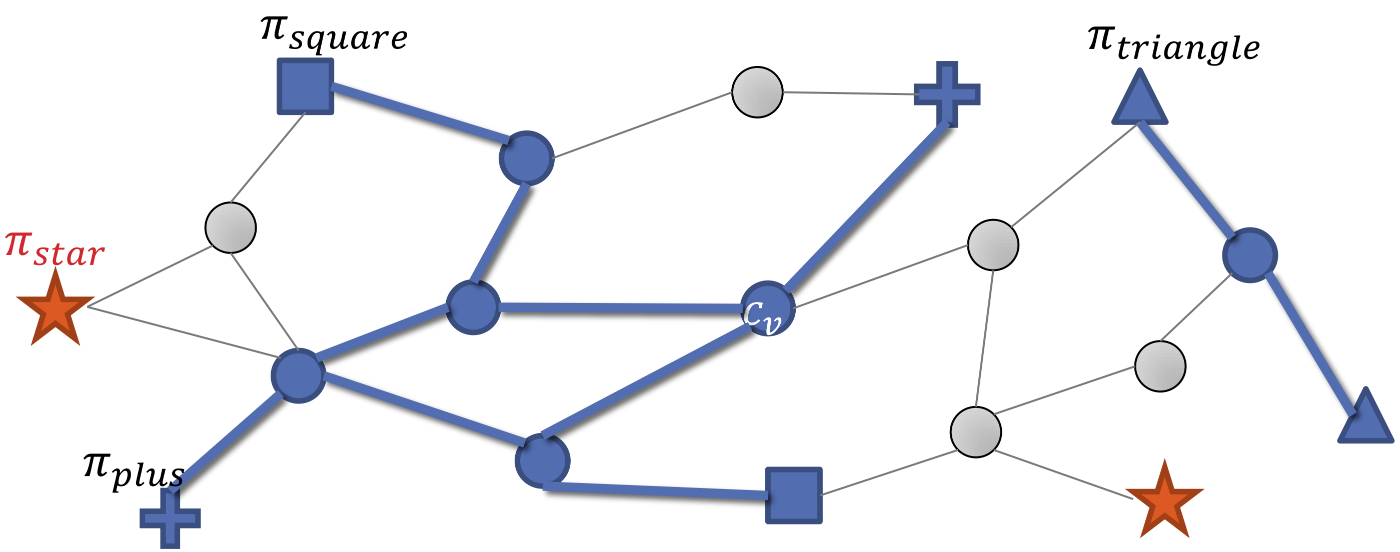
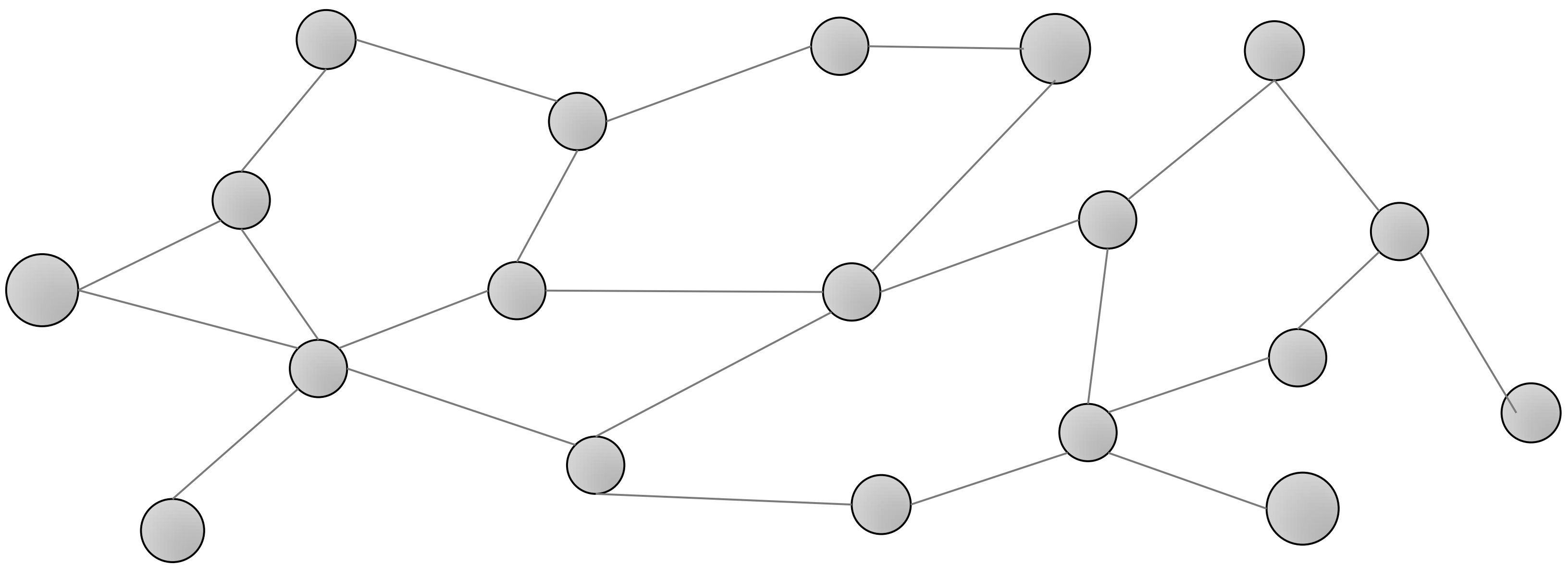
CONSTANT APPROXIMATION ALGORITHM FOR NODE-WEIGHTED PRIZE-COLLECTING STEINER FOREST ON PLANAR GRAPHS

Mateusz Lewandowski

Supervisor: Carsten Moldenhauer



Node-Weighted Prize-Collecting Steiner Forest on Planar Graphs

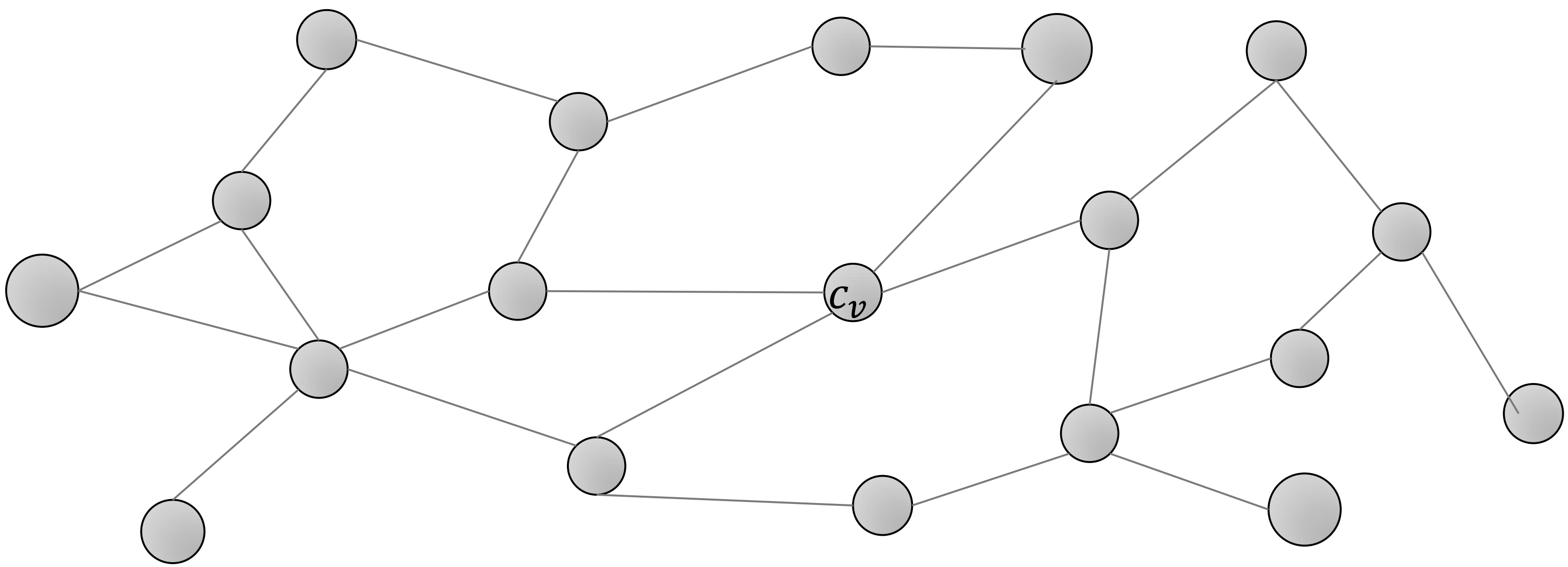


input

- **a planar graph $G = (V, E)$**

output

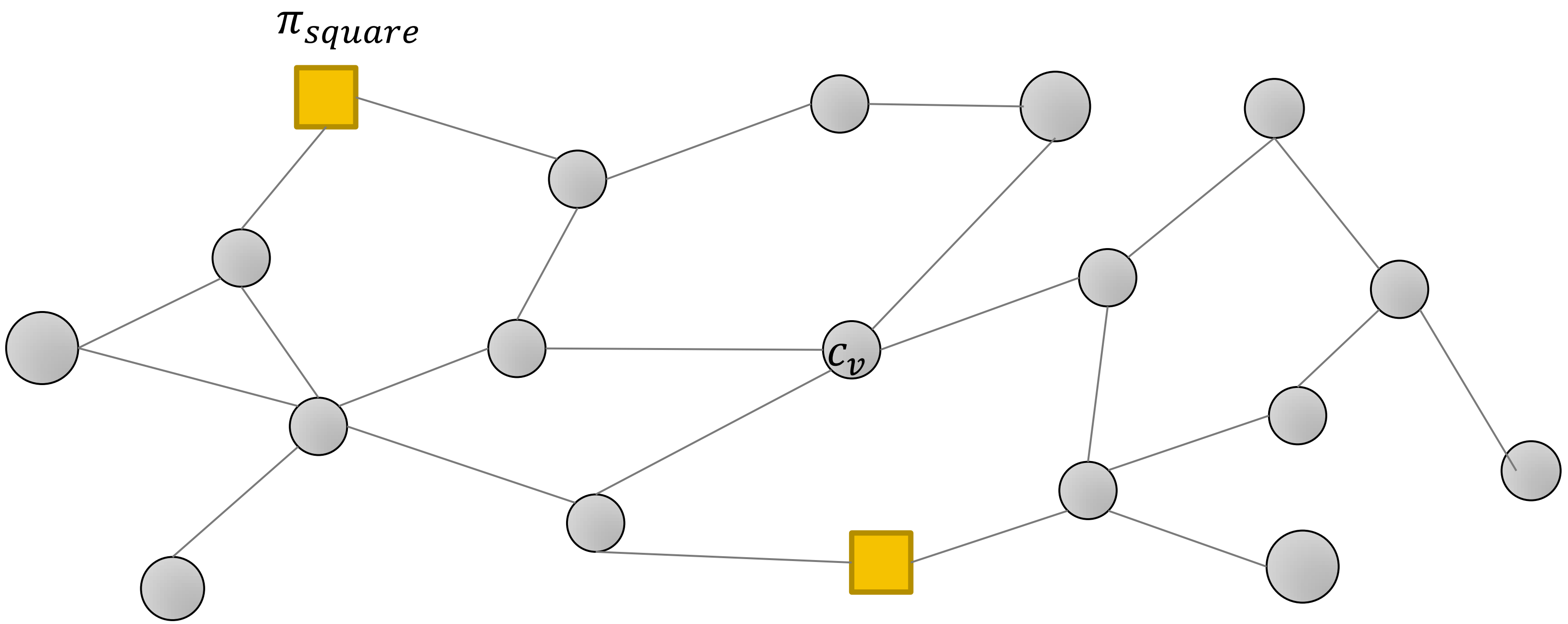
Node-Weighted Prize-Collecting Steiner Forest on Planar Graphs



input
- a planar graph $G = (V, E)$
- **non-negative costs of vertices c_v**

output

Node-Weighted Prize-Collecting Steiner Forest on Planar Graphs

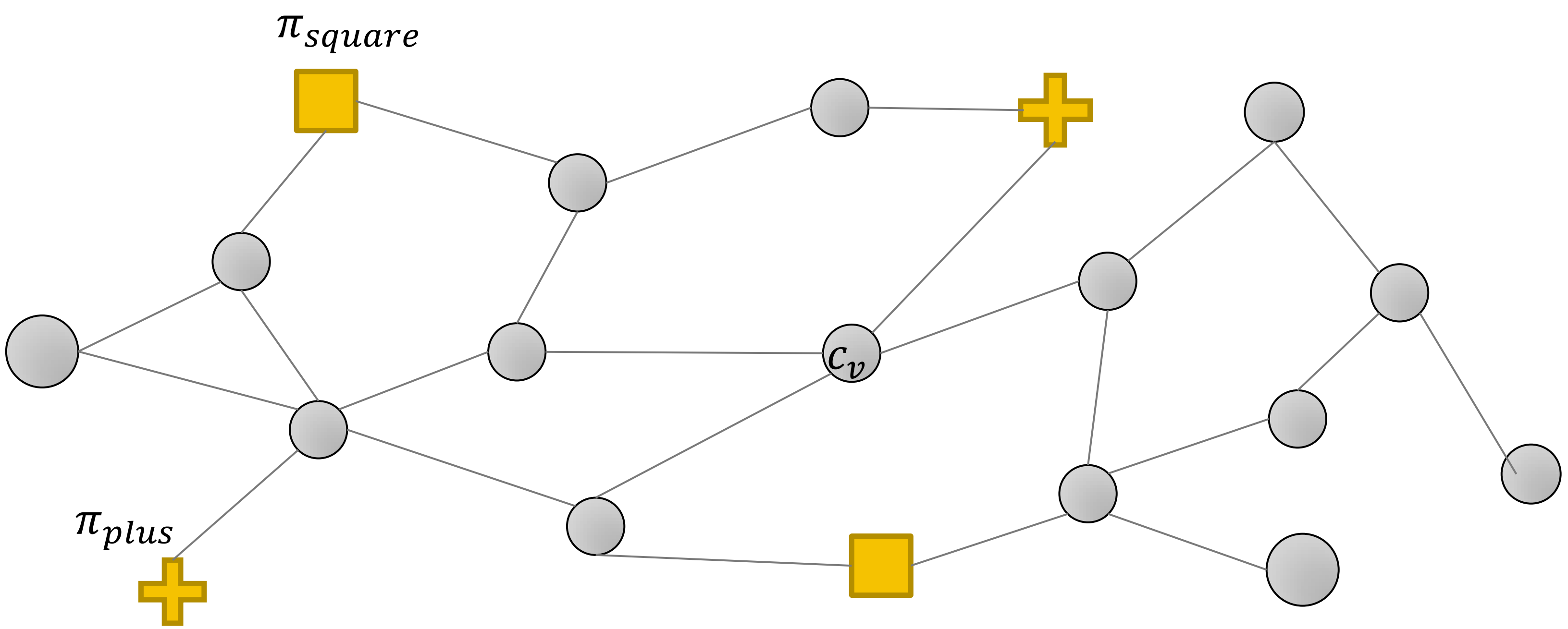


input

- a planar graph $G = (V, E)$
- non-negative costs of vertices c_v
- **demands between some pairs of vertices with penalties $\boldsymbol{\pi_{i,j}}$**

