Lewis Ho
Functional Analysis
Problem Set 3

Problem 1 Any linear functional can be fully determined by its actions on a basis by linearity:

$$f\left(\sum_k a_k e_k\right) = \sum_k a_k f(e_k) \in \mathbb{R}$$

thus any f can be defined as a sequence $\{a_n\}$ where $f(e_k) = a_n$. If $\{a_n\} \in \ell^1(\mathbb{N})$, for any $\{b_n\} \in c_0(\mathbb{N})$, $|f_a(\sum_k b_k)| = |\sum_k a_k b_k| \leq \sup(|b_n|)\sum |a_k| = M\|\{b_n\}\|$. Note that we assume the norm defined on $c_0(\mathbb{N})$ is the supremum norm, which is well defined because convergent sequences are bounded. Hence $\ell^1(\mathbb{N}) \subseteq c_0(\mathbb{N})^*$.

Conversely, given some $\{a_n\}$ such that $\sum |a_n| = \infty$, we can find a sequence of indices $n_1,\ n_2, \ldots$ such that $\sum_{i=1}^{n_k} |a_i| \geq k^2$. Consider then the sequence $\{b_n\}$, with $b_i = |1/k|$ for $n_{k-1} < i \leq n_k$, and the sign of b_i the same as the sign of a_i. Then:

$$f_a(\{b_n\}) \geq \sum_{i=1}^{n_k} a_i b_i \geq \frac{1}{k}\sum_{i=1}^{n_k} |a_i| \geq k^2/k \to \infty.$$

Note that $\|\{b_n\}\| \leq 1$. Thus f is unbounded and $\ell^1(\mathbb{N}) \supseteq c_0(\mathbb{N})^*$.

Problem 2 Suppose not. Let $X = \ell^\infty(\mathbb{N})$ contain a countable subset $\{w_i\}$ that is dense. With Gram-Schmidt we obtain a orthonormal basis $\{v_i\}$ that is countable. Consider the map $U(v_i) = e_i$, where $\{e_i\}$ is the standard "basis"—sequences of zeros, except for 1 in the ith element. For any $f = \sum_i a_i v_i$,

$$\|U(f)\| = \|U(\sum_i a_i v_i)\| = \|\sum_i a_i U(v_i)\| = \|\sum_i a_i e_i\| = \sup_i\{a_i\} = \|f\|$$

U is clearly linear and bijective, and thus X is unitarily equivalent to the closure of the space spanned by $\{e_i\}$ with the same norm, say, Y.

But Y is not separable. Let $\{a_n\}$ be the sequence with $a_n = 1$ for all n. For any finite linear combination $\sum_i \alpha_i e_{k_i}$ of $\{e_i\}$,

$$\|\{a_n\} - \sum_i^N \alpha_i e_{k_i}\|_\infty \geq 1$$

as a_n has infinite terms that are 1 compared to the finite ones in finite linear combinations of $\{e_i\}$. Hence there is always some j where the jth term of $\{a_n\}$ is 1 and that of our linear combination is 0.

The fact that X is also cannot be separable follows from unitary equivalence. Suppose not: consider $U^{-1}(\{a_i\}) = b$. By separability, $\sum_i^n c_i v_i \to b$ as $n \to \infty$, i.e. for any given ε, we can find some N such that for $n \geq N$:

$$\|b - \sum_i^n c_i v_i\| < \varepsilon.$$

By the properties of U,

$$\|U(b - \sum_i^n c_i v_i)\| = \|U(b) - U(\sum_i^n c_i v_i)\| = \|\{a_n\} - \sum_i^N c_i e_i\| < \varepsilon$$

in which case linear combinations of $\{e_i\}$ converge to $\{a_i\}$, a contradiction. Thus we've shown that no subset of X can be dense.

