Languages Not Regular

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TEXAS TECH UNIVERSITY

Non-regular languages

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  Oblivious!

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- Probably the answer is "no", but how can we show it?

Non-regular languages

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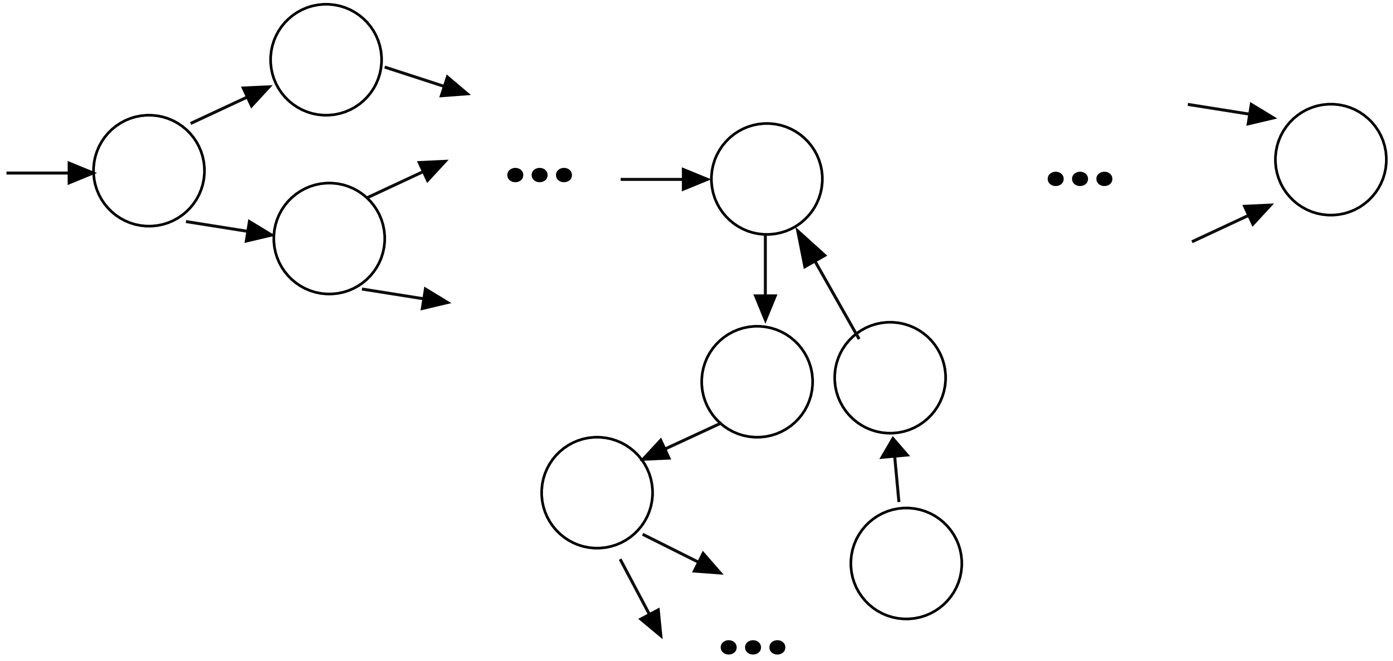
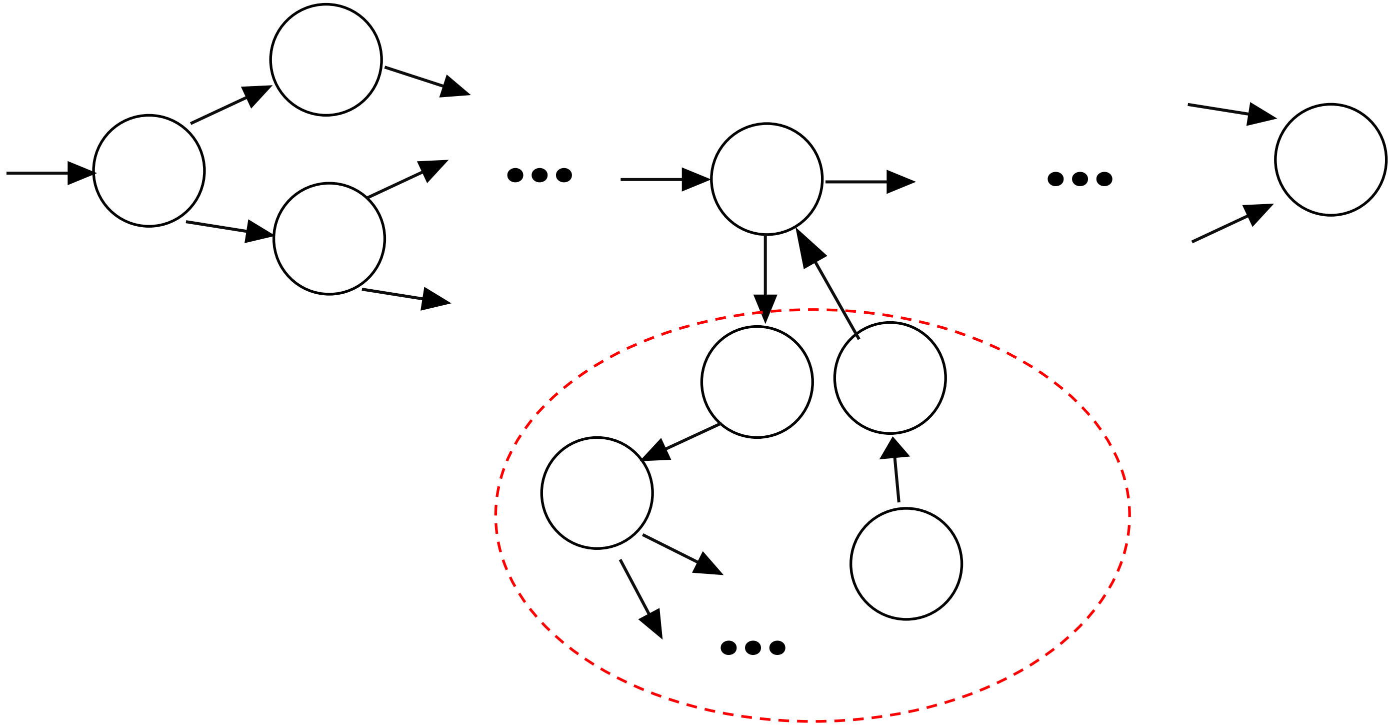


