Proofs as Programs in Classical Logic
Notes

Alexander Pluska

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1 Plan

Goal:

- Extract program from resolution proofs of $\forall\exists$-sentences over inductive datatypes.

Steps:

- Give extensions of Gödel's System **T** and HAS + EM_1 + SK_1 to arbitrary inductive datatypes (possibly GADTs) and prove (or rather check) properties, i.e. strong normalization, cut-elimination, uniqueness of normal forms.
- Adapt the results of [1] to exhibit realizers in $\mathcal{F} + \Phi$ for cut-free proofs in the extended version of HAS + EM_1 + SK_1 of formulas $\forall x \exists y Pxy$ where P is a predicate in the extended version of **T**.
- Adapt the iterative learning from [1] to extract λ-terms from realizers.
- Give a proof translation from resolution proofs to cut-free sequent calculus proofs (already done?).

Questions:

- When does the translated proof require more than EM_1?
- Using the methods in [1] the extracted term will be in simple λ-calculus. Is is possible to obtain a term in $\mathcal{F}$? (Talk with Federico Aschieri)
- Can the predicates be defined in $\mathcal{F}$ instead of $\mathcal{T}$? (probably yes)
- What happens if we add non-inductive theories?

2 Extracting constructive content from resolution proofs

2.1 The superposition calculus

First let us define the calculus from which we wish to extract programs. It will comprise a core set of rules used in the vampire theorem prover taken from [2]. Note that we pay special attention to the usually neglected (and highly non-constructive) part of CNF transformation and Skolemization. Also note that we neglect the simplification ordering which is necessary to formulate a strategy for proof search but not for our proof transformation.

Resolution.

$$\frac{A \vee B \qquad \neg A' \vee C}{(B \vee C)\theta}$$

where θ is an mgu of A and A'.

Factoring.

$$\frac{A \vee A' \vee B}{(A \vee B)\theta}$$

where θ is an mgu of A and A'.

Superposition.

$$\frac{l = r \vee B \qquad L[s] \vee C}{(L[r] \vee B \vee C)\theta} \quad \frac{l = r \vee B \qquad t[s] = t' \vee C}{(t[r] = t' \vee B \vee C)\theta} \quad \frac{l = r \vee B \qquad t[s] \neq t' \vee C}{(t[r] \neq t' \vee B \vee C)\theta}$$

where θ is an mgu of l and s, s is not a variable, $L[s]$ is not an equality literal.

Equality Resolution.

$$\frac{s \neq t \vee C}{C\theta}$$

where θ is an mgu of s and t.

Equality Factoring.

$$\frac{s = t \vee s' = t' \vee C}{(s = t \vee t \neq t' \vee C)\theta}$$

where θ is an mgu of s and s'.

First let us look at an example of the transformation we going to do. We shall look at a proof of $\forall x \exists y : f(y) = g(x)$ from $\forall x : f(x) = g(h(x))$ and $\forall x : h(h'(x)) = x$. The program we extract hopefully is h'.

[axiom]	$\forall x : f(x) = g(h(x))$	(1)
[cnf 1]	$f(x_0) = g(h(x_0))$	(2)
[axiom]	$\forall x : h(h'(x)) = x$	(3)
[cnf 3]	$h(h'(x_0)) = x_0$	(4)
[negated conjecture]	$\neg\forall x \exists y : f(y) = g(x)$	(5)
[ennf 5]	$\exists x \forall y : f(y) \neq g(x)$	(6)
[choice axiom]	$\exists x \forall y : f(y) \neq g(x) \to \forall x : f(x) \neq g(s_0)$	(7)
[skolemization 6, 7]	$\forall x : f(x) \neq g(s_0)$	(8)
[cnf 8]	$f(x_1) \neq g(s_0)$	(9)
[superposition 9, 2]$\{\}$	$g(s_0) \neq g(h(x_0))$	(10)
[superposition 10, 4]$\{x_0 \mapsto h'(x_0)\}$	$g(x_0) \neq g(s_0)$	(11)
[equality resolution 11]	$\perp$	(12)

