PhD Seminar

Computability and Complexity of Boolean Binary Tree Operations

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April 26, 2017

`My life amounts to no more than one drop in a limitless ocean. Yet what is any ocean, but a multitude of drops? - David Mitchell, Cloud Atlas (2012)`

Fractional permissions in concurrency

```
struct tree{int d; struct tree* l; struct tree* r;}

void processTree(struct tree* x) {
  if (x == 0) { return; }

  print(x -> d);           ||  print(x -> d);
  processTree(x -> l);     ||  processTree(x -> l);
  processTree(x -> r);     ||  processTree(x -> r);
}
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```

$$\mathsf{tree}(\ell, \pi) \stackrel{\text{def}}{=} (\ell = \mathsf{null} \ \wedge \ \mathsf{emp}) \vee$$
$$\exists d, \ell_l, \ell_r.\ (\ell \stackrel{\pi}{\mapsto} (d, \ell_l, \ell_r) \ \star \ \mathsf{tree}(\ell_l, \pi) \ \star \ \mathsf{tree}(\ell_r, \pi))$$

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Spec: $\forall x, \pi. \ (\ \{\mathsf{tree}(x, \pi)\} \ \mathtt{processTree(x)} \ \{\mathsf{tree}(x, \pi)\})$

An abstract example

$$
\begin{tikzcd}
& \mathrm{tree}(x, \pi) \arrow[dl] \arrow[dr] & \\
\mathrm{tree}(x, \pi_1) \arrow[dr] & \| & \mathrm{tree}(x, \pi_2) \arrow[dl] \\
& \mathrm{tree}(x, \pi) &
\end{tikzcd}
$$

A concrete example: Rationals in [0,1]

$$\begin{tikzcd} & \mathrm{tree}(x, 1) \arrow[dl] \arrow[dr] & \\ \mathrm{tree}(x, \frac{1}{2}) \arrow[dr] & \| & \mathrm{tree}(x, \frac{1}{2}) \arrow[dl] \\ & \mathrm{tree}(x, 1) & \end{tikzcd}$$

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What could go wrong?

The shortcoming of rational model

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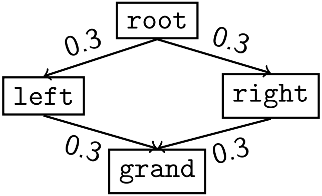
$$\exists d, \ell_l, \ell_r. \ (\ell \stackrel{\pi}{\mapsto} (d, \ell_l, \ell_r) \ * \ \mathsf{tree}(\ell_l, \pi) \ * \ \mathsf{tree}(\ell_r, \pi))$$

$$\begin{aligned}
&\mathtt{root} \stackrel{0.3}{\mapsto} (1, \mathtt{left}, \mathtt{right}) \, * \\
&\mathtt{left} \stackrel{0.3}{\mapsto} (1, \mathsf{null}, \mathtt{grand}) \ \ * \\
&\mathtt{right} \stackrel{0.3}{\mapsto} (1, \mathtt{grand}, \mathsf{null}) \, * \\
&\mathtt{grand} \stackrel{0.6}{\mapsto} (1, \mathsf{null}, \mathsf{null})
\end{aligned}$$



The shortcoming of rational model

$$\begin{aligned}\mathsf{tree}(\ell, \pi) \quad \overset{\mathtt{def}}{=} \quad & (\ell = \mathsf{null} \ \wedge \ \mathsf{emp}) \ \vee \\ & \exists d, \ell_l, \ell_r. \ (\ell \overset{\pi}{\mapsto} (d, \ell_l, \ell_r) \ \star \ \mathsf{tree}(\ell_l, \pi) \ \star \ \mathsf{tree}(\ell_r, \pi))\end{aligned}$$

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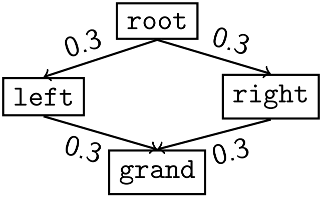

This is a dag, not a tree!

