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Functional Analysis
Problem Set 5

Problem 1

a) Consider $A = f^{-1}(U) \in \mathcal{X}^2$, where f is the addition function. Because $\mathcal{X}$ is a TVS, A is open in the product topology. Further, because 0 is in U, 0 (or really, (0,0)) is in A, and thus some neighborhood containing 0, say $X \times Y$, is in A, where X and Y are open sets of $\mathcal{X}$. Thus $V = X \cap Y$ is open, is a member of $\mathcal{U}$, $V \times V \subset X \times Y \subset A$, and $V + V \subset U$.

b) Because of the joint continuity of scalar multiplication, there exists in the preimage of U under scalar multiplication a neighborhood of 0, $[-c, c] \times W$, where W is an open set. Let $V = W/c$, hence $\alpha V \subset U$ for all $|\alpha| \leq 1$. Then define

$$\mathcal{V} = \bigcup_{|\alpha|\leq 1} \alpha V$$

For any vector $v \in \mathcal{V}$ and $|\beta| \leq 1$, $\beta v \in \mathcal{V}$, as $v \in \alpha V$ for some α, and then $|\alpha\beta| \leq 1$ so $\beta v \in \alpha\beta V \subset \mathcal{V}$. Our set thus is open (union of open sets), contains 0, and is balanced.

Problem 2 Continuity of addition: let U be an open set. I show that for every $x_0 + y_0 \in U$, there is some open neighborhood around (x_0, y_0) that maps into U.

Because U is open, it contains some neighborhood around $x_0 + y_0$:

$$V_{x_0+y_0} = \bigcap_{j=1}^{n} \{x \in \mathcal{X} \mid p_j(x - x_0 - y_0) < \varepsilon_j\}.$$

Consider the neighborhood around (x_0, y_0), $V_{x_0} \times V_{y_0}$, where V is defined as it is above for $x_0 + y_0$, except with $\varepsilon_j/2$ instead of ε. For any (x, y) in our neighborhood:

$$p_j(x + y - x_0 - y_0) \leq p_j(x - x_0) + p_j(y - y_0) < \varepsilon_j,$$

meaning $V_{x_0} \times V_{y_0}$ maps into the neighborhood around (x_0, y_0), and thus into U. Thus addition is continuous.

Continuity of scalar multiplication. Again let U be an open set; I show every (α, x_0) with $\alpha x_0 \in U$ has a neighborhood also mapping into U.

Let $V_{\alpha x_0}$ be an open neighborhood contained by U:

$$V_{\alpha x_0} = \bigcap_{j=1}^{n} \{x \in \mathcal{X} \mid p_j(x - \alpha x_0) < \varepsilon_j\}.$$

Then let us construct a neighborhood of (α, x_0). Define

$$W_{x_0} = \bigcap_{j=1}^{n} \{x \mid p_j(x - x_0) < \varepsilon_j/|2\alpha|\}, \; c = \inf_{x \in W_{x_0},\, j=1,\ldots,n} \frac{\varepsilon_j}{2|p_j(x)|},$$

and finally our neighborhood $V_{(\alpha,x_0)}$:

$$V_{(\alpha,x_0)} = (\alpha - c, \alpha + c) \times W_{x_0}.$$

Note that c is nonzero because n is finite, $|p_j(x)|$ is bounded by $|p(x_0)| + \varepsilon/2\alpha$. Given some (β, x) in our neighborhood,

$$\begin{aligned} p_j(\alpha x_0 - \beta x) &= p_j(\alpha(x_0 - x) + (\alpha - \beta)x) \\ &\leq |\alpha| p_j(x - x_0) + |\alpha - \beta| p_j(x) \\ &< \varepsilon_j. \end{aligned}$$

Thus scalar multiplication maps our neighborhood into $V_{\alpha x_0} \subset U$.

Problem 3 Let U be an open subset of $\mathcal{X}$ and $x_0 + y_0 \in U$. By the continuity of addition, there exists some neighborhood $V_{x_0} \times V_{y_0}$ such that $(x, y) \in V_{x_0} \times V_{y_0}$ satisfies $x + y \in U$. By the definition of neighborhoods in a product space, V_{y_0} is a open neighborhood of y_0 in $\mathcal{X}$. Thus $V_{y_0} + x_0 \in U$ implies our function is continuous. The continuity of the inverse is established by repeating the argument with $-x_0$ instead of x_0.

Let U be an open set in $\mathcal{X}$ and let $\alpha_0 x_0 \in U$. By continuity, there is some open neighborhood $V_{\alpha_0} \times V_{x_0}$ such that (α, x) in it satisfies $\alpha x \in U$. As above, V_{x_0} is open in $\mathcal{X}$ and satisfies $\alpha_0 V_{x_0} \in U$, thus establishing continuity. The continuity of its inverse follows with the same argument, replacing $1/\alpha$ for α.