output

Node-Weighted Prize-Collecting Steiner Forest on Planar Graphs



input
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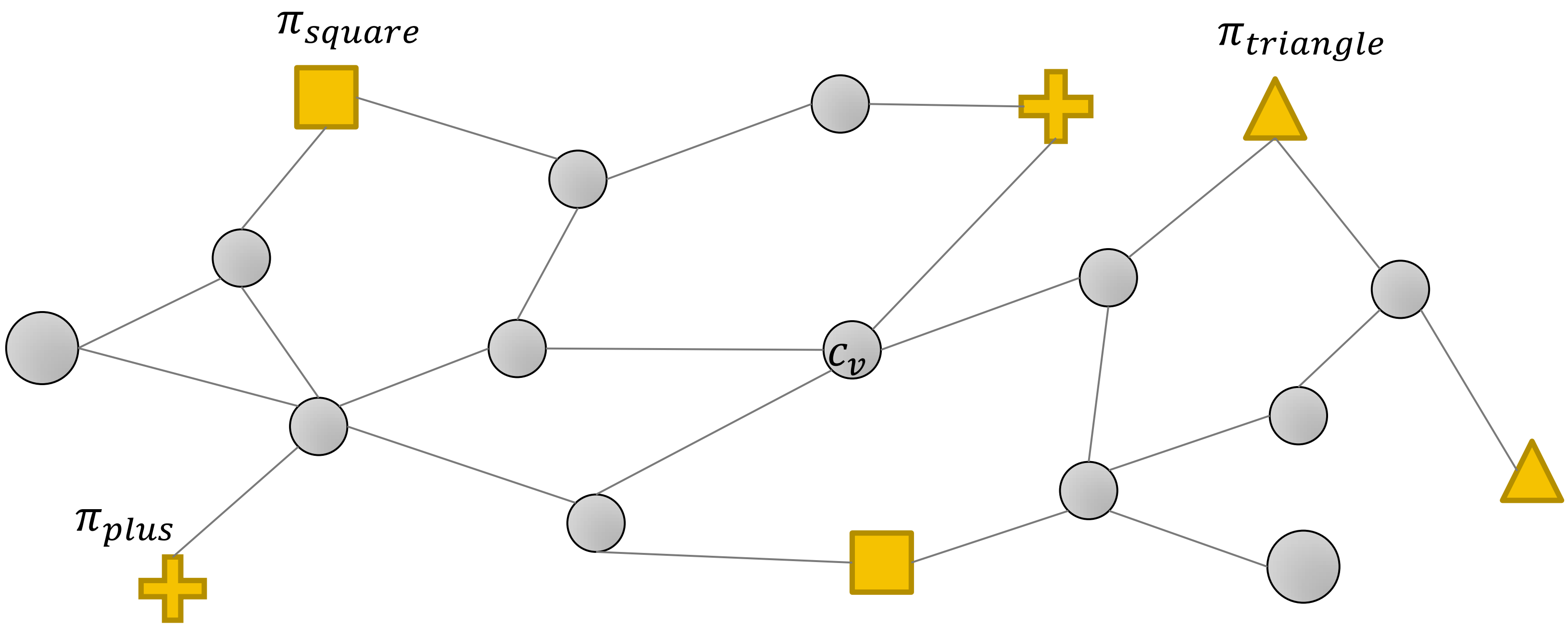

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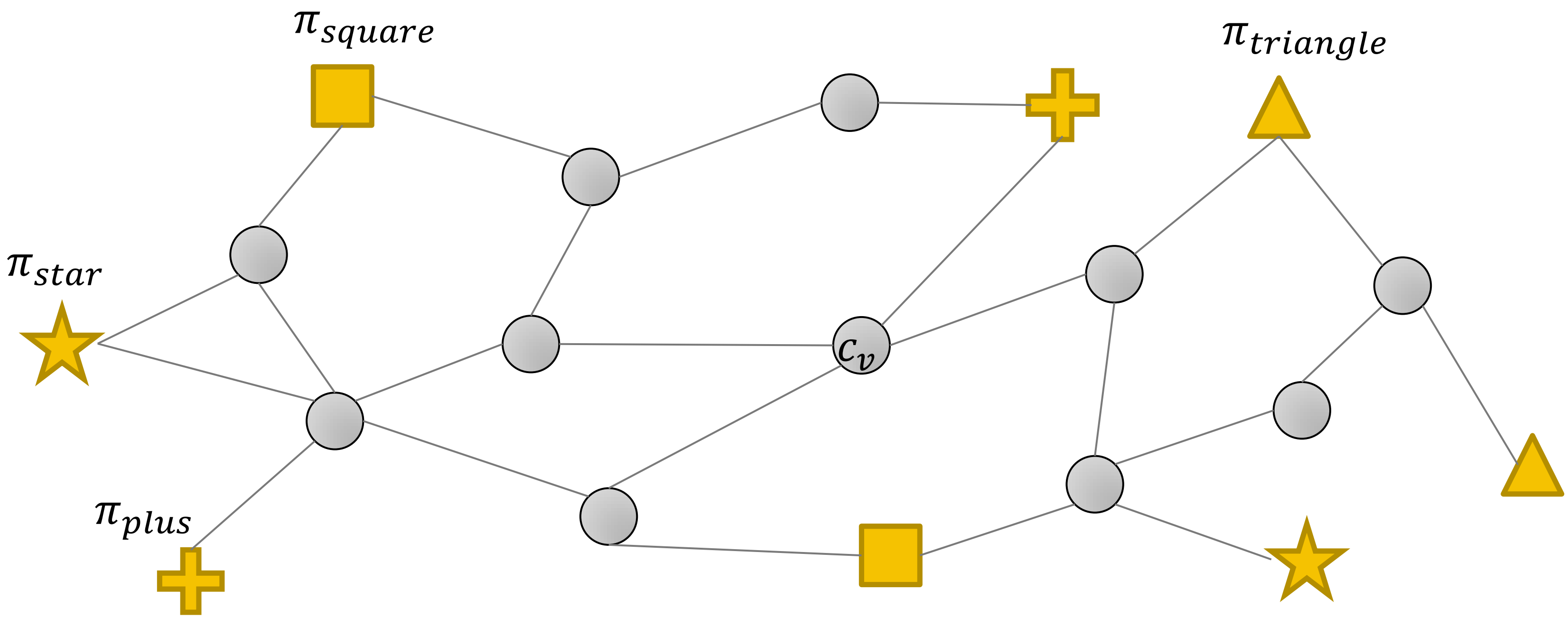

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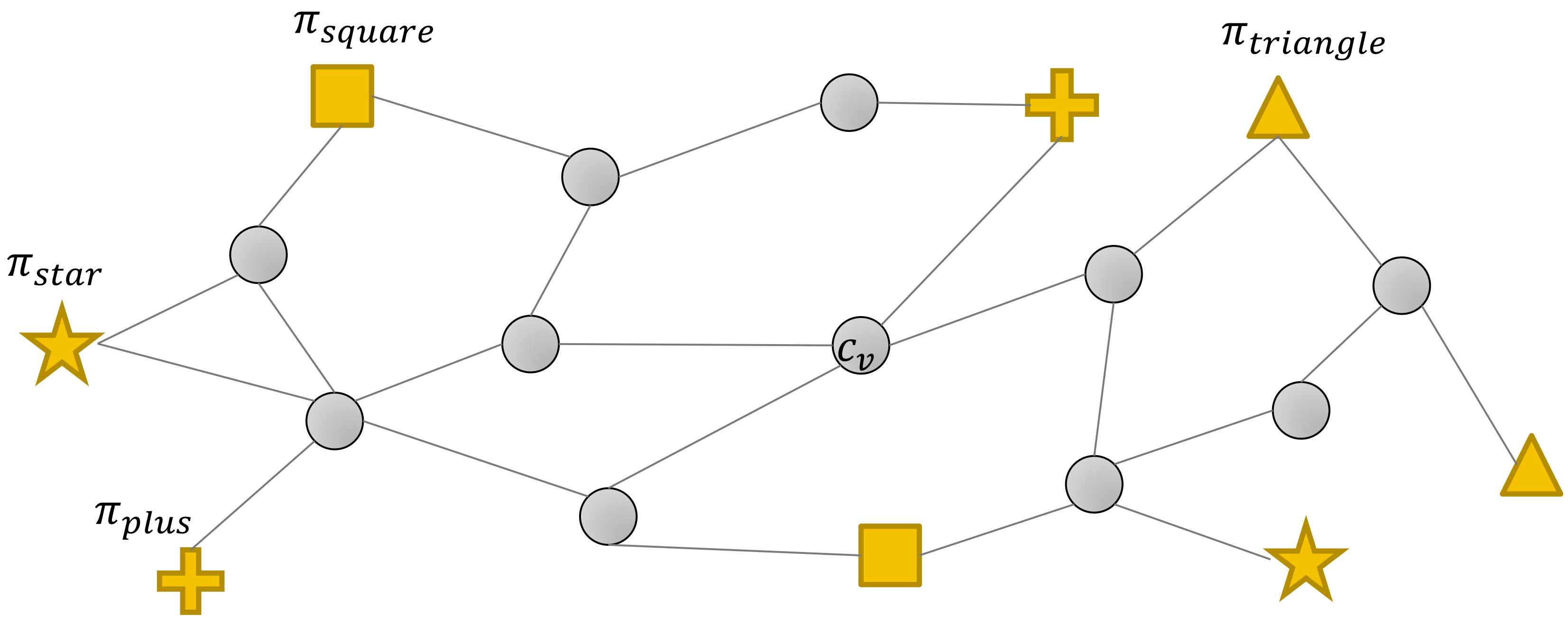
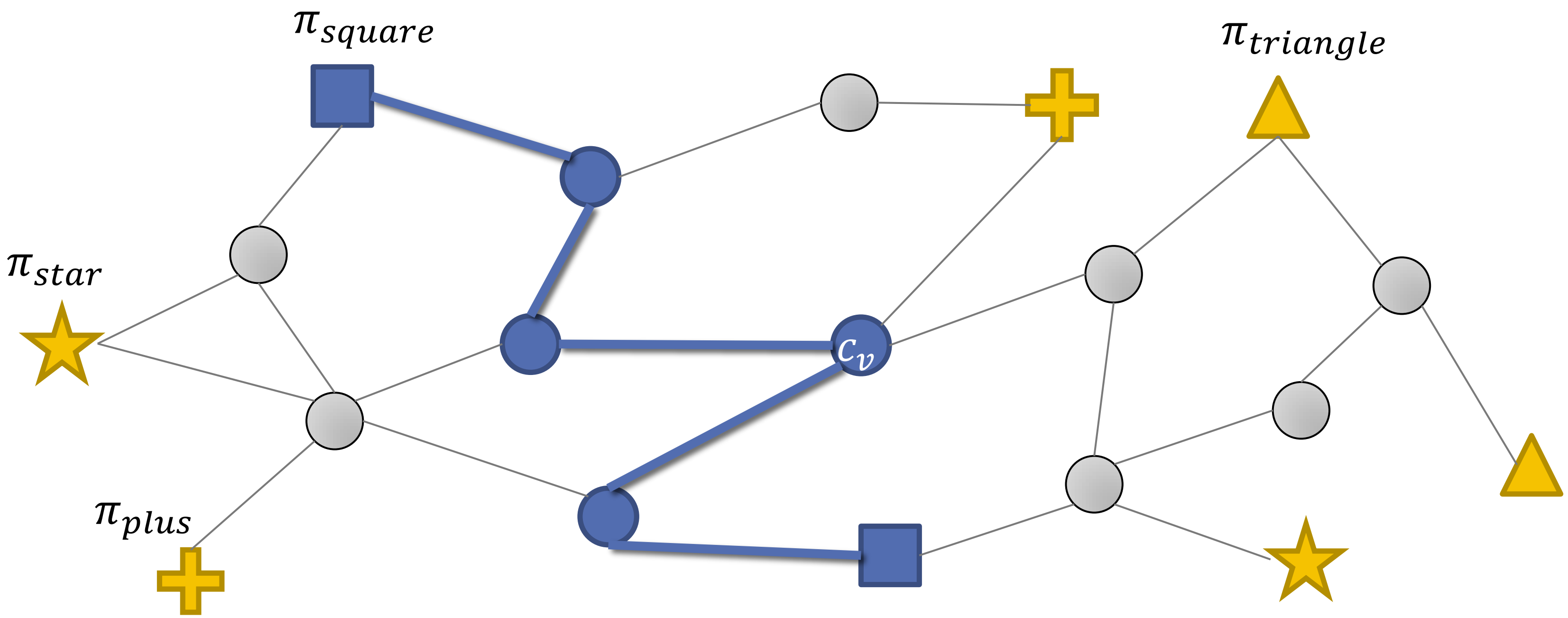


input
- a planar graph $G = (V, E)$
- non-negative costs of vertices c_v
- demands between some pairs of vertices with penalties $\pi_{i,j}$
- **buy vertices => connect demands**

output

Node-Weighted Prize-Collecting Steiner Forest on Planar Graphs



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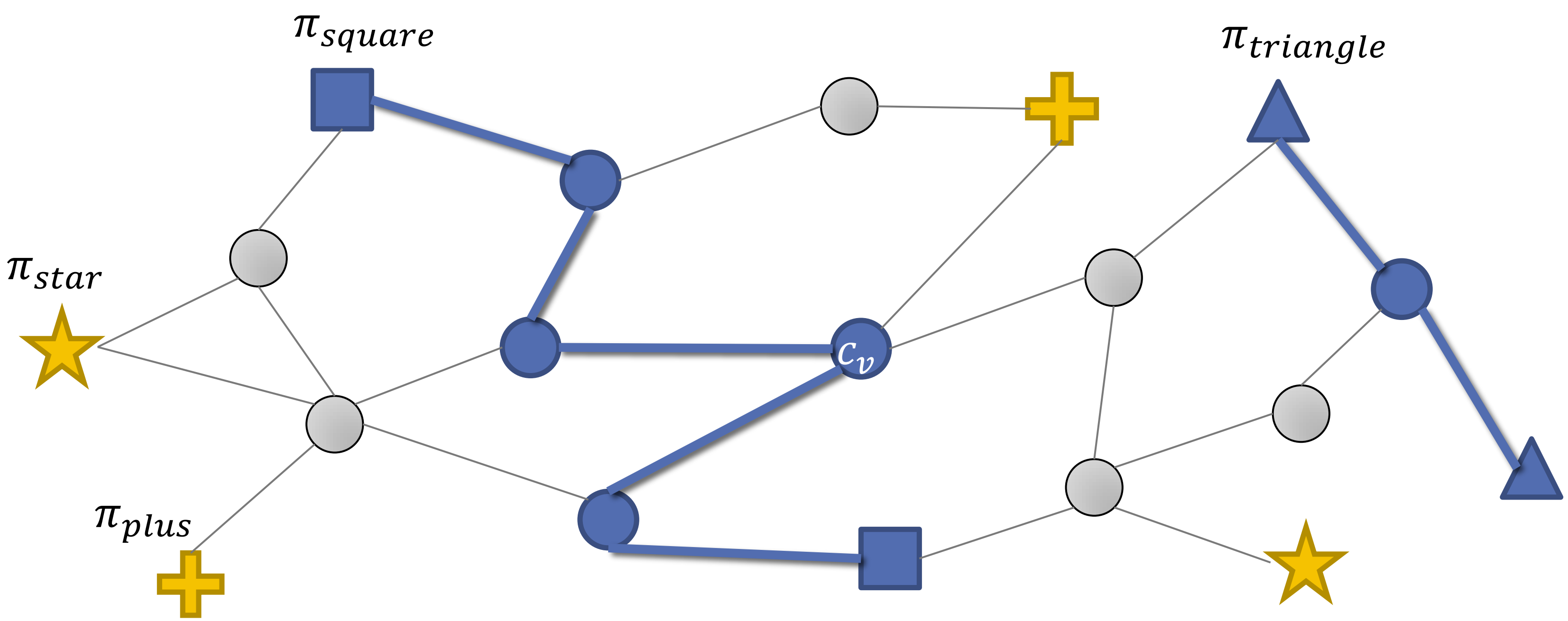

