CS1382 Discrete Computational Structures

Lecture 01: Sets

Spring 2019

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TEXAS TECH UNIVERSITY

References

The materials of this presentation is mostly from the following:

- Discrete Mathematics and Its Applications (Text book and Slides)

  By Kenneth Rosen, 7th edition

- Lecture slides of CMSC 250: Discrete Structures (Summer 2016)

  By Jason Filippou, University of Maryland

Sets

- A set is an unordered collection of objects
  - the students enrolled in this class
  - the laptops in the lab
- Sets are used to group objects together
  - Zero or more objects
  - Often, but not always, objects with similar properties
- Examples:
  - V = { a, e, i, o, u }
  - S = {1,2,3,........,99}

- Key Ideas:
  - Sets are unordered: {a, b, c} = {a, c, b} = {b, a, c}
  - Repetitions in a set description are irrelevant: {a, a, b} = {a, b}
  - Naming variations are irrelevant. If x=y, then {x, y} = {x}= {y}

Sets – Membership ($\in$)

- The objects in a set are called the elements, or members of the set.
- A set is said to contain its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$
- Example
  - $a \in \{a, b, c\}$ is read "a is an element of $\{a, b, c\}$"

Set Notions

- Roster Method
  - Curly Braces
    - S = { 1 , 3 , 5 , 7, 9 }
    - Z = { John, Mark, Jude }
  - Ellipses (...)
    - A = { a, b, c, d, … , z }
- Set Builder Method
  - State the property or properties that all members must satisfy
    - S = {x | x is a positive integer less than 100}
    - O = {x | x is an odd positive integer less than 10}
    - O = $\{x \in Z^+ \mid x$ is odd and $x < 10\}$

- Important Sets:
  - N = natural numbers = {0,1,2,3….}
  - Z = integers = {…,-3,-2,-1,0,1,2,3,…}
  - Z^+ = positive integers = {1,2,3,…..}
  - R = set of real numbers
  - R^+ = set of positive real numbers
  - C = set of complex numbers.
  - Q = set of rational numbers

Set Notions

- Interval Notation:
  - $[a,b] = \{ x \mid a \leq x \leq b \}$
  - $[a,b) = \{ x \mid a \leq x < b \}$
  - $(a,b] = \{ x \mid a < x \leq b \}$
  - $(a,b) = \{ x \mid a < x < b \}$

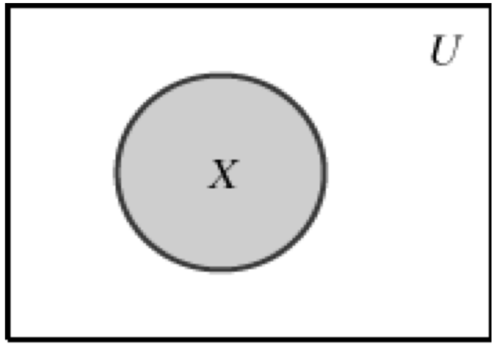
Closed interval - [a, b]

Open interval - (a, b)

- Some common confusions pre-empted…
  - Elements of sets may be themselves sets: {1, 2, {1}}
  - And in particular we note: $1 \neq \{1\} \neq \{\{1\}\}$
  - Those are three different things,
    - 1 is the number 1, (and it is not a set)
    - {1} is a set, with one element: the number 1.
    - {{1}} is a set, with one element {1}, which is a set with one element, the number 1.
    - S = { N, Z, Q, R}

Sets – Universal and Empty Sets

- Universal set U
  - Set containing everything currently under consideration.
    - Sometimes implicit
    - Sometimes explicitly stated.
    - Contents depend on the context.
- Empty or null set
  - Unique set with no elements.
  - Symbolized Ø, but {} also used.



Some practice …

- How many elements do the following sets contain?
  - Ø
  - { Ø }
  - { { } }
- Is Ø = { Ø } ?