Non-regular languages

- Suppose you have a magic DFA to accept $\mathrm{L} = \{0^n 1^n : n > 0\}$
- The machine only has finite states, so if n is sufficiently large, there must be some loop (pigeonhole principle).



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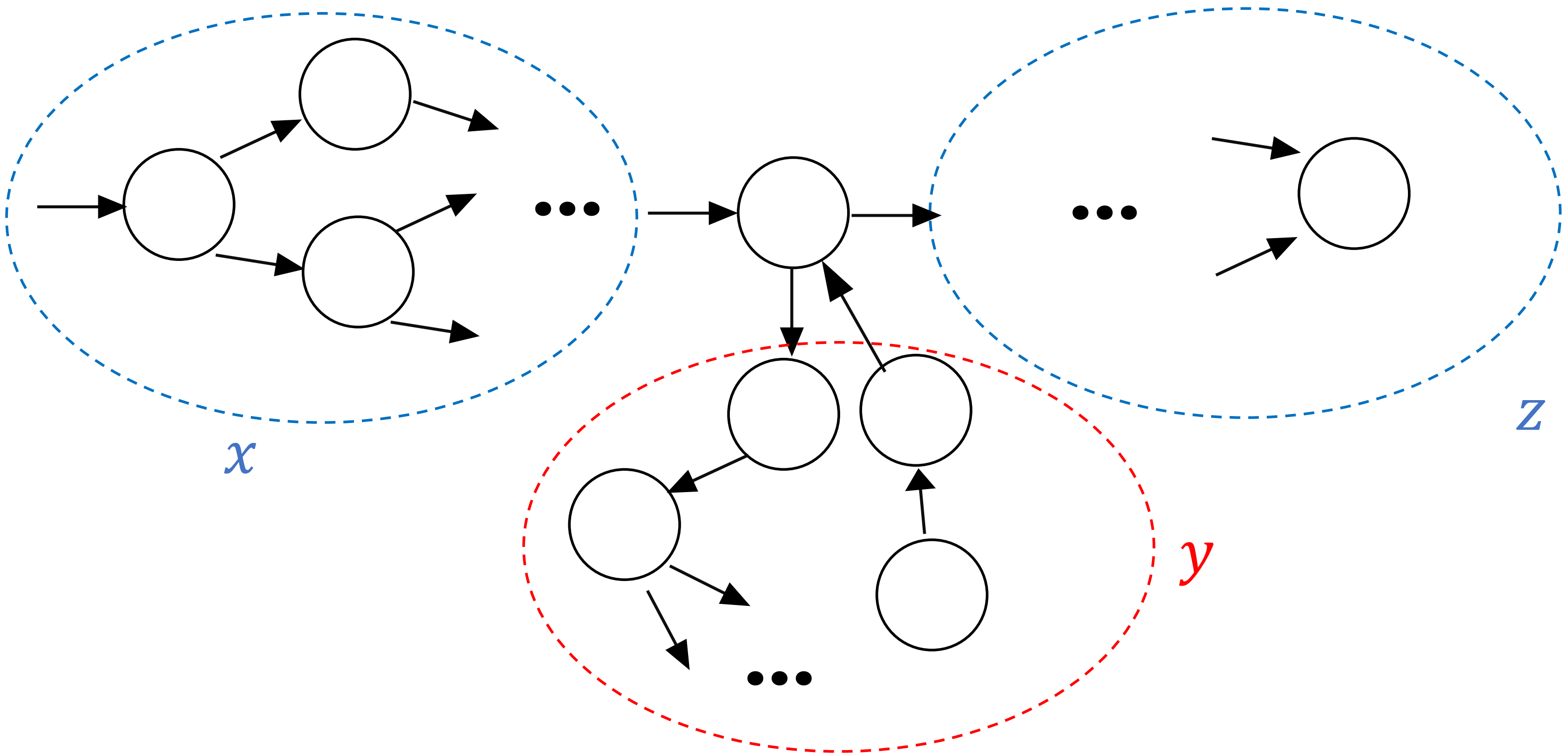

