CS1382 Discrete Computational Structures

Lecture 02: Functions

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References

The materials of this presentation is mostly from the following:

- Discrete Mathematics and Its Applications (Text book and Slides)

  By Kenneth Rosen, 7th edition

- Lecture slides of CMSC 250: Discrete Structures (Summer 2016)

  By Jason Filippou, University of Maryland

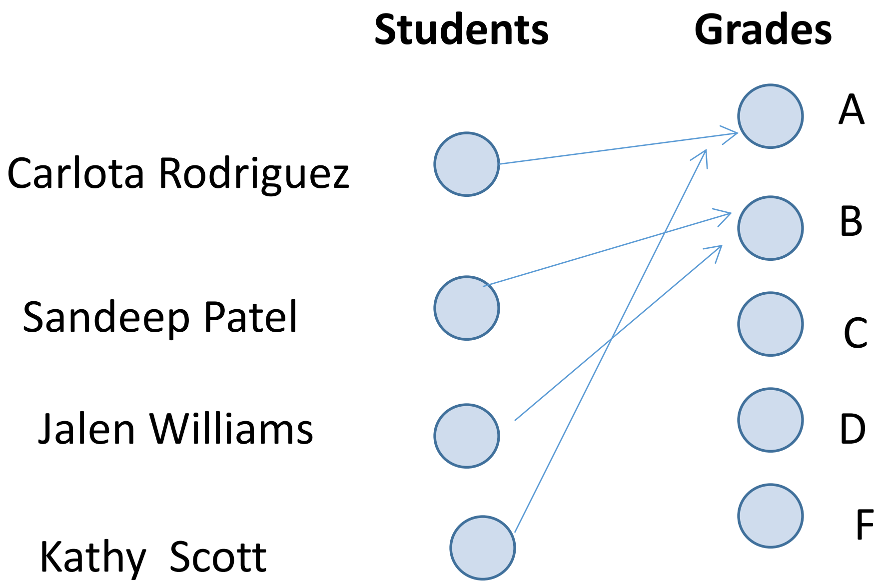
Functions

- Let A and B be nonempty sets.

  A *function* f from A to B, denoted $f : A \to B$ is an assignment of each element of A to exactly one element of B.

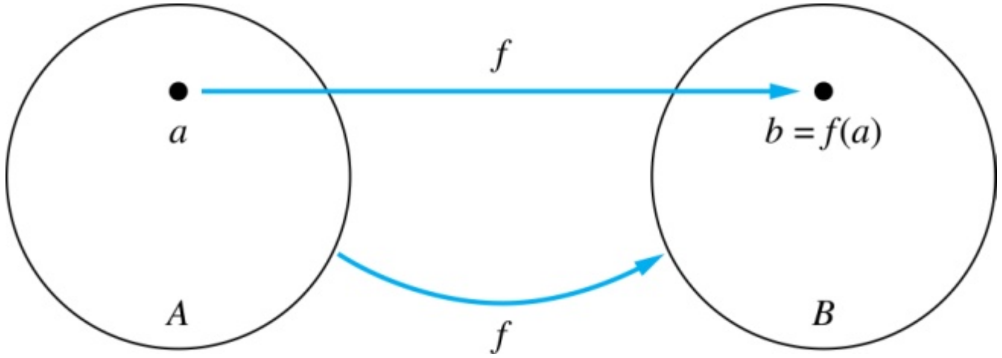
  We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called ***mappings*** or ***transformations***.



Functions

- Given a function f: $A \to B$:
  We say ***f* *maps* *A* to *B*** *or* f is a ***mapping*** from A to B.
  - A is called the ***domain*** of f.
  - B is called the ***codomain*** of f.
- If $f(a) = b$,
  - then b is called the ***image*** of a under f.
  - a is called the ***preimage*** of b.
- The **range** of f is the set of all images of points in **A** under f. We denote it by $f(\boldsymbol{A})$.
- Two functions are ***equal*** when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



Representing Functions

Functions may be specified in different ways:

- An explicit statement of the assignment.

  Students and grades example.

- A formula.

