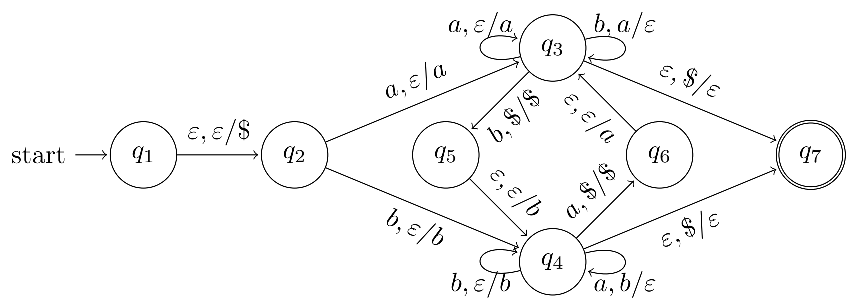
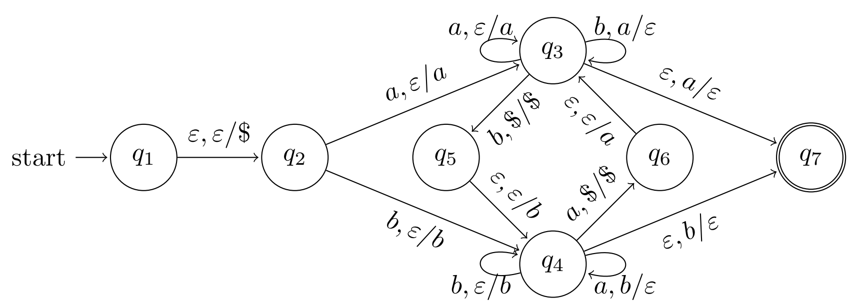
Test 2 Review Key

1. For each of the following languages, give a pushdown automaton that recognizes it.

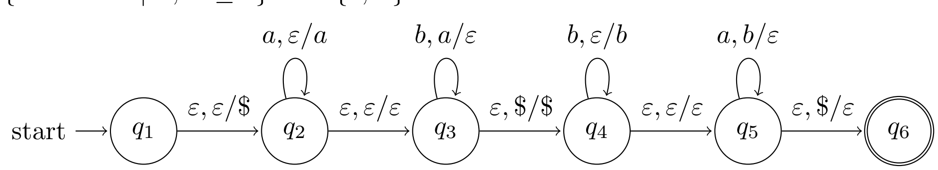
   (a) $\{\omega \mid n_a(\omega) = n_b(\omega)\}$ over {a, b}.



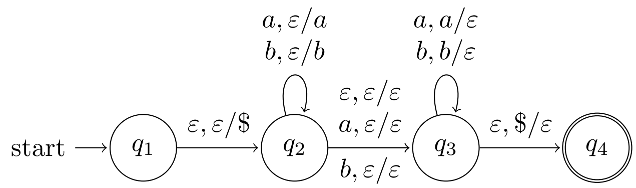
   (b) $\{\omega \mid n_a(\omega) \neq n_b(\omega)\}$ over {a, b}.



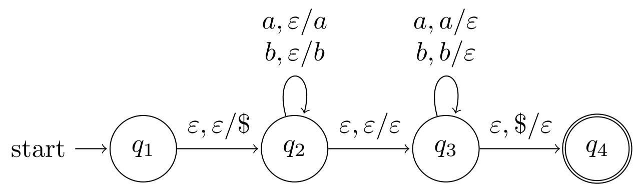
   (c) $\{a^n b^{n+m} a^m \mid n, m \geq 0\}$ over {a, b}.



