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Functional Analysis
Problem Set 7

Problem 1 Right shift operator: firstly, for all λ, $R - \lambda I$ has a trivial kernel. Proof: if λ is zero, then it has a trivial kernel. For $\lambda \neq 0$, let v be in the kernel of $R - \lambda I$. Then as $Rv = \lambda v$, and as $(Rv)_1 = 0$ by the nature of the right-shift operation, $v_1 = 0$. Additionally, if $v_n = 0$, $(Rv)_{n+1} = \lambda v_n = 0$, and hence $v_{n+1} = 0$. By induction, $v_n = 0$ for all n. This shows that there are no eigenvectors for any value of λ, and thus $\sigma_p(R) = \emptyset$, and it also means we've shown injectivity for all values of λ (for the subsequent spectra).

$s_c(R) \supset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$: first I show the range of $R - \lambda I$ is dense. Consider some $v = \sum a_i e_i \in X$. Then let w_n have the first element $-a_i/\lambda$, then subsequent terms $-((w_n)_{k-1} - a_k)/\lambda$ until the nth element. $R - \lambda I$ applied to w_n thus agrees with v on the first n elements. Further:

$$\|(R - \lambda I)w_n - v\|^2 \leq \sum_{k=n}^{\infty} \|a_k\|^2$$

which goes to zero as $n \to \infty$ as v is square summable, thus $(R - \lambda I)w_n \to v$.

Problem 2 I show no λ can exist where $A = (T - \lambda I)$ has a trivial kernel and a range that is not dense.

If A's kernel is trivial, $\lambda \notin \{\lambda_i\}$, else e_i is in the kernel of A. In this case, note further that $A(e_i/(\lambda_i - \lambda)) = e_i$, so each e_i is in the range, and finite linear combinations of $e_i/(\lambda_i - \lambda)$ are members of $\ell^2(\mathbb{N})$, thus because finite linear combinations of e_i are dense, the range is dense. Thus no lambda exists in $\sigma_r(T)$.

I now show $\lambda \in \overline{\{\lambda_i\}} \setminus \{\lambda_i\} \in \sigma_c(T)$. Let λ be a member of the set. Because λ isn't an eigenvalue, by the same argument as in the previous paragraph $A = T - \lambda I$ is injective and the range is dense. To show that the range isn't closed, let $\lambda_n \to \lambda$, with $|\lambda_n - \lambda| < 1/n^2$. Then let v be the vector with $1/n$ in the component corresponding to λ_n in T and 0 otherwise. The formal preimage of v is the sequence with $1/(n\lambda_n - n\lambda)$ in the positions corresponding to each $\lambda_n \in T$, but as $|\lambda_n - \lambda| = 1/n^2$, the preimage of v isn't in $\ell^2(\mathbb{N})$ as it isn't square-summable. Thus $\lambda \in \sigma_c(T)$.

Problem 3 All such operators map $\ell^2(\mathbb{N})$ to itself. a) 0 is a compact operator, and $0 - 0I$ has a nontrivial kernel so 0 is in the point spectrum. b) $T = T - 0I$ mapping $e_i \to e_i/k$ has all coordinate vectors in its range and thus has a dense range, but $(1, 1/2, 1/3, \ldots)$ has the formal preimage $(1, 1, 1, \ldots)$, and thus the range isn't closed. It is compact because it is the norm limit of finite rank operators sending $e_k \to e_k/k$ for $k < n$. c) $T = T - 0I$ sending $e_k \to e_{2k}/k$ is compact by the same argument as the previous, has no kernel, but also has a range that isn't dense as, for example, e_1 cannot be approximated by anything in its range.

Problem 4 Let $\lambda \in \sigma_p(T)$ but $\notin \sigma_p(T^*)$, specifically, let $Tv = \lambda v$. Then

$$T^* x^*(v) = x^*(Tv) = \lambda x^*(v).$$

I now show that the range of $T - \lambda I$ cannot be dense. Let v^* be the vector of norm 1 such that $v^*(v) = \|v\|$, which is possible by Hahn-Banach. Suppose the range is dense, and let x_n^* be a sequence of vectors such that $(T - \lambda I)x_n^* \to v^*$. Consider then

$$(T - \lambda I)x_n^*(v) = Tx_n^*(v) - \lambda x_n^*(v) = \lambda x_n^*(v) - \lambda x_n^*(v) = 0,$$

which means x_n^* doesn't converge weak*ly to v^*, and hence also not in the norm topology, a contradiction. Thus $\lambda \in \sigma_r(T^*)$.