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Functional Analysis
Practice Pset 1

Problem 1 For any point v in the range of P (i.e. S), the only point u satisfying $v-u \perp S$ is v itself as $v-u \in S$. Thus $P^2 = P$.

Let e_i be an orthogonal basis of S and f_i an orthogonal basis of $S^{\perp}$. Then any $v = \sum_i a_i e_i + \sum_i b_i f_i$, and $Pv = \sum_i a_i e_i$. Then if $u = \sum_i c_i e_i + \sum_i d_i f_i$, $(Pv, u) = \sum_i a_i c_i = (v, Pu)$, i.e. $P = P^*$.

Problem 2 Consider $(x - Px, Py)$:

$$(x - Px, Py) = (x - Px, P^2 y) = (P(x - Px), Py) = 0$$

It remains to be shown that the range of P is a closed subspace. This follows from the continuity of the function which follows from its boundedness.

Problem 3 Given that A_n is a finite rank operator, for the "if" statement it suffices to show that $\|A_n - A\| \to 0$, and we do this by showing $\|(A_n - A)v\| \to 0$ for all v. $\|(A_n - A)v\| \leq \alpha^{(n)}\|v\|$, where $\alpha^{(n)}$ is the supremum of α_i with $i \geq n$. Clearly as $|\alpha_n\| \to 0$, this quantity goes to zero too.

For the converse, suppose there always exists, for every n, some α_i greater than some epsilon. Then $\{Ae_i\}$ is not precompact.

Problem 4

$$\|Tf\| = \left(\int \left| \int K(x,y) f(y) dy \right|^2 dx \right)^{1/2} \leq \left(\int |Af(x)|^2 dx \right)^{1/2} \leq A\|f\|.$$

Problem 5 If M and N are the norms of T_1 and T_2, then $(T_1 + T_2)v = T_1 v + T_2 v \leq (M + N)\|v\|$, which means the norm of the sum must be less than or equal to $M + N$.

Problem 6 Let $\tilde{A}(v) = Av$ for $v \in \mathcal{H}_0$, and $\tilde{A} \lim v_n = \lim Av_n$ for $v_n \in \mathcal{H}_0$. To show that such a limit exists, consider $\|Av_n - Av_m\| = \|A(v_n - v_m)\|$. Because A is bounded, this is less than some $M\|v_n - v_m\| = M\varepsilon_n$, i.e. Av_n is Cauchy. Boundedness follows from the continuity of the norm: $\|Av\| = \|\lim Av_n\| = \lim \|Av_n\|$, and $\|Av_n\| \leq M\|v_n\|\ \forall v_n$.