We first use the well known trick of adding the original conjecture to the negated one from [3] to transform this into a classical proof of $\forall x \exists y : f(x) = g(y)$ rather than a refutation of its negation:

$$
\begin{array}{rlr}
\text{[axiom]} & \forall x : f(x) = g(h(x)) & (1) \\
\text{[cnf 1]} & f(x_0) = g(h(x_0)) & (2) \\
\text{[axiom]} & \forall x : h(h'(x)) = x & (3) \\
\text{[cnf 3]} & h(h'(x_0)) = x_0 & (4) \\
\text{[tautology]} & \forall x \exists y : f(y) = g(x) \vee \neg \forall x \exists y : f(y) = g(x) & (5) \\
\text{[ennf 5]} & \forall x \exists y : f(y) = g(x) \vee \exists x \forall y : f(y) \neq g(x) & (6) \\
\text{[choice axiom]} & \exists x \forall y : f(y) \neq g(x) \rightarrow \forall x : f(x) \neq g(s_0) & (7) \\
\text{[choice axiom]} & \forall x (\exists y : f(y) = g(x) \rightarrow f(s_1(x)) = g(x)) & (8) \\
\text{[skolemization 6, 7]} & \forall x \exists y : f(y) = g(x) \vee \forall x : f(x) \neq g(s_0) & (9) \\
\text{[skolemization 9, 8]} & \forall x : f(s_1(x)) = g(x) \vee \forall x : f(x) \neq g(s_0) & (10) \\
\text{[cnf 8]} & f(s_1(x_1)) = g(x_1) \vee f(x_0) \neq g(s_0) & (11) \\
\text{[superposition 11, 2]}\{\} & f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(h(x_0)) & (12) \\
\text{[superposition 12, 4]}\{x_0 \mapsto h'(x_0)\} & f(s_1(x_1)) = g(x_1) \vee g(x_0) \neq g(s_0) & (13) \\
\text{[equality resolution 13]}\{x_0 \rightarrow s_0\} & f(s_1(x_1)) = g(x_1) & (14)
\end{array}
$$

Next we eliminate free variables by propagating substitutions:

$$
\begin{array}{rlr}
\text{[axiom]} & \forall x : f(x) = g(h(x)) & (1) \\
\text{[cnf 1]} & f(h'(x_0)) = g(h(h'(x_0))) & (2) \\
\text{[axiom]} & \forall x : h(h'(x)) = x & (3) \\
\text{[cnf 3]} & h(h'(x_0)) = x_0 & (4) \\
\text{[tautology]} & \forall x \exists y : f(y) = g(x) \vee \neg \forall x \exists y : f(y) = g(x) & (5) \\
\text{[ennf 5]} & \forall x \exists y : f(y) = g(x) \vee \exists x \forall y : f(y) \neq g(x) & (6) \\
\text{[choice axiom]} & \exists x \forall y : f(y) \neq g(x) \rightarrow \forall x : f(x) \neq g(s_0) & (7) \\
\text{[choice axiom]} & \forall x (\exists y : f(y) = g(x) \rightarrow f(s_1(x)) = g(x)) & (8) \\
\text{[skolemization 6, 7]} & \forall x \exists y : f(y) = g(x) \vee \forall x : f(x) \neq g(s_0) & (9) \\
\text{[skolemization 9, 8]} & \forall x : f(s_1(x)) = g(x) \vee \forall x : f(x) \neq g(s_0) & (10) \\
\text{[cnf 8]} & f(s_1(x_1)) = g(x_1) \vee f(h'(x_0)) \neq g(s_0) & (11) \\
\text{[superposition 11, 2]}\{\} & f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(h(h'(x_0))) & (12) \\
\text{[superposition 12, 4]}\{\} & f(s_1(x_1)) = g(x_1) \vee g(x_0) \neq g(s_0) & (13) \\
\text{[equality resolution 13]}\{x_0 \rightarrow s_0\} & f(s_1(x_1)) = g(x_1) & (14)
\end{array}
$$

