Proofs as Programs in Classical Logic
Notes

Alexander Pluska

February 13, 2022

1 Plan

Goal:

- Extract program from resolution proofs of $\forall\exists$-sentences over inductive datatypes.

Steps:

- Give extensions of Gödel's System $\mathbf{T}$ and HAS + EM_1 + SK_1 to arbitrary inductive datatypes (possibly GADTs) and prove (or rather check) properties, i.e. strong normalization, cut-elimination, uniqueness of normal forms.
- Adapt the results of [1] to exhibit realizers in $\mathcal{F} + \Phi$ for cut-free proofs in the extended version of HAS + EM_1 + SK_1 of formulas $\forall x \exists y Pxy$ where P is a predicate in the extended version of $\mathbf{T}$.
- Adapt the iterative learning from [1] to extract λ-terms from realizers.
- Give a proof translation from resolution proofs to cut-free sequent calculus proofs (already done?).

Questions:

- When does the translated proof require more than EM_1?
- Using the methods in [1] the extracted term will be in simple λ-calculus. Is is possible to obtain a term in $\mathcal{F}$? (Talk with Federico Aschieri)
- Can the predicates be defined in $\mathcal{F}$ instead of $\mathcal{T}$? (probably yes)
- What happens if we add non-inductive theories?

To explain our approach let us first recall how resolution works and why it is effective for classical logic. In a nutshell the principle of resolution works as follows: From a set of formulas $A, B, C \ldots$ obtain a set of sequents $A'_1 \ldots A'_n \vdash A_1 \ldots A_n$, $B'_1 \ldots B'_n \vdash B_1 \ldots B_n$, $C'_1 \ldots C'_n \vdash C_1 \ldots C_n$ in which all occurring formulas are somehow simple. Then iteratively apply the resolution rule (and some other rules, but this is the crucial one), which is an analogon to cut, i.e.

$$\frac{A, \Gamma \vdash \Delta \qquad \Gamma' \vdash A', \Delta'}{(\Gamma \cup \Gamma')\theta \vdash (\Delta \cup \Delta')\theta}$$

where θ is the mgu of A and A', and by process of saturation (attempt to) obtain the empty Sequent.

Now for classical logic we can simply interpret $A'_1, \ldots A'_n \vdash A_1, \ldots, A_n$ as $\neg A'_1 \vee \cdots \vee \neg A'_n \vee A_1 \vee \cdots \vee A_n$. Furthermore we can normalize every formula such that each of the A'_i, A_i is atomic. The resolution calculus is then immediately refutationally complete, i.e. if $A, B, C \ldots$ are inconsistent there exists a successful resolution, giving a refutation of $A \wedge B \wedge C \ldots$, which is classically equivalent to a proof of $B \wedge C \cdots \to \neg A$. I.e. if we want to prove $B \wedge C \cdots \to A$ we can simply preform resolution on $\neg A \wedge B \wedge C \ldots$

Now there are a number of hurdles in applying this strategy to intuitionistic logic, or to extract intuitionistic proof from classical resolution proofs. First and foremost is the non-existence of normal forms. In particular with the (intuitionally speaking) strict interpretation of $A'_1, \ldots A'_n \vdash A_1, \ldots, A_n$ as $\neg A'_1 \vee \cdots \vee \neg A'_n \vee A_1 \vee \cdots \vee A_n$ we can hardly translate any formulas into such a form where the A_i are atomic, or even literals. This can be remedied of course by interpreting $A'_1, \ldots A'_n \vdash A_1, \ldots, A_n$ as $\neg(A'_1 \wedge \cdots \wedge A'_n \wedge (\neg A_1 \vee \cdots \vee \neg A_n))$, i.e. a double negation translation. But this brings us to the second problem: Even if our calculus is refutationally complete a refutation of $\neg A \wedge B \wedge C \ldots$ does not give us a proof of $B \wedge C \cdots \to A$ but rather $B \wedge C \cdots \to \neg\neg A$. This problem is especially pronounced in the second interpretation of $\vdash$ where we are not even using A but $\neg\neg A$.

