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Functional Analysis
Pset 1
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Problem 1

$h_n \to 0$. Proof: suppose not, i.e. $\exists \varepsilon > 0$ s.t. $\forall N > 0\ \exists n \geq N$ where $||h_n|| > \varepsilon$. In which case there are infinite h_n with norms greater than ε (simply make each n the next N to generate the subsequence), and $\sum_n ||h_n||$ is infinite.

Problem 2

1. If 0 is in E, it's also in $\bar{E}$.
2. Let u and v be members of $\bar{E}$. By definition there exist sequences u_n and $v_n \in E$ converging to them. $u_n + v_n$ is in E (linearity), $u_n + v_n \to u + v$, therefore $u + v \in \bar{E}$.
3. Let $c \in \mathbb{F}$, $a \in \bar{E}$. There exists some sequence a_n converging to a in E. $ca_n \in E$, thus $ca = \lim ca_n \in \bar{E}$.

Problem 3

$(E^\top)^\top \subseteq$ $\overline{\textbf{span}}$**:** Let $v_n \to v \in (E^\top)^\top$. Consider $v_n - Pv_n$, where P is projection onto the span. Let w_i be orthogonal basis vectors for $E^\top$. $(v_n - Pv_n, w_i) = (v_n, w_i) \to (v, w_i) = 0$, and $v_n - Pv_n \in E^\top$, so $||v_n - Pv_n|| \to 0$. I.e. $Pv_n \in \text{span} \to v$.

$\supseteq$**:** Let $v \in \overline{\text{span}}$, $v_n \to v$, with $v_n \in \text{span}$. For any $w \in E^\top$, $(v_n, w) = 0$, as v_n are linear combinations of vectors orthogonal to w. By continuity, $(v, w) = 0$.

Problem 4

1. If $\sum |a_n|^2 < \infty$, $\exists \sup a_n = a$. $\| \sum a_n z^n \| \leq a \| \sum z^n \| \leq \infty$ for $|z| < 1$.
2. Linearity: $L(\alpha\{a_n\} + \beta\{b_n\}) = \sum(\alpha a_n + \beta b_n)\lambda^n = \alpha \sum a_n \lambda^n + \beta \sum b_n \lambda^n$.

   Bounded: let $a = \sup\{a_n\}$. $|L(\{a_n\})| \leq \sum |a\lambda^n| = |a^2|^{\frac{1}{2}} \sum |\lambda^n| \leq \sum \lambda^n (\sum |a_n|^2)^{\frac{1}{2}}$.
3. $h = (\lambda, \lambda^2, \lambda^3, \ldots)$. (h, h_0) is maximized at $h = h_0$, so $\|L\| = \frac{(h_0,h_0)}{\|h_0\|} = \|h_0\|$.

Problem 5

a) Norm preservation:

$$\begin{aligned}
\|U[F]\| &= \int_{-\infty}^{\infty} \frac{1}{\pi^{1/2}(i+x)} F\left(\frac{i-x}{i+x}\right) \overline{\frac{1}{\pi^{1/2}(i+x)} G\left(\frac{i-x}{i+x}\right)} dx \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+x^2} F\left(\frac{i-x}{i+x}\right) \overline{G\left(\frac{i-x}{i+x}\right)} dx
\end{aligned}$$

Let $e^{i\pi} = \frac{i-x}{i+x}$. Note: $\frac{1}{1+x^2} = \frac{2}{x+i} \cdot \frac{i+x}{i-x}$, and thus $d\theta = \frac{-2}{(x+i)^2 e^{i\theta}} = \frac{2}{1+x^2}$. Substituting, we get:

$$\|U[F]\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\pi})\overline{G(e^{i\pi})}d\theta = \|F\|.$$

Bijectivity: combining the inverse maps $x \mapsto \frac{1-xi}{1+x}$ and $F \mapsto \pi^{1/2}(i+x)F$, we have our inverse mapping (both functions are defined everywhere):

$$F(x) \mapsto \pi^{1/2}(i+x)G\left(\frac{i-x}{1+x}\right)$$

Thus our mapping is bijective and hence unitary.

Problem 7

Suppose A is bounded yet $\sup|\alpha_n|$ is infinite. Let a_n be the sequence such that $\alpha_{a_n} > n$. Clearly $|Ae_{a_n}| \to \infty$, i.e. isn't bounded, a contradiction. Thus A being bounded means $\sup|\alpha_n|$ is finite. Conversely, suppose $\sup|\alpha_n| \leq M$. $\|Aa_n\| \leq \|Ma_n\| = M\|a_n\|$, i.e. A is bounded.

Problem 8

Linearity follows from the linearity of integration and multiplication. Boundedness:

$$\begin{aligned}
\|Af\|^2 &= \int_0^1 \left| \int_0^1 k(x,y)f(y)dy \right|^2 dx \\
&\leq \int_0^1 \left(\int_0^1 |k(x,y)|^2 dy \int_0^1 |\overline{f(y)}|^2 dy \right) dx \\
&= \|k\|^2 \|f\|^2.
\end{aligned}$$

Problem 9

The subspace spanned by, say, polynomials on $[0,1]$ is dense in the subspace of functions that vanish outside $[0,1]$, but it is not closed (as there are clearly non-polynomial L^2 functions defined on $[0,1]$. Likewise with the subspace of continuous square-integrable functions on $\mathbb{R}$, which is dense in L^2 (and thus whose closure is $L^2(\mathbb{R})$)—note that not all L^2 functions are equivalent to continuous functions, hence the subspace is not itself closed.

