Convexification for low-rank functions with indicator variables

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Agenda

Joint work with

Andrés Gómez
ISE, USC

A motivating example

Portfolio index tracking problem Construct a portfolio of securities to reproduce the performance of a stock market index

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\begin{aligned}
\min_{x,z}\ & (x - x_B)^\top Q (x - x_B) \\
\text{s.t. } & x \geq 0,\ \sum_{i \in [n]} x_i = 1 \\
& \|x\|_0 \leq m
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- $x_B \in \mathbb{R}^n$: benchmark index portfolio
- Q: covariance matrix of security returns
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MIQP reformulation

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- $z_i = 0 \Rightarrow x_i = 0$
- Covariances are estimated from a factor model (Bienstock (1996))

$$Q = FF^\top,$$

where $F \in \mathbb{R}^n \times \mathbb{R}^k,\ k \leq 20$ is small

Introduction

Consider

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Assumption f is a low-rank closed convex function

Definition (Rank; Rockafellar (1970))

The rank of f is the smallest integer k such that $f(x) = g(Ax) + c^\top x$ for some convex function $g : \mathbb{R}^k \to \mathbb{R}$ and matrix $A \in \mathbb{R}^{k\times n}$

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Examples

- $f(x) = c^\top x$, then $\text{rank}(f) = 0$
- $f(x) = g(a^\top x) + c^\top x$, then $\text{rank}(f) = 1$
- $f(x) = x^\top Qx + c^\top x$, then $\text{rank}(f) = \text{rank}(Q)$

To solve it efficiently, we study

$$\mathcal{Q} = \left\{ (t, x, z) \in \mathbb{R}^{n+1} \times \{0,1\}^n : \begin{array}{r} t \geq f(x), x_i \geq 0 \; \forall i \in \mathcal{I}_+ \\ x_i(1 - z_i) = 0 \; \forall i \in [n] \end{array} \right\}$$

$$\boxed{\text{Goal: Compute } \operatorname{cl}\operatorname{conv} \mathcal{Q}}$$

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Atamtürk and Gómez (2020):

- $f(x) = \left(a^\top x\right)^2$
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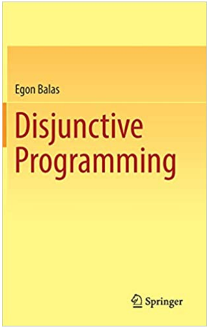
This talk: A more general approach based on disjunctive programming

Preliminaries: disjunctive programming

Egon Balas (1922∼2019)

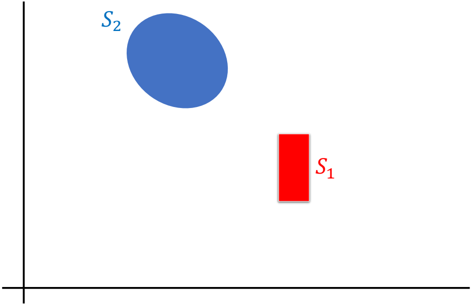


- Ceria and Soares (1999)
- Grossmann (2002)
- Aktürk et al. (2009)
- Kılınç-Karzan and Yıldız (2015)
- ...

Preliminaries: disjunctive programming

Assume one binary variable

$$\mathcal{X} \subseteq \mathbb{R}^2 \times \{0,1\}$$



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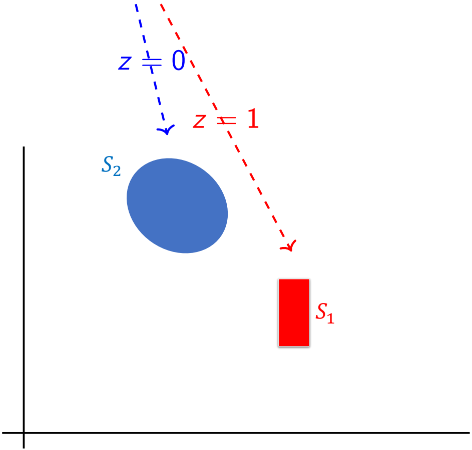