Now there are a few saving graces: First of all, if A is decidable, i.e. $\neg\neg A \to A$, we immediately obtain a proof of A. Furthermore we may attempt Friedman Translation [3] of our proof and indeed [2] gives us some quite liberal criteria when this is possible in principle, e.g. when $B, C \ldots$ are theorems of PA and A is a Π^2_0 formula. And indeed interpreting $A'_1, \ldots A'_n \vdash A_1, \ldots, A_n$ as $A'_1 \to \cdots \to A'_n \to A_1 \vee \cdots \vee A_n$, we are able to obtain an intuitionistic proof if their conditions are met and the input formulas can be transformed to such normal forms.

2 Extracting constructive content from resolution proofs

2.1 The superposition calculus

First let us define the calculus from which we wish to extract programs. It will comprise a core set of rules used in the vampire theorem prover taken from [4]. Note that we pay special attention to the usually neglected (and highly non-constructive) part of CNF transformation and Skolemization. Also note that we neglect the simplification ordering which is necessary to formulate a strategy for proof search but not for our proof transformation.

Resolution.

$$\frac{A \vee B \qquad \neg A' \vee C}{(B \vee C)\theta}$$

where θ is an mgu of A and A'.

Factoring.

$$\frac{A \vee A' \vee B}{(A \vee B)\theta}$$

where θ is an mgu of A and A'.

Superposition.

$$\frac{l = r \vee B \qquad L[s] \vee C}{(L[r] \vee B \vee C)\theta} \quad \frac{l = r \vee B \qquad t[s] = t' \vee C}{(t[r] = t' \vee B \vee C)\theta} \quad \frac{l = r \vee B \qquad t[s] \neq t' \vee C}{(t[r] \neq t' \vee B \vee C)\theta}$$

where θ is an mgu of l and s, s is not a variable, $L[s]$ is not an equality literal.

Equality Resolution.

$$\frac{s \neq t \vee C}{C\theta}$$

where θ is an mgu of s and t.

Equality Factoring.

$$\frac{s = t \vee s' = t' \vee C}{(s = t \vee t \neq t' \vee C)\theta}$$

where θ is an mgu of s and s'.

First let us look at an example of the transformation we going to do. We shall look at a proof of $\forall x \exists y : f(y) = g(x)$ from $\forall x : f(x) = g(h(x))$ and $\forall x : h(h'(x)) = x$. The program we extract hopefully is h'.

[axiom]	$\forall x : f(x) = g(h(x))$	(1)
[cnf 1]	$f(x_0) = g(h(x_0))$	(2)
[axiom]	$\forall x : h(h'(x)) = x$	(3)
[cnf 3]	$h(h'(x_0)) = x_0$	(4)
[negated conjecture]	$\neg \forall x \exists y : f(y) = g(x)$	(5)
[ennf 5]	$\exists x \forall y : f(y) \neq g(x)$	(6)
[choice axiom]	$\exists x \forall y : f(y) \neq g(x) \to \forall x : f(x) \neq g(s_0)$	(7)
[skolemization 6, 7]	$\forall x : f(x) \neq g(s_0)$	(8)
[cnf 8]	$f(x_1) \neq g(s_0)$	(9)
[superposition 9, 2]$\{\}$	$g(s_0) \neq g(h(x_0))$	(10)
[superposition 10, 4]$\{x_0 \mapsto h'(x_0)\}$	$g(x_0) \neq g(s_0)$	(11)
[equality resolution 11]	$\perp$	(12)

We first use the well known trick of adding the original conjecture to the negated one from [5] to transform this into a classical proof of $\forall x \exists y : f(x) = g(y)$ rather than a refutation of its negation:

[axiom]	$\forall x : f(x) = g(h(x))$	(1)
[cnf 1]	$f(x_0) = g(h(x_0))$	(2)
[axiom]	$\forall x : h(h'(x)) = x$	(3)
[cnf 3]	$h(h'(x_0)) = x_0$	(4)
[tautology]	$\forall x \exists y : f(y) = g(x) \vee \neg \forall x \exists y : f(y) = g(x)$	(5)
[ennf 5]	$\forall x \exists y : f(y) = g(x) \vee \exists x \forall y : f(y) \neq g(x)$	(6)
[choice axiom]	$\exists x \forall y : f(y) \neq g(x) \to \forall x : f(x) \neq g(s_0)$	(7)
[choice axiom]	$\forall x (\exists y : f(y) = g(x) \to f(s_1(x)) = g(x))$	(8)
[skolemization 6, 7]	$\forall x \exists y : f(y) = g(x) \vee \forall x : f(x) \neq g(s_0)$	(9)
[skolemization 9, 8]	$\forall x : f(s_1(x)) = g(x) \vee \forall x : f(x) \neq g(s_0)$	(10)
[cnf 8]	$f(s_1(x_1)) = g(x_1) \vee f(x_0) \neq g(s_0)$	(11)
[superposition 11, 2]$\{\}$	$f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(h(x_0))$	(12)
[superposition 12, 4]$\{x_0 \mapsto h'(x_0)\}$	$f(s_1(x_1)) = g(x_1) \vee g(x_0) \neq g(s_0)$	(13)
[equality resolution 13]$\{x_0 \to s_0\}$	$f(s_1(x_1)) = g(x_1)$	(14)

