Lewis Ho
Functional Analysis
Practice Pset 4

Problem 1 This is an application of a theorem we proved in class. I proceed with the same proof. Let B be the closure of the ball in $c_0(\mathbb{N}$ with respect to the weak* topology. Suppose the statement is false: then there exists some $x_0 \in \ell^\infty(\mathbb{N}) \setminus B$. Because B is convex and closed, by Hahn-Banach there exists some $\ell \in (\ell^\infty, wk^*)^*$ such that $|\ell(x)| \leq \alpha < \alpha + \varepsilon \leq |\ell(x_0)|$, for all x in B. Note further that generally, $(X^{**}, wk^*)^* = X^*$ (WHY?). Let ℓ be norm 1. Because $sup(\ell(x)) = 1$, $\ell(x_0) = x_0(\ell) > 1$, violating our assumption that $\|x_0\| = 1$.

Problem 2 x_n^* converging weak* to some x^* is equivalent to saying $x^*(x) \to x^*(x)$ for all x, meaning that by the uniform boundedness principle x_n^* are norm bounded.

Further, note that X is isomorphic to a subset of X^{**}, and X^* is Banach if X is. Thus any weakly convergent sequence in X is represented by a weak*ly convergent sequence in X^{**}, and is hence norm bounded.

Problem 3 Suppose $x_i \to x$ weakly. This in particular means that for any x^*, for every ε x_i is eventually in the neighborhood $\{y \in X : |(y - x, x^*)| < \varepsilon\}$, as these are neighborhoods of x in the weak topology. Thus $(x_i, x^*) \to (x, x^*)$.

Conversely, suppose $(x_i, x^*) \to (x, x^*)$ for all x^*. Then given any weak neighborhood consisted of $\{(x_i^*, \varepsilon_i)\}$, because there are finite tuples per neighborhood, we simply choose N such that $(x_i, x_j^*) < \varepsilon_j$ for all $j > N$. The arguments are exactly the same for weak* convergence.