A GENTLE INTRODUCTION TO MODULAR FORMS

LEWIS COMBES

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1. Introduction

These notes aim to be a gentle introduction to modular forms, specifically for those intending to apply for the UGRI project `Computing eta expressions attached to modular forms`. Most introductory texts on modular forms contain a great deal of detail, which is good for those looking to build a theoretical backbone in the subject, but less helpful for those looking to start learning through computation. To that end, we will explain the important points, but leave much of the theory on the cutting room floor. While it is fruitful to understand this theory, it is not practical on the timescale of the project. It is simpler to take much of it as a "black box" that we are confident others have verified, but need not verify for ourselves. No originality is claimed for any of the mathematics in these notes. Indeed, we crib regularly from a variety of sources, most notably [DS16], but also often simply from Wikipedia. Mistakes are my own, comments and corrections are always welcome at `lmcombes1(at)sheffield(dot)ac(dot)uk`.

2. Modular forms—a first look

At its most basic, a modular form is a function of a complex variable z. In particular, it is a function on values of z that lie in the **upper half-plane** in $\mathbb{C}$. This space is denoted by $\mathbb{H}$, and is defined as

$$\mathbb{H} = \{a + bi \in \mathbb{C} \mid b > 0\}.$$

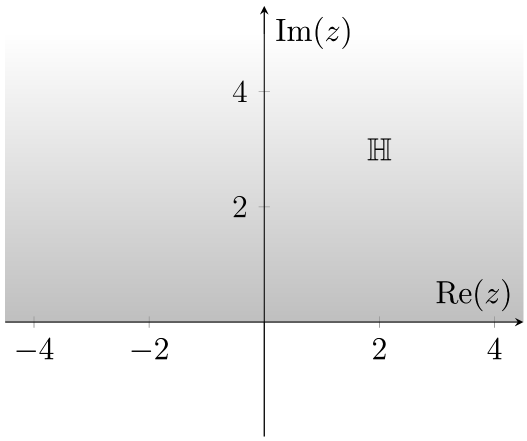


Figure 1. The upper half-plane

To justify this rather peculiar choice, we introduce the **Möbius action of** $\mathrm{SL}_2(\mathbb{Z})$ **on** $\mathbb{C}$, which takes a matrix

$$\gamma \in \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

and a point $z \in \mathbb{C}$, and returns the point

$$\gamma \cdot z = \frac{az + b}{cz + d}.$$

Perhaps this only further moves the justification for choosing $\mathbb{H}$ as the choice of domain another step, but exactly why the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane is interesting is a large can of worms best left for the moment. Assuming that we are happy to believe it *is* important to *somebody*, it gives us a reason to care about $\mathbb{H}$ specifically.

That reason is that $\mathbb{H}$ is fixed under this action of $\mathrm{SL}_2(\mathbb{Z})$. Or phrased another way, if $z \in \mathbb{H}$, then $\gamma \cdot z \in \mathbb{H}$ also. This is quite easy to show: just multiply the top and bottom of $\gamma \cdot z$ by $\overline{cz+d}$ and rearrange to get

$$\mathrm{Im}(\gamma \cdot z) = \frac{(ad-bc)\mathrm{Im}(z)}{|cz+d|}.$$

Since $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the expression $ad-bc$ is 1, and so the imaginary part of $\gamma \cdot z$ is larger than 0 exactly when the imaginary part of z is.

Now we have enough setup to give the first definition of a modular form.

Definition 2.1 (Modular forms, v.1)**.** A modular form of **weight** k is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying

(1) The function f is holomorphic on $\mathbb{H}$.
(2) For any $z \in \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the function f satisfies

$$f(\gamma \cdot z) = (cz+d)^k f(z).$$

(3) The values of $f(z)$ are bounded as $z \to i\infty$.

Let's unpack this piece by piece. The word "holomorphic" just means "complex differentiable". The notion of the derivative extends from functions defined over $\mathbb{R}$ to functions defined over $\mathbb{C}$, and all the usual customers (sufficiently generalised) are still differentiable there: $f(z) = z,\ z^n,\ \cos(z),\ \sin(z),\ \exp(z)$ are all holomorphic, and their derivatives are exactly what you expect. Condition 1 can be thought of as a "niceness" condition. It just means we're not going to end up thinking about any functions with unpleasant properties. Everything is nice and smooth, and we do as much calculus to it as we'd like.