Next we eliminate free variables by propagating substitutions:

$$\begin{array}{rlr}
\text{[axiom]} & \forall x : f(x) = g(h(x)) & (1) \\
\text{[cnf 1]} & f(h'(x_0)) = g(h(h'(x_0))) & (2) \\
\text{[axiom]} & \forall x : h(h'(x)) = x & (3) \\
\text{[cnf 3]} & h(h'(x_0)) = x_0 & (4) \\
\text{[tautology]} & \forall x \exists y : f(y) = g(x) \vee \neg \forall x \exists y : f(y) = g(x) & (5) \\
\text{[ennf 5]} & \forall x \exists y : f(y) = g(x) \vee \exists x \forall y : f(y) \neq g(x) & (6) \\
\text{[choice axiom]} & \exists x \forall y : f(y) \neq g(x) \to \forall x : f(x) \neq g(s_0) & (7) \\
\text{[choice axiom]} & \forall x (\exists y : f(y) = g(x) \to f(s_1(x)) = g(x)) & (8) \\
\text{[skolemization 6, 7]} & \forall x \exists y : f(y) = g(x) \vee \forall x : f(x) \neq g(s_0) & (9) \\
\text{[skolemization 9, 8]} & \forall x : f(s_1(x)) = g(x) \vee \forall x : f(x) \neq g(s_0) & (10) \\
\text{[cnf 8]} & f(s_1(x_1)) = g(x_1) \vee f(h'(x_0)) \neq g(s_0) & (11) \\
\text{[superposition 11, 2]\{\}} & f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(h(h'(x_0))) & (12) \\
\text{[superposition 12, 4]\{\}} & f(s_1(x_1)) = g(x_1) \vee g(x_0) \neq g(s_0) & (13) \\
\text{[equality resolution 13]}\{x_0 \to s_0\} & f(s_1(x_1)) = g(x_1) & (14)
\end{array}$$

$$\begin{array}{rlr}
\text{[axiom]} & \forall x : f(x) = g(h(x)) & (1) \\
\text{[cnf 1]} & f(h'(s_0)) = g(h(h'(s_0))) & (2) \\
\text{[axiom]} & \forall x : h(h'(x)) = x & (3) \\
\text{[cnf 3]} & h(h'(s_0)) = s_0 & (4) \\
\text{[tautology]} & \forall x \exists y : f(y) = g(x) \vee \neg \forall x \exists y : f(y) = g(x) & (5) \\
\text{[ennf 5]} & \forall x \exists y : f(y) = g(x) \vee \exists x \forall y : f(y) \neq g(x) & (6) \\
\text{[choice axiom]} & \exists x \forall y : f(y) \neq g(x) \to \forall x : f(x) \neq g(s_0) & (7) \\
\text{[choice axiom]} & \forall x (\exists y : f(y) = g(x) \to f(s_1(x)) = g(x)) & (8) \\
\text{[skolemization 6, 7]} & \forall x \exists y : f(y) = g(x) \vee \forall x : f(x) \neq g(s_0) & (9) \\
\text{[skolemization 9, 8]} & \forall x : f(s_1(x)) = g(x) \vee \forall x : f(x) \neq g(s_0) & (10) \\
\text{[cnf 8]} & f(s_1(x_1)) = g(x_1) \vee f(h'(s_0)) \neq g(s_0) & (11) \\
\text{[superposition 11, 2]\{\}} & f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(h(h'(s_0))) & (12) \\
\text{[superposition 12, 4]\{\}} & f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(s_0) & (13) \\
\text{[equality resolution 13]\{\}} & f(s_1(x_1)) = g(x_1) & (14)
\end{array}$$

