

Lewis Ho
Functional Analysis
Problem Set 2

Problem 1 Let $M = \sup\{|T(f, f)|, \|f\| = 1\}$. Clearly $M \leq \|T\|$. To show $M \geq \|T\|$, consider the polarization identity:

$$(Tf, g) = \frac{1}{4}[(T(f+g), f+g) - (T(f-g), f-g) + i(T(f+ig), f+ig) - i(T(f-ig), f-ig)].$$

Because $(Th, h) = (h, Th) = \overline{(Tf, f)}$, (Tf, f) is real, and thus:

$$\operatorname{Re}(Tf, g) = \frac{1}{4}[(T(f+g), f+g) - (T(f-g), f-g)].$$

By definition, $|(Th, h)| \leq M\|h\|^2$, so:

$$\begin{aligned} \operatorname{Re}(Tf, g) &\leq \frac{M}{4}(\|f+g\|^2 - \|f-g\|^2) \\ &\leq \frac{M}{2}(\|f\|^2 + \|g\|^2), \end{aligned}$$

by the parallelogram law. Letting $\|f\| = \|g\| = 1$, we see $|\operatorname{Re}(Tf, g)| \leq M$. Finally, substituting $e^{i\theta}g$ for g , we see that from the fact that 1) for some θ , $\operatorname{Im}(T(f, e^{i\theta}g)) = 0$, and 2) $|T(f, e^{i\theta}g)|$ is invariant under θ , we conclude that $(Tf, g) \leq M$ for f, g with norms less than 1, and thus $M = \|T\|$.

Problem 3

- a) The linearity of T follows from the linearity of multiplication. The boundedness of T can be shown by noting that:

$$\|Tf\|^2 = \int_0^1 |tf(t)|^2 dt \leq \int_0^1 |f(t)|^2 dt = \|f\|^2.$$

Compactness: take a bounded orthonormal basis spanning $\mathbb{L}^2([0.5, 1])$, (e.g. the fourier basis) say $\{e_i\}$ (define them as 0 elsewhere). Consider the sequence $f_n = e_i/t$: it is bounded as

$$\|f_n\|^2 = \int_{0.5}^1 |e_i/t|^2 dt \leq \int_{0.5}^1 \frac{M}{t^2} dt = M,$$

but $Tf_n = e_n$, which, being orthonormal, contain no convergent subsequence.

- b) Suppose λ is an eigenvalue for some f : $tf(t) = \lambda f(t)$, thus $(t - \lambda)f(t) = 0 \ \forall t \in [0, 1]$. Clearly this is possible only when $f(t) = 0$, i.e. there are no eigenvectors.

Problem 4 Boundedness: by Pythagoras,

$$\|Tf\|^2 = \left\| \sum_k \alpha_k \frac{e_{k+1}}{k} \right\|^2 = \sum_k \frac{\alpha_k^2}{k^2} \leq \sum_k \alpha_k^2 = \|f\|^2.$$

Compactness: let $a_n = \sum_k \alpha_k e_k$ have norm ≤ 1 . We can write

$$Ta_n = \sum_{k=1}^{\infty} \frac{\alpha_k e_{k+1}}{k} = \sum_{k=1}^N \frac{\alpha_k e_{k+1}}{k} + \sum_{k=N+1}^{\infty} \frac{\alpha_k e_{k+1}}{k}$$

The second term converges to zero in norm as $N \rightarrow \infty$, so for any m , we can choose N such that this term is less than $1/10m$, and then because the first term is finite dimensional, there exists a subsequence that converges in that term, and we can choose some n_i such that the distance between the first N terms of any two a_{n_j} with $j \geq i$ is also less than $1/10m$. Repeat, this time with the N -convergent subsequence, and index the resultant (sub)subsequence $\{a_m\}$. Clearly for $x, y \geq m$, $\|a_x - a_y\| \leq \frac{1}{m} \rightarrow 0$.

No eigenvectors: suppose $\sum a_k e_k$ was an eigenvector, then there exists some nonzero coefficient a_k . Because $Tf = \sum \frac{\lambda \alpha_k k e_{k+1}}{k}, \frac{\lambda \alpha_{k-1}}{k-1} = a_k$, i.e. a_{k-1} is nonzero and by induction a_1 is nonzero. But the coefficient of e_1 in Tf is 0, so no eigenvectors can exist.

Problem 5 Suppose $\lambda_k \rightarrow 0$: we can show compactness by the same argument as in the previous problem. Write:

$$Tf_k = \sum_{k=1}^N \lambda_k \alpha_k e_k + \sum_{k=N+1}^{\infty} \lambda_k \alpha_k e_k,$$

and again pick some f_m from nested N -convergent subsequences. Note that the second term decreases in norm to zero exactly because $\lambda_n \rightarrow 0$, and the first sits in a compact set as before.

Conversely, suppose λ_k doesn't converge to zero, i.e. $\exists \varepsilon$ such that for all N there exists $k \geq N$ such that $\lambda_k > \varepsilon$. Create from this a sequence K_N . Clearly $\{e_{K_N}\}$ have norm one but the image of $\{\lambda_{K_N} e_{K_N}\}$ has no convergent subsequence as they are all orthogonal with norm $> \varepsilon$, i.e. are always at least $\sqrt{2}\varepsilon$ apart, by Pythagoras.