The second condition tells us modular forms transform in a very special way. They are "almost invariant" with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, only differing by the factor $(cz+d)^k$, where k is the weight of the modular form. We have once again come across a seemingly-arbitrary choice, and while there *is* a good reason for this particular choice, it can still be confusing. Why not $(cz+d)^{-k}$?

Or $(az+b)^k$? The particulars of the answer to this question are interesting and worth thinking about, but for the moment it's best to take this choice as a given. It *does* lead to interesting mathematics, and for the moment that will do.

The third condition, like the first, is also a "niceness" condition. It just tells us that values of a modular form shouldn't be able to get too big.

Conditions 1 and 3 are reasonable conditions to impose on *any* complex function one might want to study, as they lead to a function having a lot of nice complex-analytic properties. The point of interest, then, is condition 2, which is best interpreted as telling us that modular forms encode information about the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

3. Some basic properties of modular forms

The first thing one should do after writing down a definition is to ask if the object defined actually exists. This is not an unreasonable request when it comes to modular forms, as a first-time viewer has almost no intuition as to what the properties might imply about a function. It could be that the chosen properties are so restrictive as to preclude the existence of an object that satisfies them.

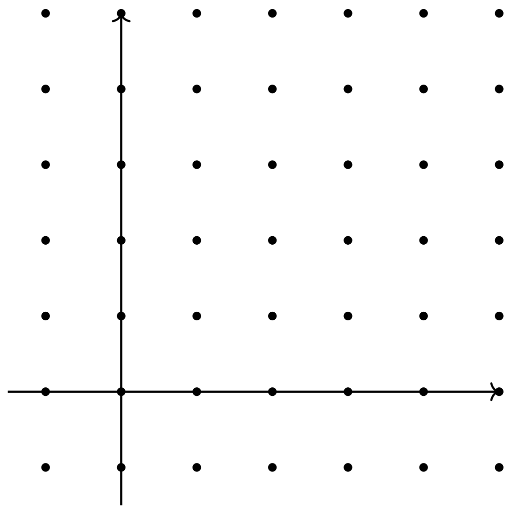
This is (luckily!) not the case with modular forms, although our first example is perhaps a stupid one. The function $f : \mathbb{H} \to \mathbb{C}$ that sends everything to 0 is a modular form. To see, we check each of the conditions:

(1) Certainly the zero function is holomorphic (recall: differentiable). It is a constant function, and the derivative of a constant function is 0. The same is true for complex functions, so this condition is satisfied.
(2) The transformation property is also satisfied: take $z \in \mathbb{H}$. Then $f(\gamma \cdot z) = 0$, and $(cz+d)^k f(z) = (cz+d)^k \cdot 0 = 0$ also.
(3) Finally, values of f are certainly bounded everywhere, as they are always 0.

For a less-stupid example, we consider the **Eisenstein series of weight** $2k$, denoted $G_{2k}(z)$, and defined as

$$G_{2k}(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{1}{(m+nz)^{2k}}.$$

This definition can seem opaque, but it is fairly straightforward when unpacked. Take your favourite value $z \in \mathbb{H}$, and form the lattice in $\mathbb{C}$ of all points of the form $m + nz$ for integers m and n. So if you chose $z = i$, the lattice looks like



The value $G_{2k}(i)$ then equals the sum of all these points to the power $-2k$, apart from the origin, which we can't invert so we leave out. There is a question as to whether this sum converges, but when $k \geq 2$ it does (and, in fact, it is absolutely convergent).