Next we remove all the skolem constants in the conjecture-tautology by unification, propagate this change, and finally reinterpet superposition as composition to yield a valid intuitionistic proof:

$$\begin{array}{rlr}
\text{[axiom]} & \forall x : f(x) = g(h(x)) & (1) \\
\text{[instantiation 1]} & f(h'(x_1)) = g(h(h'(x_1))) & (2) \\
\text{[axiom]} & \forall x : h(h'(x)) = x & (3) \\
\text{[instantiation 3]} & h(h'(x_1)) = x_1 & (4) \\
\text{[tautology]} & f(h'(x_1)) = g(x_1) \Rightarrow f(h'(x_1)) = g(x_1) & (5) \\
\text{[equality 5, 2]\{\}} & g(x_1) = g(h(h'(x_1))) \Rightarrow f(h'(x_1)) = g(x_1) & (6) \\
\text{[equality 6, 4]\{\}} & g(x_1) = g(x_1) \Rightarrow f(h'(x_1)) = g(x_1) & (7) \\
\text{[equality 13]\{\}} & f(h'(x_1)) = g(x_1) & (8)
\end{array}$$

Let us look at a second example: Consider the sentences $\forall x : p(x_0, f(x_0))$, $\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))$, $\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$. We take a look at a proof of $\forall x_0 \exists x_1 : r(x_0, x_1)$ which hopefully gives us $h \circ g \circ f$. First consider the output of vampire

$$
\begin{aligned}
&[\text{axiom}] & \forall x_0 : p(x_0, f(x_0)) && (1)\\
&[\text{axiom}] & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) && (2)\\
&[\text{axiom}] & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) && (3)\\
&[\text{negated conjecture}] & \neg\forall x_0 : \exists x_1 : r(x_0, x_1) && (4)\\
&[\text{ennf transformation 2}] & \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) && (5)\\
&[\text{ennf transformation 3}] & \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) && (6)\\
&[\text{ennf transformation 4}] & \exists x_0 \forall x_1 : \neg r(x_0, x_1) && (7)\\
&[\text{choice axiom}] & \exists x_0 \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) && (8)\\
&[\text{skolemisation 7,8}] & \forall x_1 : \neg r(s_0, x_1) && (9)\\
&[\text{cnf transformation 1}] & p(x_0, f(x_0)) && (10)\\
&[\text{cnf transformation 5}] & q(x_1, g(x_2)) \vee \neg p(x_1, x_2) && (11)\\
&[\text{cnf transformation 6}] & r(x_3, h(x_4)) \vee \neg q(x_3, x_4) && (12)\\
&[\text{cnf transformation 9}] & .\neg r(s_0, x_5) && (13)\\
&[\text{resolution 12, 13}]\{x_3 \mapsto s_0, x_5 \mapsto h(x_4)\} & \neg q(s_0, x_4) && (14)\\
&[\text{resolution 14, 11}]\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\} & \neg p(s_0, x_2) && (15)\\
&[\text{resolution 15, 10}]\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\} & \bot && (16)
\end{aligned}
$$

Again replace negated conjecture by tautology

$$
\begin{aligned}
&[\text{axiom}] & \forall x_0 : p(x_0, f(x_0)) && (1)\\
&[\text{axiom}] & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) && (2)\\
&[\text{axiom}] & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) && (3)\\
&[\text{tautology}] & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg\forall x_0 \exists x_1 : r(x_0, x_1) && (4)\\
&[\text{ennf transformation 2}] & \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) && (5)\\
&[\text{ennf transformation 3}] & \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) && (6)\\
&[\text{ennf transformation 4}] & \forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1) && (7)\\
&[\text{choice axiom}] & \exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) && (8)\\
&[\text{choice axiom}] & \forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0))) && (9)\\
&[\text{skolemisation 7,8}] & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1) && (10)\\
&[\text{skolemisation 9,10}] & \forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1) && (11)\\
&[\text{cnf transformation 1}] & p(x_0, f(x_0)) && (12)\\
&[\text{cnf transformation 5}] & q(x_1, g(x_2)) \vee \neg p(x_1, x_2) && (13)\\
&[\text{cnf transformation 6}] & r(x_3, h(x_4)) \vee \neg q(x_3, x_4) && (14)\\
&[\text{cnf transformation 11}] & r(x_6, s_1(x_6)) \vee \neg r(s_0, x_5) && (15)\\
&[\text{resolution 14, 15}]\{x_3 \mapsto s_0, x_5 \mapsto h(x_4)\} & r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4) && (16)\\
&[\text{resolution 16, 13}]\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\} & r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2) && (17)\\
&[\text{resolution 17, 12}]\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\} & r(x_6, s_1(x_6)) && (18)
\end{aligned}
$$