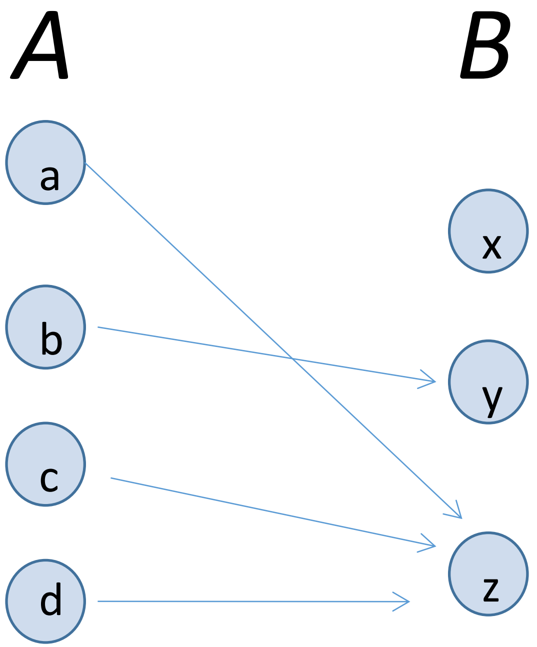
  $f(x) = x + 1$

- A computer program.

  - A Java program that when given an integer n, produces the nth Fibonacci Number (covered in the next section and also in Chapter 5).

Questions

1.	*f*(a) = ?	z
2.	The image of d is ?	z
3.	The domain of f is ?	A
4.	The codomain of f is ?	B
5.	The preimage of y is ?	B
6.	f(A) = ?	{ y , z }
7.	The preimage(s) of z is (are) ?	{ a, c, d }

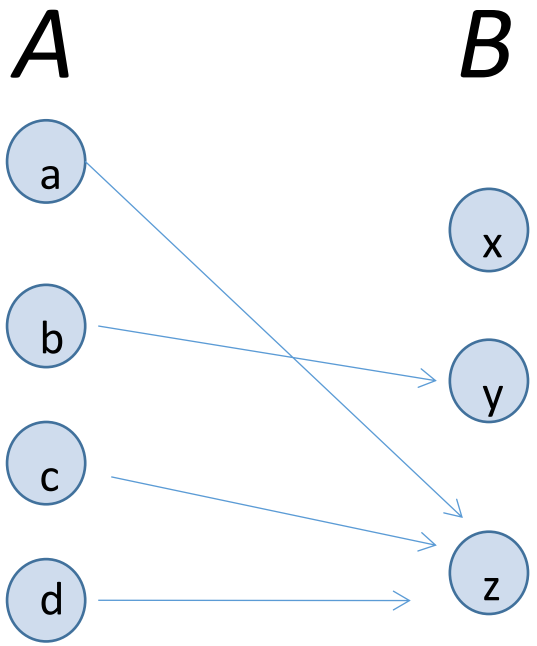


Question on Functions and Sets

If $f : A \to B$ and S is a subset of A, then $f(S) = \{f(s) | s \in S\}$

f {a,b,c,} is ? { y, z}

f {c,d} is ? { *z* }



Question on Functions and Sets

Why is f not a function from R to R

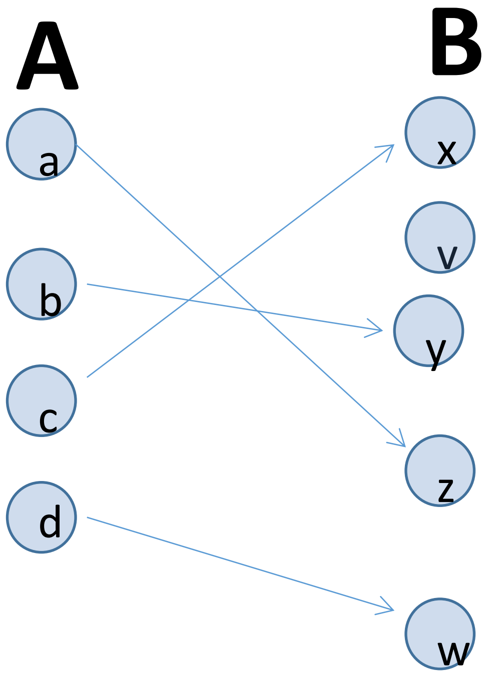
1. $f(x) = 1/x$
2. $f(x) = x^{½}$
3. $f(x) = \pm (x^2 + 1)^{1/2}$

Injections

- A function f is said to be ***one-to-one*** or ***injective***, if and only if ***f*(*a*) = *f*(*b*)** implies that ***a* = *b*** for all *a* and *b* in the domain of *f*.

