Learning FOL

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1 Multi-sorted first-order logic

1.1 Syntax

We consider a countably infinite set of sorts $S, T, \ldots$ and for each sort a countably infinite set of variables $x, y, \ldots$. Furthermore for each $n+1$-tuple of sorts $S_0, \ldots, S_n$ we define a countably infinite set of function symbols $f, g, \ldots$, we say that those functions are of sort $S_0 \times \ldots \times S_{n-1} \to S_n$. A *term* of sort S is defined inductively as follows:

- A variable x of sort S is a term of sort S.
- Given a function f of sort $S_0 \times \ldots \times S_{n-1} \to S_n$ and n terms $t_0, \ldots, t_{n-1}$ of sorts $S_0, \ldots, S_{n-1}$, $f(t_0, \ldots, t_{n-1})$ is a term of sort S_n.

Given two terms t, u of the same sort S, $t = u$ is an equation. Given equations $t_0 = u_0, \ldots, t_m = u_m$ and $v_0 = w_0, \ldots, v_n = w_n$

$$t_0 = u_0, \ldots, t_m = u_m \vdash v_0 = w_0, \ldots, v_n = w_n$$

is a sequent. A set of sequents is a *theory*.

1.2 Semantics

An *interpretation* is an assignment I that maps each sort to a set, each variable x of sort S to an element $I(x) \in I(S)$ and each function f of sort $S_0 \times \ldots \times S_{n-1} \to S_n$ to a function $I(f) : I(S_0) \times \ldots \times I(S_{n-1}) \to I(S_n)$. As usual t is canonically extended to terms. We write t^I for $I(t)$.

We say that an interpretation I *satisfies* an equation $t = u$ if $t^I = u^I$. We say that an interpretation I *satisfies* a sequent $t_0 = u_0, \ldots, t_m = u_m \vdash v_0 = w_0, \ldots, v_n = w_n$ if it does not satisfy one of $t_0^I = u_0^I, \ldots, t_m^I = u_m^I$ or it satisfies all of $v_0^I = w_0^I, \ldots, v_n^I = w_n^I$. We say that an interpretation I

satisfies a theory $\mathcal{T}$ if it satisfies all sequents in $\mathcal{T}$. A sequent is *valid* in a theory $\mathcal{T}$ if every interpretation that satisfies $\mathcal{T}$ satisfies the sequent. A theory is *consistent* if it has a model.