Lewis Ho
Functional Analysis
Practice Pset 3

Problem 1 x_0 is an extreme point of F, because there are no other points in F and so no points of which x_0 could be a convex combination of. Then recall the theorem that $\mathcal{E}(F) = F \cap \mathcal{E}A$ if F is an extreme set of A, closed, convex and compact. Thus $x_0 \in \mathcal{E}(F) \subseteq \mathcal{E}(A)$.

Problem 2 The open mapping theorem implies $T(B_1(0)$ contains a ball of radius ε. Thus $T^{-1}(x) \in B_1(0)$ for $x \in B_\varepsilon(0)$, and $\|T\| \leq 1/\varepsilon$.

Problem 3 Suppose T is an open map. $B_1(0)$ being an open set, $T(B_1(0)$ must be open as well. $T(0) = 0 \in T(B_1(0)$, thus there must be some $B_\varepsilon(0) \subseteq T(B_1(0))$. Conversely, let A be an open set. For any x in A, there is some $B_\varepsilon(x) \subseteq A$. If $T(B_1(0))$ contains an open ball of radius δ,

$$T(B_\varepsilon(x)) = x + \varepsilon T(B_1(0))$$

contains a ball of radius $\epsilon\delta$, which is a subset of $T(B_\varepsilon(x)) \subseteq T(A)$, and thus T is an open map.

Problem 4 We can see that F is nonempty because if we construct a sequence x_n where $\ell(x_n) - \sup \ell(y) \leq 1/n$, by compactness we have a convergent subsequence whose limit, x, must satisfy $\ell(x) = \sup \ell(y)$. And because B is closed, $x \in B$ and F, so F is nonempty.

Closedness follows from the continuity of ℓ and the closedness of B— specifically, if $a_n \in F$, then their limit a must satisfy $\ell(a) = \sup \ell(x)$ by continuity, and must be in B. Convexity follows from the linearity of ℓ.

Extremity: suppose for some $x \in F$ there exists a, b in A such that x is a convex combination of a and b. Then a and b must be in F because if, say, a wasn't, then $\ell(b)$ must be greater than $\ell(x)$, and that's not possible as x is the supremum of ℓ over B, and a and b must be in B because B is an extreme subset.

Problem 5 We show that f in the ball can always be written as a convex combination of two other functions in the ball. If $f = 0$, then any g and $-g$ in the ball will suffice. If $f \neq 0$, then f's support has nonzero measure. We can split that support in two and have $g = f$ on one part of that and zero elsewhere and h defined likewise for the other half. Then $g + h = f$, and clearly both must have norm less than one.

Problem 6 Our distance function satisfies non-negativity, identity of indiscernibles, symmetry, and subadditivity, so it defines a metric space.

Consider the set defined by $a_i \leq 1/i$. Compactness follows from the fact that this is the range of the closure of finite-rank operators. On the other hand, $\sum$