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Functional Analysis
Problem Set 3

**Problem 1** Any linear functional can be fully determined by its actions on a basis by linearity:

$$f\left(\sum_k a_k e_k\right) = \sum_k a_k f(e_k) \in \mathbb{R}$$

thus any $f$ can be defined as a sequence $\{a_n\}$ where $f(e_k) = a_n$. If $\{a_n\} \in \ell^1(\mathbb{N})$, for any $\{b_n\} \in c_0(\mathbb{N})$, $|f_a(\sum_k b_k)| = |\sum_k a_k b_k| \leq \sup(|b_n|) \sum |a_k| = M\|\{b_n\}\|$. Note that we assume the norm defined on $c_0(\mathbb{N})$ is the supremum norm, which is well defined because convergent sequences are bounded. Hence $\ell^1(\mathbb{N}) \subseteq c_0(\mathbb{N})^*$.

Conversely, given some $\{a_n\}$ such that $\sum |a_n| = \infty$, we can find a sequence of indices $n_1,\ n_2, \ldots$ such that $\sum_{i=1}^{n_k} |a_i| \geq k^2$. Consider then the sequence $\{b_n\}$, with $b_i = |1/k|$ for $n_{k-1} < i \leq n_k$, and the sign of $b_i$ the same as the sign of $a_i$. Then:

$$f_a(\{b_n\}) \geq \sum_{i=1}^{n_k} a_i b_i \geq \frac{1}{k} \sum_{i=1}^{n_k} |a_i| \geq k^2/k \to \infty.$$

Note that $\|\{b_n\}\| \leq 1$. Thus $f$ is unbounded and $\ell^1(\mathbb{N}) \supseteq c_0(\mathbb{N})^*$.

**Problem 2** Suppose not. Let $X = \ell^\infty(\mathbb{N})$ contain a countable subset $\{w_i\}$ that is dense. With Gram-Schmidt we obtain a orthonormal basis $\{v_i\}$ that is countable. Consider the map $U(v_i) = e_i$, where $\{e_i\}$ is the standard “basis”—sequences of zeros, except for 1 in the $i$th element. For any $f = \sum_i a_i v_i$,

$$\|U(f)\| = \|U(\sum_i a_i v_i)\| = \|\sum_i a_i U(v_i)\| = \|\sum_i a_i e_i\| = \sup_i\{a_i\} = \|f\|$$

$U$ is clearly linear and bijective, and thus $X$ is unitarily equivalent to the closure of the space spanned by $\{e_i\}$ with the same norm, say, $Y$.

But $Y$ is not separable. Let $\{a_n\}$ be the sequence with $a_n = 1$ for all $n$. For any finite linear combination $\sum_i \alpha_i e_{k_i}$ of $\{e_i\}$,

$$\|\{a_n\} - \sum_i^N \alpha_i e_{k_i}\|_\infty \geq 1$$

as $a_n$ has infinite terms that are 1 compared to the finite ones in finite linear combinations of $\{e_i\}$. Hence there is always some $j$ where the $j$th term of $\{a_n\}$ is 1 and that of our linear combination is 0.

The fact that $X$ is also cannot be separable follows from unitary equivalence. Suppose not: consider $U^{-1}(\{a_i\}) = b$. By separability, $\sum_i^n c_i v_i \to b$ as $n \to \infty$, i.e. for any given $\varepsilon$, we can find some $N$ such that for $n \geq N$:

$$\|b - \sum_i^n c_i v_i\| < \varepsilon.$$

By the properties of $U$,

$$\|U(b - \sum_i^n c_i v_i)\| = \|U(b) - U(\sum_i^n c_i v_i)\| = \|\{a_n\} - \sum_i^N c_i e_i\| < \varepsilon$$

in which case linear combinations of $\{e_i\}$ converge to $\{a_i\}$, a contradiction. Thus we've shown that no subset of $X$ can be dense.