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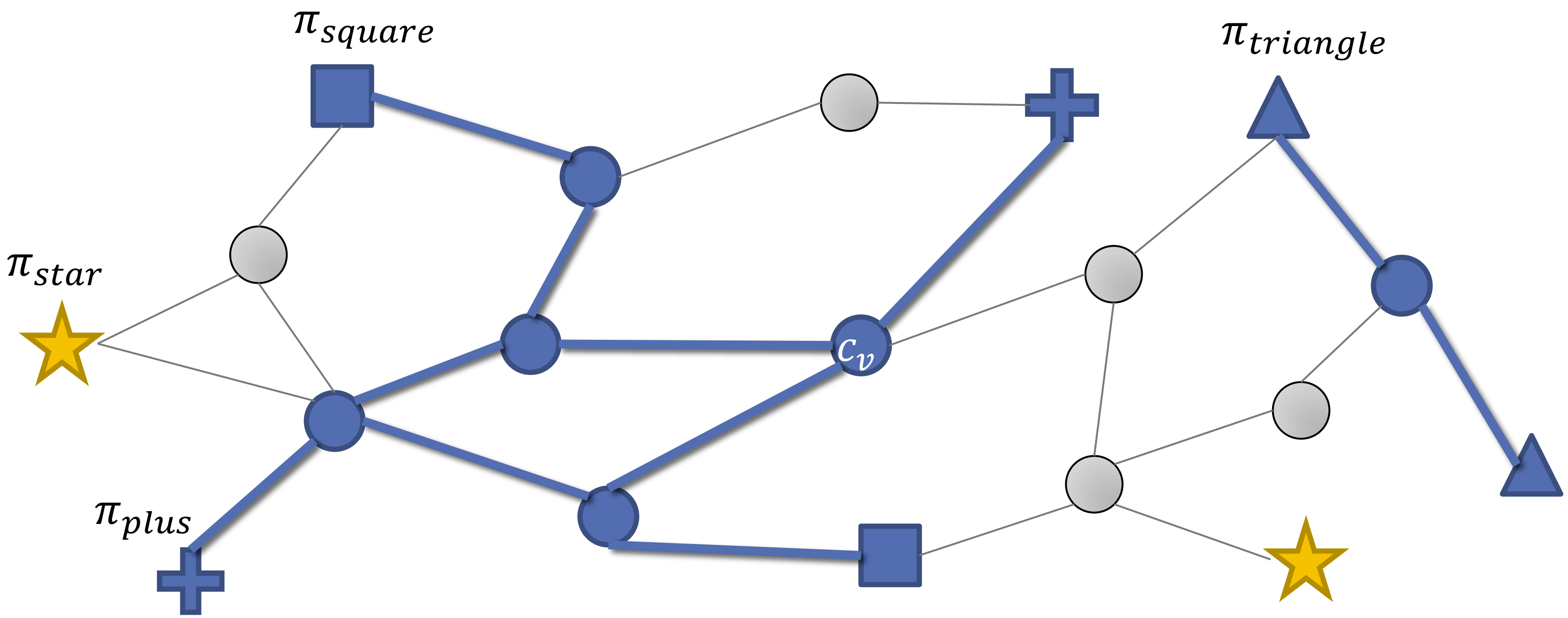

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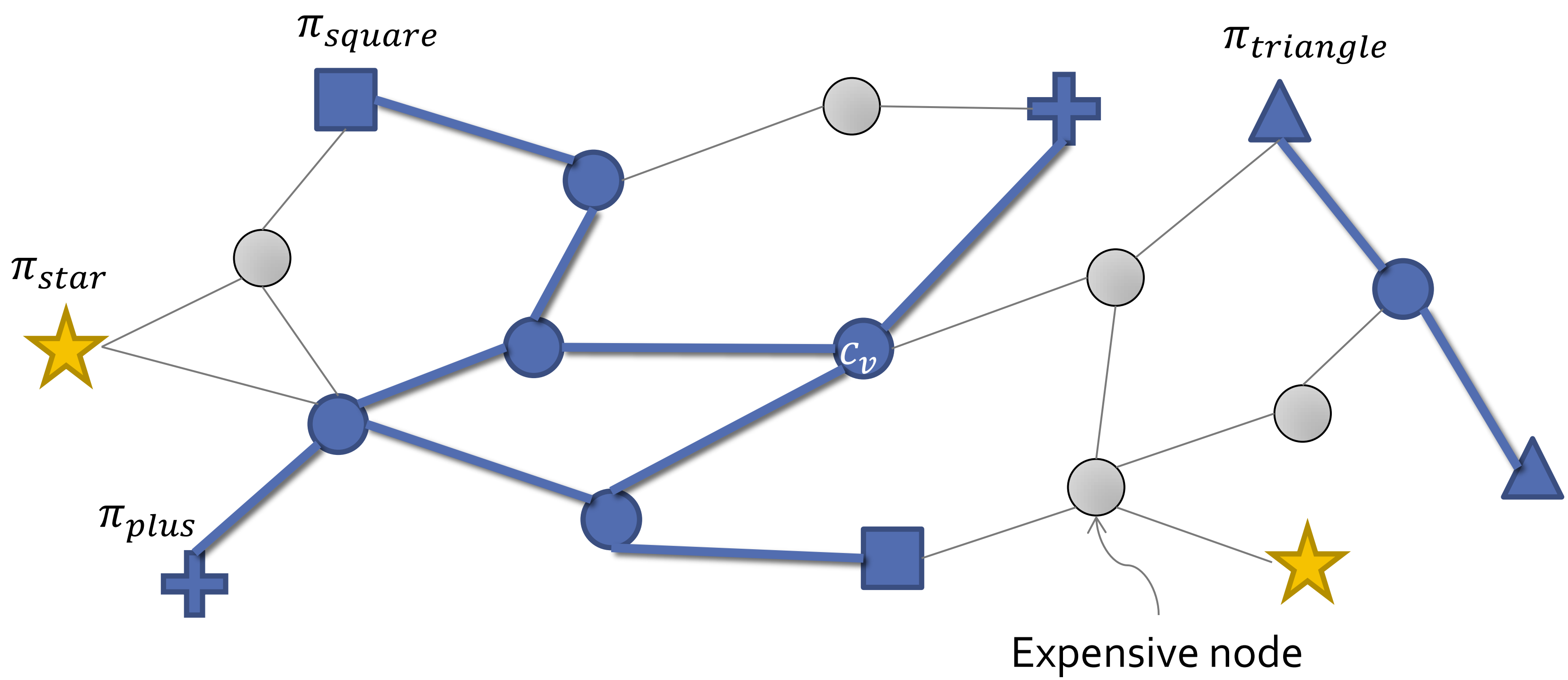

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- demands between some pairs of vertices with penalties $\pi_{i,j}$
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output

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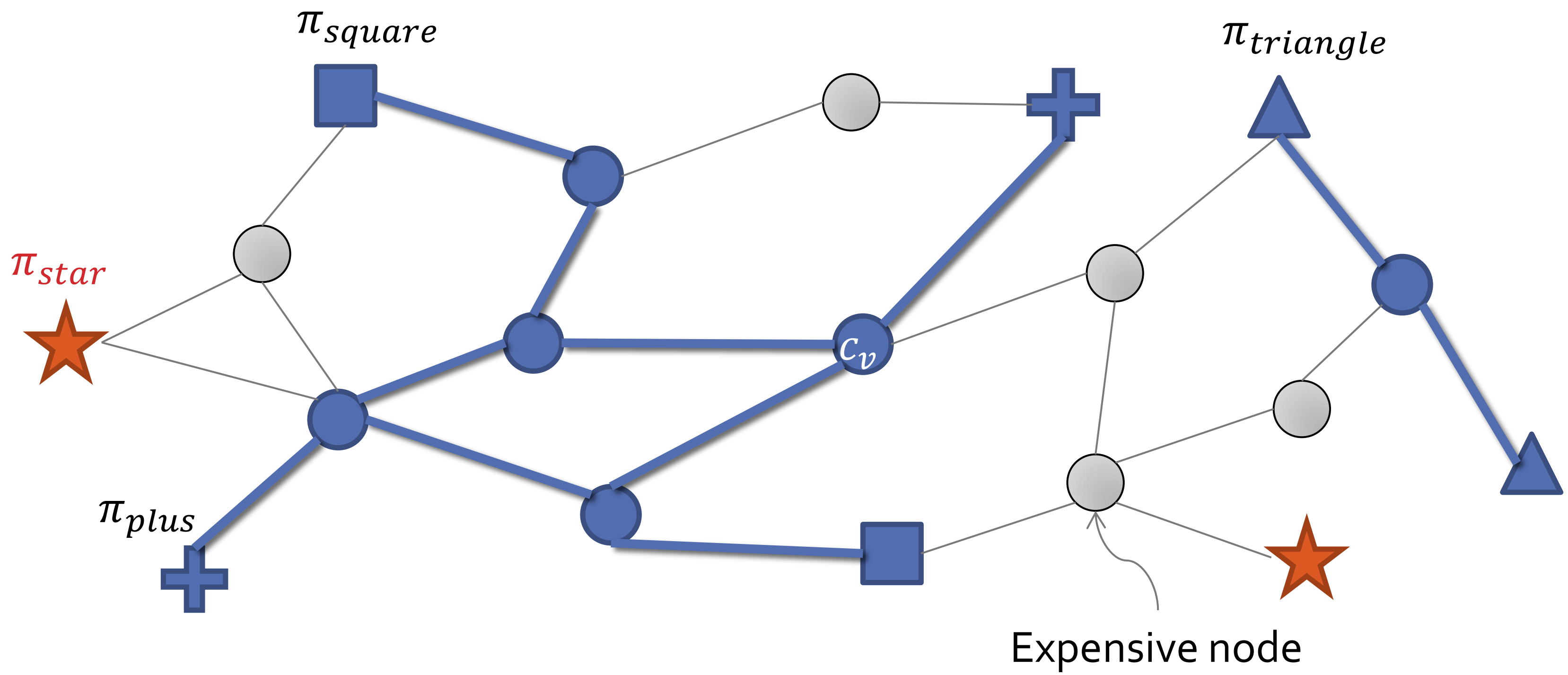
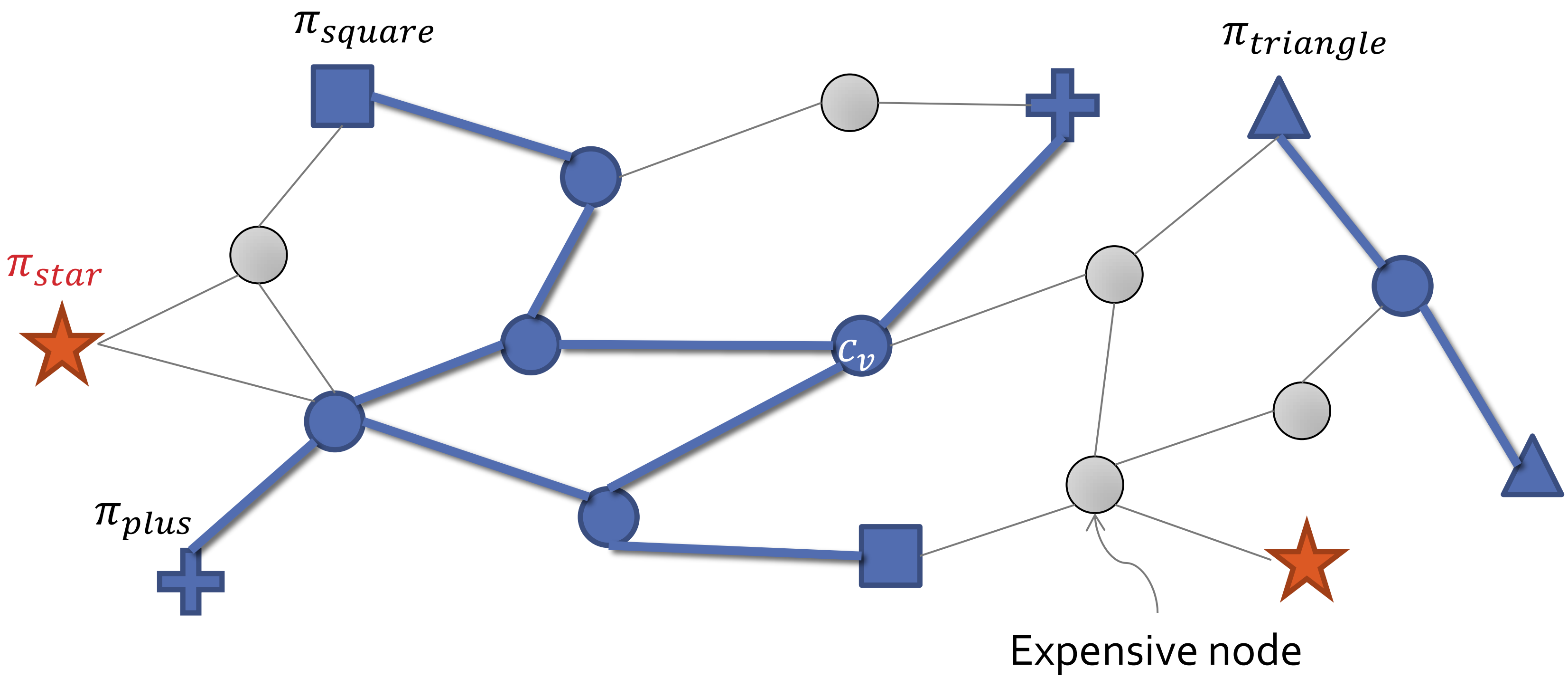


input
- a planar graph $G = (V, E)$
- non-negative costs of vertices c_v
- demands between some pairs of vertices with penalties $\pi_{i,j}$

