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Functional Analysis
Practice Pset 2

Problem 1

Problem 2 We show that the infimum of all M such that $|Tf| \leq M\|f\|$ for $\|f\| = 1$ is a norm on the space of all $T \in X^*$. Let $N(T) = \inf M$.

Positiveness follows from the definition. Likewise, $N(T) = 0$ clearly only if $Tf = 0$ for all f. Thus N is positive-definite. $N(\alpha T) = \alpha N(T)$ from the linearity of T. Finally, if $N(T) = M,\ N(R) = S$, $N(T + R) \leq M + S$, otherwise it would imply there exists some f such that $|Tf + Rf| \geq (M + N)\|f\|$, which would violate the definition of M or N.

Problem 3 Consider isosceles triangles of area 1 centered around 0.5. We can have arbitrarily high isosceles triangles by decreasing their width. We can define functions f_n as 0 outside the triangle of height n and taking value of their diagonals inside. Clearly $\|f_n\| = 1$, but $L(f_n) \to \infty$ as $n \to \infty$.

Problem 4 Suppose X is a Banach space. Let $a_i = \sum_{k=1}^{i} x_n$, and let $m > n$.

$$\|a_n - a_m\| = \| \sum_{k=m+1}^{n} x_k\| \leq \sum_{k=m+1}^{\infty} \|x_k\|$$

by the triangle inequality. Because $\sum_k \|x_k\|$ converges, the right hand side can be made arbitrarily small and thus a_i is Cauchy and converges, as X is a Banach space.

The converse: let a_n be a Cauchy sequence in X. Consider the sequence subsequence a_{n_k}, where k is chosen such that $\|a_{n_i} - a_{n_j}\| \leq \frac{1}{n^2}$, and then define $b_k = a_{n_{k+1}} - a_{n_k}$. By construction, $\sum_k \|b_k\| < \infty$, so by assumption $\sum_{n=1}^{\infty} b_n \in X$. But $\sum_n^N b_n = a_{n_k}$, so the limit of $a_n \in X$.

Problem 5 Because the domain of f in $f(\tau(x))$ is the range of $\tau \subseteq [0, 1]$,

$$\sup_{x\in[0,1]} |f(\tau(x))| \leq \sup_{x\in[0,1]} |f(x)|,$$

i.e. $\|Af\| \leq \|f\|$ for all f. There are also clearly continuous functions attaining their maximum in the range of τ, so $\|Af\| = \|f\|$ for some f. Thus $\|A\| = 1$.

Any injective τ will yield a surjective operator, as for any f there exists a continuous $\tau^{-1} \circ f$, where τ^{-1} maps the range of τ to $[0, 1]$, and $\tau\tau^{-1} = 1$. For injectivity, it is clear that τ must be surjective, else functions that differ outside its range will be mapped to the same functions. If $Af(x) = Ag(x)$, $f(\tau(x)) = g(\tau(x))$. Thus because τ is surjective, $Af(x) = Ag(x)$ for all x implies $f(x) = g(x)$ for all $x \in [0, 1]$.

Problem 6 Comparing the infinity and L^1 norms of some f, it is clear that for some $\|f\|_\infty = a$, the largest L^1 norm it can attain is when all its components are a, and the smallest when all its other components are zero. Thus $c = 1$ and $C = n$, where n is the dimension of X.