$$\begin{aligned}
&[\text{axiom}] && \forall x : f(x) = g(h(x)) && (1)\\
&[\text{cnf } 1] && f(h'(s_0)) = g(h(h'(s_0))) && (2)\\
&[\text{axiom}] && \forall x : h(h'(x)) = x && (3)\\
&[\text{cnf } 3] && h(h'(s_0)) = s_0 && (4)\\
&[\text{tautology}] && \forall x \exists y : f(y) = g(x) \vee \neg \forall x \exists y : f(y) = g(x) && (5)\\
&[\text{ennf } 5] && \forall x \exists y : f(y) = g(x) \vee \exists x \forall y : f(y) \neq g(x) && (6)\\
&[\text{choice axiom}] && \exists x \forall y : f(y) \neq g(x) \rightarrow \forall x : f(x) \neq g(s_0) && (7)\\
&[\text{choice axiom}] && \forall x (\exists y : f(y) = g(x) \rightarrow f(s_1(x)) = g(x)) && (8)\\
&[\text{skolemization } 6,\ 7] && \forall x \exists y : f(y) = g(x) \vee \forall x : f(x) \neq g(s_0) && (9)\\
&[\text{skolemization } 9,\ 8] && \forall x : f(s_1(x)) = g(x) \vee \forall x : f(x) \neq g(s_0) && (10)\\
&[\text{cnf } 8] && f(s_1(x_1)) = g(x_1) \vee f(h'(s_0)) \neq g(s_0) && (11)\\
&[\text{superposition } 11,\ 2]\{\} && f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(h(h'(s_0))) && (12)\\
&[\text{superposition } 12,\ 4]\{\} && f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(s_0) && (13)\\
&[\text{equality resolution } 13]\{\} && f(s_1(x_1)) = g(x_1) && (14)
\end{aligned}$$

Next we remove all the skolem constants in the conjecture-tautology by unification, propagate this change, and finally reinterpet superposition as composition to yield a valid intuitionistic proof:

$$\begin{aligned}
&[\text{axiom}] && \forall x : f(x) = g(h(x)) && (1)\\
&[\text{instantiation } 1] && f(h'(x_1)) = g(h(h'(x_1))) && (2)\\
&[\text{axiom}] && \forall x : h(h'(x)) = x && (3)\\
&[\text{instantiation } 3] && h(h'(x_1)) = x_1 && (4)\\
&[\text{tautology}] && f(h'(x_1)) = g(x_1) \Rightarrow f(h'(x_1)) = g(x_1) && (5)\\
&[\text{equality } 5,\ 2]\{\} && g(x_1) = g(h(h'(x_1))) \Rightarrow f(h'(x_1)) = g(x_1) && (6)\\
&[\text{equality } 6,\ 4]\{\} && g(x_1) = g(x_1) \Rightarrow f(h'(x_1)) = g(x_1) && (7)\\
&[\text{equality } 13]\{\} && f(h'(x_1)) = g(x_1) && (8)
\end{aligned}$$

Let us look at a second example: Consider the sentences $\forall x : p(x_0, f(x_0))$, $\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))$, $\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$. We take a look at a proof of $\forall x_0 \exists x_1 : r(x_0, x_1)$ which hopefully gives us $h \circ g \circ f$. First consider the output of vampire [axiom] $\forall x_0 : p(x_0, f(x_0))$ (1)

[axiom] $\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$ (2)

[axiom] $\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$ (3)

[negated conjecture] $\neg\forall x_0 : \exists x_1 : r(x_0, x_1)$ (4)

[ennf transformation 2] $\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$ (5)

[ennf transformation 3] $\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$ (6)

[ennf transformation 4] $\exists x_0 \forall x_1 : \neg r(x_0, x_1)$ (7)

[choice axiom] $\exists x_0 \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$ (8)

[skolemisation 7,8] $\forall x_1 : \neg r(s_0, x_1)$ (9)

[cnf transformation 1] $p(x_0, f(x_0))$ (10)

[cnf transformation 5] $q(x_1, g(x_2)) \vee \neg p(x_1, x_2)$ (11)

[cnf transformation 6] $r(x_3, h(x_4)) \vee \neg q(x_3, x_4)$ (12)

[cnf transformation 9] $.\neg r(s_0, x_5)$ (13)