**Problem 3** It is a subspace by the fact that if $a_n \to a$ and $b_n \to b$, $\lim_{n\to\infty} a_n + b_n$ exists, and is in fact $a + b$ (this all follows from the continuity of addition $\mathbb{R}^2 \to \mathbb{R}$). Thus $\{a_n\} + \{b_n\} \in c$. Closure: let $\{b_n\}$ be the limit of a sequence of sequences $\{a_n\}_k$. For any $\varepsilon > 0$, there exists some $c_n = \{a_n\}_i$ such that $\sup |c_n - b_n| < \varepsilon/3$, and also some $N$, because $c_n$ is convergent, such that $|a_m - a_n| < \varepsilon/3$ for all $m,\ n > N$. Thus for any $m,\ n > N$,

$$|b_n - b_m| \leq |b_n - c_n| + |b_m - c_m| + |c_n - c_m| < \varepsilon,$$

i.e. $\{b_n\}$ is Cauchy and thus in $c$ as well. $c$ is therefore closed.

**Problem 5** $\ell(f) = f(x_0) \leq \sup(f) = \|f\|$ on $[0, 1]$. Additionally, there are continuous functions that attain their maximum on $x_0$, thus the inequality is sometimes an equality and $\|\ell\| = 1$.

**Problem 6**

(a) Let $f,\ g \in E_\alpha$. For all $\lambda \in \mathbb{R}$, $\lambda f(0) + (1 - \lambda)g(0) = \lambda\alpha + (1 - \lambda)\alpha = \alpha$, thus all convex combinations of $f$ and $g$ are clearly also in $E_\alpha$.

Then, given any $f \in X$ and $\varepsilon > 0$, there exists some continuous function $g \in X$ such that $\|f - g\| < \varepsilon/2$, (given the density of the continuous functions in $L^2$). Then we can find another function $h \in E_\alpha$ such that $\|g - h\| < \varepsilon/2$, where $h = g$ in $[-1, 1] \setminus [-\delta, \delta]$, and $h$ decreases/increases linearly to $\alpha$ at 0. We can find such a close $h$ because we can, by decreasing $\delta$, decrease the measure of the set on which $h$ differs from $g$ arbitrarily small, and this difference (squared) is bounded given the continuity of both functions and the compactness of our domain. Thus $\|f - h\| < \varepsilon$, i.e. $E_\alpha$ is dense.

(b) No function can attain the value of both $\alpha$ and $\beta$ at 0. Further, because all functions we are considering are continuous, functions in $E_\alpha$ and $E_\beta$ must differ in an area of nonzero measure, thus cannot belong to the same equivalence class in $L^2$. Thus both sets are disjoint. To show they cannot be separated, we first prove a lemma.

Lemma: if $A$ is a dense set, there exist no nonzero bounded linear functionals such that $\ell(a) \leq C$. Proof: because $\ell$ is nonzero, there exists some $a$ such that $\ell(a) \neq 0$. By linearity, we can scale $a$ to find an element $b$ such that $\ell(b) > C + \varepsilon$. Because $A$ is dense, we can find some $f \in A$ such that $\|f - b\| < \varepsilon/M$, where $M$ is the norm of $\ell$. This means $\|\ell(f) - \ell(b)\| < \varepsilon$, and thus $\ell(f) > C$.

From the lemma it follows that because $E_\alpha$ is dense, $\{\ell(E_\alpha)\}$ has no supremum, and thus no such separation is possible. Geometric Hahn Banach is not possible because the sets are not compact. Note that we assume here that the question refers to nonzero functionals, because as far as I can tell such an arrangement is true for $\ell = 0$.

**Problem 7** Disjointness: a polynomial cannot both have a negative leading coefficient and all non-negative coefficients, thus they are disjoint. Convexity: the sum of two polynomials with negative leading coefficients has a negative leading coefficient, this being either the sum of both or just one of them; and the sum of two polynomials with non-negative coefficients has non-negative coefficients as well. Finally, multiplying polynomials with $0 < \lambda \leq 1$ doesn't change the sign of its coefficients, thus both $A$ and $B$ are convex.

