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Solutions to Exercise Sheet no.7

Analysis for CS

(G 18)

a) We have for every $x \in \left[\frac{1}{2}, 3\right]$ that

$$f'(x) = \frac{\cos\sqrt{x}}{2\sqrt{x}} \text{ and } f''(x) = -\frac{\cos\sqrt{x}}{2x\sqrt{x}} - \frac{\sin\sqrt{x}}{4x}.$$

Thus $f(1) = \sin 1$, $f'(1) = \frac{\cos 1}{2}$, $f''(1) = -\frac{\sin 1 + \cos 1}{4}$, hence

$$T_2(x, 1) = \sin 1 + \frac{\cos 1}{2}(x-1) - \frac{\sin 1 + \cos 1}{8}(x-1)^2.$$

b) We have for every $x \in \left[\frac{1}{2}, 3\right]$ that

$$f^{(3)}(x) = \frac{3\cos\sqrt{x}}{8x^2\sqrt{x}} + \frac{\sin\sqrt{x}}{8x^2} + \frac{\sin\sqrt{x}}{4x^2} - \frac{\cos\sqrt{x}}{8x\sqrt{x}}.$$

If $x \in \left[\frac{1}{2}, 3\right] \setminus \{1\}$, then, according to Taylor's formula, there exists c strictly between x and 1 such that

$$R_2(x, 1) = \frac{f^{(3)}(c)}{3!}(x-1)^3.$$

(G 19)

a) We have $\sin^{(2n)} = (-1)^n \sin$ and $\sin^{(2n+1)} = (-1)^n \cos$, for every $n \in \mathbb{N}$.

b) We get from a) that $\sin^{(2n)}(0) = 0$ and $\sin^{(2n+1)}(0) = (-1)^n$, for every $n \in \mathbb{N}$. Thus

$$T_{2n+1}(x, 0) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \forall\, n \in \mathbb{N}, \text{ and } T_{2n}(x, 0) = T_{2n-1}(x, 0), \forall\, n \in \mathbb{N}^*.$$

c) Let $x \in \mathbb{R}$. We know from Taylor's formula that there exists c between 0 and x such that

$$R_n(x, 0) = \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

It follows that

$$|R_n(x, 0)| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Taking into account that $\lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, we conclude that $\lim\limits_{n\to\infty} R_n(x, 0) = 0$. Taylor's theorem finally yields the following Taylor series expansion for the sine function

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \forall\, x \in \mathbb{R}.$$

We analyze the same requirements for cos.

a) We have $\cos^{(2n)} = (-1)^n \cos$ and $\cos^{(2n+1)} = (-1)^{n+1} \sin$, for every $n \in \mathbb{N}$.

b) We get from a) that $\cos^{(2n)}(0) = (-1)^n$ and $\cos^{(2n+1)}(0) = 0$, for every $n \in \mathbb{N}$. Thus

$$T_{2n}(x,0) = 1 - \frac{1}{2!}x^2 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n}, \forall\, n \in \mathbb{N}, \text{ and } T_{2n+1}(x,0) = T_{2n}(x,0), \forall\, n \in \mathbb{N}.$$

c) Let $x \in \mathbb{R}$. We know from Taylor's formula that there exists c between 0 and x such that

$$R_n(x,0) = \frac{\cos^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

It follows that

$$|R_n(x,0)| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Taking into account that $\lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, we conclude that $\lim\limits_{n\to\infty} R_n(x,0) = 0$. Taylor's theorem finally yields the following Taylor series expansion for the cosine function

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}, \forall\, x \in \mathbb{R}.$$

(G 20)

a) $(e^{3x})^{(n)} = 3^n e^{3x}, \forall\, n \in \mathbb{N},\ x \in \mathbb{R}$.

