Non-context-free language

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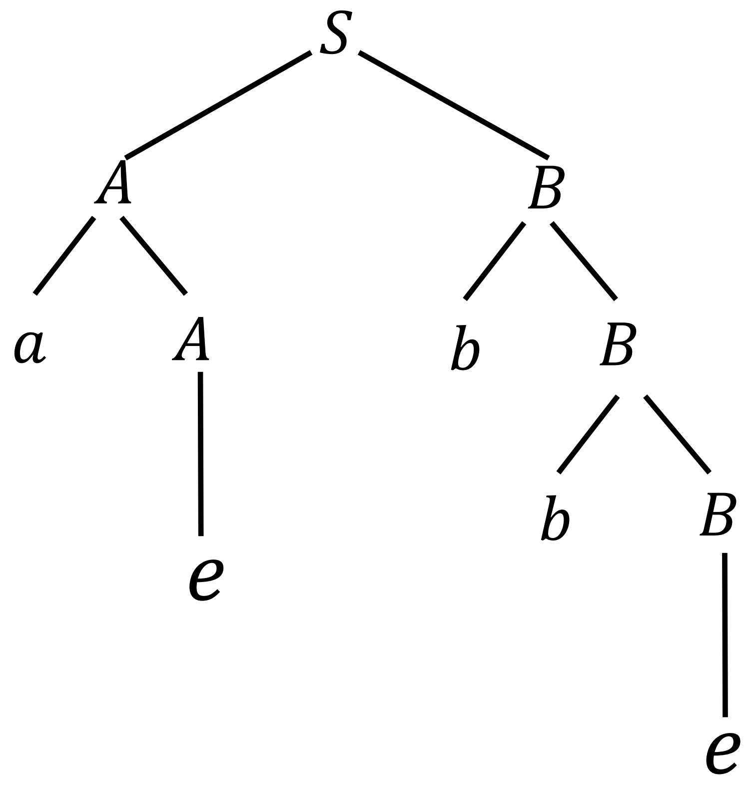
Pumping theorem

- A parse tree is a graphical representation of a derivation
  - Example:

$$S \to AB$$
$$A \to aA|e$$
$$B \to bB|e$$

$$\begin{aligned} S &\Rightarrow AB \\ &\Rightarrow aAB \\ &\Rightarrow aAbB \\ &\Rightarrow abB \\ &\Rightarrow abbB \\ &\Rightarrow abb \end{aligned}$$

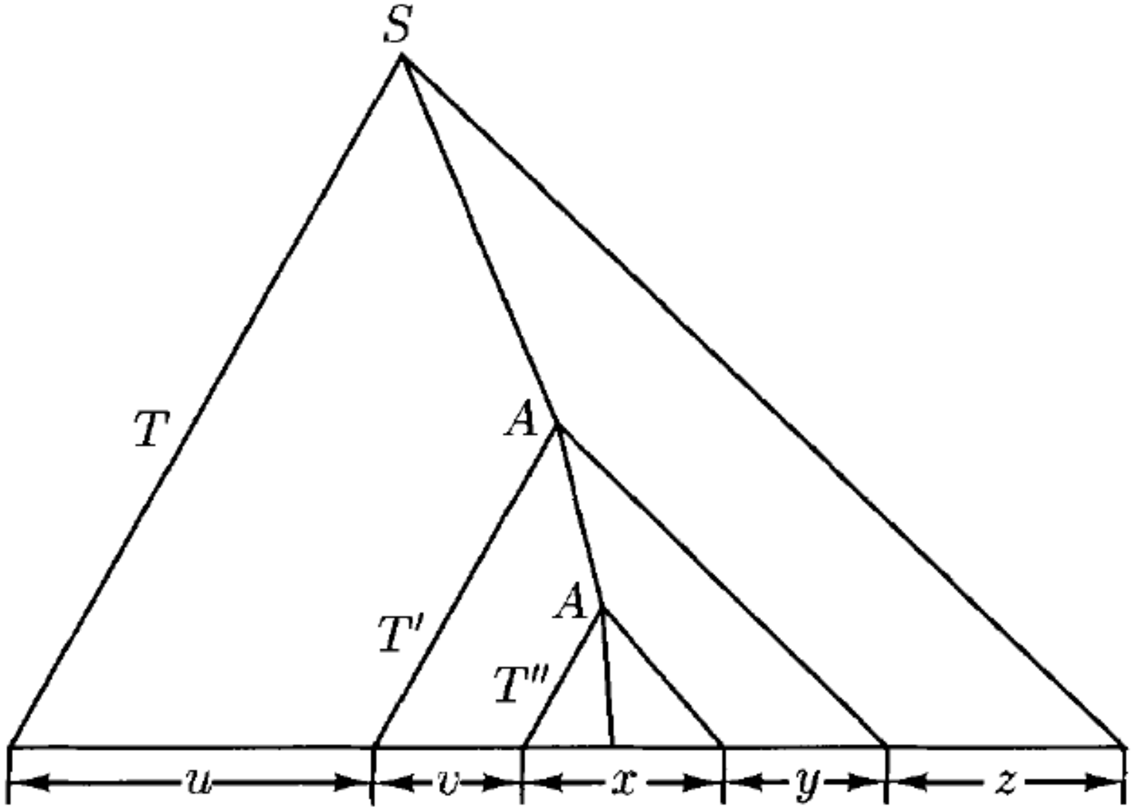


Pumping theorem

- A parse tree is a graphical representation of a derivation
  - The root of a parse tree is the start symbol S
  - A leaf of a parse tree is a terminal
  - The leaves of a parse tree, from left to right, form the string