It is a simple exercise[1] in complex analysis to show that $G_{2k}(z)$ is a holomorphic function bounded as $z \to i\infty$, and so, to show it is a modular form, we only need check the transformation law. That is, we need to show that

$$G_{2k}(\gamma \cdot z) = (cz+d)^{2k} G_{2k}(z)$$

To do so, we plug what we have into the definition, and see where it leads. Namely:

$$\begin{aligned} G_{2k}(\gamma \cdot z) = G_{2k}\left(\frac{az+b}{cz+d}\right) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{1}{\left(m + n\frac{az+b}{cz+d}\right)^{2k}} \\ &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{(cz+d)^{2k}}{\left(dm + bn + (cm+an)z\right)^{2k}} \end{aligned}$$

[1]A phrase which here means "the author is sure the statement is true, but to prove it to you would be a distracting sideshow". Note, this phrase very rarely guarantees an exercise actually *is* simple.

This doesn't look particularly nice, but a neat little trick finishes the job. The map $\mathbb{Z} \to \mathbb{Z}$ given by $(m, n) \mapsto (dm + bn, cm + an)$ is invertible, with inverse given by the map $(p, q) \mapsto (ap - bq, -cp + dq)$. To verify:

$$\begin{aligned}
(dm + bn, cm + an) &\mapsto (a(dm + bn) - b(cm + an), -c(dm + bn) + d(cm + an)) \\
&\mapsto (adm + abn - bcn - abn, -cdm - bcn + cdm + adn) \\
&\mapsto ((ad - bc)m, (ad - bc)n) \\
&\mapsto (m, n) \qquad\qquad \text{since } ad - bc = 1
\end{aligned}$$

The takehome is that the effect of the map $(m, n) \mapsto (dm + bn, cm + an)$ on $\mathbb{Z}^2$ is just that it shuffles the elements around. Since the sum we take is over all such pairs[2], we have the equality

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{1}{(dm + bn + (cm + an)z)^{2k}} = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{1}{(m + nz)^{2k}}$$

So, rewriting from above, we have

$$\begin{aligned}
G_{2k}(\gamma \cdot z) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{(cz + d)^{2k}}{(dm + bn + (cm + an)z)^{2k}} \\
&= (cz + d)^{2k} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{1}{(m + nz)^{2k}} \\
&= (cz + d)^{2k} G_{2k}(z)
\end{aligned}$$

exactly as we require.

This argument is a little lengthy, but the upshot is an important one: there are modular forms other than the zero form[3]. Further, if we take two modular forms

[2] To make this kind of "argument via shuffling" work, we also need that the series is absolutely convergent. Luckily, it is!

[3] This is something of a double-edged sword for number theorists, since it both guarantees us something to do all day, but also presents a whole host of problems that would be solved if modular forms had never existed in the first place.

f and g of weight k, we can see that the functions αf for some constant $\alpha \in \mathbb{C}$ and $f + g$ are also both modular forms, also of weight k. The unconvinced reader should take a moment to prove this to themselves.

This means the space of modular forms of weight k forms a vector space, specifically a complex vector space. The next most-obvious question is then to ask how big this vector space is. The answer comes from the powerful Riemann-Roch theorem, of complex algebraic geometry: the space of modular forms of weight k is **finite dimensional**. This means we can write down a finite basis of weight k forms whose span gives us the whole space.

We've reached a point where it would be sensible to introduce some notation, so let us do that. We write $M_k(1)$ to denote the space of modular forms of weight k—the 1 will come into play later on, for now just ignore it. The Riemann-Roch theorem also leads to the following formula for the dimension of $M_k(1)$:

Theorem 3.1 (Dimension formula)**.** Let k be an even integer, $k > 0$. Then

$$\dim(M_k(1)) = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \pmod{12} \end{cases}$$

Meanwhile, if k is odd with $k > 0$, then $M_k(1)$ has dimension 0.