b) We apply the formula of Leibniz for $f(x) = x^2$, $g(x) = \sin 2x$, and take also into account that, for all $n \in \mathbb{N}$, the following equalities hold true $(\sin 2x)^{(4n)} = 2^{4n} \sin 2x$, $(\sin 2x)^{(4n+1)} = 2^{4n+1} \cos 2x$, $(\sin 2x)^{(4n+2)} = -2^{4n+2} \sin 2x$, $(\sin 2x)^{(4n+3)} = -2^{4n+3} \cos 2x$, hence

$$\begin{aligned}(x^2 \sin 2x)^{(100)} &= -2^{99} C_{100}^{98} \sin 2x - 2^{100} C_{100}^{99} x \cos 2x + 2^{100} x^2 \sin 2x \\ &= 2^{100}(-2475 \sin 2x - 100x \cos 2x + x^2 \sin 2x).\end{aligned}$$

c) We apply the formula of Leibniz for $f(x) = x^3 + 2x - 1$, $g(x) = e^{2x}$. We notice that $f'(x) = 3x^2 + 2$, $f''(x) = 6x$, $f^{(3)}(x) = 6$ and $f^{(4)}(x) = 0$. Moreover, $f^{(n)}(x) = 0$ for all $n \ge 4$. Like in the case a) of this exercise, $(e^{2x})^{(n)} = 2^n e^{2x}$ for all $n \in \mathbb{N}$. If $n \ge 3$ we thus get

$$\begin{aligned}\left((x^3+2x-1)e^{2x}\right)^{(n)} &= C_n^{n-3} 6 \cdot 2^{n-3} \cdot e^{2x} + C_n^{n-2} 6x \cdot 2^{n-2} \cdot e^{2x} + C_n^{n-1}(3x^2+2) \cdot 2^{n-1} \cdot e^{2x} \\ &\quad + C_n^n (x^3+2x-1) \cdot 2^n \cdot e^{2x} = \\ &= 2^{n-3} e^{2x} \left(n(n-1)(n-2) + n(n-1)6x + n(3x^2+2)4 + (x^3+2x-1)8\right) = \\ &= 2^{n-3} e^{2x} \left(8x^3 + 12nx^2 + (6n(n-1)+16)\,x + n(n-1)(n-2) + 8n - 8\right).\end{aligned}$$

We further have

$$\left((x^3+2x-1)e^{2x}\right)' = (2x^3+3x^2+4x)e^{2x} \text{ and } \left((x^3+2x-1)e^{2x}\right)'' = (4x^3+12x^2+14x+4)e^{2x}.$$

Homework:

(H 18)

a) First of all bare in mind the following equalities:

$$\sin\left(x+\frac{\pi}{2}\right)=\sin x\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\cos x=\cos x$$

and

$$\cos\left(x+\frac{\pi}{2}\right)=\cos x\cos\frac{\pi}{2}-\sin x\sin\frac{\pi}{2}=-\sin x.$$

We use mathematical induction. First of all we are going to prove the proposition:

$$P(n):\sin^{(n)}(x)=\sin\left(x+n\frac{\pi}{2}\right),\ \text{ for } n\in\mathbb{N}.$$

I. P(1): $\sin' x=\cos x=\sin(x+1\cdot\frac{\pi}{2})$, thus $P(1)$ is verified.

II. $P(k)\Longrightarrow P(k+1)$. We know now that $\sin^{(k)}(x)=\sin\left(x+k\frac{\pi}{2}\right)$. Then

$$\sin^{(k+1)}(x)=\left(\sin^{(k)}(x)\right)'=\sin'\left(x+k\frac{\pi}{2}\right)=\cos\left(x+k\frac{\pi}{2}\right)=\sin\left(x+(k+1)\frac{\pi}{2}\right).$$

Thus $P(k+1)$ holds true.