A quick diagnosis

- The disjointness property of SL:
  - $x \mapsto 1 * y \mapsto 1$: x and y are disjoint
  - $x \overset{\frac{1}{2}}{\mapsto} 1 * y \overset{\frac{1}{2}}{\mapsto} 1$: Hard to say …

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- Facts about normal heap may be false in fractional heap, *e.g.*, heap's size.

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- The disjointness property of SL:
  - $x \mapsto 1 * y \mapsto 1$: x and y are disjoint
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- Facts about normal heap may be false in fractional heap, *e.g.*, heap's size.
- Desired property (Parkinson (2005)):

$$x \overset{\pi}{\mapsto} v \ * \ x \overset{\pi}{\mapsto} v \quad \vdash \quad \perp$$

A brief history of permission models

Rationals in $[0, 1]$

- John Boyland. Checking interference with fractional permissions. In *SAS*, 2003.
- Duy-Khanh Le *et al*. Threads as resource for concurrency verification. In *PEPM*, 2015.

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Others

- Marieke Muisman *et al.* A symbolic approach to permission accounting for concurrent reasoning. In *ISPDC*, 2015.

A cure for disjointness

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- Disjointness property (permission layer):

$$\forall a \forall b.\ a \oplus a = b \ \rightarrow \ a = b$$

both a, b can be derived to be identity element.

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- Disjointness property (predicate layer):

$$x \overset{\pi}{\mapsto} v * x \overset{\pi}{\mapsto} v \vdash \bot$$

.

A checklist for my PhD

Three main pillars: application, theory and system.

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Three main pillars: application, theory and system.

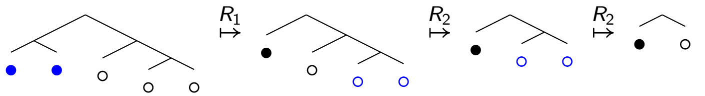
1. Application: Permission reasoning in Separation Logic.
2. Theory: Complexity and decidability of tree shares.
3. System: Certified tool in Coq, implementation and benchmarking in HIP/SLEEK.

Tree Share Definition

- Definition:

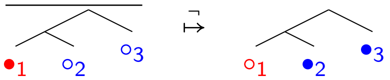
$$\tau \stackrel{\texttt{def}}{=} \circ \mid \bullet \mid \widehat{\tau \quad \tau} \qquad R_1 : \widehat{\bullet \quad \bullet} \mapsto \bullet \qquad R_2 : \widehat{\circ \quad \circ} \mapsto \circ$$

