TAYLOR POLYNOMIALS AND MACLAURIN SERIES

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Introduction:

Taylor polynomials and Maclaurin series are very useful in computer science and mathematics, where some functions are difficult to compute, integrate, or take the limit of. Calculators make use of Taylor polynomials to calculate complex functions, while mathematicians utilise Maclaurin series to represent irrational numbers and to solve complex integrals. The aim of this investigation is to explore the polynomial that best approximates a function around a point. This polynomial, named the function's Taylor polynomial, will be experimented with, and its behaviour as more terms are added will be assessed and discussed. The Maclaurin series of functions, which is an extension of its Taylor polynomials, will also be examined and discussed. The limitations encountered in this investigation will be analysed and discussed as well. Part A focuses on the best linear and quadratic estimations of the functions $y = \cos x$ and $y = \sqrt{x+3}$. The percentage error between the polynomial estimations and the actual function will be analysed, while the behaviour of the error as x varies will be evaluated. Part B introduces the concept of Taylor polynomials and the Taylor polynomials of a function centred at $x = a$, $a \in \mathbb{R}$. The section investigates the behaviour of the Taylor polynomials as more terms are added to it and attempts to find a general expression for the x^n term, $n \in \mathbb{N}$, in a Taylor polynomial. Any mathematical patterns and reoccurrences will also be discussed and explained. The concept of a Taylor polynomial of a function centred at $x = 0$ with infinitely many terms, otherwise known as a Maclaurin series, will be introduced in Part C of this investigation. This section investigates the behaviour of Taylor polynomials as infinitely many terms are added to it and explores the convergence of such a series. Some mathematical methodology, i.e., ratio test, will be explained and used to evaluate the convergence of the Maclaurin series associated with the function. Part D utilises the findings and concepts from Part A to C to find the Maclaurin series of the function $f(x) = \arctan x$. The processes used to determine its Taylor polynomials and its Maclaurin series will be discussed, and the convergence of the Maclaurin series will be solved and further investigated.

Part A:

This section investigates approximations of a graph with form $y = f(x)$ around an arbitrary point $x = a$, and discusses how polynomials of low degree can be used to approximate the curve around the point $x = a$.

Any point at $x = a$ on the curve given by $y = f(x)$ can be approximated by a polynomial. The best 1^{st} degree polynomial that can approximate the curve at the point $x = a$ is the tangent of the curve at $x = a$. This line intersects the curve at $x = a$, while having the same rate of change as the curve at that point. The best 2^{nd} degree polynomial that can approximate the curve at the point $x = a$ is the parabola that has the same y value, has the same rate of change, and the same concavity as $y = f(x)$ at $x = a$. In summary, the best quadratic that approximates $y = f(x)$ at $x = a$ is the polynomial $P(x)$ where:

- $P(a) = f(a)$, ensuring the same y value at $x = a$
- $P'(a) = f'(a)$, ensuring the same slope at $x = a$
- $P''(a) = f''(a)$, ensuring the same concavity at $x = a$

Consider the function $f(x) = \cos(x)$ around the point $x = 0$. The polynomial of degree 1 that approximates $y = f(x)$ at $x = 0$ best is the tangent of $y = f(x)$ at the point $x = 0$. The equation of the tangent to $y = f(x)$ at $x = 0$ can be given by $L(x)$:

$$L(x) = f'(a)(x-a) + f(a)$$

The values of $f(a)$ and $f'(a)$ can be obtained through deriving $f(x) = \cos(x)$ and substituting $x = a = 0$, where:

$$f(x) = \cos(x) \quad f(0) = \cos(0) \quad \therefore f(0) = 1$$

$$f'(x) = \frac{d}{dx}\cos(x) \quad f'(x) = -\sin(x) \quad f'(0) = -\sin(0) \quad \therefore f'(0) = 0$$

By substituting $f(0) = 1$ and $f'(0) = 0$ the equation of the tangent to $y = f(x)$ at $x = 0$ is:

$$L(x) = (0)(x-0) + (1)$$
$$\therefore L(x) = 1$$

Both the equation $y = 1$ and $y = \cos(x)$ have the same y value and slope at the point $x = 0$:

$f(0) = \cos(0) = 1$ and $L(0) = 1$ $\quad \therefore f(0) = L(0)$

$f'(0) = -\sin(0) = 0$ and $L'(0) = \frac{d}{dx}1 = 0$ $\quad \therefore f'(0) = L'(0)$

The similarity between $y = L(x)$ and $y = \cos(x)$ will be discussed later in this section.

The best 2^{nd} degree polynomial that approximates $y = f(x)$ at $x = 0$ is the quadratic $P(x) = A + Bx + Cx^2$ that satisfies $P(0) = f(0)$, $P'(0) = f'(0)$, and $P''(0) = f''(0)$. These 3 equations can be solved to obtain A, B and C.

Table 1.0:

$\boldsymbol{P(0) = f(0)}$	$\boldsymbol{P'(0) = f'(0)}$	$\boldsymbol{P''(0) = f''(0)}$
$\boldsymbol{A + B(0) + C(0)^2 = \cos(0)}$ $\boldsymbol{A = 1}$	$\frac{d}{dx}A + Bx + Cx^2 = \frac{d}{dx}\cos(x), x = 0$ $B + 2Cx = -\sin(x), x = 0$ $B + 2C(0) = -\sin(0)$ $B = 0$	$\frac{d}{dx}B + 2Cx = \frac{d}{dx} - \sin(x), x = 0$ $2C = -\cos(0)$ $2C = -1$ $C = -\frac{1}{2}$

From this, we get:

$$P(x) = 1 + (0)x + \left(-\frac{1}{2}\right)x^2$$

$$P(x) = 1 - \frac{x^2}{2}$$

When graphing $y = P(x), y = L(x), y = \cos(x)$, the precision of the approximations can be assessed and measured. The graph of $y = P(x), y = L(x), y = \cos(x)$ is attached below in Figure 1.

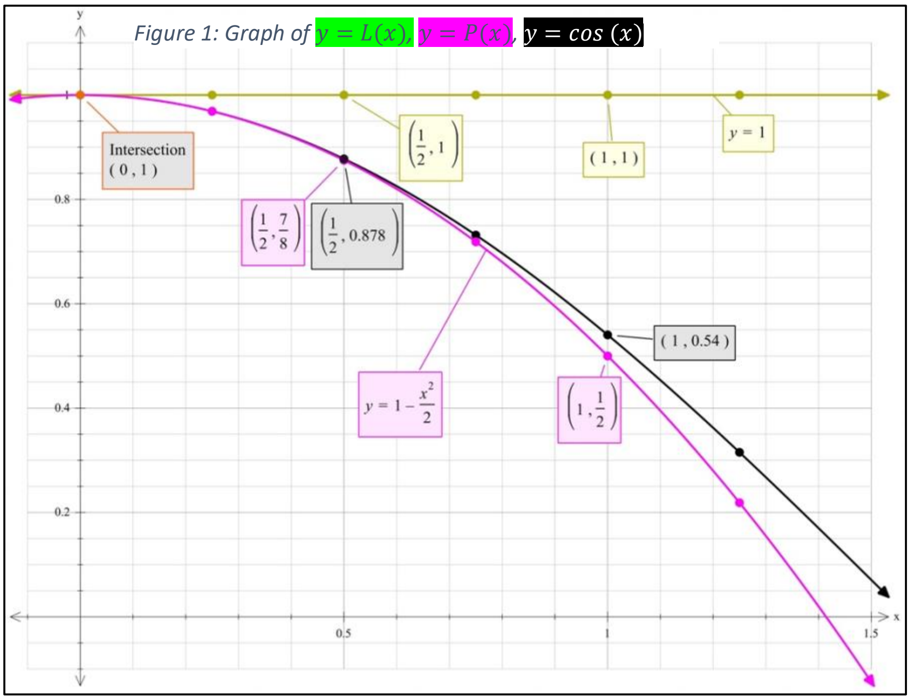


Figure 1: Graph of $y = L(x)$, $y = P(x)$, $y = \cos(x)$

Since $L(x)$ is a constant function that remains at $y = 1$ it does not provide a good estimation of $f(x) = \cos(x)$ outside of $x = 0$. $P(x)$ provides a much better estimation of $\cos(x)$. This is due to $y = P(x)$ having the same concavity and slope as $y = \cos(x)$ at the point $x = 0$. In the graph, it can clearly be seen that $y = P(x)$ approximates $y = \cos(x)$ much better than $y = L(x)$ too. The points around $x = 0$ at intervals of ± 0.5 were graphed and labelled to show the similarity between the functions around $x = 0$. Their similarity can be measured and compared using absolute percentage error, where absolute percentage error of a function $f(x)$ is abbreviated as $APE_{f(x)}$:

$APE = \left|\frac{\check{y} - y}{y}\right| \times 100\%$ with $\check{y} = P(x)$ or $\check{y} = P(x)$ and $y = \cos(x)$, and $APE < 1\%$ is a good approximation.

From figure 1, it is evident that both $L(x)$ and $P(x)$ diverge from $\cos(x)$ as x varies away from $x = 0$. Table 1.1:

Interval	$x = 0$	$x = 0.25$	$x = 0.5$	$x = 0.75$	$x = 1$	$x = 1.25$
$y = \cos(x)$	1	0.969	0.878	0.732	0.540	0.315
$y = P(x)$	1	0.969	0.875	0.719	0.5	0.219
$APE_{P(x)}$	0%	0.0168%	0.294%	1.77%	7.46%	30.6%
$y = L(x)$	1	1	1	1	1	0.540
$APE_{L(x)}$	0%	3.20%	13.9%	36.7%	85.1%	217%

Table 1.0 above summarises the similarity of $y = P(x), y = L(x)$ against $y = \cos(x)$. The measurements were rounded to 3 significant figures. Since $y = \cos(x)$, $y = P(x)$ and $y = L(x)$ are symmetrical about the y axis, they are even functions. Because of this, the graphs of the three functions were only considered for $x > 0$ since the values obtained would be the same for $x < 0$. It can be concluded that as x varies away from $x = 0$ in the positive direction, the percentage difference between the approximation functions and $\cos(x)$ increases, although $P(x)$ offers very little error ($APE < 1\%$) within the domain $\{-0.5 \leq x \leq 0.5\}$, while $L(x)$ does not give a good approximation outside of $x = 0$. This is reflected in the table where the $APE_{P(x)} = 0.0168\%$ and $APE_{P(x)} = 0.294\%$ for $x = 0.25$ and $x = 0.5$ respectively, which is a good approximation of $f(x) = \cos(x)$. However, $APE_{L(x)} = 3.20\%$ and jumps to $APE_{L(x)} = 13.9\%$ at $x = 0.25$ and $x = 0.5$ respectively, which shows that $L(x)$ is not a good approximation of $f(x) = \cos(x)$ outside of $x = 0$. As explained above, the same behaviour occurs as x deviates from $x = 0$ in the negative direction.

