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Econ 471
Problem set 1

Problem 1

(a) Let $X \to y$ denote X approaching y in that round of the algorithm. The generation of μ_M proceeds as follows:

  (i) $A \to a,\ B \to d,\ C \to d,\ D \to a,\ E \to a$, generating the incomplete matching: $\langle Aa, Bd \rangle$.
  (ii) $C \to c,\ D \to d,\ E \to b$, getting $\langle Aa, Cc, Dd, Eb \rangle$.
  (iii) $B \to b$, getting $\langle Aa, Bb, Cc, Dd \rangle$.
  (iv) $E \to d$ and getting rejected—that's the last person on his list, no changes occur and the algorithm terminates.

  Thus $\mu_M = \langle Aa, Bb, Cc, Dd, E \rangle$. μ_W goes:

  (i) $a \to B,\ b \to C,\ c \to E,\ d \to A$, getting $\langle Ad, Ba, Cb \rangle$.
  (ii) $c \to D$, getting $\langle Ad, Ba, Cb, Dc$, and thus all the women are matched.

  Hence $\mu_W = \langle Ad, Ba, Cb, Dc, E \rangle$.

(b) The procedure proceeds:

  (i) $A \to a,\ B \to d,\ C \to d,\ D \to a,\ E \to a$, generating the incomplete matching: $\langle Bd, Da \rangle$.
  (ii) $A \to b, C \to c,\ E \to b$, getting $\langle Ab, Bd, Cc, Da \rangle$.
  (iii) $E \to d$, getting $\langle Ab, Cc, Da, Ed \rangle$.
  (iv) $B \to c$, getting $\langle Ab, Bc, Da, Ed \rangle$.
  (v) $C \to a$, getting $\langle Ab, Bc, Ca, Ed \rangle$.
  (vi) $D \to d$, getting $\langle Ab, Bc, Ca, Dd \rangle$.
  (vii) E has no more people on his list so the procedure terminates.

  The matching is thus $\langle Ab, Bc, Ca, Dd, E \rangle$, which is preferable for a as she prefers B to A.

Problem 2

(a) As proved in class, the MPP and WPP yield the "boy-best" and "girl-best" matchings respectively. The MPP goes: (I've switched the order of men and women in the matching for convenience)

  (i) $U \to a,\ V \to b,\ W \to d,\ X \to c,\ Y \to a,\ Z \to d$, yielding $\langle aY, bV, cX, dZ \rangle$.
  (ii) $U \to b,\ W \to c$, yielding $\langle aY, bU, cW, dZ \rangle$.

(iii) $V \to d,\ X \to b$, yielding $\langle aY, bU, cW, dZ \rangle$.
(iv) $V \to a,\ X \to a$, yielding $\langle aV, bU, cW, dZ \rangle$.
(v) $X \to d,\ Y \to b$, yielding $\langle aV, bU, cW, dZ \rangle$.
(vi) $Y \to c$, yielding $\langle aV, bU, cY, dZ \rangle$.
(vii) $W \to b$, yielding $\langle aV, bW, cY, dZ \rangle$.
(viii) $U \to c$, yielding $\langle aV, bW, cY, dZ \rangle$.
(ix) $U \to d$, yielding $\langle aV, bW, cY, dZ \rangle$

Thus our final matching is $\langle aV, bW, cY, dZ, X, U \rangle$. For the WPP:

(i) $a \to V,\ b \to W,\ c \to V,\ d \to Z$ gives $\langle aV, bW, dZ \rangle$.
(ii) $c \to Z$ gives $\langle aV, bW, dZ \rangle$.
(iii) $c \to Y$ gives $\langle aV, bW, cY, dZ \rangle$, and as every woman is matched, the procedure finishes.

Thus the final "girl-best" matching is $\langle aV, bW, cY, dZ, U, X \rangle$, which is the same as the "boy-best" one.