Problem 6

Problem 7

- a) An operator is Hilbert-Schmidt if $\sum \|Te_i\|^2 < \infty$, where e_i is an orthonormal basis. $\{\phi_n\}$ constitutes one such basis, and $\|T\phi_n\|^2 = |\lambda_n|$. Thus T is Hilbert-Schmidt only if $\sum_k |\lambda_n|^2 < \infty$.

- b) Given $\{\phi_n\}$ is a basis for $\mathbb{L}^2([0, 1])$, we can write $K(x, y) = \sum_i \sum_j a_{ij} \phi_i(x) \phi_j(y)$. Then we find the values of a_{ij} satisfying $T\phi_n = \lambda_n$:

$$\int \sum_i \sum_j a_{ij} \phi_i(x) \phi_j(y) \phi_n(y) dy = \sum_i a_{in} \phi_i(x),$$

because ϕ_j and ϕ_n are orthogonal when $j \neq n$. Because ϕ_i are linearly independent, the only way this can equal $\lambda_n \phi_n$ is if $a_{jn} = \lambda_n$ when $j = n$ and 0 otherwise. I.e. $K(x, y) = \sum_k \lambda_k \phi_k(x) \phi_k(y)$.

- c) This is basically the same as part a): an operator is Hilbert-Schmidt if and only if $\sum_k \|Te_k\|^2 < \infty$, and $\|Te_k\|^2 = |\lambda_k|^2$ if we let $\{e_k\}$ be the set of eigenvectors of our symmetric and compact operator. Thus $\sum_k |\lambda_k|^2 < \infty$ is necessary and sufficient.

Problem 8

- a) Let $\{e_k\}$ be a basis of eigenvectors for T_1 . Then $T_1 T_2 e_k = T_2 T_1 e_k = T_2 \lambda_k e_k = \lambda_k T_2 e_k$. In other words, $T_2 e_k$ is an eigenvector for T_1 with eigenvalue λ_k . Next, to show that e_k is an eigenvector of T_2 (with possibly eigenvalue 0), we consider the alternative: that there is a set of eigenvectors $\{e_i\}$ of T_1 which all have the same eigenvalues, and T_2 takes them to each other. Clearly they have to be a closed cycle (i.e. there is some k such that $T_2^k e_k = \lambda^k e_k$), else T_2 is clearly not compact. However, this would imply there was some $k \geq 1$ such that $\sum \mu_i^k a_i f_i = C \sum \mu_i a_i f_i$, where $\sum a_i f_i = e_k$ and f_i are eigenvectors of T_2 . This is clearly impossible, unless $\mu_i = 0$ or 1 (in which case e_k would be an eigenvector), so the eigenvectors of T_1 must also be those for T_2 , and vice versa.
- b) Write $T = \frac{T+T^*}{2} + i\frac{T-T^*}{2i}$. Clearly both fractions are symmetric. Further, $(T+T^*)(T-T^*) = T^2 + T^*T + TT^* + T^{*2} = T^2 - T^{*2} = (T-T^*)(T+T^*)$, so they commute. By a), there is thus a basis $\{e_i\}$ of eigenvectors for both fractions. Thus $Te_i = (\lambda_i + \mu_i)e_i$, i.e. T is diagonalizable.
- c) U is normal: for any a, b , $(a, b) = (Ua, Ub) = (a, U^*Ub)$, so in particular given this is true for all $a, b = U^*Ub$, so $U^*U = I$. The same can be derived for UU^* , so U is normal. Additionally, by definition, U is type-2 Fredholm, i.e. compact, so by b), U must be diagonalizable.

Problem 9

- a) Because $RK = I - A$, we know that $\dim(\mathcal{N}(RK)) \leq \infty$. Clearly $\dim(\mathcal{N}(K)) \leq \dim(\mathcal{N}(RK)) \leq \infty$. Then suppose $\dim(\mathcal{N}(R)) = \infty$, then $\dim(K^{-1}(\mathcal{N}(R))) = \infty$, but that is a subset of $\mathcal{N}(RK)$, which we know to be finite dimensional. Thus both are of finite dimension.

- b) If $f \in \mathcal{N}(K^*)^\perp$, $f \in \mathcal{R}(K)$ because K is compact, thus $K\phi = f$ has solutions, as does $RK\phi = Rf$.
- c) Define $R^{-1} : \mathcal{H} \rightarrow \mathcal{R}(K)$ and $K^{-1} : \mathcal{R}(K) \rightarrow \mathcal{H}$ to be the right- and left-inverses of R and K in the following sense: R^{-1} maps any element in \mathcal{H} to its preimage in $\mathcal{R}(K)$, and K^{-1} maps any element in the range of K to its preimage in \mathcal{H} . These functions are well defined because in order for RK to be bijective, R must be injective and surjective on $\mathcal{R}(K)$, and K injective on \mathcal{H} . Clearly $S = K^{-1}R^{-1}$ is well defined on \mathcal{H} and is the inverse to RK . In this case, $KSR = KK^{-1}R^{-1}R = I$, and thus $\mathcal{R}(I - KSR) = \{0\}$. This is trivially a subset of $\mathcal{N}(K)$. Likewise the adjoint of 0 is 0, so this is true for the adjoint case too. Finally, if $f \in \mathcal{N}(K^*)^\perp$, it is in the domain of SR as defined above ($SR = K^{-1}R^{-1}R = K^{-1}$, so its domain is $\mathcal{R}(K)$), and clearly $KSRf = KK^{-1}f = f$.