Chen Zhang

① $F(s) = \dfrac{3s+\beta}{(s-2)^2}$

$= \dfrac{3s}{(s-2)^2} + \dfrac{\beta}{(s-2)^2}$

$= \dfrac{3s-6+6}{(s-2)^2} + \dfrac{\beta}{(s-2)^2}$

$= \dfrac{3(s-2)+6}{(s-2)^2} + \dfrac{\beta}{(s-2)^2}$

$= \dfrac{3\cancel{(s-2)}}{(s-2)^{\cancel{2}}} + \dfrac{6}{(s-2)^2} + \dfrac{\beta}{(s-2)^2}$

$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left(\dfrac{3}{s-2}\right) + \mathcal{L}^{-1}\left(\dfrac{6}{(s-2)^2}\right) + \mathcal{L}^{-1}\left(\dfrac{\beta}{(s-2)^2}\right)$

$= 3\mathcal{L}^{-1}\left(\dfrac{1}{s-2}\right) + 6\mathcal{L}^{-1}\left(\dfrac{1}{(s-2)^2}\right) + \beta\mathcal{L}^{-1}\left(\dfrac{1}{(s-2)^2}\right)$

$= 3e^{2t} + 6te^{2t} + \beta te^{2t}$

$\mathcal{L}^{-1}\left(\dfrac{1}{s-2}\right) = e^{2t}$ $\qquad$ $\mathcal{L}^{-1}\left(\dfrac{1}{(s-2)^2}\right) = e^{2t}\cdot t$

Thus, $\mathcal{L}^{-1}[F(s)] = 3e^{2t} + 6te^{2t} + \beta te^{2t}$

Chen Zhang

(2) $f(t) = \begin{cases} 0, & 0 \le t < \pi/2 \\ \sin(t), & \pi/2 \le t < \pi \\ 0, & \pi \le t \end{cases}$

Using the unit step function:

$$f(t) = 0 \cdot [u(t-0) - u(t-\pi/2)]$$
$$+ \sin(t) \cdot [u(t-\pi/2) - u(t-\pi)]$$
$$+ 0 \cdot [u(t-\pi)]$$
$$= \sin(t) \cdot [u(t-\pi/2) - u(t-\pi)]$$

Thus, $f(t) = \sin(t) \cdot [u(t-\pi/2) - u(t-\pi)]$

Chen Zhang

③ $f(t) = \sin(t) \cdot u(t + \frac{\pi}{2})$

$\because \mathcal{L}\{f(t)\}$

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-sa}\mathcal{L}\{f(t)\}$

$\because \sin(t) = \sin(t + \frac{\pi}{2} - \frac{\pi}{2}) = -\cos(t + \frac{\pi}{2})$

$\therefore f(t) = -\cos(t + \frac{\pi}{2}) \cdot u(t + \frac{\pi}{2})$

$= -\cos(t - (-\frac{\pi}{2}))\, u(t - (-\frac{\pi}{2}))$

$\mathcal{L}\{f(t)\} = -\mathcal{L}\{\cos(t - (-\frac{\pi}{2}))\, u(t - (-\frac{\pi}{2}))\}$

$= -e^{-(-\frac{\pi}{2})s} \cdot \mathcal{L}\{\cos(t - (-\frac{\pi}{2}) + (-\frac{\pi}{2}))\}$

$= -e^{\frac{\pi}{2}s} \cdot \mathcal{L}\{\cos(t)\}$

$= -e^{\frac{\pi}{2}\cdot s} \cdot \frac{s}{s^2+1}$ (because $\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$

Thus, $\boxed{\mathcal{L}\{f(t)\} = -\frac{e^{\frac{\pi}{2}\cdot s}\cdot s}{s^2+1}}$

Chen Zhang

(4) $ty'' - y' = 2t^2$

$\mathcal{L}\{ty''\} - \mathcal{L}\{y'\} = 2\mathcal{L}\{t^2\}$

$\because$ $t^n f(t) = (-1)^n F^{(n)}(s)$

$\therefore$ $\mathcal{L}\{ty''\} = (-1)^1 \frac{d}{ds}\{\mathcal{L}\{y''\}\}$

$= -1 \frac{d}{ds}\{s^2 Y(s) - sy(0) - y'(0)\}$

$= -(s^2 Y'(s) + 2sY(s))$

$\mathcal{L}\{y'\} = sY(s) - y(0)$

$= sY(s) - 0$

$= sY(s)$

$\mathcal{L}\{t^2\} = \frac{2}{s^3}$

Thus, $ty'' - y' = 2t^2$

$\mathcal{L}\{ty''\} - \mathcal{L}\{y'\} = 2\mathcal{L}\{t^2\}$ (laplace from both side)

$-(s^2 Y'(s) + 2sY(s)) - sY(s) = \frac{4}{s^3}$

$-s^2 Y'(s) - 3sY(s) = \frac{4}{s^3}$

$Y'(s) + \frac{3}{s}Y(s) = -\frac{4}{s^5}$ (divide $-s^2$ from both side)