- A function is said to be an ***injection*** if it is one-to-one.

- Exercise
  - Is $f(x) = x^2$ from set of integers to set of integers injective?
  - What about if its domain is a set of non-negative integers?

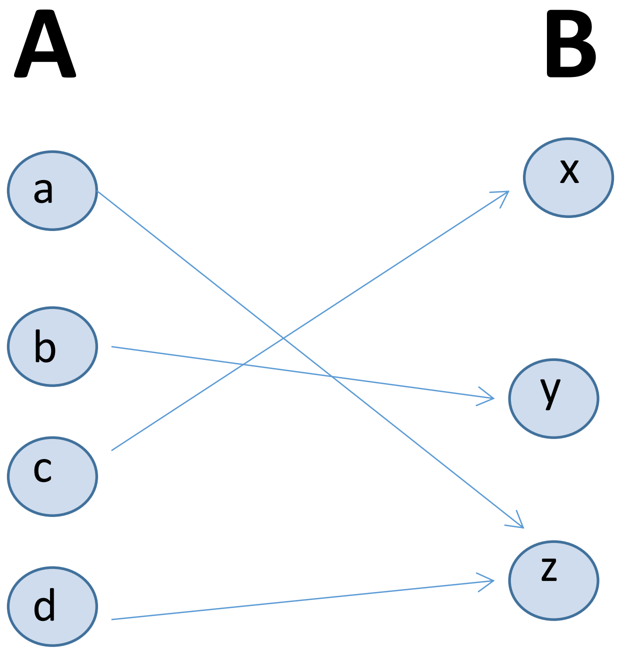


Surjections

- A function *f* from *A* to *B* is called ***onto*** or ***surjective***, if and only if for every element $a \in A$ there is an element $b \in B$ with $f(a) = b$

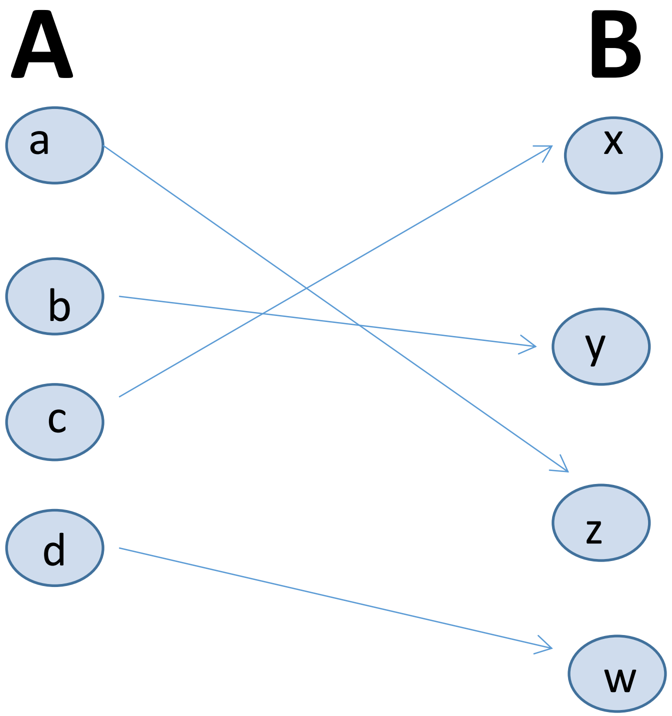
- A function *f* is called a ***surjection*** if it is ***onto***.

- Exercise
    - Is f(x) = x^2 from set of integers to set of integers surjective?



Bijections

- A function f is a ***one-to-one correspondence***, or a ***bijection***, if it is both one-to-one and onto (surjective and injective).



