MATH 3350-011/012 Final Exam

Directions: Provided on this sheet are the ten problems that you need to complete for this exam. This exam is open-book and open-note, but do **NOT** use the Internet. You will find that to be a fruitless effort on several of these problems. Please **neatly** and **clearly** show **ALL** of your work. This exam is due by **11:59 PM on Monday, 11 May 2020**. You do **NOT** need to print out this exam sheet in your submission. Work each problem on a **separate** page and scan/submit your solutions to the assignment page on Blackboard where you downloaded this exam. Ensure that all pages are of the same size and oriented correctly.

Groups: The Final Exam is split into two Groups. The Problems in Group A are all **mandatory**. The Problems in Group B are **optional** for extra credit: you are allowed to choose **ONE** of them to complete. If you do both, then we will automatically grade Problem #11 and ignore the attempt at Problem #12.

Notation: The unit step function $\coloneqq u(t-a)$. The dirac delta function $\coloneqq \delta(t-t_0)$. α and β are unknown constants.

Group A begins on the next page and consists of Pages 2-3.
Group B begins on the pages after that and consists of Pages 4-5.

Good luck!

Group A

Prob. 1 Consider the differential equation $\alpha t y'' + \beta y' = 0$.

(a) Show that no initial conditions are needed to solve the equation provided that $\alpha + \beta = 0$.

(b) Solve the differential equation under the assumption that $\alpha + \beta = 0$. Show all steps.

Prob. 2 Compute the Inverse Laplace Transform of $\frac{\alpha s+\beta}{s^2+12s+42}$. Show all steps.

Prob. 3 Solve the differential equation $y'' - 7y' + 6y = e^t + \delta(t-\pi)$ subject to the initial conditions $y(0) = y'(0) = 0$ by using any method that has been taught in this course during the Semester. Show all steps.

Prob. 4 Express the solution of the differential equation $xy'' - (x+2)y' + 2y = 0$ by using the summation of a power series.

Prob. 5 Solve the differential equation $x^2y'' - xy' - \beta y = x^3$ by using any method that has been taught in this course during the Semester. Show all steps.

Prob. 6 Solve the differential equation $y'' - 6y' - 7y = \beta e^x + \cos(x)$ by using any method that has been taught in this course during the Semester. Show all steps.

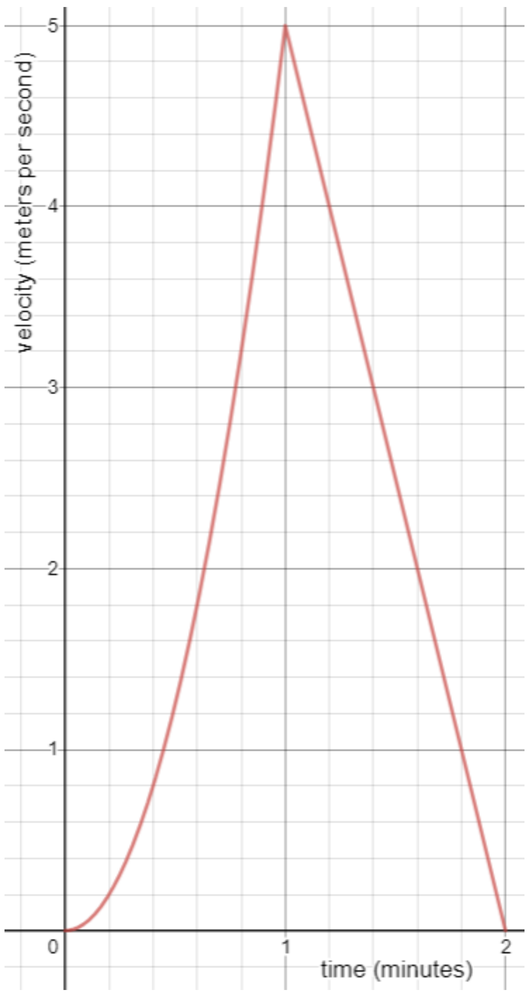
Prob. 7 Solve the differential equation $\frac{dy}{dx} + \beta y = -3(x+1)^2y^2$ by using any method that has been taught in this course during the Semester. Show all steps.

Prob. 8 Solve the differential equation $y'' - 8y' - 20y = te^t$ subject to the initial conditions $y(0) = y'(0) = 0$. Use Laplace Transforms, Inverse Laplace Transforms, and partial fraction decomposition. Show all steps.

Prob. 9 Solve the fourth-order differential equation $y'''' - 6y'' + 9y = 0$ by using any method that has been taught in this course during the Semester. Show all steps.

