Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.6

Analysis for CS

(G 15)

1) Bare in mind that $V \in \vartheta(\alpha)$ (where $\alpha \in \mathbb{R}$) $\iff$ $\exists r > 0$ such that $(\alpha - r, \alpha + r) \subseteq V$.

a) $[-1, 1) \in \vartheta(0)$ (take for example $r = 1$).

b) $\mathbb{Q} \notin \vartheta(0)$. Assume by contradiction that $\mathbb{Q} \in \vartheta(0)$. Thus, $\exists r > 0$ such that $(-r, r) \subseteq \mathbb{Q}$. From the density property of $\mathbb{R}\backslash\mathbb{Q}$, we know that there exists $p \in \mathbb{R}\backslash\mathbb{Q}$ such that $-r < p < r$, which contradicts $(-r, r) \subseteq \mathbb{Q}$. Therefore, $\mathbb{Q} \notin \vartheta(0)$.

c) We have that $\bigcap\limits_{n\in\mathbb{N}^*} \left[-\frac{1}{n}, \frac{1}{n}\right] = \{0\}$. For this observe that, on the one hand, $0 \in \bigcap\limits_{n\in\mathbb{N}^*} \left[-\frac{1}{n}, \frac{1}{n}\right]$, and, on the other hand, if $x \in \bigcap\limits_{n\in\mathbb{N}^*} \left[-\frac{1}{n}, \frac{1}{n}\right]$, then $-\frac{1}{n} \leq x \leq \frac{1}{n}$, $\forall\, n \in \mathbb{N}^*$, thus $x = 0$ (by the Sandwich Theorem). Thus $\bigcap\limits_{n\in\mathbb{N}^*} \left[-\frac{1}{n}, \frac{1}{n}\right] \notin \vartheta(0)$, since $\{0\}$ doesn't contain an open interval centered at 0.

2) a) $A = [0, 1] \Rightarrow M = (0, 1)$.

b) $A = (-\infty, -1) \Rightarrow M = A$.

c) $A = (0, 1] \cup [2, 3] \Rightarrow M = (0, 1) \cup (2, 3)$.

d) $A = \mathbb{R} \Rightarrow M = A$.

e) $A = \mathbb{N} \Rightarrow M = \emptyset$.

3) Bare in mind that $A' = \left\{x \in \overline{\mathbb{R}} \mid\ \forall\, V \in \vartheta(x),\ V \cap (A \setminus \{x\}) \neq \emptyset\right\}$.

a) $A = \mathbb{Q} \Rightarrow A' = \overline{\mathbb{R}}$.

b) $A = (-\infty, 1) \cup (2, +\infty) \Rightarrow A' = [-\infty, 1] \cup [2, +\infty]$.

c) $A = \mathbb{Z} \Rightarrow A' = \{-\infty, +\infty\}$.

Let us take a closer look to a), where $A = \mathbb{Q}$. Let $\alpha \in \mathbb{R}$ and let $V \in \vartheta(\alpha)$ be an arbitrary neighborhood of α. Then, there exists $r > 0$ such that $(\alpha - r, \alpha + r) \subseteq V$. It is easy to remark that $(\alpha - r, \alpha + r) \cap (\mathbb{Q} \setminus \{\alpha\}) \neq \emptyset$ (by the density property of $\mathbb{Q}$) and hence $V \cap (\mathbb{Q} \setminus \{\alpha\}) \neq \emptyset$. Therefore $\alpha \in A'$. Hence $\mathbb{R} \subseteq A'$.

For $\alpha = -\infty$ let $V \in \vartheta(\alpha)$ be an arbitrary neighborhood of $-\infty$. Then, there exists $t \in \mathbb{R}$ such that $[-\infty, t) \subseteq V$. It is easy to remark that $[-\infty, t) \cap (\mathbb{Q} \setminus \{-\infty\}) \neq \emptyset$ and hence $V \cap (\mathbb{Q} \setminus \{-\infty\}) \neq \emptyset$. Therefore $-\infty \in A'$. For $\alpha = \infty$ the proof is similar to the one for $-\infty$. Thus we come to the conclusion that $\mathbb{Q}' = \overline{\mathbb{R}}$.

4) We now finish the proof of **L1** in the 6th lecture.

Case 2: $x = -\infty$, $y \in \mathbb{R}$. Since $y \in \mathbb{R}$, $y - 1 < y$. Take $U := [-\infty, y-1) \in \vartheta(-\infty)$ and $V := (y-1, y+1) \in \vartheta(y)$. Then $U \cap V = \emptyset$.

Case 3: $x \in \mathbb{R}$, $y = \infty$. Since $x \in \mathbb{R}$, $x < x + 1$. Take $U := (x-1, x+1) \in \vartheta(x)$ and $V := (x+1, \infty] \in \vartheta(\infty)$. Then $U \cap V = \emptyset$.

Case 4: $x = -\infty$, $y = \infty$. Take $U := [-\infty, -1) \in \vartheta(-\infty)$ and $V := (1, \infty] \in \vartheta(\infty)$. Then $U \cap V = \emptyset$.