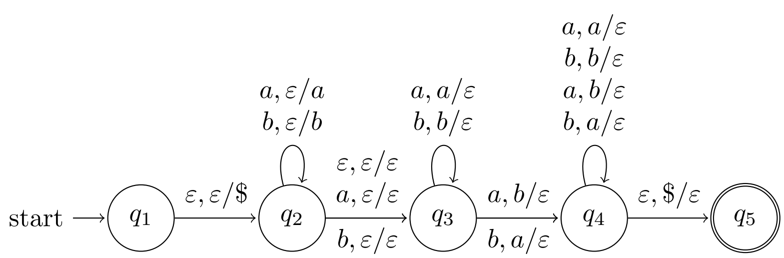
(d) The language of palindromes over {a, b}.



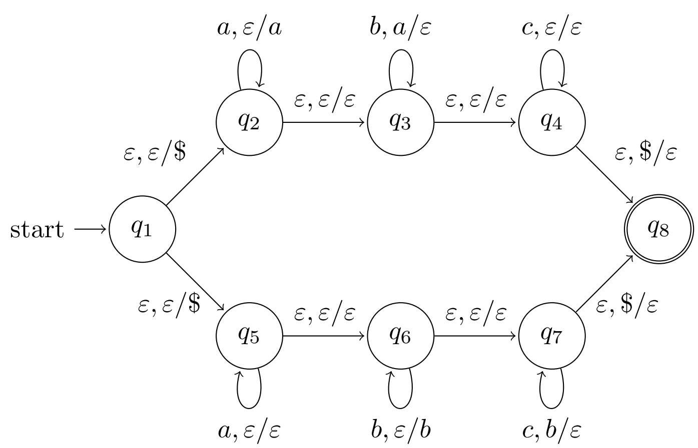
(e) The language of even-length palindromes over {a ,b}.



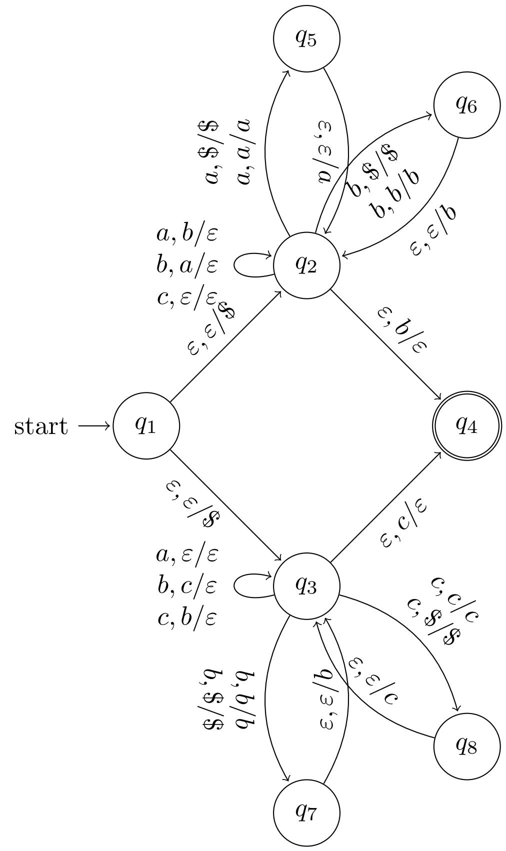
(f) The language of all non-palindromes over {a, b}.



(g) $\{a^i b^j c^k |\ i, j, k \geq 0 \text{ and } j = i \text{ or } j = k\}$ over {a, b, c}.



(h) $\{\omega \mid n_a(\omega) < n_b(\omega) \text{ or } n_a(\omega) < n_c(\omega)\}$ over {a, b, c}.