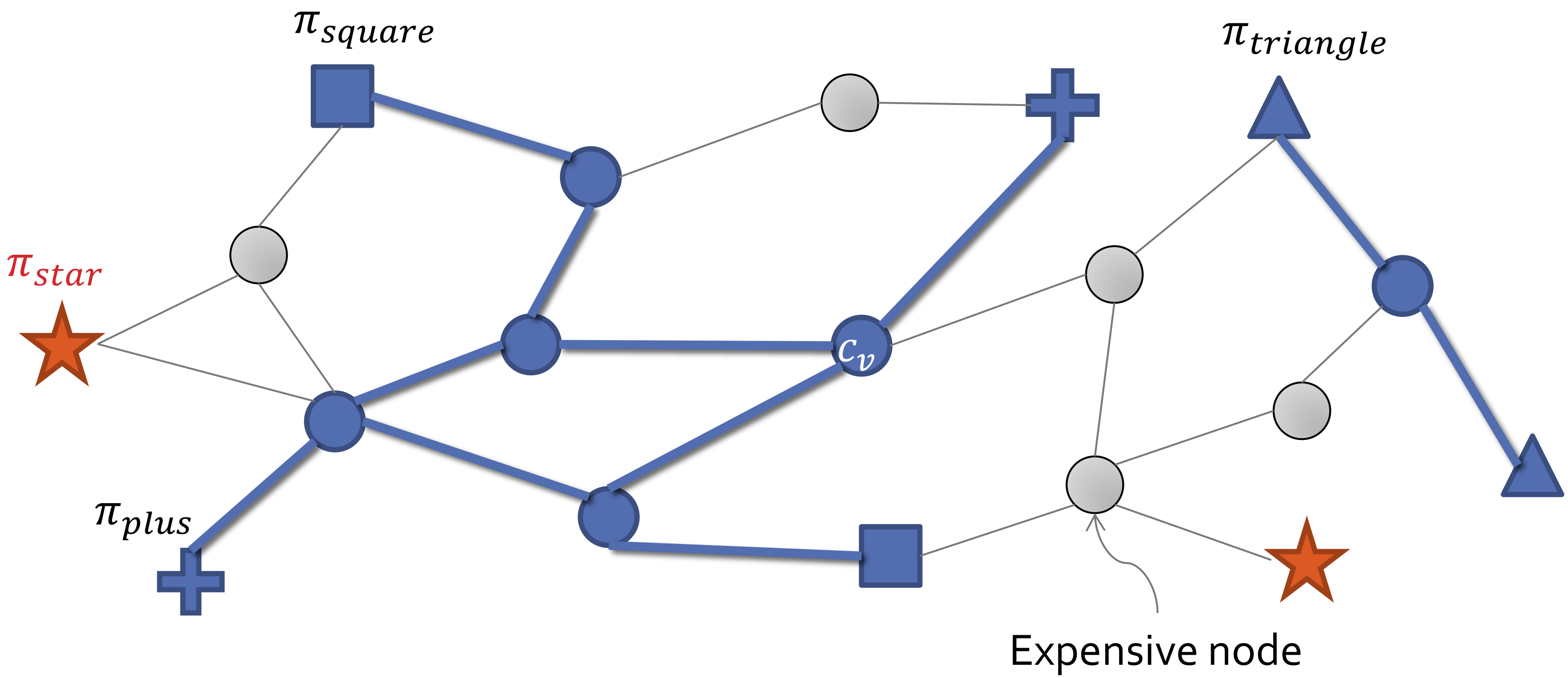
output
- buy vertices => connect demands
- **pay a penalty for not connected demands**

Node-Weighted Prize-Collecting Steiner Forest on Planar Graphs



Problem: minimize cost of nodes + penalties

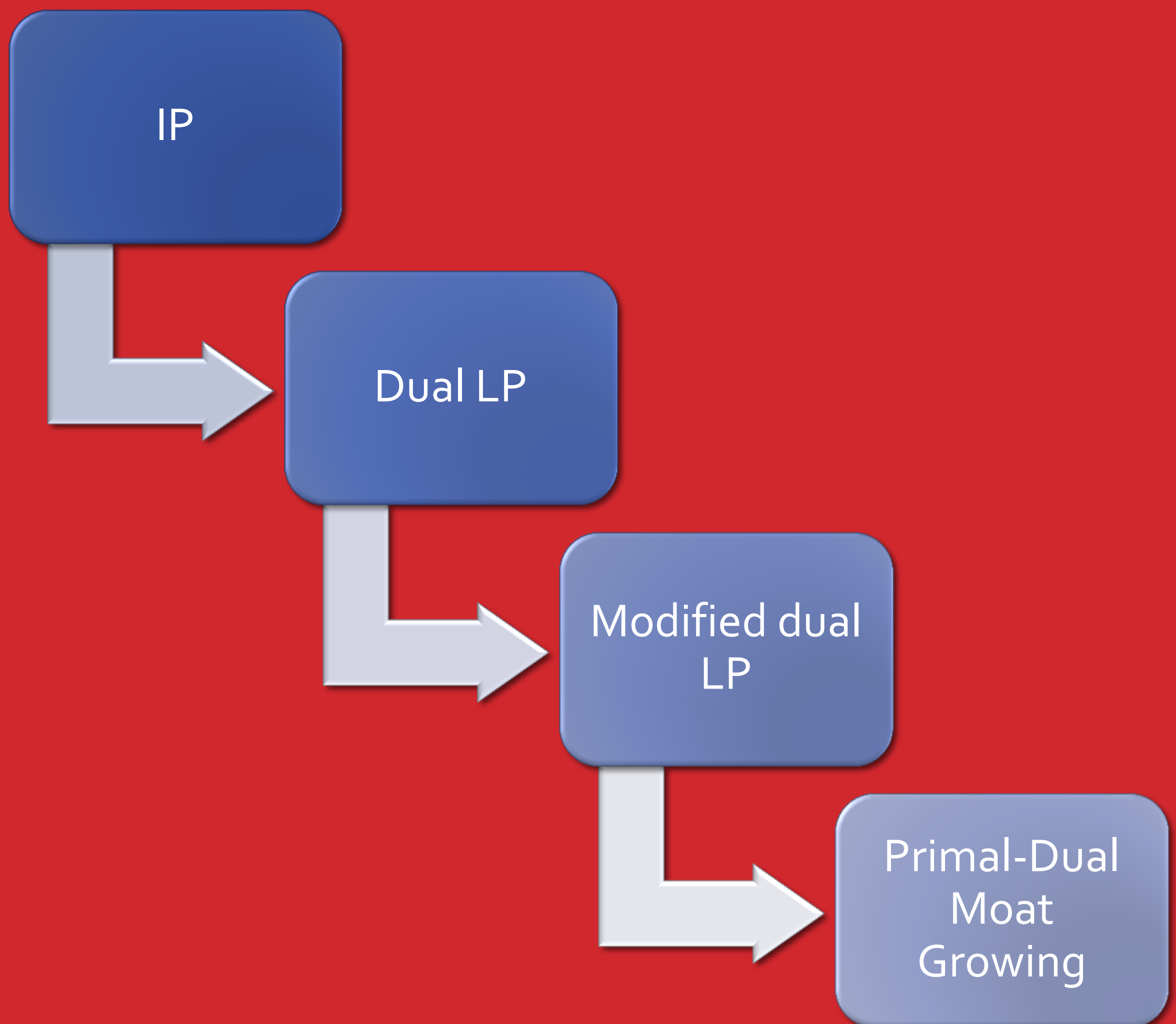
Node-Weighted Prize-Collecting Steiner Forest on Planar Graphs



Problem: minimize cost of nodes + penalties

My result: 4 approximation

OUTLINE



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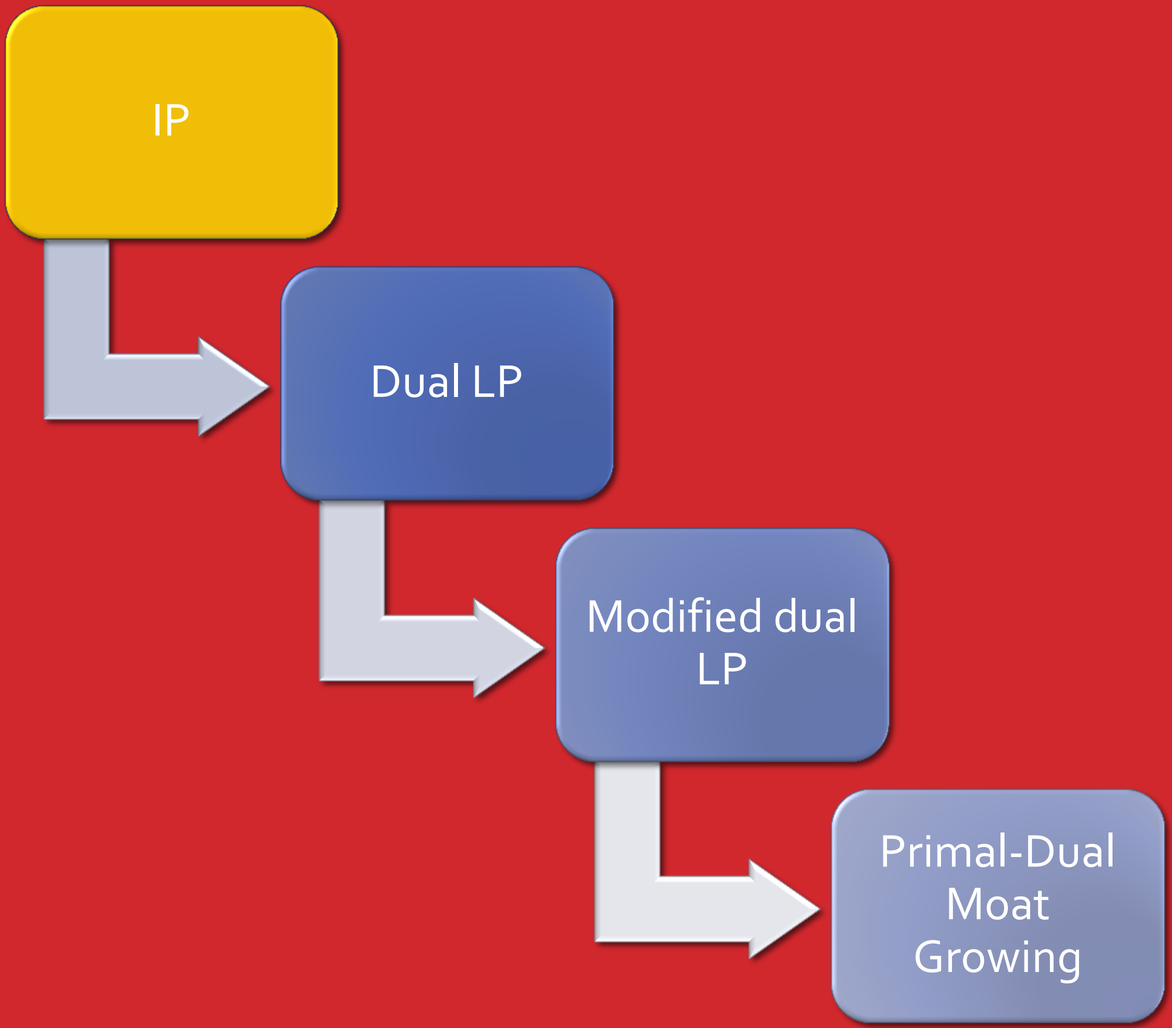

Integer programming formulation

$$min \qquad \sum_{v} c_v x_v + \sum_{(i,j)} \pi_{i,j} z_{i,j}$$

$$s.t.$$

$$\sum_{v \in \Gamma(S)} x_v + z_{i,j} \geq 1 \qquad \begin{array}{c} \forall\, demand(i,j) \\ \forall\, S \odot (i,j) \end{array}$$

$$x_v \in \{0,1\}$$

$$z_{i,j} \in \{0,1\}$$

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Integer programming formulation

nodes

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Integer programming formulation

nodes penalties

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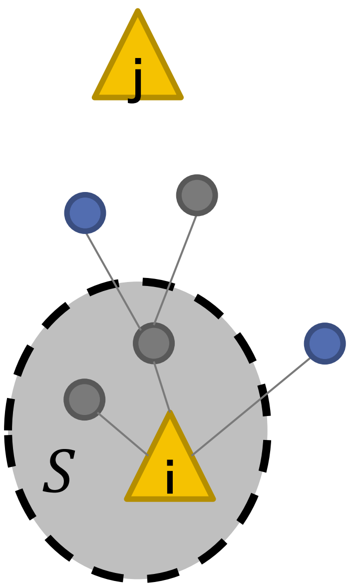
Integer programming formulation



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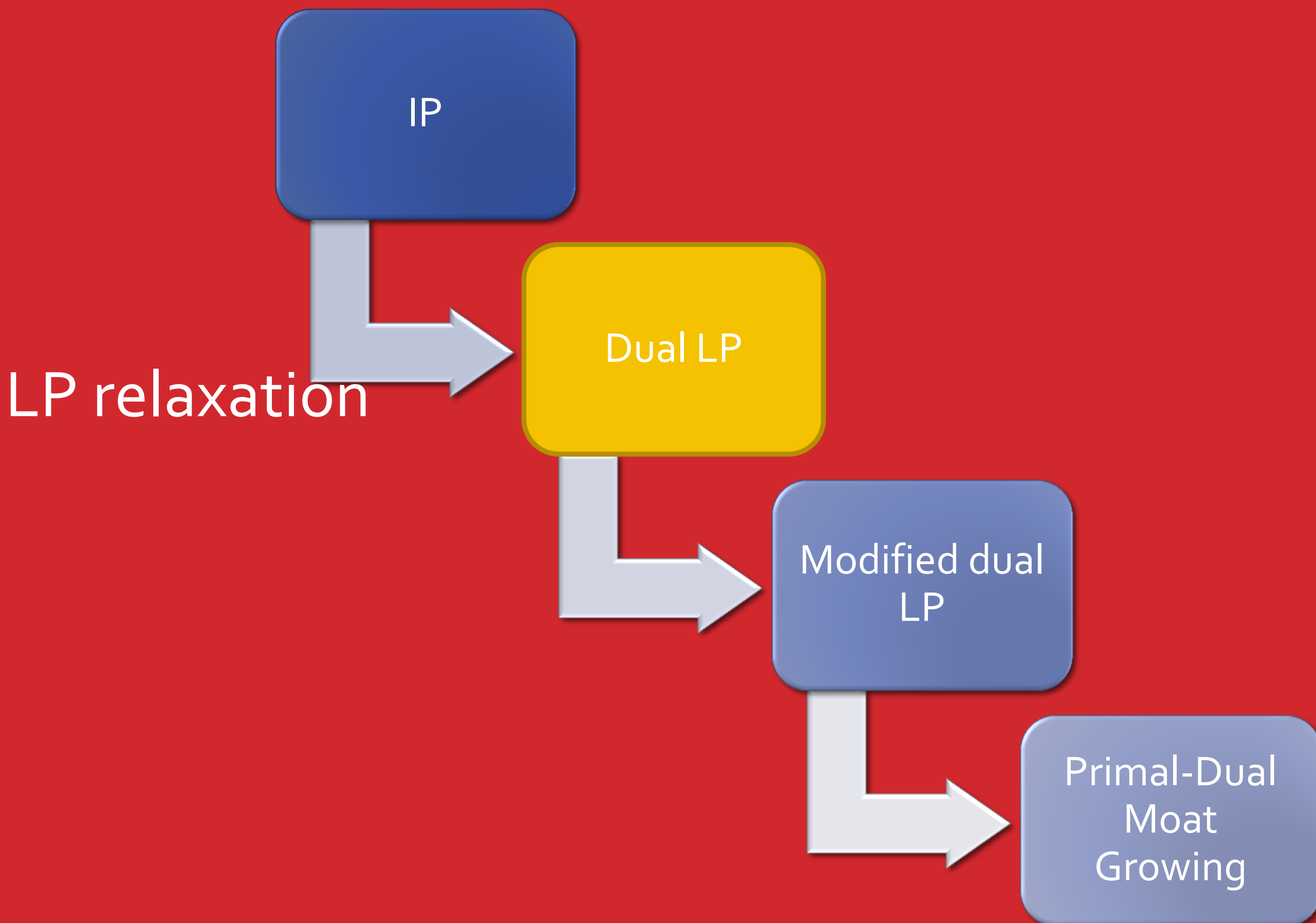
IP
Dual LP
LP relaxation
Modified dual LP
Primal-Dual Moat Growing

