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Functional Analysis
Problem Set 7
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Problem 1 Lemma: if $\lambda \in \sigma_r(T)$, $\lambda \in \sigma_p(T^*)$.

$\sigma_p(R)$, $\sigma_r(L) = \emptyset$: Right shift operator: firstly, for all λ, $R - \lambda I$ has a trivial kernel. Proof: if λ is zero, then it has a trivial kernel. For $\lambda \neq 0$, let v be in the kernel of $R - \lambda I$. Then as $Rv = \lambda v$, and as $(Rv)_1 = 0$ by the nature of the right-shift operation, $v_1 = 0$. Additionally, if $v_n = 0$, $(Rv)_{n+1} = \lambda v_n = 0$, and hence $v_{n+1} = 0$. By induction, $v_n = 0$ for all n. This shows that there are no eigenvectors for any value of λ, and thus $\sigma_p(R) = \sigma_r(L) = \emptyset$, and it also means we've shown injectivity for all values of λ (for the subsequent spectra).

$\sigma_p(L)$, $\sigma_r(R) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$: Consider the vector $(1, \lambda, \lambda^2, \lambda^3, \ldots)$. $L - \lambda I$ applied to this vector gives 0. Further, the norm of this vector is a geometric series with ratio λ^2, and thus has a finite sum when $|\lambda| < 1$. Hence our operator has a nontrivial kernel and $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_p(L)$. To show equality, note that $|\lambda| > 1$ yields a set in the resolvent set. Thus we consider $|\lambda| = 1$. Suppose then that $L - \lambda I$ has a nontrivial kernel. Then v in the kernel has nonzero element v_k. As $Lv = \lambda v$, $v_{k+1} = \lambda v_k$. And by induction all subsequent elements v_n have $|v_n| = |v_k|$, which is not a square-summable sequence, a contradiction.

$\rho(L), \rho(R) = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$: Let $T = L$ or R. This follows from the Neumann series: T/λ has norm less than one so $T/\lambda - I$ has a bounded inverse. Thus $\rho(L), \rho(R) \supset \{\lambda \in \mathbb{C} : |\lambda| > 1\}$.

To show equality, note that the spectrum of an operator is closed. $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ is in the closure of $\sigma_r(L) \cup \sigma_p(R)$, thus $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ firstly cannot be in ρ of either operator, and secondly must be the continuous spectrum of both operators by elimination.

Problem 2 I show no λ can exist where $A = (T - \lambda I)$ has a trivial kernel and a range that is not dense.

If A's kernel is trivial, $\lambda \notin \{\lambda_i\}$, else e_i is in the kernel of A. In this case, note further that $A(e_i/(\lambda_i - \lambda)) = e_i$, so each e_i is in the range, and finite linear combinations of $e_i/(\lambda_i - \lambda)$ are members of $\ell^2(\mathbb{N})$, thus because finite linear combinations of e_i are dense, the range is dense. Thus no lambda exists in $\sigma_r(T)$.

I now show $\lambda \in \overline{\{\lambda_i\}} \setminus \{\lambda_i\} \in \sigma_c(T)$. Let λ be a member of the set. Because λ isn't an eigenvalue, by the same argument as in the previous paragraph $A = T - \lambda I$ is injective and the range is dense. To show that the range isn't closed, let $\lambda_n \to \lambda$, with

$|\lambda_n - \lambda| < 1/n^2$. Then let v be the vector with $1/n$ in the component corresponding to λ_n in T and 0 otherwise. The formal preimage of v is the sequence with $1/(n\lambda_n - n\lambda)$ in the positions corresponding to each $\lambda_n \in T$, but as $|\lambda_n - \lambda| = 1/n^2$, the preimage of v isn't in $\ell^2(\mathbb{N})$ as it isn't square-summable. Thus $\lambda \in \sigma_c(T)$.

Problem 3 All such operators map $\ell^2(\mathbb{N})$ to itself.

a) 0 is a compact operator, and $0 - 0I$ has a nontrivial kernel so 0 is in the point spectrum.

b) $T = T - 0I$ mapping $e_i \to e_i/k$ has all coordinate vectors in its range and thus has a dense range, but $(1, 1/2, 1/3, \ldots)$ has the formal preimage $(1, 1, 1, \ldots)$, and thus the range isn't closed. It is compact because it is the norm limit of finite rank operators sending $e_k \to e_k/k$ for $k < n$.

c) $T = T - 0I$ sending $e_k \to e_{2k}/k$ is compact by the same argument as the previous, has no kernel, but also has a range that isn't dense as, for example, e_1 cannot be approximated by anything in its range.

Problem 4 Let $\lambda \in \sigma_p(T)$ but $\notin \sigma_p(T^*)$, specifically, let $Tv = \lambda v$. Then

$$T^* x^*(v) = x^*(Tv) = \lambda x^*(v).$$

I now show that the range of $T - \lambda I$ cannot be dense. Let v^* be the vector of norm 1 such that $v^*(v) = \|v\|$, which is possible by Hahn-Banach. Suppose the range is dense, and let x_n^* be a sequence of vectors such that $(T - \lambda I)x_n^* \to v^*$. Consider then

$$(T - \lambda I)x_n^*(v) = Tx_n^*(v) - \lambda x_n^*(v) = \lambda x_n^*(v) - \lambda x_n^*(v) = 0,$$

which means x_n^* doesn't converge weak*ly to v^*, and hence also not in the norm topology, a contradiction. Thus $\lambda \in \sigma_r(T^*)$.