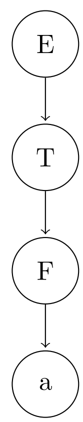


2. Let G_1 be the following grammar:

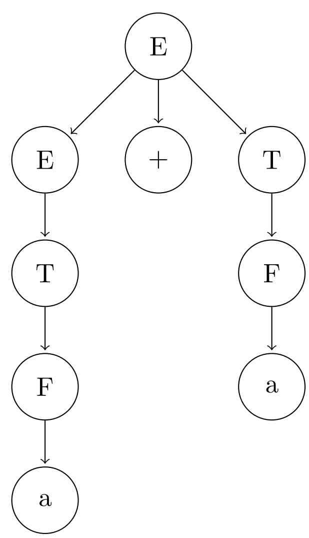
$$\begin{aligned} E &\to E + T \mid T \\ T &\to T \times F \mid F \\ F &\to (E) \mid a \end{aligned}$$

where E is the start variable. Give the parse trees for each of the following strings using G_1:

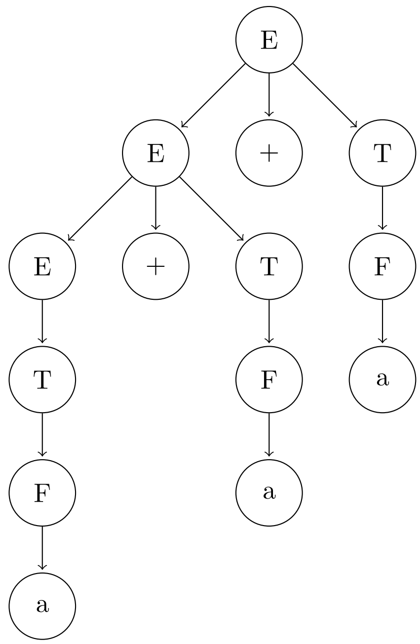
(a) a



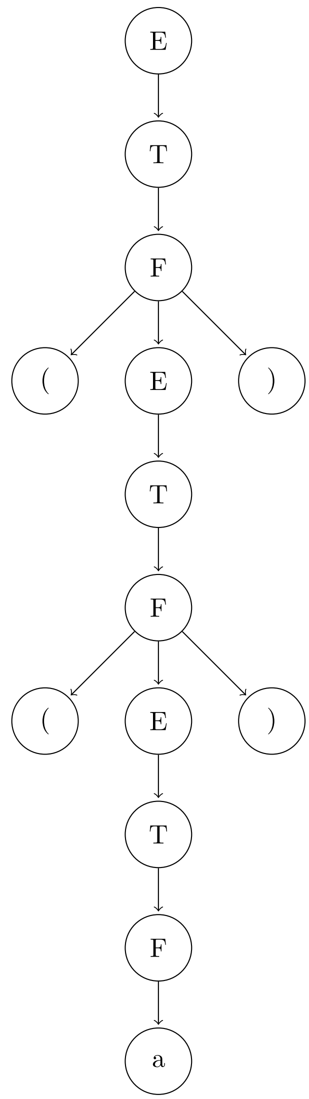
(b) a+a



(c) a+a+a



(d) ((a))