Dual of the linear relaxation

$$\max \quad \sum_{S \subseteq V} \sum_{(i,j):\, S \odot (i,j)} y_{S_{i,j}}$$

$s.t.$

$$\sum_{\substack{S:\, v \in \Gamma(S) \\ S \odot (i,j)}} y_{S_{i,j}} \leq c_v \qquad \forall\, v \in V$$

$$\sum_{S:\, S \odot (i,j)} y_{S_{i,j}} \leq \pi_{i,j} \qquad \forall\, demand\ (i,j)$$

$$y_{S_{i,j}} \geq 0$$

Dual of the linear relaxation

$$\max \sum_{S \subseteq V} \sum_{(i,j):\, S \odot (i,j)} y_{S_{i,j}}$$

Problem:
Many different dual variables

$$s.t.$$

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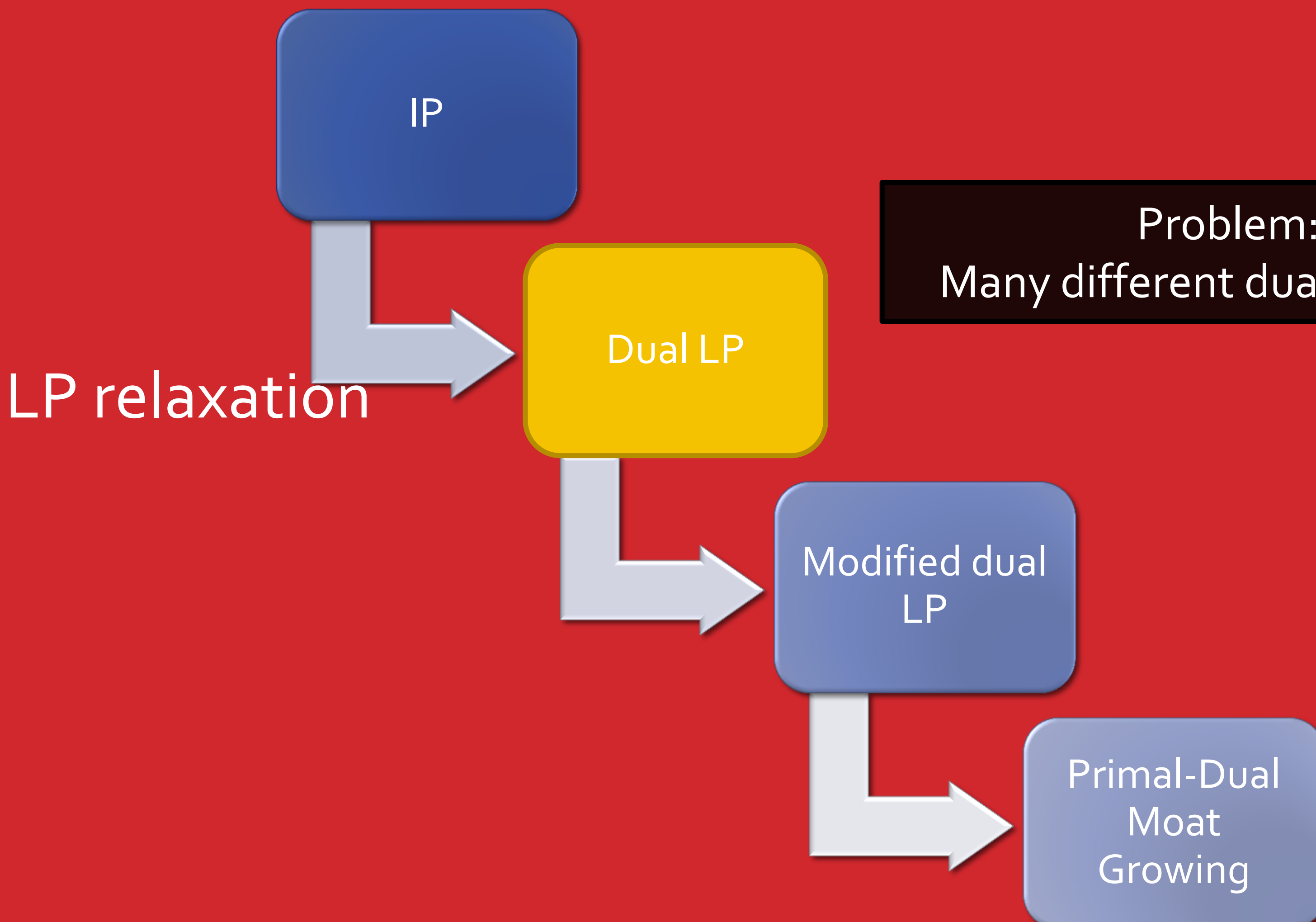


Problem:
Many different dual variables

OUTLINE

IP

LP relaxation

Dual LP

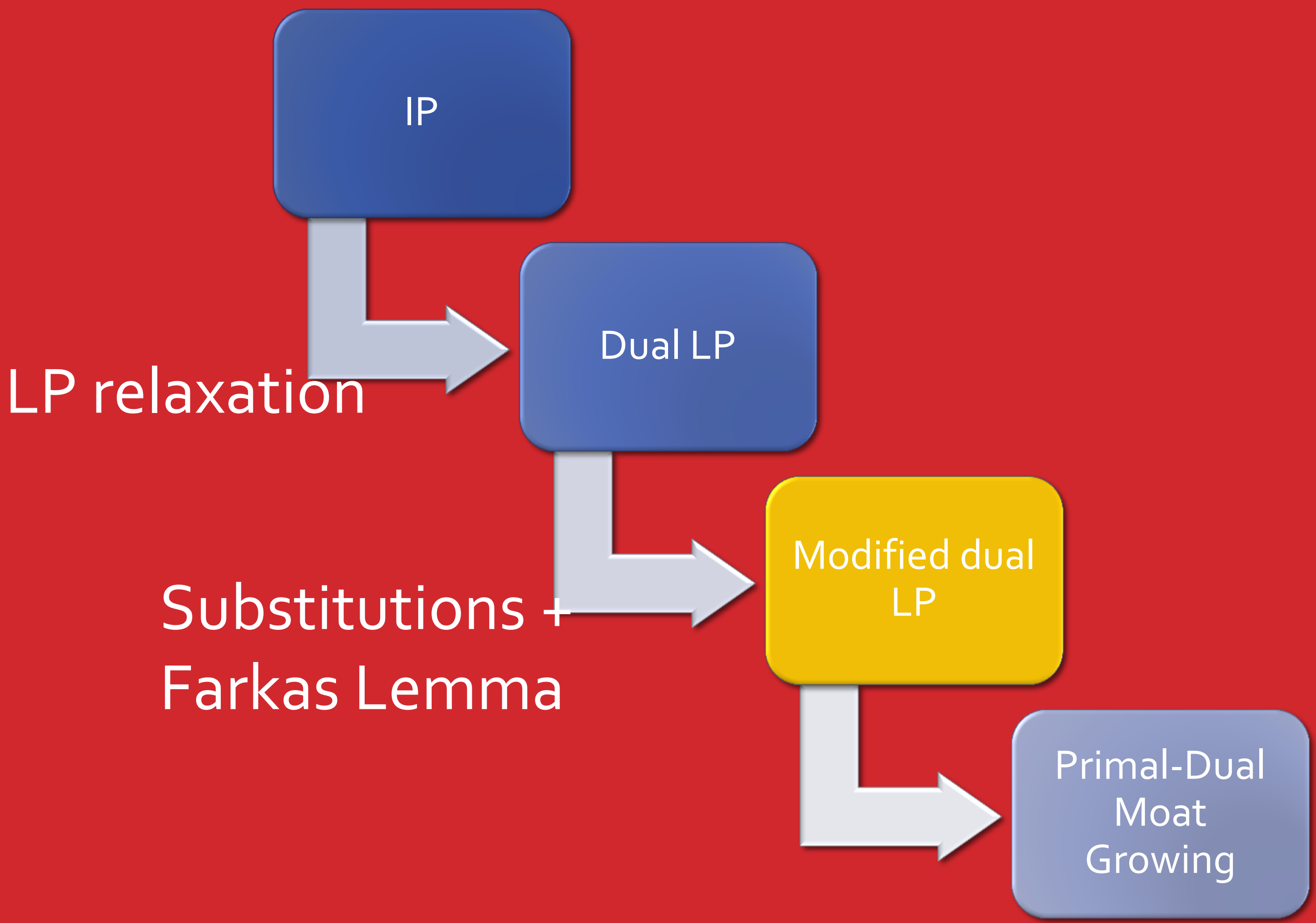
Substitutions + Farkas Lemma

Modified dual LP

Primal-Dual Moat Growing

Problem:
Many different dual variables

OUTLINE


IP
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Farkas Lemma

Modified dual program

$$\max \quad \sum_{S \subseteq V} y_S$$

$$s.t.$$

$$\sum_{S:\, v \in \Gamma(S)} y_S \leq c_v \qquad \forall\, v \in V$$

$$\sum_{S \in \mathcal{F}} y_S \leq \mathrm{g}(\mathcal{F}) \qquad \forall\, \mathcal{F} \in 2^{2^V}$$

$$y_S \geq 0$$

where $\mathrm{g}(\mathcal{F}) = \sum_{\substack{S \in \mathcal{F} \\ (i,j) \odot S}} \pi_{i,j}$

Modified dual program

$$max \quad \sum_{S\subseteq V} y_S$$

Problem simplified

$s.t.$

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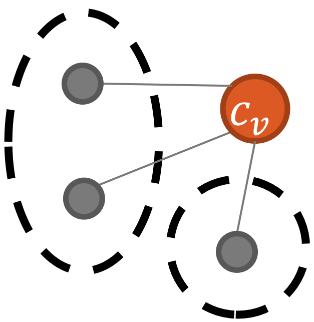

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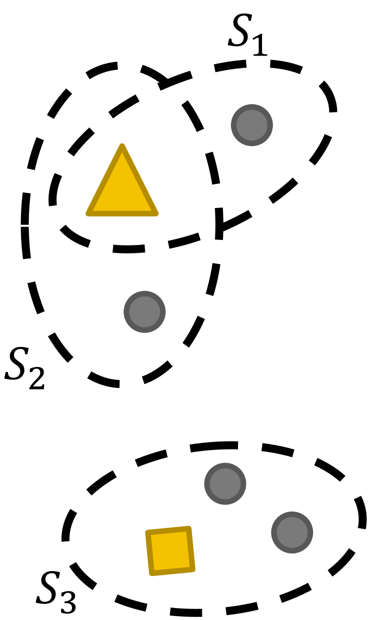

$$\mathcal{F} = \{S_1, S_2, S_3\}$$

where $g(\mathcal{F}) = \sum_{\substack{S \in \mathcal{F} \\ (i,j) \odot S}} \pi_{i,j}$

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New problem:
Double exponential numer of constraints

where $\mathrm{g}(\mathcal{F}) = \sum_{\substack{S \in \mathcal{F} \\ (i,j) \odot S}} \pi_{i,j}$

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New problem:
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Can be resolved (see later)

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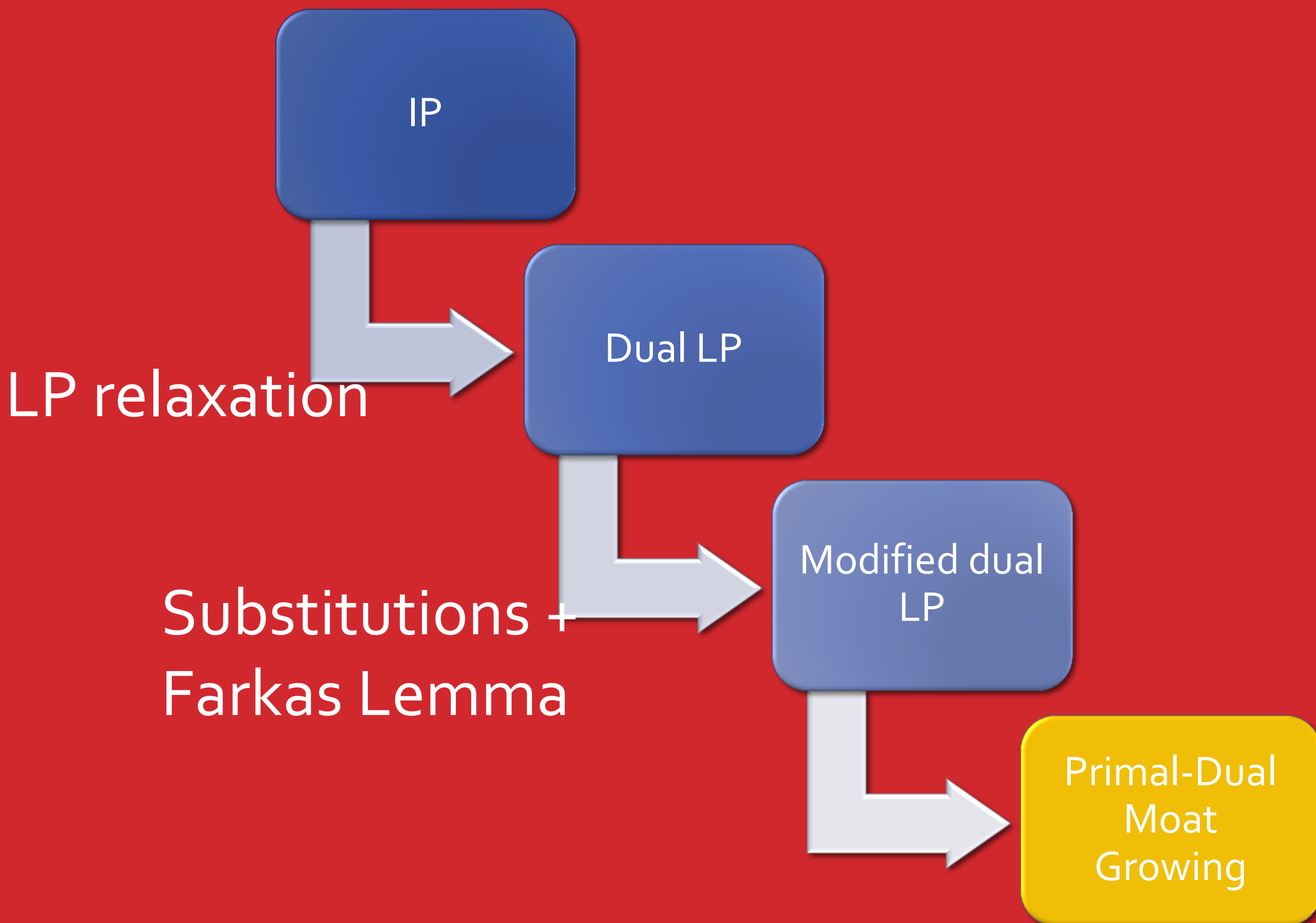
OUTLINE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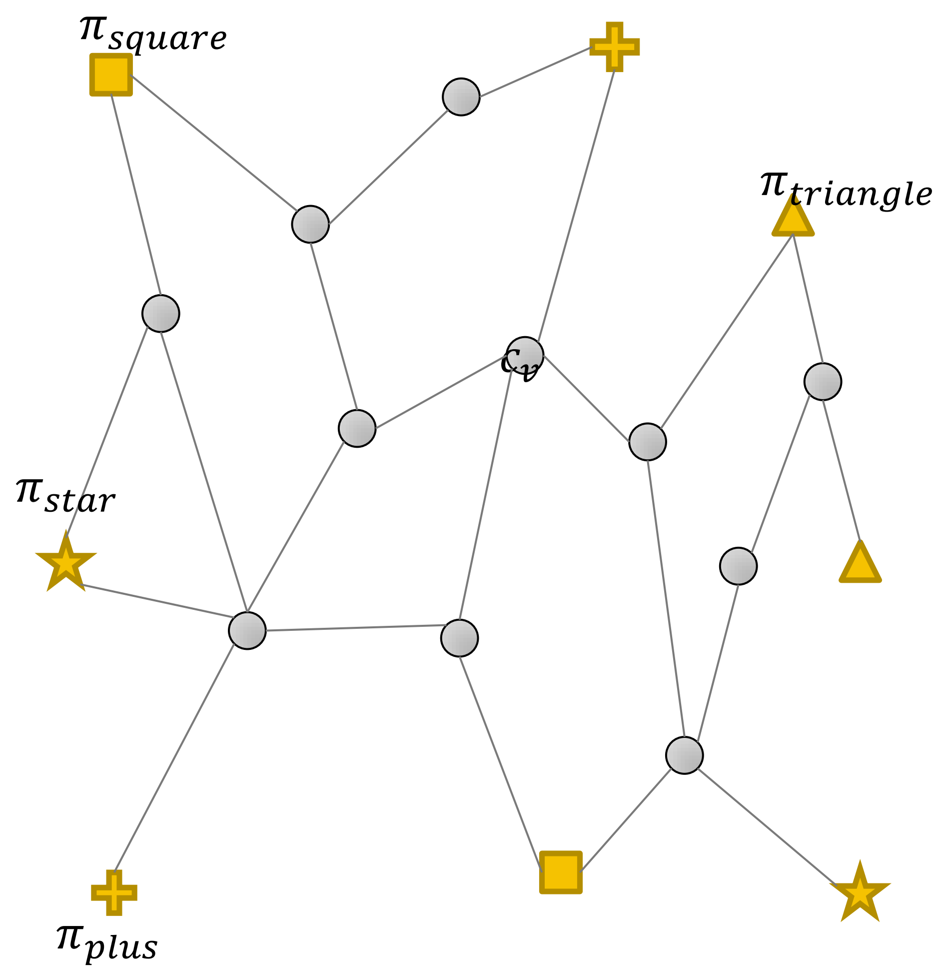

