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Solutions to Warming up Exercises

Analysis for CS

GROUPWORK:

(G 1)

a) We prove (using mathematical induction) that the statement $P(n)$: "Every real numbers $x_1, \dots, x_n \geq 1$ such that $x_i x_j \geq 0$, for $i, j \in \{1, \dots, n\}$, satisfy the inequality

$$(1 + x_1) \dots (1 + x_n) \geq 1 + x_1 + \dots + x_n."$$

holds true, for all $n \in \mathbb{N}^*$.

I (The first step): Proposition $P(1)$ does obviously hold.

II (The second step): Assume that $P(n)$ does hold for some $n \in \mathbb{N}^*$. Let $x_1, \dots, x_n, x_{n+1} \geq 1$ be such that $x_i x_j \geq 0$, for $i, j \in \{1, \dots, n+1\}$. According to our assumption, we have that

$$(1 + x_1) \dots (1 + x_n) \geq 1 + x_1 + \dots + x_n.$$

Multiplying both sides of the above inequality by the nonnegative number $1 + x_{n+1}$, we get that

$$(1 + x_1) \dots (1 + x_n)(1 + x_{n+1}) \geq 1 + x_1 + \dots + x_n + x_{n+1} + x_1 x_{n+1} + \dots + x_n x_{n+1}.$$

Since $x_i x_{n+1} \geq 0$ for all $i \in \{1, \dots, n\}$, we obtain

$$(1 + x_1) \dots (1 + x_n)(1 + x_{n+1}) \geq 1 + x_1 + \dots + x_n + x_{n+1}.$$

Hence $P(n+1)$ holds true. Thus proposition $P(n)$ holds true, for all $n \in \mathbb{N}^*$.

b) Taking $x_1 = \dots = x_n = x$ in the generalized Bernoulli-inequality, we get the Bernoulli-inequality.

(G 2)

This exercise has been discussed in detail in the exercise-class.

(G 3)

a) The inequality is a direct consequence of the AM-GM inequality.

b) The inequality follows from the AM-GM inequality, taking into account Remark 2) and the fact that $1 + 2 + \dots + n = \frac{n(n+1)}{2}$.

HOMEWORK:

(H 1)

(a) We start by computing in a direct manner the sum

$$S := 1^2 + 2^2 + ... + n^2.$$

Consider the following difference

$$\begin{aligned}\sum_{k=1}^{n}[(1+k)^3 - k^3] &= (2^3 - 1^3) + (3^3 - 2^3) + ... + (n^3 - (n-1)^3) + ((n+1)^3 - n^3) \\ &= (n+1)^3 - 1 \\ &= n^3 + 3n^2 + 3n.\end{aligned} \tag{1}$$

On the other hand, by changing the approach, we have

$$\begin{aligned}\sum_{k=1}^{n}[(1+k)^3 - k^3] &= \sum_{k=1}^{n}(3k^2 + 3k + 1) = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n \\ &= 3S + \tfrac{n(n+1)}{2} + n = 3S + 3\tfrac{n^2}{2} + 5\tfrac{n}{2}\end{aligned}. \tag{2}$$

From (1) and (2) we get

$$n^3 + 3n^2 + 3n = 3S + 3\frac{n^2}{2} + 5\frac{n}{2},$$

hence

$$S = \frac{n(n+1)(2n+1)}{6}.$$

Let us proceed by proving through mathematical induction that the statement

$$P(n) : 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

holds for all $n \in \mathbb{N}^*$.

I. Statement $P(1)$ is clearly true.

II. We prove that $P(k) \Longrightarrow P(k+1)$. As we assume that $P(k)$ holds for some $k \in \mathbb{N}^*$, we have

$$1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

By adding to the above equality the term $(k+1)^2$ we get

$$\begin{aligned}1^2 + 2^2 + ... + k^2 + (k+1)^2 &= \tfrac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)\tfrac{2k^2+k+6k+6}{6} = (k+1)\tfrac{2k^2+7k+6}{6} \\ &= \tfrac{(k+1)(k+2)(2(k+1)+1)}{6}.\end{aligned}$$

Thus statement $P(k+1)$ holds true.

From the two steps of mathematical induction we have that $P(n)$ holds true, for all $n \in \mathbb{N}^*$.

(b) We proceed by computing in a direct way the sum

$$1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n!.$$

First of all let us notice that for an arbitrary $k \in \mathbb{N}^*$, we have

$$(k+1)! = (k+1)k! = k \cdot k! + k!,$$

hence

$$k \cdot k! = (k+1)! - k!.$$

Then, we obtain the following equality

$$1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! = (2! - 1!) + (3! - 2!) + ...(n! - (n-1)!) + ((n+1)! - n!) = (n+1)! - 1.$$

We continue by proving through mathematical induction that the statement

$$P(n) : 1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! = (n+1)! - 1.$$

holds true for every $n \in \mathbb{N}^*$.

I. Statement $P(1)$ is obviously true.

II. We prove that $P(k) \Longrightarrow P(k+1)$ for $k \in \mathbb{N}^*$. From $P(k)$ we know that

$$1 \cdot 1! + 2 \cdot 2! + ... + k \cdot k! = (k+1)! - 1.$$

By adding $(k+1) \cdot (k+1)!$ to the equality above we get

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + ... + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+2)(k+1)! - 1 = (k+2)! - 1. \end{aligned} \tag{3}$$

Hence $P(k+1)$ does hold. Thus, due to mathematical induction, statement $P(n)$ holds true, for all $n \in \mathbb{N}^*$.

(H 2)

The proofs are almost similar to those done in (G 1).

(H 3)

These inequalities are proved in the third course.

(H 4)

Consider an arbitrary rectangle with edges having the length l and the width w. It is well known that its area is $A = l \cdot w$ and its perimeter is $P = 2(l + w)$. By applying the AM-GM inequality for $n = 2$, $x_1 = l$ and $x_2 = w$, we get

$$\sqrt{l \cdot w} \leq \frac{l + w}{2}.$$

The above inequality can be rewritten in terms of the area and the perimeter as

$$\sqrt{A} \leq \frac{P}{4}.$$

In the particular case of squares we have $l = w$ and

$$\sqrt{A_{square}} = \frac{P_{square}}{4}.$$

Taking now rectangles with $A = A_{square}$, we get from the above relations that $\frac{P_{square}}{4} \leq \frac{P}{4}$, implying that the square has the smallest perimeter amongst all rectangles of equal area.