CS1382 Discrete Computational Structures

Lecture 03: Sequences and Summations

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References

The materials of this presentation is mostly from the following:

- Discrete Mathematics and Its Applications (Text book and Slides)

  By Kenneth Rosen, 7th edition

Sequences

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8 , …
  - 1, 3, 9, 27, 81, …….
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

- A **Sequence** is a function from a subset of the integers (usually either the set { 0, 1, 2, 3, 4, ... } or { 1, 2, 3, 4, ... }) to a set S.
  - The notation a_n is used to denote the image of the integer n.
  - We can think of a_n as the equivalent of f(n)
    - where f is a function from {0,1,2,.....} to S.
  - We call a_n a term of the sequence.
  - To denote the sequence as a whole, we often write { an }.

Note that { a_n } unfortunately conflicts with the notation for sets introduced earlier.

Sequences - Example

- Consider the sequence $\{a_n\}$ where $a_n = \dfrac{1}{n}$

$$\{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Geometric Progression

A geometric progression is a sequence of the form: $a, ar, ar^2, \ldots, ar^n, \ldots$

where a – initial term and r – common ratio and a, r are real numbers

Examples:

- Let a = 1 and r = -1, then

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

- Let a = 2 and r = 5, then

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

- Let a = 6 and r = 1/3, then

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic Progression

An arithmetic progression is a sequence of the form: $a, a + d, a + 2d, \ldots, a + nd, \ldots$

where a – initial term and d – common difference and a, d are real numbers

Examples:

- Let a = -1 and d = 4, then

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

- Let a = 7 and d = -3, then

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

- Let a = 1 and d = 2, then

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Recurrence Relations

- A **recurrence relation** for the sequence $\{ a_n \}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.
- The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Recurrence Relations

Example:

- Let { a_n } be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1,2,3,4,…. and suppose that $a_0 = 2$. What are a_1 , a_2 and a_3? [Here $a_0 = 2$ is the initial condition]

- Solution: We see from the recurrence relation that

$a_1 = a_0 + 3 = 2 + 3 = 5$

$a_2 = 5 + 3 = 8$

$a_3 = 8 + 3 = 11$

Exercise

- Let $\{ a_n \}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} – a_{n-2}$ for $n = 2,3,4,\ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3 ?

  [Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

- Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 – 3 = 2$$

$$a_3 = a_2 – a_1 = 2 – 5 = –3$$

Fibonacci Sequence

- We define the ***Fibonacci sequence***, $f_0, f_1, f_2, \ldots$ by:
  - Initial Conditions: $f_0 = 0,\ f_1 = 1$
  - Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6.

Answer:

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$

$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$

$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$

$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$

$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

Solving Recurrence Relations

- Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called **solving the recurrence relation**.

- Such a formula is called a **closed formula**.

- Various methods for solving recurrence relations will be covered in Chapter 8.

Iterative Solution Example

Method 1: Working upward, forward substitution

- Let { a_n } be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 2,3,4,…. and suppose that $a_1 = 2$.

$a_2 = 2 + 3$

$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$

$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3$

.

$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$

Exercise:

Solve the recurrence relation $a_n = 2\, a_{n-1} + 1$ for n = 2,3,4,…. and suppose that $a_1 = 1$

Exercise

- Consider the recurrence relation $a_n = 2\,a_{n-1} - a_{n-2}$ for $n >= 2$.

  Does $a_n = 3n$ satisfy this relation?

  $$\begin{aligned} a_n &= 2\,a_{n-1} - a_{n-2} \\ &= 2(3(n-1)) - 3(n-2) \\ &= 6n - 6 - 3n + 6 \\ &= 3n \qquad \text{OK!} \end{aligned}$$

- Verify that $a_n = 3^{n+4}$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$

Exercise

1. Find formulae for the sequences with the following first five terms: 1, ½, ¼, 1/8, 1/16

   ***Solution**: Note that the denominators are powers of 2. The sequence with $a_n = 1/2^n$ is a possible match. This is a geometric progression with $a = 1$ and $r = ½$.*

2. Consider 1, 3, 5, 7, 9

   ***Solution:** Note that each term is obtained by adding 2 to the previous term. A possible formula is $a_n = 2n + 1$. This is an arithmetic progression with $a = 1$ and $d = 2$.*

3. 1, -1, 1, -1,1

   ***Solution:** The terms alternate between 1 and -1. A possible sequence is $a_n = (-1)n$. This is a geometric progression with $a = 1$ and $r = -1$.*

Useful Sequences

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Applications of Integer Sequences

Integer sequences appear in a wide range of contexts:

- Sequence of prime numbers (Chapter 4)
- The number of ways to order n discrete objects (Chapter 6)
- The number of moves needed to solve the Tower of Hanoi puzzle with n disks (Chapter 8)
- The number of rabbits on an island after n months (Chapter 8)

Example – Compound Interest

- Suppose a person deposits $1,000 in a savings account yielding 3% per year with interest compounded annually. How much is in the account after 20 years?
  - Let P_n denote amount after n years
  - $P_n = P_{n-1} + 0.03\ P_{n-1} = (1.03)\ P_{n-1}$
  - The initial condition $P_0 = 1000$.
  - $P_1 = (1.03)\ P_0$, …, $P_n = (1.03)\ P_{n-1} = (1.03)^n\ P_0$
  - $P_{20} = (1.03)^{20}\ 1000 = 1,806$

Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation: $$\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{m \leq j \leq n} a_j$$

  represents $a_m + a_{m+1} + \cdots + a_n$

- The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

Summations

- More generally for a set S: $\sum_{j \in S} a_j$

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{0}^{n} r^j$$

- Examples:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{1}^{\infty} \frac{1}{i}$$

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation: $$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{m \le j \le n} a_j$$

represents $a_m \times a_{m+1} \times \cdots \times a_n$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)

Questions?

Thank You!