The general 2^{nd} degree polynomial $P(x) = A + Bx + Cx^2$ that best approximates any curve $y = f(x)$ around any point $x = a$ can be expressed from the three equalities: $P(a) = f(a), P'(a) = f'(a), P''(a) = f''(a)$, where $y = P(x)$ has the same y value, rate of change, and concavity as $y = f(x)$ at $x = a$, which ensures $y = P(x)$ moves similarly to $y = f(x)$ around $x = a$. To simplify calculations and derivatives, $P(x)$ can be translated right by a units, such that $P(x) = A + B(x - a) + C(x - a)^2$. This results in straightforward cancellations when considering $P(a)$ and its derivatives, where derivatives including non-constant terms of x^n become zero at $x = a$, since $(a - a)^n = 0$. This leaves behind a derivative that is a constant at $x = a$.

Table 1.2:

$P(x) = f(x), x = a$	$P'(x) = f'(x), x = a$	$P''(x) = f''(x), x = a$
$A + B(x - a) + C(x - a)^2 = f(x)$ $A + B(a - a) + C(a - a)^2 = f(a)$ $A + B(0) + C(0) = f(a)$ $\therefore A = f(a)$	$\frac{d}{dx} A + B(x - a) + C(x - a)^2 = f'(x)$ $B + 2C(x - a) = f'(x)$ $B + 2C(a - a) = f'(a)$ $\therefore B = f'(a)$	$\frac{d}{dx} B + 2C(x - a) = f''(x)$ $2C = f''(a)$ $2C = f''(a)$ $\therefore C = \frac{1}{2} f''(a)$

By substituting the respective values of A, B, C into the equation $P(x) = A + B(x - a) + C(x - a)^2$, the quadratic that best approximates any curve $y = f(x)$ at $x = a$ is:

$$\therefore P(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2$$

The quadratic that best approximates any function $f(x)$ can be easily found using the above expression.

Now, consider the curve $y = \sqrt{x+3}$ near $x = a = 1$. The linear equation $L(x)$ that best estimates the curve at $x = 1$ is the tangent to the curve at $x = 1$. The equation of the tangent is given by the equation:

$$L(x) = f'(a)(x-a) + f(a)$$

By substituting the $a = 1$ into $f'(a)$ and $f(a)$:

$f'(1)$	$f(1)$
$f'(1) = \frac{d}{dx}\sqrt{x+3}, x = 1$ $f'(1) = \frac{1}{2\sqrt{x+3}}, x = 1$ $f'(1) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$	$f(1) = \sqrt{x+3}, x = 1$ $f(1) = \sqrt{1+3}$ $f(1) = \sqrt{4} = 2$

By substituting $f'(1) = \frac{1}{4}$ and $f(1) = 2$ into $L(x)$:

$$L(x) = f'(1)(x-1) + f(1)$$
$$L(x) = \frac{1}{4}(x-1) + 2$$
$$\therefore L(x) = \frac{1}{4}x + \frac{7}{4}$$

To find the best quadratic equation that approximates $y = \sqrt{x+3}$ around $x = 1$, the general expression, $P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$, can be used, where $a = 1$. $f(1), f'(1)$ and $f''(1)$ can be found by deriving the equations:

$f(1) = \sqrt{1+3}$ $f(1) = 2$	$f''(1) = \frac{d}{dx}f'(x), x = 1$ $f''(1) = \frac{d}{dx}\frac{1}{2\sqrt{x+3}}, x = 1$ $f''(1) = -\frac{1}{4}(x+3)^{-\frac{3}{2}}, x = 1$ $f''(1) = -\frac{1}{4}(1+3)^{-\frac{3}{2}}$ $\therefore f''(1) = -\frac{1}{32}$
$f'(1) = \frac{d}{dx}\sqrt{x+3}, x = 1$ $f'(1) = \frac{1}{2\sqrt{x+3}}, x = 1$ $\therefore f'(1) = \frac{1}{4}$	

The values of $f(1), f'(1)$ and $f''(1)$ can be substituted into $P(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$:

$$P(x) = 2 + \frac{1}{4}(x-1) + \frac{1}{2}\left(-\frac{1}{32}\right)(x-1)^2$$
$$P(x) = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x^2 - 2x + 1)$$
$$P(x) = 2 + \frac{1}{4}x - \frac{1}{4} - \frac{1}{64}x^2 + \frac{1}{32}x - \frac{1}{64}$$
$$\therefore P(x) = -\frac{1}{64}x^2 + \frac{9}{32}x + \frac{111}{64}$$

To verify that the quadratic satisfies $P(a) = f(a), P'(a) = f'(a)$, and $P''(a) = f''(a)$ for $a = 1$, the following 3 equations can be solved to obtain A, B and C in $P(x) = A + B(x-1) + C(x-1)^2$, as shown in Table 1.2 below: Table 1.3:

$P(1) = f(1)$	$P'(1) = f'(1)$	$P''(1) = f''(1)$
$A + B(x-1) + C(x-1)^2 = \sqrt{x+3}$ $A + B(1-1) + C(1-1)^2 = \sqrt{1+3}$ $A + B(0) + C(0)^2 = \sqrt{4}$ $A = 2$	$\frac{d}{dx}(A + B(x-1) + C(x-1)^2) = \frac{d}{dx}\sqrt{x+3}$ $B + 2C(x-1) = \frac{1}{2\sqrt{x+3}}$ $B = \frac{1}{4}$	$\frac{d}{dx}(B + 2C(x-1)) = \frac{d}{dx}\frac{1}{2\sqrt{x+3}}$ $2C = -\frac{1}{4}(1+3)^{-\frac{3}{2}}$ $C = -\frac{1}{64}$

By substituting the respective values of A, B, C into the equation $P(x) = A + B(x-a) + C(x-a)^2$:

$$P(x) = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2$$
$$\therefore P(x) = -\frac{1}{64}x^2 + \frac{9}{32}x + \frac{111}{64}$$

Both methods generate the same expression for $P(x)$, which verifies the validity of the general expression for the quadratic that best approximates any function $f(x)$ around $x = a$ found in the previous section of Part A.

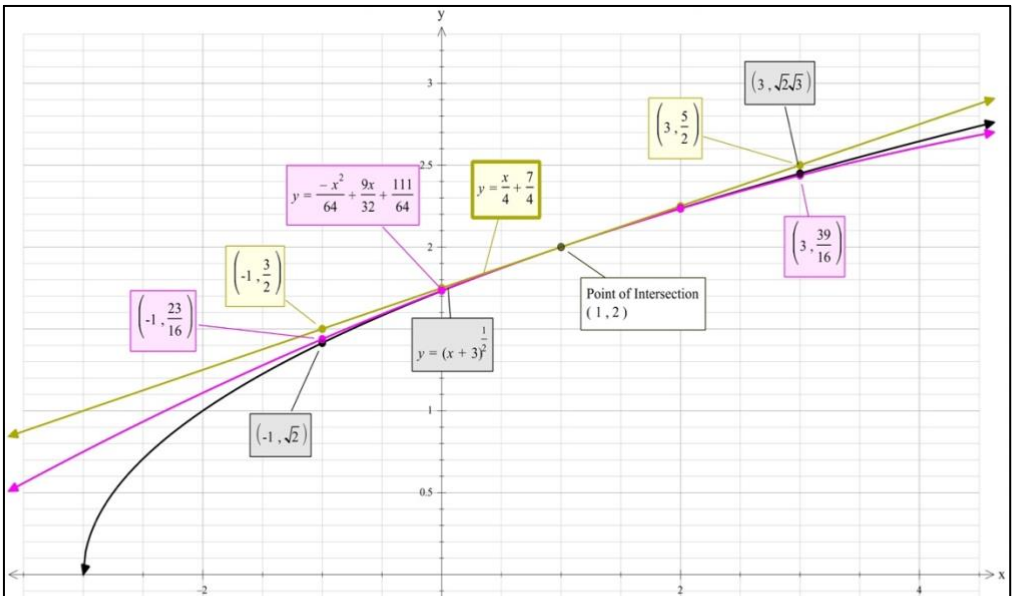


Figure 2: Graph of $y = L(x)$, $y = P(x)$, $y = \cos(x)$

The functions $y = P(x), y = L(x)$, $y = \sqrt{x+3}$ are graphed in Figure 2. It can be seen that $L(x)$ and $P(x)$ both give a good approximation of $\sqrt{x+3}$ around $x = 1$. However, $P(x)$ maintains a good approximation within the domain $\{-1 \leq x \leq 3\}$, whereas $L(x)$ loses correlation with $\sqrt{x+3}$ outside of the domain $\{0 \leq x \leq 2\}$. This is because, $y = L(x)$ and $y = \sqrt{x+3}$ only have the same y-value and slope at $x = 1$, while $y = P(x)$ and $y = \sqrt{x+3}$ have the same y-value, slope, and concavity at $x = 1$, providing better estimation. The points around $x = 1$ at intervals of ± 1 were graphed and labelled to show the similarity between the functions around

$x = 1$. Their difference can be measured and compared using absolute percentage error, with $APE < 1\%$ considered as a good approximation. Table 1.4:

Interval	$x = -1$	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$y = \sqrt{x+3}$	$\sqrt{2} \approx 1.41$	$\sqrt{3} \approx 1.73$	2	$\sqrt{5} \approx 2.24$	$\sqrt{6} \approx 2.45$
$y = P(x)$	1.4375	1.734375	2	2.234375	2.4375
$APE_{P(x)}$	1.65%	0.134%	0%	0.0757%	0.489%
$y = L(x)$	1.5	1.75	2	2.25	2.5
$APE_{L(x)}$	6.07%	1.04%	0%	0.623%	2.06%

Table 1.4 summarises the behaviour of $P(x)$ and $L(x)$ and its similarity to $y = \sqrt{x+3}$. It can be concluded as x deviates further from $x = 1$, the absolute percentage error between the approximation functions and $\sqrt{x+3}$ increases. It is evident in the table and the graph that $P(x)$ offers a good estimation of $\sqrt{x+3}$ in the range $0 \leq x \leq 3$, where $APE < 1\%$. $P(x)$ provides a better approximation than that of $L(x)$, where $L(x)$ provides a good approximation of $\sqrt{x+3}$ within $x = 1 \pm 1$, and the average absolute percentage error between $P(x)$ and $\sqrt{x+3}$ is less than that between $L(x)$ and $\sqrt{x+3}$.

Part B:

This section of the investigation discusses the Taylor Series of a function and its n^{th} degree Taylor polynomials. The general form of a n^{th} degree Taylor polynomial will be investigated, while the behaviour of the Taylor polynomial as n increases will also be assessed.

Consider the n^{th} degree polynomial $T(x)$ that best approximates a function $f(x)$ around the point $x = a$. As discussed in Part A of this investigation, the polynomial should have the same y value, slope, and concavity. To increase the similarity between how the two functions move and behave, their n^{th} derivatives at $x = a$ should be equal, that is:

$$T(a) = f(a)$$
$$T'(a) = f'(a)$$
$$T''(a) = f''(a)$$
$$\vdots$$
$$T^{(n)}(a) = f^{(n)}(a)$$

Where $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$.