Showing that f is one-to-one or onto

1. Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

   Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

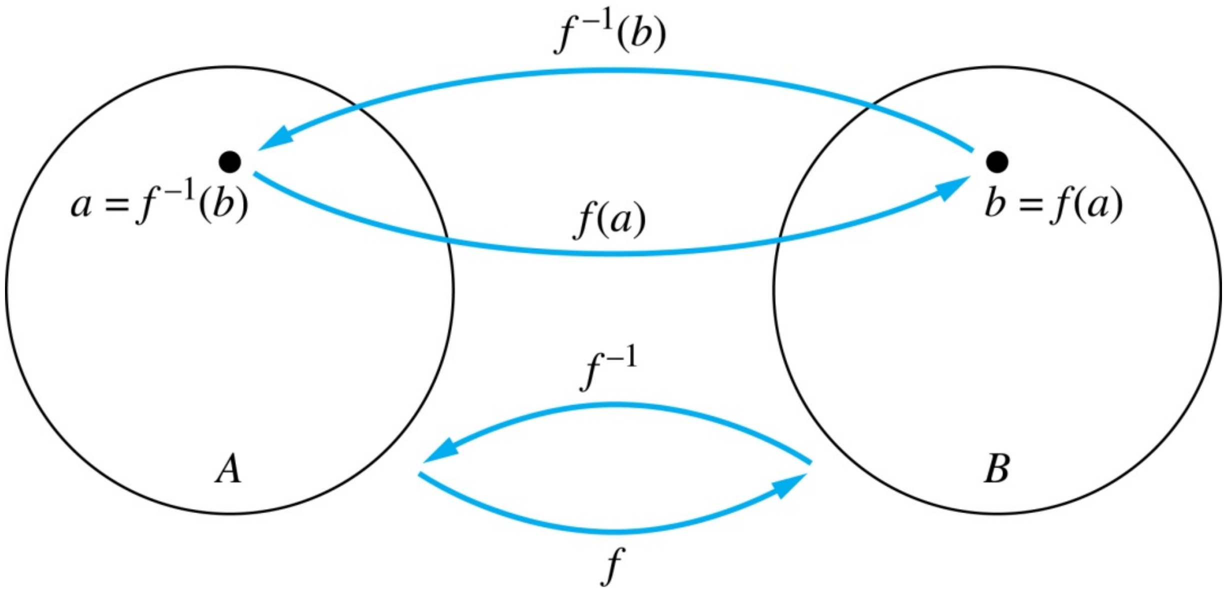
2. Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

   Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

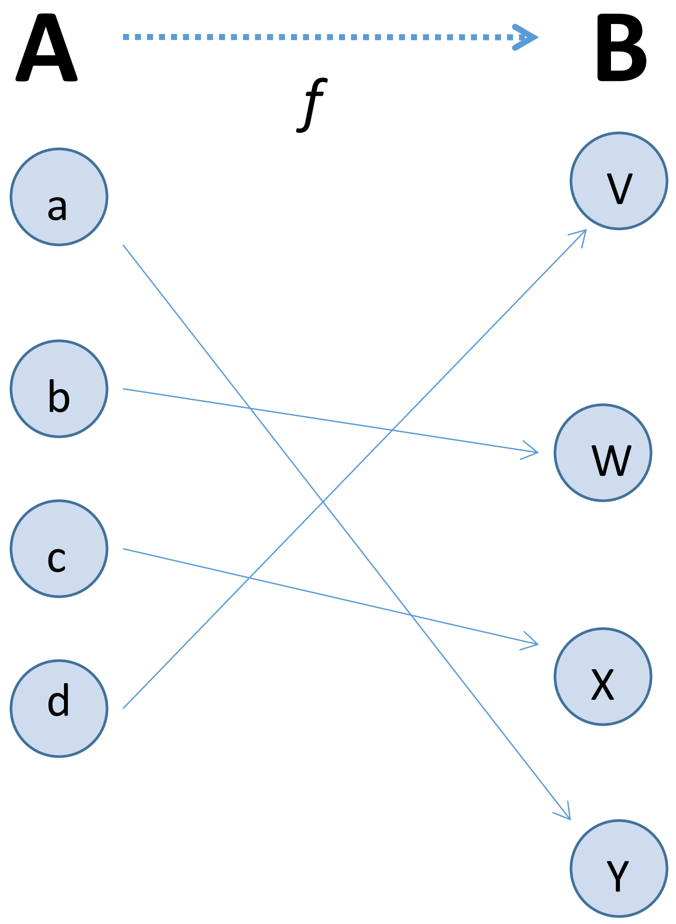
Inverse Functions

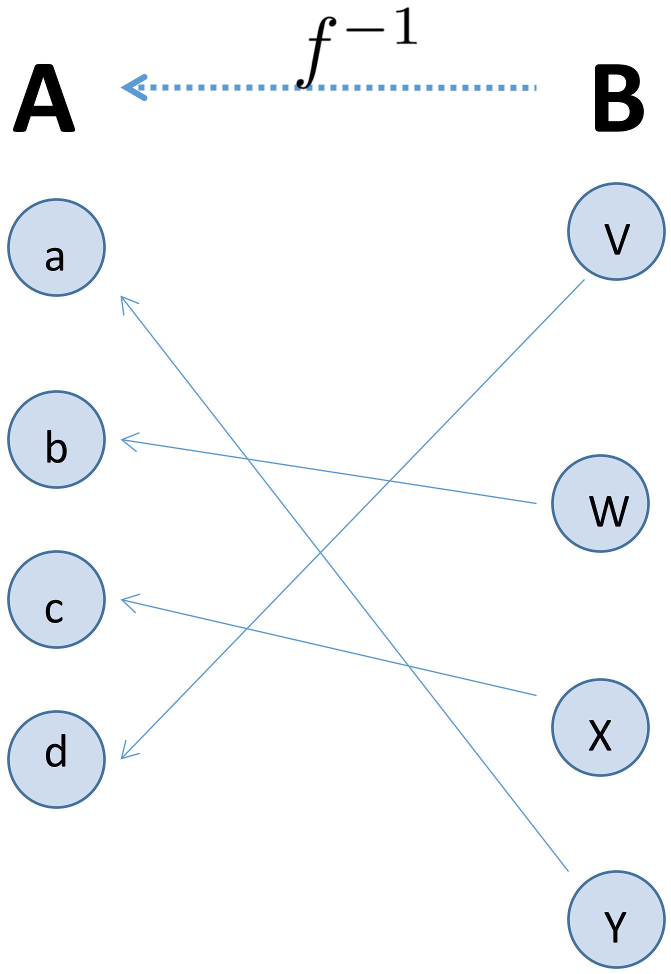
- Let *f* be a bijection from *A* to *B*. Then the ***inverse*** of *f*, denoted f^{-1} is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

- No inverse exists unless *f* is a bijection. Why?



Inverse Functions





Questions

1. Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.

   Is f invertible and if so what is its inverse?

   Solution: The function f is invertible because it is a one-to-one correspondence.