Pumping theorem

- Consider a parse tree of a long enough string
  - some of the rule is reused

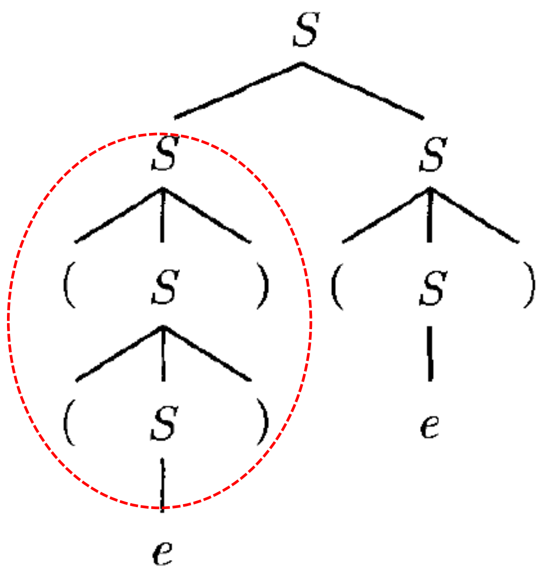


Pumping theorem

- Consider a parse tree of a long enough string
  - some of the rule is reused

$$
\begin{aligned}
V &= \{S, (,)\},\\
\Sigma &= \{(,)\},\\
R &= \{S \to e,\\
&\qquad S \to SS,\\
&\qquad S \to (S)\}.
\end{aligned}
$$

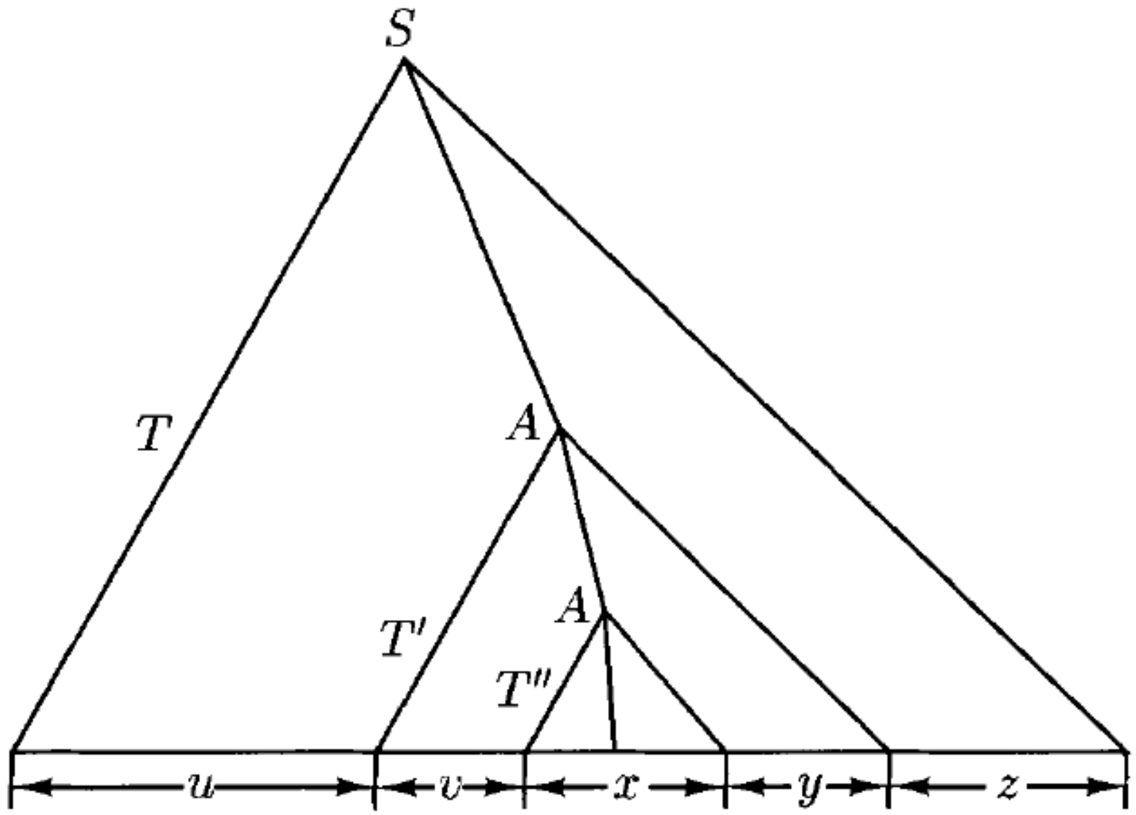
$$D_1 = S \Rightarrow SS \Rightarrow (S)S \Rightarrow ((S))S \Rightarrow (())S \Rightarrow (())(S) \Rightarrow (())()$$



Pumping theorem

- Consider a parse tree of a long enough string
  - some of the rule is reused
  - If $A \to \cdots \to vAy$, then

    $A \to \cdots \to vAy \to \cdots \to vvAyy$



Pumping theorem

Pumping lemma for context-free languages If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,

2. $|vy| > 0$, and

3. $|vxy| \leq p$.

