Proofs as Programs in Classical Logic
Notes

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1 Plan

Goal:

- Extract program from resolution proofs of $\forall\exists$-sentences over inductive datatypes.

Steps:

- Give extensions of Gödel's System $\mathbf{T}$ and HAS + EM_1 + SK_1 to arbitrary inductive datatypes (possibly GADTs) and prove (or rather check) properties, i.e. strong normalization, cut-elimination, uniqueness of normal forms.
- Adapt the results of [1] to exhibit realizers in $\mathcal{F} + \Phi$ for cut-free proofs in the extended version of HAS + EM_1 + SK_1 of formulas $\forall x \exists y Pxy$ where P is a predicate in the extended version of $\mathbf{T}$.
- Adapt the iterative learning from [1] to extract λ-terms from realizers.
- Give a proof translation from resolution proofs to cut-free sequent calculus proofs (already done?).

Questions:

- When does the translated proof require more than EM_1?
- Using the methods in [1] the extracted term will be in simple λ-calculus. Is is possible to obtain a term in $\mathcal{F}$? (Talk with Federico Aschieri)
- Can the predicates be defined in $\mathcal{F}$ instead of $\mathcal{T}$? (probably yes)
- What happens if we add non-inductive theories?

2 Extracting constructive content from resolution proofs

2.1 The superposition calculus

First let us define the calculus from which we wish to extract programs. It will comprise a core set of rules used in the vampire theorem prover taken from [2]. Note that we pay special attention to the usually neglected (and highly non-constructive) part of CNF transformation and Skolemization. Also note that we neglect the simplification ordering which is necessary to formulate a strategy for proof search but not for our proof transformation.

Resolution.

$$\frac{A \vee B \qquad \neg A' \vee C}{(B \vee C)\theta}$$

where θ is an mgu of A and A'.

Factoring.

$$\frac{A \vee A' \vee B}{(A \vee B)\theta}$$

where θ is an mgu of A and A'.

Superposition.

$$\frac{l = r \vee B \qquad L[s] \vee C}{(L[r] \vee B \vee C)\theta} \quad \frac{l = r \vee B \qquad t[s] = t' \vee C}{(t[r] = t' \vee B \vee C)\theta} \quad \frac{l = r \vee B \qquad t[s] \neq t' \vee C}{(t[r] \neq t' \vee B \vee C)\theta}$$

where θ is an mgu of l and s, s is not a variable, $L[s]$ is not an equality literal.

Equality Resolution.

$$\frac{s \neq t \vee C}{C\theta}$$

where θ is an mgu of s and t.

Equality Factoring.

$$\frac{s = t \vee s' = t' \vee C}{(s = t \vee t \neq t' \vee C)\theta}$$

where θ is an mgu of s and s'.

First let us look at an example of the transformation we going to do. We shall look at a proof of $\forall x \exists y : f(y) = g(x)$ from $\forall x : f(x) = g(h(x))$ and $\forall x : h(h'(x)) = x$. The program we extract hopefully is h'.

[axiom]	$\forall x : f(x) = g(h(x))$	(1)
[cnf 1]	$f(x_0) = g(h(x_0))$	(2)
[axiom]	$\forall x : h(h'(x)) = x$	(3)
[cnf 3]	$h(h'(x_0)) = x_0$	(4)
[negated conjecture]	$\neg \forall x \exists y : f(y) = g(x)$	(5)
[ennf 5]	$\exists x \forall y : f(y) \neq g(x)$	(6)
[choice axiom]	$\exists x \forall y : f(y) \neq g(x) \to \forall x : f(x) \neq g(s_0)$	(7)
[skolemization 6, 7]	$\forall x : f(x) \neq g(s_0)$	(8)
[cnf 8]	$f(x_1) \neq g(s_0)$	(9)
[superposition 9, 2]{}	$g(s_0) \neq g(h(x_0))$	(10)
[superposition 10, 4]$\{x_0 \mapsto h'(x_0)\}$	$g(x_0) \neq g(s_0)$	(11)
[equality resolution 11]	$\perp$	(12)

We first use the well known trick of adding the original conjecture to the negated one from [3] to transform this into a classical proof of $\forall x \exists y : f(x) = g(y)$ rather than a refutation of its negation:

$$\begin{array}{lr}
\text{[axiom]} & \forall x : f(x) = g(h(x)) \quad (1) \\
\text{[cnf 1]} & f(x_0) = g(h(x_0)) \quad (2) \\
\text{[axiom]} & \forall x : h(h'(x)) = x \quad (3) \\
\text{[cnf 3]} & h(h'(x_0)) = x_0 \quad (4) \\
\text{[tautology]} & \forall x \exists y : f(y) = g(x) \vee \neg \forall x \exists y : f(y) = g(x) \quad (5) \\
\text{[ennf 5]} & \forall x \exists y : f(y) = g(x) \vee \exists x \forall y : f(y) \neq g(x) \quad (6) \\
\text{[choice axiom]} & \exists x \forall y : f(y) \neq g(x) \rightarrow \forall x : f(x) \neq g(s_0) \quad (7) \\
\text{[choice axiom]} & \forall x (\exists y : f(y) = g(x) \rightarrow f(s_1(x)) = g(x)) \quad (8) \\
\text{[skolemization 6, 7]} & \forall x \exists y : f(y) = g(x) \vee \forall x : f(x) \neq g(s_0) \quad (9) \\
\text{[skolemization 9, 8]} & \forall x : f(s_1(x)) = g(x) \vee \forall x : f(x) \neq g(s_0) \quad (10) \\
\text{[cnf 8]} & f(s_1(x_1)) = g(x_1) \vee f(x_0) \neq g(s_0) \quad (11) \\
\text{[superposition 11, 2]}\{\} & f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(h(x_0)) \quad (12) \\
\text{[superposition 12, 4]}\{x_0 \mapsto h'(x_0)\} & f(s_1(x_1)) = g(x_1) \vee g(x_0) \neq g(s_0) \quad (13) \\
\text{[equality resolution 13]}\{x_0 \rightarrow s_0\} & f(s_1(x_1)) = g(x_1) \quad (14)
\end{array}$$

