ISyE 7406: Ball Weight Example in Weekly Knowledge Check #2

(Ball Weights). A GT student was asked to find the weights of two balls, A and B, using a scale with random measurement errors. When the student measure one ball at a time, the weights of A and B were 2 lbs and 3 lbs, respectively. However, if the student measured both balls simultaneously, then the total weights (A+B) were 4 lbs. The poor student was confused, and decided to repeat the above measurements. The new observed weights of A, B, and A+B, are 2, 2.2, and 4.6 lbs, respectively.

(1) Use the method of least squares to find the best estimates of the weights of balls A and B.

(2) In the context of this question of measuring the balls 6 times, What is the estimation of the variance σ^2?

(3) In the context of this question of measuring the balls 6 times, we have a point estimate of (A+B) is 4.4 lbs with the estimated standard error $\hat{\sigma} = \sqrt{0.14} = 0.3742$. Suppose the student plans to measure both balls simultaneously one more time, and know that a 70% prediction interval on the observed weight of $(A+B)$ is of the form $4.4 \pm C \cdot 0.3742$ What is the constant C?

Solution: Before we answer the questions, we need to do some preparation on data.

(i) For your information, the observed weights are summarized in the following table:

	A	B	$A+B$
First Time	2	3	4
Second Time	2	2.2	4.6

(ii) In this question, there are $n = 6$ observations, and $p = 2$ unknown ball weights. Thus we can write the measured weights through the additive noise models as

$$\begin{aligned} 2 &= A + \epsilon_1 \\ 3 &= B + \epsilon_2 \\ 4 &= A + B + \epsilon_3 \\ 2 &= A + \epsilon_4 \\ 2.2 &= B + \epsilon_5 \\ 4.6 &= A + B + \epsilon_6 \end{aligned} \tag{1}$$

(iii) We can write the observed data in the matrix form $Y_{n\times 1} = X_{n\times p}\beta_{p\times 1} + \epsilon_{n\times 1}$ with $n = 6$ and $p = 2$. Based on the relation (1), we can write the observed data in matrix form $Y_{6\times 1} = X_{6\times 2}\beta_{2\times 1} + \epsilon_{6\times 1}$, where

$$Y = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \\ 2.2 \\ 4.6 \end{bmatrix}, \; X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \; \beta = \begin{bmatrix} A \\ B \end{bmatrix}, \text{and } \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}. \tag{2}$$

(iv) we need to have some understandings of the matrix X. From the above definition of the matrix X, we have

$$X^T X = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \tag{3}$$

and thus

$$(X^TX)^{-1} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{12}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \tag{4}$$

Now we are ready to answer the questions.

(1) Note that the coefficient β includes the weight of ball A and B, respectively. Thus by (2), the least square estimates of ball weights are

$$\begin{aligned} \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix} = \hat{\beta} &= (X^TX)^{-1}X^TY \qquad (5)\\ &= \frac{1}{12}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \\ 2.2 \\ 4.6 \end{bmatrix} \\ &= \frac{1}{12}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2+4+2+4.6 \\ 3+4+2.2+4.6 \end{bmatrix} \\ &= \frac{1}{12}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 12.6 \\ 13.8 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 4 * 12.6 - 2 * 13.8 \\ -2 * 12.6 + 4 * 13.8 \end{bmatrix} \\ &= \frac{1}{12}\begin{bmatrix} 22.8 \\ 30 \end{bmatrix} = \begin{bmatrix} 22.8/12 \\ 30/12 \end{bmatrix} \\ &= \begin{bmatrix} 1.9 \\ 2.5 \end{bmatrix}. \end{aligned}$$

Therefore, we estimate the ball weights as $\hat{A} = 1.9$ lbs and $\hat{B} = 2.5$ lbs. In particular, we estimate $\hat{A} + \hat{B} = 1.9 + 2.5 = 4.4$ lbs.

(2) Let $e_1 = 2 - \hat{A}, e_2 = 3 - \hat{B}, \cdots, e_n = 4.6 - (\hat{A} + \hat{B})$ to be the estimates of the measure errors ϵ_i when the true weights are replaced by the least square estimates. Then we can estimate (here $n = 6$ and $p = 2$)

$$\begin{aligned} \hat{\sigma}^2 &= \frac{e_1^2 + \cdots + e_n^2}{n-p} \\ &= \frac{(2-1.9)^2 + (3-2.5)^2 + (4-(1.9+2.5))^2 + (2-1.9)^2 + (2.2-2.5)^2 + (4.6-(1.9+2.5))^2}{6-2} \\ &= \frac{0.1^2 + 0.5^2 + (-0.4)^2 + (0.1)^2 + (-0.3)^2 + (0.2)^2}{6-2} \\ &= \frac{0.56}{4} = 0.14. \qquad (6) \end{aligned}$$

Thus we estimate σ^2 as 0.14 and estimate σ as $\sqrt{0.14} = 0.3742$.

(3) Suppose the student plans to measure both balls simultaneously one more time and we are interested in finding a 70% prediction interval on the observed weight of $(A + B)$.

First, we need to translate this to the problem of finding a $(1-\alpha)$ prediction interval at x_{new}, where the general formula for such prediction interval is

$$x_{new}^T\beta \pm \hat{\sigma}t_{\alpha/2,n-p}\sqrt{1 + x_{new}^T(X^TX)^{-1}x_{new}}. \tag{7}$$

Clearly in our context, $n = 6, p = 2$, and $\alpha = 0.3$. The key is to find the corresponding x_{new} value. Note that

$$A + B = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T\beta = x_{new}^T\beta, \tag{8}$$

Thus, in this part,

$$x_{new} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{9}$$

Second, combining (4) with this x_{new} value in (9), we have

$$\begin{aligned} x_{new}^T (X^T X)^{-1} x_{new} &= \begin{bmatrix} 1 & 1 \end{bmatrix} \left(\frac{1}{12} \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{12} \begin{bmatrix} 4-2 \\ -2+4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{12} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \frac{1}{12}(1*2+1*2) = \frac{4}{12} = \frac{1}{3}. \end{aligned} \tag{10}$$

Third, plugging this into the general formula in (8) for the prediction level, we have a 70% prediction interval on the observed weight of $(A+B)$ is

$$\begin{aligned} &\hat{A} + \hat{B} \pm \hat{\sigma} t_{\alpha/2, n-p} \sqrt{1 + \frac{1}{3}} \\ &1.9 + 2.5 \pm \sqrt{0.14}(t_{0.15, 6-2})\sqrt{4/3} \\ &4.4 \pm (0.3742)(t_{0.15,4}\sqrt{4/3}). \end{aligned}$$

This is of the form $4.4 \pm C \cdot 0.3742$ with the constant

$$C = t_{0.15,4}\sqrt{4/3}. \tag{11}$$

In R, if you type "-qt(0.15, 4)" or "qt(1-0.15, 4)", it returns "1.1896". This implies that

$$t_{0.15,4} = 1.1896.$$

From here it is clear that $C = 1.1896\sqrt{4/3} = 1.3736$, and you should be able to find the numerical value of the 70% prediction interval on $(A+B)$:

$$4.4 \pm (0.3742) * (1.3736) = 4.4 \pm 0.514 = [3.886, 4.914].$$