CS1382 Discrete Computational Structures

Lecture 12: Counting

Spring 2019

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TEXAS TECH UNIVERSITY

Introduction

- **Combinatorics** is the study of arrangements of objects
  - Enumeration – counting of objects with certain properties
- Counting objects is used to solve many problems
  - Determine the complexity of algorithms
  - Determine whether there are enough telephone numbers or Internet protocol addresses
- Counting techniques are used extensively when computing probabilities of events

- We cover:
  - Basic rules of counting
  - Permutations and combinations
  - Generating all the arrangements of a specified kind. Useful in computer simulations
- *Objective isn't JUST about learning the formula BUT **how to abstract** what you are trying to figure out and then applying the formula*

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Basics of Counting

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Basic Counting Principles: The Product Rule

The Product Rule:

A procedure can be broken down into a sequence of two tasks.

- There are $\boldsymbol{n_1}$ ways to do the first task and $\boldsymbol{n_2}$ ways to do the second task.
- Then there are $\boldsymbol{n_1 \cdot n_2}$ ways to do the procedure.

Examples:

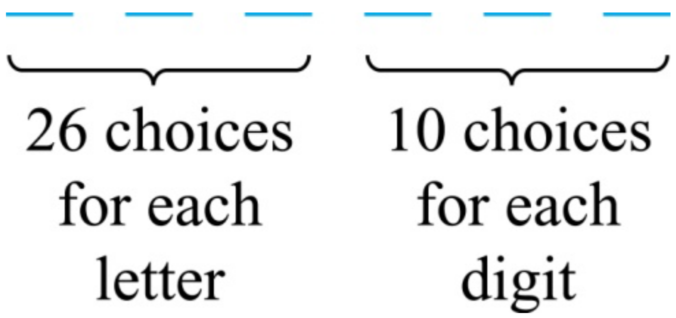
- How many bit strings of length seven are there?
- **Solution**: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

The Product Rule

- How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

- **Solution**:

  By the product rule, there are 26 · 26 · 26 · 10 · 10 · 10 = 17,576,000 different possible license plates.



Example: Telephone Numbering Plan

- The *North American numbering plan* (*NANP*) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.
  - Let *X* denote a digit from 0 through 9.
  - Let *N* denote a digit from 2 through 9.
  - Let *Y* denote a digit that is 0 or 1.
  - In the old plan (in use in the 1960s) the format was *NYX-NNX-XXX*.
  - In the new plan, the format is *NXX-NXX-XXX*.
- How many different telephone numbers are possible under the old plan and the new plan?

- **Solution**:

  Use the Product Rule.
  - There are $8 \cdot 2 \cdot 10 = 160$ area codes (*NYX*)
  - There are $8 \cdot 10 \cdot 10 = 800$ area codes (*NXX*)
  - There are $8 \cdot 8 \cdot 10 = 640$ office codes (*NNX*)
  - There are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes (*XXXX*).
- Number of old plan telephone numbers:

  $160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000$.
- Number of new plan telephone numbers:

  $800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$.

Basic Counting Principles: The Sum Rule

The Sum Rule:

If a task can be done either in one of $\boldsymbol{n_1}$ ways or in one of $\boldsymbol{n_2}$, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $\boldsymbol{n_1 + n_2}$ ways to do the task.

Example:

The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution:

By the sum rule, it follows that there are **37 + 83 = 120** possible ways to pick a representative.

Combining the Sum and Product Rule

Examples:

- Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

- **Solution**:

  $26 + 26 \cdot 10 = 286$

Example: Counting Passwords

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution:

Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.
- To find each of P_6, P_7, and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:
  - $P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560$.
  - $P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920$.
  - $P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880$.
- Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360$.

Basic Counting Principles: Subtraction Rule

Subtraction Rule:

If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ **minus the number of ways to do the task that are common to the two different ways**.

Also known as, the ***principle of inclusion-exclusion***:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

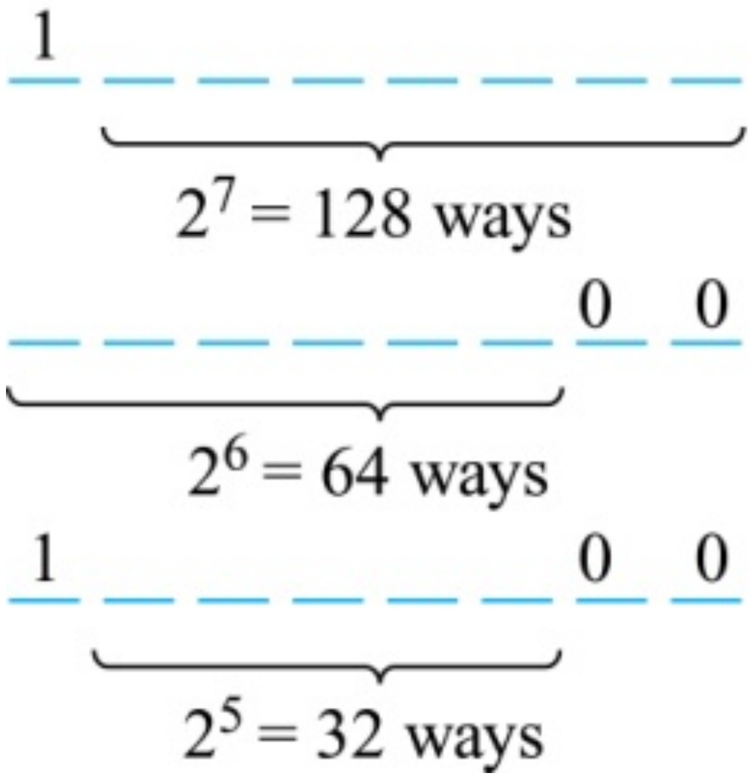
Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.