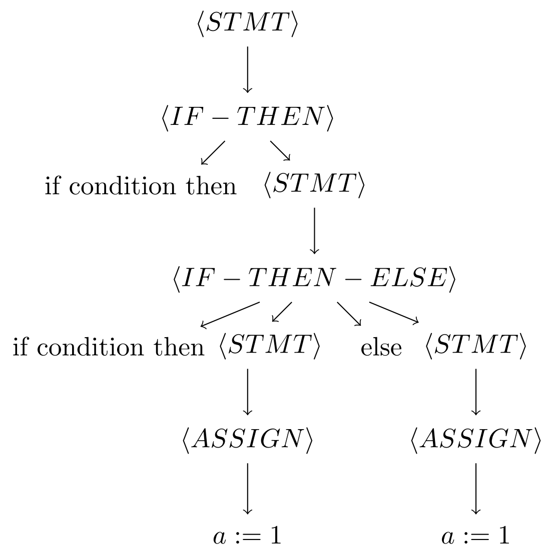


3. Let G_2 be the following grammar:

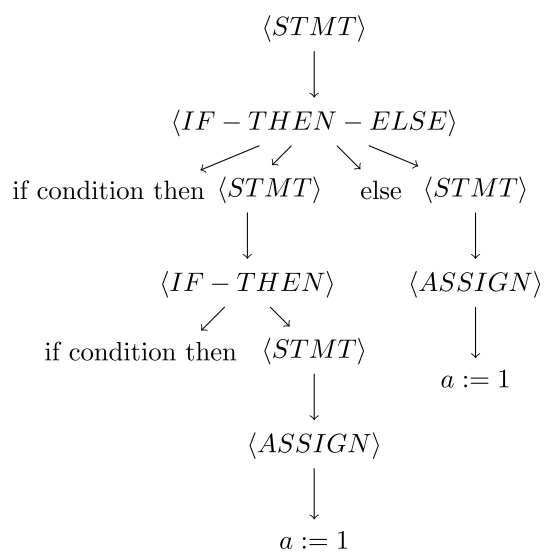
$$\begin{aligned}
\langle STMT\rangle &\to \langle ASSIGN\rangle \mid \langle IF-THEN\rangle \mid \langle IF-THEN-ELSE\rangle \\
\langle IF-THEN\rangle &\to if\ condition\ then\ \langle STMT\rangle \\
\langle IF-THEN-ELSE\rangle &\to if\ condition\ then\ \langle STMT\rangle\ else\ \langle STMT\rangle \\
\langle ASSIGN\rangle &\to a := 1
\end{aligned}$$

where $\langle STMT\rangle$ is the start variable. Show that G_2 is ambiguous (i.e. give two different derivations for a string.

First derivation:



Second derivation:



4. Let G_3 be the following grammar:

$$S \to SS \mid T$$
$$T \to aTb \mid ab$$

Show that G_3 is ambiguous (i.e. give two different derivations for a string).
First derivation:

$$\begin{tikzcd}
 & & S \arrow[dl] \arrow[dr] & \\
 & S \arrow[dl] \arrow[dr] & & S \arrow[d] \\
S \arrow[d] & & S \arrow[d] & T \arrow[d] \\
T \arrow[d] & & T \arrow[d] & ab \\
ab & & ab &
\end{tikzcd}$$

Second derivation:

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] & & \\
S \arrow[d] & & S \arrow[dl] \arrow[dr] & \\
T \arrow[d] & S \arrow[d] & & S \arrow[d] \\
ab & T \arrow[d] & & T \arrow[d] \\
 & ab & & ab
\end{tikzcd}$$

5. For each of the following languages, give a context-free grammar that describes it.

   (a) $\{\omega|\ n_a(\omega) = n_b(\omega)\}$ over {a, b}.
   $S \to aSb \mid bSa \mid SS \mid \varepsilon$

   (b) $\{\omega|\ n_a(\omega) \neq n_b(\omega)\}$ over {a, b}.
   $S \to AaA \mid BbB$
   $A \to aAb \mid bAa \mid AA \mid a \mid \varepsilon$
   $B \to aBb \mid bBa \mid BB \mid b \mid \varepsilon$

   (c) $\{a^n b^{n+m} a^m|\ n, m \geq 0\}$ over {a, b}.
   $S \to S1\ S2$
   $S1 \to aS1b \mid \varepsilon$
   $S2 \to bS2a \mid \varepsilon$

(d) The language of all non-palindromes over {a, b}.
$S \to aSa \mid bSb \mid aS1b \mid bS1a$
$S1 \to aS1a \mid bS1b \mid aS1b \mid bS1a \mid a \mid b \mid \varepsilon$

(e) $\{a^i b^j c^k \mid i, j, k \geq 0$ and $j = i$ or $j = k\}$ over {a, b, c}.
$S \to S1\ C \mid A\ S2$
$S1 \to aS1b \mid \varepsilon$
$S2 \to bS2c \mid \varepsilon$
$A \to aA \mid \varepsilon$
$C \to cC \mid \varepsilon$