[resolution 12, 13]$\{x_3 \mapsto s_0, x_5 \mapsto h(x_4)\}$ $\neg q(s_0, x_4)$ (14)

[resolution 14, 11]$\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\}$ $\neg p(s_0, x_2)$ (15)

[resolution 15, 10]$\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$ $\perp$ (16)

Again replace negated conjecture by tautology

[axiom] $\forall x_0 : p(x_0, f(x_0))$ (1)

[axiom] $\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$ (2)

[axiom] $\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$ (3)

[tautology] $\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg\forall x_0 \exists x_1 : r(x_0, x_1)$ (4)

[ennf transformation 2] $\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$ (5)

[ennf transformation 3] $\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$ (6)

[ennf transformation 4] $\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$ (7)

[choice axiom] $\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$ (8)

[choice axiom] $\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$ (9)

[skolemisation 7,8] $\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$ (10)

[skolemisation 9,10] $\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$ (11)

[cnf transformation 1] $p(x_0, f(x_0))$ (12)

[cnf transformation 5] $q(x_1, g(x_2)) \vee \neg p(x_1, x_2)$ (13)

[cnf transformation 6] $r(x_3, h(x_4)) \vee \neg q(x_3, x_4)$ (14)

[cnf transformation 11] $r(x_6, s_1(x_6)) \vee \neg r(s_0, x_5)$ (15)

[resolution 14, 15]$\{x_3 \mapsto s_0, x_5 \mapsto h(x_4)\}$ $r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4)$ (16)

[resolution 16, 13]$\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\}$ $r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$ (17)

[resolution 17, 12]$\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$ $r(x_6, s_1(x_6))$ (18)

We propagate the substitutions

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(x_0, f(x_0))$	(12)
[cnf transformation 5]	$q(x_1, g(x_2)) \vee \neg p(x_1, x_2)$	(13)
[cnf transformation 6]	$r(s_0, h(x_4)) \vee \neg q(s_0, x_4)$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, h(x_4))$	(15)
[resolution 14, 15]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4)$	(16)
[resolution 16, 13]$\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\}$	$r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$	(17)
[resolution 17, 12]$\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$r(x_6, s_1(x_6))$	(18)

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(x_0, f(x_0))$	(12)
[cnf transformation 5]	$q(s_0, g(x_2)) \vee \neg p(s_0, x_2)$	(13)
[cnf transformation 6]	$r(s_0, h(g(x_2))) \vee \neg q(s_0, g(x_2))$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(x_2)))$	(15)
[resolution 14, 15]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg q(s_0, g(x_2))$	(16)
[resolution 16, 13]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$	(17)
[resolution 17, 12]$\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$r(x_6, s_1(x_6))$	(18)

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(s_0, f(s_0))$	(12)
[cnf transformation 5]	$q(s_0, g(f(s_0))) \vee \neg p(s_0, f(s_0))$	(13)
[cnf transformation 6]	$r(s_0, h(g(f(s_0)))) \vee \neg q(s_0, g(f(s_0)))$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(f(s_0))))$	(15)
[resolution 14, 15]{}	$r(x_6, s_1(x_6)) \vee \neg q(s_0, g(f(s_0)))$	(16)
[resolution 16, 13]{}	$r(x_6, s_1(x_6)) \vee \neg p(s_0, f(s_0))$	(17)
[resolution 17, 12]{}	$r(x_6, s_1(x_6))$	(18)

Then unification at 15 gives $s_1 = h \circ g \circ f$ and $s_0 = x_6$ and the final transformation gives:

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[instantiation 1]	$p(x_6, f(x_6))$	(4)
[instantiation 2]	$p(x_6, f(x_6)) \Rightarrow q(x_6, g(f(x_6)))$	(5)
[instantiation 3]	$q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6))))$	(6)
[tautology]	$r(x_6, h(g(f(x_6)))) \Rightarrow r(x_6, h(g(f(x_6))))$	(7)
[composition 6, 7]{}	$q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6))))$	(8)
[composition 5, 8]{}	$p(x_6, f(x_6)) \Rightarrow r(x_6, h(g(f(x_6))))$	(9)
[composition 4, 9]{}	$r(x_6, h(g(f(x_6))))$	(10)

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