Empty Set Drill

Let $A = \{1,2,3\}$, $B = \{1,2,3,\varnothing\}$, $C = \{1,2,3,\{\varnothing\}\}$ [remember: $\varnothing = \{\}$]

1. $\varnothing \subseteq A$
2. $\varnothing \in A$
3. $\varnothing \subseteq B$
4. $\varnothing \in B$
5. $\{\varnothing\} \subseteq A$
6. $\{\varnothing\} \in A$
7. $\{\varnothing\} \subseteq B$
8. $\{\varnothing\} \in B$
9. $\{\varnothing\} \subseteq C$
10. $\{\{\varnothing\}\} \subseteq C$
11. $\{\varnothing\} \in C$
12. $\varnothing \subseteq C$
13. $\varnothing \in C$

Set Equality

- Two sets are equal if and only if they have the same elements.
- Therefore if A and B are sets, then A and B are equal if and only if

$$\forall x(x \in A \leftrightarrow x \in B)$$

- We write A = B if A and B are equal sets.
  - {1,3,5} = {3, 5, 1}
  - {1,5,5,5,3,3,1} = {1,3,5}

Set Cardinality

- Let S be a finite set, the number of distinct elements in S is the **cardinality** of S.
- The cardinality of S is denoted by $|S|$.
- Examples
  - $| \emptyset | = 0$
  - Let S be the letters of the English alphabet. Then $|S| = 26$
  - $| \{ 1, 2, 3 \} | = 3$
  - $| \{ \emptyset \} | = 1$

(Absolute) Set Complement

- Let S be a set in the universal domain U, the **absolute complement** of S, denoted S’, is defined as the set:

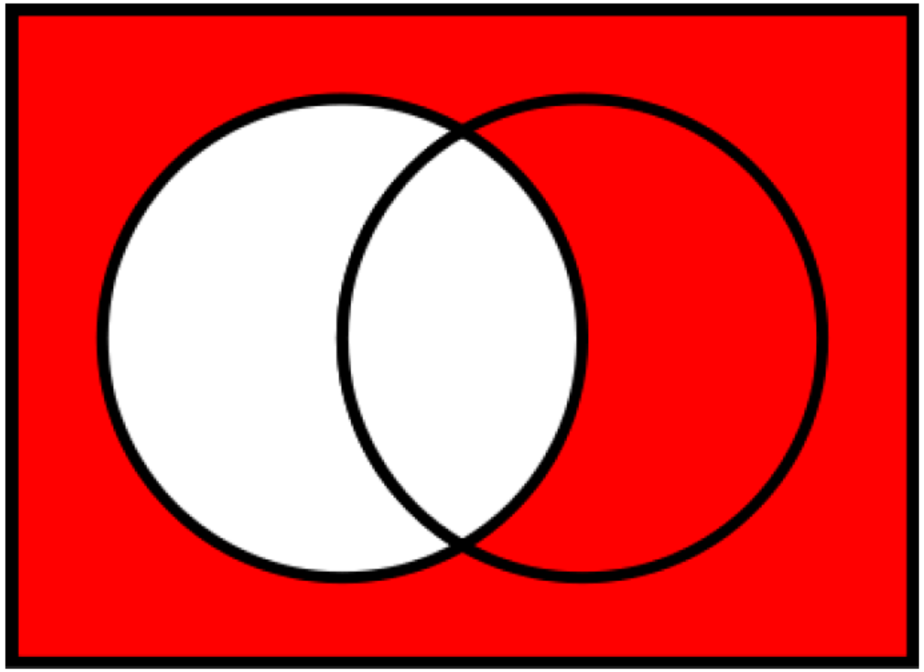
  $\{ x \in U \mid x \notin A \}$

- Complement of an empty set

  $\emptyset ' = U$

- Complement of universal domain

  $U ' = \emptyset$



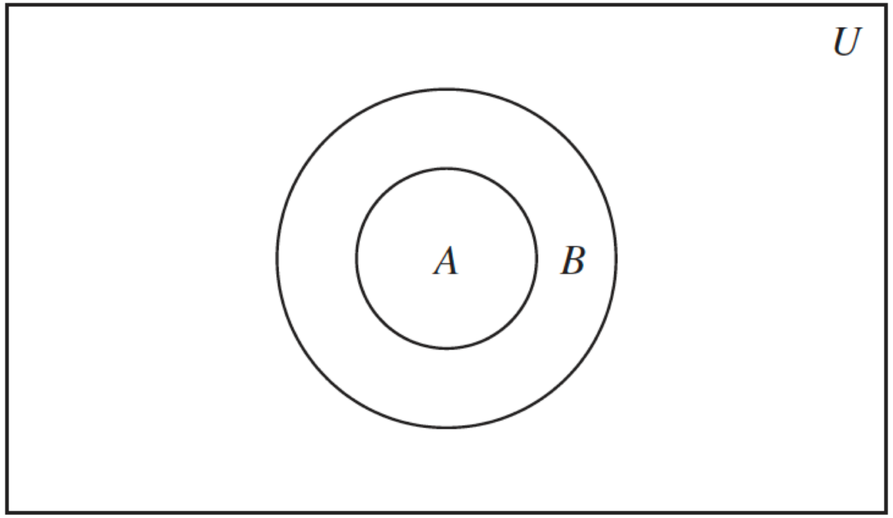
Subsets

- The set A is a subset of B, if and only if every element of A is also an element of B.
- The notation A ⊆ B is used to indicate that A is a subset of the set B.
- A ⊆ B holds if and only if

$$\forall x(x \in A \rightarrow x \in B)$$

  is true.

- For every set S:
  - ∅ ⊆ S
  - S ⊆ S
- Examples: N ⊆ Z , { 1, 3, 5 } ⊆ { 1, 3, 5 }



Superset

- The set A is a superset of set B, denoted A ⊇ B, if and only if B ⊆ A

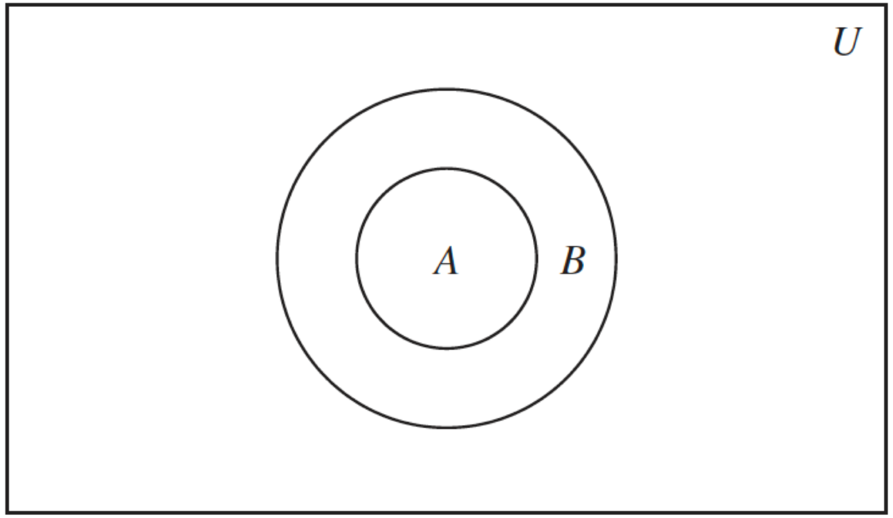
Proper Subsets

- If A ⊆ B, but A ≠B, then we say A is a proper subset of B, denoted by A ⊂ B.
- If A ⊂ B, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

- Examples:
  - Q ⊂ R
  - { 1 } ⊂ { 1, 3, 5, 7, 9 }



Proper Superset

- The set A is a superset of set B, denoted A ⊃ B, if and only if B ⊂ A

Subsets and Membership

Are the following statements true or false ?

1. $\{ 1 \} \subseteq \{ 1, 2, 3 \}$?
2. $\{ \{ 1 \} \} \subseteq \{ 1, 2, 3 \}$?
3. $\{ \{ 1 \} \} \subseteq \{ 1, 2, 3, \{ 1 \} \}$?
4. $\{ \{ 1 \} \} \in \{ 1, 2, 3, \{ 1 \} \}$?
5. $\{ 1 \} \in \{ 1, 2, 3, \{ 1 \} \}$?
6. $\emptyset \subseteq Z$?
7. $\emptyset \subseteq \{ \}$?
8. $\emptyset \in \{ \}$?
9. $\emptyset \in \{ \{ \} \}$?
10. $\emptyset \in \{ \{ \emptyset \} \}$?