We propagate the substitutions

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(x_0, f(x_0))$	(12)
[cnf transformation 5]	$q(x_1, g(x_2)) \vee \neg p(x_1, x_2)$	(13)
[cnf transformation 6]	$r(s_0, h(x_4)) \vee \neg q(s_0, x_4)$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, h(x_4))$	(15)
[resolution 14, 15]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4)$	(16)
[resolution 16, 13]$\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\}$	$r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$	(17)
[resolution 17, 12]$\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$r(x_6, s_1(x_6))$	(18)

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(x_0, f(x_0))$	(12)
[cnf transformation 5]	$q(s_0, g(x_2)) \vee \neg p(s_0, x_2)$	(13)
[cnf transformation 6]	$r(s_0, h(g(x_2))) \vee \neg q(s_0, g(x_2))$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(x_2)))$	(15)
[resolution 14, 15]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg q(s_0, g(x_2))$	(16)
[resolution 16, 13]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$	(17)
[resolution 17, 12]$\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$r(x_6, s_1(x_6))$	(18)

$$\begin{aligned}
&[\text{axiom}] & \forall x_0 : p(x_0, f(x_0)) && (1)\\
&[\text{axiom}] & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) && (2)\\
&[\text{axiom}] & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) && (3)\\
&[\text{tautology}] & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1) && (4)\\
&[\text{ennf transformation 2}] & \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) && (5)\\
&[\text{ennf transformation 3}] & \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) && (6)\\
&[\text{ennf transformation 4}] & \forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1) && (7)\\
&[\text{choice axiom}] & \exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) && (8)\\
&[\text{choice axiom}] & \forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0))) && (9)\\
&[\text{skolemisation 7,8}] & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1) && (10)\\
&[\text{skolemisation 9,10}] & \forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1) && (11)\\
&[\text{cnf transformation 1}] & p(s_0, f(s_0)) && (12)\\
&[\text{cnf transformation 5}] & q(s_0, g(f(s_0))) \vee \neg p(s_0, f(s_0)) && (13)\\
&[\text{cnf transformation 6}] & r(s_0, h(g(f(s_0)))) \vee \neg q(s_0, g(f(s_0))) && (14)\\
&[\text{cnf transformation 11}] & r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(f(s_0)))) && (15)\\
&[\text{resolution 14, 15}]\{\} & r(x_6, s_1(x_6)) \vee \neg q(s_0, g(f(s_0))) && (16)\\
&[\text{resolution 16, 13}]\{\} & r(x_6, s_1(x_6)) \vee \neg p(s_0, f(s_0)) && (17)\\
&[\text{resolution 17, 12}]\{\} & r(x_6, s_1(x_6)) && (18)
\end{aligned}$$

Then unification at 15 gives $s_1 = h \circ g \circ f$ and $s_0 = x_6$ and the final transformation gives:

$$\begin{aligned}
&[\text{axiom}] & \forall x_0 : p(x_0, f(x_0)) && (1)\\
&[\text{axiom}] & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) && (2)\\
&[\text{axiom}] & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) && (3)\\
&[\text{instantiation 1}] & p(x_6, f(x_6)) && (4)\\
&[\text{instantiation 2}] & p(x_6, f(x_6)) \Rightarrow q(x_6, g(f(x_6))) && (5)\\
&[\text{instantiation 3}] & q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6)))) && (6)\\
&[\text{tautology}] & r(x_6, h(g(f(x_6)))) \Rightarrow r(x_6, h(g(f(x_6)))) && (7)\\
&[\text{composition 6, 7}]\{\} & q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6)))) && (8)\\
&[\text{composition 5, 8}]\{\} & p(x_6, f(x_6)) \Rightarrow r(x_6, h(g(f(x_6)))) && (9)\\
&[\text{composition 4, 9}]\{\} & r(x_6, h(g(f(x_6)))) && (10)
\end{aligned}$$