Problem 4 We've shown in class that x^* is continuous in the weak topology. It remains to be shown that weaker topologies are insufficient.

Let τ be a strict subset of wk, i.e there is some open set in wk that is not in τ. This implies there's some x in the set with a neighborhood in wk but not τ, i.e.

$$V_{x_0} = \bigcap_{i=0}^{n} \{x \in \mathcal{X} \mid |\ell_i(x - x_0)| < \varepsilon_i\} \notin \tau$$

This in turn implies for one of the ℓ_i, its ε_i cylinder is not an open set of τ, (otherwise V_{x_0}, a finite intersection of all such cylinders, would too be open). The ε_i cylinder can be rewritten as $\ell_i^{-1}((\ell(x_0) - \varepsilon_i, \ell(x_0) + \varepsilon_i))$, i.e. $\ell_i \in \mathcal{X}^*$ is not continuous in τ as our interval is an open set in our field.

Problem 5 We've shown in class that $(\mathcal{X}^*, wk^*)^* = \mathcal{X}$. Let τ be a yet weaker topology; thus there is an open set and within it a neighborhood of some ℓ_0^* that is not in τ, i.e.

$$V_{\ell_0^*} = \bigcap_{i=0}^{n} \{\ell \in \mathcal{X}^* \mid |(\ell - \ell^*)(x_i)| < \varepsilon_i\} \notin \tau.$$

This means that some ε_i cylinder of some ℓ_i isn't an open set in τ. Note that:

$$\{\ell \in \mathcal{X}^* \mid |(\ell - \ell^*)(x_i)| < \varepsilon_i\} = x_i^{-1}((\ell^*(x_i) - \varepsilon_i, \ell^*(x_i) + \varepsilon_i)) \notin \tau.$$

Thus x_i isn't continuous in τ.

Problem 6 With some algebra:

$$\begin{aligned}
\|h_n - h\|^2 &= \sum_i ((h_n)_i - h_i)^2 \\
&= \sum_i (h_n)_i^2 - 2\sum_i (h_n)_i h_i + \sum_i h_i^2 \\
&= \sum_i (h_n)_i^2 - \sum_i h_i^2 + 2\sum_i h_i^2 - 2\sum_i (h_n)_i h_i \\
&= (\|h_n\|^2 - \|h\|^2) + 2((h,h) - (h_n, h)).
\end{aligned}$$

The first converges to zero by the convergence *of* norms, and the second by weak convergence.

Problem 7 Let $\bar{S}$ denote the weak closure of S. Let $R = \{x : \|x\| \leq 1\}$.

$\bar{S} \subset R$: to show that every element in the closure of S must have norm less than one, suppose it isn't true: there is some $x_i \to x$ where x_i are norm 1 but $\|x\| > 1$. By Hahn-Banach, however, there is some linear functional ℓ for which for all $y \in \overline{B_1(0)}$, $|\ell(y) - \ell(x)| > \varepsilon$. Note that x_i in particular are members of $\overline{B_1(0)}$, thus they never enter the ε neighborhood of x induced by ℓ, contradicting our assumption that $x_i \to x$ weakly.

$R \subset \bar{S}$: I first prove a lemma: any weak neighborhood in an infinite dimensional vector space has elements of any norm greater than that of the point on which it is defined.

Proof: let $\{\ell_i\}_{i=1}^n = A \in X^*$ define the neighborhood. First we show the intersection of their kernels is nontrivial. This is because X can be written, for any ℓ_i as $M_i \oplus N_i$, where M_i is the infinite dimensional kernel and N a one-dimensional subspace isomorphic to X/M_i. Then given some ℓ_j, $M_i \cap M_j$ is at most $|M_i| - |N_j|$ dimensional, which is still infinite. Thus the finite intersection of all M_i must be infinite dimensional and nontrivial.

Then let $y \neq 0$ be in the intersection of kernels of $\{\ell_i\}$. $x + \alpha y$ is in our neighborhood, and because the map $f : F \to F$ taking α to $\|x + \alpha y\|$ is continuous, and f can be arbitrarily large by making α arbitrarily large, by the intermediate value theorem we can find some $x + \alpha y$ of any norm larger than $\|x\|$ inside the neighborhood of x.

From this lemma, it follows that every neighborhood of x_0 in $\bar{S}$ (i.e. with norm less than 1) contains a vector of norm 1. Thus x_0 is a limit point of S.

Problem 8 The lemma proved in the previous problem shows that any weak neighborhood has vectors of arbitrarily large norm, thus no nonempty bounded set can contain a neighborhood, let alone weak neighborhoods for all its elements. Thus no bounded set can be weakly open.