Suppose there does exist some nonzero $\ell$ such that $\ell(a) \leq \ell(b)$, for all $a$ in A and $b$ in $B$, then by the completeness of $\mathbb{R}$, there exists a real $C$ (e.g. the supremum of $\ell(a)$) such that $\ell(a) \leq C \leq \ell(b)$ for all $a$ and $b$. Because $0 \in B$, by the linearity of $\ell$, $C \leq \ell(0) = 0$. However, because for any monomial $a \in A$, $a/n \in A$ also, for $n > 0$, thus $\ell(a/n) = \ell(a)/n \to 0$ as $n \to \infty$. Thus $C \geq 0$, i.e. $C = 0$.

If $\ell$ is nonzero, there must be some monomial $x^n$ such that $\ell(x^n) \neq 0$, as all polynomials are finite sums of monomials, and thus if they all mapped to 0, $\ell$ would be 0 on all of $\mathcal{P}$. Clearly the sign of $\ell(ax^n)$ is the same as the sign of $a$. Consider then the polynomial $p = -x^{n+1} + ax^n \in A$: because we can make $a$ and thus $\ell(ax^n)$ arbitrarily large, $\ell(-x^{n+1})$ cannot be finite and yet keep $\ell(p) = \ell(-x^{n+1}) + \ell(ax^n)$ to remain $\leq 0$ for all $a > 0$ i.e. no such linear functional can exist.

**Problem 8** Let $K_1$ be the half space with $y \leq 0$, and let $K_2 = \{(x, y) \in \mathbb{R}^2 \mid x \geq 1,\ y \geq 1/x\}$. Both sets are convex: the half space is convex, and $\frac{1}{x}$ is a convex function, and truncating it like so doesn't hurt this convexity. Because the boundary of $K_1$ is the $x$ axis, the only hyperplanes (lines) not intersecting $K_1$ must be parallel to the $x$ axis and $\geq 0$, i.e. $\ell(x, y) = ay$, where $a \geq 0$. However, the infimum for $y$ where

$(x, y) \in K_2$ is 0, thus:

$$\sup_{x \in K_1} \ell(x) = 0 = \inf_{y \in K_2} \ell(y),$$

with this construction. I.e. no such $\ell$ exists that satisfies our requirement.

**Problem 9**

(a) Let $a^* \in Z^*$ and $b \in X$. Because $T$ and $S$ are bounded, their adjoints $T^*$ and $S^*$ exist and are bounded. Note that for any $A : X \to Y$, $y^* \in Y^*$, $x \in X$, $(y^*, Tx)_Y = y^*Tx = T^*y^*x = (T^*y^*, x)_X$ by definition. Consider then $(z^*, STx)_Z$:

$$(z^*, STx)_Z = (S^*z^*, Tx)_Y = (T^*S^*z^*, x)_X,$$

i.e. $T^*S^*z^* = z^*ST$ for all $z^* \in Z^*$, which is the domain of both $(ST)^*$ and $T^*S^*$. This last equation is how we define $(ST)^*$, thus $(ST)^*z^* = T^*S^*z^*\ \forall z^*$, hence $(ST)^* = T^*S^*$.

(b) Again we need to show that $(aS^* + bT^*)y^* = y^*(aS + bT)$ for all $y^*$ in $Y^*$, i.e. $(aS^* + bT^*)y^*(x) = y^*(aS + bT)(x)$ for all $y^* \in Y^*$, $x \in X$:

$$(aS^* + bT^*)y^*(x) = aS^*y^*(x) + bT^*y^*(x)$$

then by the definition of $T^*$ and $S^*$:

$$= ay^*S(x) + by^*T(x) = y^*aS(x) + y^*bT(x) = y^*(aS + bT)(x)$$

by linearity.

(c) We show $(T^{-1})^* : X^* \to Y^*$ is the inverse map of $T^* : Y^* \to X^*$:

$$(T^{-1})^*T^*y^*(a) = (T^{-1})^*y^*T(a) = y^*T^{-1}T(x) = y^*(a)$$

likewise,

$$T^*(T^{-1})^*x^*(b) = T^*x^*T^{-1}(b) = x^*TT^{-1}(b) = x^*(b)$$

for all $a \in Y,\ b \in X,\ x^* \in X^*\ y^* \in Y^*$, by repeated application of the $A^*b^* = b^*A$ rule. Thus $(T^{-1})^* = (T^*)^{-1}$.