   The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

2. Let f: **Z** $\rightarrow$ **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

   Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

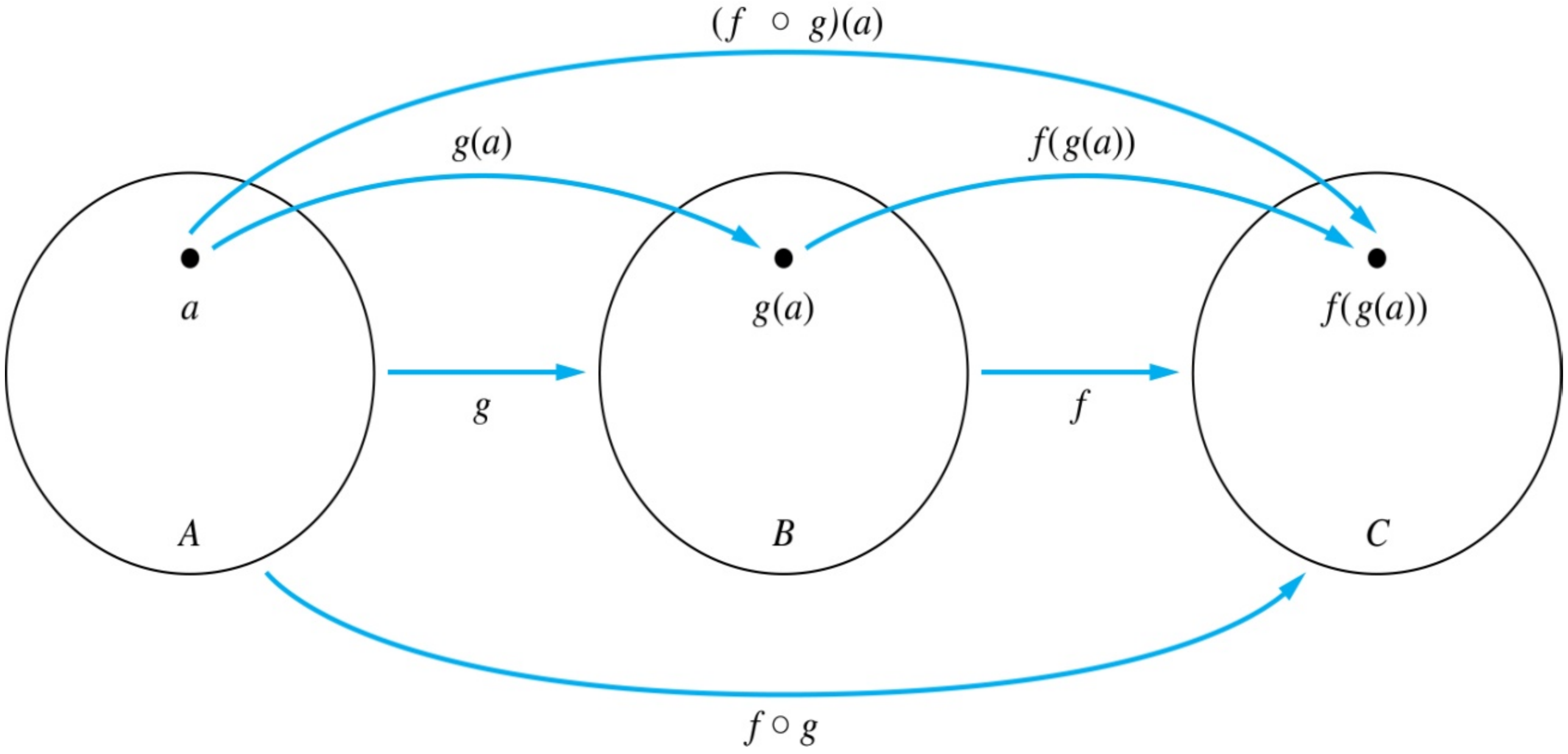
Questions

3. Let *f*: **R** → **R** be such that $f(x) = x^2$. Is *f* invertible, and if so, what is its inverse?

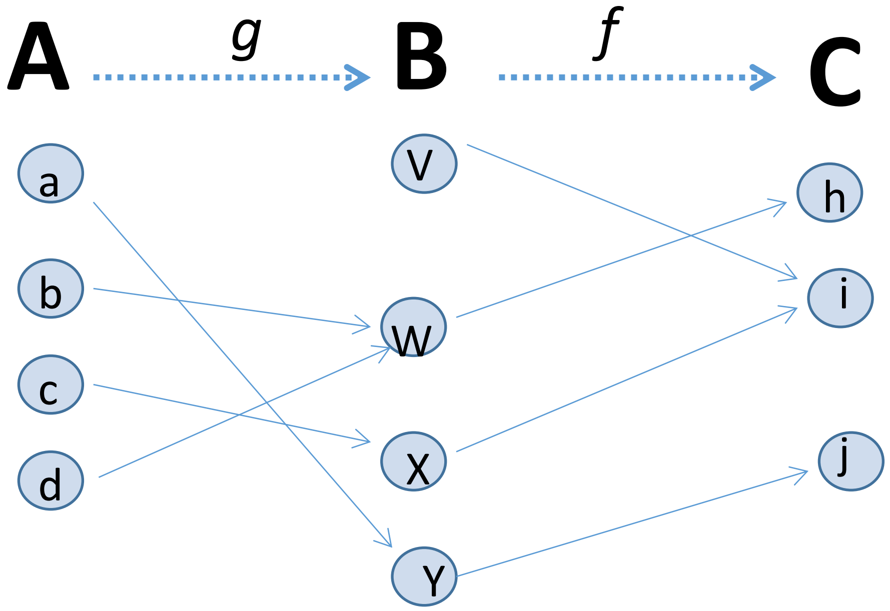
   Solution: The function *f* is not invertible because it is not one-to-one .