$$\begin{aligned}
&\text{[axiom]} && \forall x_0 : p(x_0, f(x_0)) && (1)\\
&\text{[axiom]} && \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) && (2)\\
&\text{[axiom]} && \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) && (3)\\
&\text{[negated conjecture]} && \neg\forall x_0 : \exists x_1 : r(x_0, x_1) && (4)\\
&\text{[ennf transformation 2]} && \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) && (5)\\
&\text{[ennf transformation 3]} && \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) && (6)\\
&\text{[ennf transformation 4]} && \exists x_0 \forall x_1 : \neg r(x_0, x_1) && (7)\\
&\text{[choice axiom]} && \exists x_0 \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) && (8)\\
&\text{[skolemisation 7,8]} && \forall x_1 : \neg r(s_0, x_1) && (9)\\
&\text{[cnf transformation 1]} && p(x_0, f(x_0)) && (10)\\
&\text{[cnf transformation 5]} && q(x_1, g(x_2)) \vee \neg p(x_1, x_2) && (11)\\
&\text{[cnf transformation 6]} && r(x_3, h(x_4)) \vee \neg q(x_3, x_4) && (12)\\
&\text{[cnf transformation 9]} && .\neg r(s_0, x_5) && (13)\\
&\text{[resolution 12, 13]}\{x_3 \mapsto s_0, x_5 \mapsto h(x_4)\} && \neg q(s_0, x_4) && (14)\\
&\text{[resolution 14, 11]}\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\} && \neg p(s_0, x_2) && (15)\\
&\text{[resolution 15, 10]}\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\} && \bot && (16)
\end{aligned}$$

Again replace negated conjecture by tautology

$$\begin{aligned}
&\text{[axiom]} && \forall x_0 : p(x_0, f(x_0)) && (1)\\
&\text{[axiom]} && \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) && (2)\\
&\text{[axiom]} && \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) && (3)\\
&\text{[tautology]} && \forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg\forall x_0 \exists x_1 : r(x_0, x_1) && (4)\\
&\text{[ennf transformation 2]} && \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) && (5)\\
&\text{[ennf transformation 3]} && \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) && (6)\\
&\text{[ennf transformation 4]} && \forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1) && (7)\\
&\text{[choice axiom]} && \exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) && (8)\\
&\text{[choice axiom]} && \forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0))) && (9)\\
&\text{[skolemisation 7,8]} && \forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1) && (10)\\
&\text{[skolemisation 9,10]} && \forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1) && (11)\\
&\text{[cnf transformation 1]} && p(x_0, f(x_0)) && (12)\\
&\text{[cnf transformation 5]} && q(x_1, g(x_2)) \vee \neg p(x_1, x_2) && (13)\\
&\text{[cnf transformation 6]} && r(x_3, h(x_4)) \vee \neg q(x_3, x_4) && (14)\\
&\text{[cnf transformation 11]} && r(x_6, s_1(x_6)) \vee \neg r(s_0, x_5) && (15)\\
&\text{[resolution 14, 15]}\{x_3 \mapsto s_0, x_5 \mapsto h(x_4)\} && r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4) && (16)\\
&\text{[resolution 16, 13]}\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\} && r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2) && (17)\\
&\text{[resolution 17, 12]}\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\} && r(x_6, s_1(x_6)) && (18)
\end{aligned}$$

We propagate the substitutions

$$
\begin{array}{rlr}
\text{[axiom]} & \forall x_0 : p(x_0, f(x_0)) & (1)\\
\text{[axiom]} & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) & (2)\\
\text{[axiom]} & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) & (3)\\
\text{[tautology]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1) & (4)\\
\text{[ennf transformation 2]} & \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) & (5)\\
\text{[ennf transformation 3]} & \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) & (6)\\
\text{[ennf transformation 4]} & \forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1) & (7)\\
\text{[choice axiom]} & \exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) & (8)\\
\text{[choice axiom]} & \forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0))) & (9)\\
\text{[skolemisation 7,8]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1) & (10)\\
\text{[skolemisation 9,10]} & \forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1) & (11)\\
\text{[cnf transformation 1]} & p(x_0, f(x_0)) & (12)\\
\text{[cnf transformation 5]} & q(x_1, g(x_2)) \vee \neg p(x_1, x_2) & (13)\\
\text{[cnf transformation 6]} & r(s_0, h(x_4)) \vee \neg q(s_0, x_4) & (14)\\
\text{[cnf transformation 11]} & r(x_6, s_1(x_6)) \vee \neg r(s_0, h(x_4)) & (15)\\
\text{[resolution 14, 15]}\{\} & r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4) & (16)\\
\text{[resolution 16, 13]}\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\} & r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2) & (17)\\
\text{[resolution 17, 12]}\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\} & r(x_6, s_1(x_6)) & (18)
\end{array}
$$