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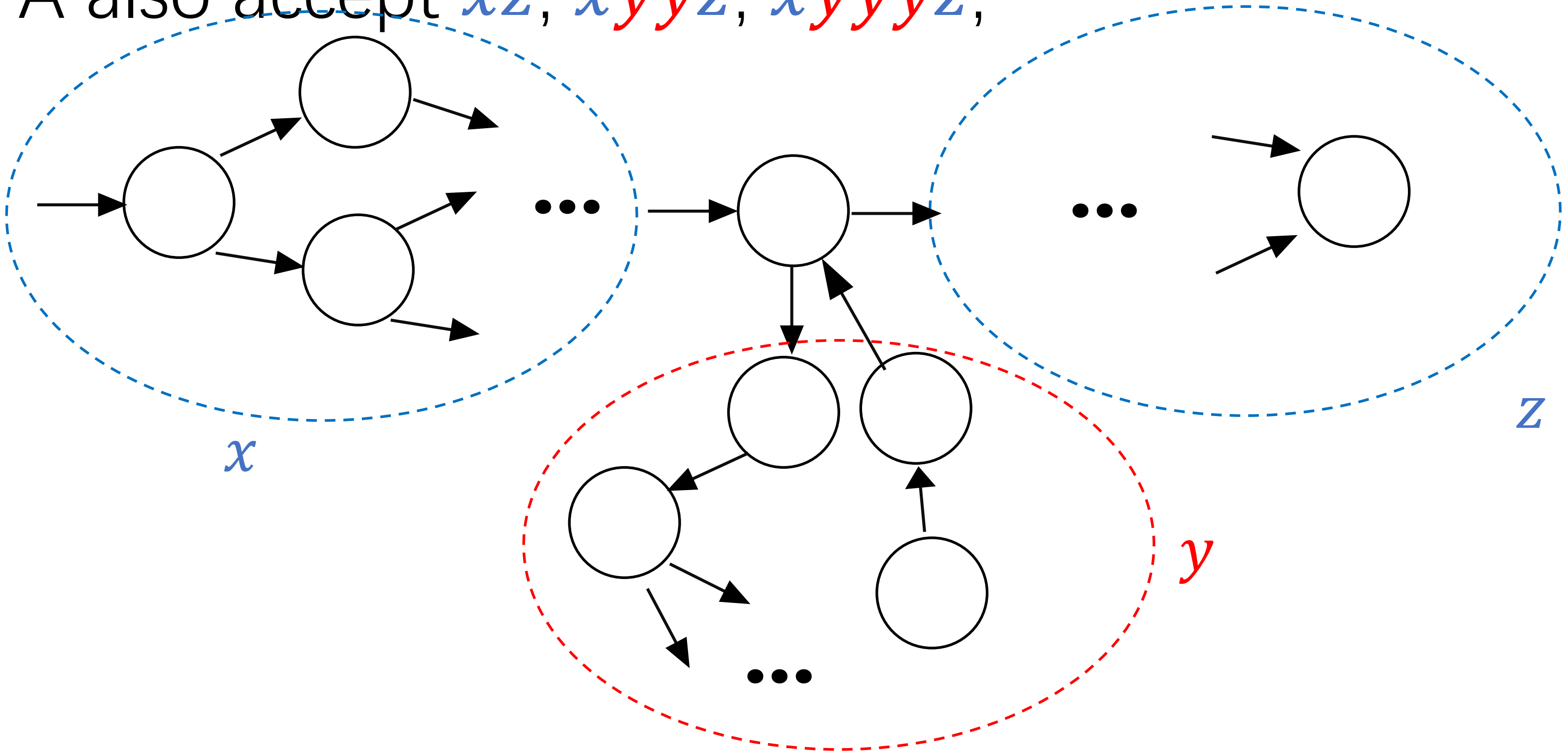

