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Cooperative Game Theory
Problem Set 2

Problem 1

a) The core is the set of points in $\mathbb{R}^3$ satisfying:

$$x_1 + x_2 + x_3 = 1$$
$$x_1 \leq \frac{1}{3},\ x_2 \leq \frac{1}{3},\ x_3 \leq \frac{1}{2}$$

The shapley values are:

$$x_1^* = \frac{1}{6}(0 + 0 + \frac{1}{2} + \frac{2}{3} + \frac{1}{3} + \frac{1}{3}) = 0.30\bar{5}$$
$$x_2^* = \frac{1}{6}(0 + 0 + \frac{1}{2} + \frac{2}{3} + \frac{1}{3} + \frac{1}{3}) = 0.30\bar{5}$$
$$x_3^* = \frac{1}{6}(0 + 0 + \frac{2}{3} + \frac{2}{3} + \frac{1}{2} + \frac{1}{2}) = 0.3\bar{8}$$

b) The other values keep the following conditions the same:

$$x_1 + x_2 + x_3 = 1$$
$$x_1 \leq \frac{1}{3},\ x_2 \leq \frac{1}{3}$$

x gives us a upper bound on what we can offer 3. Clearly the most we can offer them, given our other upper bounds, is $1/3$. Therefore if $x \leq 2/3$, the core will be nonempty.

Problem 2 Because this game is symmetric, our requirement for y is that $y/3 \geq (3/4)/2$. Thus $y \geq 9/8$ produces a nonempty core.

Problem 3 Core $v = \{x \in \mathbb{R}^n_+ \ : \ \sum_{i \in T(v)} x_i = 1\}$ is equivalent to saying $x_i = 0$ for all $v \notin T(v)$, and that all combinations that satisfy this are in the core. To show this condition is necessary, suppose it is false, i.e we have an allocation where some $x_i \neq 0$ where $i \notin T(v)$. Because i is not in $T(v)$, there exists some coalition S where $v(S) = 1$. This coalition can block the allocation: we take i's share, and the share of anyone else not in $T(V)$, and distribute it amongst S. The share of $v(S)$ adds to one so it's feasible, and everyone is better than off in the new allocation. I.e. $x_i = 0$ for all $i \notin T(v)$ is a necessary condition.

To show sufficiency, suppose we have an allocation where only those in $T(v)$ have nonzero payoffs. Because of this fact, any blocking coalition must contain everyone with nonzero payoff, but in this case the coalition cannot distribute the payoff in a way that is better for everyone in the coalition: for someone to be better off, we must take away from someone with a nonzero payoff in the original allocation, but all such people are also in the coalition, so no such superior allocations exist for the coalition, and original allocation is stable.

Problem 4 I will assume this means that any coalition without the landlord produces 0 units of food.

For the landlord, note that we can account for all S without the landlord by counting all S of different sizes, as all workers are the same. For each s, there are 10 choose s coalitions of that size, and s ranges from 1 to 10 (we leave out zero because $f(0) = 0$). Finally, note that $v(S \cup L) - v(S) = f(s)$, where L is the landlord, as the workers by themselves produce nothing. Thus the Shapley value for L is:

$$\sum_{s=1}^{10} \binom{10}{s} \frac{s!(11-s-1)!}{11!} f(s) = \sum_{s=1}^{10} \frac{10!}{s!(10-s)!} \frac{s!(10-s)!}{11!} f(s) = \sum_{s=1}^{10} \frac{f(s)}{11}.$$

The other Shapley values can be calculated by efficiency and symmetry. If $\phi_L(v)$ is the Shapley value for the landlord calculated above, then $\phi_i(v)$ where $i \neq L$ is $(f(10) - \phi_L(v))/10$.

Problem 5 In the first game, a coalition wins a majority if and only if it contains the first player. Thus for $i = 1$, $v(S \cup i) - v(S) = 1$, and $= 0$ for all others. Thus the Shapley value $\phi_i(v) = 1$ for $i = 1$, and 0 otherwise. For the second, let's consider $\phi_1(v)$ first. Of all the different ways of ordering 1, 2, 3, 4 and 5, there are 24 orderings in which $v(S \cup i) - v(S) = 0$, namely those in which 1 comes last. It is equal to 1 in all other cases, and thus the Shapley value is:

$$\phi_1(v) = \frac{5! - 24}{5!} = 0.8$$

Then, by efficiency and symmetry, $\phi_i = 0.05$ for $i = 2,\ 3,\ 4,\ 5$.

Problem 6 For permanent members, the coalitions S for which $v(S \cup i) - v(S) = 1$ are those including all other permanent members and at least 4 non-permanent members, say k of them (this value is 0 for all other coalitions). For any k, there are 4 choose 4 times 10 choose k coalitions S of that specification. Thus the Shapley value for a security council member is:

$$\sum_{k=4}^{10} \binom{10}{k} \frac{(4+k)!(15-(4+k)-1)!}{15!},$$

which, with a little help from python, evaluates to 0.196. Then, using symmetry and efficiency, we infer that the Shapley value for non-permanent members is $(1 - 5 \cdot 0.195)/10 = 0.00186$.