D	M	A

Ecuación de onda 2D en coordenadas cilíndricas

③ Demostrar 4.50:

$$u_{i,j}^{l+1} = \nu^2\left[u_{i+1,j}^{l} - 2u_{i,j}^{l} + u_{i-1,j}^{l} + \frac{\Delta\rho}{\rho_{[i]}}\left(u_{i,j}^{l} - u_{i-1,j}^{l}\right) + \left(\frac{\lambda}{\rho[i]}\right)^2\left(u_{i,j+1}^{l} - 2u_{i,j}^{l} + u_{i,j-1}^{l}\right)\right]$$
$$+2u_{i,j}^{l} - u_{i,j}^{l-1}$$

Con: $\lambda = \frac{\Delta\rho}{\Delta\phi}$ y $\nu = \alpha\frac{\Delta t}{\Delta\rho}$

Segun la sección 4.1.1, el operador de laplace en coordenadas cilíndricas es:

$$\nabla^2 u(\rho,\phi) = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta\rho)^2} + \frac{1}{\rho_{[i]}}\left(\frac{u_{i,j} - u_{i-1,j}}{\Delta\rho}\right) + \frac{1}{\rho_{[i]}^2}\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta\phi)^2}\right)$$

Para la ecuacion de onda:

$$\partial_{tt} u(\rho,\phi) = \alpha^2\nabla^2 u(\rho,\phi)$$

Al juntar ambas formulas y expresar en forma de diferencias finitas se obtiene:

$$\frac{u_{i,j}^{l+1} - 2u_{i,j}^{l} + u_{i,j}^{l-1}}{(\Delta t)^2} = \alpha^2\left[\frac{u_{i+1,j}^{l} - 2u_{i,j}^{l} + u_{i-1,j}^{l}}{(\Delta\rho)^2} + \frac{1}{\rho_{[i]}}\left(\frac{u_{i,j}^{l} - u_{i-1,j}^{l}}{\Delta\rho}\right) + \frac{1}{\rho_{[i]}^2}\left(\frac{u_{i,j+1}^{l} - 2u_{i,j}^{l} + u_{i,j-1}^{l}}{(\Delta\phi)^2}\right)\right]$$

Despejando $u_{i,j}^{l+1}$:

$$u_{i,j}^{l+1} = (\Delta t)^2\alpha^2\left[\text{"} \qquad \text{"}\right] + 2u_{i,j}^{l} - u_{i,j}^{l-1}$$

Usando la tecnica usada en la sección 4.1.1 se multiplica en ambos lados del igual $(\Delta\rho)^2$ y posteriormente se usa el factor $\lambda = \frac{\Delta\rho}{\Delta\phi}$

↳ De la forma $\frac{\partial^2 u}{\partial t^2}\Delta\rho^2 = \alpha^2\nabla^2 u\,\Delta\rho^2$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\alpha^2}{\Delta\rho^2}\left[\nabla^2 u\,\Delta\rho^2\right]$$

→ Para que $\Delta\rho^2$ no afecte $+2u_{i,j}^{l} - u_{i,j}^{l-1}$

$$u_{i,j}^{l+1} = \frac{(\Delta t)^2 \alpha^2}{(\Delta \rho)^2}\left[u_{i+1,j}^{l} - 2u_{i,j}^{l} + u_{i-1,j}^{l} + \frac{\Delta \rho}{\rho[i]}\left(u_{i,j}^{l} - u_{i-1,j}^{l}\right) + \frac{\Delta \rho^2}{\rho[i]^2 \Delta \Phi^2}\left(u_{i,j+1}^{l} - 2u_{i,j}^{l} + u_{i,j-1}^{l}\right)\right]$$
$$+ 2u_{i,j}^{l} - u_{i,j}^{l-1}$$

Con la finalidad de expresar la formula de forma explicita para el tiempo se usa $\nu = \alpha \frac{\Delta t}{\Delta \rho}$

$$\Rightarrow u_{i,j}^{l+1} = \nu^2\left[u_{i+1,j}^{l} - 2u_{i,j}^{l} + u_{i-1,j}^{l} + \frac{\Delta \rho}{\rho[i]}\left(u_{i,j}^{l} - u_{i-1,j}^{l}\right) + \frac{\lambda^2}{\rho[i]^2}\left(u_{i,j+1}^{l} - 2u_{i,j}^{l} + u_{i,j-1}^{l}\right)\right]$$
$$+ 2u_{i,j}^{l} - u_{i,j}^{l-1}$$