Power Sets

- Given a set S, the power set of S is the set of all subsets of the set S.
- The power set of S is denoted by $\mathcal{P}(\{S\})$
- For every set S,
  - $\emptyset \in \mathcal{P}(\{S\})$
  - $S \in \mathcal{P}(\{S\})$
- Examples
  - $A = \{a, b\}$, then $\mathcal{P}(\{A\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Some practice …

- $\mathcal{P}(\{a, b, c\})$
- $\mathcal{P}(\{-1\})$

Tuples

- The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has
  - a_1 as its first element
  - a_2 as its second element
  - and so on until a_n as its last element
- Two n-tuples are equal if and only if their corresponding elements are equal
- 2-tuples are called ordered pairs.
- The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product

- The Cartesian Product of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

- Example
  - A = { a, b } , B = {1, 2, 3}
  - A × B = { (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3) }
- Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product A × B and how can it be used?

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Set Operations

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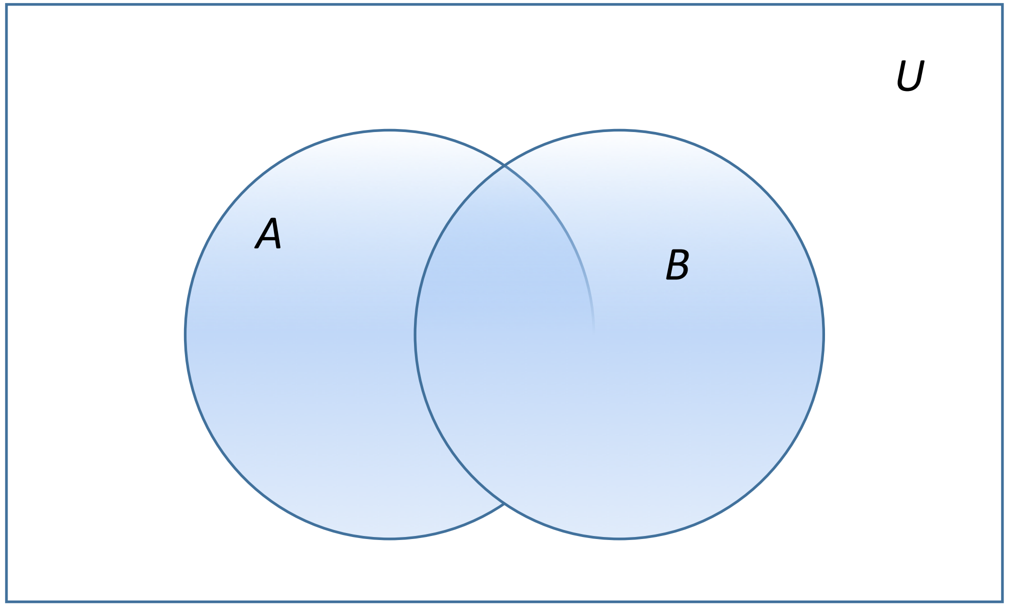
Union

Let A and B be sets. The union of the sets A and B, denoted by A ∪ B, is the set:

$$\{x | x \in A \vee x \in B\}$$

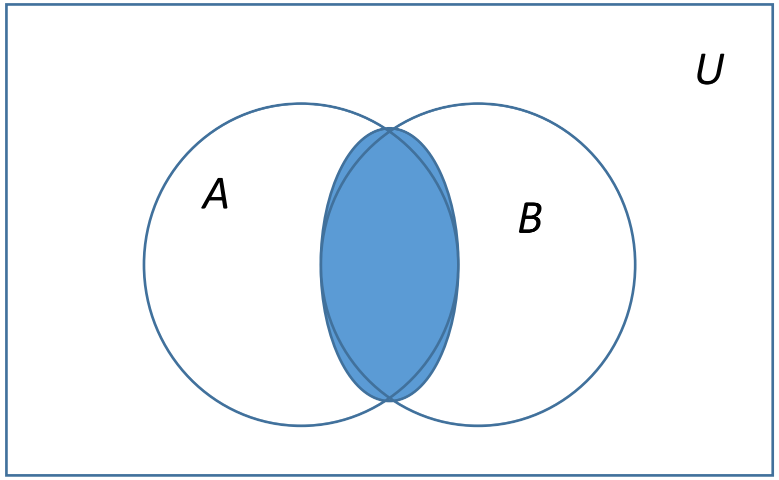


- Examples:
  - { -10, -7, 6 } U { 4 , 1 } = { -10, -7, 6, 4, 1 }
  - { -3, 5, 10 } U { 5, 7 } = { -3, 5, 10, 7 }
  - { {-1}, { 1 }, 0 } U { -1, 0 } = { 0, -1, {-1} , {1} }

Intersection

The intersection of sets A and B, denoted by A ∩ B, is $\{x | x \in A \land x \in B\}$

Note if the intersection is empty, then A and B are said to be disjoint.



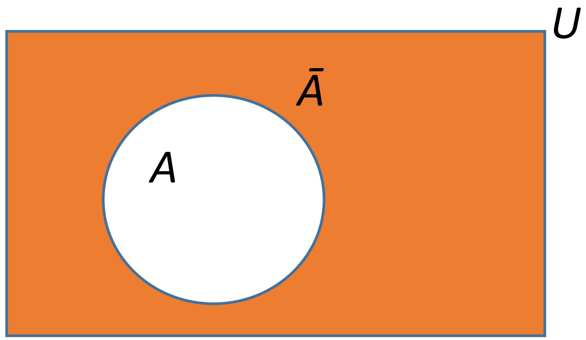
- Examples:
  - { -1, 4, 1 } ∩ { 4 , 1 } = { 4, 1 }
  - { -3, 10 } ∩ { 5, 7 } = Ø

Complement

If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set U – A

$$\bar{A} = \{x \in U \mid x \notin A\}$$

The complement of A is sometimes denoted by A^c



- Examples:
  - If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$?

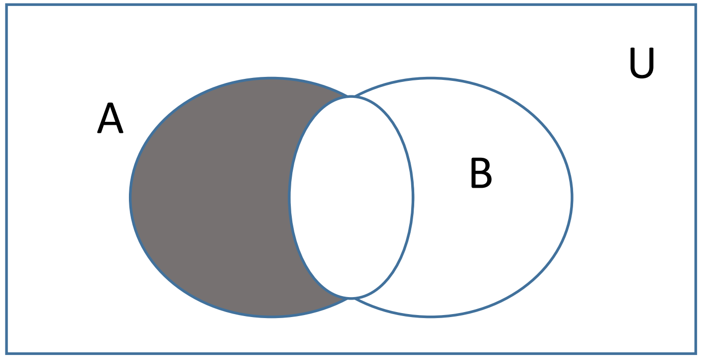
    $\{x \mid x \leq 70\}$

Difference

Let A and B be sets. The difference of A and B, denoted by A – B, is the set containing the elements of A that are not in B.

The difference of A and B is also called the complement of B with respect to A.

