Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

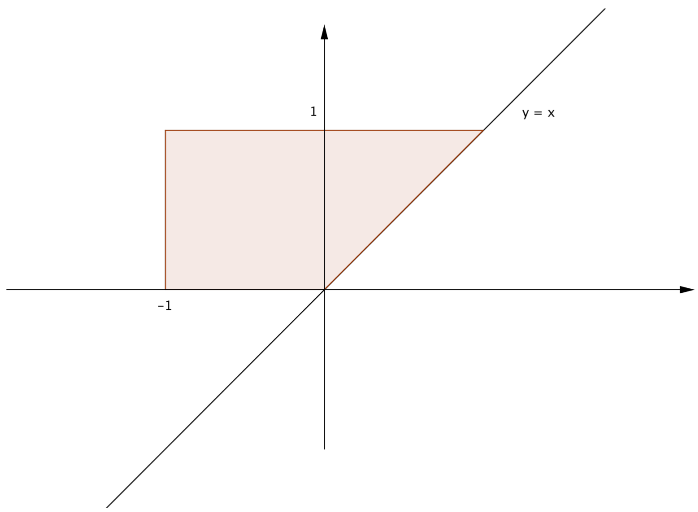
Winter semester 2013-2014

Solutions to Exercise Sheet no.14

Analysis for CS

(G 33)

a)

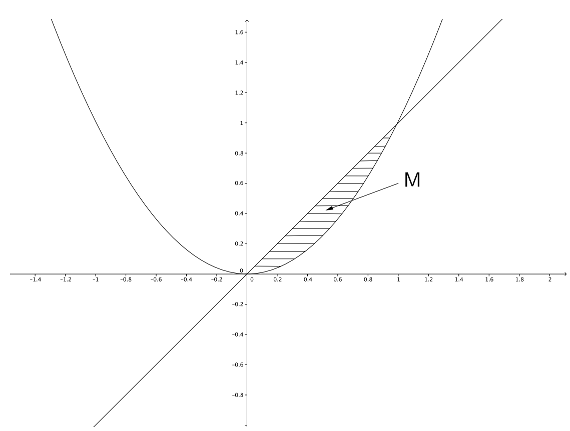


We regard M as a normal domain with respect to the x-axis. Applying **Th2** in the exercise-class no. 13, we get that $I = \int_0^1 dy \int_{-1}^{y} (xy - y^3)dx$. Since for every $y \in [0, 1]$

$$\int_{-1}^{y} (xy - y^3)dx = \left(\frac{1}{2}x^2 y - xy^3 \right) \Big|_{-1}^{y} = -\frac{1}{2}y - \frac{1}{2}y^3 - y^4,$$

we finally obtain that $I = -\int_0^1 \left(\frac{1}{2}y + \frac{1}{2}y^3 + y^4 \right) dy = -\frac{23}{40}$.

b)



We regard M as a normal domain with respect to the y-axis, i.e.,

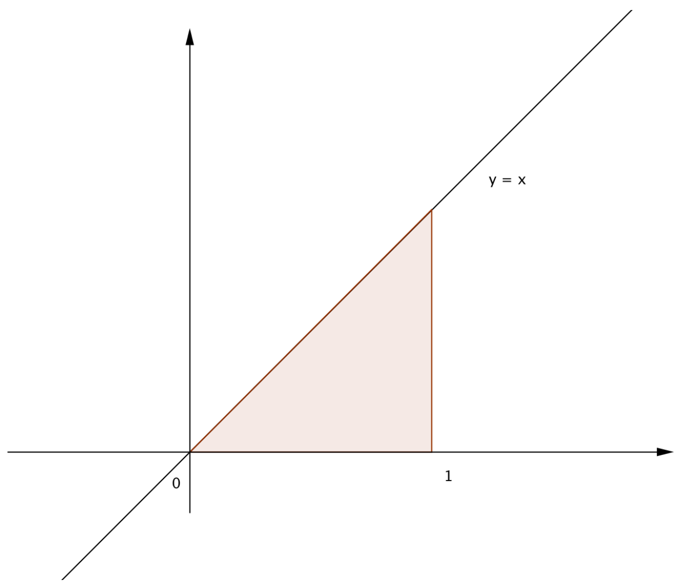
$$M = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ x^2 \leq y \leq x\}.$$