Problem 3 It is a subspace by the fact that if $a_n \to a$ and $b_n \to b$, $\lim_{n\to\infty} a_n + b_n$ exists, and is in fact $a + b$ (this all follows from the continuity of addition $\mathbb{R}^2 \to \mathbb{R}$). Thus $\{a_n\} + \{b_n\} \in c$. Closure: let $\{b_n\}$ be the limit of a sequence of sequences $\{a_n\}_k$. For any $\varepsilon > 0$, there exists some $c_n = \{a_n\}_i$ such that $\sup |c_n - b_n| < \varepsilon/3$, and also some N, because c_n is convergent, such that $|a_m - a_n| < \varepsilon/3$ for all $m,\ n > N$. Thus for any $m,\ n > N$,

$$|b_n - b_m| \leq |b_n - c_n| + |b_m - c_m| + |c_n - c_m| < \varepsilon,$$

i.e. $\{b_n\}$ is Cauchy and thus in c as well. c is therefore closed.

Problem 5 $\ell(f) = f(x_0) \leq \sup(f) = \|f\|$ on $[0, 1]$. Additionally, there are continuous functions that attain their maximum on x_0, thus the inequality is sometimes an equality and $\|\ell\| = 1$.

Problem 6

(a) Let $f,\ g \in E_\alpha$. For all $\lambda \in \mathbb{R}$, $\lambda f(0) + (1-\lambda)g(0) = \lambda\alpha + (1-\lambda)\alpha = \alpha$, thus all convex combinations of f and g are clearly also in E_α.

Then, given any $f \in X$ and $\varepsilon > 0$, there exists some continuous function $g \in X$ such that $\|f - g\| < \varepsilon/2$, (given the density of the continuous functions in L^2). Then we can find another function $h \in E_\alpha$ such that $\|g - h\| < \varepsilon/2$, where $h = g$ in $[-1, 1] \setminus [-\delta, \delta]$, and h decreases/increases linearly to α at 0. We can find such a close h because we can, by decreasing δ, decrease the measure of the set on which h differs from g arbitrarily small, and this difference (squared) is bounded given the continuity of both functions and the compactness of our domain. Thus $\|f - h\| < \varepsilon$, i.e. E_α is dense.

(b) No function can attain the value of both α and β at 0. Further, because all functions we are considering are continuous, functions in E_α and E_β must differ in an area of nonzero measure, thus cannot belong to the same equivalence class in L^2. Thus both sets are disjoint. To show they cannot be separated, we first prove a lemma.

Lemma: if A is a dense set, there exist no nonzero bounded linear functionals such that $\ell(a) \leq C$. Proof: because ℓ is nonzero, there exists some a such that $\ell(a) \neq 0$. By linearity, we can scale a to find an element b such that $\ell(b) > C+\varepsilon$. Because A is dense, we can find some $f \in A$ such that $\|f-b\| < \varepsilon/M$, where M is the norm of ℓ. This means $\|\ell(f) - \ell(b)\| < \varepsilon$, and thus $\ell(f) > C$.

From the lemma it follows that because E_α is dense, $\{\ell(E_\alpha)\}$ has no supremum, and thus no such separation is possible. Geometric Hahn Banach is not possible because the sets are not compact.

Problem 7 Disjointness: a polynomial cannot both have a negative leading coefficient and all non-negative coefficients, thus they are disjoint. Convexity: the sum of two polynomials with negative leading coefficients has a negative leading coefficient, this being either the sum of both or just one of them; and the sum of two polynomials with non-negative coefficients has non-negative coefficients as well. Finally, multiplying polynomials with $0 < \lambda \leq 1$ doesn't change the sign of its coefficients, thus both A and B are convex.

Suppose there does exist some nonzero ℓ such that $\ell(a) \leq \ell(b)$, for all a in A and b in B, then by the completeness of $\mathbb{R}$, there exists a real C (e.g. the supremum of $\ell(a)$) such that $\ell(a) \leq C \leq \ell(b)$ for all a and b. Because $0 \in B$, by the linearity of ℓ, $C \leq \ell(0) = 0$. However, because for any monomial $a \in A$, $a/n \in A$ also, for $n > 0$, thus $\ell(a/n) = \ell(a)/n \to 0$ as $n \to \infty$. Thus $C \geq 0$, i.e. $C = 0$.

