Geometry[1]
First Year, Computer science

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[1] These notes are not in a final form. They are continuously being improved

Content

The coplanarity condition of two straight lines

Using the notations of the previous section, observe that the straight lines d_1, d_2 are coplanar if and only if the vectors $\overrightarrow{A_1A_2}, d_1, d_2$ are linearly dependent (coplanar), or equivalently $(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2) = 0$. Consequently the stright lines d_1, d_2 are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0 \qquad (1.1)$$

The equations of the projection parallel to a given direction

Let π be a plan, let d be a straight line unparallel to π and let M be an arbitrary point in $\mathcal{P}$. Consider the straight line through M which is also parallel to d. This line punctures the plane π in a point denoted by $p_{\pi,d}(M)$. We call the map $\mathcal{P} \longrightarrow \pi, M \longrightarrow p_{\pi,d}(M)$ the *projection of the space on the plane* π *parallel to the line* d, or, the *projection of the space on the plane* π *parallel to the direction* $\overrightarrow{d}$, where $\overrightarrow{d}$ stands for a direction vector of the line d. We shall fine here the equations of some projections. More precisely, we shall find the coordinates of the point $p_{\pi,d}(M)$ in terms of the coordinates of M.

Assume that the space related to an orthonormal cartesian coordinate system $R = (0, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$, with respect to which the equations of the plane π and the straight line d are:

$$\pi : F(x, y, z) = Ax + By + Cz + D = 0$$

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

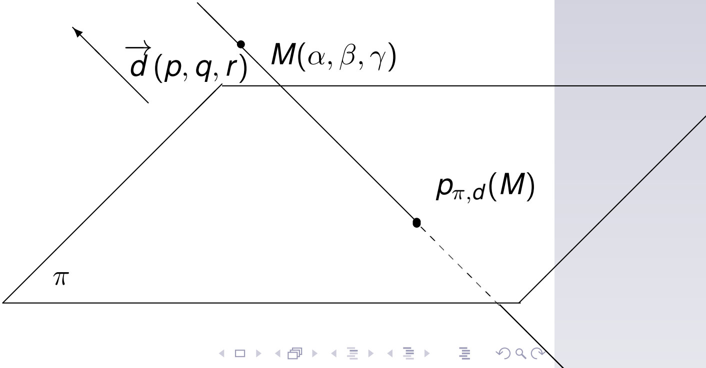
The unparallelism between the plane π and the straight line d is equivalent to $\overrightarrow{d} \cdot \overrightarrow{n}_\pi \neq 0$, where $\overrightarrow{d}(p, q, r)$ is the director vector of the line d, and $\overrightarrow{n}_\pi(A, B, C)$ is the normal vector of the plane π.

The equation of the line through the point $M(\alpha, \beta, \gamma)$ which has the same direction with the line d, is

$$\frac{x-\alpha}{p} = \frac{y-\beta}{q} = \frac{z-\gamma}{r},$$

and the parametric equations are:

$$\begin{cases} x = \alpha + pt \\ y = \beta + qt \\ z = \gamma + rt \end{cases}$$



In order to find the intersection point of this line with the plane π we shall solve the equation in t

$$A(\alpha + pt) + B(\beta + qt) + C(\gamma + rt) + D = 0.$$

This is equivalent to

$$A\alpha + B\beta + C\gamma + D + t(Ap + Bq + Cr) = 0$$

which shows that

$$t = -\frac{F(\alpha, \beta, \gamma)}{\overrightarrow{n}_\pi \cdot \overrightarrow{d}}.$$

Thus, the coordinate of the point $p_{\pi,d}(M)$ are:

$$\begin{cases} x = \alpha - \frac{F(\alpha,\beta,\gamma)}{\overrightarrow{n}_\pi \cdot \overrightarrow{d}} p \\ y = \beta - \frac{F(\alpha,\beta,\gamma)}{\overrightarrow{n}_\pi \cdot \overrightarrow{d}} q \\ z = \gamma - \frac{F(\alpha,\beta,\gamma)}{\overrightarrow{n}_\pi \cdot \overrightarrow{d}} r \end{cases}$$

Therefore, by identifying the point M with the triplet of its coordinates, we obtain:

$$p_{\pi,d}(\alpha,\beta,\gamma) = (\alpha,\beta,\gamma) - \frac{F(\alpha,\beta,\gamma)}{\overrightarrow{n}_\pi \cdot \overrightarrow{d}}(p,q,r).$$

If the vectors d and n are linearly dependent (collinear), then the projection is orhogonal and, in this case, its equations are:

$$p_{\pi,d}(\alpha,\beta,\gamma) = (\alpha,\beta,\gamma) - \frac{F(\alpha,\beta,\gamma)}{||\overrightarrow{n}_\pi||^2}(A,B,C).$$