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- The DFA also accept xz, $xyyz$, $xyyyz$,⋯
- What can be the possible y?
  - $y = 000 \cdots 0$
  - $y = 111 \cdots 1$
  - $y = 000 \cdots 0111 \cdots 1$
- Impossible to have such a DFA, $\mathrm{L} = \{0^n 1^n : n > 0\}$ is non-regular

Pumping lemma

- If a language L is regular, then there exists some $n \geq 1$ such that
  - for any string $w \in L$, $|w| \geq n$, we have $w = xyz$ such that
    - $y \neq e$
    - $|xy| < n$
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Repeating started once the length exceeds the total number of states

Pumping lemma

- Suppose you have a magic DFA to accept $\mathrm{L} = \{0^n 1^n : n > 0\}$
- Exists some n, such that if $2m \geq n$, then $0^m 1^m = xyz$ such that
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- Choose $m = n$
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- The DFA should also accept xz, contradiction.

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- Choose $v = a^nba^nb$
  - $x = a^i$, $y = a^k$ for some i, k
  - $xy^2z = a^{n+k}ba^nb \notin L$

The way to prove- apply Pumping lemma

- You have an adversary who thinks L is regular. You need to prove that your adversary is wrong.
  - you: Language L is not regular!
  - adv: Yes it is! I have a DFA to prove it!
  - you: Oh really? How many states are in your DFA?
  - adv: n
  - you: OK, here's a string $w \in L$ with $|w| > n$. Your machine must accept w – but since $|w| > n$, there must be a loop in your computation. Where's the loop?
  - adv: Right here! (breaks w into xyz, where y is the part of the string that goes through the loop)
  - you: Ah hah! If we go through the loop 2 times instead of 1, we get a string not in L that your machine will accept!
  - adv: ⋯

The way to prove- apply Pumping lemma

- Your adversary picks an n
  - because you do not know this n
- You pick a $w \in L$ (such that $|w| > n$)
- Your adversary breaks w into xyz (subject to $|xy| \leq n, |y| > 0$)
  - because you do not know how exactly it is split
- You pick an i such that $xy^iz \notin L$

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- Pick $|w| = n + 2$
  - $x = a^\alpha, y = a^\beta, z = a^\gamma$ such that
    - $\beta \geq 1$, $\alpha + \beta < n$, $\alpha + i\beta + \gamma$ is always prime for any $i \geq 0$
    - choose $i = \alpha + \gamma$, then $\alpha + i\beta + \gamma = (\alpha + \gamma)(\beta + 1)$

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- What happens if we try to apply Pumping lemma to a regular language
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    - $xy^iz \in L$ for any $i \geq 0$
  - Choose some $w = a^{2k}$ for $2k \geq n$
  - $x = a^\alpha, y = a^\beta, z = a^\gamma$ such that
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This can be correct if β is even – we have no control on the values of α, β, γ, so no contradiction

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- Not able to apply pumping lemma to show a language is non-regular does not necessarily mean this language is regular

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  - To show it is regular, create a regular expression or DFA/NFA

- Not able to create a DFA/NFA for a language does not necessarily mean this language is non-regular

  - To show it is non-regular, apply pumping lemma suitably

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    Given a DFA $M = (K, \Sigma, \delta, s, F)$ for L_{REG}, we create $M' = (K, \Sigma, \delta, s, K - F)$
  - Is L_{REG} closed under intersection?

    $A \cap B = \overline{\bar{A} \cup \bar{B}}$

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- Prove that $L = \{w \in \{a, b\}^* : w \text{ has an equal number of } a \text{ and } b\}$
  - a^*b^* is regular
  - $L \cap a^*b^* = \{a^n b^n : n \geq 0\}$
  - we have shown $\{a^n b^n : n \geq 0\}$ is not regular
  - if L is regular, then by closure property $L \cap a^*b^*$ is regular
  - so L is not regular