Group A

Prob. 10 Data for the velocities of a kiddie rollercoaster at Cedar Point in Sandusky, Ohio are shown on the projection below. Do not worry about the units for this Problem.



(a) The projection has the general form of

$$f(t) = \begin{cases} g(t), & 0 \le t < 1 \\ h(t), & 1 \le t < 2 \\ 0, & 2 \le t \end{cases},$$

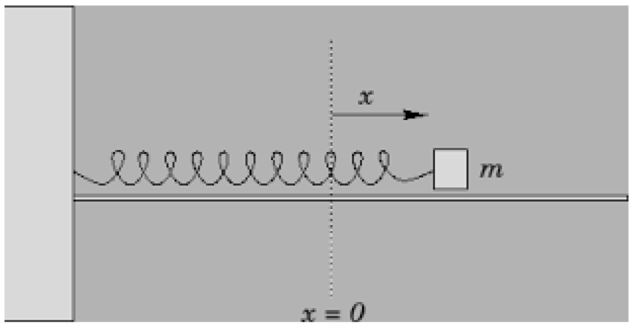
where $g(t)$ and $h(t)$ are elementary polynomial functions. What are $g(t)$ and $h(t)$?

(b) Express $f(t)$ in terms of only t, the unit step function, and numerical constants, as appropriate. Your expression cannot be explicitly piecewise like the above definition.

(c) Let $F(s) = \mathcal{L}[f(t)]$. Find $F(s)$.

Group B

Prob. 11 Suppose that a spring of length L is connected to a wall with a constant mass m attached. Define the origin $x = 0$ to occur at the location where the spring is neither compressed nor stretched. This situation is depicted below.



Suppose that the spring-mass system is governed by Newton's Law, Hooke's Law, and a drag (retarding) force. If this is the case, then we can describe the spring-mass system via the differential equation

$$m\frac{d^2x}{dt^2} = -\alpha x - \beta\frac{dx}{dt}$$

(a) Hooke's Law states that the force exerted by the spring is proportional to the distance that the spring is stretched or compressed. With this in mind, which of the three terms in the above differential equation represents Hooke's Law? **Justify your answer**.

(b) The drag (retarding) force for the spring-mass system is roughly proportional to the velocity of the spring-mass system. With this in mind, which of the three terms in the above differential equation represents Hooke's Law? **Justify your answer**.

(c) Solve the differential equation under the assumption that $\beta^2 = 4m\alpha$. Show all steps.

(d) Suppose that an external force $f(t)$ is introduced into the spring-mass system. Place this $f(t)$ into the above differential equation such that the spring-mass system is governed by Newton's Law, Hooke's Law, a drag (retarding) force, and the external force $f(t)$. **Justify your answer**.

(e) Suppose that the solution of the differential equation in (d) has the form $x(t) = x_h + x_p$. Show that if $\beta^2 = 4m\alpha$, then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x_p$.

Group B

Prob. 12 In this Problem, we will model biochemical reactions with differential equations by using mass-action kinetics. The rate of a reaction reflects how fast or slow it takes place. Mass-action kinetics is a microscopic approach to reaction kinetics and is one of the more common methods. You will be using it in this Problem.

Reactions can be classified based on their reactants. The reaction $A \to P$ is called uni-molecular because one reactant A becomes a product P. Let us consider a uni-molecular reaction $A \xrightarrow{k} P$, where k is a kinetic rate constant that describes how likely it is for this reaction to occur and produce the product P. Denote $[A]$ and $[P]$ as the concentrations of A and P, respectively. By mass-action kinetics, the rate of this reaction can be written as

$$\frac{d[P]}{d[A]} = k[A][P].$$

(a) Using the change of variables $x = [A]$ and $y = [P]$, express the above biochemical differential equation into an easily-recognizable and easily-solvable differential equation.

(b) Solve the differential equation found in part (a) by using any method that has been taught in this course during the Semester.

Assume that a certain number of A molecules are present along with a certain number of P molecules. Suppose that a sudden and very brief surge of A molecules starts a chain of biochemical reactions and that this surge can be modelled by $\delta([A] - \alpha)$. By mass-action kinetics, the rate of this reaction can be written as

$$[P]' = k\delta([A] - \alpha)$$

(c) Using the change of variables $t = [A]$, $y = [P]$, and $\alpha = \beta$, express the above biochemical differential equation into a recognizable and solvable initial-value problem. Include the initial condition. (Hint: $[P](0) = \alpha$.)

(d) Solve the differential equation found in part (c) by using any method that has been taught in this course during the Semester.