BRAD ALEXANDRU
Gr. 911

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Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems.
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1. Find the general solutions of the following differential equations:

$$x' = -x\,, \quad x' = 3x + 2 - 3t + e^{-3t}\,, \quad x'' - x' + 2x = 0\,, \quad x''' = 0.$$

2. We consider the differential equation

$$y' = \frac{1 - \sqrt[3]{y}}{1 - xy}$$

and three Initial Value Problems for it with the conditions: $y(0) = 1$, $y(1) = 1$ and $y(0) = 0$, respectively. Here the unknown function is $y = y(x)$.

a) Are the above Initial Value Problems well–defined?

b) If they are well–defined, decide whether or not the Local Existence and Uniqueness Theorem is applicable.

c) If the Local Existence and Uniqueness Theorem is applicable, find the solution.

3. Find the differential equation of the family of planar curves described by $x^2 + 9y^2 = c$, $c \in \mathbb{R}$. Find also a planar autonomous system whose trajectories are these curves.

4. We consider the logistic map $f_\lambda : [0,1] \to [0,1]$ $f_\lambda(x) = \lambda x(1-x)$, where $\lambda \in (0,4)$ is a parameter. Find the fixed points of the logistic map and study their stability (discuss with respect to the parameter λ).

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Written exam

1) a) $x' = -x$

$x' + x = 0$

The ch. equation: $r + 1 = 0 \Rightarrow r = -1 \Rightarrow e^{-t}$

$x = c \cdot e^{-t}, c \in \mathbb{R}$

b) $x' = 3x + 2 - 3t + e^{-3t}$

$x' - 3x = 2 - 3t + e^{-3t}$

The lin. homogeneous diff. eq. associated:

$x' - 3x = 0$

The ch. equation:

$r - 3 = 0 \Rightarrow r = 3 \Rightarrow e^{3t}$

$x_0 = c \cdot e^{3t}, c \in \mathbb{R}$

Now we consider the differential equations:

$x' - 3x = 2 - 3t$

$x' - 3x = e^{-3t}$

For each of them we use the method of undetermined coefficients.

In the first case:

$x_{p_1} = at + b$

$x_{p_1}' = a$

$a - 3at - 3b = 2 - 3t$

$(a - 3b - 2) + 3(1-a)t = 0$

$\begin{cases} a - 3b - 2 = 0 & \Rightarrow 3b = -1 \Rightarrow b = -\frac{1}{3} \\ 1 - a = 0 & \Rightarrow a = 1 \end{cases}$

$x_{p_1} = t - \frac{1}{3}$

In the second case:

$x_{p_2} = a e^{-3t}$

$x_{p_2}' = -3a e^{-3t}$

$-3ae^{-3t} - 3ae^{-3t} = e^{-3t} \;/ : e^{-3t}$

$-6a = 1 \Rightarrow a = -\frac{1}{6}$

$x_{p_2} = -\frac{1}{6} e^{-3t}$

Now, applying the superposition principle, we obtain that $x_p = t - \frac{1}{3} - \frac{1}{6} e^{-3t}$ is a particular solution of the diff. eq. $x' = 3x + 2 - 3t + e^{-3t}$.

The general solution is:

$$x = c \cdot e^{3t} + t - \frac{1}{3} - \frac{1}{6} e^{-3t}, \quad c \in \mathbb{R}$$

c) $x'' - x' + 2x = 0$

The ch. equation: $r^2 - r + 2 = 0$

$\Delta = 1 - 8 = -7$

$r_{1,2} = \frac{1 \pm i\sqrt{7}}{2} \mapsto e^{\frac{t}{2}} \sin \frac{t\sqrt{7}}{2}, \; e^{\frac{t}{2}} \cos \frac{t\sqrt{7}}{2}$

The general solution: $x = c_1 e^{\frac{t}{2}} \sin \frac{t\sqrt{7}}{2} + c_2 e^{\frac{t}{2}} \cos \frac{t\sqrt{7}}{2}, \; c_1, c_2 \in \mathbb{R}$.

d) $x''' = 0$

The ch. eq: $r^3 = 0 \Rightarrow r = 0$ - triple root $\mapsto 1, t, t^2$

The general solution: $x = c_1 + c_2 t + c_3 t^2, \; c_1, c_2, c_3 \in \mathbb{R}$