Problem 10

Let P_1 and P_2 be orthogonal projections. Note that:

$$(Px, (y - Py)) = (x - Px, Py) = 0 \Rightarrow (P, y) = (x, Py)$$

for any projection P (i.e. they are self-adjoint). Assume P_1P_2 is a projection, then $P_1P_2 = (P_1P_2)^* = P_2^*P_1^* = P_2P_1$.

Conversely, let P_1 and P_2 commute. Thus $P_1P_2 = P_2P_1 = P_2^*P_1^* = (P_2P_1)^*$. I.e. P_1P_2 is self-adjoint. Further, note that $P_1P_2P_1P_2 = P_1P_1P_2P_2 = P_1P_2$, i.e. $(P_1P_2)^2 = P_1P_2$. Let P be a linear operator that is self-adjoint and satisfies $P^2 = P$. Let Px be any element in the range of P.

$$(Px, y - Py) = (P^2x, y - Py) = (Px, Py - P^2y) = (Px, Py - Py) = 0.$$

Thus P_1P_2 is a projection onto its range.

Problem 11

a)

$$\mathcal{F}[\chi_{[-k_0,k_0]}] = \int_{-k_0}^{k_0} e^{i2\pi xy} dy = \frac{e^{i2\pi xk_0} - e^{-i2\pi xk_0}}{2\pi ix} = \frac{\sin(2\pi xk_0)}{\pi x}$$

b) We use the substitution $u = z - y$:

$$\int_{-\infty}^{\infty} K(x-z)K(z-y)dz = \int_{-\infty}^{\infty} K((x-y)-u)K(u)du = \mathcal{F}[\chi] * \mathcal{F}[\chi](x-y) = \mathcal{F}[\chi \cdot \chi](x-y)$$

which is $K(x-y)$.

c) With Cauchy-Schwarz and the square integrability of K,

$$\|\mathcal{K}(f)\|^2 = \left| \int_{-\infty}^{\infty} K(x-y)f(y)dy \right|^2 \leq \|K(x-y)\|^2 \|f(x)\|^2.$$

d)

$$\begin{aligned} \mathcal{K}[\mathcal{K}[f]] &= \int_{-\infty}^{\infty} k(x-z) \int_{-\infty}^{\infty} k(z-y)f(y)dydz \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x-z)k(z-y)f(y)dydz \\ &= \int_{-\infty}^{\infty} k(x-y)f(y)dy. \end{aligned}$$

e) Linearity: because $\mathcal{F}$ is a linear operator, if f and $g = 0$ at x, $\mathcal{F}[\alpha f + \beta g](x) = 0$ also, and thus the subspace is linear. Closedness: let $f_n \to f$ with $f_n \in \mathcal{H}_0$. Let f_k be the subsequence that is dominated by $2f$, say. Thus at $|x| > k_0$,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{2\pi ixy} f_n(y)dy &= 0 \\ \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{2\pi ixy} f_n(y)dy &= 0 \\ \int_{-\infty}^{\infty} e^{2\pi xy} f(y)dy = \mathcal{F}[f] &= 0. \end{aligned}$$

f) Lemma: $F[K] = \mathcal{F}[\mathcal{F}[\chi_{[-k_0,k_0]}]] = \chi_{[-k_0,k_0]}$. Proof:

$$\int_{-\infty}^{\infty} e^{2\pi ixy} \frac{\sin(2\pi k_0 y)}{\pi y} dx = -\int_{\infty}^{-\infty} e^{-2\pi ixz} \frac{\sin(2\pi k_0 z)}{\pi z} dz = \mathcal{F}^{-1}[\mathcal{F}[\chi]].$$

Lemma 2: for all $g \in \mathcal{H}_0$, $\mathcal{K}[g] = g$. Proof:

$$\mathcal{K}[g] = \mathcal{F}^{-1}[\mathcal{F}[K * g]] = \mathcal{F}^{-1}[\mathcal{F}[K] \cdot \mathcal{F}[g]] = \mathcal{F}^{-1}[\chi \cdot \mathcal{F}[g]].$$

As $F[g] = 0$ for all $x \notin [-k_0, k_0]$, the last term $= \mathcal{F}^{-1}[\mathcal{F}[g]] = g$.

Proof of statement: (note that $K(x) = K(-x)$) we show $f - \mathcal{K}[f] \perp g$ for all $g \in \mathcal{H}_0$.

$$\begin{aligned}
\int_{-\infty}^{\infty}(f - \mathcal{K}[f])(x)g(x)dx &= \int_{-\infty}^{\infty} fg - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-y)f(y)dy \cdot g(x)dx \\
&= \int_{-\infty}^{\infty} fg - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(y-x)g(x)f(y)dxdy \\
&= \int_{-\infty}^{\infty} fg - \int_{-\infty}^{\infty} \mathcal{K}[g](y)f(y)dy = 0.
\end{aligned}$$