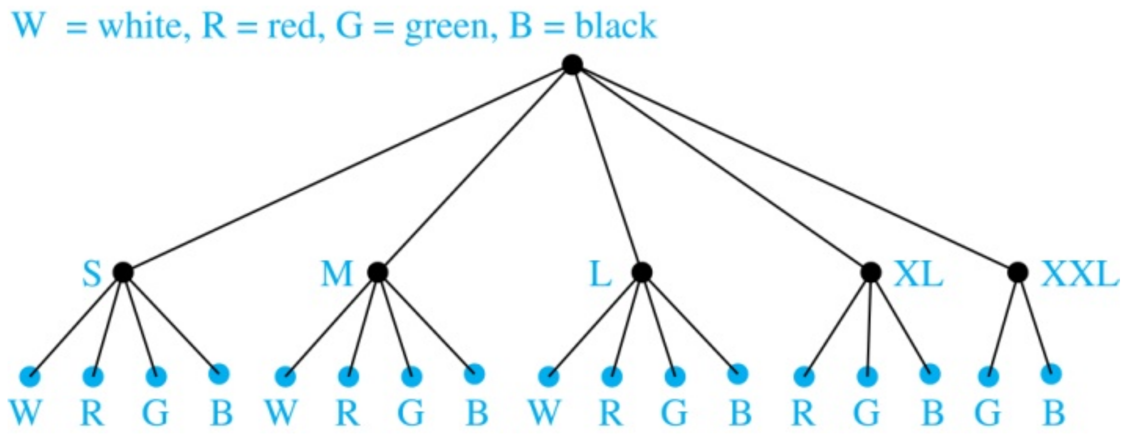
- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.



Tree Diagrams

- Can solve many counting problems
  - Branch represents a possible choice
  - Leaves represent possible outcomes.
- **Example**: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?
- **Solution**:



The store must stock 17 T-shirts.

Exercise

1. How many ways are there to roll a 5 or 6 on a die?
2. How many ways to draw a Face card in a deck? (Assume 52 without jokers)
3. How many ways are there if I want to roll a 5 or 6 on a die, or face card in a deck?
4. How many ways can we pick a license plate with 3 letters and then 4 numbers?
5. How many ways can you make an ice-cream cone (one scoop) if there are 3 types of cones, 30 flavors of ice cream and 5 toppings (this includes NONE)?
6. How many ways can we pick a license plate with 2 even numbers, 3 vowels, 1 number and then 2 letters? (assume uppercase letters)

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Permutations and Combinations

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Factorials

If $n \in Z^+$, then the factorial of n is denoted by

$n! = n\,(n-1)\,(n-2)\,\ldots\ldots\,(3)\,(2)\,(1)$

Examples:

- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $0! = 1$
- $6! = 6 \times 5!$

- How many ways can we list 7 numbers without repetition?
  - $7 \times 6 \times \ldots \times 1 = 7! = 5040$
- What if the first three numbers have to be odd?
  - $(4 \times 3 \times 2) \times (4 \times 3 \times 2 \times 1)$

    $= 4! \times 4! = 24 \times 24 = 576$

Permutations

A ***permutation*** of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an ***r-permutation***.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.

- The ordered arrangement 3,2 is a 2-permutation of S.

The number of r-permutations of a set with n elements is denoted by $P(n,r)$.

- The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

- **Theorem 1**:

  If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

  $$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

  r-permutations of a set with n distinct elements.

- Note that $P(n,0) = 1$, since there is only one way to order zero elements.

- **Corollary 1**: If n and r are integers with $1 \leq r \leq n$, then

  $$P(n,r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

- How many ways can we list 4 numbers out of a list of 7 numbers?
  - **Solution:** $P(7,4) = 7 \times 6 \times 5 \times 4 = 42 \times 40 = 840$
- How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?
  - **Solution**: $P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$
- How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?
  - **Solution**: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

    $P(6,6) = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

Solving Counting Problems by Counting Permutations

- How many permutations can we make with the word BASE ?
  - **Solution**: P(4,4) = 4! / (4 – 4) ! = 4! / 0! = 4! = 24
- How many permutations can we make with the word BALL ?
  - **Solution:** 4! / 2 ! = 4 x 3 = 12
- How many arrangements can we make with the word DATABASES ?
  - Solution: 9 ! / (2 ! x 3 !) = 36 x 840 = 30,240

Combinations

- An ***r-combination*** of elements of a set is an **unordered selection** of *r* elements from the set.