Next we eliminate free variables by propagating substitutions:

$$\begin{array}{lr}
\text{[axiom]} & \forall x : f(x) = g(h(x)) \quad (1) \\
\text{[cnf 1]} & f(h'(x_0)) = g(h(h'(x_0))) \quad (2) \\
\text{[axiom]} & \forall x : h(h'(x)) = x \quad (3) \\
\text{[cnf 3]} & h(h'(x_0)) = x_0 \quad (4) \\
\text{[tautology]} & \forall x \exists y : f(y) = g(x) \vee \neg \forall x \exists y : f(y) = g(x) \quad (5) \\
\text{[ennf 5]} & \forall x \exists y : f(y) = g(x) \vee \exists x \forall y : f(y) \neq g(x) \quad (6) \\
\text{[choice axiom]} & \exists x \forall y : f(y) \neq g(x) \rightarrow \forall x : f(x) \neq g(s_0) \quad (7) \\
\text{[choice axiom]} & \forall x (\exists y : f(y) = g(x) \rightarrow f(s_1(x)) = g(x)) \quad (8) \\
\text{[skolemization 6, 7]} & \forall x \exists y : f(y) = g(x) \vee \forall x : f(x) \neq g(s_0) \quad (9) \\
\text{[skolemization 9, 8]} & \forall x : f(s_1(x)) = g(x) \vee \forall x : f(x) \neq g(s_0) \quad (10) \\
\text{[cnf 8]} & f(s_1(x_1)) = g(x_1) \vee f(h'(x_0)) \neq g(s_0) \quad (11) \\
\text{[superposition 11, 2]}\{\} & f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(h(h'(x_0))) \quad (12) \\
\text{[superposition 12, 4]}\{\} & f(s_1(x_1)) = g(x_1) \vee g(x_0) \neq g(s_0) \quad (13) \\
\text{[equality resolution 13]}\{x_0 \rightarrow s_0\} & f(s_1(x_1)) = g(x_1) \quad (14)
\end{array}$$

[axiom]	$\forall x : f(x) = g(h(x))$	(1)
[cnf 1]	$f(h'(s_0)) = g(h(h'(s_0)))$	(2)
[axiom]	$\forall x : h(h'(x)) = x$	(3)
[cnf 3]	$h(h'(s_0)) = s_0$	(4)
[tautology]	$\forall x \exists y : f(y) = g(x) \vee \neg \forall x \exists y : f(y) = g(x)$	(5)
[ennf 5]	$\forall x \exists y : f(y) = g(x) \vee \exists x \forall y : f(y) \neq g(x)$	(6)
[choice axiom]	$\exists x \forall y : f(y) \neq g(x) \to \forall x : f(x) \neq g(s_0)$	(7)
[choice axiom]	$\forall x (\exists y : f(y) = g(x) \to f(s_1(x)) = g(x))$	(8)
[skolemization 6, 7]	$\forall x \exists y : f(y) = g(x) \vee \forall x : f(x) \neq g(s_0)$	(9)
[skolemization 9, 8]	$\forall x : f(s_1(x)) = g(x) \vee \forall x : f(x) \neq g(s_0)$	(10)
[cnf 8]	$f(s_1(x_1)) = g(x_1) \vee f(h'(s_0)) \neq g(s_0)$	(11)
[superposition 11, 2]{}	$f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(h(h'(s_0)))$	(12)
[superposition 12, 4]{}	$f(s_1(x_1)) = g(x_1) \vee g(s_0) \neq g(s_0)$	(13)
[equality resolution 13]{}	$f(s_1(x_1)) = g(x_1)$	(14)

Next we remove all the skolem constants in the conjecture-tautology by unification, propagate this change, and finally reinterpet superposition as composition to yield a valid intuitionistic proof:

[axiom]	$\forall x : f(x) = g(h(x))$	(1)
[instantiation 1]	$f(h'(x_1)) = g(h(h'(x_1)))$	(2)
[axiom]	$\forall x : h(h'(x)) = x$	(3)
[instantiation 3]	$h(h'(x_1)) = x_1$	(4)
[tautology]	$f(h'(x_1)) = g(x_1) \Rightarrow f(h'(x_1)) = g(x_1)$	(5)
[equality 5, 2]{}	$g(x_1) = g(h(h'(x_1))) \Rightarrow f(h'(x_1)) = g(x_1)$	(6)
[equality 6, 4]{}	$g(x_1) = g(x_1) \Rightarrow f(h'(x_1)) = g(x_1)$	(7)
[equality 13]{}	$f(h'(x_1)) = g(x_1)$	(8)

Let us look at a second example: Consider the sentences $\forall x : p(x_0, f(x_0))$, $\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))$, $\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$. We take a look at a proof of $\forall x_0 \exists x_1 : r(x_0, x_1)$ which hopefully gives us $h \circ g \circ f$. First consider the output of vampire

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[negated conjecture]	$\neg \forall x_0 : \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[skolemisation 7,8]	$\forall x_1 : \neg r(s_0, x_1)$	(9)
[cnf transformation 1]	$p(x_0, f(x_0))$	(10)
[cnf transformation 5]	$q(x_1, g(x_2)) \vee \neg p(x_1, x_2)$	(11)
[cnf transformation 6]	$r(x_3, h(x_4)) \vee \neg q(x_3, x_4)$	(12)
[cnf transformation 9]	$.\neg r(s_0, x_5)$	(13)
[resolution 12, 13] $\{x_3 \mapsto s_0, x_5 \mapsto h(x_4)\}$	$\neg q(s_0, x_4)$	(14)
[resolution 14, 11] $\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\}$	$\neg p(s_0, x_2)$	(15)
[resolution 15, 10] $\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$\perp$	(16)