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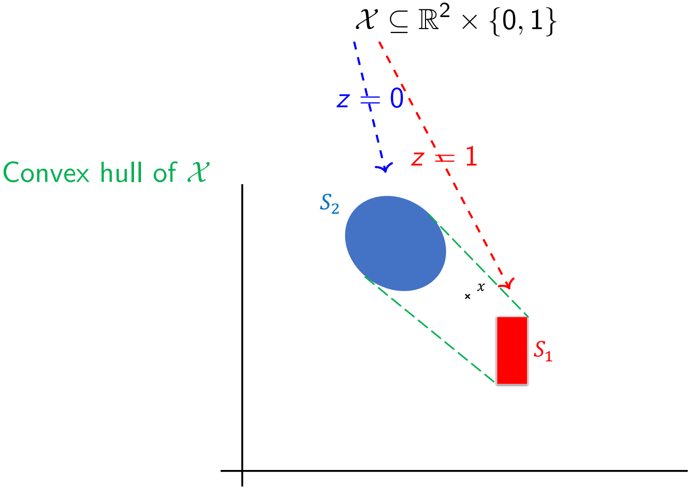

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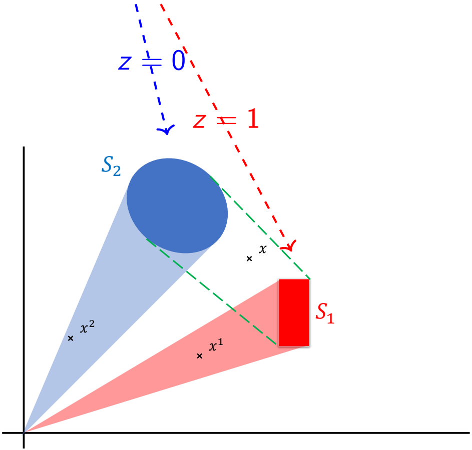
Convex hull of $\mathcal{X}$



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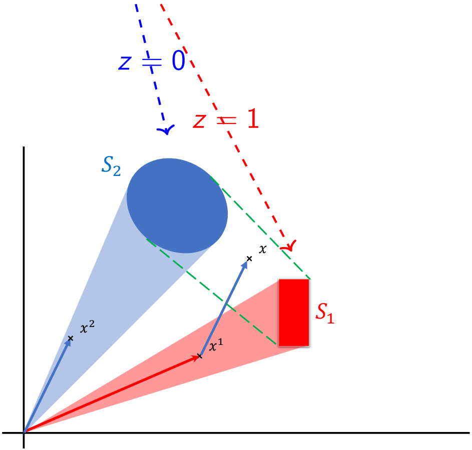

Preliminaries: disjunctive programming

For any mixed-binary set $\mathcal{X} \subseteq \mathbb{R}^m \times \{0,1\}^n$,

$$\mathcal{X} = \bigcup_{\bar{z} \in \{0,1\}^n} [\mathcal{X} \cap (\mathbb{R}^m \times \{\bar{z}\})]$$

- $\#$ of disjunctions $= 2^n$
- Disjunctive programming $\Rightarrow$ introduce additional variables to describe $\operatorname{conv} \mathcal{X}$ in a lifted space

$$\# \text{ of additional vars} \approx \dim(\mathcal{X}) \times \# \text{ of disjunctions} = \mathcal{O}((n+m)2^n)$$

$\Rightarrow$ Only applicable in practice when there are few binary variables

Agenda

1. Introduction
2. Main results - convex hull description
3. Conclusions

Homogeneous cases

Consider

$$\mathcal{Q} = \left\{ (t, x, z) \in \mathbb{R}^{n+1} \times \{0,1\}^n : \begin{array}{r} t \geq f(x), x_i \geq 0 \ \forall i \in \mathcal{I}_+ \\ x_i(1 - z_i) = 0 \ \forall i \in [n] \end{array} \right\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $\mathcal{I}_+ \subseteq [n]$

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Proposition (Han and Gómez (2021))