If ℓ is nonzero, there must be some monomial x^n such that $\ell(x^n) \neq 0$, as all polynomials are finite sums of monomials, and thus if they all mapped to 0, ℓ would be 0 on all of $\mathcal{P}$. Clearly the sign of $\ell(ax^n)$ is the same as the sign of a. Consider then the polynomial $p = -x^{n+1} + ax^n \in A$: because we can make a and thus $\ell(ax^n)$ arbitrarily large, $\ell(-x^{n+1})$ cannot be finite and yet keep $\ell(p) = \ell(-x^{n+1}) + \ell(ax^n)$ to remain ≤ 0 for all $a > 0$ i.e. no such linear functional can exist.

Problem 8 Let K_1 be the half space with $y \leq 0$, and let $K_2 = \{(x,y) \in \mathbb{R}^2 \mid x \geq 1,\ y \geq 1/x\}$. Both sets are convex: the half space is convex, and $\frac{1}{x}$ is a convex function, and truncating it like so doesn't hurt this convexity. Because the boundary of K_1 is the x axis, the only hyperplanes (lines) not intersecting K_1 must be parallel to the x axis and ≥ 0, i.e. $\ell(x,y) = ay$, where $a \geq 0$. However, the infimum for y where $(x,y) \in K_2$ is 0, thus:

$$\sup_{x \in K_1} \ell(x) = 0 = \inf_{y \in K_2} \ell(y),$$

with this construction. I.e. no such ℓ exists that satisfies our requirement.

Problem 9

(a) Let $a^* \in Z^*$ and $b \in X$. Because T and S are bounded, their adjoints T^* and S^* exist and are bounded. Note that for any $A : X \to Y$, $y^* \in Y^*$, $x \in X$, $(y^*, Tx)_Y = y^*Tx = T^*y^*x = (T^*y^*, x)_X$ by definition. Consider then $(z^*, STx)_Z$:

$$(z^*, STx)_Z = (S^*z^*, Tx)_Y = (T^*S^*z^*, x)_X,$$

i.e. $T^*S^*z^* = z^*ST$ for all $z^* \in Z^*$, which is the domain of both $(ST)^*$ and T^*S^*. This last equation is how we define $(ST)^*$, thus $(ST)^*z^* = T^*S^*z^*\ \forall z^*$, hence $(ST)^* = T^*S^*$.

(b) Again we need to show that $(aS^* + bT^*)y^* = y^*(aS + bT)$ for all y^* in Y^*, i.e. $(aS^* + bT^*)y^*(x) = y^*(aS + bT)(x)$ for all $y^* \in Y^*$, $x \in X$:

$$(aS^* + bT^*)y^*(x) = aS^*y^*(x) + bT^*y^*(x)$$

then by the definition of T^* and S^*:

$$= ay^*S(x) + by^*T(x) = y^*aS(x) + y^*bT(x) = y^*(aS + bT)(x)$$

by linearity.

(c) We show $(T^{-1})^* : X^* \to Y^*$ is the inverse map of $T^* : Y^* \to X^*$:

$$(T^{-1})^*T^*y^*(a) = (T^{-1})^*y^*T(a) = y^*T^{-1}T(x) = y^*(a)$$

likewise,

$$T^*(T^{-1})^*x^*(b) = T^*x^*T^{-1}(b) = x^*TT^{-1}(b) = x^*(b)$$

for all $a \in Y,\ b \in X,\ x^* \in X^*\ y^* \in Y^*$, by repeated application of the $A^*b^* = b^*A$ rule. Thus $(T^{-1})^* = (T^*)^{-1}$.