- Example:



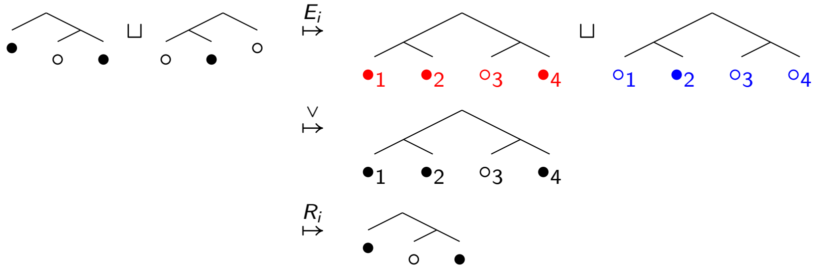
Tree Share Operators

The complement $\bar{\square}$:



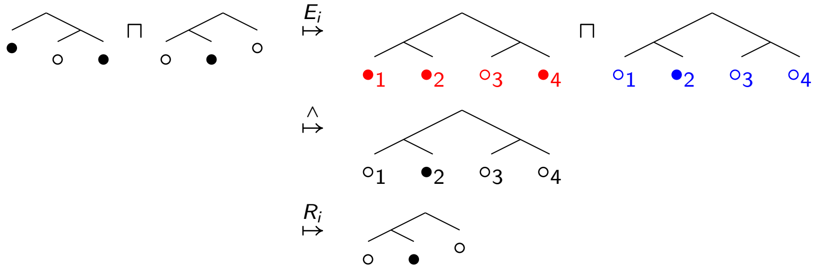
Tree Share Operators

The union $\sqcup$:



Tree Share Operators

The intersection $\sqcap$:



Properties of $\sqcup$, $\sqcap$ and $\bar{\square}$

$\mathcal{M} = (\sqcup, \sqcap, \bar{\square}, \bullet, \circ)$ is a Boolean Algebra (Dockins et al. (2009)):

$B1a.\ (\tau_1 \sqcap \tau_2) \sqcap \tau_3 = \tau_1 \sqcap (\tau_2 \sqcap \tau_3)$	$B1b.\ (\tau_1 \sqcup \tau_2) \sqcup \tau_3 = \tau_1 \sqcup (\tau_2 \sqcup \tau_3)$	(associativity)
$B2a.\ \tau_1 \sqcap \tau_2 = \tau_2 \sqcap \tau_1$	$B2b.\ \tau_1 \sqcup \tau_2 = \tau_2 \sqcup \tau_1$	(commutativity)
$B3a.\ \tau_1 \sqcap (\tau_2 \sqcup \tau_3) = (\tau_1 \sqcap \tau_2) \sqcup (\tau_1 \sqcap \tau_3)$	$B3b.\ \tau_1 \sqcup (\tau_2 \sqcap \tau_3) = (\tau_1 \sqcup \tau_2) \sqcap (\tau_1 \sqcup \tau_3)$	(distributivity)
$B4a.\ \tau_1 \sqcap (\tau_1 \sqcup \tau_2) = \tau_1$	$B4b.\ \tau_1 \sqcup (\tau_1 \sqcap \tau_2) = \tau_1$	(absorption)
$B5a.\ \tau \sqcap \bullet = \tau$	$B5b.\ \tau \sqcup \circ = \tau$	(identity)
$B6a.\ \tau \sqcap \bar{\tau} = \circ$	$B6b.\ \tau \sqcup \bar{\tau} = \bullet$	(complement)

Tree Share Operators(cont.)

The join relation $\oplus$:

$$\tau_1 \oplus \tau_2 = \tau_3 \quad \stackrel{\mathtt{def}}{=} \quad \tau_1 \sqcup \tau_2 = \tau_3 \ \wedge \ \tau_1 \sqcap \tau_2 = \circ$$

Tree Share Operators(cont.)

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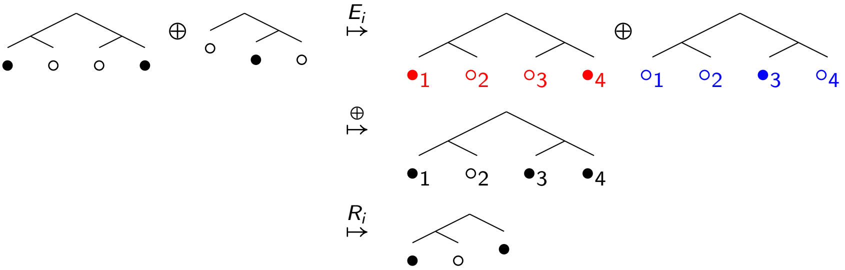


Properties of $\oplus$

$\mathcal{O} = (\mathbb{T}, \oplus)$ is a model for fractional permission in SL (Dockins et al. (2009)):

$J1.\ \tau_1 \oplus \tau_2 = \tau_3 \Rightarrow \tau_1 \oplus \tau_2 = \tau_3' \Rightarrow \tau_3 = \tau_3'$ (functionality)

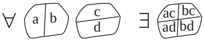
$J2.\ \tau_1 \oplus \tau_2 = \tau_2 \oplus \tau_1$ (commutativity)

$J3.\ \tau_1 \oplus (\tau_2 \oplus \tau_3) = (\tau_1 \oplus \tau_2) \oplus \tau_3$ (associativity)

$J4.\ \tau_1 \oplus \tau_2 = \tau_3 \Rightarrow \tau_1' \oplus \tau_2 = \tau_3 \Rightarrow \tau_1 = \tau_1'$ (cancellation)

