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Functional Analysis  
Problem Set 4

**Problem 1**  $\overline{A(B_1(0))}$  must contain a ball, else the range of  $A$  will be  $\bigcap_{n=1}^{\infty} A(B_n(0)) = \bigcap_{n=1}^{\infty} nA(B_1(0))$  which will then be meagre. Because  $\overline{A(B_1(0))}$  is symmetric and convex, it will in particular contain a ball  $B_{\varepsilon_0/2}(0)$ , as if the original ball is centered around some  $y_0$ , then  $(-y_0 + y_0 + B_{\varepsilon_0}(0))/2$  will be a set of convex combinations of points we know to be in the closure.

Then we show  $\overline{A(B_1(0))} \subset A(B_2(0))$ , following the proof of the open mapping theorem. Suppose  $y \in \overline{A(B_1(0))}$ , i.e.  $\exists x_1 \in B_1(0)$  such that  $\|y - Ax_1\| < \varepsilon/4$ , i.e.  $y - Ax_1 \in B_{\varepsilon_0/4} \subset A(B_{1/2}(0))$ . Because it's contained in this ball, there exists some  $x_2 \in B_{1/2}(0)$  such that  $\|y - Ax_1 - Ax_2\| < \varepsilon_0/8$ , i.e.  $y - Ax_1 - Ax_2 \in B_{\varepsilon_0/8}(0) \subset \overline{A(B_{1/4}(0))}$ . Generally, there exists  $x_n \in B_{1/2^{n-1}}(0)$  such that  $\|y - \sum_{i=1}^n x_i\| < \varepsilon_0/2^{n+1}$ .

We know that in Banach spaces if  $\sum \|x_n\| < \infty$  then  $\sum x_n$  converges, and  $\sum \|x_n\|$  in this case  $\leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \leq 2$ . Thus the limit  $x$  of this sum exists, and by the triangle inequality, is in  $B_2(0)$ , and finally satisfies  $Ax = y$ . Hence  $y \in A(B_2(0))$ . From this we conclude (by scaling) that  $A(B_1(0))$  contains a ball of  $\varepsilon_0/4$  radius, and thus by multiplying  $B_1(0)$  by arbitrarily large numbers, can conclude that  $A$  is surjective and thus has a closed range.

**Problem 2** The statement is false. We can construct arbitrarily tall isosceles triangles of  $\varepsilon$  area. Center those triangles at 0.5, and define functions  $f_h$  to be the functions that are 0 outside the base of the triangle of height  $h$ , and take the value of the diagonals up and down inside.  $\|f_h\|$  is unbounded, but there is always a function  $F_h$  whose supremum (and thus norm) is  $\varepsilon$  and has derivative  $f_h$ , namely the antiderivative of  $f_h$ .

**Problem 3** If there exists some  $c$  such that  $\|Ax\| \geq c\|x\|$  for all  $x$ , then  $\mathcal{N}(A)$  must be empty, otherwise arbitrarily large vectors in the null space can have images with norm zero, a contradiction. Closedness: let  $Af_n$  be a Cauchy sequence in the range. For any  $\|Af_n - Af_m\| < \varepsilon$ , by the assumption,

$$\|Af_n - Af_m\| = \|A(f_n - f_m)\| \geq c\|f_n - f_m\|,$$

in other words,  $f_n$  must also be Cauchy, converging to some  $f$  in the domain. Then by continuity,  $Af = A \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} Af_n$ , i.e. the range is closed.

Conversely, if both  $\mathcal{N}(A) = \{0\}$  and the range is closed, then if there were to not exist a  $c$  such that  $\|Ax\| \geq c\|x\|$ , we could construct a sequence  $Af_n \rightarrow 0$ , where  $\|f_n\| = 1$ . Because the range is closed, there is some  $f$  with norm 1 such that  $Af = 0$ , violating our  $\mathcal{N}(A) = \{0\}$  assumption.

**Problem 4** By a modification of Hölder's inequality, we get:  $(x_n, y) \leq \|x_n\|_p \|y\|_q$ .

Suppose our assumptions hold  $\sup_n \|x_n\|_p < \infty$  and  $x_n^j \rightarrow 0$ . I will show that we can make  $(x_n, y)$  arbitrarily small. Firstly, note that we can bound  $(x_n, y)$  by decomposing it and applying "Hölder's" as follows:

$$|(x_n, y)| = \left| \sum_{|y^j| > \delta} x_n^j y^j + \sum_{|y^j| \leq \delta} x_n^j y^j \right| \leq \|y\|_q \left( \sum_{|y^j| > \delta} |x_n^j|^p \right)^{\frac{1}{p}} + \left( \sum_{|y^j| < \delta} |y^j|^q \right)^{\frac{1}{q}} \sup_k \|x_k\|_p$$

Note that we can always find an  $N$  such that for all  $n > N$  both parts are arbitrarily small. First we can fix  $\delta$  to make the latter term small, e.g. because  $\|y\|_q$  is finite, we can look at the first  $N$  terms such that their  $q$  sum is arbitrarily close to  $\|y\|_q^q$ , and then set  $\delta$  to be their infimum. With this fixed  $\delta$ , there are now finite elements in the sum of  $x_n^j$ s in the first term, and thus we can pick some  $N$  such that for  $n > N$  all  $x_n^j$  are arbitrarily small, and thus their finite  $p$  sum is also arbitrarily small.

