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Functional Analysis
Problem Set 6
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Problem 1 Norm closed $\Rightarrow$ weakly closed: I show the complement of a norm-closed ball is weakly open. Let x not be in the norm-closed ball, i.e have norm greater than 1. By Hahn Banach, there exists some ℓ with norm 1 such that $\ell(x) = \|x\|$. Let $\varepsilon = (\|x\| - 1)/2$. Let y be a vector inside the ℓ, ε neighborhood of x. Then

$$\varepsilon > |\ell(y - x)| = |\ell(y) - \|x\||.$$

Because $\ell(y)$ is bounded by $\|y\|$ from above, this means $|\|y\| - \|x\|| < \varepsilon$, i.e. $\|y\| > 1$. Thus every element outside the norm-closed ball is contained in a weak neighborhood also outside the ball, meaning that the norm-closed ball is also weakly closed.

Weakly closed $\Rightarrow$ norm closed: let x_n with norm ≤ 1 be Cauchy. For all $\ell \in \mathcal{X}^*$ and $\varepsilon > 0$, there exists some N for which $m, n > N$ implies $\|x_n - x_m\| < \varepsilon/\|\ell\|$. Thus $\|\ell(x_n - x_m)\| < \varepsilon$, x_n is weakly Cauchy, and converges to some x in the weak closure of the ball. Suppose x is not the norm limit of the sequence: then there is some closed ε ball containing all x_m for $m > N$ large enough that x isn't in. But then by Hahn Banach, some linear functional strictly separates x from all x_m with $m > N$, so x cannot be the weak limit of x_n, a contradiction. Thus if the unit ball is weakly closed, it is norm closed.

Problem 2 Suppose not. Let b_{n_k} be a pointwise convergent subsequence. Then let $x \in \ell^\infty$ be defined as alternating 1 and -1 for each n_kth element and 0 otherwise. Then $b_{n_k}(x)$ is not Cauchy, a contradiction.

Problem 3 Lower semicontinuity of $\|\cdot\|_X$ in the weak topology: given some $x_0 \in X$ and $\varepsilon > 0$, by Hahn-Banach, there exists some $\ell \in X^*$ such that $\ell(x_0) = \|x_0\|$. Consider V, the ε neighborhood of x_0 defined by ℓ. For all $x \in V$,

$$\varepsilon > |\ell(x - x_0)| = |\ell(x) - \|x_0\||.$$

$\ell(x)$ is bounded from above by $\|x\|$, so $\|x\|$ is at most ε less than $\|x_0\|$.

Lower semicontinuity of $\|\cdot\|_{X^*}$ in the weak* topology: by the above argument, $\|\cdot\|_{X^*}$ is lower semicontinuous in the weak topology. Because X is reflexive, $(X^*, wk^*) = (X^*, wk)$, and in particular, every weak neighborhood of X^* is a weak* neighborhood; thus $\|\cdot\|_{X^*}$ is lower semicontinuous in the weak* topology as well.

Minima for lower semicontinuous functions on compact sets: suppose the statement is false, i.e. some lower semicontinuous f gets arbitrarily large (negatively) on a weakly compact set (note: both norms are weakly lower semicontinous). Then let x_n be a sequence of points with $f(x_n) < -n$. Because weakly compact sets are weakly sequentially compact, there exists some weakly convergent subsequence $x_{n_k} \to x$. This means there exists arbitrarily negative values of f in every neighborhood of x, and thus f can't both be well defined and lower semicontinuous, a contradiction.

Proof of statement: let d be the distance between x_0 and M, the subspace it is not a member of. Let B be the closed $2d$ ball around x_0, and consider $B \cap M$. Because both are closed and B is weakly compact by the reflexivity of X, the lowersemicontinuous function $f(y) = \|x_0 - y\|$ attains its minimum in $B \cap M$ at some y_0 (closed subsets of compact sets are compact). Note then that:

$$f(y_0) = \min_{y \in B \cap M} \|x_0 - y\| = \min_{y \in M} \|x_0 - y\| = \inf_{y \in M} \|x_0 - y\|,$$

as no points of M outside B can be closer to x_0 than the ones within.

Problem 4 Weak convergence $\Rightarrow$ norm bounded, pointwise convergent: for pointwise convergence, note that evaluation at some point x defines a bounded linear functional. Thus if f_n converge for all linear functionals they must converge pointwise. Additionally, as all vectors $f \in C[0,1]$ can be identified with elements in the double dual $\hat{f}$, if f_n converges weakly, $\hat{f}_n$ converges weak*ly, and by the uniform boundedness condition must be norm bounded in the double dual and hence the space itself.

Norm bounded, pointwise convergence $\Rightarrow$ weak convergence: let f_n be a norm bounded, pointwise converging sequence in $C[0,1]$, and $\mu \in BV[0,1]$, the dual of $C[0,1]$. Because f_n are collectively norm bounded, we can apply the dominated convergence theorem as follows:

$$\lim_{n\to\infty} \mu(f_n) = \lim_{n\to\infty} \int_0^1 f_n d\mu = \int_0^1 \lim_{n\to\infty} f_n d\mu = \int_0^1 f d\mu = \mu(f).$$

Unfortunately I don't know enough measure theory to come up with an example.