$J5.\ \exists u.\ \forall \tau.\ \tau \oplus u = \tau$ (unit)

$J6.\ \tau_1 \oplus \tau_1 = \tau_2 \Rightarrow \tau_1 = \tau_2$ (disjointness)



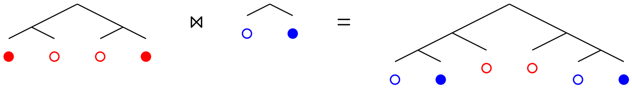
$J7.\ a \oplus b = z \wedge c \oplus d = z \Rightarrow \exists ac, ad, bc, bd.$

$ac \oplus ad = a \wedge bc \oplus bd = b \wedge ac \oplus bc = c \wedge ad \oplus bd = d$ (cross split)

$J8.\ \tau \neq \circ \Rightarrow \exists \tau_1, \tau_2.\ \tau_1 \neq \circ\ \wedge\ \tau_2 \neq \circ\ \wedge\ \tau_1 \oplus \tau_2 = \tau$ (infinite split)

Tree Share Operators(cont.)

The injective bowtie function $\bowtie$ replaces $\bullet$ with tree:



Properties of $\bowtie$

$\mathcal{S} = (\bowtie, \bullet)$ is a cancellative monoid (Dockins et al. (2009)):

*M*1. $(\tau_1 \bowtie \tau_2) \bowtie \tau_3 = \tau_1 \bowtie (\tau_2 \bowtie \tau_3)$	(associativity)
*M*2. $\tau \bowtie \bullet = \bullet \bowtie \tau = \tau$	(identity)
*M*3. $\tau \bowtie \tau_1 = \tau \bowtie \tau_2 \Rightarrow \tau \neq \circ \Rightarrow \tau_1 = \tau_2$	(left cancellation)
*M*4. $\tau_1 \bowtie \tau = \tau_2 \bowtie \tau \Rightarrow \tau \neq \circ \Rightarrow \tau_1 = \tau_2$	(right cancellation)
*M*5. $\tau \bowtie \circ = \circ \bowtie \tau = \circ$	(collapse point)
*M*6. $\tau_1 \bowtie (\tau_2 \diamond \tau_3) = (\tau_1 \diamond \tau_2) \bowtie (\tau_1 \diamond \tau_3),\ \diamond \in \{\sqcap, \sqcup, \oplus\}$	(distributivity)

Outline

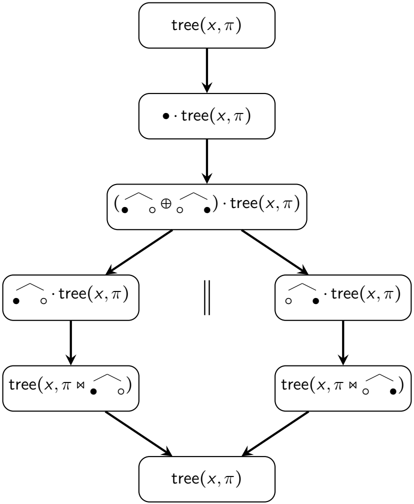
Scaling permissions for scaling predicates

$\mathcal{T} = (\mathbb{T}, \oplus, \bowtie)$ builds the *scaling permissions* for scaling predicates:

$$
\begin{array}{rclr}
\bullet \cdot P & \dashv\vdash & P & (\textsc{DotFull}) \\
(\pi_1 \oplus \pi_2) \cdot P & \dashv\vdash & (\pi_1 \cdot P) \ \ast \ (\pi_2 \cdot P) & (\textsc{DotPlus}) \\
\pi \cdot (P_1 \ast P_2) & \dashv\vdash & (\pi \cdot P_1) \ \ast \ (\pi \cdot P_2) & (\textsc{DotStar}) \\
\pi_1 \cdot (\pi_2 \cdot P) & \dashv\vdash & (\pi_2 \bowtie \pi_1) \cdot P & (\textsc{DotDot}) \\
\pi \cdot \mathsf{emp} & \dashv\vdash & \mathsf{emp} & (\textsc{DotEmp}) \\
\pi \cdot (x \overset{\pi'}{\mapsto} v) & \dashv\vdash & x \overset{\pi' \bowtie \pi}{\mapsto} v & (\textsc{DotMap})
\end{array}
$$

. . . with some side conditions.