For the converse, note that  $x_n$  define bounded linear operators from  $\ell^q$  to  $\mathbb{R}$ . If  $(x_n, y) \rightarrow 0$  for all  $y$ , then the set  $E = \{y \in \ell^q \mid \sup_n \|(x_n, y)\| < \infty\} = \ell^q$ , and by the uniform boundedness principle  $\sup_n \|x_n\|_p$  must be bounded. Finally, if  $x_n^j \rightarrow 0$  is violated, the sequence with  $j$ th element = 1 has image not converging to 0.

**Problem 5** In the previous problem set we showed that  $c_0^* = \ell^1$ . NEED TO SHOW THE NORM OF ELL IS THE NORM OF THE SPACE.

To show  $(x_n, y) \rightarrow 0$ , we decompose  $(x_n, y)$  again as follows:

$$|(x_n, y)| = \left| \sum_{|y^j| > \delta} x_n^j y^j + \sum_{|y^j| \leq \delta} x_n^j y^j \right| \leq \|y\|_\infty \sum_{|y^j| > \delta} |x_n^j| + \delta \sup_n \|x_n\|_1$$

As previously, we can choose  $\delta$  such that the second term becomes arbitrarily small, and then given this delta, choose  $n$  such that the first term becomes arbitrarily small.

To show the converse, note that  $x_n$  define linear operators from  $c_0 \rightarrow \mathbb{R}$ . If  $(x_n, y) \rightarrow 0$  for all  $y$ , then the set  $E$  for  $\{x_n\}$  is all of  $c_0$ , and thus  $\sup_n \|x_n\|$  is finite. Additionally if we have some  $x_n^j$  not converging to 0, then  $e_j$  has image not converging to zero.

**Problem 6** Let  $h = \sum_i a^i e_i$ , and  $h_n = \sum_i h_n^i e_i$ . If our assumptions are satisfied, we can write  $|(h_n, h)|$  as

$$\left| \sum_{|a^i| > \delta} a^i h_n^i + \sum_{|a^i| \leq \delta} a^i h_n^i \right| \leq \|h\| \left( \sum_{|a^i| > \delta} |h_n^i|^2 \right)^{\frac{1}{2}} + \left( \sum_{|a^i| < \delta} |a^i|^2 \right)^{\frac{1}{2}} \sup_k \|h_k\|$$

using the Cauchy-Schwarz inequality to bound it. Then as before, the latter term is made arbitrarily small with our choice of  $\delta$ , and the former with a sufficiently large choice of  $N$ .

For the converse, we use the uniform boundedness principle, as all  $h_n$  define operators  $\mathcal{H} \rightarrow \mathbb{R}$ , and  $E$  is all of space for  $\{h_n\}$ . Likewise if some  $b_n^j$  doesn't go to zero,  $(e_j, h)$  won't converge to zero either.

**Problem 7** Proof by contradiction: suppose this statement is false, i.e. there exists some  $\varepsilon_0$  such that for all  $N$  there exists some  $n > N$  such that  $\sup_f \|A_n f - A f\| > \varepsilon_0$ . From this we can construct a sequence  $\{f_n\}$  where  $\|A_n f_n - A f_n\| > \varepsilon_0$ , and then find the limit  $f$  of a convergent subsequence  $\{f_{n_k}\}$  using compactness.

Consider  $\|A_n f - A f\|$ . Using our subsequence  $\{f_{n_k}\}$  and their accompanying  $A_{n_k}$ , we can write

$$\begin{aligned} \|A_{n_k} f - A f\| &= \|(A_{n_k} - A)(f_{n_k} + f - f_{n_k})\| \\ &= \|A_{n_k} f_{n_k} - A f_{n_k} + (A_{n_k} - A)(f - f_{n_k})\| \\ &\geq \varepsilon_0 - (\sup_n \|A_n\| + \|A\|)\|f - f_{n_k}\| \end{aligned}$$

and that we can choose large enough  $n_k$  such  $f$  and  $f_{n_k}$  are very close and this difference is bounded from below by, say,  $\varepsilon_0/2$ , meaning that it doesn't converge. Note that  $\sup_n \|A_n\|$  and  $\|A\|$  exist by the uniform boundedness principle.

**Problem 8** Let  $M = \sup_{A \in \mathcal{A}} \|A\|$ , which exists because the image set of the unit ball is compact (and hence bounded), and thus by the uniform boundedness principle the supremum of their norms is finite. From the previous problem and both the pointwise convergence of  $L_n$  and the collective compactness of  $\mathcal{A}$ , we know that:

$$\begin{aligned} \sup_{g \in \overline{\mathcal{A}(B_1(0))}} \|L_n g - L g\| &\rightarrow 0 \\ \Rightarrow \sup_{\|f\| \leq 1, A \in \mathcal{A}} \|(L_n - L)A f\| &\rightarrow 0 \end{aligned}$$

the first implying the second because:

$$\overline{\mathcal{A}(B_1(0))} \subseteq \mathcal{A}(B_1(0)) = \{A f \mid A \in \mathcal{A}, f \in B_1(0)\}.$$

Now we invoke the definition of the norm of an operator  $T$  as  $\sup \|T f\|$  for  $\|f\| \leq 1$  to infer

$$\sup_{\|f\| \leq 1, A \in \mathcal{A}} \|(L_n - L)A f\| = \sup_{A \in \mathcal{A}} (\sup_{\|f\| \leq 1} \|(L_n - L)A f\|) = \sup_{A \in \mathcal{A}} \|(L_n - L)A\| \rightarrow 0.$$