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Functional Analysis
Problem Set 2

Problem 1 Let $M = \sup\{|T(f,f)|,\ \|f\| = 1\}$. Clearly $M \leq \|T\|$. To show $M \geq \|T\|$, consider the polarization identity:

$$(Tf,g) = \frac{1}{4}[(T(f+g),f+g)-(T(f-g),f-g)+i(T(f+ig),f+ig)-i(T(f-ig),f-ig)].$$

Because $(Th,h) = (h,Th) = \overline{(Tf,f)}$, (Tf,f) is real, and thus:

$$\mathrm{Re}(Tf,g) = \frac{1}{4}[(T(f+g),f+g)-(T(f-g),f-g)].$$

By definition, $|(Th,h)| \leq M\|h\|^2$, so:

$$\begin{aligned}\mathrm{Re}(Tf,g) &\leq \frac{M}{4}(\|f+g\|^2 - \|f-g\|^2)\\ &\leq \frac{M}{2}(\|f\|^2+\|g\|^2),\end{aligned}$$

by the parallelogram law. Letting $\|f\| = \|g\| = 1$, we see $|\mathrm{Re}(Tf,g)| \leq M$. Finally, substituting $e^{i\theta}g$ for g, we see that from the fact that 1) for some θ, $\mathrm{Im}(T(f,e^{i\theta}g)) = 0$, and 2) $|T(f,e^{i\theta}g)|$ is invariant under θ, we conclude that $(Tf,g) \leq M$ for f, g with norms less than 1, and thus $M = \|T\|$.

Problem 2 All $\mathbb{L}^2$ functions can be approximated by continuous functions, all values of α for which K is square integrable will yield viable kernels. Specifically, we're worried about singularities at $y = x$. Given that square integrating our bound over x and y yields the "antiderivative":

$$\frac{C'}{|x-y|^{2(\alpha-n)}},$$

where n is the dimension of G, we can see this integral converges when $\alpha \leq n$.

(Sorry—didn't have time to find counterexamples for $\alpha > n$.)

Problem 3

a) The linearity of T follows from the linearity of multiplication. The boundedness of T can be shown by noting that:

$$\|Tf\|^2 = \int_0^1 |tf(t)|^2 dt \leq \int_0^1 |f(t)|^2 dt = \|f\|^2.$$

Compactness: take a bounded orthonormal basis spanning $\mathbb{L}^2([0.5,1])$, (e.g. the fourier basis) say $\{e_i\}$ (define them as 0 elsewhere). Consider the sequence $f_n = e_i/t$: it is bounded as

$$\|f_n\|^2 = \int_{0.5}^{1} |e_i/t|^2 dt \leq \int_{0.5}^{1} \frac{M}{t^2} dt = M,$$

but $Tf_n = e_n$, which, being orthonormal, contain no convergent subsequence.

b) Suppose λ is an eigenvalue for some f: $tf(t) = \lambda f(t)$, thus $(t-\lambda)f(t) = 0\ \forall t \in [0,1]$. Clearly ths is possible only when $f(t) = 0$, i.e. there are no eigenvectors.

Problem 4 Boundedess: by Pythagoras,

$$\|Tf\|^2 = \|\sum_k \alpha_k \frac{e_{k+1}}{k}\|^2 = \sum_k \frac{\alpha_k^2}{k^2} \leq \sum_k \alpha_k^2 = \|f\|^2.$$

Compactness: let $a_n = \sum_k \alpha_k e_k$ have norm ≤ 1. We can write

$$Ta_n = \sum_{k=1}^{\infty} \frac{\alpha_k e_{k+1}}{k} = \sum_{k=1}^{N} \frac{\alpha_k e_{k+1}}{k} + \sum_{k=N+1}^{\infty} \frac{\alpha_k e_{k+1}}{k}$$

The second term converges to zero in norm as $N \to \infty$, so for any m, we can choose N such that this term is less than $1/10m$, and then because the first term is finite dimensional, there exists a subsequence that converges in that term, and we can choose some n_i such that the distance between the first N terms of any two a_{n_j} with $j \geq i$ is also less than $1/10m$. Repeat, this time with the N-convergent subsequence, and index the resultant (sub)subsequence $\{a_m\}$. Clearly for $x, y \geq m$, $\|a_x - a_y\| \leq \frac{1}{m} \to 0$.