Again replace negated conjecture by tautology

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(x_0, f(x_0))$	(12)
[cnf transformation 5]	$q(x_1, g(x_2)) \vee \neg p(x_1, x_2)$	(13)
[cnf transformation 6]	$r(x_3, h(x_4)) \vee \neg q(x_3, x_4)$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, x_5)$	(15)
[resolution 14, 15] $\{x_3 \mapsto s_0, x_5 \mapsto h(x_4)\}$	$r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4)$	(16)
[resolution 16, 13] $\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\}$	$r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$	(17)
[resolution 17, 12] $\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$r(x_6, s_1(x_6))$	(18)

We propagate the substitutions

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(x_0, f(x_0))$	(12)
[cnf transformation 5]	$q(x_1, g(x_2)) \vee \neg p(x_1, x_2)$	(13)
[cnf transformation 6]	$r(s_0, h(x_4)) \vee \neg q(s_0, x_4)$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, h(x_4))$	(15)
[resolution 14, 15]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4)$	(16)
[resolution 16, 13]$\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\}$	$r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$	(17)
[resolution 17, 12]$\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$r(x_6, s_1(x_6))$	(18)

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
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[cnf transformation 1]	$p(x_0, f(x_0))$	(12)
[cnf transformation 5]	$q(s_0, g(x_2)) \vee \neg p(s_0, x_2)$	(13)
[cnf transformation 6]	$r(s_0, h(g(x_2))) \vee \neg q(s_0, g(x_2))$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(x_2)))$	(15)
[resolution 14, 15]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg q(s_0, g(x_2))$	(16)
[resolution 16, 13]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$	(17)
[resolution 17, 12]$\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$r(x_6, s_1(x_6))$	(18)

$$
\begin{array}{rrr}
\text{[axiom]} & \forall x_0 : p(x_0, f(x_0)) & (1)\\
\text{[axiom]} & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) & (2)\\
\text{[axiom]} & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) & (3)\\
\text{[tautology]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg\forall x_0 \exists x_1 : r(x_0, x_1) & (4)\\
\text{[ennf transformation 2]} & \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) & (5)\\
\text{[ennf transformation 3]} & \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) & (6)\\
\text{[ennf transformation 4]} & \forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1) & (7)\\
\text{[choice axiom]} & \exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) & (8)\\
\text{[choice axiom]} & \forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0))) & (9)\\
\text{[skolemisation 7,8]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1) & (10)\\
\text{[skolemisation 9,10]} & \forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1) & (11)\\
\text{[cnf transformation 1]} & p(s_0, f(s_0)) & (12)\\
\text{[cnf transformation 5]} & q(s_0, g(f(s_0))) \vee \neg p(s_0, f(s_0)) & (13)\\
\text{[cnf transformation 6]} & r(s_0, h(g(f(s_0)))) \vee \neg q(s_0, g(f(s_0))) & (14)\\
\text{[cnf transformation 11]} & r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(f(s_0)))) & (15)\\
\text{[resolution 14, 15]\{\}} & r(x_6, s_1(x_6)) \vee \neg q(s_0, g(f(s_0))) & (16)\\
\text{[resolution 16, 13]\{\}} & r(x_6, s_1(x_6)) \vee \neg p(s_0, f(s_0)) & (17)\\
\text{[resolution 17, 12]\{\}} & r(x_6, s_1(x_6)) & (18)
\end{array}
$$

Then unification at 15 gives $s_1 = h \circ g \circ f$ and $s_0 = x_6$ and the final transformation gives:

$$
\begin{array}{rrr}
\text{[axiom]} & \forall x_0 : p(x_0, f(x_0)) & (1)\\
\text{[axiom]} & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) & (2)\\
\text{[axiom]} & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) & (3)\\
\text{[instantiation 1]} & p(x_6, f(x_6)) & (4)\\
\text{[instantiation 2]} & p(x_6, f(x_6)) \Rightarrow q(x_6, g(f(x_6))) & (5)\\
\text{[instantiation 3]} & q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6)))) & (6)\\
\text{[tautology]} & r(x_6, h(g(f(x_6)))) \Rightarrow r(x_6, h(g(f(x_6)))) & (7)\\
\text{[composition 6, 7]\{\}} & q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6)))) & (8)\\
\text{[composition 5, 8]\{\}} & p(x_6, f(x_6)) \Rightarrow r(x_6, h(g(f(x_6)))) & (9)\\
\text{[composition 4, 9]\{\}} & r(x_6, h(g(f(x_6)))) & (10)
\end{array}
$$

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[negated conjecture]	$\neg \forall x_0 : \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[skolemisation 7,8]	$\forall x_1 : \neg r(s_0, x_1)$	(9)
[cnf transformation 1]	$p(x_0, f(x_0))$	(10)
[cnf transformation 5]	$q(x_1, g(x_2)) \vee \neg p(x_1, x_2)$	(11)
[cnf transformation 6]	$r(x_3, h(x_4)) \vee \neg q(x_3, x_4)$	(12)
[cnf transformation 9]	$.\neg r(s_0, x_5)$	(13)
[resolution 12, 13] $\{x_3 \mapsto s_0, x_5 \mapsto h(x_4)\}$	$\neg q(s_0, x_4)$	(14)
[resolution 14, 11] $\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\}$	$\neg p(s_0, x_2)$	(15)
[resolution 15, 10] $\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$\perp$	(16)