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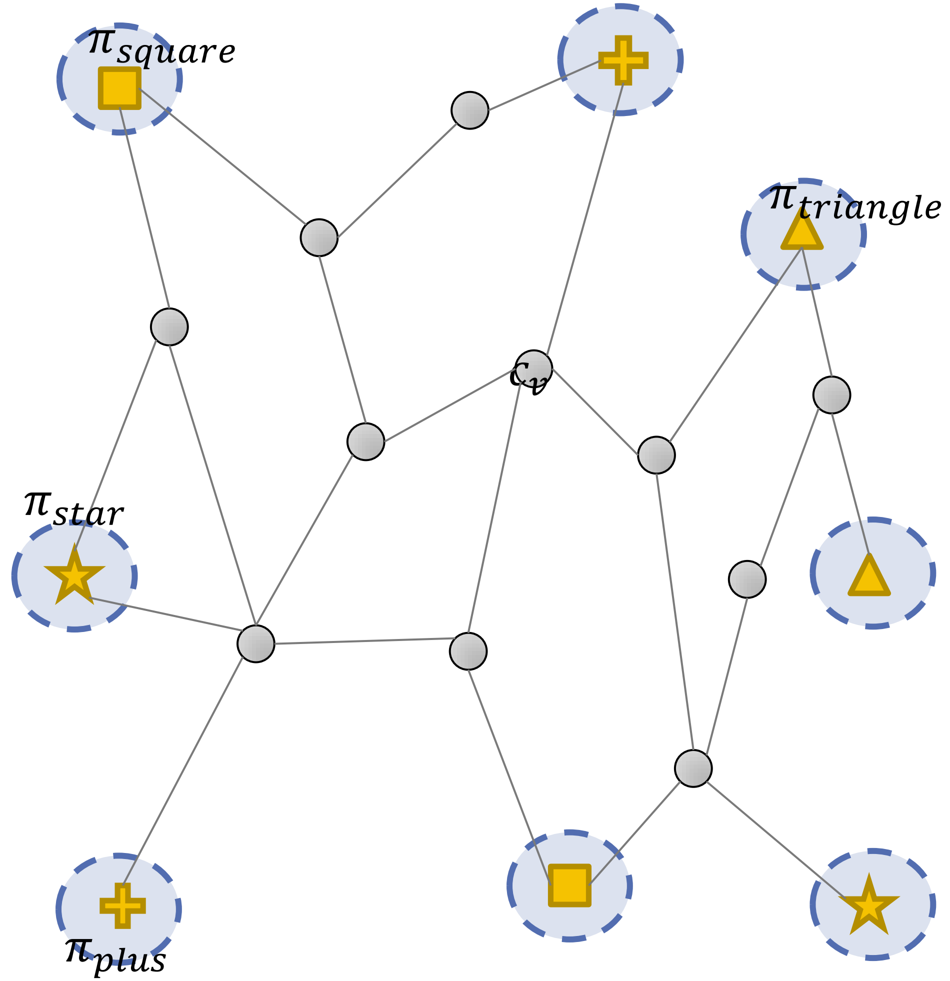

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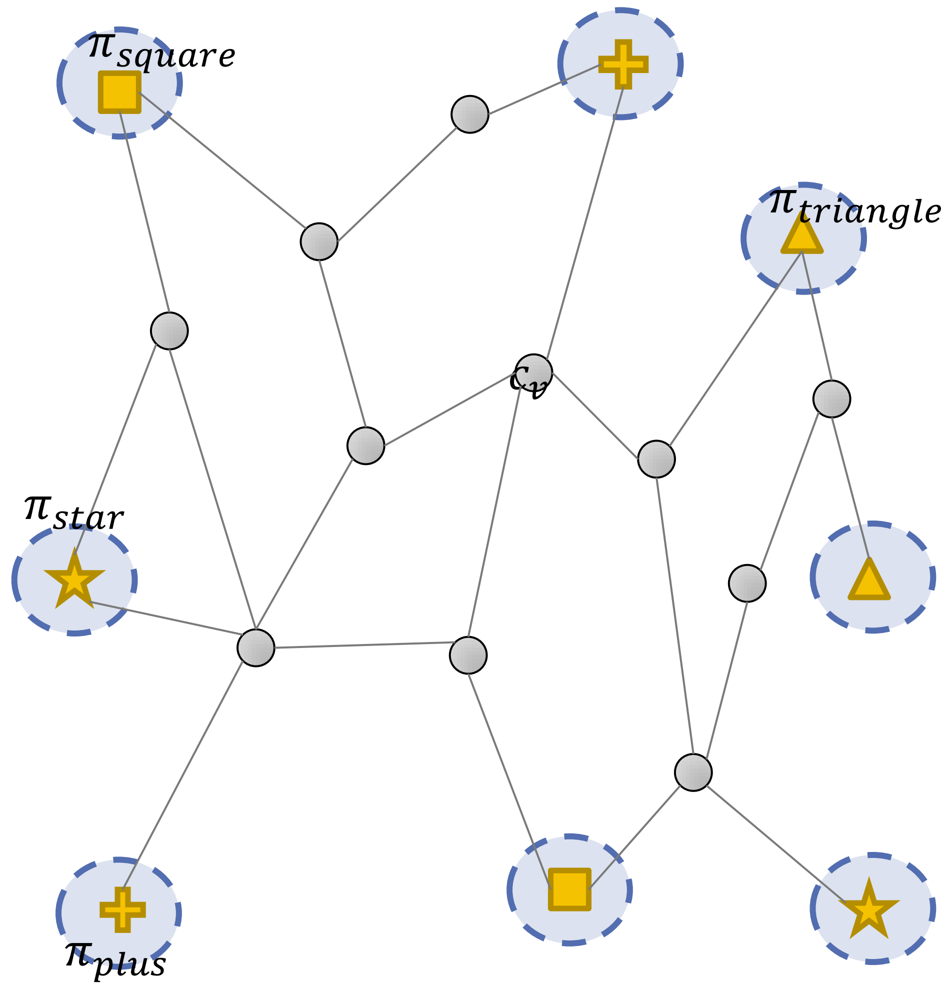

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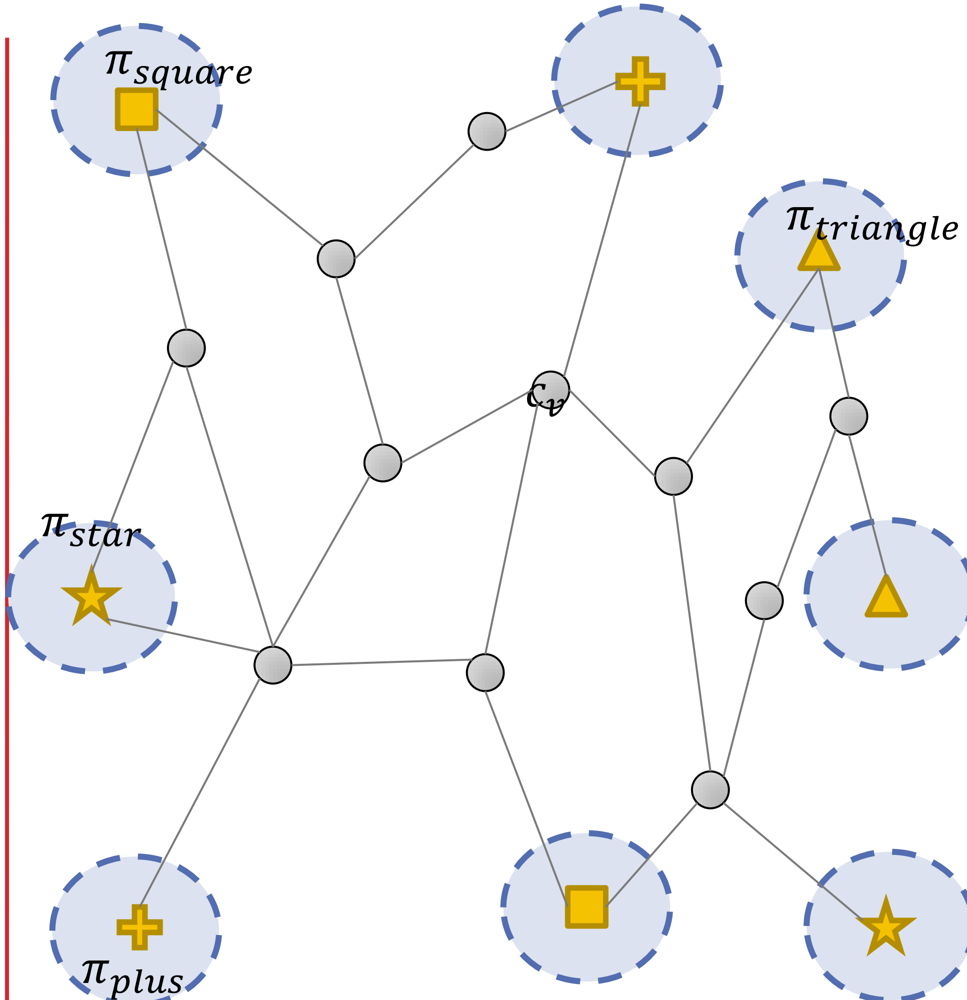

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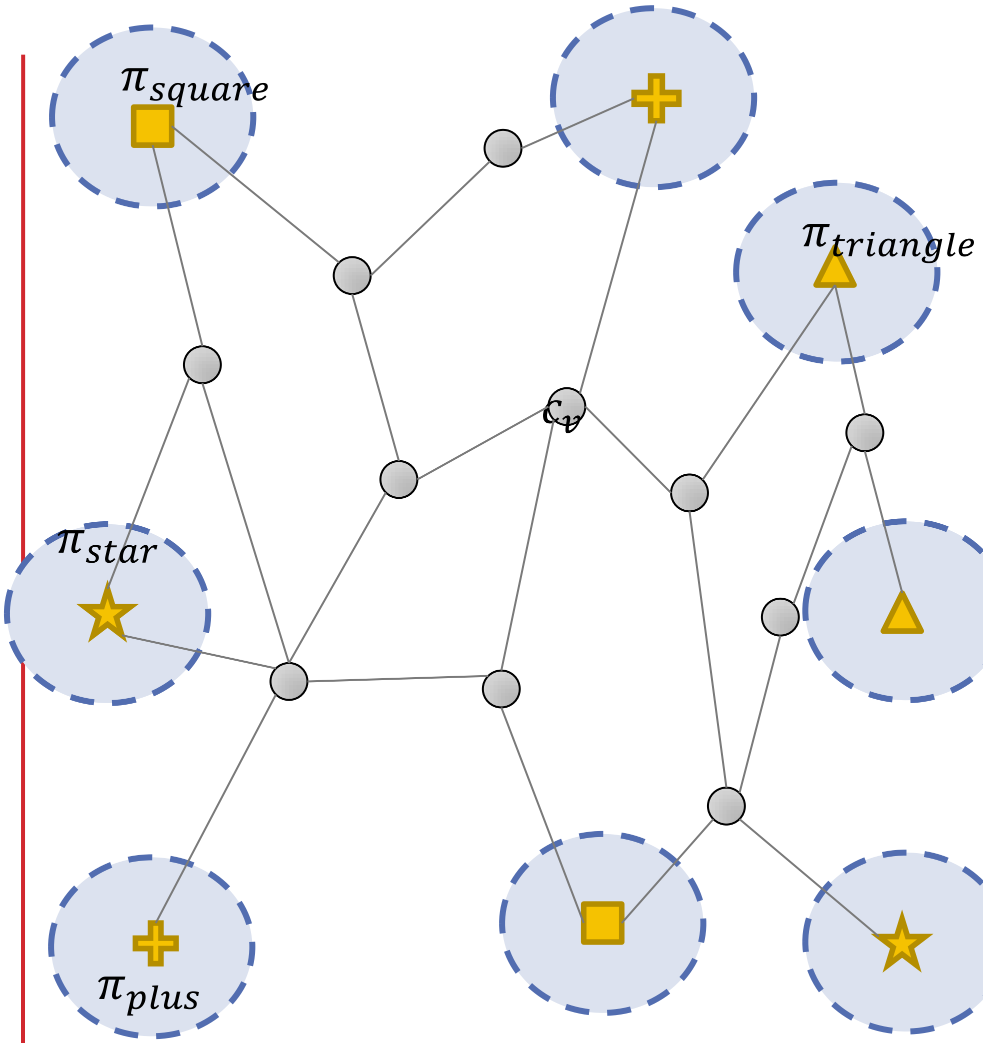