2) The domain of the diff. eq. is $D_f = \mathbb{R}^2 \setminus \{(x,y) \mid xy = 1\}$

~~0 · 1 = 0~~

(1) $\begin{cases} y' = \dfrac{1 - \sqrt[3]{y}}{1 - xy} \\ y(0) = 1 \end{cases}$ (2) $\begin{cases} y' = \dfrac{1 - \sqrt[3]{y}}{1 - xy} \\ y(1) = 1 \end{cases}$ (3) $\begin{cases} y' = \dfrac{1 - \sqrt[3]{y}}{1 - xy} \\ y(0) = 0 \end{cases}$

a) $0 \cdot 1 = 0 \neq 1 \Rightarrow$ ~~the IVP~~ $(0,1) \in D_f \Rightarrow$ the IVP (1) is well-defined

$1 \cdot 1 = 1 \Rightarrow (1,1) \notin D_f \Rightarrow$ the IVP (2) is not well-defined

$0 \cdot 0 = 0 \neq 1 \Rightarrow (0,0) \in D_f \Rightarrow$ the IVP (3) is well-defined

b) $f(x,y) = \dfrac{1 - \sqrt[3]{y}}{1 - xy}$

$\dfrac{\partial f}{\partial x}(x,y) = \dfrac{y(1 - \sqrt[3]{y})}{(1 - xy)^2}$

$\dfrac{\partial f}{\partial y}(x,y) = \dfrac{-\frac{1}{3} y^{-\frac{2}{3}}(1 - xy) + x(1 - y^{\frac{1}{3}})}{(1 - xy)^2}$

Since $y^{-\frac{2}{3}}$ is not defined in $(x,0)$, $x\in\mathbb{R}$, it follows the f is differentiable only on $D_f \setminus \{(x,0) \mid x\in\mathbb{R}\}$.

Denote $U = D_f \setminus \{(x,0) \mid x\in\mathbb{R}\}$

~~$(0,1)\in U$~~

~~$f\in C^1(U)$~~

~~U open and connected~~

$U = U_1 \cup U_2 \cup U_3 \cup U_4$, where

$U_1 = \{(x,y) \mid xy<1,\ x\in\mathbb{R},\ y\in(-\infty,0)\}$

$U_2 = \{(x,y) \mid xy<1,\ x\in\mathbb{R},\ y\in(0,\infty)\}$

$U_3 = \{(x,y) \mid xy>1,\ x\in\mathbb{R},\ y\in(-\infty,0)\}$

$U_3 = \{(x,y) \mid xy>1,\ x\in\mathbb{R},\ y\in(0,\infty)\}$

$(0,1)\in U_2$

$f\in C^1(U_2)$

U_2 open and connected

$\Rightarrow$ according to T∃! the IVP (1) has a unique local solution

$(0,0)\in U \Rightarrow f$ is not differentiable in $(0,0) \Rightarrow$ the T∃! cannot be applied for IVP (3)

c) $\begin{cases} y' = \dfrac{1-\sqrt[3]{y}}{1-xy} \\ y(0)=1 \end{cases}$

We notice that $y=1$ is a solution of this IVP. Since we have proven at (b) that T∃! applies to this IVP, $y=1$ is the only solution of this IVP.

~~The solution can only be defined on~~

The solution is defined on $(-\infty,1)\cup(1,\infty)$, and $0\in(-\infty,1)$.

In conclusion, the solution of the IVP (3) is $\varphi:(-\infty,1)\to\mathbb{R}$, $\varphi(x)=1$.