Again replace negated conjecture by tautology

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(x_0, f(x_0))$	(12)
[cnf transformation 5]	$q(x_1, g(x_2)) \vee \neg p(x_1, x_2)$	(13)
[cnf transformation 6]	$r(x_3, h(x_4)) \vee \neg q(x_3, x_4)$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, x_5)$	(15)
[resolution 14, 15] $\{x_3 \mapsto s_0, x_5 \mapsto h(x_4)\}$	$r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4)$	(16)
[resolution 16, 13] $\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\}$	$r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$	(17)
[resolution 17, 12] $\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$r(x_6, s_1(x_6))$	(18)

We propagate the substitutions

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(x_0, f(x_0))$	(12)
[cnf transformation 5]	$q(x_1, g(x_2)) \vee \neg p(x_1, x_2)$	(13)
[cnf transformation 6]	$r(s_0, h(x_4)) \vee \neg q(s_0, x_4)$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, h(x_4))$	(15)
[resolution 14, 15]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4)$	(16)
[resolution 16, 13]$\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\}$	$r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$	(17)
[resolution 17, 12]$\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$r(x_6, s_1(x_6))$	(18)

[axiom]	$\forall x_0 : p(x_0, f(x_0))$	(1)
[axiom]	$\forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1)))$	(2)
[axiom]	$\forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1)))$	(3)
[tautology]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1)$	(4)
[ennf transformation 2]	$\forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1))$	(5)
[ennf transformation 3]	$\forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1))$	(6)
[ennf transformation 4]	$\forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1)$	(7)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1)$	(8)
[choice axiom]	$\forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0)))$	(9)
[skolemisation 7,8]	$\forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1)$	(10)
[skolemisation 9,10]	$\forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1)$	(11)
[cnf transformation 1]	$p(x_0, f(x_0))$	(12)
[cnf transformation 5]	$q(s_0, g(x_2)) \vee \neg p(s_0, x_2)$	(13)
[cnf transformation 6]	$r(s_0, h(g(x_2))) \vee \neg q(s_0, g(x_2))$	(14)
[cnf transformation 11]	$r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(x_2)))$	(15)
[resolution 14, 15]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg q(s_0, g(x_2))$	(16)
[resolution 16, 13]$\{\}$	$r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2)$	(17)
[resolution 17, 12]$\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\}$	$r(x_6, s_1(x_6))$	(18)

$$
\begin{array}{rrr}
\text{[axiom]} & \forall x_0 : p(x_0, f(x_0)) & (1)\\
\text{[axiom]} & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) & (2)\\
\text{[axiom]} & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) & (3)\\
\text{[tautology]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1) & (4)\\
\text{[ennf transformation 2]} & \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) & (5)\\
\text{[ennf transformation 3]} & \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) & (6)\\
\text{[ennf transformation 4]} & \forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1) & (7)\\
\text{[choice axiom]} & \exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) & (8)\\
\text{[choice axiom]} & \forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0))) & (9)\\
\text{[skolemisation 7,8]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1) & (10)\\
\text{[skolemisation 9,10]} & \forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1) & (11)\\
\text{[cnf transformation 1]} & p(s_0, f(s_0)) & (12)\\
\text{[cnf transformation 5]} & q(s_0, g(f(s_0))) \vee \neg p(s_0, f(s_0)) & (13)\\
\text{[cnf transformation 6]} & r(s_0, h(g(f(s_0)))) \vee \neg q(s_0, g(f(s_0))) & (14)\\
\text{[cnf transformation 11]} & r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(f(s_0)))) & (15)\\
\text{[resolution 14, 15]\{\}} & r(x_6, s_1(x_6)) \vee \neg q(s_0, g(f(s_0))) & (16)\\
\text{[resolution 16, 13]\{\}} & r(x_6, s_1(x_6)) \vee \neg p(s_0, f(s_0)) & (17)\\
\text{[resolution 17, 12]\{\}} & r(x_6, s_1(x_6)) & (18)
\end{array}
$$

Then unification at 15 gives $s_1 = h \circ g \circ f$ and $s_0 = x_6$ and the final transformation gives:

$$
\begin{array}{rrr}
\text{[axiom]} & \forall x_0 : p(x_0, f(x_0)) & (1)\\
\text{[axiom]} & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) & (2)\\
\text{[axiom]} & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) & (3)\\
\text{[instantiation 1]} & p(x_6, f(x_6)) & (4)\\
\text{[instantiation 2]} & p(x_6, f(x_6)) \Rightarrow q(x_6, g(f(x_6))) & (5)\\
\text{[instantiation 3]} & q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6)))) & (6)\\
\text{[tautology]} & r(x_6, h(g(f(x_6)))) \Rightarrow r(x_6, h(g(f(x_6)))) & (7)\\
\text{[composition 6, 7]\{\}} & q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6)))) & (8)\\
\text{[composition 5, 8]\{\}} & p(x_6, f(x_6)) \Rightarrow r(x_6, h(g(f(x_6)))) & (9)\\
\text{[composition 4, 9]\{\}} & r(x_6, h(g(f(x_6)))) & (10)
\end{array}
$$

Now a third example including induction:

[input]	$\forall x_0 : z + x_0 = x_0$	(1)
[input]	$\forall x_0, x_1 : s(x_0) + x_1 = s(x_0 + x_1)$	(2)
[input]	$\forall x_0, x_1 : x_0 + x_1 = x_1 + x_0$	(3)
[input]	$\forall x_0, x_1 : (p(x_0, x_1) \Leftrightarrow (s(x_1 + x_1) = x_0 \vee x_1 + x_1 = x_0))$	(4)
[input]	$(\forall x_0 : (\exists x_1 : p(x_0, x_1) \Rightarrow \exists x_1 : p(s(x_0), x_1)) \wedge \exists x_1 : p(z, x_1)) \Rightarrow \forall x_0 : \exists x_1 : p(x_0, x_1)$	(5)
[negated conjecture]	$\neg\forall x_0 : \exists x_1 : p(x_0, x_1)$	(6)
[rectify 5]	$(\forall x_0 : (\exists x_1 : p(x_0, x_1) \Rightarrow \exists x_2 : p(s(x_0), x_2)) \wedge \exists x_3 : p(z, x_3)) \Rightarrow \forall x_4 : \exists x_5 : p(x_4, x_5)$	(7)
[ennf transformation 7]	$\forall x_4 : \exists x_5 : p(x_4, x_5) \vee (\exists x_0 : (\forall x_2 : \neg p(s(x_0), x_2) \wedge \exists x_1 : p(x_0, x_1)) \vee \forall x_3 : \neg p(z, x_3))$	(8)
[flattening 8]	$\forall x_4 : \exists x_5 : p(x_4, x_5) \vee \exists x_0 : (\forall x_2 : \neg p(s(x_0), x_2) \wedge \exists x_1 : p(x_0, x_1)) \vee \forall x_3 : \neg p(z, x_3)$	(9)
[ennf transformation 6]	$\exists x_0 : \forall x_1 : \neg p(x_0, x_1)$	(10)
[nnf transformation 4]	$\forall x_0, x_1 : ((p(x_0, x_1) \vee (s(x_1 + x_1) \neq x_0 \wedge x_1 + x_1 \neq x_0)) \wedge ((s(x_1 + x_1) = x_0 \vee x_1 + x_1 = x_0) \vee \neg p(x_0, x_1)))$	(11)
[flattening 11]	$\forall x_0, x_1 : ((p(x_0, x_1) \vee (s(x_1 + x_1) \neq x_0 \wedge x_1 + x_1 \neq x_0)) \wedge (s(x_1 + x_1) = x_0 \vee x_1 + x_1 = x_0 \vee \neg p(x_0, x_1)))$	(12)
[rectify 9]	$\forall x_0 : \exists x_1 : p(x_0, x_1) \vee \exists x_2 : (\forall x_3 : \neg p(s(x_2), x_3) \wedge \exists x_4 : p(x_2, x_4)) \vee \forall x_5 : \neg p(z, x_5)$	(13)
[choice axiom]	$\forall x_0 : (\exists x_1 : p(x_0, x_1) \Rightarrow p(x_0, s_0(x_0)))$	(14)
[choice axiom]	$\exists x_2 : (\forall x_3 : \neg p(s(x_2), x_3) \wedge \exists x_4 : p(x_2, x_4)) \Rightarrow (\forall x_3 : \neg p(s(s_1), x_3) \wedge \exists x_4 : p(s_1, x_4))$	(15)
[choice axiom]	$\exists x_4 : p(s_1, x_4) \Rightarrow p(s_1, s_2)$	(16)
[skolemisation 13,16,15,14]	$\forall x_0 : p(x_0, s_0(x_0)) \vee (\forall x_3 : \neg p(s(s_1), x_3) \wedge p(s_1, s_2)) \vee \forall x_5 : \neg p(z, x_5)$	(17)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg p(x_0, x_1) \Rightarrow \forall x_1 : \neg p(s_3, x_1)$	(18)
[skolemisation 9,17]	$\forall x_1 : \neg p(s_3, x_1)$	(19)
[cnf transformation 1]	$z + x_0 = x_0$	(20)
[cnf transformation 2]	$s(x_0) + x_1 = s(x_0 + x_1)$	(21)
[cnf transformation 3]	$x_0 + x_1 = x_1 + x_0$	(22)
[cnf transformation 11]	$\neg p(x_0, x_1) \vee x_1 + x_1 = x_0 \vee s(x_1 + x_1) = x_0$	(23)
[cnf transformation 11]	$p(x_0, x_1) \vee x_1 + x_1 \neq x_0$	(24)
[cnf transformation 11]	$p(x_0, x_1) \vee s(x_1 + x_1) \neq x_0$	(25)

[cnf transformation 17]	$\neg p(z, x_5) \vee p(s_1, s_2) \vee p(x_0, s_0(x_0))$	(26)
[cnf transformation 17]	$\neg p(s(s_1), x_3) \vee p(x_0, s_0(x_0)) \vee \neg p(z, x_5)$	(27)
[cnf transformation 19]	$\neg p(s_3, x_1)$	(28)
[equality resolution 25]	$p(s(x_1 + x_1), x_1)$	(29)
[equality resolution 24]	$p(x_1 + x_1, x_1)$	(30)
[superposition 30,20]$\{x_0 \mapsto x, x_1 \mapsto z\}$	$p(z, z)$	(31)
[superposition 21,22]$\{x_0 \mapsto x_3, x_1 \mapsto x_2\}$	$x_3 + s(x_2) = s(x_2 + x_3)$	(32)
[superposition 30,21]$\{x_0 \mapsto x_1\}$	$p(s(x_1 + s(x_1)), s(x_1))$	(33)
[forward demodulation 33,32]$\{x_2 \mapsto x_1, x_3 \mapsto x_1\}$	$p(s(s(x_1 + x_1)), s(x_1))$	(34)
[resolution 26,31]$\{x_5 \mapsto z\}$	$p(x_0, s_0(x_0)) \vee p(s_1, s_2)$	(35)
[resolution 35,28]$\{x_0 \mapsto s_3, x_1 \mapsto s_0(s_3)\}$	$p(s_1, s_2)$	(36)
[resolution 23,36]$\{x_0 \mapsto s_1, x_1 \mapsto s_2\}$	$s_1 = s(s_2 + s_2) \vee s_1 = s_2 + s_2$	(37)
[superposition 24,37]$\{x_0 \mapsto s_1, x_1 \mapsto s_2\}$	$p(s(s_1), s(s_2)) \vee s_1 = s_2 + s_2$	(38)
[resolution 38,27]$\{x_2 \mapsto s(s_2), x_5 \mapsto x_1\}$	$\neg p(z, x_1) \vee p(x_0, s_0(x_0)) \vee s_1 = s_2 + s_2$	(39)
[resolution 39,31]$\{x_1 \mapsto z\}$	$p(x_0, s_0(x_0)) \vee s_1 = s_2 + s_2$	(40)
[resolution 40,28]$\{x_0 \mapsto s_3, x_1 \mapsto s_0(s_3)\}$	$s_1 = s_2 + s_2$	(41)
[superposition 41,29]$\{x_1 \mapsto s_2\}$	$p(s(s_1), s_2)$	(42)
[resolution 42,27]$\{x_3 \mapsto s_2, x_5 \mapsto x_1\}$	$\neg p(z, x_1) \vee p(x_0, s_0(x_0))$	(43)
[resolution 43,31]$\{x_1 \mapsto z\}$	$p(x_0, s_0(x_0))$	(44)
[resolution 44,28]$\{x_0 \mapsto s_3, x_1 \mapsto s_0(s_3)\}$	$\perp$	(45)