Concrete example with tree shares



Ongoing work

We show that:

1. The scaling permissions work well with inductive predicates.
2. Reasonable side conditions can be checked syntactically.
3. A general induction framework for reasoning SL formulas with fractional permissions.
4. Our set of rules for scaling permissions is sound (Coq check).

Outline

Tree Shares as Countable Atomless Boolean Algebra

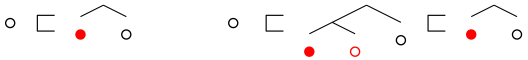
- $\mathcal{M} = (\mathbb{T}, \sqcup, \sqcap, \bar{\square}, \bullet, \circ)$ is Countable Boolean Algebra.
- Strict partial order:

$$\tau_1 \sqsubset \tau_2 \quad \stackrel{\text{def}}{=} \quad \tau_1 \sqcup \tau_2 = \tau_2 \wedge \tau_1 \neq \tau_2$$

- Atomless:

$$\circ \sqsubset \tau \ \rightarrow \ \exists \tau_1 .\ \circ \sqsubset \tau_1 \sqsubset \tau$$

- Example:



Decidability of $\mathcal{M} = (\mathbb{T}, \sqcup, \sqcap, \bar{\square}, \bullet, \circ)$

The first-order theory of $\mathcal{M}$ is decidable. Furthermore, it is:

1. $\bigcup_{c<\omega} \mathsf{STA}(*, 2^{cn}, n)$-complete if only $\{\bullet, \circ\}$ are allowed (Le et al. (2016)).
2. Still $\bigcup_{c<\omega} \mathsf{STA}(*, 2^{cn}, n)$-complete with arbitrary constants (under submission).

Decidability of $\bowtie$

Decidability of $\mathcal{S} = (\mathbb{T}, \bowtie)$ (Le et al. (2016))

Let $\mathcal{S} = (\mathbb{T}, \bowtie)$ then:

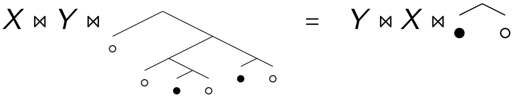
- The $\exists$-theory of $\mathcal{S}$ is decidable in PSPACE.
- The $\exists$-theory of $\mathcal{S}$ is NP-hard.
- The first-order theory of $\mathcal{S}$ is undecidable.

Main idea: Reduction to word equation.

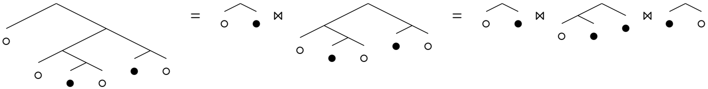
Isomorphism between $\bowtie$ and $\cdot$

Proof skech:

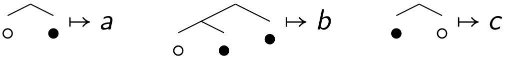
1. Tree equation:



2. Factorize tree into prime trees:



3. Prime trees are the alphabet:



4. Word equation: $XYabc = YXc$

Key observations

Unique tree factorization

Let $\tau \in \mathbb{T} \backslash \{\circ, \bullet\}$ then there exists a unique sequence of prime trees $\tau_1, \ldots, \tau_n$ such that:

$$\tau = \tau_1 \bowtie \ldots \bowtie \tau_n$$

Furthermore, the factorization problem is PTIME.

