# Error of a ML Detector for a M-ary PSK Modulated Signal Over an AWGN Channel

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***Abstract*—This paper compares the symbol and bit error probabilities for M-ary Phase-Shift Keying (M-PSK) modulation obtained through the use of a Monte Carlo simulation with the theoretical values.**

***Index Terms*—M-PAM, BPSK, QPSK, 8-PSK, Bit Error Probability.**

## I. Introduction

This simulation will show the symbol and bit error probabilities that a Maximum Likelihood Detector would incur when demodulating M-PSK (BPSK, QPSK, and 8PSK) symbols that have been transmitted through an AWGN channel. The accepted theoretical bit and symbol error probabilities of M-ary PSKs for varying SNRs will validate the results generated in this simulation.

## II. Implementation

### A. Monte Carlo

The simulation is performed by utilizing the Monte Carlo method. The Monte Carlo method focuses on independent random sampling to solve deterministic problems. In this simulation, the bit sequence to be transmitted is generated through independent random sampling, this randomization allows for the imitation of the stochastic process that is, what the information source is feeding to the modulator.

### B. Bit Generator

The length of the sequence determines the type of modulation used, the relationship between sequence length L and number of symbols M is $M = 2^L$, meaning if the sequence length were 2 bits long then the number of symbols that need to be generated to represent these sequences is 8 symbols. Moreover, the bit generator was utilized in the simulator as it randomly generated the bit sequence to map to a symbol, which said symbol was later corrupted by the channel, and interpreted by a ML detector. This will be discussed in the next sections.

### C. Modulator and Modulation Scheme

Phase Shift Keying or PSK is a two dimensional modulation scheme that utilizes a phase offset as a method of fitting more information on the medium. The $m^{th}$ signal waveform, out of M waveforms, can be represented as,

$$s_m(t) = g(t)cos[2\pi f_c t + \frac{2\pi(m-1)}{M}]$$

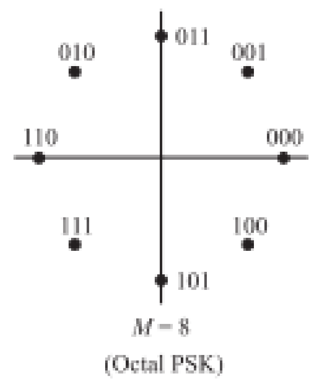


Fig. 1. Gray Encoding for a Octal-PSK Modulation Scheme

where the possible phases are represented by,

$$\theta_m = 2\pi\frac{(m-1)}{M}$$

Phase Shift Keying consists of equal energy symbols with a phase offset determined by the number of symbols as shown above. The mapping of bit sequences to symbols is done in such a way that the most likely symbol errors will result in only one bit error, referred to as Gray Encoding. This is shown in figure 1 [1].

Modulation was achieved in the simulation by representing every generated symbol as a two-dimensional vector, one component being real, the other imaginary. Each symbols' energy was kept to be unit energy, therefore all symbols lie on a circle of radius 1 as shown in figure 1. An alphabet of symbols whose size was determined from the length of the bit sequence was created and it defined the mapping from bit sequence to symbol. The relationship from sequence to symbol was established using Gray Encoding, as stated above. In the simulation, each symbol received it's sequence based off of the closest symbol's sequence to insure the hamming distance between those two symbols is 1, minimizing bit error probability.

### D. Channel

The channel is an AWGN (Additive White Gaussian Noise) channel. AWGN noise has constant power spectral density, can be represented as a Gaussian random variable, $n \sim N(0, N_0)$, and is added to the transmitted signal,