The regions of the space devided by two unparallel planes

Consider the nonorthogonal planes

$$(\pi_1) P_1(x, y, z) = A_1 x + B_1 y + C_1 z + D_1 = 0$$

$$(\pi_2) P_2(x, y, z) = A_2 x + B_2 y + C_2 z + D_2 = 0$$

such that

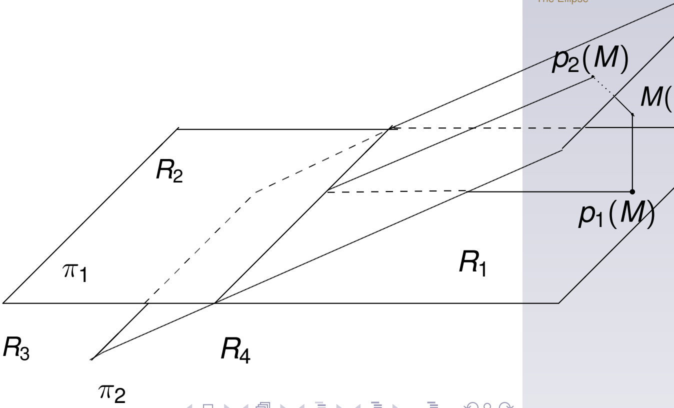
$$\mathrm{rang}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$$

(The space is related to a cartesian orthogonal reference system). The planes π_1, π_2 devide the space into four regions, two of which, say $\mathcal{R}_1$ and $\mathcal{R}_3$, correspond to the acute dihedral angle of the two planes.

Proposition 1.1

The poin $M(x, y, z)$ belong to these regions, that is to the union $\mathcal{R}_1 \cup \mathcal{R}_3$, if nad only if

$$P_1(x, y, z) \cdot P_2(x, y, z)(A_1A_2 + B_1B_2 + C_1C_2) < 0.$$



Proof.

$$\begin{aligned} M(x,y,z) \in \mathcal{R}_1 \cup \mathcal{R}_3 \quad &\Leftrightarrow m(\widehat{\overrightarrow{Mp_1}(M), \overrightarrow{Mp_2}(M)}) > 90^\circ \\ &\Leftrightarrow \overrightarrow{Mp_1}(M) \cdot \overrightarrow{Mp_2}(M) < 0, \end{aligned}$$

where $p_1(M), p_2(M)$ are the projections of the point M on the planes π_1 and π_2 respectively. Thus, the equations are:

$$p_i(x,y,z) = (x,y,z) - \frac{P_i(x,y,z)}{||\overrightarrow{n}_{\pi_i}||^2}(A_i, B_i, C_i), i \in \{1,2\},$$

which implies that

$$M\overrightarrow{p_i}(M) = -\frac{P_i(x,y,z)}{||\overrightarrow{n}_{\pi_i}||^2} n_{\pi_i}, i \in \{1,2\}.$$

Therefore we have:

$$M(x,y,z) \in \mathcal{R}_1 \cup \mathcal{R}_3 \Leftrightarrow \left(-\frac{P_i(x,y,z)}{||\overrightarrow{n}_{\pi_1}||^2}\overrightarrow{n}_{\pi_1}\right) \cdot \left(-\frac{P_i(x,y,z)}{||\overrightarrow{n}_{\pi_2}||^2}\overrightarrow{n}_{\pi_2}\right) < 0$$

$$\Leftrightarrow \frac{P_1(x,y,z) \cdot P_2(x,y,z)}{||\overrightarrow{n}_{\pi_1}||^2 \cdot ||\overrightarrow{n}_{\pi_2}||^2}(n_{\pi_1} \cdot \overrightarrow{n}_{\pi_2}) < 0$$

$$\Leftrightarrow P_1(x,y,z) \cdot P_2(x,y,z)(\overrightarrow{n}_{\pi_1} \cdot \overrightarrow{n}_{\pi_2}) < 0$$

$$\Leftrightarrow P_1(x,y,z) \cdot P_2(x,y,z)(A_1A_2 + B_1B_2 + C_1C_2) < 0. \square$$

Conics (This chapter is done following [AnTo]) The Ellipse

Definition 2.1

An ellipse is the locus of points in a plane whose sum of distances to two fixed points F_1 and F_2, called foci, *is constant.*

The distance between the two fixed points is called the *focal distance*

Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$. If M is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

One may chose a Cartesian system of coordinates centered at the midpoint of the segment $[F'F]$, so that $F(c, 0)$ and $F'(-c, 0)$.

Remark 2.2

In $\Delta MFF'$ the following inequality $|MF| + |MF'| > |FF'|$ holds. Hence $2a > 2c$. Thus, the constants a and c must verify $a > c$.

Let us determine the equation of an ellipse. Starting with the definition, $|MF| + |MF'| = 2a$, or

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

This is equivalent to

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

and thus

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2.$$

We therefore obtain

$$a\sqrt{(x+c)^2 + y^2} = cx + a^2,$$

which leads us to

$$a^2(x^2+2xc+c^2)+a^2y^2=c^2x^2+2a^2cx+a^2,$$

and therefore to

$$(a^2-c^2)x^2+a^2y^2-a^2(a^2-c^2)=0.$$

Denoting $a^2-c^2=b^2$ $(a>c)$, we deduce that

$$b^2x^2+a^2y^2-a^2b^2=0.$$

Dividing by a^2b^2, we obtain the equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0. \tag{2.1}$$

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