Using **Th1** in the exercise-class no. 13, we get that $I = \int_0^1 dx \int_{x^2}^x xydy$. Since for every $x \in [0,1]$

$$\int_{x^2}^x xydy = \frac{1}{2}xy^2\bigg|_{x^2}^x = \frac{1}{2}x^3 - \frac{1}{2}x^5,$$

we finally obtain that $I = \int_0^1 \left(\frac{1}{2}x^3 - \frac{1}{2}x^5\right) dx = \frac{1}{24}$.

c)



We regard M as a normal domain with respect to the y-axis. Applying **Th1** in the exercise-class no. 13, we get that $I = \int_0^1 dx \int_0^x (y + \sin \pi x^2)dy$. Since for every $x \in [0,1]$

$$\int_0^x (y + \sin \pi x^2)dy = \left(\frac{1}{2}y^2 + y \sin \pi x^2\right)\bigg|_0^x = \frac{1}{2}x^2 + x \sin \pi x^2,$$

we finally obtain that $I = \int_0^1 \left(\frac{1}{2}x^2 + x \sin \pi x^2\right) dx = \left(\frac{1}{6}x^3 - \frac{1}{2\pi} \cos \pi x^2\right)\Big|_0^1 = \frac{1}{6} + \frac{1}{\pi}$.

(G 34)

a) The function $F\colon \mathbb{R} \to \mathbb{R}$, defined by $F(x) = -e^{-x}$, is an antiderivative of f. Since $\lim\limits_{x \to -\infty} F(x) = -\infty$, assertion 1° of **Th3** in the exercise-class no. 11 yields that f is not improperly integrable on $\mathbb{R}$.

b) The function $F\colon [2,\infty) \to \mathbb{R}$, defined, for every $x \geq 2$, by

$$F(x) = \begin{cases} \frac{1}{1-\alpha}(\ln x)^{1-\alpha}, & \alpha \neq 1 \\ \ln(\ln x), & \alpha = 1, \end{cases}$$

is an antiderivative of f. Since

$$\lim_{x \to \infty} F(x) = \begin{cases} \infty, & 1 \geq \alpha \\ 0, & \alpha > 1, \end{cases}$$

assertion 1° of **Th2** in the exercise-class no. 11 implies that f is improperly integrable on $[2,\infty)$ if and only if $\alpha > 1$. In this case, assertion 2° of the same theorem yields that $\int_2^\infty f(x)dx = \frac{1}{(\alpha-1)(\ln 2)^{\alpha-1}}$.

(G 35)

1) We apply in each cases **Th2** in lecture no. 9.

a) Since $\lim_{n\to\infty} f\left(\frac{1}{n}, 0\right) = 0$ and $\lim_{n\to\infty} f\left(0, \frac{1}{n}\right) = 1$, we conclude that f doesn't have a limit at 0_2.

b) Since $\lim_{n\to\infty} f\left(\frac{1}{n}, 0\right) = 1$ and $\lim_{n\to\infty} f\left(0, \frac{1}{n}\right) = -1$, we conclude that f doesn't have a limit at 0_2.

2) The inequality $|xy| \leq \frac{1}{2}(x^2 + y^2)$ implies that

$$|g(xy)| \leq \frac{y^2}{2}, \forall\ (x, y) \in \mathbb{R}^2 \setminus \{0_2\}.$$

If $((x_n, y_n))_{n\in\mathbb{N}^*}$ is a sequence in $\mathbb{R}^2 \setminus \{0_2\}$ converging to 0_2, then the above inequality and the Sandwich-Theorem yield that $\lim_{n\to\infty} g(x_n, y_n) = 0$. Applying **Th2** in lecture no. 9, we conclude that $\lim_{(x,y)\to 0_2} g(x, y) = 0$.

(G 36)

Since (by the definition of the scalar product) $\langle x, y\rangle = \langle y, x\rangle$, we have that $\langle y, x\rangle = 0$. Using the definition of the Euclidean norm and the properties of the scalar product, we thus obtain

$$||x + y||^2 = \langle x + y, x + y\rangle = ||x||^2 + \langle x, y\rangle + \langle y, x\rangle + ||y||^2 = ||x||^2 + ||y||^2.$$