This theorem doesn't tell us about every possible weight, although it comes close. Modular forms of weight 0 are excluded, as are modular forms of negative weight. Weight 0 is excluded because it is so simple: these are just the constant functions $\mathbb{H} \to \mathbb{C}$. And there are no modular forms of negative weight.[4]

Now that we know the sizes of the spaces, we want to know what the bases actually look like. We already have the Eisenstein series $G_{2k}(z)$ for $k > 1$, and looking at a table of values we see this generates the whole of $M_k(1)$ for the first few non-zero spaces:

[4]The reason for both of these statements is to do with the proof of the theorem. The interested reader should consult Cohen's introductory course on modular forms, [Coh18], §3.2, for details.

k	4	5	6	7	8	9	10	11	12	13	14
$\dim(M_k(1))$	1	0	1	0	1	0	1	0	2	0	1
Basis	$G_4(z)$		$G_6(z)$		$G_8(z)$		$G_{10}(z)$		$G_{12}(z)$, ?		$G_{14}(z)$

The first "interesting" case is $k = 12$, which is the first time the dimension formula returns a value bigger than 1. Certainly $G_{12}(z)$ is in the space, but there is another dimension unaccounted for. The missing element is the **modular discriminant**, denoted $\Delta(z)$. Modular forms are generally hard to write down explicitly, but setting $g_2(z) = 60G_4(z)$ and $g_3(z) = 140G_6(z)$, we can write the modular discriminant as:

$$\Delta(z) = g_2(z)^3 - 27g_3(z)^2.$$

This is not particularly illuminating, as the majority of the theory behind it has been stripped away for brevity[5]. Nevertheless, it is easy to see that Δ is indeed a modular form of weight 12: sums and products of holomorphic (resp. bounded) functions are holomorphic (resp. bounded), and the transformation property is satisfied due to the following result.

Proposition 3.2. Take $f \in M_k(1)$ and $g \in M_l(1)$. Then $f \cdot g \in M_{k+l}(1)$.

Proof. We only need to show that $f \cdot g$ satisfies the transformation property. Take $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\begin{aligned}(f \cdot g)(\gamma \cdot z) &= f(\gamma \cdot z)g(\gamma \cdot z) \\ &= (cz+d)^k f(z)(cz+d)^l g(z) \\ &= (cz+d)^{k+l} f(z)g(z) \\ &= (cz+d)^{k+l}(f \cdot g)(z)\end{aligned}$$

□

[5]Again, the interested reader is encouraged to read into this further, as its proof winds a path through some beautiful mathematics underpinning connections between modular forms and elliptic curves.

From this we can then see that, since $G_4(z)$ is a modular form of weight 4, $G_4(z)^3$ is a modular form of weight 12. Similarly $G_6(z)^2$ is weight 12, and so the difference $\Delta(z) = g_2(z)^3 - 27g_3(z)^2$ is weight 12 as well.

Of course, just because we have found another vector in our space, it does not mean we know for sure that $\{G_{12}(z), \Delta(z)\}$ is a basis for $M_{12}(1)$. It could be the case that $\Delta(z) = \alpha G_{12}(z)$ for some $\alpha \in \mathbb{C}$, in which case we would still only have one linearly independent vector. The proof that this *not* the case, that $\Delta(z)$ really is different from $G_{12}(z)$, requires the examination of the two functions' q-**expansions**.

4. Modular forms as Fourier series

Let $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. A modular form of weight k has to satisfy $f(\gamma \cdot z) = (cz+d)^k f(z)$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $z \in \mathbb{H}$, so using $\gamma = T$ and noting $T \cdot z = z+1$, we get

$$f(z+1) = f(z).$$

That is, a modular form is periodic on $\mathbb{H}$ with period 1. Fourier's theorem tells us that a (sufficiently well-behaved) periodic function with period 1 can be written as an infinite series in $\exp(2\pi i z)$ and $\exp(-2\pi i z)$. The good-behaviour of modular forms guaranteed by property (1) in Definition 2.1 meets Fourier's criteria, and so modular forms have Fourier series. We write $q = \exp(2\pi i z)$, and note $q^{-1} = \exp(-2\pi i z)$, so a modular form can be written

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n$$

This is called the **q-expansion** of the modular form, and it gives us a way to write down modular forms in a systematic way. We can never write down all the coefficients a_n of a modular form, but this is not much of a problem. Since the space of modular forms of a given weight is finite-dimensional, there will be some finite number of terms we need in order to tell elements of a basis apart. Writing down coefficients up to this bound at least gives us a way to specify a basis.