(b) Using the same process:

(i) The first stage gives: $\langle Xabc, Yeg, Zhik \rangle$.
(ii) $d \to X,\ f \to Z,\ j \to Y \rangle$, thus $\rangle Xabcd, Yeg, Zhik \rangle$.
(iii) $f \to X, j \to X$, thus $\langle Xacdf, Yeg, Zhik \rangle$.
(iv) $b \to Y$, thus $\langle Xacdf, Yeg, Zhik \rangle$.
(v) $b \to Z$, thus $\langle Xacdf, Yeg, Zhik \rangle$, and the algorithm terminates.

Thus the final matching is $\langle Xacdf, Yeg, Zhik, b, j \rangle$.

Problem 3

(a) Suppose not: there exist A and B, both pointing at x. x either prefers A or B (say A), but then the matching in which Bx supposedly exists is unstable, as in that one x prefers A and A prefers x (as by assumption, A prefers x of the two wives from the two matchings).

(b) Let μ^* be the matching in (a), and A be a man in the profile. Any woman x who prefers A to her current husband is matched to that husband in a stable matching, and in that matching A prefers his wife to her, else it wouldn't be stable. A's current wife is his preferred one out of the two he's matched with in μ and $\tilde{\mu}$, so is preferred over x. Thus no man is preferred by a woman he doesn't prefer, and μ^* is stable.

(c) Suppose not: suppose in μ^* some woman x is matched with a husband she prefers out of the two; in other words, let's say μ^* contains the couple Ax, A preferring x to the other woman y out of the two matchings (by the definition of μ^*), and x preferring A over B, the other husband out of the two matchings. Then in the matching in which A and x aren't matched, they both prefer each other to their husband/wife, which violates our assumption that it's a stable matching.

Problem 4

Lemma: Problem 3 implies the uniqueness of men- and women-optimal matchings. Suppose not, and let μ_1 and μ_2 be men-optimal matchings. Then by the process in 3a, a new matching can be generated that is better than both for men—each man being paired with the preferred wife out of the two. If the final matching is the same as μ_1 or μ_2, it means that one is superior to the other, violating our assumption. The same argument can be made for women-optimal matchings.

Proof of statement: suppose not, that there exists some μ_2 differing from our men- and women-optimal matching μ_1; that means there exists some matching in μ_1, say Aa, that is not in μ_2. Who are A and a matched to in μ_2? By the optimality of μ_1, they must both be matched to worse people, otherwise μ_1 isn't uniquely optimal for men or women (at least one of them would be matched with a preferred person and be better off). But then μ_2 is unstable, as A and a prefer each other to their spouses. Thus μ_1 is unique.

Problem 5

(a) (i) The cycles for this procedure are (CFG), (A), (BE), (DH), in each step, giving us the allocation

$$\langle Aa, Be, Cf, Dh, Eb, Fg, Gc, Hd \rangle.$$

(ii) The cycles are (C), (E), (F), (D), (AB), (again, one in each step), and the final allocation is

$$\langle Ab, Ba, Cc, Dd, Ee, Ff \rangle.$$

(b) We want to show that no coalition can block a' on earth. A and C get their top choices, so they won't be part of any coalitions, as on earth participants join coalitions only if they're better off through them. Thus the only remaining possible coalition is $\{B, D\}$, which gives allocations $\langle Bb, Dd \rangle$ or $\langle Bd, Db \rangle$. But both are better off in a', so these coalitions also don't form. Thus a' is in the core.

(c) Clearly A and C are just as well off, and B and D prefer c and a better than a and c respectively, so a' is weakly dominated by the allocation giving traders their first choice.

(d) TTC gives the cycles (A), (C), (BD) in the first step, and terminates. As proved in class, the resultant allocation, $\langle Ab, Bc, Cd, Da \rangle$ is the only strictly stable solution and thus the only solution in the core.

The core consists of all allocations that wouldn't be blocked by earth given a' as a starting allocation; i.e. it's an allocation in which there is no coalition which in a' is strictly better off or in which a cycle through the coalition starting from a' leaves them strictly better off. Note, however, that because A and C have their ideal houses, the only allocation that weakly dominates a' is that given by the TTC (B and D can only swap houses, if they are not to make A or C worse off). Finally, a' itself isn't in the core, as B and D are better by swapping. Thus our core also consists of only the TTC allocation.