(f) $\{\omega \mid n_a(\omega) < n_b(\omega)$ or $n_a(\omega) < n_c(\omega)\}$ over {a, b, c}.
$S \to S1bS1 \mid S2cS2$
$S1 \to S1\ S1 \mid aS1b \mid bS1a \mid bS1 \mid cS1 \mid \varepsilon$
$S2 \to S2\ S2 \mid aS2c \mid cS2a \mid bS2 \mid cS2 \mid \varepsilon$

6. Select at least 5 of the following problems. Prove whether the language is context-free or non-context-free.

(a) $\{a^i b^j \mid i \leq j^2\}$ over {a, b}.
Language is not context-free:
Assume language is context free and P is the pumping length.
Let $S = a^{P^2} b^P$. By pumping lemma $S = UVXYZ$ where $|VXY| \leq P$ and $|VY| \geq 1$.
Case1: VY contains at least one a and no b's.
if we pump with $i = 2$ we add at least 1 a so number of a's is $>$ number of b's squared.
Case2: VY contains at least 1 b (may contain a's).
if we pump with $i = 0$ we lose at least 1 b and at most $P-1$ a's. But then the remaining number of a is at least $P^2 - (P-1) = P^2 - P + 1$ and the remaining number of b's squared is at most $(P-1)^2 = P^2 - 2P + 1$ so a's is $>$ number of b's squared.
This is all possible cases and pumping doesn't work in any of them. This violates the assumption, hence language is not context-free.

(b) $\{a^i b^j c^k \mid k = ji\}$ over {a, b, c}.
Language is not context-free:
Assume language is context free and P is the pumping length.
Let $S = a^P b^P c^{P^2}$. By pumping lemma $S = UVXYZ$ where $|VXY| \leq P$ and $|VY| \geq 1$.
Case1: VY contains at least one a and/or b but no c's.
if we pump with $i = 2$ we add at least 1 a and/or b so number of a's times number of b's is $>$ number of c's.
Case2: VY contains at least 1 b and at least 1 c.
if we pump with $i = 0$ we lose at least 1 b and at most $P-1$ c's. But then the number of a times the remaining number of b's is at most $P * (P-1) = P^2 - P$ and the remaining number of c's is a least $P^2 - (P-1) = P^2 - P + 1$, so they cannot be equal.

Case3: VY contains only c's. Similar to case1, if we pump with $i = 0$, we lose c's but no a's or b's and the number of a's times number of b's is $>$ number of c's.
This is all possible cases and pumping doesn't work in any of them. This violates the assumption, hence language is not context-free.

(c) $\{a^i b^j c^k |\ i < j < k\}$ over {a, b, c}.
Language is not context-free:
Assume language is context free and P is the pumping length.
Let $S = a^P b^{P+1} c^{P+2}$.
By pumping lemma $S = UVXYZ$ where $|VXY| \leq P$ and $|VY| \geq 1$.
Case1: VY contains at least one a (may contain b's).
If we pump with $i = 3$ we add at least 2 a's and the number of a's in the resulting string is $\geq$ to the number of c's, so not in language.
Case2: VY contains at least one b but no a's (may contain c's).
if we pump with $i = 0$ we lose at least 1 b but no a's and the number of a's in the resulting string is $\geq$ to the number of b's, so not in language.
Case3: VY contains at least one c and only c's.
if we pump with $i = 0$ we lose at least 1 c but no b's and the number of b's in the resulting string is $\geq$ to the number of v's, so not in language.
This is all possible cases and pumping doesn't work in any of them. This violates the assumption, hence language is not context-free.

(d) $\{\omega\omega |\ \omega \in \{a, b\}^*\}$.
Language is not context-free:
Assume language is context free and P is the pumping length.
Let $S = a^P b^P a^P b^P$.
Notice that any string in the language must be even length so if VY were odd length then pumping with i even would result in an odd length string hence not in language. So we can assume VY must be even. If VY is all before the center of the string then pumping with $i = 0$ will shift the center right and hence the first half of the string would end with an a while the second half would end with a b hence not in language. Similarly, if VY is all after the center of the string then pumping with $i = 0$ will shift the center left and hence the first half of the string would begin with an a while the second half would begin with a b hence not in language. The final case is VY contains b's from the first half of the sting and a's from the second half. If there were not equal number of each we would shift the center and have one of the previous cases. But if they are equal and we pump with $i = 0$ we lose b's in the first half and a in the second and so it is not of the form $\omega\omega$ and hence not in the language. This is all possible cases and pumping doesn't work in any of them. This violates the assumption, hence language is not context-free.

