Dissolving a Partnership

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Introduction

John Harsanyi, in the 1959 paper "A bargaining model for the cooperative n-person game," proposed a procedure for dissolving a partnership in which the overall value of the partnership is more than the sum of what they each contributed to it. Given that each partner may have brought something of a different worth to the table, and different groups within the partnership may also have made their own collective contributions, the question of how the overall assets are to be distributed to each is a complicated one.

Harsanyi's approach rests on the intuition that each partner, or subgroup, should recoup what they contributed, and what remains should be distributed evenly to all. His procedure, however, appears to be a fairly inexact implementation of this idea: coalitions are chosen at random, their contributions are distributed evenly and subtracted from the remaining assets to be distributed, and this process continues until there is nothing left.

There is, in fact, intention behind this apparently haphazard approach: in fact the procedure works no matter how one decides to pick coalitions, and the outcome always coincides exactly with the ideal put forward, known in game theory as the Shapley value of the game. In this paper I describe and demonstrate the procedure and prove that the outcome coincides with the Shapley value of the partnership.

The Procedure

I formalize the notion of a partnership and its contributors as follows: let $N = \{1, 2, \ldots, n\}$ be partners, and let $v(S)$ with $S \subseteq N$ be the worth or contribution of all members of S collectively. This means, for example, that

$v(1,2)$ is the sum of all the contributions of 1 and 2 as individuals, and of their contribution as a pair. Thus $v(N)$ is the total worth of the partnership being dissolved, and is the amount that must be divided amongst the partners. Note that it is possible for the contributions of a collective to be negative, or less than the sum of its parts, because the collective may have made a poor decision, so both $v(A)+v(B) \leq v(A \cup B)$ and $v(A)+v(B) \geq v(A \cup C)$ are possible. Finally, note that v is a vector in $\mathbb{R}^{P(N)}$, where $P(N)$ is the power set of N (we exclude the null set from $P(N)$ for convenience here), and that other "games," i.e. other situations, can be obtained for the same set of partners with other vectors $v' \in \mathbb{R}^{P(N)}$.

The procedure works as follows: any coalition $S \subset N$ is chosen, $v(S)$ is divided equally among the members of S, and $v(S)$ is subtracted from all components of v containing S to yield some v_2.[1] This procedure repeats, with $v_2, v_3 \ldots v_k$ replacing v and coalitions S with $v_i(S) \neq 0$ selected until some $v_k = 0$ is reached.

Example I dissolve the partnership represented by:

S	1	2	3	1,2	1,3	2,3	1,2,3
v(S)	30	12	6	36	36	30	90

I use the same notation as in the reading: on the left I indicate the chosen coalition ("claimant") leading to the game on that line (this is a slight deviation from the reading, but I think it makes more intuitive sense), in the middle the value of each successive game, and on the right how much each partner gets from each distribution.

Claimant	$v_i(1)$	$v_i(2)$	$v_i(3)$	$v_i(1,2)$	$v_i(1,3)$	$v_i(2,3)$	$v_i(1,2,3)$	1	2	3
	30	12	6	36	36	30	90			
1	0	12	6	6	6	30	60	30	0	0
2	0	0	6	-6	6	18	48	0	12	0
3	0	0	0	-6	0	12	42	0	0	6
2,3	0	0	0	-6	0	0	30	0	6	6
1,2	0	0	0	0	0	0	36	-3	-3	0
1,2,3	0	0	0	0	0	0	0	12	12	12
Total								39	27	24

[1] As in $v_2(R) = v(R) - v(S)$ if $S \subseteq R$, and $v(R)$ otherwise.

And with a different order of coalitions chosen:

Claimant	v_i(1)	v_i(2)	v_i(3)	v_i(1,2)	v_i(1,3)	v_i(2,3)	v_i(1,2,3)	1	2	3
	30	12	6	36	36	30	90			
2	30	0	6	24	36	18	78	0	12	0
3	30	0	0	24	30	18	72	0	0	6
1	0	0	0	6	0	12	42	30	0	0
1,2	0	0	0	0	0	12	48	-3	-3	0
2,3	0	0	0	0	0	0	36	0	6	6
1,2,3	0	0	0	0	0	0	0	12	12	12
Total								39	27	24

I proceed to prove two propositions about the procedure.

Termination

I first show that the procedure terminates regardless of which coalitions are chosen at each step. The proof proceeds by contradiction. Suppose the statement is false, then there exists an infinite sequence of coalitions $\{S_n\}_{n=1}^{\infty}$ such that $v_n(S_n) \neq 0$, with (for convenience we let $v_1 = v$, the original game)—as the procedure terminates when no coalition with nonzero value exists.

As $|P(N)|$ is finite, there must be coalitions appearing infinitely many times in $\{S_n\}$, and there must be a smallest (though not necessarily strictly smallest) coalition appearing infinitely many times. Note then two facts:

1. During some round i of a procedure, for all $S \subseteq N$ with $|S| \leq |S_i|$ and $S \neq S_i$, $v_{i+1}(S) = v_i(S)$, where S_i is the chosen coalition for that round. This is because given those conditions $S_i \not\subseteq S$, thus $v_{i+1}(S) = v_i(S)$ by the definition of v_{i+1}.