Again replace negated conjecture by tautology

[input]	$\forall x_0 : z + x_0 = x_0$	(1)
[input]	$\forall x_0, x_1 : s(x_0) + x_1 = s(x_0 + x_1)$	(2)
[input]	$\forall x_0, x_1 : x_0 + x_1 = x_1 + x_0$	(3)
[input]	$\forall x_0, x_1 : (p(x_0, x_1) \Leftrightarrow (s(x_1 + x_1) = x_0 \vee x_1 + x_1 = x_0))$	(4)
[input]	$(\forall x_0 : (\exists x_1 : p(x_0, x_1) \Rightarrow \exists x_1 : p(s(x_0), x_1)) \wedge \exists x_1 : p(z, x_1)) \Rightarrow \forall x_0 : \exists x_1 : p(x_0, x_1)$	(5)
[tautology]	$\forall x_0 : \exists x_1 : p(x_0, x_1) \vee \neg \forall x_0 : \exists x_1 : p(x_0, x_1)$	(6)
[rectify 5]	$(\forall x_0 : (\exists x_1 : p(x_0, x_1) \Rightarrow \exists x_2 : p(s(x_0), x_2)) \wedge \exists x_3 : p(z, x_3)) \Rightarrow \forall x_4 : \exists x_5 : p(x_4, x_5)$	(7)
[ennf transformation 7]	$\forall x_4 : \exists x_5 : p(x_4, x_5) \vee (\exists x_0 : (\forall x_2 : \neg p(s(x_0), x_2) \wedge \exists x_1 : p(x_0, x_1)) \vee \forall x_3 : \neg p(z, x_3))$	(8)
[flattening 8]	$\forall x_4 : \exists x_5 : p(x_4, x_5) \vee \exists x_0 : (\forall x_2 : \neg p(s(x_0), x_2) \wedge \exists x_1 : p(x_0, x_1)) \vee \forall x_3 : \neg p(z, x_3)$	(9)
[ennf transformation 6]	$\exists x_0 : \forall x_1 : \neg p(x_0, x_1)$	(10)
[nnf transformation 4]	$\forall x_0, x_1 : ((p(x_0, x_1) \vee (s(x_1 + x_1) \neq x_0 \wedge x_1 + x_1 \neq x_0)) \wedge ((s(x_1 + x_1) = x_0 \vee x_1 + x_1 = x_0) \vee \neg p(x_0, x_1)))$	(11)
[flattening 11]	$\forall x_0, x_1 : ((p(x_0, x_1) \vee (s(x_1 + x_1) \neq x_0 \wedge x_1 + x_1 \neq x_0)) \wedge (s(x_1 + x_1) = x_0 \vee x_1 + x_1 = x_0 \vee \neg p(x_0, x_1)))$	(12)
[rectify 9]	$\forall x_0 : \exists x_1 : p(x_0, x_1) \vee \exists x_2 : (\forall x_3 : \neg p(s(x_2), x_3) \wedge \exists x_4 : p(x_2, x_4)) \vee \forall x_5 : \neg p(z, x_5)$	(13)
[choice axiom]	$\forall x_0 : (\exists x_1 : p(x_0, x_1) \Rightarrow p(x_0, s_0(x_0)))$	(14)
[choice axiom]	$\exists x_2 : (\forall x_3 : \neg p(s(x_2), x_3) \wedge \exists x_4 : p(x_2, x_4)) \Rightarrow (\forall x_3 : \neg p(s(s_1), x_3) \wedge \exists x_4 : p(s_1, x_4))$	(15)
[choice axiom]	$\exists x_4 : p(s_1, x_4) \Rightarrow p(s_1, s_2)$	(16)
[skolemisation 13,16,15,14]	$\forall x_0 : p(x_0, s_0(x_0)) \vee (\forall x_3 : \neg p(s(s_1), x_3) \wedge p(s_1, s_2)) \vee \forall x_5 : \neg p(z, x_5)$	(17)
[choice axiom]	$\exists x_0 : \forall x_1 : \neg p(x_0, x_1) \Rightarrow \forall x_1 : \neg p(s_3, x_1)$	(18)
[choice axiom]	$\forall x_0 (\exists x_1 : p(x_0, x_1) \Rightarrow p(x_0, s_4(x_0)))$	(19)
[skolemisation 9,17,18]	$\forall x_0 : p(x_0, s_4(x_0)) \vee \forall x_1 : \neg p(s_3, x_1)$	(20)
[cnf transformation 1]	$z + x_0 = x_0$	(21)
[cnf transformation 2]	$s(x_0) + x_1 = s(x_0 + x_1)$	(22)
[cnf transformation 3]	$x_0 + x_1 = x_1 + x_0$	(23)
[cnf transformation 11]	$\neg p(x_0, x_1) \vee x_1 + x_1 = x_0 \vee s(x_1 + x_1) = x_0$	(24)
[cnf transformation 11]	$p(x_0, x_1) \vee x_1 + x_1 \neq x_0$	(25)
[cnf transformation 11]	$p(x_0, x_1) \vee s(x_1 + x_1) \neq x_0$	(26)