(e) $\{a^n \omega \omega^R a^n |\ n \geq 0, \omega \in \{a, b\}^*\}$.

Language is context-free:
Note that the center part is simply an even length palindrome and then the outsides are just the same number of a's at the beginning and end, and since the a's are the same at each end, this language is equivalent to just $\{\omega\omega^R | \ \omega \in \{a, b\}^*\}$ which is:
$S \to aSa \mid bSb \mid \varepsilon$.

(f) $\{a^i b^j a^i b^j | \ i, j \geq 0\}$ over {a, b}.
Language is not context-free:
The proof is the same as it was for (d).

(g) $\{a^i b^j a^j b^i | \ i, j \geq 0\}$ over {a, b}.
Language is context-free:
$S \to aSb \mid S1$
$S1 \to bS1a \mid \varepsilon$

(h) $\{\omega |\ n_a(\omega) < n_b(\omega) < n_c(\omega)\}$ over {a, b, c}.
Language is not context-free:
While order of letters does not matter in this string, the proof from (c) still works for this language as well.

(i) $\{\omega |\ n_a(\omega)/n_b(\omega) = n_c(\omega)\}$ over {a, b, c}.
Language is not context-free:
Note that this one is probably easier to think of as $n_a(\omega) = n_b(\omega) * n_c(\omega)\}$ and the proof is similar to that for (a).
Let $S = a^{P^2} b^P c^P$.
Case1: If VY only a's (must be at least 1) then pumping with $i = 0$ we lose at least $1a$ so number of a's no longer equals the number of b's times number of c's.
Case2: If VY only b's and/or c's (must be at least 1 of one of the two), then the case is similar to case 1.
Case3: If VY is contains both a's and b's, if we pump with $i = 0$ we lost at minimum 1 b and at most $P-1$ a's. As a result we would have the remaining number of a is at least $P^2 - (P-1) = P^2 - P + 1$ and the remaining number of b's times c's is at most $(P-1)*P = P^2 - P$ so a's is $>$ number of b's times c's.

(j) $\{a^i b^j |\ i$ is prime or j is prime$\}$ over {a, b}.
Language is not context-free:
Note, j can be 0 so long as i is prime so let $S = a^Q$ where Q is the smallest prime greater than $P + 1$. By the pumping lemma VY is a^K for some $1 \leq K \leq P$ and therefore UXZ is a^M for some $2 \leq M \leq Q - 1$. if we pump with $i = M$ the resulting string is $a^M a^{M*K} = a^{M*(K+1)}$ which is not prime, Hence the language is not context-free.

(k) $\{a^n |\ n \geq 0$ and n is a prime number$\}$ over {a}.
Language is not context-free:
This is a simpler case of problem (j) and my proof for (j) works for this as well.

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9. attached at end of file

10. attached at end of file

11. The Church-Turing Thesis states that Turing's idea of an algorithm (the rules of a Turing Machine) is equivalent to the intuitive idea of a algorithm. If true, the implications of this are that nothing can be computed algorithmically that can't be computed by a Turing machine. It also means that if a Turing machine can be simulated in a computer language, that language can compute anything that can be computed algorithmically.