3) $x^2+9y^2=c$, $c\in\mathbb{R}$

The differential equation we are looking for has as first integral $H(x,y)=x^2+9y^2$

$$\left.\begin{array}{l} dH = 2x\,dx + 18y\,dy \\ dH=0 \end{array}\right| \Rightarrow 2x\,dx+18y\,dy=0 /:2 \Rightarrow$$

$$\Rightarrow x\,dx = -9y\,dy \Rightarrow \frac{dy}{dx} = -\frac{x}{9y} \Rightarrow y' = -\frac{x}{9y}$$

The diff eq. $y' = -\frac{x}{9y}$ is equivalent to the system $\begin{cases} \frac{dy}{dt} = -x \\ \frac{dx}{dt} = 9y \end{cases}$

In conclusion, the planar autonomous system whose trajectories are the curves $x^2+9y^2=c$, $c\in\mathbb{R}$ is $\begin{cases} \dot{y} = -x \\ \dot{x} = 9y \end{cases}$

4) $f_\lambda : [0,1] \to [0,1]$, $f_\lambda(x) = \lambda x(1-x)$, $\lambda\in(0,4)$

In order to determine the fixed points of f_λ we must intersect its graph with the line $y=x$.

$$f_\lambda(x) = x \Rightarrow \lambda x - \lambda x^2 = x \Rightarrow \lambda x^2 + (1-\lambda)x = 0 \Rightarrow$$

$$\Rightarrow x(\lambda x + 1 - \lambda) = 0$$

$$\eta_1^* = 0$$

$$\eta_2^* = \frac{\lambda-1}{\lambda} = 1-\frac{1}{\lambda}$$

$$f_\lambda'(x) = (\lambda x - \lambda x^2)' = \lambda - 2\lambda x$$

<u>if $\lambda\in(0,1)$</u>

$$\left.\begin{array}{l} \eta_1^* = 0 \in [0,1] \\ \eta_2^* = 1-\frac{1}{\lambda} \notin [0,1] \end{array}\right| \Rightarrow f_\lambda \text{ has only one fixed point, } \eta^*=0$$

$$f_\lambda'(0) = \lambda < 1 \Rightarrow \eta^*=0 \text{ is as. stable}$$

<u>if $\lambda = 1$</u>

$\eta_1^* = 0 \in [0,1]$

$\eta_2^* = 1 - \frac{1}{\lambda} = 0 = \eta_1^*$ $\Big| \Rightarrow f_\lambda$ has only one fixed point, $\eta^* = 0$

$f'_\lambda(0) = 1 \Rightarrow \eta^* = 0$ is non-hyperbolic

<u>if $\lambda \in (1,3)$</u>

$\eta_1^* = 0 \in [0,1]$

$\eta_2^* = 1 - \frac{1}{\lambda} \in [0,1]$ $\Big| \Rightarrow f_\lambda$ has two fixed points, $\eta_1^* = 0$, $\eta_2^* = 1 - \frac{1}{\lambda}$

$f'_\lambda(0) = \lambda > 1 \Rightarrow \eta_1^* = 0$ is unstable

$f'_\lambda\left(1 - \frac{1}{\lambda}\right) = \lambda - 2\lambda + 2 = 2 - \lambda \in (-1,1) \Rightarrow \eta_2^* = 1 - \frac{1}{\lambda}$ is as. stable

$\left(\cancel{\lambda - 2\lambda + 2 = 0} \right.$
$\left. \cancel{\Delta = 4 - 8 = -4} \right)$

<u>if $\lambda = 3$</u>

$\eta_1^* = 0 \in [0,1]$

$\eta_2^* = \frac{2}{3} \in [0,1]$ $\Big| \Rightarrow f_\lambda$ has two fixed points, $\eta_1^* = 0$, $\eta_2^* = \frac{2}{3}$

$f'_\lambda(0) = 3 > 1 \Rightarrow \eta_1^* = 0$ is unstable

$f'_\lambda\left(\frac{2}{3}\right) = 3 - 4 = -1 \Rightarrow \eta_2^* = \frac{2}{3}$ is non-hyperbolic

<u>if $\lambda \in (3,4)$</u>

$\eta_1^* = 0 \in [0,1]$

$\eta_2^* = 1 - \frac{1}{\lambda} \in [0,1]$ $\Big| \Rightarrow f_\lambda$ has two fixed points, $\eta_1^* = 0$, $\eta_2^* = 1 - \frac{1}{\lambda}$

$f'_\lambda(0) = \lambda > 1 \Rightarrow \eta_1^* = 0$ is unstable

$f'_\lambda\left(1 - \frac{1}{\lambda}\right) = 2 - \lambda \in (-\infty, -1) \Rightarrow \eta_2^* = 1 - \frac{1}{\lambda}$ is unstable