Through a series of clever algebraic manipulations[6], it is possible to write the Eisenstein series $G_{2k}(z)$ as the Fourier series:

$$G_{2k}(z) = 2\left(\zeta(2k) + (-1)^k \frac{(2\pi)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n\right)$$

where $\sigma_m(n) = \sum_{d|n} d^m$ is the sum-of-divisor-powers function. There a couple of properties of this expression that it is worth noting, as they tell us about the shape of things to come.

[6]Due to Apostol, communicated perhaps more accessibly by the StackExchange answer [hs].

First, there are no negative powers of q in the q-expansion. This is always the case for modular forms, and is a consequence of properties (1) and (3) in Definition 2.1.

Secondly, up to the common factor of π^{2k}, all the coefficients of the q-expansion are rational numbers. This is not a coincidence, and while it doesn't happen for *all* modular forms that such a scaling is possible, special types of modular forms called **eigenforms** guarantee this property[7].

It is this expression for the Eisenstein series that lets us show that $\Delta(z)$ is not simply a multiple of $G_{12}(z)$, and we do it by the ancient and elegant art of comparing coefficients. Writing down all the relevant series, we have

$$G_4(z) = \frac{16\pi^4}{3}\left(\frac{1}{240} + q + 9q^2 + 73q^3 + ...\right)$$
$$G_6(z) = \frac{16\pi^6}{15}\left(\frac{1}{504} - q - 33q^2 - 244q^3 + ...\right)$$
$$G_{12}(z) = \frac{16\pi^{12}}{155925}\left(\frac{691}{32760} + q + 2049q^2 + 177148q^3 + ...\right)$$

Scaling and multiplying everything out, we see that

$$\begin{aligned}\Delta(z) &= g_2(z)^3 - 27g_3(z)^2 \\ &= \pi^{12}\left(\left(\frac{4}{3} + 320q + 2880q^2 + ...\right)^3 - 27\left(\frac{8}{27} - \frac{448}{3}q - 4928q^2 - ...\right)^2\right) \\ &= \pi^{12}\left(\left(\frac{64}{27} + \frac{5120}{3}q + 424960q^2 + ...\right) - \left(\frac{64}{27} - \frac{7168}{3}q + 523264q^2 - ...\right)\right) \\ &= \pi^{12}\left(4096q - 98304q^2 + 1032192q^3 - ...\right) \\ &= (2\pi)^{12}\left(q - 24q^2 + 252q^3 - ...\right)\end{aligned}$$

Comparing constant the coefficients of $\Delta(z)$ and $G_{12}(z)$, we can see that the first is 0 and the second is not. So if $\Delta(z)$ is to be a multiple of $G_{12}(z)$, it would have to be $0 \cdot G_{12}(z) = 0$. But $\Delta(z)$ is not zero, as its q-expansion's coefficients would

[7]This is technically not the whole truth; the coefficients of eigenforms will always be *algebraic integers*, which means they live in the ring of integers of some algebraic number field.

have to all be 0 if it were. So we really do have a bona fide basis of $M_{12}(1)$, given by $\{G_{12}, \Delta\}$.

Our winding path, having shown us all we need to see for the moment, has led us back to bases of the spaces $M_k(1)$. The only thing left, then, is to justify the appearance of the number 1 in the notation for the space $M_k(1)$.

5. Modular forms of level $N > 1$

So far we have looked at modular forms with weight k, which satisfy a transformation law for elements in the group $\mathrm{SL}_2(\mathbb{Z})$. But there is nothing stopping us considering the action of a different group on $\mathbb{H}$. In particular, subgroups $H \leq \mathrm{SL}_2(\mathbb{Z})$ are a good choice, since much of what we've already done should carry over nicely. To that end, we introduce the **congruence subgroup**

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\},$$

pronounced "gamma-nought of N", or simply "gamma-nought N". The group $\Gamma_0(N)$ is not the only congruence subgroup number theorists care about—there are also $\Gamma(N)$ and $\Gamma_1(N)$, defined similarly in terms of congruences on the entries of their elements. The number N in all these groups (although we will only talk about $\Gamma_0(N)$ in these notes) is called the **level** of the group.