Primal-Dual Moat Growing

Vertex goes tight

$$\max \sum_{S \subseteq V} y_S$$

$$s.t. \quad \sum_{S:\, v \in \Gamma(S)} y_S \leq c_v \qquad \forall\, v \in V$$

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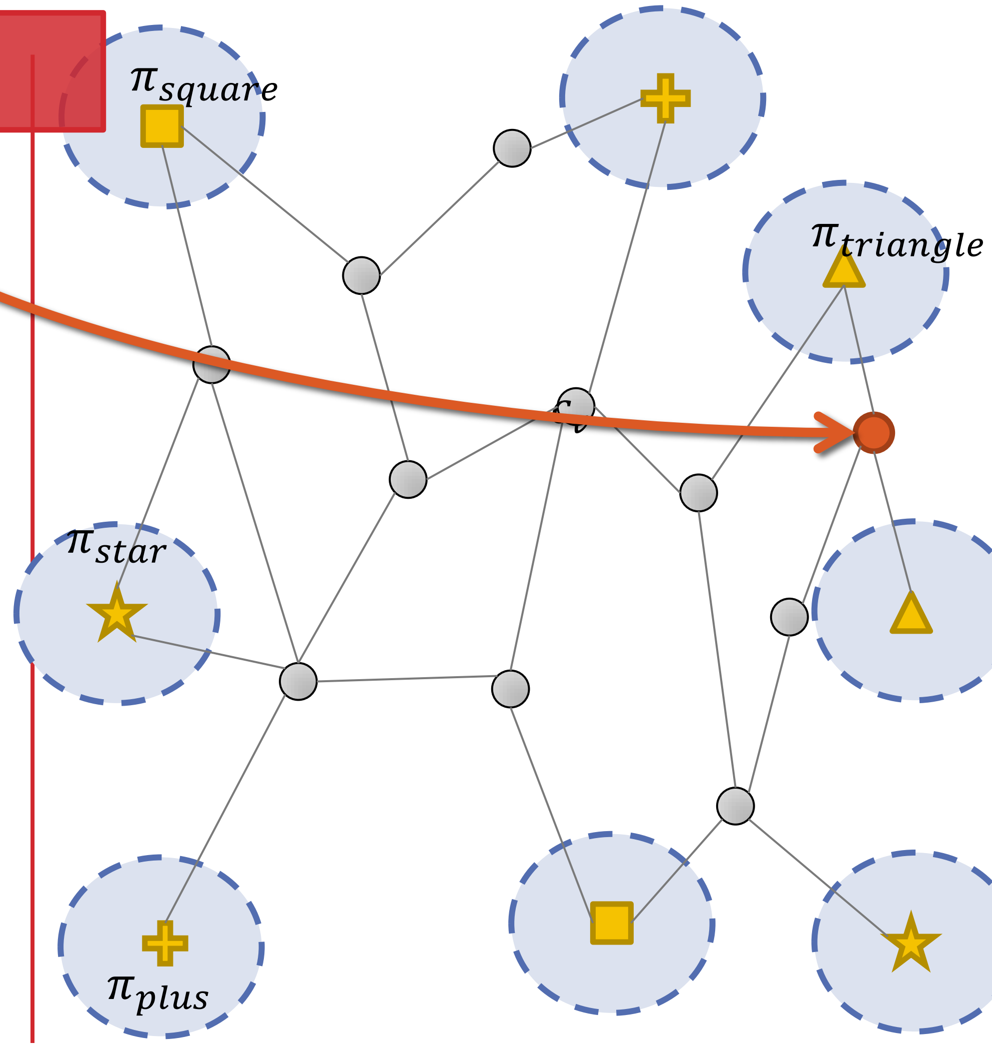

Primal-Dual Moat Growing

Vertex goes tight

Buy vertex

$$\max \sum_{S\subseteq V} y_S$$

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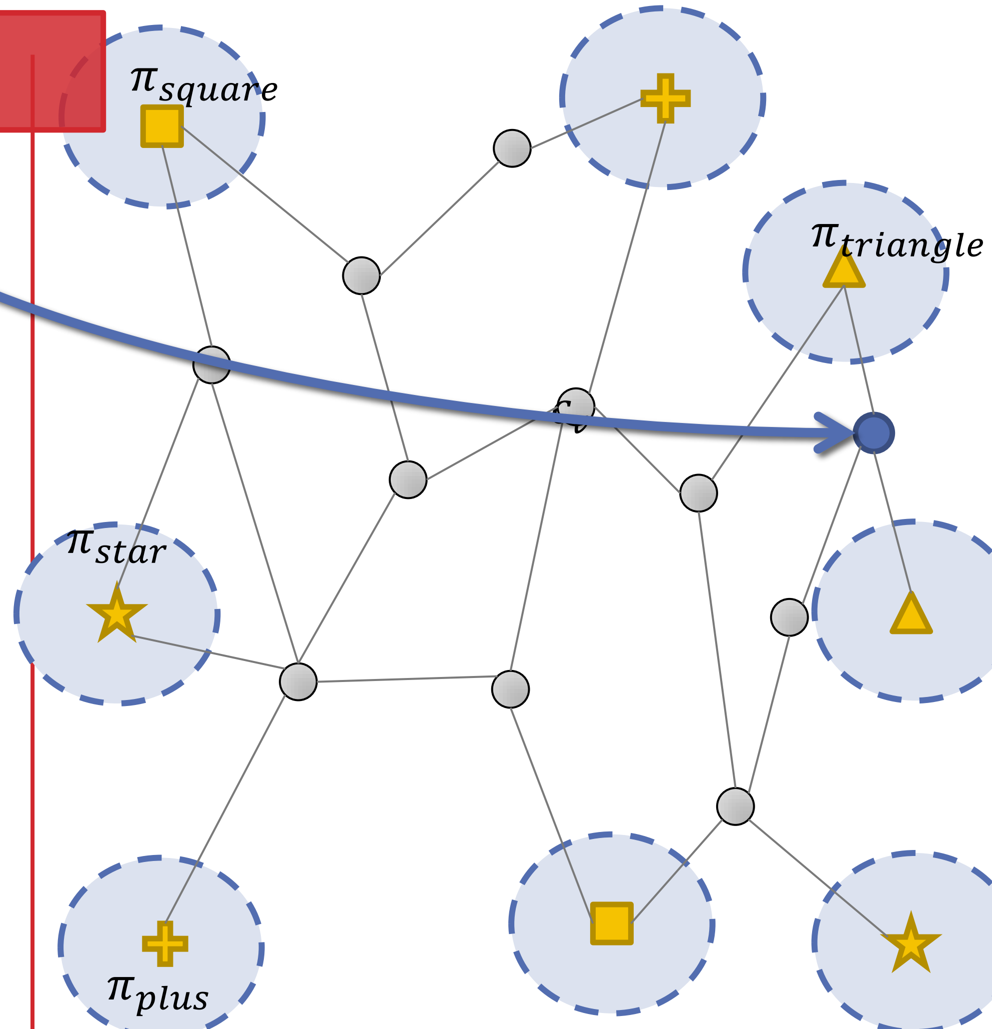
$$y_S \ge 0$$

where $g(\mathcal{F}) = \sum_{\substack{S\in\mathcal{F}\\ (i,j)\odot S}} \pi_{i,j}$



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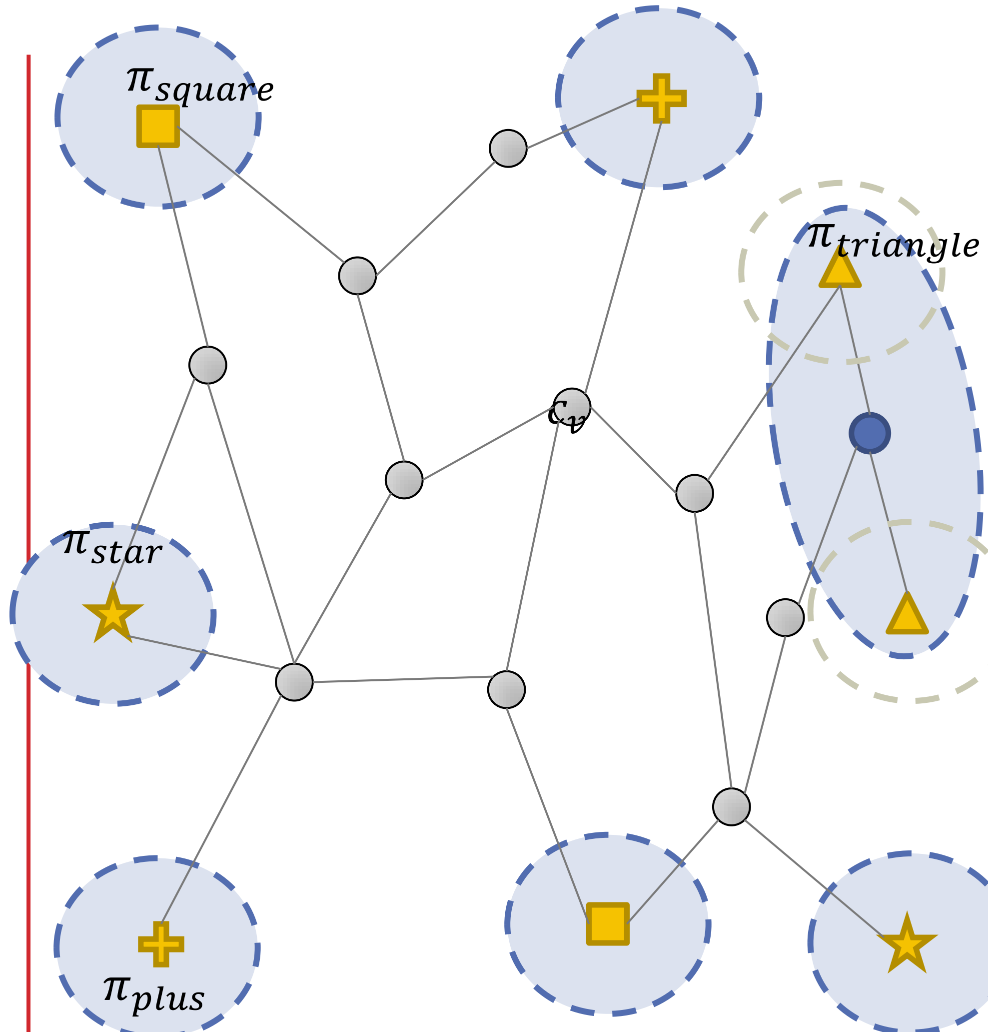

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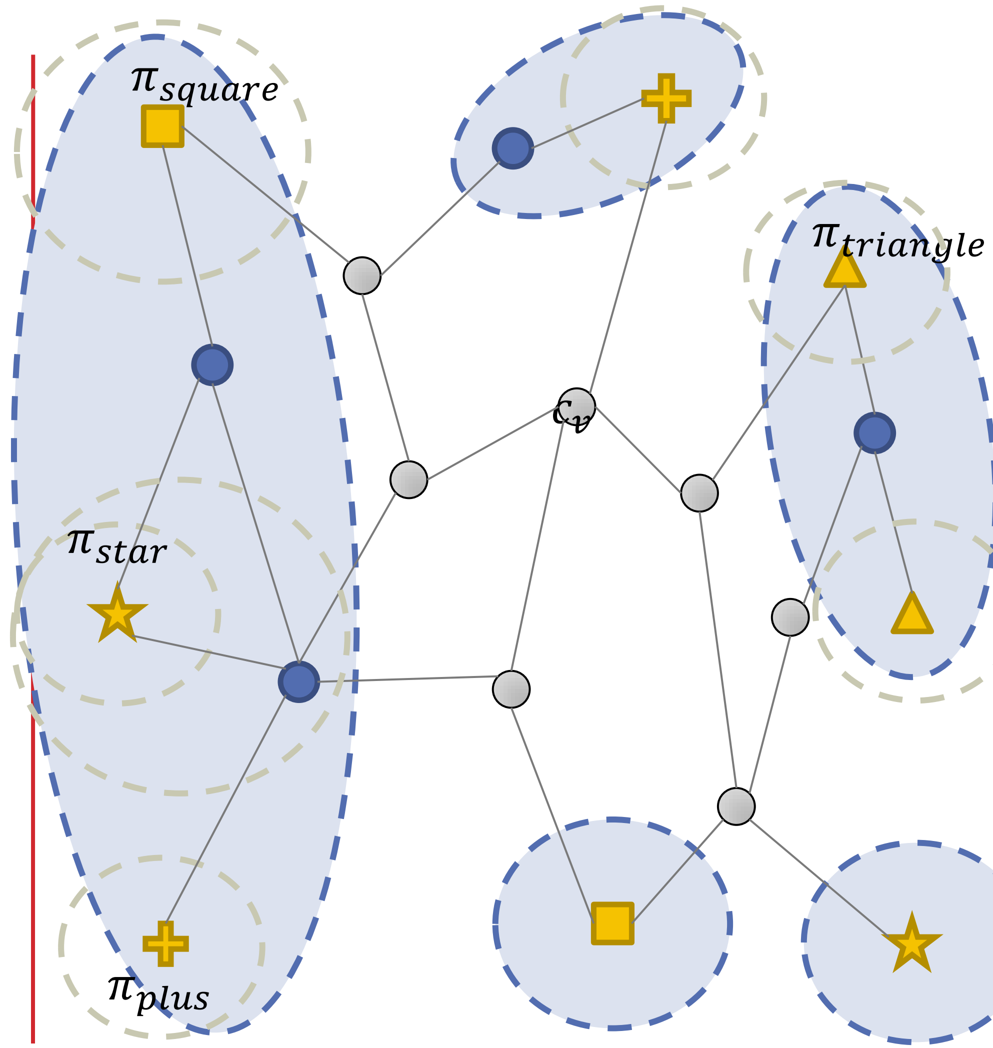


Primal-Dual Moat Growing

Family goes tight

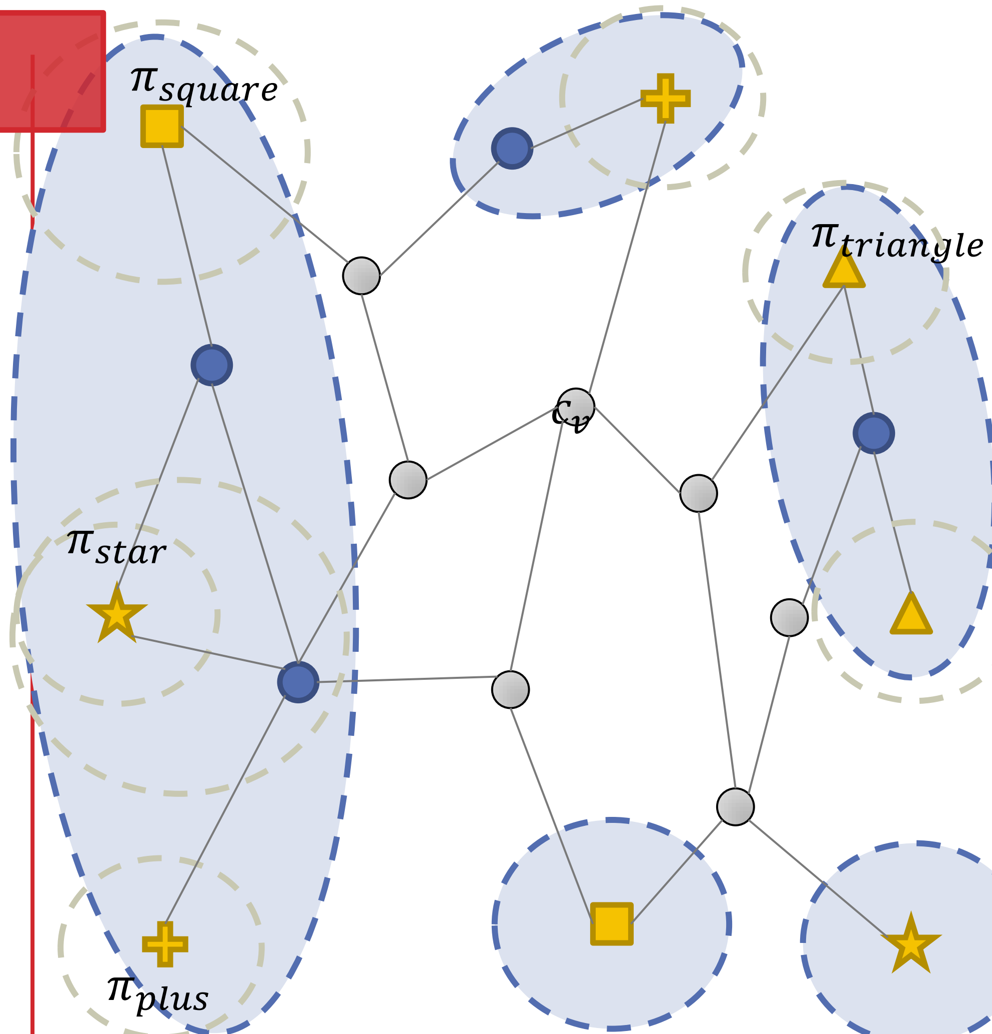
$$\max \sum_{S \subseteq V} y_S$$

$$s.t. \quad \sum_{S:\, v \in \Gamma(S)} y_S \leq c_v \qquad \forall\, v \in V$$

$$\boxed{\sum_{S \in \mathcal{F}} y_S \leq g(\mathcal{F})} \qquad \forall\, \mathcal{F} \in 2^{2^V}$$

$$y_S \geq 0$$

where $g(\mathcal{F}) = \sum_{\substack{S \in \mathcal{F} \\ (i,j) \odot S}} \pi_{i,j}$



