Error of a Maximum Likelihood Detector for a Bipolar Signal Over an AWGN Channel

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This paper explores the error a maximum likelihood detector incurs when extrapolating symbols passed through a AWGN channel. This simulation is done by recreating the a ML detector and using the Monte Carlo Method in MATLAB. These results are then compared to the theoretical results using the Q-function.

I. Introduction

This simulation will show the probability error that a Maximum Likelihood Detector would incur when implemented with a transmitter transmitting a bipolar signal across a AWGN channel. This detector will be shown to act as if it were randomly guessing when the SNR is near 0, and more certain when the SNR is large.

II. Implementation

A. Algorithm

The algorithm used is the Monte Carlo method. The Monte Carlo method is an algorithm that focuses on independent random sampling to solve deterministic problems. The transmitted symbols and channel noise is randomly sampled and added together which represents the received signal at the detector. The Maximum Likelihood Detector will then classify the received signal. The process above takes place N times, where the algorithm will count how many times a received signal was misclassified, per signal to noise ratio, for a range of signal to noise ratios. This algorithm will tell us the probability of error given a signal to noise ratio. (come back and check)

B. Transmitter

The transmitter transmits a bipolar signal, meaning the signal is represented by the alphabet or set of possible symbols $S = \{-1, 1\}$, where the symbol transmitted is either -1, or 1.

C. Channel

The channel is an AWGN (Additive White Gaussian Noise) channel. AWGN noise has constant power spectral density, can be represented as a Gaussian random variable, $n \sim \mathcal{N}(0, N)$, and is added to the transmitted signal,

$$y = x + n$$

where x is the transmitted symbol and n is the noise. If the SNR is low, meaning that the noise is louder than the signal, then the received signal may look like another symbol altogether.

D. Detector

A maximum likelihood detector assumes both transmit symbols to be equally likely,

$$P(X = 1) = P(x = -1)$$

what determines which symbol is more likely is the distance of the received symbol from possible transmitted symbols. The received symbol y, is most likely to be the transmitted signal that it is closest to when using the euclidean distance metric. Furthermore, the probability of error is the probability that the transmitted symbol is decoded as the wrong symbol from the addition of noise shifting it closer to the other transmit symbol.

$$P(Error) = P(X = 1)P(Y < 0|X = 1)$$
$$+P(X = -1)P(Y >= 0|X = -1)$$

Furthermore,

$$P(Y|X = -1) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(y+1)^2}$$

$$ln(P(Y|X = -1)) = \frac{-1}{2\sigma^2}(y + 1)^2$$

$$P(Y|X = 1) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(y-1)^2}$$

$$ln(P(Y|X = 1)) = \frac{-1}{2\sigma^2}(y - 1)^2$$

$(y \pm 1)^2$ is the euclidean distance formula. The value of y that brings the euclidean distance closer to the value of 0 will maximize the probability of the the conditional density.

III. Solution

The Monte Carlo simulation showed that when the SNR approached 0, that the probability of error was 50%. In other words, when there is too much noise in the system the likelihood of the detector extrapolating a correct symbol is equivalent to randomly guessing. This is shown in figure 2, and figure 3. Note: $-\infty < log_{10}(P_{error}) <= 0$ when probability of error is certain, the log probability of error is 0, and when error is impossible, log probability of error is $-\infty$. Analyzing figure 3, it is shown that when the SNR is increasing, or the signal is much greater than the noise, that the probability of error is small. As opposed to, when the SNR is decreasing, it is shown that the probability of error approaches $log_{10}(\frac{1}{2})$, which states that when there is too much noise, the decoder

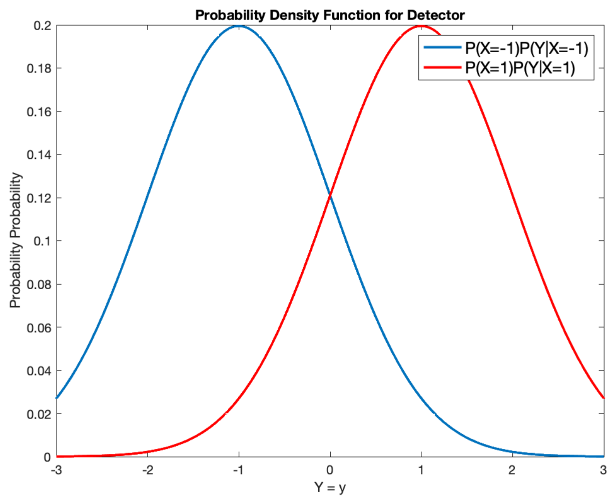

Fig. 1. Conditional Density Function for Detector: This shows the likelihood of a correct decision by the detector depending on the value that y takes.

