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Functional Analysis
Pset 2

Problem 1 Let $M = \sup\{|T(f,f)|, \ \|f\| = 1\}$. Clearly $M \le \|T\|$. To show $M \ge \|T\|$, consider the polarization identity:

$$(Tf,g) = \frac{1}{4}[(T(f+g), f+g) - (T(f-g), f-g) + i(T(f+ig), f+ig) - i(T(f-ig), f-ig)].$$

Because $(Th,h) = (h,Th) = \overline{(Tf,f)}$, (Tf,f) is real, and thus:

$$\mathrm{Re}(Tf,g) = \frac{1}{4}[(T(f+g), f+g) - (T(f-g), f-g)].$$

By definition, $|(Th,h)| \le M\|h\|^2$, so:

$$\begin{aligned} \mathrm{Re}(Tf,g) &\le \frac{M}{4}(\|f+g\|^2 - \|f-g\|^2) \\ &\le \frac{M}{2}(\|f\|^2 + \|g\|^2), \end{aligned}$$

by the parallelogram law. Letting $\|f\| = \|g\| = 1$, we see

Problem 4 Boundedess: by Pythagoras,

$$\|Tf\|^2 = \|\sum_k \alpha_k \frac{e_{k+1}}{k}\|^2 = \sum_k \frac{\alpha_k^2}{k^2} \le \sum_k \alpha_k^2 = \|f\|.$$

Compactness: let $a_n = \sum_k \alpha_k e_k$ have norm ≤ 1. We can write

$$Ta_n = \sum_{k=1}^{\infty} \frac{\alpha_k e_{k+1}}{k} = \sum_{k=1}^{N} \frac{\alpha_k e_{k+1}}{k} + \sum_{k=N+1}^{\infty} \frac{\alpha_k e_{k+1}}{k}$$

The second term converges to zero in norm as $N \to \infty$, so for any m, we can choose N such that this term is less than $1/10m$, and then because the first term is finite dimensional, there exists a subsequence that converges in that term, and we can choose some n_i such that the distance between the first N terms of any two a_{n_j} with $j \ge i$ is also less than $1/10m$. Repeat, this time with the N-convergent subsequence, and index the resultant (sub)subsequence $\{a_m\}$. Clearly for $x, y \ge m$, $\|a_x - a_y\| \le \frac{1}{m} \to 0$.

No eigenvectors: suppose $\sum a_k e_k$ was an eigenvector, then there exists some nonzero coefficient a_k. Because $Tf = \sum \frac{\lambda \alpha_k k e_{k+1}}{k}$, $\frac{\lambda \alpha_{k-1}}{k-1} = a_k$, i.e. a_{k-1} is nonzero and by induction a_1 is nonzero. But the coefficient of e_1 in Tf is 0, so no eigenvectors can exist.

Problem 5 Suppose $\lambda_k \to 0$: we can show compactness by the same argument as in the previous problem. Write:

$$Tf_k = \sum_{k=1}^{N} \lambda_k \alpha_k e_k + \sum_{k=N+1}^{\infty} \lambda_k \alpha_k e_k,$$

and again pick some f_m from nested N-convergent subsequences.

Conversely, suppose λ_k doesn't converge to zero, i.e. $\exists \varepsilon$ such that for all N there exists $k \geq N$ such that $\lambda_k > \varepsilon$. Create from this a sequence K_N. Clearly $\{e_{K_N}\}$ have norm one but $\mathrm{Im}\{\lambda_{K_N} e_{K_N}\}$, which has no convergent subsequence as they are all orthogonal with norm $> \varepsilon$, i.e. are always at least $\sqrt{2}\varepsilon$ apart, by Pythagoras.

Problem 6 We first show an eigenvec