This ensures that both functions not only have the same slope and concavity at $x = a$, but also have concavities that change at the same rate, and that the rate of change of the concavities change at the same rate and so forth. When the polynomial $T(x)$ satisfies the above equalities, it behaves very similarly to $f(x)$ around $x = a$.

Consider $T(x)$, where c_n are coefficients of the n^{th} term of the polynomial:

$$T(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$$

The constants c_n can be solved using the equality:

$$T^{(n)}(a) = f^{(n)}(a)$$

Solving c_n for $0 \leq n \leq 2, n \in \mathbb{Z}$:

$T(a) = f(a)$	$T'(a) = f'(a)$
$T(a) = f(a)$ $c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots + c_n(x-a)^n = f(a)$ $\therefore c_0 = f(a)$	$T'(a) = f'(a)$ $\frac{d}{dx}c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n = f'(x)$ $c_1 + 2c_2(x-a) + \cdots + nc_n(x-a)^{n-1} = f'(x)$ $c_1 + 2c_2(a-a) + \cdots + nc_n(a-a)^{n-1} = f'(a)$ $\therefore c_1 = f'(a)$

$T''(a) = f''(a)$
$T''(a) = f''(a)$ $\frac{d}{dx}c_1 + 2c_2(x-a) + 3c_3(x-a)^2 \ldots + nc_n(x-a)^{n-1} = f''(x)$ $2c_2 + 6c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2} = f''(x)$ $2c_2 + 6c_3(a-a) + \cdots + n(n-1)c_n(a-a)^{n-2} = f''(a)$ $2c_2 = f''(a)$ $c_2 = \frac{f''(a)}{2}$

From the above working out, it can be seen that each term of c_n is proportional to only the n^{th} derivative of $f(x)$ at $x = a$. This is due to the chain rule when calculating derivatives. When calculating the 0^{th} derivative at $x = a$, the constant c_0 remains unchanged and the other terms become zero. When calculating the 2^{nd} derivative, the constant term, and the linear term of $f(x)$ become zero, while the quadratic term of $f(x)$ becomes a constant term.

This pattern of the n^{th} derivative of the polynomial at $x = a$ being solely dependent on the x^n term continues when the polynomial is extended to more terms. For example, the fourth derivative of $T_n(x)$ is given by:

$$T^{(4)}(x) = \frac{d^4}{dx^4}(c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots + c_n(x-a)^n)$$
$$= \frac{d^3}{dx^3}(c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1})$$
$$= \frac{d^2}{dx^2}(2c_2 + 2 \times 3c_3(x-a) + 3 \times 4c_4(x-a)^2 + \cdots + (n-1) \times nc_n(x-a)^{n-2})$$
$$= \frac{d}{dx}(2 \times 3c_3 + 2 \times 3 \times 4c_4(x-a) + \cdots + (n-2) \times (n-1) \times nc_n(x-a)^{n-3})$$

$$= 1 \times 2 \times 3 \times 4c_4 + \cdots + (n-3) \times (n-2) \times (n-1) \times nc_n(x-a)^{n-4}$$

$$T^{(4)}(a) = 1 \times 2 \times 3 \times 4c_4 + \cdots + (n-3) \times (n-2) \times (n-1) \times nc_n(a-a)^{n-4}$$
$$T^{(4)}(a) = 1 \times 2 \times 3 \times 4c_4$$
$$\therefore T^{(4)}(a) = 4!\, c_4$$

To solve for c_4, the fourth derivative of $T_n(x)$ should be the same as the fourth derivative of $f(x)$ at $x = a$, which can be expressed as $T_n^{(4)}(a) = f^{(4)}(a)$. By substituting the above into $T_n^{(4)}(a) = f^{(4)}(a)$:

$$4!\, c_4 = f^{(4)}(a)$$
$$\therefore c_4 = \frac{f^{(4)}(a)}{4!}$$

As shown, the factorial notation comes very naturally because of the chain rule when deriving terms of x^n. For example, when deriving the sixth derivative of $E(x) = kx^6$, where k is an arbitrary coefficient, the chain rule multiplies the power of x into the coefficient of x^6, and reduces x^6 to x^5 in the first derivative to produce $E'(x) = 6 \times kx^5$. The same process is completed to find the second derivative, where $E''(x) = 5 \times 6 \cdot kx^4$. This continues into the sixth derivative where:
$E^{(6)}(x) = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times kx^0$. This shows how the factorial term appears as a result of cascading the chain rule. Generally, the m^{th} derivative of $E(x) = kx^n$ is $E^{(m)}(x) = \frac{n!}{m!}kx^{n-m}$.

The same process can be completed to find an expression for the k^{th} term, where $k \in \mathbb{Z}$ and $k \leq n$, in an n^{th}-degree Taylor Polynomial where $y = T_n(x)$:

$$T_n^{(k)}(x) = \frac{d^k}{dx^k}[c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_k(x-a)^k + \cdots + c_n(x-a)^n]$$
$$= \frac{d^{k-1}}{dx^{k-1}}[c_1 + 2c_2(x-a) + \cdots + k \times c_k(x-a)^{k-1} + \cdots + n \times c_n(x-a)^{n-1}]$$
$$= \frac{d^{k-2}}{dx^{k-2}}[2c_2 + \cdots + (k-1) \times k \times c_k(x-a)^{k-2} + \cdots + (n-1) \times n \times c_n(x-a)^{n-2}]$$
$$\vdots$$
$$= \frac{d^2}{dx^2}[(k-2)!\, c_{k-2} + (k-1)!\, c_{k-1}(x-a) + (3)(4) \times \cdots \times (k-1)(k)c_k(x-a)^2 + \cdots + \frac{n!}{(n-k+2)!}c_n(x-a)^{n-k+2}]$$
$$= \frac{d}{dx}[(k-1)!\, c_{k-1} + (2)(3) \times \cdots \times (k-1)(k)c_k(x-a) + \cdots + \frac{n!}{(n-k+1)!}c_n(x-a)^{n-k+1}]$$
$$= (1)(2)(3) \times \cdots \times (k-1)(k)c_k + \cdots + \frac{n!}{(n-k)!}(x-a)^{n-k}$$
$$= k!\, c_k + \cdots + \frac{n!}{(n-k)!}c_n(x-a)^{n-k}$$

Then, when substituting $x = a$ into the equation, an expression for $T^{(k)}(x)$ at $x = a$ that is solely dependent on c_k can be found:

$$T^{(k)}(a) = k!\, c_k + \cdots + \frac{n!}{(n-k)!}c_n(a-a)^{n-k}$$
$$\therefore T^{(k)}(a) = k!\, c_k$$

By substituting $T^{(k)}(a) = k!\, c_k$ into $T^{(k)}(a) = f^{(k)}(a)$, an expression for c_k can be found, where:

$$T^{(k)}(a) = f^{(k)}(a)$$
$$k!\, c_k = f^{(k)}(a)$$
$$\therefore c_k = \frac{f^{(k)}(a)}{k!}$$

In the working out above, it can be observed that the k^{th} derivative of the Taylor polynomial at $x = a$ is solely dependent on the x^k term of the polynomial. This is because the x^m terms, $m \in \mathbb{Z}^+$, where $m < k$, become zero after deriving, and the x^m terms where $m > k$, become zero since $(a-a)^m = 0$.

From this, an expression for an n^{th} degree Taylor polynomial $T(x)$ that estimates $f(x)$ around $x = a$ can be obtained:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$$
$$T_n(x) = \frac{f(a)}{0!} + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Now, the n^{th} degree Taylor polynomial of a function can be easily found by substituting in the respective derivatives.
This also means that the $(n+1)^{th}$ degree Taylor Polynomial centred at $x = a$ is the same as the n^{th} degree Taylor Polynomial appended with an x^{n+1} term, or:

$$T_{n+1}(x) = \sum_{k=0}^{n+1} \frac{f^{(k)}(a)(x-a)^k}{k!} = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{f^{(n+1)}(a)(x-a)^{n+1}}{(n+1)!} = T_n(x) + \frac{f^{(n+1)}(a)(x-a)^{n+1}}{(n+1)!}$$

Consider $f(x) = \cos(x)$ and its $2^{nd}, 4^{th}, 6^{th}, 8^{th}$-degree Taylor polynomials centred at $x = 0$.
The 2^{nd}-degree Taylor polynomial centred at $x = 0$ can be denoted as $T_2(x)$, where $T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(a)(x-a)^k}{k!}$, therefore:

$$T_2(x) = \frac{f(a)}{0!} + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!}$$
$$\therefore T_2(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!}$$

$f(0), f'(0)$ and $f''(0)$ can be solved through finding the first and second derivative of $y = \cos(x)$, shown in Table 2.0

$f(0)$	$f'(0)$	$f''(0)$
$f(0) = \cos(0)$ $f(0) = 1$	$f'(0) = \frac{d}{dx}\cos(x), x = 0$ $f'(0) = -\sin(x), x = 0$ $f'(0) = -\sin(0)$ $f'(0) = 0$	$f''(0) = \frac{d}{dx}f'(x), x = 0$ $f''(0) = \frac{d}{dx} - \sin(x), x = 0$ $f''(0) = -\cos(x), x = 0$ $f''(0) = -\cos(0)$ $f''(0) = -1$

Through substituting the values of $f(0), f'(0)$ and $f''(0)$ into $T_2(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!}$, $T_2(x)$ can be found:

$$T_2(x) = \frac{(1)}{0!} + \frac{(0)x}{1!} + \frac{(-1)x^2}{2!}$$

$$\therefore T_2(x) = 1 - \frac{x^2}{2}$$

The 4^{th}-degree Taylor polynomial centred at $x = 0$ can be denoted as $T_4(x)$, where $T_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(a)(x-a)^k}{k!}$, therefore:

$$T_4(x) = \frac{f(a)}{0!} + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f^{(3)}(a)(x-a)^4}{3!} + \frac{f^{(4)}(a)(x-a)^4}{4!}$$

$$\therefore T_4(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!}$$

$f(0), f'(0)$ and $f''(0)$ have been solved while finding $T_2(x)$, so only $f^{(3)}(0)$ and $f^{(4)}(0)$ need to be solved, as shown in Table 2.1:

$f^{(3)}(0)$	$f^{(4)}(0)$
$f^{(3)}(0) = \frac{d}{dx}f''(x), x = 0$ $f^{(3)}(0) = \frac{d}{dx} - \cos(x), x = 0$ $f^{(3)}(0) = \sin(x), x = 0$ $f^{(3)}(0) = \sin(0)$ $\therefore f^{(3)}(0) = 0$	$f^{(4)}(0) = \frac{d}{dx}f^{(3)}(x), x = 0$ $f^{(4)}(0) = \frac{d}{dx}\sin(x), x = 0$ $f^{(4)}(0) = cos(x), x = 0$ $f^{(4)}(0) = cos(0)$ $\therefore f^{(4)}(0) = 1$

It can be seen that $f^{(4)}(x) = cos(x)$ and $f(x) = \cos(0)$. This means that $f^{(5)}(x) = f'(x), f^{(6)} = f''(x), f^{(7)}(x) = f^{(3)}(x)$, and $f^{(n)}(x) = f^{(n \bmod 4)}(x),\ n \in \mathbb{Z}^+$. This pattern will be useful further into this section of the investigation as it will simplify and streamline the calculations involving higher-degree derivatives.