Primal-Dual Moat Growing

Family goes tight

Max-flow computations

$$\max \sum_{S \subseteq V} y_S$$

$$s.t. \quad \sum_{S:\, v \in \Gamma(S)} y_S \leq c_v \qquad \forall\, v \in V$$

$$\boxed{\sum_{S \in \mathcal{F}} y_S \leq g(\mathcal{F})} \qquad \forall\, \mathcal{F} \in 2^{2^V}$$

$$y_S \geq 0$$

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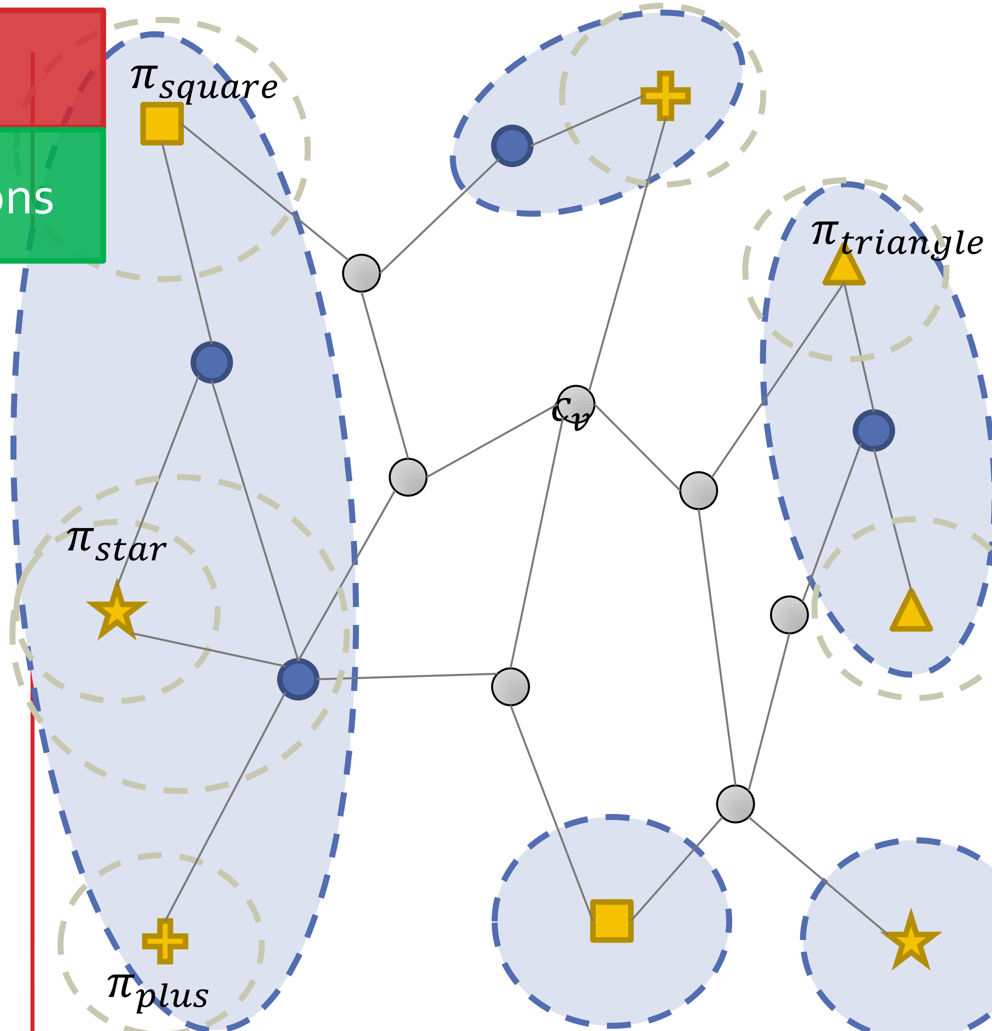


Primal-Dual Moat Growing

Family goes tight

Max-flow computations

$$\max \sum_{S\subseteq V} y_S$$

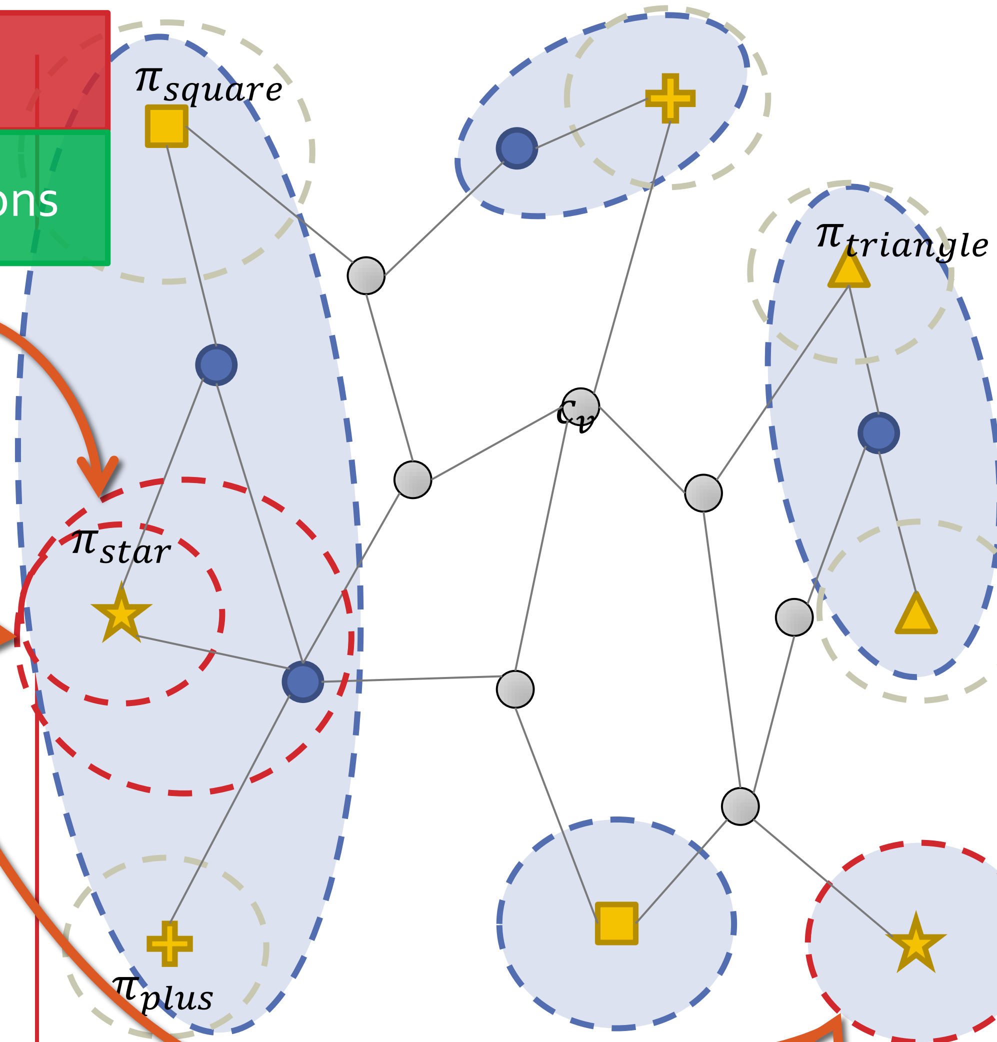
s.t.

$$\sum_{S:\, v\in\Gamma(S)} y_S \le c_v \qquad \forall\, v \in V$$

$$\boxed{\sum_{S\in\mathcal{F}} y_S \le g(\mathcal{F})} \qquad \forall\, \mathcal{F} \in 2^{2^V}$$

$$y_S \ge 0$$

where $g(\mathcal{F}) = \sum_{\substack{S\in\mathcal{F}\\ (i,j)\odot S}} \pi_{i,j}$



Primal-Dual Moat Growing

Family goes tight

Max-flow computations

Pay penalty

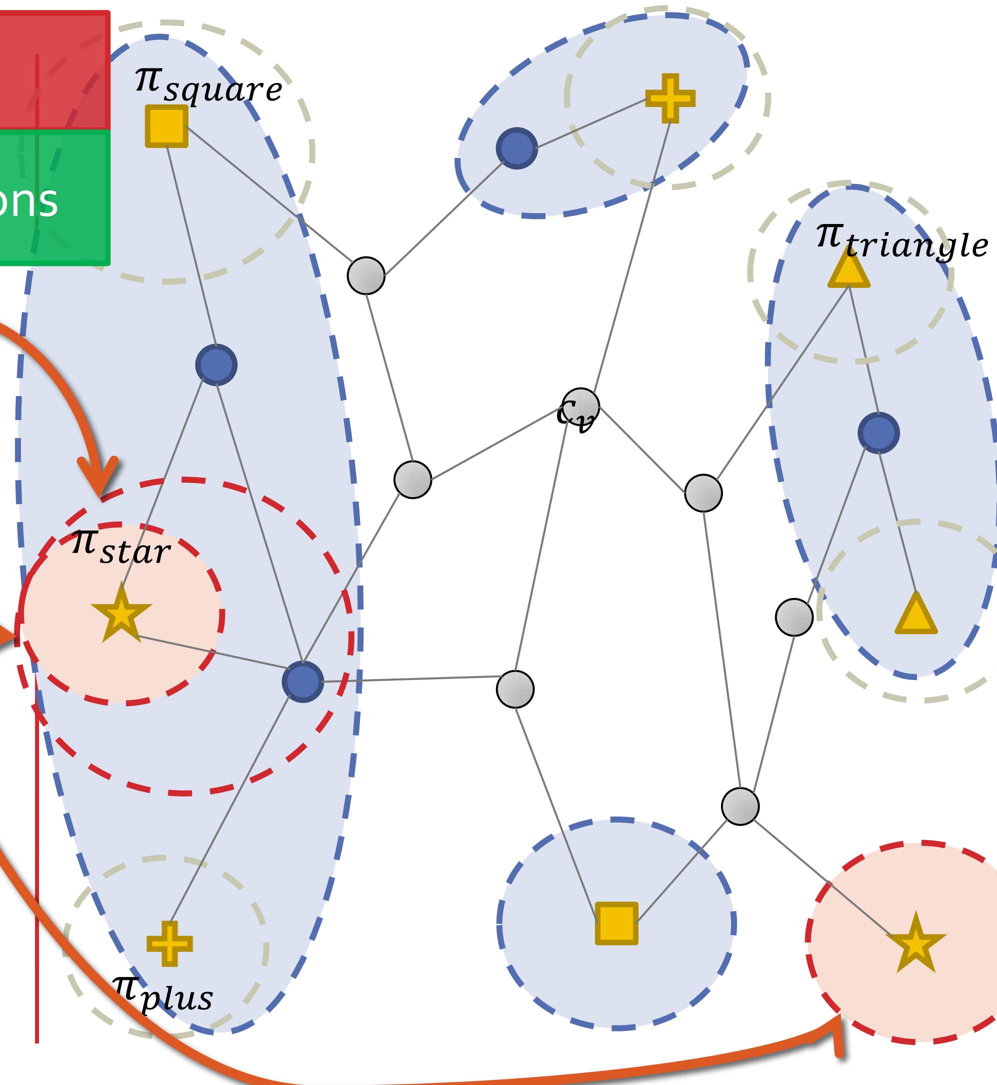
$$\max \sum_{S\subseteq V} y_S$$

$$\text{s.t.} \quad \sum_{S:\, v\in\Gamma(S)} y_S \le c_v \qquad \forall v\in V$$

$$\sum_{S\in\mathcal{F}} y_S \le g(\mathcal{F}) \qquad \forall\, \mathcal{F}\in 2^{2^V}$$

$$y_S \ge 0$$

where $g(\mathcal{F}) = \sum_{\substack{S\in\mathcal{F}\\ (i,j)\odot S}} \pi_{i,j}$



Analysis

nodes penalties

$$SOLUTION = \sum_{v} c_v x_v + \sum_{(i,j)} \pi_{i,j} z_{i,j}$$

Analysis

$$SOLUTION = \underbrace{\sum_{v} c_v x_v}_{\text{nodes}} + \underbrace{\sum_{(i,j)} \pi_{i,j} z_{i,j}}_{\text{penalties}}$$

Lemma 1 $\leq$

$$3 \sum_{S \subseteq V} y_S$$

Analysis

$$\begin{array}{rcccc}
 & & \text{nodes} & & \text{penalties} \\
SOLUTION = & & \sum_{v} c_v x_v & + & \sum_{(i,j)} \pi_{i,j} z_{i,j} \\
 & \textit{Lemma 1} & \leq & & \leq \quad \textit{Lemma 2} \\
 & & 3\sum_{S\subseteq V} y_S & & \sum_{S\subseteq V} y_S
\end{array}$$

Analysis

$$\overbrace{}^{\text{nodes}} \qquad \overbrace{}^{\text{penalties}}$$

$$SOLUTION = \sum_{v} c_v x_v + \sum_{(i,j)} \pi_{i,j} z_{i,j}$$

Lemma 1 $\leq$ $\leq$ *Lemma 2*

$$3 \sum_{S \subseteq V} y_S \qquad \sum_{S \subseteq V} y_S$$

$$SOLUTION \leq 4 \sum_{S \subseteq V} y_S$$

Analysis

nodes penalties

$$SOLUTION = \sum_{v} c_v x_v + \sum_{(i,j)} \pi_{i,j} z_{i,j}$$

Lemma 1 $\leq$ $\leq$ *Lemma 2*

$$3 \sum_{S \subseteq V} y_S \qquad \sum_{S \subseteq V} y_S$$

$$SOLUTION \leq 4 \sum_{S \subseteq V} y_S \leq 4\, OPT$$

weak duality

Analysis

Lemma 1 (nodes)

$$\sum_{v} c_v x_v \leq 3 \sum_{S \subseteq V} y_S$$

Node-Weighted Steiner Forest on Planar Graphs

3 approximation (C. Moldenhauer, 2011)

Analysis

Lemma 1 (nodes)

$$\sum_{v} c_v x_v \leq 3 \sum_{S \subseteq V} y_S$$

Node-Weighted Steiner Forest on Planar Graphs

3 approximation (C. Moldenhauer, 2011)

Lemma 2 (penalties)

$$\sum_{(i,j)} \pi_{i,j} z_{i,j} \leq \sum_{S \subseteq V} y_S$$

Edge-Weighted Prize-Collecting Steiner Forest

3 approximation (M. Hajiaghayi, K. Jain, 2006)

$$3 = \underbrace{2}_{\text{edges}} + \underbrace{1}_{\text{penalties}}$$

More on this

- Implementation (c++)

  n – number of vertices

  k – number of demands

  $M(a, b)$ - complexity of max-flow with a vertices and b edges

  Complexity $O\big(n \cdot k \cdot M(n + k,\ n \cdot k)\big) \approx O(n^2 k^2)$

Open questions

- Improve 4 - approximation

  $\approx$ 2,93 with threshold rounding (but requires solving LP)

- Node-Weighted Prize-Collecting Steiner Tree on planar graphs

  maybe even PTAS?

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Merry Christmas!