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Functional Analysis
Problem Set 4

Problem 1 $\overline{A(B_1(0))}$ must contain a ball of radius ε_0, else the range of A will be $\bigcap_{n=1}^{\infty} A(B_n(0)) = \bigcap_{n=1}^{\infty} nA(B_1(0))$ which will then be meagre. Because $\overline{A(B_1(0))}$ is symmetric and convex, it will in particular contain a ball $B_{\varepsilon_0/2}(0)$, as if the original ball is centered around some y_0, then $(-y_0 + y_0 + B_{\varepsilon_0}(0))/2$ will be a set of convex combinations of points we know to be in the closure.

Then we show $\overline{A(B_1(0))} \subset A(B_2(0))$, following the proof of the open mapping thoerem. Suppose $y \in \overline{A(B_1(0))}$, then $\exists x_1 \in B_1(0)$ such that $\|y - Ax_1\| < \varepsilon_0/4$, i.e. $y - Ax_1 \in B_{\varepsilon_0/4} \in \overline{A(B_{1/2}(0)}$. Because it's contained in this ball, there exists some $x_2 \in B_{1/2}(0)$ such that $\|y - Ax_1 - Ax_2\| < \varepsilon_0/8$, i.e. $y - Ax_1 - Ax_2 \in B_{\varepsilon_0/8}(0) \subset \overline{A(B_{1/4})}$. Generally, there exists $x_n \in B_{1/2^{n-1}}(0)$ such that $\|y - \sum_{i=1}^{n} Ax_i\| < \varepsilon_0/2^{n+1}$.

We know that in Banach spaces if $\sum \|x_n\| < \infty$ then $\sum x_n$ converges, and $\sum \|x_n\|$ in this case $\leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \leq 2$. Thus the limit x of this sum exists, and by the triangle inequality, is in $B_2(0)$, and finally satisfies $Ax = y$. Hence $y \in A(B_2(0)$. From this we conclude (by scaling) that $A(B_1(0))$ contains a ball of $\varepsilon_0/4$ radius, and thus by multiplying $B_1(0)$ by arbitrarily large numbers, can conclude that A is surjective and thus has a closed range.

Problem 2 The statement is false. We can construct arbitrarily tall isosceles triangles of ε area. Center those triangles at 0.5, and define functions f_h to be the functions that are 0 outside the base of the triangle of height h, and take the value of the diagonals up and down inside. $\|f_h\|$ is unbounded, but there is always a function F_h whose supremum (and thus norm) is ε and has derivative f_h, namely the antiderivative of f_h.

Problem 3 If there exists some c such that $\|Ax\| \geq c\|x\|$ for all x, then $\mathcal{N}(A)$ must empty, otherwise arbitrarily large vectors in the null space can have images with norm zero, a contradiction. Closedness: let Af_n be a Cauchy sequence in the range. For any $\|Af_n - Af_m\| < \varepsilon$, by the assumption,

$$\|Af_n - Af_m\| = \|A(f_n - f_m)\| \geq c\|f_m - f_n\|,$$

in other words, f_n must also be Cauchy, converging to some f in the domain. Then by continuity, $Af = A\lim_{n\to\infty} f_n = \lim_{n\to\infty} Af_n$, i.e. the range is closed.

Conversely, if both $\mathcal{N}(A) = \{0\}$ and the range is closed, then if there were to not exist a c such that $\|Ax\| \geq c\|x\|$, we could construct a sequence $Af_n \to 0$, where $\|f_n\| = 1$. Because the range is closed, there is some f with norm 1 such that $Af = 0$, violating our $\mathcal{N}(A) = \{0\}$ assumption.

Problem 4 Let us define (x, y) to be $\sum_i x_i y_i$. Lemma: (almost Hölder's inequality) $|(x, y)| \leq \|x\|_p \|y\|_q$ if $x \in \ell^p$, $y \in \ell^q$ and $1/p + 1/q = 1$. Proof: if either x or $y = 0$, then then the lemma holds. If not, define:

$$a_k = \frac{x_k}{\|x\|_p}, \ b_k = \frac{y_k}{\|y\|_q}.$$

By Young's inequality, $\sum_i |a_i b_i| \leq \sum_i (|a_k|^p/p + |b_k|^q/q)$, and $\sum_i |a_i| = \sum_i |x_i|^p / \|x\|^p = 1$, and likewise for w. Thus $\sum_i (|a_k|^p/p + |b_k|^q/q) = 1 \geq \sum_i |x_i y_i|$. Multiplying both sides by $\|x\|$ and $\|y\|$ we get:

$$\|x\|_p \|y\|_q \geq \sum_i |x_i y_i| \geq |\sum_i x_i y_i| = |(x, y)|.$$

Suppose our assumptions hold: $\sup_n \|x_n\|_p < \infty$ and $x_n^j \to 0$. I will show that we can make (x_n, y) arbitrarily small. Firstly, note that we can bound (x_n, y) by decomposing it and applying "Hölder's" as follows:

$$|(x_n, y)| = |\sum_{|y^j| > \delta} x_n^j y^j + \sum_{|y^j| \leq \delta} x_n^j y^j| \leq \|y\|_q \left(\sum_{|y^j| > \delta} |x_n^j|^p \right)^{\frac{1}{p}} + \left(\sum_{|y^j < \delta|} |y^j|^q \right)^{\frac{1}{q}} \sup_k \|x_k\|_p$$