By substituting the values of $f(0), f'(0), f''(0), f^{(3)}(0)$ and $f^{(4)}(0)$ into $T_4(x)$:

$$T_4(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!}$$

$$T_4(x) = \frac{(1)}{0!} + \frac{(0)x}{1!} + \frac{(-1)x^2}{2!} + \frac{(0)x^3}{3!} + \frac{(1)x^4}{4!}$$

$$\therefore T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{24}$$

From the above calculation steps, the x^n term where n is an odd number seems to become zero. This is due to the pattern of the derivatives of $f(x) = \cos(x)$, where $f^{(i+1)}(x) = -\sin(x)$, $f^{(i+2)}(x) = -\cos(x)$, $f^{(i+3)}(x) = \sin(x)$, $f^{(i)}(x) = \cos(x)$ and i is a positive multiple of 4. When $x = 0$, $f^{(i+1)}(0) = 0$, $f^{(i+2)}(0) = -1$, $f^{(i+3)}(0) = 0$, $f^{(i)}(0) = 1$. This is why the x^n term where n is an odd number is always equal to zero, since $\sin(0) = 0$. Aside from this, it is evident that the fourth Taylor polynomial is very similar to the second Taylor polynomial, where the only difference is that the fourth Taylor Polynomial has an extra x^4 term. This is due to how $T_n(x)$ is calculated, where the n^{th} derivative of the polynomial at $x = a$ is solely dependent on the x^n term. When the polynomial is extended to have higher degrees, such as the polynomial $T_{n+1}(x)$, the $(n+1)^{th}$ derivative of $T_{n+1}(x)$ and $f(x)$ at $x = a$ will be the same, and since the only thing influencing the $(n+1)^{th}$ derivative of $T_{n+1}(x)$ at $x = a$ is the x^{n+1} term, the only change to the polynomial is the change to the x^{n+1} term. Hence, $T_{n+1}(x)$ can be expressed as:

$$T_{n+1}(x) = T_n(x) + \frac{f^{(n+1)}(a)(x-a)^{n+1}}{(n+1)!}$$

This dramatically simplifies the calculation of high degree Taylor polynomials since previously calculated expressions can be reused.

The 6^{th} degree Taylor polynomial centred at $x = 0$ can be denoted as $T_6(x)$. Using the expression found above for $T_{n+1}(x)$, $T_6(x)$ can be written as:

$$T_6(x) = T_5(x) + \frac{f^{(6)}(0)(x)^6}{6!}$$

Since $T_5(x) = T_4(x)$, because the odd terms of the Taylor polynomial of $f(x) = \cos(x)$ is zero, $T_6(x)$ can be expressed in terms of $T_4(x)$:

$$T_6(x) = T_4(x) + \frac{f^{(6)}(0)(x)^6}{6!}$$

Now, only $f^{(6)}(a)$ needs to be solved by calculating the 6^{th} derivative. This can be done by using the pattern of the n^{th} derivatives of $f(x) = \cos(x)$, where $f^{(6)}(0) = f^{(4+2)}(0)$, and $f^{(n+2)}(0) = -1$ where n is a multiple of four, as discussed above. Therefore, $f^{(6)}(0) = -1$. Substituting this into $T_6(x) = T_4(x) + \frac{f^{(6)}(a)(x-a)^6}{6!}$ for $a = 0$:

$$T_6(x) = T_4(x) + \frac{f^{(6)}(0)(x-0)^6}{6!}$$
$$T_6(x) = T_4(x) + \frac{(-1)(x)^6}{6!}$$
$$T_6(x) = T_4(x) - \frac{x^6}{720}$$
$$\therefore T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{24} - \frac{x^6}{720}$$

The same process can be used to find the eighth-degree Taylor polynomial of $f(x) = \cos(x)$, where $T_8(x) = T_6(x) + \frac{f^{(8)}(0)(x-0)^8}{8!}$, and $f^{(8)}(0) = f^{(4\times2)}(0)$. Since $f^{(n)}(0) = 1$ when n is a multiple of four, as explained above, $f^{(8)}(0) = 1$. Substituting these:

$$T_8(x) = T_6(x) + \frac{f^{(8)}(0)(x-0)^8}{8!}$$
$$T_8(x) = T_6(x) + \frac{(1)(x)^8}{8!}$$
$$T_8(x) = T_6(x) + \frac{x^8}{40320}$$
$$\therefore T_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}$$

The graphs of $y = T_2(x), y = T_4(x), y = T_6(x), y = T_8(x)$ can be graphed and compared to the graph of $y = \cos(x)$ to assess how well they approximate $f(x) = \cos(x)$ around $x = 0$.

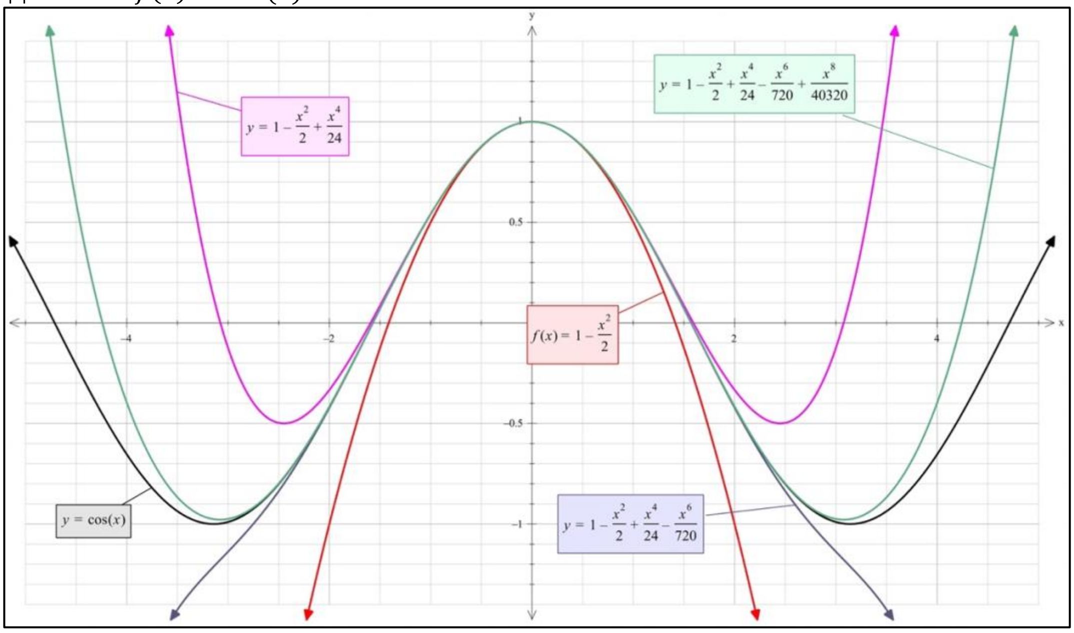


Figure 3 :Graph of $y = T_2(x), y = T_4(x), y = T_6(x), y = T_8(x), y = cos(x)$

As shown in Figure 3, higher degree Taylor polynomials of $f(x) = \cos(x)$ centred at $x = 0$ provide a more accurate approximation of $y = \cos(x)$ over a larger domain. The accuracy of the Taylor polynomials' estimation of $f(x) = \cos(x)$ can be assessed by finding the absolute percentage error between the $y = T_n(x)$ and $y = \cos(x)$ at intervals of 1 in the domain $\{0 \le x \le 4\}$. The domain $\{x \le 0\}$ can be ignored since $y = T_2(x), y = T_4(x), y = T_6(x), y = T_8(x), y = \cos(x)$ are all even functions where $f(-x) = f(x)$. Table 2.2 below summarises the results with calculations rounded to three significant figures. This ensures that insignificant information is lost when rounding the data. Table 2.2

	$y = T_2(x)$	$y = T_4(x)$	$y = T_6(x)$	$y = T_8(x)$	$y = \cos x$
$x = 0$	1	1	1	1	1
$APE_{T_n(x)}$	0%	0%	0%	0%	0%
$x = 1$	0.5	0.542	0.540	0.540	0.540
$APE_{T_n(x)}$	7.46%	0.253%	0.000454%	0.0000506%	0%
$x = 2$	−1	−0.333	−0.422	−0.416	−0.416
$APE_{T_n(x)}$	140%	19.9%	1.46%	0.0658%	0%
$x = 3$	−3.5	−0.125	−1.14	−0.975	−0.990
$APE_{T_n(x)}$	253%	87.4%	14.9%	1.54%	0%
$x = 4$	−7	3.67	−2.02	−0.397	−0.654
$APE_{T_n(x)}$	971%	661%	209%	39.3%	0%

From the table above, it is again evident that the approximations become less accurate as x deviates from $x = 0$. However, it is obvious from Figure 3 and the table that the higher degree Taylor polynomials provide far better estimations of $f(x) = \cos(x)$ for a larger domain. $T_2(x)$ has a percentage error larger than 1% at $x = 1$, so it does not give a good approximation of $f(x) = \cos(x)$, whereas $T_4(x), T_6(x), T_8(x)$ all give good approximations with percentage error less than 1% at $x = 1$. $T_6(x), T_8(x)$ greatly outperformed the other Taylor polynomials with insignificant percentage error (percentage error less than 3 decimal places). At $x = 2$, $T_2(x)$ becomes an unreliable estimator of $f(x) = \cos(x)$, while $T_4(x), T_6(x)$ continue to give a rough estimation, and $T_8(x)$ continues to give a good approximation with $APE < 1\%$. At $x = 3$ and $x = 4$, none of the polynomials give a good estimation of $f(x) = \cos(x)$. $T_2(x), T_4(x)$ begin to completely deviate from $f(x) = \cos(x)$ while $T_6(x), T_8(x)$ maintain some correlation with $f(x) = \cos(x)$ at $x = 3$. This is due to the graph of $y = \cos(x)$ changing direction at $x = \pi$, which the second and fourth derivates of $f(x) = \cos(x)$ do not account for enough. $y = T_8(x)$ maintains the least percentage error with $y = \cos(x)$ across all Taylor polynomials assessed. This shows that increasing the degree of a Taylor polynomial also increases the accuracy of the approximation made by the polynomial.