Decidability and Complexity of Word Equation (Marchenkov (1982); Plandowski (2006))

The satisfiability problem of word equation is decidable with:

- Lower bound: NP-hard
- Upper bound: PSPACE

Also, the first-order theory of word equation is undecidable.

Under submission

Restricted bowtie with constants on the left/right:

$$\bowtie_{\tau}(x) \overset{\text{def}}{=} x \bowtie \tau \qquad {}_{\tau}\!\bowtie(x) \overset{\text{def}}{=} \tau \bowtie x$$

Decidability of restricted $\bowtie$

The first-order theory of $(\mathbb{T}, \bowtie_{\tau}, {}_{\tau}\!\bowtie)$ is:

1. Lower bound: PSPACE-hard
2. Upper bound: 2EXSPACE

Proof sketch: Reduction to automatic structure with bounded degree.

Connection to Tree Automatic Structures

Restrict $\bowtie$ with constants on the right: $\bowtie_{\mathcal{T}}$.

Tree automatic structure (Le et al. (2016))

Let $\mathcal{T} = (\mathbb{T}, \sqcup, \sqcap, \bar{\square}, \bowtie_{\mathcal{T}})$ then:

- $\mathcal{T}$ is tree automatic
- The first-order theory of $\mathcal{T}$ is decidable [Blumensath (1999); Blumensath and Gradel (2004)].

Important for scaling permissions.

Under submission

- The FO of $\mathcal{T} = (\mathbb{T}, \sqcup, \sqcap, \bar{\square}, \bowtie_\tau)$ is non-elementary.
  - Key idea: Reduce to FO of two successors + prefix relation.

Under submission

- The FO of $\mathcal{T} = (\mathbb{T}, \sqcup, \sqcap, \bar{\square}, \bowtie_\tau)$ is non-elementary.
  - Key idea: Reduce to FO of two successors + prefix relation.
- No shame using tree automatic solvers (*e.g.* MONA)!

Outline

Integrate tree shares into HIP/SLEEK

- Extract share constraint from SL entailment:

$$x \overset{\bullet}{\mapsto} 1 \vdash x \overset{\widehat{\bullet \quad \circ}}{\mapsto} 1 \quad \leadsto \quad v_1 = \bullet \vdash \exists v_2.\ v_1 = \widehat{\bullet \quad \circ} \oplus v_2$$

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- Let $\Sigma = \{v_1 \oplus v_2 = v_3\}$. Two procedures (Le et al. (2012)):
  1. SAT(Σ)
  2. $\Sigma_1 \vdash \Sigma_2$

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- Let $\Sigma = \{v_1 \oplus v_2 = v_3\}$. Two procedures (Le et al. (2012)):
  1. SAT(Σ)
  2. $\Sigma_1 \vdash \Sigma_2$
- Sound and complete. Main idea: Small model property.

Implementation in Coq (under submission)

- Can handle negative constraints: $\neg(a \oplus b = c)$.
- Implementation + certified proof: 35k+ LOC. Can run entirely in Coq.
- Some bugs in the old tool and HIP/SLEEK ...

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  2. Heuristic rules + bound algorithm (Hobor and Gherghina (2012)).
  3. Faster: 4.6%.

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- Extracted to Ocaml or Haskell.

Outline

Contributions

- We show how model tree shares for scaling permission in SL.
- We investigate the decidability and complexity of tree shares.
- We certify tree share decision procedures in Coq and integrate them into HIP/SLEEK.

Future Work

- Generalize the tree share model to "generic binary trees".
- Close the complexity gap in certain tree share sub-structures.
- Develop decision procedures for $\mathcal{T} = (\mathbb{T}, \sqcup, \sqcap, \bar{\square}, \bowtie_{\mathcal{T}})$.
- Integrate the scaling permissions in HIP/SLEEK for benchmarking.

Thank you! ☺

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