To get a feel for $\Gamma_0(N)$, we look at three example matrices.

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}, \quad V = \begin{pmatrix} 7 & -5 \\ 10 & -7 \end{pmatrix}$$

A quick check show all have determinant 1, and so live in $\mathrm{SL}_2(\mathbb{Z})$. By examining the lower-left entry of T, which is 0, we can see that $T \in \Gamma_0(N)$ for *any* N.

The lower-left entry of U is -3, which is $0 \pmod 3$, and so $U \in \Gamma_0(3)$. Meanwhile, the lower-left entry of V is 10, which is 0 when taken mod 2, mod 5 and mod 10, so $V \in \Gamma_0(2)$, $\Gamma_0(5)$ and $\Gamma_0(10)$. It should also be clear that the identification $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$ holds.

These kinds of identifications are easy enough to do, but they don't really tell us what congruence subgroups are for. One small strand of the web connecting modular forms and elliptic curves[8] is the theory of **modular curves**—curves defined in terms of congruence subgroups, whose points carry information about

[8]The most dramatic aspect of which is certainly Andrew Wiles' use of **modular elliptic curves** to prove Fermat's Last Theorem, a result no introductory text on modular forms written after 1995 should ever leave unmentioned.

elliptic curves with prescribed structure coming from the level of the group. This is all a bit vague, but by now the reader is hopefully used to this. There *is* a vast cache of information about these groups and why they're interesting, but we won't get into it here.

Our main reason for caring about $\Gamma_0(N)$ is that it gives us a fresh source of modular forms. In the original definition, a modular form had to obey the transformation law with respect to all matrices in the group $\mathrm{SL}_2(\mathbb{Z})$, but we can easily relax this to be only those matrices in $\Gamma_0(N)$ for some N. So we now venture the new, improved definition of a modular form.

Definition 5.1 (Modular forms, v.2)**.** A modular form of **weight** k and **level** N is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying

(1) The function f is holomorphic on $\mathbb{H}$.
(2) For any $z \in \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, the function f satisfies
$$f(\gamma \cdot z) = (cz + d)^k f(z).$$
(3) The values of $f(z)$ are bounded as $z \to i\infty$.

This definition is almost exactly the same as the previous one, only now we have an extra parameter to play with. We write $M_k(N)$ for the space of modular forms of weight k and level N, which should make clear the reasoning for denoting the space of modular forms of weight k (and, looking back, level 1) with $M_k(1)$.

Informally, the groups $\Gamma_0(N)$ are "smaller" than the group $\mathrm{SL}_2(\mathbb{Z})$, and so we expect more modular forms in $M_k(N)$ as we make the level larger. This is because the overall number of relations of the form $f(\gamma \cdot z) = (cz + d)^k f(z)$ that f has to satisfy gets smaller, making the list of checks a given function has to pass in order to be a modular form less restrictive. Both $\Gamma_0(N)$ and $\mathrm{SL}_2(\mathbb{Z})$ are infinite groups, so this is not really a rigorous idea, but the heuristic is borne out in the data. In the table below we list the dimension of $M_k(N)$, with levels N as the column headings and weights k as the row headings.

	1	2	3	4	5	6	7	8	9	10	11	12
2	0	1	1	2	1	3	1	3	3	3	2	5
4	1	2	2	3	3	5	3	5	5	7	4	9
6	1	2	3	4	3	7	5	7	7	9	6	13
8	1	3	3	5	5	9	5	9	9	13	8	17
10	1	3	4	6	5	11	7	11	11	15	10	21
12	2	4	5	7	7	13	9	13	13	19	12	25

FIGURE 2. Dimensions of $M_k(N)$ for various k, N. Notice, the larger one makes N or k, the larger the dimension tends to get.

There is not much new to say about modular forms of level N, other than to restate the properties already covered for forms of weight 1. The space $M_k(N)$ is still finite-dimensional, elements $f(z) \in M_k(N)$ still have q-expansions $\sum_{n=1}^{\infty} a_n q^n$, and the weight-addition trick of Proposition 3.2 still works.