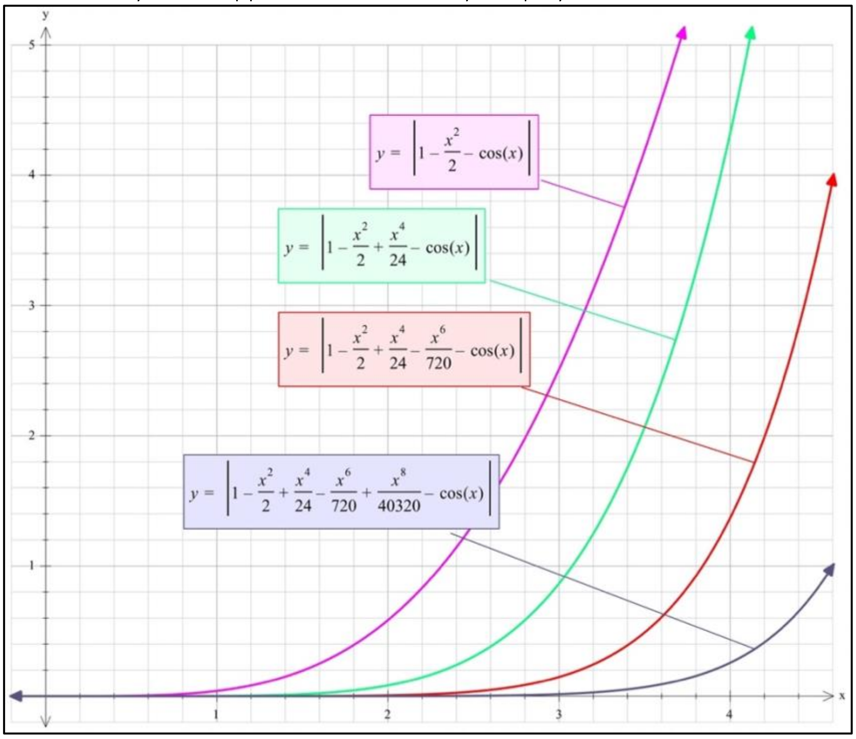


Figure 4: $y = |T_2(x) - \cos(x)|$, $y = |T_4(x) - \cos(x)|$, $y = |T_6(x) - \cos(x)|$, $y = |T_8(x) - \cos(x)|$

The difference between the Taylor polynomials and $f(x) = \cos(x)$ can be visually demonstrated by graphing $y = |T_n(x) - \cos(x)|$, where $T_n(x)$ is the n^{th} degree Taylor polynomial of $f(x) = \cos(x)$ centred at $x = 0$. Figure 4 depicts: $y = |T_2(x) - \cos(x)|$, $y = |T_4(x) - \cos(x)|$, $y = |T_6(x) - \cos(x)|$, and $y = |T_8(x) - \cos(x)|$, and how the difference between $y = T_n(x)$ and $y = \cos(x)$ varies around $x = 0$.

As seen in Figure 4, the difference between $y = T_2(x)$ and $y = \cos(x)$ is the greatest in the domain: $\{-4 \leq x \leq 4\}$ (since the functions are even where $f(-x) = f(x)$). The difference of $y = T_4(x)$ is less prominent and stays insignificant in the domain $\{-1.5 \leq x \leq 1.5\}$. $y = T_6(x)$ is an even better estimation of $y = \cos(x)$, where the difference between the two is insignificant in the domain $\{-2 \leq x \leq 2\}$. $y = T_8(x)$ provides the best estimation where an insignificant difference between the two functions is maintained in the domain $\{-3 \leq x \leq 3\}$. It is evident that all four Taylor polynomials begin to diverge from $y = \cos(x)$, and that the higher degree Taylor polynomials maintain insignificant difference for a larger domain.

Part C:

This section of the investigation discusses Taylor polynomials with infinite terms, otherwise known as Taylor series, and Taylor series that are centred at $x = 0$, otherwise known as Maclaurin Series. This section also investigates the divergence or convergence of Maclaurin series.

From Part B, it was stated that the n^{th} Taylor polynomial $T(x)$, of an arbitrary function $f(x)$, is given by the expression, $T(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$. Hence, the Taylor polynomial with infinite terms, $T_\infty(x)$ is given by the expression:

$$T_\infty(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

When the Taylor series is centred at $x = 0$, $T_\infty(x)$, otherwise known as the Maclaurin series, can be given by the expression:

$$T_\infty(x) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!} = M(x)$$

Since $T_\infty(x)$ is an infinite sum of finite terms, it can either approach negative or positive infinity, or it can approach a limit. That is, the Maclaurin series of a function, $f(x)$, either diverges from the function, or converges to it. When the series does not converge, it may not be a good approximation for a function beyond a certain threshold. The domain where the series is a good approximation is called the interval of convergence, and if the interval of convergence is the set of all real numbers, the series can be used to approximate the entire curve.

Consider the function $f(x) = e^x$, and $M(x) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}$. Now, only the derivatives of $f(x) = e^x$ at $x = 0$ need to be solved. Since the derivative of $f(x) = e^x$ is e^x, it can be concluded that $f^{(n)}(x) = e^x$ for all n. Hence, $f^{(n)}(0) = e^0 = 1$. Substituting $f^{(n)}(0) = 1$ into the equation for $M(x)$:

$$M(x) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}$$

$$M(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\therefore M(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$$

The Taylor polynomials of $f(x)$, $T(x)$, can be graphed with $y = f(x)$ to illustrate how $y = T(x)$ estimates $y = f(x)$ and whether $y = T(x)$ converges or diverges from $y = f(x)$ when more terms are added to $T(x)$. If modern computers could compute $M(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, it would be relatively simple to determine whether $y = M(x)$ converges or diverges from $y = f(x)$, however, it is computationally infeasible to compute the infinite terms in $M(x)$. Instead, the n^{th} degree polynomials of $f(x)$ can be graphed, where $y = T_n(x)$ and $3 \leq n \leq 5$. The behaviour of $y = T_n(x)$ will be assessed for the values of n. If $y = T_n(x)$ converges towards $y = f(x)$ for values of x that are far away from $x = 0$ as n increases, it can be induced that $y = T_n(x)$ converges towards $y = f(x)$ for all $x \in \mathbb{R}$. Otherwise, if $y = T_n(x)$ diverges from $y = f(x)$ for values of x that are far away from $x = 0$ as n increases, it can be induced that $y = T_n(x)$ diverges from $y = f(x)$ as x deviates from $x = 0$.

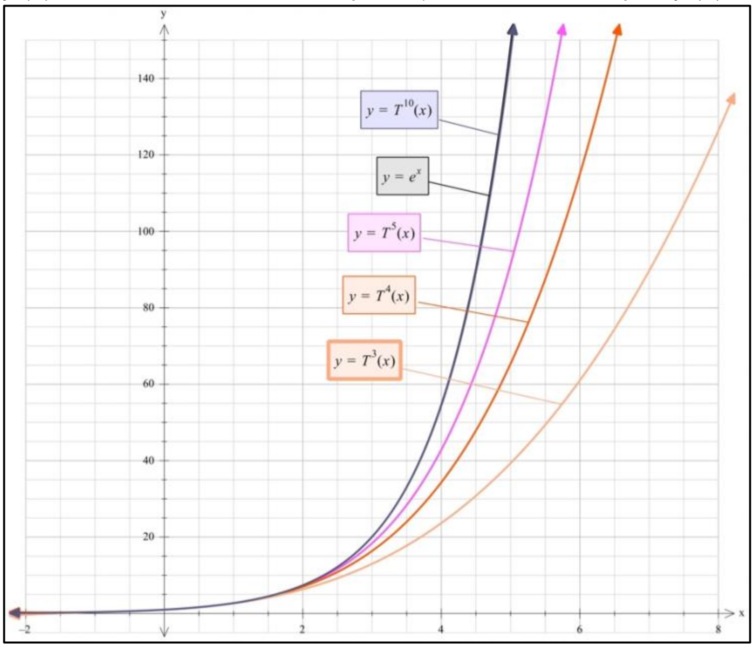


Figure 5: Graph of $y = T_3(x), y = T_4(x), y = T_5(x), y = T_{10}(x), y = e^x$

From Figure 5, it is shown that $y = T_5(x)$ diverges from $y = f(x)$ for values of x that are far away from $x = 0$ less than $y = T_4(x)$ does. It can also be seen that $y = T_4(x)$ diverges from $y = f(x)$ more than $y = T_3(x)$ for values of x far away from $x = 0$. From this, it can be induced that as more terms are appended to the Taylor polynomial centred at $x = 0$ of $f(x)$, the higher-degree Taylor polynomial diverges from $y = f(x)$ less, and thus, the Maclaurin series of $f(x) = e^x$, denoted $M(x)$ converges towards $y = f(x)$ as more and more terms are summated to it. This is visually illustrated by the graph of the tenth-degree Taylor polynomial of $y = f(x)$ centred at $x = 0$. It is evident that $y = T_{10}(x)$ is very similar to $y = f(x)$, even when x is far away from $x = 0$, such as at $x = 5$. In conclusion, it can be said that when infinitely many terms are added to the $M(x)$, it becomes equal to $f(x)$, or:

$$M(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

Using the same processes, it can be determined whether the Maclaurin Series of $h(x) = \ln(x+1)$ converges or diverges. The Maclaurin series of $h(x)$ can be denoted by $K(x)$, where:

$$K(x) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!}$$

The first, second, third, and fourth derivatives of $h(x) = \ln(x+1)$ are given by the following: Table 3.0:

$\boldsymbol{h'(x)}$	$\boldsymbol{h''(x)}$	$\boldsymbol{h^{(3)}(x)}$	$\boldsymbol{h^{(4)}(x)}$
$\boldsymbol{h'(x) = \frac{d}{dx}\ln(x+1)}$ $\boldsymbol{\therefore h'(x) = (x+1)^{-1}}$	$h''(x) = \frac{d}{dx}h'(x)$ $h''(x) = \frac{d}{dx}(x+1)^{-1}$ $\therefore h''(x) = -(x+1)^{-2}$	$h^{(3)}(x) = \frac{d}{dx}h''(x)$ $h^{(3)}(x) = \frac{d}{dx} - (x+1)^{-2}$ $\therefore h^{(3)}(x) = 2(x+1)^{-3}$	$h^{(4)}(x) = \frac{d}{dx}h^{(n)}(x)$ $h^{(4)}(x) = \frac{d}{dx}2(x+1)^{-3}$ $\therefore h^{(4)}(x) = -6(x+1)^{-4}$

From the above, a conjecture for the n^{th} derivative of $h(x)$ can be made, where:

$$h^{(n)}(x) = (-1)^{n-1}(n-1)!\,(x+1)^{-n}$$

This conjecture can be proved using Mathematical Induction. (Mathematical induction | Definition, Principle, & Proof | Britannica 2023). Mathematical Induction is based on the concept that if a proposition, $P(n)$ is true for $P(1)$, and that $P(k+1)$ is proved to be true for some $k \in \mathbb{N}$, then $P(n)$ is true for all $n \in \mathbb{N}$.

The proposition in this case is $h^{(n)}(x) = (-1)^{n-1}(n-1)!\,(x+1)^{-n}$.

This is true for $h'(x)$, where:

$$h'(x) = (-1)^0(0)!\,(x+1)^{-1}$$
$$\therefore h'(x) = (x+1)^{-1} \textit{ as required}$$

Now, assume that $h^{(k)}(x) = (-1)^{k-1}(k-1)!\,(x+1)^{-k}$ for some $k \in \mathbb{N}$. Since $h^{(k+1)}(x)$ is the derivative of $h^{(k)}(x)$, the L.H.S is obtained by deriving $h^{(k)}(x)$, while the R.H.S is obtained by using the formula $h^{(k+1)}(x) = (-1)^k(k)!\,(x+1)^{-k-1}$.