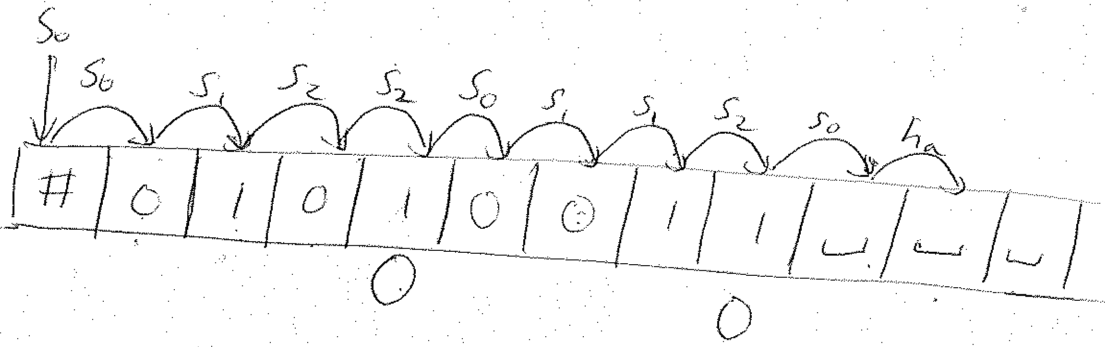
12. If a language is Turing recognizable it means a Turing machine can be created that halts in the accepting state on any input which is a member of the language. If the input is not in the language, the Turing machine may either halt rejecting or not halt at all. If a language is Turing decidable it means a Turing machine can be created that halts in the accepting state on any input which is a member of the language. If the input is not in the language, the Turing machine halts in the rejecting state. Turing decidable languages are a subset of Turing recognizable languages as the Turning machine for a Turing recognizable languages may not halt on a string not in a language, while there exists a Turing machine for a recognizable language that always halts on any input.

7.
$(q_0, \#, q_0, \#, R)$ (q_0, a, q_1, a, R) (q_0, b, q_0, b, R)
$(q_0, \sqcup, h_a, \sqcup, R)$
(q_1, a, q_0, c, R) (q_1, b, q_1, b, R) $(q_1, \sqcup, h_a, \sqcup, R)$

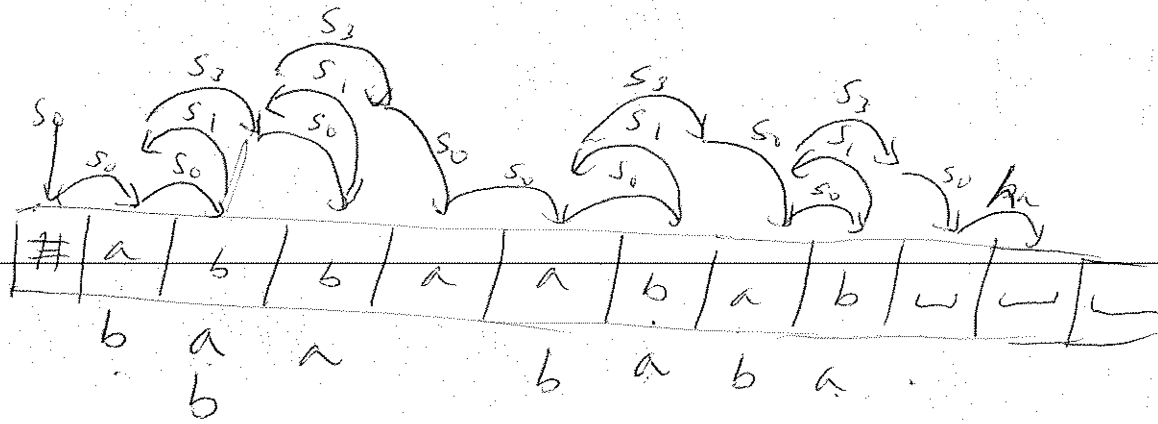
8. $(q_0, \#, q_0, \#, R)$ $(q_0, 1, q_1, 1, R)$ $(q_0, \sqcup, h_a, \sqcup, R)$
$(q_1, 1, q_2, 1, R)$ $(q_1, \sqcup, h_R, \sqcup, R)$
$(q_2, 1, q_0, 1, R)$, $(q_2, \sqcup, h_R, \sqcup, R)$