No eigenvectors: suppose $\sum a_k e_k$ was an eigenvector, then there exists some nonzero coefficient a_k. Because $Tf = \sum \frac{\lambda \alpha_k k e_{k+1}}{k}$, $\frac{\lambda \alpha_{k-1}}{k-1} = a_k$, i.e. a_{k-1} is nonzero and by induction a_1 is nonzero. But the coefficient of e_1 in Tf is 0, so no eigenvectors can exist.

Problem 5 Suppose $\lambda_k \to 0$: we can show compactness by the same argument as in the previous problem. Write:

$$Tf_k = \sum_{k=1}^{N} \lambda_k \alpha_k e_k + \sum_{k=N+1}^{\infty} \lambda_k \alpha_k e_k,$$

and again pick some f_m from nested N-convergent subsequences. Note that the second term decreases in norm to zero exactly because $\lambda_n \to 0$, and the first sits in a compact set as before.

Conversely, suppose λ_k doesn't converge to zero, i.e. $\exists \varepsilon$ such that for all N there exists $k \geq N$ such that $\lambda_k > \varepsilon$. Create from this a sequence K_N. Clearly $\{e_{K_N}\}$ have norm one but the image of $\{\lambda_{K_N} e_{K_N}\}$ has no convergent subsequence as they are all orthogonal with norm $> \varepsilon$, i.e. are always at least $\sqrt{2}\varepsilon$ apart, by Pythagoras.

Problem 6 Let $Tv = \lambda v$. The operator $T - \lambda I$ is type 2 Fredholm as T is compact, and has a nontrivial kernel that includes v. Thus its adjoint $T^* - \bar{\lambda} I$ also has a nontrivial kernel. I.e. there exists some w such that $T^* w = \bar{\lambda} w$.

Problem 7

a) An operator is Hilbert-Schmidt if $\sum \|Te_i\|^2 < \infty$, where e_i is an orthonormal basis. $\{\phi_n\}$ constitutes one such basis, and $\|T\phi_n\|^2 = |\lambda_n|$. Thus T is Hilbert-Schmidt only if $\sum_k |\lambda_n|^2 < \infty$.

b) Given $\{\phi_n\}$ is a basis for $\mathbb{L}^2([0,1])$, we can write $K(x,y) = \sum_i \sum_j a_{ij}\phi_i(x)\phi_j(y)$. Then we find the values of a_{ij} satisfying $T\phi_n = \lambda_n$:

$$\int \sum_i \sum_j a_{ij}\phi_i(x)\phi_j(y)\phi_n(y)dy = \sum_i a_{in}\phi_i(x),$$

because ϕ_j and ϕ_n are orthogonal when $j \neq n$. Because ϕ_i are linearly independent, the only way this can equal $\lambda_n \phi_n$ is if $a_{jn} = \lambda_n$ when $j = n$ and 0 otherwise. I.e. $K(x,y) = \sum_k \lambda_k \phi_k(x)\phi_k(y)$.

c) This is basically the same as part a): an operator is Hilbert-Schmidt if and only if $\sum_k \|Te_k\|^2 < \infty$, and $\|Te_k\|^2 = |\lambda_k|^2$ if we let $\{e_k\}$ be the set of eigenvectors of our symmetric and compact operator. Thus $\sum_k |\lambda_k|^2 < \infty$ is necessary and sufficient.