$$\bar{y} = \bar{x} + \bar{n}$$

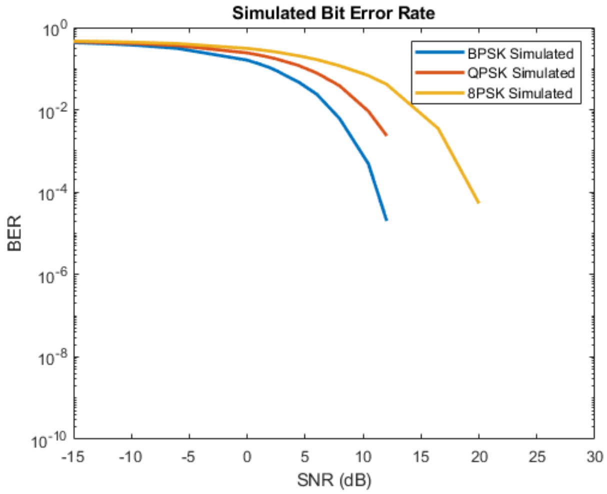


Fig. 2. Bit Error Probability of Simulated Results

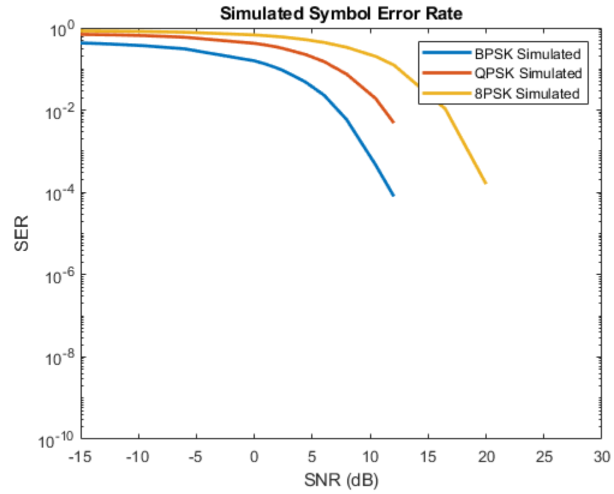


Fig. 3. Symbol Error Probability of Simulated Results

where x is the transmitted symbol and $\bar{n}$ is the noise vector comprising of real and imaginary noise components. If the Signal to Noise Ratio (SNR) is small (the noise is louder than the signal) then the received symbol may be mistaken for another symbol.

### *E. Maximum Likelihood Detector*

The maximum likelihood detector receives the attenuated and rotated symbol from the channel, and finds $\bar{x}$, the most likely symbol, by calculating the euclidean distance, $d_{mn}$, from itself to all other symbols. The symbol that has the smallest distance is assumed to be the most likely transmitted symbol, and the associated bit sequence for said symbol $\bar{x}$ is also assumed to be the most likely bit sequence.

This is implemented by taking the received symbol that has been attenuated and rotated by the channel, and iterating through the alphabet of symbols. Once again, note that this alphabet of symbols holds the relationship between the symbol and the sequence, so if given a sequence the symbol can be found, and vice versa. At each iteration the distance between what was received and the symbol being considered is calculated. At the end of this process there will be a minimum distance, as well as a most likely symbol, and a most likely sequence returned.

## III. Results

The following figures show the relationship between signal to noise ratio to bit error rate and symbol error rate for both the simulated and theoretical results.

The theoretical results follow these equations respectively, firstly, for symbol error rate:

$$P(Error)_{symbol} = 2Q(\sqrt{2log_2(M)\rho_b}sin(\frac{\pi}{M}))$$

Secondly, Bit Error Rate:

$$P(Error)_bit = \frac{1}{k}P(Error)_{symbol}$$

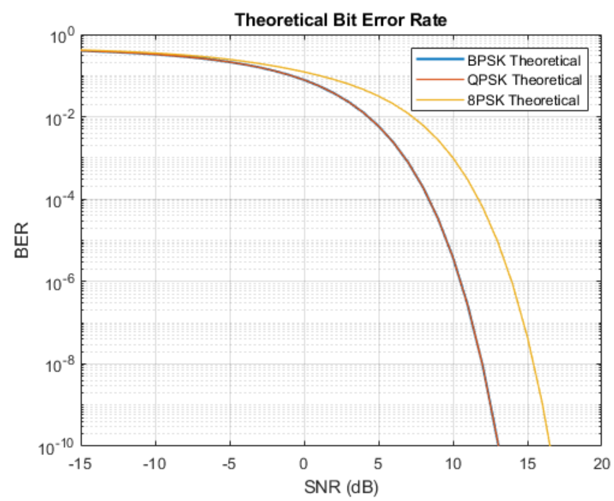


Fig. 4. Bit Error Probability of Theoretical Results

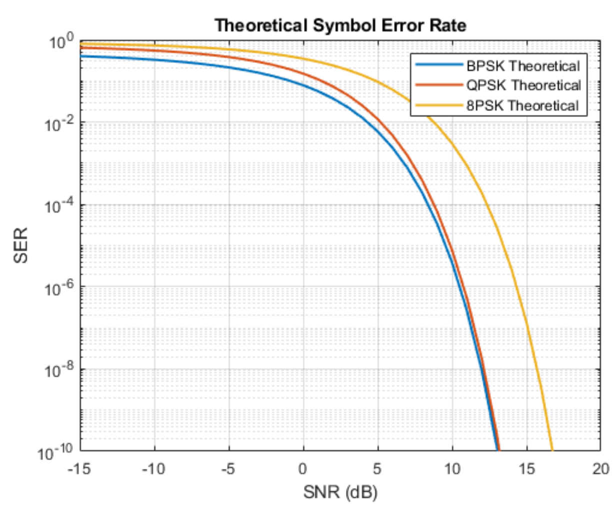


Fig. 5. Symbol Error Probability of Theoretical Results

Finally, comparing the corresponding graphs, figure 2 with figure 4, and figure 3 with figure 5, simulated and theoretical respectively. It is shown that the algorithm did indeed correctly approximate the theoretical within reasonable error for both the bit error rate and the symbol error rate. Note that the equations above are lower bounds, and the equations approximated show very near probabilities of error. As M and SNR increase so does probability of error.

## IV. Conclusion

In conclusion, the results shown above are a satisfactory approximation of the theoretical results that have been widely accepted. Monte Carlo simulation is an effective way of approximating deterministic processes through independent random sampling, for various applications especially those regarding calculating probability of error given modulation schemes.

## Appendix A
## Theoretical Bit Error Probability of M-PAM

Firstly, find the probability of symbol error:

The probability of symbol error is the probability of transmitting a symbol $m_{i,sent}$ and not receiving said symbol. Given that $m_{i,sent}$ was sent, the detector detecting anything but $m_{i,sent}$ is erroneous.

$$P(SymbolError) = \frac{1}{M}\sum_{m=1}^{M} P(error|m_{i,sent})$$

The detector compares the output with the M-1 boundary regions or thresholds located between the symbols. Therefore, the average amount of error the detector will incur is the probability that the noise added to the symbol will exceed its the decision boundary. Note that there are two different types of symbols in these constellations, a small portion that holds one directional error which are the two ”outside” symbols, and the majority that holds bi-directional error, which are the ”inside” symbols.

$$d_{min} = \sqrt{\frac{12log_2(M)\epsilon_{avg}}{M^2-1}}$$

$$P(error|m_{i,sent})_{inner} = 2Q(\frac{d_{min}}{\sqrt{2N_0}})$$

$$P(error|m_{i,sent})_{outer} = Q(\frac{d_{min}}{\sqrt{2N_0}})$$

$$P(SymbolError) = \frac{2(M-1)}{M}Q(\frac{d_{min}}{\sqrt{2N_0}})$$

For Gray Encoding:

$$P(BitError) = \frac{2(M-1)}{Mlog_2(M)}Q(\frac{d_{min}}{\sqrt{2N_0}})$$

The probability of bit error is an lower bounding on the bit error probability when gray encoding is used. This is due to the fact that the most likely errors which are symbols next to each other will results in a hamming distance of 1. Due to the Gray encoding, the error of a symbol is associated with that of a single bit error.

## References

[1] John Proakis, *Digital Communications*, 5th ed. New York, United States of America: McGraw-Hill Education, 2000, pp 254-300.

[2] John Proakis, Masoud Salehi *Communication Systems Engineering*, 2nd ed. New Jersey, United States of America: Prentice Hall, 2001, pp 254-300.