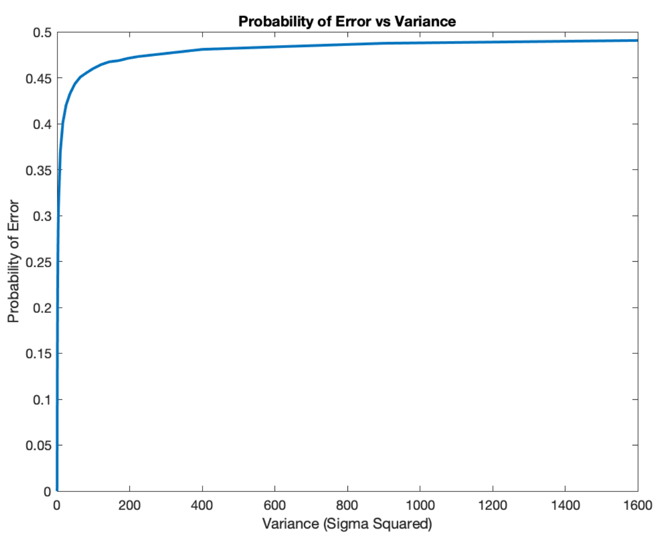

Fig. 2. Probability of Error vs Variance shows that the more noisy the channel the more the detector will behave as it is guessing.

is as accurate as a random guess. Furthermore, figure 4 is a theoretical approach using the Q function, while figure 3 is evaluating simulated data, both the experimental data and theoretical conclusions line up.

IV. Conclusion

$$P_b = Q(\sqrt{\frac{E_b}{N_0/2}}), Q(a) = \int_a^\infty e^{-x^2},$$

A. In Conclusion

It is shown as the Noise increases relative to the Signal, the uncertainty increases as well, and the detector starts to behave in a manner that has no better accuracy than randomly guessing the symbol. So, it is ideal to utilize a channel with a larger signal to noise ratio, in order to combat the entropy due to noise.

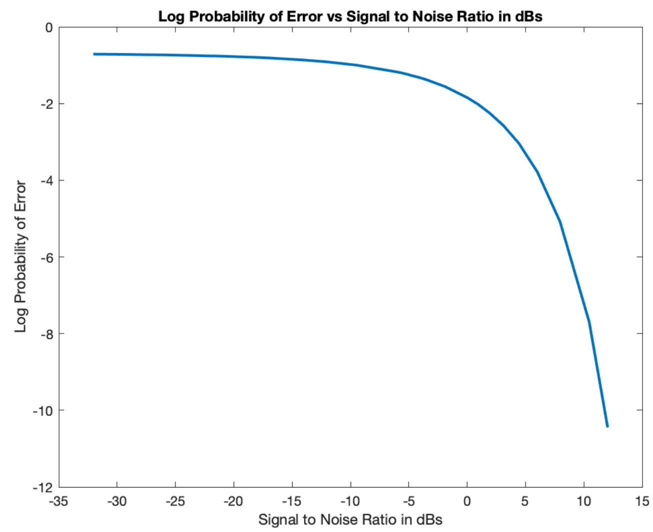

Fig. 3. Log Probability vs Signal to Noise Ratio in dBs shows the direction the probability takes when faced with a high SNR vs a low SNR

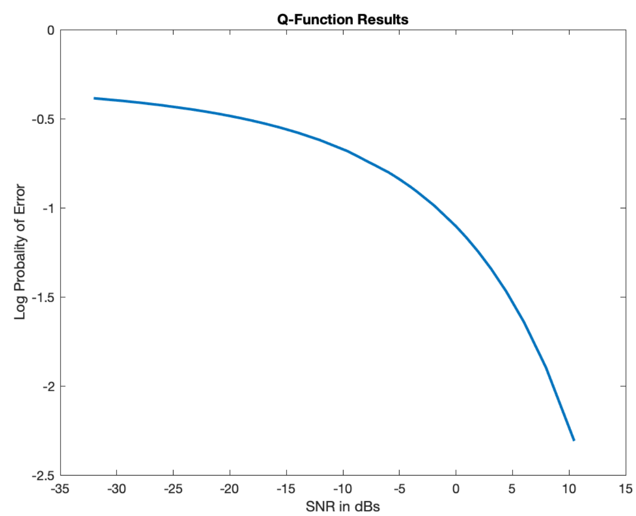

Fig. 4. Log Probability vs Signal to Noise Ratio in dBs using the Q-Function shows the direction the probability takes when faced with a high SNR vs a low SNR

References

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[2] Proakis, John G. Digital Communications. 4th ed.

Appendix

$$P_{error} = \frac{1}{2}[P(err|X=-1) + P(err|X=1)]$$

$$Power_B = \frac{A^2}{2}$$

$$P(err|X=A) = \frac{1}{N_o\sqrt{2\pi}}\int_A^\infty e^{-(x-A)^2/2N_o^2}$$

$$P_{error} = \frac{1}{2}[\frac{1}{N_o\sqrt{2\pi}}\int_A^\infty e^{-(x-A)^2/2_o^2} + \frac{1}{N_o\sqrt{2\pi}}\int_A^\infty e^{-(x+A)^2/2N_o^2}]$$

$$P_{error} = \frac{1}{N_o\sqrt{2\pi}} \int_A^\infty e^{-x^2/2N_o^2}$$

$$P_{error} = Q(\sqrt{\frac{E_g}{N_0/2}})$$