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Exercise Sheet no.3

Analysis for CS

GROUPWORK:

(G 9)

Compute the limit of the sequences having the general term defined as follows:

a) $\left(1+\frac{1}{-n^3+3n}\right)^{n^2-n^3}$, b) $(3n^2+5)\ln\left(1+\frac{1}{n^2}\right)$, c) $\frac{n^n}{1+2^2+3^3+\cdots+n^n}$,

d) $\frac{x_1+2x_2+\cdots+nx_n}{n^2}$, where $(x_n)_{n\geq 1}$ is a sequence converging to $x \in \mathbb{R}$.

(G 10) (A sequence approximating $\frac{1}{a}$)

Let $a > 0$, and fix $x_0 \in \mathbb{R}$ such that $0 < x_0 < \frac{1}{a}$. Define $(x_n)_{n\in\mathbb{N}}$ recursively as

$$x_{n+1} = 2x_n - ax_n^2, \ \forall n \in \mathbb{N}.$$

Prove that $(x_n)_{n\in\mathbb{N}}$ converges to $\frac{1}{a}$, keeping in mind the following steps:

(i) Prove (using mathematical induction) that $x_n < \frac{1}{a}$, $\forall n \in \mathbb{N}$.

(ii) Prove (using mathematical induction) that $0 < x_n$, $\forall n \in \mathbb{N}$.

(iii) Using (i) and (ii), prove that $(x_n)_{n\in\mathbb{N}}$ is strictly increasing.

(iv) Finally conclude that $(x_n)_{n\in\mathbb{N}}$ is convergent and that $\lim\limits_{n\to\infty} x_n = \dfrac{1}{a}$.

(G 11) (Train your brain)

Prove Th 6 (concerning limits and boundedness properties) in the third course.

Homework:

(H 11) (To be delivered in the next exercise-class)

1) Compute the limit of the sequences having the general term defined as follows:

a) $\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{n}$, b) $\frac{\sqrt{1+2^2}+\sqrt{1+3^2}+\cdots+\sqrt{1+n^2}}{1+n^2}$,

c) $\frac{x_0+2^1x_1+2^2x_2+\cdots+2^nx_n}{2^{n+1}}$, where $(x_n)_{n\geq 0}$ is a sequence converging to $x \in \mathbb{R}$.

2) Prove statement 2° of Th 7 in the third course: If $(x_n)_{n\in\mathbb{N}^*}$ is a decreasing sequence and if X is the set consisting of all its terms, then $\lim\limits_{n\to\infty} x_n = \inf X$.

(H 12)

a) Prove (either by a direct computation or by mathematical induction) that the following equalities hold for every natural number $n \geq 2$

$$\left(1+\frac{1}{1}\right)^1\left(1+\frac{1}{2}\right)^2\left(1+\frac{1}{3}\right)^3\cdots\left(1+\frac{1}{n-1}\right)^{n-1} = \frac{n^n}{n!},$$

$$\left(1+\frac{1}{1}\right)^2\left(1+\frac{1}{2}\right)^3\left(1+\frac{1}{3}\right)^4\cdots\left(1+\frac{1}{n-1}\right)^{n} = \frac{n^n}{(n-1)!}.$$

b) Using a) and the following inequalities (proved in the third course)

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}, \ \forall n \in \mathbb{N}^*,$$

show that

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n, \ \forall n \in \mathbb{N}^*.$$

c) Using b) and the Sandwich-Theorem, prove that $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.