L.H.S.	R.H.S.
$\boldsymbol{P(k+1) = \frac{d}{dx}P(k)}$ $\boldsymbol{P(k+1) = \frac{d}{dx}(-1)^{k-1}(k-1)!\,(x+1)^{-k}}$ $\boldsymbol{P(k+1) = (-1)^{k-1}(k-1)! \times \frac{d}{dx}(x+1)^{-k}}$ $\boldsymbol{P(k+1) = (-1)^{k-1}(k-1)! \times -k(x+1)^{-k-1}}$ $\boldsymbol{P(k+1) = (-1)^{k-1}(k-1)!\,(k)(-1)(x+1)^{-k-1}}$ $\boldsymbol{P(k+1) = (-1)^{k}(k)!\,(x+1)^{-k-1}}$	$P(k+1) = (-1)^{(k+1)-1}((k+1)-1)!\,(x+1)^{-(k+1)}$ $P(k+1) = (-1)^{k}(k)!\,(x+1)^{-k-1}$

Therefore, $h^{(k+1)}(x) = (-1)^k (k)! (x+1)^{-k-1}$ is true since L.H.S is equal to R.H.S
Since the proposition is true for $h'(x)$, and true for $h^{(k+1)}(x)$ whenever $h^{(k)}(x)$ is true, the proposition is true for all $n \in \mathbb{N}$.
Hence, $h^{(n)}(0) = (-1)^{n-1}(n-1)!(1)^{-n} = (-1)^{n-1}(n-1)!$

From this, a new expression for $K(x)$ can be obtained, where:

$$K(x) = \sum_{k=0}^{\infty} (-1)^{k-1}(k-1)! \frac{x^k}{k!}$$

$$\therefore K(x) = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

However, the first term where $k = 0$ is not defined, as $\frac{x^k}{0}$ is undefined, and $(-1)!$ is undefined. Instead, the first term where $k = 0$ can be replaced with a term such that $K(0) = h(0)$. Since the 0^{th} derivative of $h(x)$ is solely dependent on the $k = 0$ term of its Maclaurin series, the first term is $\ln(1+0) = 0$. This gives the new expression for $K(x)$:

$$K(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

The graph of $y = T_n(x)$ and $y = \ln(x+1)$ where $3 \le n \le 5, n \in \mathbb{Z}$, can illustrate whether $y = T_n(x)$ converges or diverges from $y = \ln(x+1)$ for larger values of n.

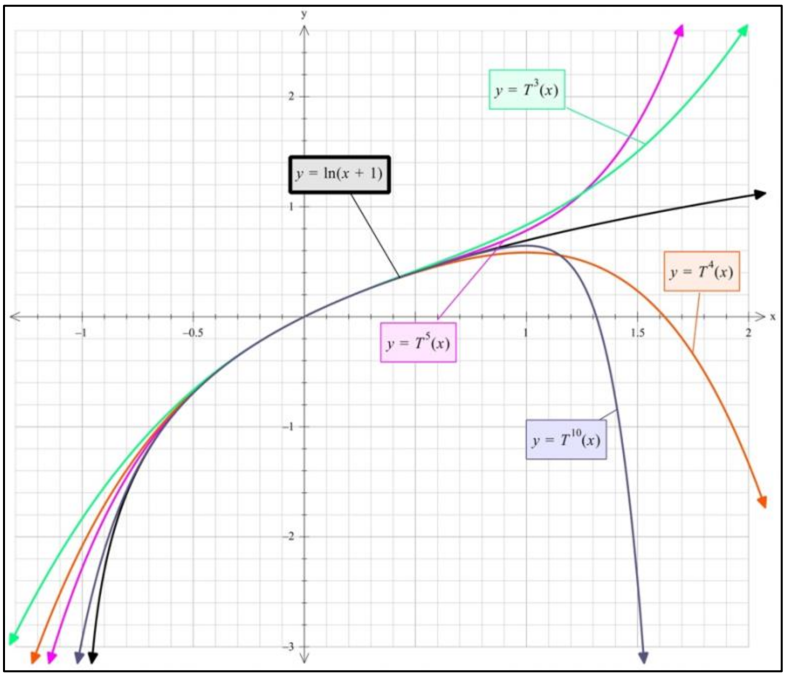


Figure 6: Graph of $y = T_3(x), y = T_4(x), y = T_5(x), y = T_{10}(x), y = ln(x+1)$

As shown in Figure 6, the graph of $y = T_n(x)$ provides a good approximation of $y = h(x)$ near $x = 0$, but quickly diverges away from $y = \ln(x)$ when x deviates far away from $x = 0$. It can also be seen that $y = T_3(x)$ does not diverge as rapidly as $y = T_4(x)$, and that the graph of $y = T_4(x)$ does not diverge as rapidly as that of $y = T_5(x)$. From this, it can be induced that when more terms are added into the Taylor polynomial centred at $x = 0$, the resulting graph diverges from the graph of $y = h(x)$ more rapidly than Taylor polynomials with fewer terms. Hence, it can also be said that the Maclaurin series of $h(x)$ provides a good approximation only within a certain domain, and that the Maclaurin series diverges away from $y = h(x)$ beyond that threshold, or $K(x) = h(x),\ \{l \le x \le u\}$ where l and u are the lower and upper limits of the threshold respectively. By observation, it is evident that the graphs of $y = T_n(x)$ diverge from the graph of $y = h(x)$ outside the domain $\{-1 \le x \le 1\}$. The interval of convergence for $K(x)$., defined as "an interval that is associated with a given power series such that the series converges for all values of the variable inside the interval and diverges for all values outside it." (Interval of convergence Definition & Meaning | Dictionary.com 2023), can be deduced to be $\{-1 < x < 1\}$.
However, this method does not provide an exact value for the interval of convergence and does not give a definitive answer to whether the Maclaurin series of a function is convergent or divergent. This approach where the high-degree Taylor polynomials are used to determine the convergence or divergence of Maclaurin series is also not efficient. Instead, the ratio test (Smith 2023) can be utilised to generate an unambiguous conclusion for a Maclaurin series' interval of convergence.
The ratio test takes the absolute ratio between the $(k+1)^{th}$ term and the k^{th} term in a series and takes the limit of that ratio as k approaches infinity. This limit represents the ratio between each consecutive term as infinitely many terms are added to the series. If the ratio between each consecutive term is greater than one, the series is divergent, since each term is getting larger, and approaches infinity. If the ratio between each consecutive term is less than one, the series is convergent, since each term is getting smaller, and the series will eventually reach a finite limit. If the ratio between each term is equal to one, its convergence is inconclusive.
Consider the general Maclaurin series of an arbitrary function, where:

$$E(x) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!}$$

This can be re-written to:

$$E(x) = \sum_{k=0}^{\infty} c_k x^k$$

Where c_k is an arbitrary constant value. Hence, the ratio between any two consecutive terms can be given by the expression:

$$Ratio = \left|\frac{c_{k+1}x^{k+1}}{c_k x^k}\right| \qquad \therefore Ratio = \left|x\frac{c_{k+1}}{c_k}\right|$$

The limit of the ratio as k approaches infinity can be solved by taking its limit:

$$\lim_{k\to\infty}\left|x\frac{c_{k+1}}{c_k}\right| = |x|\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|$$

From this, it can be concluded that the ratio between each consecutive term is proportional to the input x, that is, different values of x have differing convergence. To find the values of x where the series is convergent, the limit of the ratio must be less than one, where each consecutive term is smaller than the previous one, meaning the series does not grow to infinity.

$$\therefore |x|\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| < 1$$

By re-writing $\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|$ as an arbitrary limit L, an expression for x such that the series is convergent can be obtained:

$$|x| < \frac{1}{L}$$

This can be rewritten as a domain for x, where $x < \frac{1}{L}$ for $x \geq 0$ and $x > -\frac{1}{L}$ for $x < 0$, or $\{x: -\frac{1}{L} < x < \frac{1}{L}\}$. Since the ratio test is inconclusive for $x \times L = 1$, the endpoints of the domain should be checked to verify its convergence.

Now, a simple method can be used for determining the interval of convergence of any Maclaurin series of any function.

To find the interval of convergence of the Maclaurin series of $f(x) = e^x$, the ratio between each consecutive term in the Maclaurin series should be evaluated. Since $M(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}$, the general expression for any two consecutive terms is $\frac{x^k}{k!}$ and $\frac{x^{k+1}}{(k+1)!}$. Therefore, the ratio between the consecutive terms is:

$$Ratio = \left|\frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}}\right| = \left|\frac{x^{k+1}}{(k+1)!}\times\frac{k!}{x^k}\right| = \left|\frac{x}{k+1}\right|$$

As infinitely many terms are added into the series, the ratio between the terms can be found by taking the limit as k approaches infinity:

$$\lim_{k\to\infty}\left|\frac{x}{k+1}\right| = |x|\lim_{k\to\infty}\left|\frac{1}{k+1}\right|$$

And: $\lim_{k\to\infty}\left|\frac{1}{k+1}\right| = 0$

$$\therefore \lim_{k\to\infty}\left|\frac{x}{k+1}\right| = |x|\times 0$$

$$\therefore \lim_{k\to\infty}\left|\frac{x}{k+1}\right| = 0$$

From this, the ratio between consecutive terms as infinitely many terms are added to the Maclaurin series of $f(x) = e^x$ approaches zero. To solve for the interval of convergence of the Maclaurin series, the ratio must be less than one:

$$|x|\lim_{k\to\infty}\left|\frac{1}{k+1}\right| < 1$$

$$|x| < \frac{1}{0}$$

$$\therefore -\infty < x < \infty$$

Hence, the interval of convergence of the Maclaurin series of $f(x) = e^x$ is all $x \in \mathbb{R}$. This means that the Maclaurin series of $f(x) = e^x$ converges for every real value of x.