$IF = e^{\int \frac{3}{s} ds} = e^{3\ln s} = s^3$

$\therefore$ $s^3 Y'(s) + 3s^2 Y(s) = -\frac{4}{s^2}$ (multiply IF from both side)

$\frac{d}{ds}(s^3 Y(s)) = -\frac{4}{s^2}$

$Y(s^3) = \int \frac{-4s^3}{s^5} ds + C$

$= \frac{4}{s} + C$

$Y(s) = \frac{4}{s^4} + \frac{C}{s^3}$

$\therefore$ $y = \mathcal{L}^{-1}(\frac{4}{s^4}) + \mathcal{L}^{-1}(\frac{C}{s^3}) \Rightarrow y' = 2t^2 + Ct$

$= \frac{2}{3}t^3 + C$ $\qquad y'' = 4t + C$

Thus, $ty'' - y' = 2t^2 \Leftrightarrow 4t^2 + Ct - 2t^2 - Ct = 2t^2$

LHS = RHS

Chen Zhang

(5) $f(t) = \cos(t) + \int_0^t e^{-\tau} f(t-\tau)\,d\tau$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos(t) + \int_0^t e^{-\tau} f(t-\tau)\,d\tau\}$$

$$= \mathcal{L}\{\cos(t)\} + \mathcal{L}\{\int_0^t e^{-\tau} f(t-\tau)\,d\tau\}$$

$$= \frac{s}{s^2+1} + \mathcal{L}\{e^{-t} f(t)\}$$

$$= \frac{s}{s^2+1} + \mathcal{L}\{e^{-t}\} \cdot \mathcal{L}\{f(t)\}$$

$$F(s) = \frac{s}{s^2+1} + \left(\frac{1}{s+1}\right) \cdot F(s)$$

$$F(s) - \left(\frac{1}{s+1}\right) \cdot F(s) = \frac{s}{s^2+1}$$

$$\left(1 - \frac{1}{s+1}\right) F(s) = \frac{s}{s^2+1}$$

$$F(s) = \frac{s}{s^2+1} \cdot \frac{1}{\left(1-\frac{1}{s+1}\right)}$$

$$= \frac{s}{s^2+1} \cdot \frac{1}{\left(\frac{s}{s+1}\right)}$$

$$= \frac{s}{s^2+1} \cdot \frac{s+1}{s}$$

$$= \frac{s+1}{s^2+1}$$

$$= \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

Thus, $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$

$$f(t) = \cos(t) + \sin(t)$$

Chen Zhang

⑥ $y'' + y = u(t-2\pi)$, $y(0) = 1$, $y'(0) = 0$

$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(u(t-2\pi))$

$\because$ $\mathcal{L}(y'') = s^2 Y_{(s)} - s\,y(0) - y'(0) = s^2 Y_{(s)} - s$

$\mathcal{L}(y) = Y_{(s)}$

$\mathcal{L}(u(t-2\pi)) = \frac{e^{-2\pi s}}{s}$

$\therefore$ $s^2 Y_{(s)} - s + Y_{(s)} = e^{-2\pi s}/s$

$(s^2+1) Y_{(s)} = e^{-2\pi s}/s + s$

$Y_{(s)} = \frac{s^2 + e^{-2\pi s}}{s(s^2+1)}$

$Y(s) = \frac{s}{s^2+1} + \frac{e^{-2\pi s}}{s(s^2+1)}$

$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + \mathcal{L}^{-1}\left(e^{-2\pi s} \cdot \frac{1}{s(s^2+1)}\right)$

$\mathcal{L}^{-1}(Y(s)) = \cos(t) + (1 - \cos(t-2\pi))\,u(t-2\pi)$

since $\mathcal{L}^{-1}(e^{-cs}F(s)) = u_c(t)\,f(t-c)$

Chen Zhang

(7) $y''+y=\delta(t-2\pi)$, $y(0)=1$, $y'(0)=0$

$\mathcal{L}(y'')+\mathcal{L}(y)=\mathcal{L}(\delta(t-2\pi))$

$\because \mathcal{L}(y'')=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)-s$

$\mathcal{L}(y)=Y(s)$

$\mathcal{L}(\delta(t-2\pi))=e^{-2\pi s}$

$\therefore s^2Y(s)-s+Y(s)=e^{-2\pi s}$

$(s^2+1)Y(s)=e^{-2\pi s}+s$

$Y(s)=\frac{s}{s^2+1}+\frac{e^{-2\pi s}}{s^2+1}$

$\mathcal{L}^{-1}(Y(s))=\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)+\mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{s^2+1}\right)$

$=\cos(t)+\mathcal{L}^{-1}\left(e^{-2\pi s}\cdot\frac{1}{s^2+1}\right)$

$=\cos(t)+\cancel{\sin(t)u(t-2\pi)}$

$\sin(t-2\pi)u(t-2\pi)$

because $\mathcal{L}^{-1}\{F(s)e^{-cs}\}=f(t-a)u(t-a)$

and

$\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right)=\sin(at)$