We move on to the proposition:

$$Q(n):\cos^{(n)}(x)=\cos\left(x+n\frac{\pi}{2}\right),\ \text{ for } n\in\mathbb{N}.$$

I. Q(1): $\cos' x=-\sin x=\cos(x+1\cdot\frac{\pi}{2})$, thus $Q(1)$ is verified.

II. $P(k)\Longrightarrow P(k+1)$. We know now that $\cos^{(k)}(x)=\cos\left(x+k\frac{\pi}{2}\right)$. Then

$$\cos^{(k+1)}(x)=\left(\cos^{(k)}(x)\right)'=\cos'\left(x+k\frac{\pi}{2}\right)=-\sin\left(x+k\frac{\pi}{2}\right)=\cos\left(x+(k+1)\frac{\pi}{2}\right).$$

Thus $Q(k+1)$ holds true.

b) We use the Leibniz formula. In order to do that, first bare in mind that $(e^x)^{(n)}=e^x$ and $e^{(-2x)}=(-2)^n e^x$ for all $n\in\mathbb{N}$. Then

$$(e^x\sin x)^{(n)}=\sum_{k=0}^{n}C_n^k(e^x)^{(n-k)}(\sin x)^{(k)}=\sum_{k=0}^{n}C_n^k e^x\sin\left(x+k\frac{\pi}{2}\right)=e^x\sum_{k=0}^{n}C_n^k\sin\left(x+k\frac{\pi}{2}\right)$$

and

$$\begin{aligned}(e^{-2x}\cos x)^{(n)}&=\sum_{k=0}^{n}C_n^k(e^{-2x})^{(n-k)}(\cos x)^{(k)}=\sum_{k=0}^{n}C_n^k(-2)^{n-k}e^x\cos\left(x+k\frac{\pi}{2}\right)\\&=e^x\sum_{k=0}^{n}C_n^k(-2)^{n-k}\cos\left(x+k\frac{\pi}{2}\right).\end{aligned}$$

(H 19)

a) Since $\lim\limits_{x\to\infty}e^{\alpha x}=\lim\limits_{x\to\infty}x=\infty$ and since $\lim\limits_{x\to\infty}\dfrac{(e^{\alpha x})'}{x'}=\lim\limits_{x\to\infty}\alpha e^{\alpha x}=\infty$, L'Hospital's rules yield

$$\lim_{x\to\infty}\frac{e^{\alpha x}}{x}=\infty.$$

b) Since the limit obtained at a) holds true for every positive α, we get that

$$\lim_{x\to\infty} \frac{e^{\alpha x}}{x^{\beta}} = \lim_{x\to\infty} \left(\frac{e^{\frac{\alpha}{\beta}x}}{x} \right)^{\beta} = \infty.$$

c) Since $\lim\limits_{x\to\infty} \ln x = \lim\limits_{x\to\infty} x^{\alpha} = \infty$ and since $\lim\limits_{x\to\infty} \dfrac{(\ln x)'}{(x^{\alpha})'} = \lim\limits_{x\to\infty} \dfrac{1}{\alpha x^{\alpha}} = 0$, L'Hospital's rules yield

$$\lim_{x\to\infty} \frac{\ln x}{x^{\alpha}} = 0.$$

d) Since the limit obtained at c) holds true for every positive α, we get that

$$\lim_{x\to\infty} \frac{(\ln x)^{\beta}}{x^{\alpha}} \lim_{x\to\infty} \left(\frac{\ln x}{x^{\frac{\alpha}{\beta}}} \right)^{\beta} = 0.$$

e) Let $x = \frac{1}{y}$. Using the result obtained at c), we then get

$$\lim_{\substack{x\to 0\\ x>0}} x^{\alpha} \ln x = \lim_{y\to\infty} \frac{-\ln y}{y^{\alpha}} = 0.$$

f) Using the result obtained at e), we get

$$\lim_{\substack{x\to 0\\ x>0}} x^{x} = \lim_{\substack{x\to 0\\ x>0}} e^{\ln x^{x}} = \lim_{\substack{x\to 0\\ x>0}} e^{x \ln x} = e^{0} = 1.$$