6. Modular forms as eta expressions

As an application of what we've got so far, we turn our attention to so-called **eta expressions**. These are expressions for modular forms in terms of the **Dedekind eta function**, defined as:

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

The eta function is itself a modular form, although it is a form of weight $\frac{1}{2}$, not an integer as we've focussed on so far[9]. It is also a **modular form with character**, meaning it comes with an axillary function known as a **Dirichlet character**. This is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying $\chi(ab) = \chi(a)\chi(b)$, and it appears in the transformation formula:

$$\eta(\gamma \cdot z) = \chi(d)(cz+d)^{\frac{1}{2}}\eta(z)$$

So far, we've only considered modular forms without character, although secretly these have always been modular forms equipped with the **trivial character**, the function $\mathbb{Z} \to \mathbb{C}$ that sends everything to 1. This gives us our final definition of a modular form:

Definition 6.1 (Modular forms, v.3)**.** A modular form of **weight** k, **level** N and **character** χ is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying

(1) The function f is holomorphic on $\mathbb{H}$.
(2) For any $z \in \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, the function f satisfies

$$f(\gamma \cdot z) = \chi(d)(cz+d)^k f(z).$$

(3) The values of $f(z)$ are bounded as $z \to i\infty$.

[9]So far we have not thought about non-integral values for the weight, mostly because that question is its own complicated and nuanced sub-topic of the study of modular forms, and the author doesn't have much of a clue about any of it.

The eta function is useful to us because we can use it to write down simple expressions for other modular forms. For example, we have the equality

$$\Delta(z) = (2\pi)^{12}\eta(z)^{24}$$

If we are happy to imagine a weight $\frac{1}{2}$ modular form[10], then its 24^{th} power should be weight $24 \cdot \frac{1}{2} = 12$. Multiplying out the first few terms $\eta(z)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ we get $q - 24q^2 + ...$, matching the expansion for $\Delta(z)$ we've already seen.

The levels, weights, and q-expansions match, so we only need to check that the characters match as well. The character associated to the eta function takes its values in the group μ_{24}—the 24^{th} roots of unity, generated by $e^{2\pi i/24}$. Any element of this group to the power 24 will be 1, as this is exactly what it means to be a 24^{th} root of unity, and so χ^{24} will be the trivial character. Thus the equality is proved.

There are many other modular forms that have eta expressions. For example, another famous modular form, with LMFDB label `11.2.a.a` (clickable[11]), is expressible as the product

$$f(z) = \eta(z)^2\eta(11z)^2.$$

The form with LMFDB label `48.2.a.a` is the quotient of several copies of η:

$$f(z) = \frac{\eta(4z)^4\eta(12z)^4}{\eta(2z)\eta(6z)\eta(8z)\eta(24z)}$$

and (for an example with weight $\neq 2$) the form with label `8.6.a.a` has the expression

$$f(z) = \eta(z)^8\eta(4z)^4 + 8\eta(4z)^{12}.$$

[10]Which we certainly *should* be happy to do. There is a slight wrinkle, in that this requires taking a square root, and a choice needs to be made when doing so. I am reliably informed this end up not being a problem, though.

[11]The linked website is the LMFDB—the L-functions and modular forms database. It is a great resource for data on modular forms and many surrounding areas. It can be a little overwhelming at first, but it has a great deal of information to help beginners and experts alike.

The first and second expressions can be found in [MO97], and the third in [OSZ18]. All are on their respective second pages.

The recent paper [AAH+20] outlines an algorithm for finding eta expressions for modular forms of weight 2, level $N = pq$ (where p and q are odd primes satisfying some technical conditions) and trivial character. The first example they provide is the simple expression

$$f(z) = \eta(z)^2\eta(35z)^2 + \eta(5z)^2\eta(7z)^2,$$

where $f(z)$ is the modular form with LMFDB label `35.2.a.a`. Their second example is less visually appealing: it is the form `55.2.a.a`, but the length of the sum and the size of the coefficients makes it far too large to reproduce here. Nonetheless, there are many such cases covered by their algorithm where no eta expression has yet been found.

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