$A - B = \{x \mid x \in A \cap x \notin B\} = A \cap \overline{B}$

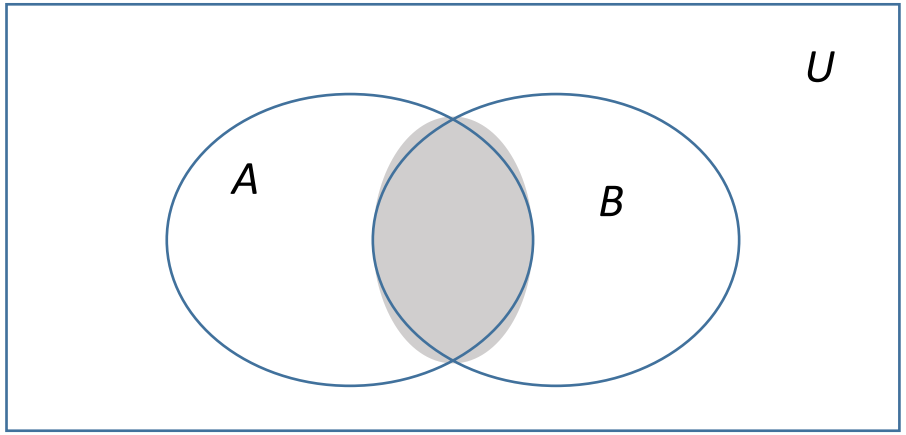


- Examples:
  - { 1, 3, 5, 7 } – { 1, 3 } = { 5, 7 }
  - { 2, 4, 6 } – { 1, 3, 5 } = { 2, 4, 6 }
  - { 0, 1 } – { -1, 0, 1 } = ?

The Cardinality of the Union of Two Sets

Inclusion - Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$



- Examples:
  - $|\{1, 3, 5, 7\} \cup \{1, 3\}| = 4$
  - $|\{2, 4, 6\} \cup \{1, 3, 5\}| = 6$
  - $|\{0, 1\} \cup \{-1, 0, 1\}| = ?$
  - Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Review Questions

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$

1. $A \cup B$
2. $A \cap B$
3. $\bar{A}$
4. B'
5. $A - B$
6. $B - A$

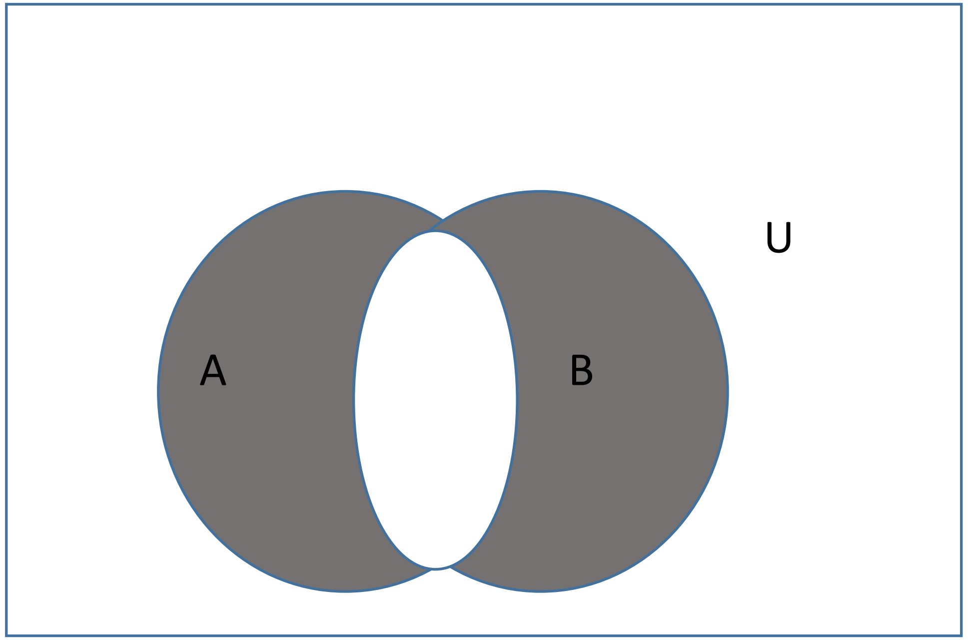
Symmetric Difference

The symmetric difference of A and B, denoted

by $A \oplus B$

is the set

$$(A - B) \cup (B - A)$$



- Examples:

  U = {0,1,2,3,4,5,6,7,8,9,10}

  A = {1,2,3,4,5}

  B = {4,5,6,7,8}

  What is the symmetric difference:

  - { 1, 2, 3, 6, 7, 8 }

Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

Set Identities

- Commutative laws

$$A \cup B = B \cup A \quad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Questions?

Thank You!