CS1382 Discrete Computational Structures

Lecture 09: Predicate Logic

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Propositional Logic Not Enough

- If we have:

  “All men are mortal.”

  “Socrates is a man.”

  - Does it follow that “Socrates is mortal?”
- Can’t be represented in propositional logic.
- Need a language that talks about objects, their properties, and their relations.
- Later we’ll see how to draw inferences.

Predicates and Quantifiers

- **Predicate Logic** involves statements with variables
- Examples:
  - x > 3
  - Computer x is functioning properly
- These statements are neither true nor false when the values of the variables are not specified
- Propositions can be produced from such statements

- x is greater than 3
  - **Variable:** The subject of the statement
  - **Predicate:** Property that the subject of the statement can have
  - Can be denoted by P(x)
    - P denotes the predicate "is greater than 3"
    - x is the variable

Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: ***x*, *y*, *z***
  - Predicates: ***P*(*x*), *M*(*x*)**
  - Quantifiers (*to be covered in a few slides*)
- ***Propositional functions e.g., P(x)***
  - They contain variables and a predicate, e.g. *P*(*x*)
  - Variables can be replaced by elements from their *domain*.

Examples – Propositional Functions

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier).
- For example, let *P(x)* denote "$x > 0$" and the domain be the integers. Then:
  - P(-3) is false.
  - P(0) is false.
  - P(3) is true.
- Often the domain is denoted by *U*. So in this example *U* is the integers.

Examples – Propositional Functions

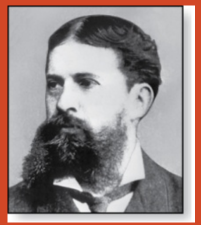
- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:
  - R(2,-1,5)
    - **Solution: F**
  - R(3,4,7)
    - **Solution: T**
  - $R(x, 3, z)$
    - **Solution: Not a Proposition**

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:
  - Q(2,-1,3)
    - **Solution: T**
  - Q(3,4,7)
    - **Solution: F**
  - $Q(x, 3, z)$
    - **Solution: Not a Proposition**

Compound Expressions

- Connectives carry over to predicate logic.
- If *P(x)* denotes "$x > 0$," find these truth values:
  - P(3) ∨ P(-1)
    - **Solution**: T
  - P(3) ∧ P(-1)
    - **Solution**: F
  - P(3) → P(-1)
    - **Solution**: F
  - P(3) → ¬P(-1)
    - **Solution**: T
- Expressions with variables are not propositions. For example,
  - P(3) ∧ P(*y*)
  - P(*x*) → P(*y*)
- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers



Charles Peirce
(1839-1914)

- ***Quantifiers*** - Express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier,*
    - **"**For all," symbol: $\forall$
  - *Existential Quantifier*,
    - "There exists," symbol: $\exists$
- We write as in
  - $\forall x\, P(x)$
  - $\exists\, x\, P(x)$.
- $\forall x\, P\,(x)$ asserts $P(x)$ is true for **every** x in the *domain*.
- $\exists\, x\, P\,(x)$ asserts $P(x)$ is true for **some** x in the *domain*.
- The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

- $\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

- Examples:
  - If $P(x)$ denotes "$x > 0$" and U is the integers
    - $\forall x\, P(x)$ is false.
  - If $P(x)$ denotes "$x > 0$" and U is the positive integers
    - $\forall x\, P(x)$ is true.
  - If $P(x)$ denotes "x is even" and U is the integers
    - $\forall x\, P(x)$ is false.

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."
- Examples:
    - If *P(x)* denotes "$x > 0$" and *U* is the integers
        - $\exists\, x\, P(x)$ is true
        - It is also true if U is the positive integers.
    - If *P(x)* denotes "$x < 0$" and *U* is the positive integers
        - $\exists x\, P(x)$ is false.
    - If *P(x)* denotes "*x* is even" and *U* is the integers
        - $\exists\, x\, P(x)$ is true.

Properties of Quantifiers

- The truth value of *Quantifiers* depend on both the **propositional function *P(x)*** and on the **domain** *U*.
- **Examples**:
  - If *U* is the positive integers and *P(x)* is the statement "$x < 2$",
    - $\exists x \ P(x)$ is true BUT $\forall x \ P(x)$ is false.
  - If *U* is the negative integers and *P(x)* is the statement "$x < 2$"
    - Both $\exists x \ P(x)$ and $\forall x \ P(x)$ are true.
  - If *U* consists of 3, 4, and 5, and *P(x)* is the statement "$x > 2$"
    - Both $\exists x \ P(x)$ and $\forall x \ P(x)$ are true
  - What if *P(x)* is the statement "$x < 2$"
    - Both $\exists x \ P(x)$ and $\forall x \ P(x)$ are false

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- Example
  - $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
  - $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Translating from English to Logic

Example 1: “Every student in this class has studied calculus”

Solution 1: **Domain - Students in this class**

- Rewrite the statement

  For every student in this class, that student has studied calculus

- Introduce a variable x

  For every student x in this class, x has studied calculus

- C(x) - “x has studied calculus.” and *domain* for x to be *“the students in the class”*
- We can translate our statement as

  $\forall$**x C(x)**.