[cnf transformation 17]	$\neg p(z, x_5) \vee p(s_1, s_2) \vee p(x_0, s_0(x_0))$	(27)
[cnf transformation 17]	$\neg p(s(s_1), x_3) \vee p(x_0, s_0(x_0)) \vee \neg p(z, x_5)$	(28)
[cnf transformation 20]	$p(x_0, s_4(x_0)) \vee \neg p(s_3, x_1)$	(29)
[equality resolution 26]	$p(s(x_1 + x_1), x_1)$	(30)
[equality resolution 25]	$p(x_1 + x_1, x_1)$	(31)
[superposition 31,21]$\{x_0 \mapsto x, x_1 \mapsto z\}$	$p(z, z)$	(32)
[superposition 22,23]$\{x_0 \mapsto x_3, x_1 \mapsto x_2\}$	$x_3 + s(x_2)) = s(x_2 + x_3))$	(33)
[superposition 31,22]$\{x_0 \mapsto x_1\}$	$p(s(x_1 + s(x_1)), s(x_1))$	(34)
[forward demodulation 34,33]$\{x_2 \mapsto x_1, x_3 \mapsto x_1\}$	$p(s(s(x_1 + x_1)), s(x_1))$	(35)
[resolution 27,32]$\{x_5 \mapsto z\}$	$p(x_0, s_0(x_0)) \vee p(s_1, s_2)$	(36)
[resolution 36,29]$\{x_0 \mapsto s_3, x_1 \mapsto s_0(s_3)\}$	$p(x_0, s_4(x_0)) \vee p(s_1, s_2)$	(37)
[resolution 24,37]$\{x_0 \mapsto s_1, x_1 \mapsto s_2\}$	$p(x_0, s_4(x_0)) \vee s_1 = s(s_2 + s_2) \vee s_1 = s_2 + s_2$	(38)
[superposition 25,38]$\{x_0 \mapsto s_1, x_1 \mapsto s_2\}$	$p(x_0, s_4(x_0)) \vee p(s(s_1), s(s_2)) \vee s_1 = s_2 + s_2$	(39)
[resolution 39,28]$\{x_2 \mapsto s(s_2), x_5 \mapsto x_1\}$	$p(x_0, s_4(x_0)) \vee \neg p(z, x_1) \vee p(x_0, s_0(x_0)) \vee s_1 = s_2 + s_2$	(40)
[resolution 40,32]$\{x_1 \mapsto z\}$	$p(x_0, s_4(x_0)) \vee p(x_0, s_0(x_0)) \vee s_1 = s_2 + s_2$	(41)
[resolution 41,29]$\{x_0 \mapsto s_3, x_1 \mapsto s_0(s_3)\}$	$p(x_0, s_4(x_0)) \vee s_1 = s_2 + s_2$	(42)
[superposition 42,30]$\{x_1 \mapsto s_2\}$	$p(x_0, s_4(x_0)) \vee p(s(s_1), s_2)$	(43)
[resolution 43,28]$\{x_3 \mapsto s_2, x_5 \mapsto x_1\}$	$p(x_0, s_4(x_0)) \vee \neg p(z, x_1) \vee p(x_0, s_0(x_0))$	(44)
[resolution 44,32]$\{x_1 \mapsto z\}$	$p(x_0, s_4(x_0)) \vee p(x_0, s_0(x_0))$	(45)
[resolution 45,29]$\{x_0 \mapsto s_3, x_1 \mapsto s_0(s_3)\}$	$p(x_0, s_4(x_0))$	(46)

We propagate the substitutions

$$
\begin{array}{lrr}
\text{[axiom]} & \forall x_0 : p(x_0, f(x_0)) & (1) \\
\text{[axiom]} & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) & (2) \\
\text{[axiom]} & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) & (3) \\
\text{[tautology]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1) & (4) \\
\text{[ennf transformation 2]} & \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) & (5) \\
\text{[ennf transformation 3]} & \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) & (6) \\
\text{[ennf transformation 4]} & \forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1) & (7) \\
\text{[choice axiom]} & \exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) & (8) \\
\text{[choice axiom]} & \forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0))) & (9) \\
\text{[skolemisation 7,8]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1) & (10) \\
\text{[skolemisation 9,10]} & \forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1) & (11) \\
\text{[cnf transformation 1]} & p(x_0, f(x_0)) & (12) \\
\text{[cnf transformation 5]} & q(x_1, g(x_2)) \vee \neg p(x_1, x_2) & (13) \\
\text{[cnf transformation 6]} & r(s_0, h(x_4)) \vee \neg q(s_0, x_4) & (14) \\
\text{[cnf transformation 11]} & r(x_6, s_1(x_6)) \vee \neg r(s_0, h(x_4)) & (15) \\
\text{[resolution 14, 15]}\{\} & r(x_6, s_1(x_6)) \vee \neg q(s_0, x_4) & (16) \\
\text{[resolution 16, 13]}\{x_1 \mapsto s_0, x_4 \mapsto g(x_2)\} & r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2) & (17) \\
\text{[resolution 17, 12]}\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\} & r(x_6, s_1(x_6)) & (18)
\end{array}
$$

