CS1382 Discrete Computational Structures

Lecture 11: Rules of Inference

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Revisiting the Socrates Example

- We have the two premises:
  - “All men are mortal.”
  - “Socrates is a man.”
- And the conclusion:
  - “Socrates is mortal.”

How do we get the conclusion from the premises?

The Argument

- Sequence of statements that end with a conclusion

$$\forall x(Man(x) \rightarrow Mortal(x))$$
$$Man(Socrates)$$
$$\rule{10cm}{0.4pt}$$
$$\therefore \quad Mortal(Socrates)$$

- Premises – above the line
- Conclusion – below the line

Valid Arguments

- If premises imply the conclusion
- Construct valid arguments
  - Propositional logic
  - Predicate logic
  - Use rules of inference
- If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then

  $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ *is a tautology*

Rules of Inference for Propositional Logic: Modus Ponens

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$(p \land (p \to q)) \to q$$

Example:

Let p be "It is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$$(\neg q \land (p \to q)) \to \neg p$$

Example:

Let p be "it is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"I will not study discrete math."

"Therefore, it is not snowing."

Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:

$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example:

Let p be "it snows."

Let q be "I will study discrete math."

Let r be "I will get an A."

"If it snows, then I will study discrete math."

"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$(\neg p \wedge (p \vee q)) \rightarrow q$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math or I will study English literature."

"I will not study discrete math."

"Therefore , I will study English literature."

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$$p \rightarrow (p \vee q)$$

Example:

Let p be "I will study discrete math."

Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Simplification

$$\frac{p \wedge q}{\therefore q}$$

Corresponding Tautology:

$$(p \wedge q) \rightarrow p$$

Example:

Let *p* be “I will study discrete math.”

Let *q* be “I will study English literature.”

“I will study discrete math and English literature”

“Therefore, I will study discrete math.”

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology:

$((p) \wedge (q)) \rightarrow (p \wedge q)$

Example:

Let p be “I will study discrete math.”

Let q be “I will study English literature.”

“I will study discrete math.”

“I will study English literature.”

“Therefore, I will study discrete math and I will study English literature.”

Resolution

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology:

$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Example:

Let *p* be "I will study discrete math."

Let *r* be "I will study English literature."

Let q be "I will study databases."

"I will not study discrete math or I will study English literature."

"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will study English literature."

Valid Arguments - Example

From the single proposition $p \wedge (p \to q)$

Show that *q* is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \to q)$	Premise
2. p	Simplification using (1)
3. $p \to q$	Simplification using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments

- With these hypotheses:
  - “It is not sunny this afternoon and it is colder than yesterday.”
  - “If we will go swimming, then it is sunny.”
  - “If we do not go swimming, then we will take a canoe trip.”
  - “If we take a canoe trip, then we will be home by sunset.”
- Using the inference rules, construct a valid argument for the conclusion:

  “We will be home by sunset.”

Solution:

1. Choose propositional variables:
   - p : “It is sunny this afternoon.”
   - r : “We will go swimming.”
   - t : “We will be home by sunset.”
   - q : “It is colder than yesterday.”
   - s : “We will take a canoe trip.”
2. Translation into propositional logic:

Continued on next slide →

Valid Arguments

Hypotheses: $\neg p \land q$, $r \to p$, $\neg r \to s$, $s \to t$
Conclusion: t

3. Construct the Valid Argument

Step	**Reason**
1. $\neg p \land q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

Handling Quantified Statements

- Valid arguments for quantified statements are a sequence of statements.
- Each statement is either a premise or follows from previous statements by rules of inference which include:
  - Rules of Inference for Propositional Logic
  - Rules of Inference for Quantified Statements
- The rules of inference for quantified statements are introduced in the next several slides.

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

- Our domain consists of all students and John is a student.
- “All students are wise”
- “Therefore, John is wise.”

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

- Used often implicitly in Mathematical Proofs.
- c must identify an arbitrary subject and could be done by
  - Introducing it with universal instantiation
  - With an assumption

Example

- For every number x, if x > 1, then x - 1 > 0. Also for every number x, x > 1.
  - We make hypothetical argument (of course, every number is not greater than 1).
  - Hence we conclude that for every number x, x - 1 > 0.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

- “There is someone who got an A in the course.”
- “Let’s call her *a* and say that *a* got an A”

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

- “Michelle got an A in the class.”
- “Therefore, someone got an A in the class.”

Using Rules of Inference - Example

Using the rules of inference, construct a valid argument to show that

"John Smith has two legs"

is a consequence of the premises:

- "Every man has two legs."
- "John Smith is a man."

Solution:

Let $M(x)$ denote "x is a man" and $L(x)$ " x has two legs" and let John Smith be a member of the domain.

Valid Argument:

Step	**Reason**
1. $\forall x(M(x) \rightarrow L(x))$	Premise
2. $M(J) \rightarrow L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference

Example 2:

Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

- "A student in this class has not read the book."
- "Everyone in this class passed the first exam."

Solution:

Let All students in university be the domain

$C(x)$ - "x is in this class,"

$B(x)$ - "x has read the book"

$P(x)$ - "x passed the first exam"

First we translate the premises and conclusion into symbolic form.

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

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Using Rules of Inference

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

Valid Argument:

Step	**Reason**
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	EG from (8)

Returning to the Socrates Example

$$\forall x(Man(x) \rightarrow Mortal(x))$$

$$Man(Socrates)$$

$$\therefore \quad Mortal(Socrates)$$

Valid Argument

Step	**Reason**
1. $\forall x(Man(x) \rightarrow Mortal(x))$	Premise
2. $Man(Socrates) \rightarrow Mortal(Socrates)$	UI from (1)
3. $Man(Socrates)$	Premise
4. $Mortal(Socrates)$	MP from (2) and (3)

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ P(a), \text{where } a \text{ is a particular} \\ \quad \text{element in the domain} \\ \hline \therefore Q(a) \end{array}$$

This rule could be used in the Socrates example.

Questions?

Thank You!