- Thus, an *r*-combination is simply a subset of the set with *r* elements.

- The number of *r*-combinations of a set with n distinct elements is denoted by $C(n, r)$.

The notation $\binom{n}{r}$ is also used and is called a ***binomial coefficient***.

Combinations - Example

- Let S be the set $\{a, b, c, d\}$, then
  - $\{a, c, d\}$ is a 3-combination from S.

    It is the same as $\{d, c, a\}$ since the order listed does not matter.
  - 2-combinations of $\{a, b, c, d\}$
    - $C(4,2) = 6$
    - Six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$
    - Answers “How many different subsets can we make with 2 elements?”

Combinations

Theorem 2:

The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n, r) = \frac{n!}{(n-r)!r!}.$$

- Example:

  How many different ways can you select 2 letters from the set of letters X, Y, Z ?

  C(3, 2) = 3! / (2! * (3 – 2)! = 3! / 2! = 3

Combinations

- **Examples**:
  - How many poker hands of five cards can be dealt from a standard deck of 52 cards?
  - How many ways are there to select 47 cards from a deck of 52 cards?
- **Solution**: Since the order in which the cards are dealt does not matter
  - The number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!} \quad = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

  - The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

Combinations

- How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.
  - **Solution**: $C(10,5) = \frac{10!}{5!5!} = 252.$
- A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?
  - **Solution**: $C(30,6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775 .$

Exercise

1. How many ways of picking a team of 3 people from a group of 10.

2. How many ways of picking a President, VP and Waterboy from a group of 10.

3. How many ways of listing your 3 favorite desserts, in order, from a menu of 10.

4. How many ways of choosing 3 desserts from a menu of 10.

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Binomial Coefficients and Identities

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Powers of Binomial Expressions

- A ***binomial* expression** is the sum of two terms, such as $x + y$.

  (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.

- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.

  - $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.

  - Terms of the form x^3, x^2y, $x y^2$, y^3 arise.

  - The question is what are the coefficients?

Powers of Binomial Expressions

- To obtain x^3 , an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
- To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
- To obtain xy^2, an x must be chosen from one of the sums and a y from the other two . There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
- To obtain y^3 , a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.

We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3x y^2 + y^3$.

Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Binomial Theorem

Let *x* and *y* be variables, and *n* a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Example:

Something we know!!!

$(x + y)^2 = (x + y)(x + y) = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$

Using the binomial theorem

$(x + y)^2 =$

Using the Binomial Theorem

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j .$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

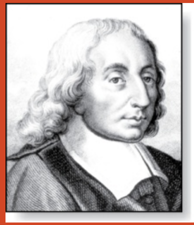
$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13} .$$

Exercise:

What is $(2a - b)^4$?

Pascal's Identity and Triangle

Blaise Pascal (1623-1662)



Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Pascal's Triangle

The nth row in the triangle consists of the binomial coefficients
,
k = 0,1,….,n.

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\ldots
\end{array}$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\begin{array}{c}
1 \\
1\quad 1 \\
1\quad 2\quad 1 \\
1\quad 3\quad 3\quad 1 \\
1\quad 4\quad 6\quad 4\quad 1 \\
1\quad 5\quad 10\quad 10\quad 5\quad 1 \\
1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1 \\
1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1 \\
1\quad 8\quad 28\quad 56\quad 70\quad 56\quad 28\quad 8\quad 1 \\
\ldots
\end{array}$$

(b)

By Pascal's identity, adding two adjacent binomial coefficients results is the binomial coefficient in the next row between these two coefficients.

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\cdots
\end{array}
$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}
$$

(b)

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Generalized Permutations and Combinations

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Permutations with Repetition

Theorem 1:

The number of *r*-permutations of a set of *n* objects with repetition allowed is $\boldsymbol{n}^{r}$.

- **Example**:

  How many strings of length *r* can be formed from the uppercase letters of the English alphabet?

- **Solution**:

  The number of such strings is 26^r, which is the number of *r*-permutations of a set with 26 elements.

Combinations with Repetition

Theorem 2:

The number of *r*-combinations from a set with *n* elements when repetition of elements is allowed is

$C(n + r - 1, r)$.

Example:

- Suppose n = 4 and r = 3, then C (4 + 3 – 1 , 3) = C(6 , 3)
- In a donut shop, there are 20 types of donuts. How many ways can we select 12 donuts to take home?
  - n = 20 , r = 12
  - C(20 + 12 – 1 , 12) = C(31 , 12)

Combinations with Repetition

How many solutions does the equation $x_1 + x_2 + x_3 = 7$ have, where x_1 , x_2 and x_3 are nonnegative integers?

Solution:

- Each solution corresponds to a way to select 7 items from a set with three elements;

  x_1 elements of type one, x_2 of type two, and x_3 of type three.

- By Theorem 2 it follows that there are

  C(3 + 7 – 1, 7) = C(9 , 7) = C (9 , 2) = 36 solutions.

Exercise: How many ways can we put 10 identical balls into 6 distinct bins?

Questions?

Thank You!