Note that we can always find an N such that for all $n > N$ both parts are arbitrarily small. First we can fix δ to make the latter term small, e.g. because $\|y\|_q$ is finite, we can look at the first N terms such that their q sum is arbitarily close to $\|y\|_q^q$, and then set δ to be their infimum. With this fixed δ, there are now finite elements in the sum of x_n^js in the first term, and thus we can pick some N such that for $n > N$ all x_n^j are arbitrarily small, and thus their finite p sum is also arbitrarily small.

For the converse, note that x_n define bounded linear operators from ℓ^q to $\mathbb{R}$. If $(x_n, y) \to 0$ for all y, then the set $E = \{y \in \ell^q \mid \sup_n \|(x_n, y)\| < \infty\} = \ell^q$, and by the uniform boundedness principle $\sup_n \|x_n\|_p$ must be bounded. Finally, if $x_n^j \to 0$ is violated, the sequence with jth element $= 1$ has image not converging to 0.

Problem 5 In the previous problem set we showed that ${c_0}^* = \ell^1$. To show $(x_n, y) \to 0$, we decompose (x_n, y) again as follows:

$$|(x_n, y)| = |\sum_{|y^j| > \delta} x_n^j y^j + \sum_{|y^j| \leq \delta} x_n^j y^j| \leq \|y\|_\infty \sum_{|y^j| > \delta} |x_n^j| + \delta \sup_n \|x_n\|_1$$

As previously, we can choose δ such that the second term becomes arbitrarily small, and then given this delta, choose n such that the first term becomes arbitrarily small.

To show the converse, note that x_n define linear operators from $c_0 \to \mathbb{R}$. If $(x_n, y) \to 0$ for all y, then the set E for $\{x_n\}$ is all of c_0, and thus $sup_n \|x_n\|$ is finite. Additionally if we have some x_n^j not converging to 0, then e_j has image not converging to zero.

Problem 6 Let $h = \sum_i a^i e_i$, and $h_n = \sum_i h_n^i e_i$. If our assumptions are satisfied, we can write $|(h_n, h)|$ as

$$| \sum_{|a^i|>\delta} a^i h_n^i + \sum_{|a^i|>\delta} a^i h_n^i| \leq \|h\| \left(\sum_{|a^i|>\delta} |h_n^i|^2 \right)^{\frac{1}{2}} + \left(\sum_{|a^i|<\delta} |a^i|^2 \right)^{\frac{1}{2}} \sup_k \|h_k\|$$

using the Cauchy-Schwarz inequality to bound it. Then as before, the latter term is made arbitrarily small with our choice of δ, and the former with a sufficiently large choice of N.

For the converse, we use the uniform boundedness principle, as all h_n define operators $\mathcal{H} \to \mathbb{R}$, and E is all of space for $\{h_n\}$. Likewise if some b_n^j doesn't go to zero, (e_j, h) won't converge to zero either.

Problem 7 Proof by contradiction: suppose this statement is false, i.e. there exists some ε_0 such that for all N there exists some $n > N$ such that $\sup_f \|A_n f - Af\| > \varepsilon_0$. From this we can construct a sequence $\{f_n\}$ where $\|A_n f_n - Af_n\| > \varepsilon_0$, and then find the limit f of a convergent subsequence $\{f_{n_k}\}$ using compactness.

Consider $\|A_n f - Af\|$. Using our subsequence $\{f_{n_k}\}$ and their accompanying A_{n_k}, we can write

$$\begin{aligned} \|A_{n_k} f - Af\| &= \|(A_{n_k} - A)(f_{n_k} + f - f_{n_k})\| \\ &= \|A_{n_k} f_{n_k} - Af_{n_k} + (A_{n_k} - A)(f - f_{n_k})\| \\ &\geq \varepsilon_0 - (\sup_n \|A_n\| + \|A\|)\|f - f_{n_k}\| \end{aligned}$$

and that we can choose large enough n_k such f and f_{n_k} are very close and this difference is bounded from below by, say, $\varepsilon_0/2$, meaning that it doesn't converge. Note that $\sup_n \|A_n\|$ and $\|A\|$ exist by the uniform boundedness principle.