2. If S_i is the chosen coalition in round i, $v_{i+1}(S_i) = 0$.

Thus if S is the smallest coalition that repeats infinitely, then there is some point in $\{S_n\}$ in which all the coalitions subsequently chosen are either the same size or greater than S. Thus the next time S is chosen, say in turn i, $v_{i+1}(S) = 0$, and because each subsequent S_k is of equal size or larger than S, $v_k(S)$ remains as 0 for all $k > i$, and thus cannot be chosen again, contradicting the assumption that S repeats infinitely. Thus our assumption cannot be true, i.e. our procedure must always terminate.

Uniqueness and the Shapley Value

Another useful concept for considering the distribution of value across a collective is the Shapley value, a quantity that allocates amongst members of a group each individual/coalition's contribution and then divides the remainder evenly. This is the ideal distribution that realizes Harsanyi's intuition. In fact, I show in this section that the outcome of the procedure is the same regardless of which coalition is chosen at any stage, and that the outcome is always the Shapley value.

Example To first demonstrate this correspondence, I shall calculate the Shapley value for each partner in the example given above.

For partner 1:

$$\begin{aligned}\phi_1(v) &= \sum_{S \subseteq N \setminus \{1\}} \frac{|S|!(3-|S|-1)!}{3!}(v(S \cup \{1\}) - v(S)) \\ &= \frac{2 \cdot 30}{6} + \frac{36-12}{6} + \frac{36-6}{6} + \frac{2(90-30)}{6} = 39.\end{aligned}$$

For partner 2:

$$\begin{aligned}\phi_1(v) &= \sum_{S \subseteq N \setminus \{2\}} \frac{|S|!(3-|S|-1)!}{3!}(v(S \cup \{2\}) - v(S)) \\ &= \frac{2 \cdot 12}{6} + \frac{36-30}{6} + \frac{30-6}{6} + \frac{2(90-36)}{6} = 27.\end{aligned}$$

And finally because $\sum_i \phi_i(v) = v(N)$, $\phi_3(v) = 24$, which is what we got from the procedure above.

Proof I now show that the procedure always gives each partner their Shapley value. Let $u_S \in \mathbb{R}^{P(N)}$ be a vector/game with components $(u_S)_T = 1$ if $S \subseteq T$ and 0 otherwise. Note that at each step i, we generate our new game v_{i+1} by subtracting $u_{S_i} \cdot v_i(S_i)$ from v_i. With this we can express the games

$\{v_i\}$ resulting from each stage of the procedure recursively:

$$
\begin{aligned}
v_2 &= v_1 - u_{S_1} \cdot v_1(S_1) \\
v_3 &= v_2 - u_{S_2} \cdot v_2(S_2) = v_1 - u_{S_1} \cdot v_1(S_1) - u_{S_2} \cdot v_2(S_2) \\
&\vdots \\
v_k &= v_{k-1} - u_{S_{k-1}} \cdot v_{k-1}(S_{k-1}) = v_1 - \sum_{i=1}^{k-1} u_{S_i} \cdot v_i(S_i).
\end{aligned}
$$

Because the procedure terminates after some turn $k-1$, we know the resultant game $v_k = 0$. Thus, rearranging our last equation, we get

$$\sum_{i=1}^{k-1} u_{S_i} \cdot v_i(S_i) = v_1 = v.$$

By the additivity of the Shapley value, we have:

$$\phi(v) = \phi\left(\sum_{i=1}^{k-1} u_{S_i} \cdot v_i(S_i)\right) = \sum_{i=1}^{k-1} \phi(u_{S_i}) \cdot v_i(S_i). \tag{1}$$

To complete the proof, I write the allocations from each round in a similar manner. In each turn i, if S_i is the chosen coalition, each player j gets $v_i(S_i)/|S_i|$ if they are in S_i and nothing otherwise. Note then that in the game u_{S_i}, all $j \notin S_i$ are dummies and the rest are symmetric, and thus $\phi(u_{S_i})_j = 1/|S_i|$ for $j \in S_i$ and zero otherwise. Putting the two together, the allocation for each player j in round i can be written as $\phi(u_{S_i})_j \cdot v_i(S_i)$. Over the course of the $k-1$ turns of the procedure, each player j thus receives

$$\sum_{i=1}^{k-1} \phi(u_{S_i})_j \cdot v_i(S_i) = \left(\sum_{i=1}^{k-1} \phi(u_{S_i}) \cdot v_i(S_i)\right)_j = \phi(v)_j$$

by equation (1). In other words each player receives his Shapley value, thus completing the proof.

Conclusion

The procedure for dissolving a partnership examined above appears to be a haphazard implementation of the idea that each individual/coalition should receive what they put in, and the surplus allocated evenly to those who have a claim to it. However, from the examination above it is clear that this seemingly random process is actually an exact implementation of that intuition in that it always terminates no matter how it is used, and it always delivers the desired outcome, namely the Shapley value of the partnership.