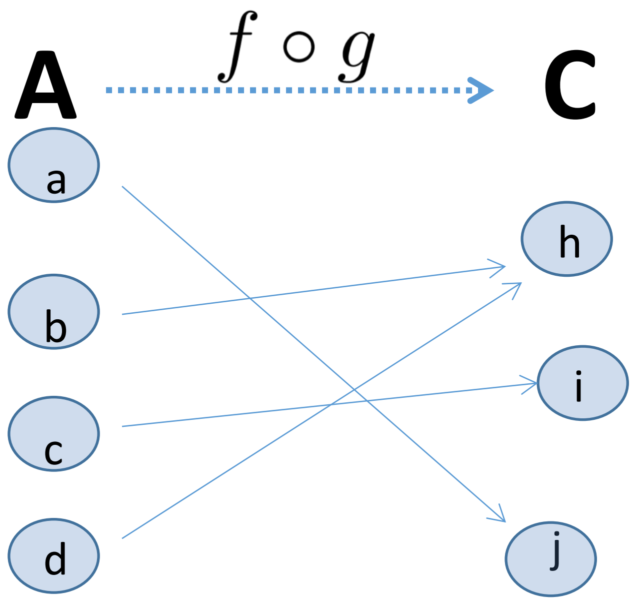
Composition

- Let f: $B \rightarrow C$, g: $A \rightarrow B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by $f \circ g(x) = f(g(x))$



Composition





Composition Questions

1. Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$.

   Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

   What is the composition of f and g, and what is the composition of g and f.

- **Solution:**

  The composition $f \circ g$ is defined by

  $f \circ g\ (a) = f(g(a)) = f(b) = 2.$

  $f \circ g\ (b) = f(g(b)) = f(c) = 1.$

  $f \circ g\ (c) = f(g(c)) = f(a) = 3.$

- Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

2. Let f and g be functions from the set of integers to the set of integers defined by

   $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

   What is the composition of *f* and *g*, and also the composition of *g* and *f* ?

- **Solution:**

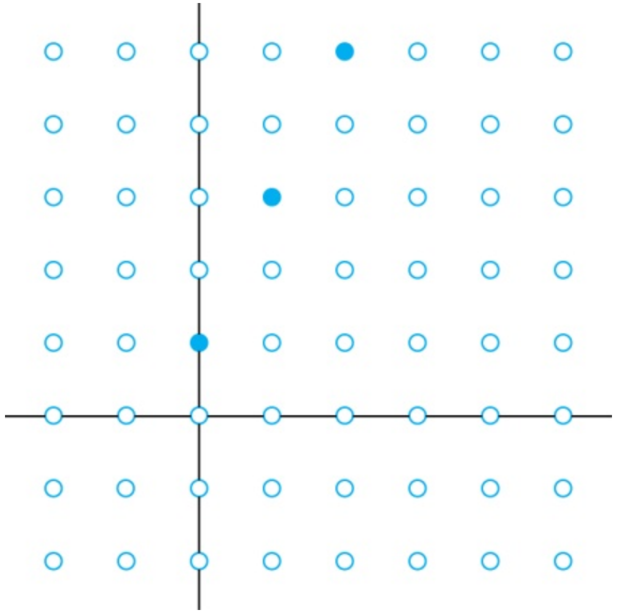
  $f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

  $g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

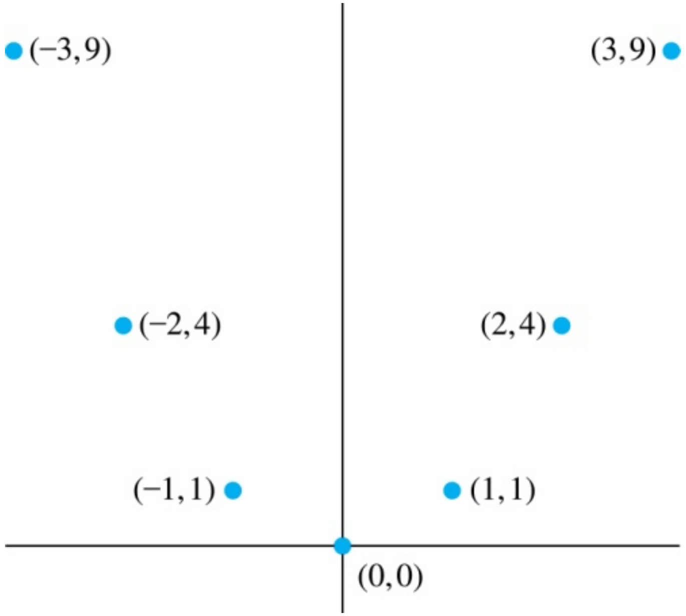
Graphs of Functions

- Let f be a function from the set A to the set B.

