) * np.sqrt((vx*vx)+(vy*vy)) * vx

  #differential equation
  return (thrust * np.cos(angle) - drag) / (mass)


def dvydt(t, vx, vy, sx, sy):
  #helper functions
  gravity = g * (6378100 / (6378100 + sy))**2
  drag = 0
  mass = 0
  thrust = 0
  if(t < boost1 and t >= 0):
    mass = m0 - f0 * t
    thrust = thrust0
    drag = k0 * np.e**(-sy/10400) * np.sqrt((vx*vx)+(vy*vy)) * vy
  elif(t < boost2 and t >= boost1):
    mass = m1 - f1 * (t - boost1)
    thrust = thrust1
```

```
    drag = k1 * np.e**(-sy/10400) * np.sqrt((vx*vx)+(vy*vy)) * vy
  elif(t < boost3 and t >= boost2):
    mass = m2 - f2 * (t - boost2)
    thrust = thrust2
    drag = k2 * np.e**(-sy/10400) * np.sqrt((vx*vx)+(vy*vy)) * vy
  else:
    mass = mf
    thrust = thrustf
    drag = kf * np.e**(-sy/10400) * np.sqrt((vx*vx)+(vy*vy)) * vy

  #diff eq
  return (thrust * np.sin(angle) - drag) / (mass) - gravity

def dsxdt(t, vx, vy, sx, sy):
  return vx

def dsydt(t, vx, vy, sx, sy):
  return vy

#calculating values
tmax = 0
ymax = 0
for i in range (1, n_steps + 1):
  t = t_arr[i-1]
  vx = vx_arr[i-1]
  sx = sx_arr[i-1]
  vy = vy_arr[i-1]
  sy = sy_arr[i-1]
  ax = ax_arr[i-1]
  ay = ay_arr[i-1]
  vx_arr[i], vy_arr[i], sx_arr[i], sy_arr[i], ax_arr[i], ay_arr[i] = rk.RK4_4(t, vx, vy, sx, sy, dt, dvxdt, dvydt, dsxdt, dsydt)
  t_arr[i] = t + dt
```