Pumping theorem

Use Pumping theorem to show the followings are not context-free:

- a). $\{a^n b^n c^n : n \geq 0\}$
- b). $\{a^p : p \text{ is prime}\}$
- c). $\{a^{n^2} : n \geq 0\}$
- d). $\{a^n b^n a^n b^n : n \geq 0\}$
- d). $\{ww : w \in \{a, b\}^*\}$
- e). $\{a^n b a^{2n} b a^{3n} b : n \geq 0\}$
- f). $\{w_1 \# w_2 : w_1, w_2 \in \{a, b\}^*, w_1 \text{ is a substring of } w_2\}$

$\{a^n b^n c^n : n \geq 0\}$

- a). Suppose on the contrary that $L = \{a^n b^n c^n : n \geq 0\}$ is CFG, then there exists some sufficiently large number N, for any $n \geq N$, we have $a^n b^n c^n = uvxyz$ such that $|vy| > 0$, $|vxy| \leq N$, and $uv^i xy^i z \in L$ for any $i \geq 0$.
- Pick $n = N$ and consider $a^N b^N c^N = uvxyz$. $|vxy| \leq N$, so there are 5 different possibilities.
- i). $vxy = a \cdots a,\ or\ b \cdots b, or\ c \cdots c$, i.e., it only consists one symbol
- ii).$vxy = a \cdots ab \cdots b$ or $vxy = b \cdots bc \cdots c$, i.e., vxy contains both a, b or b, c.

$\{a^n b^n c^n : n \geq 0\}$

- a). Suppose on the contrary that $L = \{a^n b^n c^n : n \geq 0\}$ is CFG, then there exists some sufficiently large number N, for any $n \geq N$, we have $a^n b^n c^n = uvxyz$ such that $|vy| > 0$, $|vxy| \leq N$, and $uv^i xy^i z \in L$ for any $i \geq 0$.
- Pick $n = N$ and consider $a^N b^N c^N = uvxyz$. $|vxy| \leq N$, so there are 5 different possibilities.
- i). $vxy = a \cdots a,\ or\ b \cdots b, or\ c \cdots c$, i.e., it only consists one symbol
- We show the case of $vxy = a \cdots a$, the other two cases are the same. Since $|vy| > 0$, we know $v^2 xy^2$ contains exactly $|vy|$ more a's than vxy. That is, $uv^2 xy^2 z$ will contain $N + |vy| > N$ copies of a, i.e., $uv^2 xy^2 z = a^{N+|vy|} b^N c^N \notin L$, contradicting that $uv^i xy^i z \in L$ for any $i \geq 0$.
- ii).$vxy = a \cdots ab \cdots b$ or $vxy = b \cdots bc \cdots c$, i.e., vxy contains both a, b or b, c.
- We show that case of $vxy = a \cdots ab \cdots b$, the other case is the same. Since $|vy| > 0$, we assume $vy = a^\alpha b^\beta$ for some $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$. Now we have

  $uv^2 xy^2 z = a^{N+\alpha} b^{N+\beta} c^N \notin L$, contradicting that $uv^i xy^i z \in L$ for any $i \geq 0$
- (Note that since $|vxy| \leq N$, it is impossible for vxy to contain all a, b, c. Thus we have exhausted all the possibilities.)

Closure property

- Regular language is closed under
  - Union
  - Concatenation
  - Kleene star
  - Complementation
  - Intersection

Closure property

- Context-free language is closed under
  - Union
  - Concatenation
  - Kleene star
- Context-free language is not closed under
  - Complementation
  - Intersection

Closure property

- Context-free language is closed under
  - Union

Union. Let S be a new symbol and let $G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R, S)$, where $R = R_1 \cup R_2 \cup \{S \to S_1, S \to S_2\}$. Then we claim that $L(G) = L(G_1) \cup L(G_2)$. For the only rules involving S are $S \to S_1$ and $S \to S_2$, so $S \Rightarrow_G^* w$ if and only if either $S_1 \Rightarrow_G^* w$ or $S_2 \Rightarrow_G^* w$; and since G_1 and G_2 have disjoint sets of nonterminals, the last disjunction is equivalent to saying that $w \in L(G_1) \cup L(G_2)$.

Closure property

- Context-free language is closed under
  - Concatenation

Concatenation. The construction is similar: $L(G_1)L(G_2)$ is generated by the grammar

$$G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \to S_1 S_2\}, S).$$

Closure property

- Context-free language is closed under
  - Kleene star

Kleene Star. $L(G_1)^*$ is generated by

$$G = (V_1 \cup \{S\}, \Sigma_1, R_1 \cup \{S \to e, S \to SS_1\}, S).$$