Translating from English to Logic

Example 1: “Every student in this class has studied calculus”

Solution 2: Domain – “All People”

- Rewrite the statement:

  For every person x, if person x is a student in this class then x has studied calculus.

- S(x) – “person x is in this class” and C(x) – “person x has studied calculus”

- Our statement can be expressed as

  $\forall x \, (S(x) \rightarrow C(x))$

- $\forall x \, (S(x) \land C(x))$ ***is not correct. What does it mean?***

Translating from English to Logic

Example 1: “Some student in this class has visited Mexico”

Solution 1: **Domain – Students in this class**

- Rewrite the statement

  There is a student in this class with the property that the student has visited Mexico

- Introduce a variable x

  There is a student x in this class having the property that x has visited Mexico

- M(x) - “x has visited Mexico.” and *domain* for x to be *“the students in the class”*
- We can translate our statement as

  $\exists x\, M(x)$.

Translating from English to Logic

Example 1: “Some student in this class has visited Mexico”

Solution 1: Domain – “all people”

- Rewrite the statement:

  There is a person x having the properties that x is a student in this class and x has visited Mexico.

- S(x) – “person x is in this class”

  M(x) – “x has visited Mexico”

- Our statement can be expressed as

  $\exists x\ (\ S(x) \wedge M(x)\ )$

- ***$\exists x\ (\ S(x) \rightarrow M(x)\ )$ is not correct. What does it mean?***

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are ***logically equivalent*** if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example**:

$$\forall x \neg\neg S(x) \equiv \forall x\, S(x)$$

Quantifiers as Conjunctions and Disjunctions

- If the domain is finite,
  - A **universally** quantified proposition is equivalent to a **conjunction of propositions** without quantifiers
  - An **existentially** quantified proposition is equivalent to a **disjunction of propositions** without quantifiers.

- If *U* consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$
$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- In general, for U with { 1, 2, 3, …. , n }
  - $\forall x\ P(x) <=> P(1) \wedge P(2) \wedge \ldots \wedge P(n)$
  - $\exists x\ \ P(x) <=> P(1)\ \vee P(2)\ \vee \ldots\ \vee P(n)$

Negating Quantified Expressions

- Show that $\neg \forall x\, P(x) \Leftrightarrow \exists x \neg P(x)$

  - $\forall x\, P(x) = P(1) \wedge P(2) \wedge \ldots \wedge P(n)$
  - $\neg \forall x\, P(x) = \neg\, (P(1) \wedge P(2) \wedge \ldots \wedge P(n)\,)$
  - $\neg \forall x\, P(x) = \neg\, P(1) \vee \neg\, P(2) \vee \ldots \vee \neg\, P(n)$
  - $\neg \forall x\, P(x) = \exists x \neg P(x)$

- Show that $\forall x\, P(x) \Leftrightarrow \neg \exists x\, (\neg P(x)\,)$

- **All Equivalencies**

  - $\forall x\, P(x) \Leftrightarrow \neg \exists x\, (\neg P(x)\,)$
  - $\exists x\, P(x) \Leftrightarrow \neg \forall x\, (\neg P(x)\,)$
  - $\neg \forall x\, P(x) \Leftrightarrow \exists x\, (\neg P(x)\,)$
  - $\neg \exists x\, P(x) \Leftrightarrow \forall x\, (\neg P(x)\,)$

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- These are important and we will use these.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$
  - "Every student in your class has taken a course in Java."
  - $J(x)$ is "x has taken a course in Java"
  - the domain is students in your class.
- What is $\neg \forall x\, J(x)$?

Negation

- "It is not the case that every student in your class has taken Java"
- This implies that
  "There is a student in your class who has not taken Java
- *Symbolically,*
  - $\neg \forall x\, J(x)$ and $\exists x\, \neg J(x)$

Translation from English to Logic

"Every student in this class has visited Canada or Mexico."

Solution:

- $M(x)$ denote "x has visited Mexico"
- $S(x)$ denote "x is a student in this class"
- $C(x)$ denoting "x has visited Canada."
- Domain be all people.

- $\forall x\ (S(x) \rightarrow (M(x) \lor C(x)))$

System Specification Example

- Predicate logic is used for specifying properties that systems must satisfy.
- For example, translate into predicate logic:
  - "Every mail message larger than one megabyte will be compressed."
  - "If a user is active, at least one network link will be available."

- Decide on predicates and domains (left implicit here) for the variables:
  - *L*(*m*, *y*): "Mail message *m* is larger than *y* megabytes."
  - *C*(*m*) : "Mail message *m* will be compressed."
  - *A*(*u*): "User *u* is active."
  - *S*(*n, x*) : "Network link *n* is state *x*.
- Now we have:

$$\forall m(L(m,1) \to C(m))$$

$$\exists u\, A(u) \to \exists n\, S(n, available)$$

Questions?

Thank You!