If f is positively homogeneous, i.e. $f(\lambda x) = \lambda f(x)$ for all $\lambda \geq 0$, then

$$\operatorname{cl}\operatorname{conv} \mathcal{Q} = \left\{ (t,x,z) \in \mathbb{R}^{n+1} \times [0,1]^n : \begin{array}{c} t \geq f(x), x_i \geq 0 \ \forall i \in \mathcal{I}_+ \\ \cancel{x_i(1-z_i) = 0 \ \forall i \in [n]} \end{array} \right\}.$$

New disjunctive representation in non-homogeneous cases

General (non-homogeneous) cases

Observe x_i and z_i are linked only through $x_i(1 - z_i) = 0$

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$$\mathcal{Q} = \bigcup_{\mathcal{I} \subseteq [n]} \mathcal{Q} \cap \{(t, x, z) : z_i = 1 \; \forall i \in \mathcal{I}, \; x_i = 0 \; \forall i \notin \mathcal{I}\}$$

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Still 2^n disjunctions! $\Rightarrow$ Exploit the low-rank structure $f(x) = g(Ax)$

Convex hull description of $\mathcal{Q}$

Theorem (Han and Gómez (2021))

Assume $\operatorname{rank}(f) \leq k$ *and* $f(0) = 0$. *Then*

$$\operatorname{cl}\operatorname{conv}(\mathcal{Q}) = \operatorname{cl}\operatorname{conv}\left(\left(\bigcup_{\mathcal{I}:|\mathcal{I}|\leq k} \mathcal{V}(\mathcal{I}) \cup \mathcal{R}\right)\right),$$

where

$$\mathcal{R} = \{(t, x, z) : t \geq 0, Ax = 0, x_i \geq 0,\ \forall i \in \mathcal{I}_+, z_i = 1,\ \forall i \in [n]\}$$

- $\mathcal{V}(\mathcal{I})$: "extreme points" of $\operatorname{cl}\operatorname{conv}\mathcal{Q}$
- $\mathcal{R}$: "extreme rays" of $\operatorname{cl}\operatorname{conv}\mathcal{Q}$
- $\mathcal{O}(n^k)$ number of disjunctions

Convex hull description of $\mathcal{Q}$

Proof outline.

Consider

$$\begin{aligned} &\min\ a^\top x + c^\top z + g(Ax) \\ &\text{s.t. } x_i \geq 0 \quad \forall i \in \mathcal{I}_+ \\ &\qquad x \circ (1 - z) = 0 \end{aligned} \tag{MINLP}$$

Assume $(\bar{x}, \bar{z})$ is the optimal solution to (MINLP).

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$$\begin{aligned} \min\ & a^\top x + g(A\bar{x}) \\ \text{s.t.}\ & Ax = A\bar{x} \\ & \bar{x}_i x_i \geq 0 \quad \forall i \\ & x_i = 0 \quad \forall i : \bar{z}_i = 0 \end{aligned} \tag{LP}$$

has an optimal solution $\hat{x}$ with at most $\text{rank}(A) = k$ nonzero entries. Moreover, $(\hat{x}, \bar{z})$ is optimal to (MINLP).

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Moreover, $(\hat{x}, \bar{z})$ is optimal to (MINLP).
$\Rightarrow (\hat{x}, \bar{z}) \in$ a certain $\mathcal{V}(\mathcal{I})$ with $|\mathcal{I}| \leq k$. $\square$

Preliminaries: Perspective function

Definition (Perspective function)

Given a closed convex function $f : \mathbb{R}^n \to \mathbb{R}$, its perspective function $f^\pi(x, \lambda)$ is defined as

$$f^\pi(x,\lambda) = \begin{cases} \lambda f\left(\frac{x}{\lambda}\right) & \text{if } \lambda > 0 \\ \lim\limits_{\lambda \to 0} \lambda f\left(\frac{x}{\lambda}\right) & \text{if } \lambda = 0 \\ +\infty & \text{o.w.} \end{cases}$$

- f^π is closed, convex, positively homogeneous

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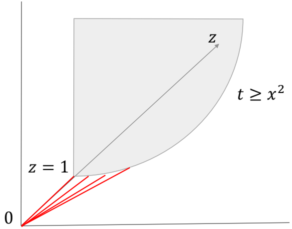
Example $f(x) = x^2$

$$\Rightarrow f^\pi(x,\lambda) = \begin{cases} x^2/\lambda & \text{if } \lambda > 0 \\ 0 & \text{if } (x,\lambda) = 0 \\ +\infty & \text{o.w.} \end{cases}$$