  The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

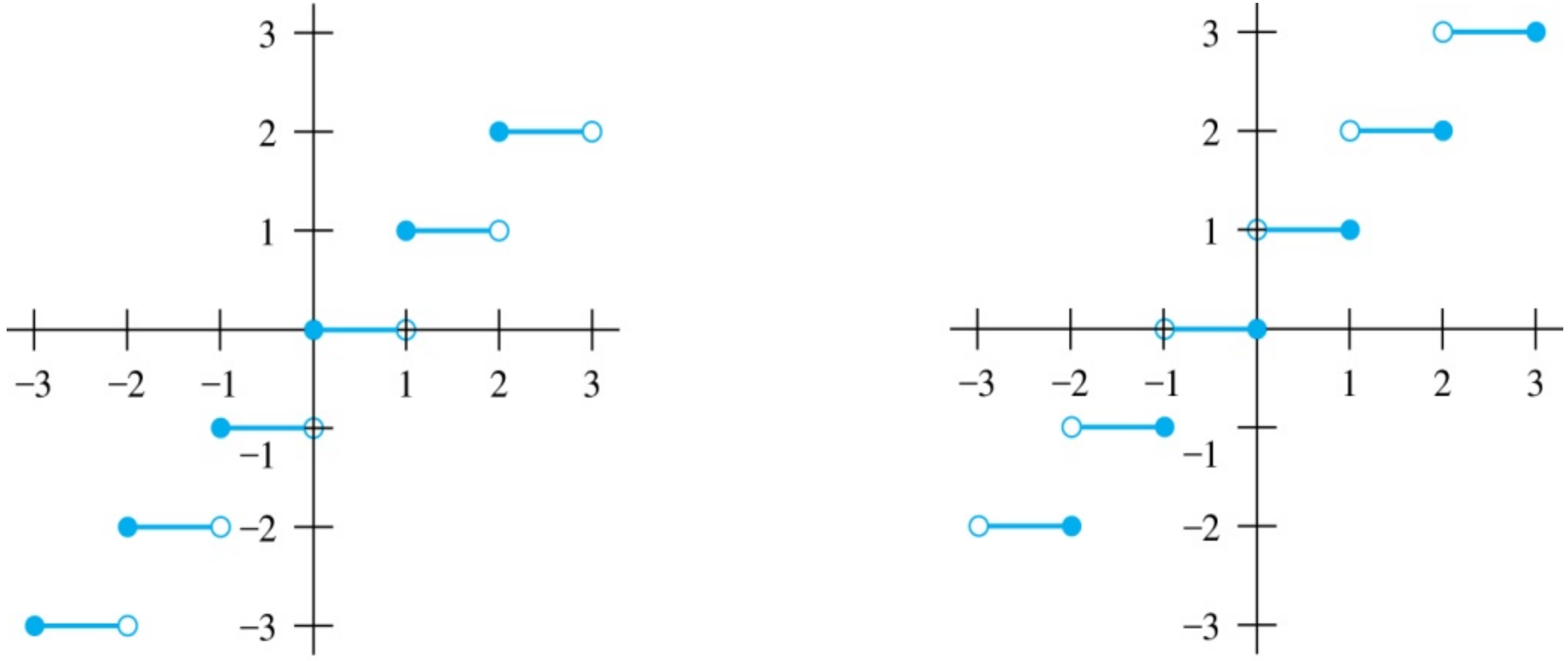
- The *floor* function, denoted $f(x) = \lfloor x \rfloor$ is the largest integer less than or equal to *x*.
- The *ceiling* function, denoted $f(x) = \lceil x \rceil$ is the smallest integer greater than or equal to *x*

Example:

$$\lceil 3.5 \rceil = 4 \qquad \lfloor 3.5 \rfloor = 3$$

$$\lceil -1.5 \rceil = -1 \qquad \lfloor -1.5 \rfloor = -2$$

Floor and Ceiling Functions



(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

Graph of (a) Floor and (b) Ceiling Functions

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Factorial Function

- f: **N** → **Z⁺** , denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer
- $f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$
- **Examples:**
  - $f(1) = 1! = 1$
  - $f(2) = 2! = 1 \cdot 2 = 2$
  - $f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$
  - $f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$

Questions?

Thank You!