Problem 8

a) Let $\{e_k\}$ be a basis of eigenvectors for T_1. Then $T_1T_2e_k = T_2T_1e_k = T_2\lambda_k e_k = \lambda_k T_2 e_k$. In other words, T_2e_k is an eigenvector for T_1 with eigenvalue λ_k. Next, to show that e_k is an eigenvector of T_2 (with possibly eigenvalue 0), we consider the alternative: that there is a set of eigenvectors $\{e_i\}$ of T_1 which all have the same eigenvalues, and T_2 takes them to each other. Clearly they have to be a closed cycle (i.e. there is some k such that $T_2^k e_k = \lambda^k e_k$), else T_2 is clearly not compact. However, this would imply there was some $k \geq 1$ such that $\sum \mu_i^k a_i f_i = C \sum \mu_i a_i f_i$, where $\sum a_i f_i = e_k$ and f_i are eigenvectors of T_2. This is clearly impossible, unless $\mu_i = 0$ or 1 (in which case e_k would be an eigenvector), so the eigenvectors of T_1 must also be those for T_2, and vice versa.

b) Write $T = \frac{T+T^*}{2} + i\frac{T-T^*}{2i}$. Clearly both fractions are symmetric. Further, $(T+T^*)(T-T^*) = T^2 + T^*T + TT^* + T^{*2} = T^2 - T^{*2} = (T-T^*)(T+T^*)$, so they commute. By a), there is thus a basis $\{e_i\}$ of eigenvectors for both fractions. Thus $Te_i = (\lambda_i + \mu_i)e_i$, i.e. T is diagonalizable.

c) U is normal: for any a, b, $(a, b) = (Ua, Ub) = (a, U^*Ub)$, so in particular given this is true for all a, $b = U^*Ub$, so $U^*U = I$. The same can be derived for UU^*, so U is normal. Next, consider T in $U = \lambda I - T$. Because U is normal:

$$UU^* = (\lambda I - T)(\lambda I - T)^* = \lambda^2 I - \lambda T^* - \bar{\lambda} T - TT^*.$$

Comparing this with $U^*U = \lambda^2 I - \lambda T^* - \bar{\lambda} T - T^*T$, we see T must be normal too. Because T is compact, by b) there exists some basis of eigenvectors $\{e_i\}$ of T, and $Ue_i = \lambda I e_i - Te_i = (\lambda - \mu_i)e_i$, i.e. $\{e_i\}$ is a basis of eigenvectors of U.

Problem 9

a) Because $RK = I - A$, we know that $\dim(\mathcal{N}(RK)) \leq \infty$. Clearly $\dim(\mathcal{N}(K)) \leq \dim(\mathcal{N}(RK)) \leq \infty$. Then suppose $\dim(\mathcal{N}(R)) = \infty$, then $\dim(K^{-1}(\mathcal{N}(R))) = \infty$, but that is a subset of $\mathcal{N}(RK)$, which we know to be finite dimensional. Thus both are of finite dimension.

b) If $f \in \mathcal{N}(K^*)^\perp$, $f \in \mathcal{R}(K)$ because K is compact, thus $K\phi = f$ has solutions, as does $RK\phi = Rf$.

c) Define $R^{-1} : \mathcal{H} \to \mathcal{R}(K)$ and $K^{-1} : \mathcal{R}(K) \to \mathcal{H}$ to be the right- and left-inverses of R and K in the following sense: R^{-1} maps any element in $\mathcal{H}$ to its preimage in $\mathcal{R}(K)$, and K^{-1} maps any element in the range of K to its preimage in $\mathcal{H}$. These functions are well defined because in order for RK to be bijective, R must be injective and surjective on $\mathcal{R}(K)$, and K injective on $\mathcal{H}$. Clearly $S = K^{-1}R^{-1}$ is well defined on $\mathcal{H}$ and is the inverse to RK. In this case, $KSR = KK^{-1}R^{-1}R = I$, and thus $\mathcal{R}(I - KSR) = \{0\}$. This is trivially a subset of $\mathcal{N}(K)$. Likewise the adjoint of 0 is 0, so this is true for the adjoint case too. Finally, if $f \in \mathcal{N}(K^*)^\perp$, it is in the domain of SR as defined above ($SR = K^{-1}R^{-1}R = K^{-1}$, so its domain is $\mathcal{R}(K)$), and clearly $KSRf = KK^{-1}f = f$.