Problem 8 Let $M = \sup_{A \in \mathcal{A}} \|A\|$, which exists because the image set of the unit ball is compact (and hence bounded), and thus by the uniform boundedness principle the supremum of their norms is finite. From the previous problem and both the pointwise convergence of L_n and the collective compactness of $\mathcal{A}$, we know that:

$$\begin{aligned} &\sup_{g \in \overline{\mathcal{A}(B_1(0))}} \|L_n g - Lg\| \to 0 \\ \Rightarrow &\sup_{\|f\| \leq 1,\ A \in \mathcal{A}} \|(L_n - L)Af\| \to 0 \end{aligned}$$

the first implying the second because:

$$\overline{\mathcal{A}(B_1(0))} \subseteq \mathcal{A}(B_1(0)) = \{Af \mid A \in \mathcal{A},\ f \in B_1(0)\}.$$

Now we invoke the definition of the norm of an operator T as $\sup \|Tf\|$ for $\|f\| \leq 1$ to infer

$$\sup_{\|f\| \leq 1,\ A \in \mathcal{A}} \|(L_n - L)Af\| = \sup_{A \in \mathcal{A}} (\sup_{\|f\| \leq 1} \|(L_n - L)Af\|) = \sup_{A \in \mathcal{A}} \|(L_n - L)A\| \to 0.$$

Problem 9 We first show that $I - A_n$ is injective (and hence bijective and invertible, by Fredholm theory) for sufficiently large n. Proof: suppose not, i.e. for each n there exists some f_n of norm 1 in the kernel of $I - A_n$. Let $M = \{f_n\}$. Further, f_n is in the kernel of $I - A_n$ if $If_n - Af_n = 0$, i.e. $Af_n = f_n$. This means $\mathcal{A}(M) \supseteq M$. Suppose then that $\mathcal{A}(M)$ is precompact, so some $f_{n_k} \to f$. Then

$$\|(I - A_n)f\| = \|(I - A_n)(f - f_n) + (I - A_n)f_n\| \lesssim \|f - f_n\|,$$

as $(I - A_n)$ converge pointwise to $(I - A)$ and are thus uniformly bounded. This however means that $(I - A_{n_k})f \to 0$, but $(I - A)f \neq 0$ by injectivity ($\|f\| = 1$), thus violating our assumption of pointwise convergence. Thus for sufficiently large n, $I - A_n$ is injective and thus invertible by the Fredholm alternative (each A_n must be compact if they are collectively compact as subsets of precompact sets are precompact).

Boundedness of the inverse: for invertible $I - A_n$, the nullspace is trivial and the range is closed. Thus $\|(I - A_n)f\| \geq c_n\|f\|$. Then

$$\|(I - A_n)(I - A_n)^{-1}f\| \geq c_n\|(I - A_n)^{-1}f\| \Rightarrow \frac{1}{c_n}\|f\| \geq \|(I - A_n)^{-1}f\|$$

Thus we need to bound $\|(I - A_n)f\|$ from below:

$$\begin{aligned}
\|(I - A_n)f\| &= \frac{\|(I - A_n)f\|(1 + \|(I - A)^{-1}A_n\|)}{1 + \|(I - A_n)^{-1}A_n\|} && (1)\\
&\geq \frac{\|(I - A_n)f + (I - A)^{-1}A_n(I - A_n)f\|}{1 + \|(I - A_n)^{-1}A_n\|} && (2)\\
&= \frac{\|f - (I - A)^{-1}(I - A)A_nf + (I - A)^{-1}(I - A_n)A_nf\|}{1 + \|(I - A_n)^{-1}A_n\|} && (3)\\
&= \frac{\|f - (I - A)^{-1}(A - A_n)A_nf\|}{1 + \|(I - A_n)^{-1}A_n\|} && (4)\\
&= \frac{\|f\| - \|(I - A)^{-1}(A - A_n)A_nf\|}{1 + \|(I - A_n)^{-1}A_n\|} && (5)\\
&= \frac{\|f\| - \|(I - A)^{-1}(A - A_n)A_n\|\|f\|}{1 + \|(I - A_n)^{-1}A_n\|} && (6)\\
&= \frac{1 - \|(I - A)^{-1}(A - A_n)A_n\|}{1 + \|(I - A_n)^{-1}A_n\|} \cdot \|f\| = c_n\|f\| && (7)
\end{aligned}$$

Note: (2) follows from the triangle inequality, (3) because $(I - A_n)$ and A_n commute, (5) by the reverse triangle inequality because the first term is larger than the second by assumption. $1/c_n$ then gives us the specified bound on $\|(I - A_n)^{-1}\|$.

Unfortunately even after a great deal of effort I couldn't bound $\|\phi_n - \phi\|$ as specified.