$$
\begin{array}{lrr}
\text{[axiom]} & \forall x_0 : p(x_0, f(x_0)) & (1) \\
\text{[axiom]} & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) & (2) \\
\text{[axiom]} & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) & (3) \\
\text{[tautology]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1) & (4) \\
\text{[ennf transformation 2]} & \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) & (5) \\
\text{[ennf transformation 3]} & \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) & (6) \\
\text{[ennf transformation 4]} & \forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1) & (7) \\
\text{[choice axiom]} & \exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) & (8) \\
\text{[choice axiom]} & \forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0))) & (9) \\
\text{[skolemisation 7,8]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1) & (10) \\
\text{[skolemisation 9,10]} & \forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1) & (11) \\
\text{[cnf transformation 1]} & p(x_0, f(x_0)) & (12) \\
\text{[cnf transformation 5]} & q(s_0, g(x_2)) \vee \neg p(s_0, x_2) & (13) \\
\text{[cnf transformation 6]} & r(s_0, h(g(x_2))) \vee \neg q(s_0, g(x_2)) & (14) \\
\text{[cnf transformation 11]} & r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(x_2))) & (15) \\
\text{[resolution 14, 15]}\{\} & r(x_6, s_1(x_6)) \vee \neg q(s_0, g(x_2)) & (16) \\
\text{[resolution 16, 13]}\{\} & r(x_6, s_1(x_6)) \vee \neg p(s_0, x_2) & (17) \\
\text{[resolution 17, 12]}\{x_0 \mapsto s_0, x_2 \mapsto f(s_0)\} & r(x_6, s_1(x_6)) & (18)
\end{array}
$$

$$
\begin{array}{rrr}
\text{[axiom]} & \forall x_0 : p(x_0, f(x_0)) & (1) \\
\text{[axiom]} & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) & (2) \\
\text{[axiom]} & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) & (3) \\
\text{[tautology]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \neg \forall x_0 \exists x_1 : r(x_0, x_1) & (4) \\
\text{[ennf transformation 2]} & \forall x_0, x_1 : (q(x_0, g(x_1)) \vee \neg p(x_0, x_1)) & (5) \\
\text{[ennf transformation 3]} & \forall x_0, x_1 : (r(x_0, h(x_1)) \vee \neg q(x_0, x_1)) & (6) \\
\text{[ennf transformation 4]} & \forall x_0 : \exists x_1 : r(x_0, x_1) \vee \exists x_0 \forall x_1 : \neg r(x_0, x_1) & (7) \\
\text{[choice axiom]} & \exists x_0 : \forall x_1 : \neg r(x_0, x_1) \Rightarrow \forall x_1 : \neg r(s_0, x_1) & (8) \\
\text{[choice axiom]} & \forall x_0 (\exists x_1 : r(x_0, x_1) \Rightarrow r(x_0, s_1(x_0))) & (9) \\
\text{[skolemisation 7,8]} & \forall x_0 \exists x_1 : r(x_0, x_1) \vee \forall x_1 : \neg r(s_0, x_1) & (10) \\
\text{[skolemisation 9,10]} & \forall x_0 : r(x_0, s_1(x_0)) \vee \forall x_1 : \neg r(s_0, x_1) & (11) \\
\text{[cnf transformation 1]} & p(s_0, f(s_0)) & (12) \\
\text{[cnf transformation 5]} & q(s_0, g(f(s_0))) \vee \neg p(s_0, f(s_0)) & (13) \\
\text{[cnf transformation 6]} & r(s_0, h(g(f(s_0)))) \vee \neg q(s_0, g(f(s_0))) & (14) \\
\text{[cnf transformation 11]} & r(x_6, s_1(x_6)) \vee \neg r(s_0, h(g(f(s_0)))) & (15) \\
\text{[resolution 14, 15]\{\}} & r(x_6, s_1(x_6)) \vee \neg q(s_0, g(f(s_0))) & (16) \\
\text{[resolution 16, 13]\{\}} & r(x_6, s_1(x_6)) \vee \neg p(s_0, f(s_0)) & (17) \\
\text{[resolution 17, 12]\{\}} & r(x_6, s_1(x_6)) & (18)
\end{array}
$$

Then unification at 15 gives $s_1 = h \circ g \circ f$ and $s_0 = x_6$ and the final transformation gives:

$$
\begin{array}{rrr}
\text{[axiom]} & \forall x_0 : p(x_0, f(x_0)) & (1) \\
\text{[axiom]} & \forall x_0, x_1 : (p(x_0, x_1) \Rightarrow q(x_0, g(x_1))) & (2) \\
\text{[axiom]} & \forall x_0, x_1 : (q(x_0, x_1) \Rightarrow r(x_0, h(x_1))) & (3) \\
\text{[instantiation 1]} & p(x_6, f(x_6)) & (4) \\
\text{[instantiation 2]} & p(x_6, f(x_6)) \Rightarrow q(x_6, g(f(x_6))) & (5) \\
\text{[instantiation 3]} & q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6)))) & (6) \\
\text{[tautology]} & r(x_6, h(g(f(x_6)))) \Rightarrow r(x_6, h(g(f(x_6)))) & (7) \\
\text{[composition 6, 7]\{\}} & q(x_6, g(f(x_6))) \Rightarrow r(x_6, h(g(f(x_6)))) & (8) \\
\text{[composition 5, 8]\{\}} & p(x_6, f(x_6)) \Rightarrow r(x_6, h(g(f(x_6)))) & (9) \\
\text{[composition 4, 9]\{\}} & r(x_6, h(g(f(x_6)))) & (10)
\end{array}
$$

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