Rank-one case

Assume $f = g\left(\sum_{i=1}^{n} a_i x_i\right)$

Proposition (Han and Gómez (2021))

Point $(t, x, z) \in \operatorname{cl}\operatorname{conv} \mathcal{Q}$ *if and only if there exists* $\lambda, \tau \in \mathbb{R}^n$ *such that the following inequality system is consistent*

$$
\begin{aligned}
& t \geq \sum_{i=1}^{n} g^{\pi}(a_i(x_i - \tau_i), \lambda_i), \\
& a^\top \tau = 0,\ 0 \leq \tau_i \leq x_i\ \forall i \in \mathcal{I}_+, \\
& \lambda_i \leq z_i \leq 1\ \forall i \in [n], \\
& \lambda \geq 0,\ \sum_{i=1}^{n} \lambda_i \leq 1
\end{aligned}
$$

- Improve the results of Atamtürk and Gómez (2020) in several ways

More discussion over the rank-one convexification results

Comparison with Atamtürk and Gómez (2020)

- $\mathcal{Q}$ is defined by a quadratic function $g(t) = t^2$ and $\mathcal{I}_+ = [n]$
  Our results: g is an arbitrary convex function

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- $\operatorname{cl}\operatorname{conv}\mathcal{Q}$ is described by cutting surfaces with each cut requiring $\mathcal{O}(n)$ additional vars
  Our results: more compact extended formulation ($\mathcal{O}(n)$ in total)

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- $\operatorname{cl}\operatorname{conv}\mathcal{Q}$ is described by cutting surfaces with each cut requiring $\mathcal{O}(n)$ additional vars
  Our results: more compact extended formulation ($\mathcal{O}(n)$ in total)
- More efficient implementation $\Rightarrow$

Experimental results

Cutting surface implementation v.s. Extended formulation

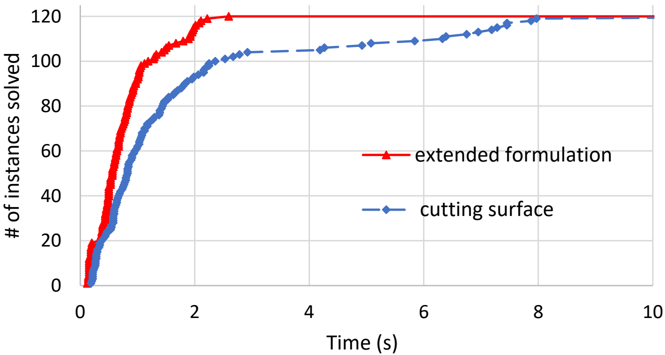

Figure: Number of instances solved as a function of time.

- Average time: cutting surface 1.79s v.s. extended formulation 0.78s
- Maximum time: cutting surface 13.3s v.s. extended formulation 2.6s

Extension - constraints over continuous variables

Additional constraints over continuous variables x?

Proposition (Han and Gómez (2021))

The box-constrained rank-one MIQP

$$\begin{aligned}\min_{x,z}\ & \left(a^\top x\right)^2 + b^\top x + c^\top z \\ \text{s.t. } & 0 \le x \le z, \\ & z \in \{0,1\}^n\end{aligned}$$

is $\mathcal{NP}$-hard.

Extension - constraints over continuous variables

Additional constraints over continuous variables x?

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The box-constrained rank-one MIQP

$$\begin{aligned} \min_{x,z} \ & \left(a^\top x\right)^2 + b^\top x + c^\top z \\ s.t.\ & 0 \leq x \leq z, \\ & z \in \{0,1\}^n \end{aligned}$$

is $\mathcal{NP}$-hard.

$\Rightarrow$ $\mathcal{NP}$-hard to describe the convex hull

Agenda

Take home message

- New DP representation for low-rank functions with indicator variables
- Compact extended formulation for convex hull description
- More efficient implementation in practice

Our paper is available at: https://arxiv.org/abs/2110.14884

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Thank You!

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