$$
\begin{array}{rlr}
\text{[axiom]} & \forall x_0 : p(x_0, f(x_0)) & (1)\\
\text{[axiom]} & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) & (2)\\
\text{[axiom]} & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) & (3)\\
\text{[tautology]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1) & (4)\\
\text{[ennf transformation 2]} & \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) & (5)\\
\text{[ennf transformation 3]} & \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) & (6)\\
\text{[ennf transformation 4]} & \forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1) & (7)\\
\text{[choice axiom]} & \exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) & (8)\\
\text{[choice axiom]} & \forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0))) & (9)\\
\text{[skolemisation 7,8]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1) & (10)\\
\text{[skolemisation 9,10]} & \forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1) & (11)\\
\text{[cnf transformation 1]} & p(x_0, f(x_0)) & (12)\\
\text{[cnf transformation 5]} & q(s_0, g(x_2)) \vee \neg p(s_0, x_2) & (13)\\
\text{[cnf transformation 6]} & r(s_0, h(g(x_2))) \vee \neg q(s_0, g(x_2)) & (14)\\
\text{[cnf transformation 11]} & r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(x_2))) & (15)\\
\text{[resolution 14, 15]}\{\} & r(x_6, s_1(x_6)) \vee \neg q(s_0, g(x_2)) & (16)\\
\text{[resolution 16, 13]}\{\} & r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2) & (17)\\
\text{[resolution 17, 12]}\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\} & r(x_6, s_1(x_6)) & (18)
\end{array}
$$

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(s_0, f(s_0))$	(12)
[cnf transformation 5]	$q(s_0, g(f(s_0))) \vee \neg p(s_0, f(s_0))$	(13)
[cnf transformation 6]	$r(s_0, h(g(f(s_0)))) \vee \neg q(s_0, g(f(s_0)))$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(f(s_0))))$	(15)
[resolution 14, 15]{}	$r(x_6, s_1(x_6)) \vee \neg q(s_0, g(f(s_0)))$	(16)
[resolution 16, 13]{}	$r(x_6, s_1(x_6)) \vee \neg p(s_0, f(s_0))$	(17)
[resolution 17, 12]{}	$r(x_6, s_1(x_6))$	(18)

Then unification at 15 gives $s_1 = h \circ g \circ f$ and $s_0 = x_6$ and the final transformation gives:

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[instantiation 1]	$p(x_6, f(x_6))$	(4)
[instantiation 2]	$p(x_6, f(x_6)) \Rightarrow q(x_6, g(f(x_6)))$	(5)
[instantiation 3]	$q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6))))$	(6)
[tautology]	$r(x_6, h(g(f(x_6)))) \Rightarrow r(x_6, h(g(f(x_6))))$	(7)
[composition 6, 7]{}	$q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6))))$	(8)
[composition 5, 8]{}	$p(x_6, f(x_6)) \Rightarrow r(x_6, h(g(f(x_6))))$	(9)
[composition 4, 9]{}	$r(x_6, h(g(f(x_6))))$	(10)

References

[1] Aschieri, F. Interactive realizability for second-order heyting arithmetic with em1 and sk1. *Mathematical Structures in Computer Science 24*, 6 (2014).

[2] Berger, U., Buchholz, W., and Schwichtenberg, H. Refined program extraction from classical proofs. *Annals of Pure and Applied Logic 114*, 1-3 (2002), 3–25.

[3] Friedman, H. Classically and intuitionistically provably recursive functions. In *Higher Set Theory* (Berlin, Heidelberg, 1978), G. H. Müller and D. S. Scott, Eds., Springer Berlin Heidelberg, pp. 21–27.

[4] Kovács, L., and Voronkov, A. First-order theorem proving and vampire. In *Computer Aided Verification*. Springer Berlin Heidelberg, 2013, pp. 1–35.

[5] Luckham, D., and Nilsson, N. J. Extracting information from resolution proof trees. *Artificial Intelligence 2*, 1 (1971), 27–54.