The same process can be repeated to find the interval of convergence of the Maclaurin series of $h(x) = \ln(x+1)$. The Maclaurin series can be denoted as $K(x)$, where $K(x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$, as discussed previously in this section. The k^{th} term in the series is $(-1)^{k-1}\frac{x^k}{k}$, and its consecutive term is $(-1)^k\frac{x^{k+1}}{k+1}$. Therefore, the absolute ratio between any two consecutive terms is:

$$Ratio = \left|\frac{(-1)^{k-1}\frac{x^{k+1}}{k+1}}{(-1)^k\frac{x^k}{k}}\right|$$

$$Ratio = \left|(-1)\frac{x^{k+1}}{k+1}\times\frac{k}{x^k}\right|$$

$$\therefore Ratio = \left|\frac{x\times k}{k+1}\right|$$

When infinitely many terms are appended into the series, the ratio between consecutive terms can be obtained by taking the limit of the ratio as k approaches infinity:

$$\lim_{k\to\infty}\left|\frac{x\times k}{k+1}\right| = |x|\lim_{k\to\infty}\left|\frac{k}{k+1}\right| \text{ and } \lim_{k\to\infty}\left|\frac{k}{k+1}\right| = 1$$

To solve for the interval of convergence of the Maclaurin series, the ratio must be less than one:

$$|x|\lim_{k\to\infty}\left|\frac{k}{k+1}\right| < 1$$

$$|x| < 1$$

$$\therefore -1 < x < 1$$

Hence, the interval of convergence of the Maclaurin series of $h(x) = \ln(x+1)$ is the domain $\{x: -1 < x < 1\}$. The endpoints of the interval, $x = \pm 1$, can be checked for convergence by observing the behaviour of the Maclaurin series as more terms are added to it. If the higher-degree polynomial with more terms converges towards $\ln(2)$ or $\ln(0)$ more than the lower-degree polynomial for $x = 1$ and $x = -1$ respectively, it can be concluded that $M(x)$ converges towards $\ln(x+1)$ for $x = \pm 1$. Since the Taylor polynomial derived from the first 3 terms of the Maclaurin series at $x = 1$, i.e., $\sum_{k=1}^{3}(-1)^{k-1}\frac{1^k}{k} = 0.833$, and the polynomial derived from the first 6 terms, i.e., $\sum_{k=1}^{6}(-1)^{k-1}\frac{1^k}{k} = 0.617$, and $\ln(2) = 0.693$, it is evident that the Taylor polynomial with more terms converges towards $\ln(x+1)$ more at $x = 1$ than the polynomial with fewer terms. The same behaviour occurs for $M(-1)$, where the Taylor polynomial derived from the first 3 terms at $x = -1$, $\sum_{k=1}^{3}(-1)^{k-1}\frac{(-1)^k}{k} = -1.83$, and the polynomial derived from the first 6 terms at $x = -1$ $\sum_{k=1}^{6}(-1)^{k-1}\frac{(-1)^k}{k} = -2.45$, while $\ln(0) = -\infty$. This shows that the Taylor polynomial with more terms converges towards $\ln(x+1)$ at $x = -1$ more than the polynomial with fewer terms. Therefore, the Maclaurin series converges for $x = \pm 1$, and thus the interval of convergence is $\{x: -1 \le x \le 1\}$. This is reflected in Figure 6, where higher-degree Taylor polynomials of $y = h(x)$ provide better approximations only in the domain $\{x: -1 \le x \le 1\}$, but diverge away from $y = h(x)$ when outside of the interval of convergence.

Part D:

This section of the investigation focuses on the Taylor polynomials of $f(x) = \arctan x$ centred at $x = 0$, and the Maclaurin series of $f(x)$. The behaviour of the Taylor polynomials as more terms are added to it will be examined, and the divergence or convergence of the Maclaurin series will be investigated and determined.

The n^{th} Taylor polynomial of $f(x)$ centred at $x = 0$, denoted $T_n(x)$ is given by the expression found in Part B of the investigation:

$$T_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)x^k}{k!}$$

Hence, only the derivatives of $f(x)$ need to be substituted into the expression to find the Taylor polynomial. The first derivative of $f(x)$, denoted $f'(x)$ can be calculated using implicit differentiation, where:

$$y = \arctan x$$

Step 1:
By taking the tangent of both sides of the equation:

$$\tan y = x$$

Step 2: By taking the derivative of both sides with respect to x:

$$\frac{d}{dx}\tan y = \frac{d}{dx}x$$

$$\therefore \frac{dy}{dx}\sec^2 y = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Step 3:
By using the identity: $\sec^2 x = \tan^2 x + 1$:

$$\frac{dy}{dx} = \frac{1}{\tan^2 y + 1}$$

Step 4:
And substituting in $\tan y = x$:

$$\frac{dy}{dx} = \frac{1}{x^2 + 1}$$

Step 5:
From this, $f'(x)$ and $f'(0)$ can be obtained, where:

$$f'(x) = \frac{1}{x^2+1}$$

$$\therefore f'(0) = 1$$

Hence, the first-degree Taylor polynomial of $f(x)$ centred at $x = 0$ can be found, where $f'(0) = 1$ and $f(0) = 0$:

$$T_1(x) = \sum_{k=0}^{1}\frac{f^{(k)}(0)x^k}{k!}$$

$$\therefore T_1(x) = x$$

The first seven derivatives of $f(x)$ and the seven five terms of its Taylor polynomial can be solved to find a pattern in the n^{th} terms of the polynomial. Table 4.0:

$f^{(n)}(x)$	$f^{(n)}(0)$	n^{th} term of $T_n(x)$
$f''(x) = \frac{d}{dx}f'(x)$ $f''(x) = -\frac{2x}{(x^2+1)^2}$	$f''(0) = -\frac{2(0)}{(0^2+1)^2}$ $f''(0) = 0$	$\frac{f''(0)x^2}{2!} = 0$
$f^{(3)}(x) = \frac{d}{dx}f''(x)$ $f^{(3)}(x) = \frac{2(3x^2-1)}{(x^2+1)^3}$	$f^{(3)}(0) = \frac{2(3(0)^2-1)}{(0^2+1)^3}$ $f^{(3)}(0) = -2$	$\frac{f^{(3)}(0)x^3}{3!} = -\frac{x^3}{3}$
$f^{(4)}(x) = \frac{d}{dx}f^{(3)}(x)$ $f^{(4)}(x) = -\frac{24x(x^2-1)}{(x^2+1)^4}$	$f^{(4)}(0) = -\frac{24(0)(0^2-1)}{(0^2+1)^4}$ $f^{(4)}(0) = 0$	$\frac{f^{(4)}(0)x^4}{4!} = 0$
$f^{(5)}(x) = \frac{d}{dx}f^{(5)}(x)$ $f^{(5)}(x) = \frac{24(5x^4-10x^2+1)}{(x^2+1)^5}$	$f^{(5)}(0) = \frac{24(5(0)^4-10(0)^2+1)}{((0)^2+1)^5}$ $f^{(5)}(0) = 24$	$\frac{f^{(5)}(0)x^5}{5!} = \frac{x^5}{5}$
$f^{(6)}(x) = \frac{d}{dx}f^{(5)}(x)$ $f^{(6)}(x) = -\frac{240x(3x^4-10x^2+3)}{(x^2+1)^6}$	$f^{(6)}(0) = \frac{240(0)(3(0)-10(0)+3)}{(0+1)^6}$ $f^{(6)}(0) = 0$	$\frac{f^{(6)}(0)x^6}{6!} = 0$

$f^{(7)}(x) = \frac{d}{dx} f^{(7)}(x)$ $f^{(7)}(x) = \frac{720(7x^6 - 35x^4 + 21x^2 - 1)}{(x^2+1)^7}$	$f^{(7)}(0) = \frac{720(7(0) - 35(0) + 21(0) - 1)}{(0+1)^7}$ $f^{(7)}(0) = -720$	$\frac{f^{(7)}(0)x^7}{7!} = -\frac{x^7}{7}$

Through finding the first seven Taylor polynomials of $f(x)$ centred at $x = 0$, a general pattern of the n^{th} term of the Taylor is evident, with the terms of x with an even degree being zero, and the terms of x with an odd degree being $\frac{x^k}{k}$ for $k \in \mathbb{Z}$ with alternating sign. Hence, the expression:

$$T_n(x) = \sum_{\frac{k}{2}=0}^{n} (-1)^{\frac{k}{2}} \frac{x^{k+1}}{k+1}$$

This can be shown and supported by analysing and manipulating the geometric sum of squares, where the infinite sum S is:

$$S = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{k=0}^{\infty} x^k$$

S is a geometric series in the form $S = \sum_{k=0}^{\infty} ar^k$, where $a = 1$ and the common ratio $r = x$ in this case. All geometric series with common ratio less than one can be expressed in closed form as $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. Therefore, for $|x| < 1$, S can be expressed as:

$$S = \frac{1}{1-x}$$

Now, if the common ratio is $-x$ instead of x, the geometric series $S' = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots = 1 - x + x^2 - x^3 + \cdots$, and can be expressed as:

$$S' = \frac{1}{1-(-x)}$$

$$\therefore S' = \frac{1}{1+x}$$

By replacing the x with x^2, a new geometric series S'' is made, where $S'' = 1 - x^2 + x^4 - x^6 + x^8 + \cdots$and can be expressed as:

$$S'' = \frac{1}{1+x^2}$$

This is the same expression as the derivative of $\arctan x$. By using the integration rule $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$:

$$\arctan x = \int S''\,dx$$

$$\arctan x = \int 1 - x^2 + x^4 - x^6 + x^8 + \cdots dx$$

$$\arctan x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \quad \text{Where } C \text{ is an arbitrary constant.}$$

$$\therefore \arctan x = C + \sum_{\frac{k}{2}=0}^{\infty} (-1)^{\frac{k}{2}} \frac{x^{k+1}}{k+1}$$

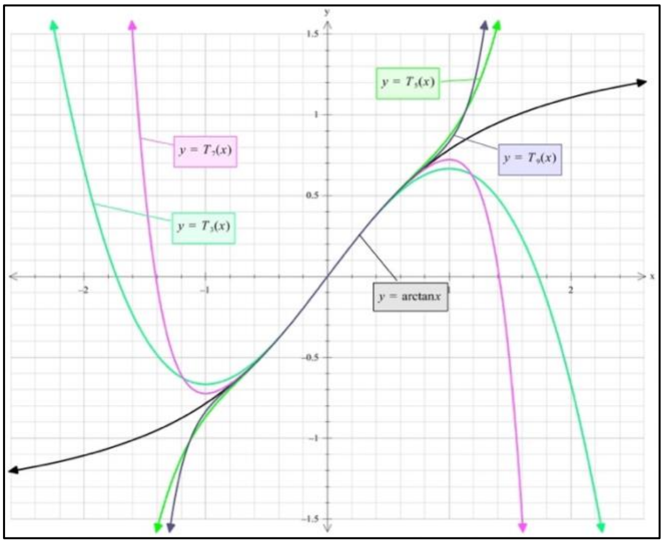
Since $\arctan 0 = 0$, $C = 0$ as well. Therefore:

$$\arctan x = \sum_{\frac{k}{2}=0}^{\infty} (-1)^{\frac{k}{2}} \frac{x^{k+1}}{k+1}$$

This supports the statement that the n^{th} degree Taylor polynomial $T_n(x)$, is expressed by $T_n(x) = \sum_{\frac{k}{2}=0}^{n} (-1)^{\frac{k}{2}} \frac{x^{k+1}}{k+1}$.

This expression also suggests that the n^{th} term of the Taylor polynomial is always zero. Hence, only the odd-degree Taylor polynomials will be investigated and examined.

Figure 7: Graph of $y = \arctan x$, $y = T_3(x), y = T_5(x), y = T_7(x), y = T_9(x)$



The four odd Taylor polynomials of $f(x)$ will be investigated, namely $T_3(x), T_5(x), T_7(x)$ and $T_9(x)$. Their behaviour as x deviates from zero will be assessed to determine the convergence of the Taylor polynomials as more terms are added, or as higher degree Taylor polynomials are used. The graph of $y = \arctan x$, $y = T_3(x), y = T_5(x), y = T_7(x)$ and $y = T_9(x)$ are graphed in Figure 7. From Figure 7, it can be observed that all the Taylor polynomials provide a sound approximation of $y = f(x)$ in the domain: $\{-1 < x < 1\}$. However, it is evident that all the Taylor polynomials quickly diverge away from $y = \arctan x$ when x is outside of the domain: $\{-1 < x < 1\}$. It is also observable that the higher-degree Taylor polynomials diverge quicker than the lower-degree Taylor polynomials. This suggests that $y = T_n(x)$ only converges in the domain $\{-1 < x < 1\}$ for $n \in \mathbb{Z}$.