9. $(q_0, \#, q_0, \#, R)$ $(q_0, a, q_1, \bar{a}, R)$ $(q_0, b, q_2, \bar{b}, R)$
$(q_0, \sqcup, h_a, \sqcup, R)$ (q_1, a, q_1, a, R) (q_1, b, q_1, b, R)
$(q_1, \sqcup, q_3, \sqcup, L)$ $(q_1, \bar{a}, q_3, \bar{a}, L)$ $(q_1, \bar{b}, q_3, \bar{b}, L)$
(q_2, a, q_2, a, R) (q_2, b, q_2, b, R) $(q_2, \sqcup, q_4, \sqcup, L)$
$(q_2, \bar{a}, q_4, \bar{a}, L)$ $(q_2, \bar{b}, q_4, \bar{b}, L)$ $(q_3, a, q_5, \bar{a}, L)$
$(q_3, b, q_6, \bar{a}, L)$ $(q_3, \bar{a}, q_7, a, R)$ $(q_4, a, q_5, \bar{b}, L)$
$(q_4, b, q_6, \bar{b}, L)$ $(q_4, \bar{b}, q_7, b, R)$ (q_5, a, q_5, a, L)
(q_5, b, q_5, b, L) $(q_5, \bar{a}, q_0, a, R)$ $(q_5, \bar{b}, q_0, a, R)$
(q_6, a, q_6, a, L) (q_6, b, q_6, b, L) $(q_6, \bar{a}, q_0, b, R)$
$(q_6, \bar{b}, q_0, b, R)$ $(q_7, \bar{a}, q_7, a, R)$ $(q_7, \bar{b}, q_7, b, R)$
$(q_7, \sqcup, h_a, \sqcup, R)$ $(q_0, \bar{a}, q_7, a, R)$ $(q_0, \bar{b}, q_7, b, R)$

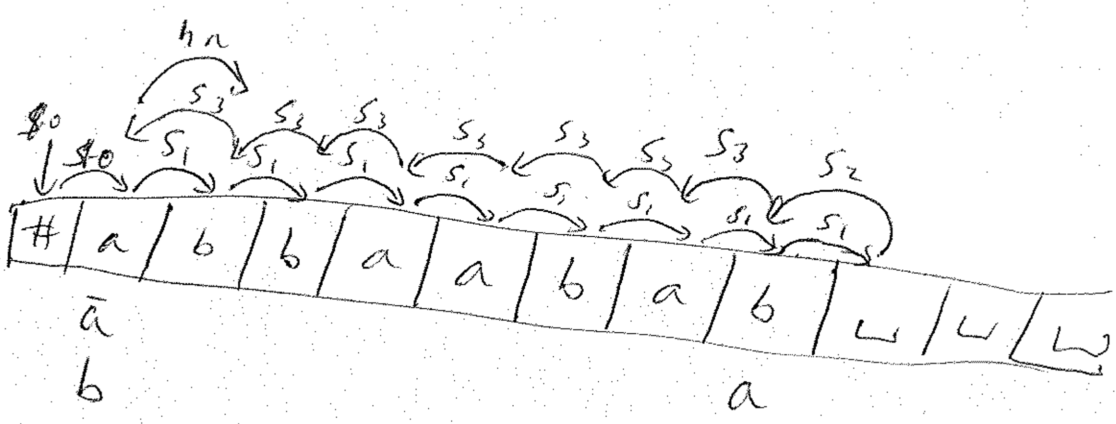
10. a)


s_0
s_0 s_1 s_2 s_2 s_0 s_1 s_1 s_2 s_0 h_a
0 1 0 1 0 0 1 1 ⊔ ⊔ ⊔
0 0

b)


a b b a a b a b ⊔ ⊔ ⊔
b a a b a b a
b

c)


h_a
a b b a a b a b ⊔ ⊔ ⊔
$\bar{a}$
b a