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Solutions to Exercise Sheet no.4

Analysis for CS

(G 12)

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \infty$, being a generalized harmonic series with $\alpha := \frac{1}{3} \leq 1$.

b) Using the rules of calculation for convergent series and the formula for the sum of the geometric series, we get $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^n} = \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{3}{4}\cdot\frac{1}{1-\frac{1}{4}} = 1.$

c) Using the rules of calculation for convergent series and the formula for the sum of the geometric series, we get, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{3^{n-1}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}} = \frac{1}{2}.$

d) The following equalities hold for every $n \geq 1$

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

Put $a_n := \frac{1}{2}\cdot\frac{1}{2n-1}$ for $n \geq 1$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty}(a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_n = \frac{1}{2}.$$

e) Since for every $n \geq 1$

$$\frac{2n+1}{n!} = \frac{2}{(n-1)!} + \frac{1}{n!},$$

we get, using the rules of calculation for convergent series,

$$\sum_{n=1}^{\infty} \frac{2n+1}{n!} = 2\sum_{n=0}^{\infty}\frac{1}{n!} + \sum_{n=1}^{\infty}\frac{1}{n!} = 2e + e - 1 = 3e - 1.$$

f) The following equalities hold for every $n \geq 1$

$$\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})} = \sqrt{n+1}-\sqrt{n}.$$

Set $a_n := \sqrt{n}$ for $n \geq 1$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}} = \sum_{n=1}^{\infty}(a_{n+1} - a_n) = \lim_{n\to\infty} a_n - a_1 = \infty.$$

g) The following equalities hold for every $n \geq 1$

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)\frac{1}{n+1} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right).$$

Let $a_n := \frac{1}{2} \cdot \frac{1}{n(n+1)}$ for $n \geq 1$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_n = \frac{1}{4}.$$

h) The following equalities hold for every $n \geq 1$

$$\frac{1}{n! + (n+1)!} = \frac{1}{n!(n+2)} = \frac{n+1}{(n+2)!} = \frac{1}{(n+1)!} - \frac{1}{(n+2)!}.$$

Put $a_n := \frac{1}{(n+1)!}$ for $n \geq 0$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=0}^{\infty} \frac{1}{n! + (n+1)!} = \sum_{n=0}^{\infty} (a_n - a_{n+1}) = a_0 - \lim_{n\to\infty} a_n = 1.$$

Homework:

(H 13)

a) Using the rules of calculation for convergent series and the formula for the sum of the geometric series, we get $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{-3}{4}\right)^n = \frac{1}{1+\frac{3}{4}} = \frac{4}{7}$.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \infty$, being a generalized harmonic series with $\alpha := \frac{1}{5} \leq 1$.

c) The following equalities hold for every $n \geq 2$

$$\begin{aligned}\ln\left(1 - \frac{1}{n^2}\right) &= \ln(n^2-1) - \ln n^2 = \ln(n+1)(n-1) - 2\ln n = \ln(n+1) + \ln(n-1) - 2\ln n \\ &= (\ln(n+1) - \ln n) - (\ln n - \ln(n-1)).\end{aligned}$$

Let $a_n := \ln n - \ln(n-1) = \ln \frac{n}{n-1}$ for $n \geq 2$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty} (a_{n+1} - a_n) = \lim_{n\to\infty} a_n - a_2 = -\ln 2.$$

d) Using the rules of calculation for convergent series and the formula for the sum of the geometric series, we get

$$\sum_{n=0}^{\infty} \left(-\frac{2}{(n+1)!} + \frac{(-1)^{n+1}}{3^{n+2}}\right) = (-2)\sum_{n=1}^{\infty} \frac{1}{n!} - \frac{1}{9}\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n = -2(e-1) - \frac{1}{12} = \frac{23}{12} - 2e.$$

(H 14)

a) The following equalities hold for every $n \geq 1$

$$\begin{aligned}\frac{n}{(n+1)(n+2)(n+3)} &= \frac{n+1-1}{(n+1)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)} \\ &= \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \frac{1}{(n+1)(n+2)(n+3)}.\end{aligned}$$

Let $a_n := \frac{1}{n+2}$ for $n \geq 1$. Using the rules of calculation for convergent series, (G 12) g) and the formula for the sum of a telescopic series, we get

$$\sum_{n\geq 1} \frac{n}{(n+1)(n+2)(n+3)} = \sum_{n=1}^{\infty}(a_n - a_{n+1}) - \sum_{n=2}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{3} - \left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{4}.$$

b) The following equalities hold for every $n \geq 2$

$$\frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)} = \frac{\ln(n+1) - \ln n}{\ln n \cdot \ln(n+1)} = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}.$$

Put $a_n := \frac{1}{\ln n}$ for $n \geq 2$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)} = \sum_{n=2}^{\infty}(a_n - a_{n+1}) = a_2 - \lim_{n\to\infty} a_n = \frac{1}{\ln 2}.$$