The accuracy of the approximations made by the Taylor polynomials can be calculated and examined using the absolute percentage error, abbreviated APE, between the approximation and the actual value of $\arctan x$. This is given by the equation: $APE = \left|\frac{\bar{y}-y}{y}\right| \times 100\%$. The approximation is considered a good approximation when $APE < 1\%$. Since $\arctan x$ and its Taylor polynomials are all even functions, where $T_n(x) = -T_n(-x)$ and $f(-x) = -f(x)$, the APE is the same for $T_n(x)$ and $T_n(-x)$. Therefore, only the APE of positive values of x will be investigated.

The table below illustrates the APE between the approximations and the actual value for values of x in $\{0 \leq x \leq 1.25\}$ in increments of 0.25. Table 4.1:

	$y = T_3(x)$	$y = T_5(x)$	$y = T_7(x)$	$y = T_9(x)$	$y = \arctan x$
$x = 0$	1	1	1	1	1
$APE_{T_n(x)}$	0%	0%	0%	0%	0%
$x = 0.25$	0.244	0.244	0.244	0.244	0.244
$APE_{T_n(x)}$	0.076%	0.00339%	0.000165%	0.00000840%	0%
$x = 0.5$	0.458	0.465	0.463	0.464	0.464
$APE_{T_n(x)}$	1.15%	0.201%	0.0389%	0.00791%	0%
$x = 0.75$	0.609	0.657	0.638	0.646	0.644
$APE_{T_n(x)}$	5.30%	2.07%	0.891%	0.405%	0%
$x = 1$	0.667	0.867	0.724	0.835	0.785
$APE_{T_n(x)}$	15.1%	10.3%	7.84%	6.31%	0%
$x = 1.25$	0.599	1.21	0.528	1.36	0.896
$APE_{T_n(x)}$	33.2%	35.0%	41.1%	51.3%	0%

From table 4.1, $y = T_3(x)$ provides a good estimation with $APE < 1\%$ up to $x = 0.5$, $y = T_5(x)$ provides a good estimation up to $x = 0.75$, $y = T_7(x)$ provides a good estimation up to $x = 1$, and $y = T_9(x)$ provides an even better estimation up to $x = 1$. From this, it is evident that the higher-degree Taylor polynomials provide a more accurate approximation for a larger domain. However, this trend does not continue when $x = 1.25$. The table shows that the APE increases for the higher-degree Taylor polynomials, and that the APE increases at an increasing rate at $x = 1.25$. This supports the previous statement that the Taylor polynomials $T_n(x)$ of $f(x)$, centred at $x = 0$ only converge towards $y = f(x)$ in the domain $\{-1 \leq x \leq 1\}$, and diverges outside that domain.

Since the n^{th} degree Taylor polynomial of $f(x)$ centred at $x = 0$ is $\sum_{\frac{k}{2}=0}^{n}(-1)^{\frac{k}{2}}\frac{x^{k+1}}{k+1}$, the Maclaurin series, $M(x)$, is given by:

$$M(x) = \sum_{\frac{k}{2}=0}^{\infty}(-1)^{\frac{k}{2}}\frac{x^{k+1}}{k+1}$$

$$\therefore M(x) = \sum_{k=0}^{\infty}(-1)^{k}\frac{x^{2k+1}}{2k+1}$$

By analysing the first few Taylor polynomials of $f(x)$, it can be induced that $M(x)$ exhibits similar behaviour and only converges towards $y = f(x)$ within the domain $\{-1 \leq x \leq 1\}$, and diverges outside of it. This can be tested and shown using the ratio test which is discussed in Part C of this investigation.

The absolute ratio between each consecutive term, that is the ratio between the $(k+1)^{th}$ and k^{th} term, is given by:

$$Ratio = \left|(-1)^{k+1}\frac{x^{2k+3}}{2k+3}\right| : \left|(-1)^{k}\frac{x^{2k+1}}{2k+1}\right|$$

$$Ratio = \frac{x^{2k+3}}{2k+3} \times \frac{2k+1}{x^{2k+1}}$$

$$\therefore Ratio = \frac{x^2(2k+1)}{2k+3}$$

The ratio between each consecutive term as more terms are added to the Maclaurin series can be obtained by taking the limit of the ratio as k approaches infinity:

$$\lim_{k\to\infty}\left|\frac{x^2(2k+1)}{2k+3}\right| = \left|x^2 \lim_{k\to\infty}\frac{2k+1}{2k+3}\right|$$

$$= |x^2|\left|\lim_{k\to\infty}\frac{2+\frac{1}{k}}{2+\frac{3}{k}}\right|$$

$$\therefore \lim_{k\to\infty}\left|\frac{x^2(2k+1)}{2k+3}\right| = |x^2|$$

This concludes that the ratio between each consecutive term in $M(x)$ as infinitely many terms are added to it is given by $|x^2|$. For the series to be convergent, the ratio must be less than one. This ensures that each next term is smaller than the previous one, which makes the series finite and convergent. Hence, $y = M(x)$ is convergent when $|x^2| < 1$.

$\therefore x^2 < 1 \; and \; x^2 > -1$	For $x^2 < 1$: $\therefore \{x: -1 < x < 1\}$	For $x^2 > -1$: $\therefore \{x: x \in \mathbb{R}\}$	Hence, the intersection of both solutions is: $\{x: -1 < x < 1\}$

The endpoints of this interval, that is $x = \pm 1$, should be checked to determine whether $M(x)$ converges for $x = \pm 1$. From the table of absolute percentage errors mentioned previously in this section of the investigation, the absolute percentage error between the $y = T_n(1)$ and $y = f(1)$ decreased as n increased. This means that the higher-degree Taylor polynomials converged towards $y = f(x)$ at $x = 1$ more than the lower-degree Taylor polynomials. Therefore, it can be concluded that the Taylor polynomial centred at $x = 0$, with infinitely many terms, or the Maclaurin series of $f(x)$ converges at $x = 1$. Since $M(x)$ is an even function, it also converges for $x = -1$.

Therefore, the interval of convergence of $y = M(x)$ is the interval $\{-1 \leq x \leq 1\}$. From this, it can be concluded that the Maclaurin series of $\arctan x$ is equal to it in the domain $\{-1 \leq x \leq 1\}$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \{-1 \leq x \leq 1\}$$

The Maclaurin series and Taylor polynomials of $f(x) = \arctan x$ have various applications, such as providing a power series for the irrational number π. This is useful in computer science where scientists attempt to find π with high precision.

Since $\tan\left(\frac{\pi}{4}\right) = 1$, it can be said that $\frac{\pi}{4} = \arctan 1$. Using the Maclaurin series:

$$\arctan 1 = M(1) = \frac{\pi}{4}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

$$\therefore \pi = 4 \times \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

This is also known as the Leibniz formula for π.

The Taylor polynomials of $f(x)$ are also vastly useful in calculating the integrals of $y = \arctan x$. This is because the integral of x^n can easily be solved using the integration rule $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$, meaning the terms in the Taylor polynomial can be easily integrated, while integrating $y = \arctan x$ would require much more working out and computation.

Limitations and Assumptions:

There were several assumptions that were made in this investigation. It was assumed that the graphing software, FX Graph, was accurate, and that the intersections between curves and the values of the curve were precise. In Part A and Part B, conclusions and observations were obtained from data shown in the graphs. If the graphs were not precise, the conclusions drawn may be inaccurate. Table 1.1, 1.4 were used to determine the percentage error between the Taylor polynomials, and those results were used to make conclusions on the trend of the accuracy of the Taylor polynomials as more terms were added, while the observations made from Figure 1,2,3,4 were used directly to create a conclusion on the effect of increasing the number of terms in the Taylor polynomials on the accuracy of the approximation. Furthermore, the percentage errors calculated in the investigation were rounded to 3 significant figures. Although this still maintains sufficient precision, the error is not exact, and small differences in the calculations may have been lost due to the rounding. In Part C, the interval of convergence of the functions were originally determined by recognising a trend in the graphs depicted in Figures 5 and 6. However, it was realised that the trends may not continue for higher-degree Taylor polynomials, and it was computationally infeasible to test it, since the graphing software would begin severely lagging and glitching for large functions. Therefore, it was decided to use the ratio test to form a definitive, unambiguous value for the interval of convergence. Part C also introduced a conjecture for the n^{th} derivative of the function $f(x) = \ln(x+1)$. This conjecture had to be validated using Mathematical Induction to ensure that the Taylor polynomials and Maclaurin series derived were correct and true. The limitation of rounding to 3 significant figures was encountered again in Part D, where the percentage error was analysed. Since some percentage errors were equal when rounded as shown in table 4.1, some precision was lost, which caused some confusion. The biggest limitation is that Taylor polynomials cannot give a perfect approximation of a function, since it is computationally impossible to compute infinitely many terms of a Maclaurin series.

Conclusion:

This investigation has successfully explored Taylor polynomials, Maclaurin series, and the interval of convergence of various Maclaurin series. Part A has delved into the derivation of the best quadratic and linear approximations of $f(x) = \cos x$ and $h(x) = \sqrt{x+3}$, and has investigated the behaviour of quadratic and linear approximations. It was concluded that the quadratic approximations provided a better representation of the actual function for a larger domain. Part B has expanded on this concept and explored the general form of a Taylor polynomial centred at $x = a$ of any function, where it was concluded that the general form of the Taylor polynomial $T_n(x)$ is $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$. This enabled the investigation of Taylor series that extend to infinitely many terms, otherwise known as Maclaurin series. Part C focused on the Maclaurin series of $f(x) = e^x$ and the series of $h(x) = \ln(x+1)$. Patterns in the derivatives of $f(x) = e^x$ and $h(x) = \ln(x+1)$ were discussed, which allowed for the formation of an expression for a general term in the Taylor polynomial of $f(x) = e^x$ and $h(x) = \ln(x+1)$ centred at $x = 0$, which aided in finding the Maclaurin series of both functions. These series were further investigated on their convergence, and the domain for which the Maclaurin series converged, also known as the series' interval of convergence, were explored, and determined using the ratio test. It was concluded that the Maclaurin series of $f(x) = e^x$ converged for every value of x, while the Maclaurin series of $h(x) = \ln(x+1)$ only converged in the domain $\{-1 \leq x \leq 1\}$. These findings were applied in Part D, which focused on the function $f(x) = \arctan x$. Its first several Taylor polynomials centred at $x = 0$ were found, and a pattern in the terms was obtained. This was used to form an expression for the n^{th} degree Taylor polynomial and was extended to an expression for the Maclaurin series. The interval of convergence of the Maclaurin series was determined using the ratio test, where it was concluded that the series converged in the domain $\{-1 \leq x \leq 1\}$.

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