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Functional Analysis
Problem Set 2

Problem 1 Let $M = \sup\{|T(f,f)|,\ \|f\| = 1\}$. Clearly $M \leq \|T\|$. To show $M \geq \|T\|$, consider the polarization identity:

$$(Tf,g) = \frac{1}{4}[(T(f+g),f+g)-(T(f-g),f-g)+i(T(f+ig),f+ig)-i(T(f-ig),f-ig)].$$

Because $(Th,h) = (h,Th) = \overline{(Tf,f)}$, (Tf,f) is real, and thus:

$$\mathrm{Re}(Tf,g) = \frac{1}{4}[(T(f+g),f+g) - (T(f-g),f-g)].$$

By definition, $|(Th,h)| \leq M\|h\|^2$, so:

$$\begin{aligned}\mathrm{Re}(Tf,g) &\leq \frac{M}{4}(\|f+g\|^2 - \|f-g\|^2)\\ &\leq \frac{M}{2}(\|f\|^2 + \|g\|^2),\end{aligned}$$

by the parallelogram law. Letting $\|f\| = \|g\| = 1$, we see $|\mathrm{Re}(Tf,g)| \leq M$. Finally, substituting $e^{i\theta}g$ for g, we see that from the fact that 1) for some θ, $\mathrm{Im}(T(f,e^{i\theta}g)) = 0$, and 2) $|T(f,e^{i\theta}g)|$ is invariant under θ, we conclude that $(Tf,g) \leq M$ for f, g with norms less than 1, and thus $M = \|T\|$.

Problem 4 Boundedess: by Pythagoras,

$$\|Tf\|^2 = \|\sum_k \alpha_k \frac{e_{k+1}}{k}\|^2 = \sum_k \frac{\alpha_k^2}{k^2} \leq \sum_k \alpha_k^2 = \|f\|^2.$$

Compactness: let $a_n = \sum_k \alpha_k e_k$ have norm ≤ 1. We can write

$$Ta_n = \sum_{k=1}^{\infty} \frac{\alpha_k e_{k+1}}{k} = \sum_{k=1}^{N} \frac{\alpha_k e_{k+1}}{k} + \sum_{k=N+1}^{\infty} \frac{\alpha_k e_{k+1}}{k}$$

The second term converges to zero in norm as $N \to \infty$, so for any m, we can choose N such that this term is less than $1/10m$, and then because the first term is finite dimensional, there exists a subsequence that converges in that term, and we can choose some n_i such that the distance between the first N terms of any two a_{n_j} with $j \geq i$ is also less than $1/10m$. Repeat, this time with the N-convergent subsequence, and index the resultant (sub)subsequence $\{a_m\}$. Clearly for $x, y \geq m$, $\|a_x - a_y\| \leq \frac{1}{m} \to 0$.

No eigenvectors: suppose $\sum a_k e_k$ was an eigenvector, then there exists some nonzero coefficient a_k. Because $Tf = \sum \frac{\lambda \alpha_k k e_{k+1}}{k}$, $\frac{\lambda \alpha_{k-1}}{k-1} = a_k$, i.e. a_{k-1} is nonzero and by induction a_1 is nonzero. But the coefficient of e_1 in Tf is 0, so no eigenvectors can exist.

Problem 5 Suppose $\lambda_k \to 0$: we can show compactness by the same argument as in the previous problem. Write:

$$Tf_k = \sum_{k=1}^{N} \lambda_k \alpha_k e_k + \sum_{k=N+1}^{\infty} \lambda_k \alpha_k e_k,$$

and again pick some f_m from nested N-convergent subsequences. Note that the second term decreases in norm to zero exactly because $\lambda_n \to 0$, and the first sits in a compact set as before.

Conversely, suppose λ_k doesn't converge to zero, i.e. $\exists \varepsilon$ such that for all N there exists $k \geq N$ such that $\lambda_k > \varepsilon$. Create from this a sequence K_N. Clearly $\{e_{K_N}\}$ have norm one but the image of $\{\lambda_{K_N} e_{K_N}\}$ has no convergent subsequence as they are all orthogonal with norm $> \varepsilon$, i.e. are always at least $\sqrt{2}\varepsilon$ apart, by Pythagoras.

Problem 6

Problem 7 Proof by contradiction: consider the matrix form of T, with components:

$$a_{jk} = \int_{[0,1]^2} K(x, y) e^{2\pi i j y} e^{2\pi i k x} dx dy.$$

Because T is diagonalizable, this matrix A can be written BDB^{-1}, where D is a diagonal matrix of eigenvalues and B transitions the eigenvector basis $\{\phi_n\}$ to the fourier basis. Note that the columns of B and B^{-1} square sum to 1, as both the fourier basis and $\{\phi_n\}$ are bases with norm 1.

Problem 8

Problem 9

a) Because $RK = I - A$, we know that $\dim(\mathcal{N}(RK)) \leq \infty$. Clearly $\dim(\mathcal{N}(K)) \leq \dim(\mathcal{N}(RK)) \leq \infty$. Then suppose $\dim(\mathcal{N}(R)) = \infty$, then $\dim(K^{-1}(\mathcal{N}(R))) = \infty$, but that is a subset of $\mathcal{N}(RK)$, which we know to be finite dimensional. Thus both are of finite dimension.

b) If $f \in \mathcal{N}(K^*)^{\perp}$, $f \in \mathcal{R}(K)$ because K is compact, thus $K\phi = f$ has solutions, as does $RK\phi = Rf$.

c) Define $R^{-1} : \mathcal{H} \to \mathcal{R}(K)$ and $K^{-1} : \mathcal{R}(K) \to \mathcal{H}$ to be the right- and left-inverses of R and K in the following sense: R^{-1} maps any element in $\mathcal{H}$ to its preimage in $\mathcal{R}(K)$, and K^{-1} maps any element in the range of K to its preimage in $\mathcal{H}$. These functions are well defined because in order for RK to be bijective, R must be injective and surjective on $\mathcal{R}(K)$, and K injective on

$\mathcal{H}$. Clearly $S = K^{-1}R^{-1}$ is well defined on $\mathcal{H}$ and is the inverse to RK. In this case, $KSR = KK^{-1}R^{-1}R = I$, and thus $\mathcal{R}(I - KSR) = \{0\}$. This is trivially a subset of $\mathcal{N}(K)$. Likewise the adjoint of 0 is 0, so this is true for the adjoint case too. Finally, if $f \in \mathcal{N}(K^*)^\perp$, it is in the domain of SR as defined above ($SR = K^{-1}R^{-1}R = K^{-1}$, so its domain is $\mathcal{R}(K)$), and clearly $KSRf = KK^{-1}f = f$.