(G 16)

(1) The one-sided limits of f at 1 are

$$\lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x<1}} e^{\frac{1}{x^2-1}} = 0 \text{ and } \lim_{\substack{x\to 1\\ x>1}} f(x) = \lim_{\substack{x\to 1\\ x>1}} e^{\frac{1}{x^2-1}} = \infty.$$

(2) The one-sided limits of f at 1 are

$$\lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x<1}} e^{\frac{x^2-2}{x-1}} = \infty \text{ and } \lim_{\substack{x\to 1\\ x>1}} f(x) = \lim_{\substack{x\to 1\\ x>1}} e^{\frac{x^2-2}{x-1}} = 0.$$

(3) The one-sided limits of f at 1 are

$$\lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x<1}} e^{1+\frac{2}{|x-1|}} = \infty \text{ and } \lim_{\substack{x\to 1\\ x>1}} f(x) = \lim_{\substack{x\to 1\\ x>1}} e^{1+\frac{2}{|x-1|}} = \infty$$

(4) The one-sided limits of f at 1 are

$$\lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x<1}} \frac{|x|-1}{x-1} = \lim_{\substack{x\to 1\\ x<1}} \frac{x-1}{x-1} = 1 \text{ and } \lim_{\substack{x\to 1\\ x>1}} f(x) = \lim_{\substack{x\to 1\\ x>1}} \frac{|x|-1}{x-1} = \lim_{\substack{x\to 1\\ x>1}} \frac{x-1}{x-1} = 1.$$

(G 17)

We get

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ \frac{1}{2}, & \text{if } x = 0 \\ 0, & \text{if } x < 0. \end{cases}$$

The function f is continuous at all points $x \in \mathbb{R} \setminus \{0\}$, and 0 is a jump discontinuity.

We have

$$g(x) = \begin{cases} 0, & \text{if } x > 1 \\ x, & \text{if } -1 < x \leq 1 \\ 0, & \text{if } x < -1. \end{cases}$$

The function g is continuous at all points $x \in \mathbb{R} \setminus \{-1, 1\}$, the points -1 and 1 are both jump discontinuities.

Homework:

(H 17)

(1) $\lim_{x\to 4}(-x^3 + 5x) = -44$.

(2) $\lim_{x\to -\infty}(-x^3 + 2x) = \infty$.

(3) We have $\frac{x^2-9}{(x+3)^2} = \frac{x-3}{x+3}$ for all $x \in \mathbb{R} \setminus \{-3\}$. Since

$$\lim_{\substack{x\to -3\\ x>-3}} \frac{x-3}{x+3} = -\infty \text{ and } \lim_{\substack{x\to -3\\ x<-3}} \frac{x-3}{x+3} = \infty,$$

we conclude that the limit $\lim\limits_{x\to -3} \dfrac{x^2-9}{(x+3)^2}$ doesn't exist.

(4) $L := \lim\limits_{x\to\infty} \dfrac{3x^k+5}{8x^3-2} = \lim\limits_{x\to\infty} \dfrac{3x^k}{8x^3}$, hence

$$L = \begin{cases} 0, & \text{if } k<3 \\ \infty, & \text{if } k>3 \\ \frac{3}{8}, & \text{if } k=3. \end{cases}$$

(5) $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{x} = 2.$

(6) $\lim\limits_{x\to 0} \left(\dfrac{1+4x+x^2}{1+x}\right)^{\frac{1}{x}} = \lim\limits_{x\to 0} \left(1+\dfrac{3x+x^2}{1+x}\right)^{\frac{1}{x}} = e^3.$

(7) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x^2+x-2} = \lim\limits_{x\to 1} \dfrac{(x+1)(x-1)}{(x-1)(x+2)} = \dfrac{2}{3}.$

(8) $\lim\limits_{\substack{x\to 1\\ x>1}} \left(\dfrac{1}{1-x} - \dfrac{1}{x^3-1}\right) = \lim\limits_{\substack{x\to 1\\ x>1}} \left(\dfrac{1}{1-x} + \dfrac{1}{1-x^3}\right) = -\infty.$

(9) $\lim\limits_{x\to\infty} (x-\sqrt{x^2-1}) = \lim\limits_{x\to\infty} \dfrac{1}{x+\sqrt{x^2-1}} = 0.$

(10) $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{x}{x\sqrt{1+\frac{1}{x^2}}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{x^2}}} = 1.$

(11) $\lim\limits_{x\to -\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to -\infty} \dfrac{x}{-x\sqrt{1+\frac{1}{x^2}}} = -\lim\limits_{x\to -\infty} \dfrac{1}{\sqrt{1+\frac{1}{x^2}}} = -1.$

(12) $\lim\limits_{x\to 1} \dfrac{x^3+x^2-x-1}{x^2-1} = \lim\limits_{x\to 1} \dfrac{x(x^2-1)+x^2-1}{x^2-1} = \lim\limits_{x\to 1}(x+1) = 2.$

(13) $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x^2} = \lim\limits_{x\to 0} \dfrac{(1+\sqrt{1-x^2})(1-\sqrt{1-x^2})}{(1+\sqrt{1-x^2})x^2} = \lim\limits_{x\to 0} \dfrac{x^2}{(1+\sqrt{1-x^2})x^2} = \dfrac{1}{2}.$

(14) Since $\left|\frac{x^2}{|x|}\right| = |x|$ for all $x \in \mathbb{R}^*$, we get that $\lim\limits_{x\to 0} \dfrac{x^2}{|x|} = 0$. The same result follows from the equalities

$$\lim_{\substack{x\to 0\\ x>0}} \frac{x^2}{|x|} = \lim_{\substack{x\to 0\\ x>0}} \frac{x^2}{x} = \lim_{\substack{x\to 0\\ x>0}} x = 0 \text{ and } \lim_{\substack{x\to 0\\ x<0}} \frac{x^2}{|x|} = \lim_{\substack{x\to 0\\ x<0}} \frac{x^2}{-x} = \lim_{\substack{x\to 0\\ x<0}} -x = 0.$$

(15) $\lim\limits_{x\to\infty} \sqrt{x}(\sqrt{x+1}-\sqrt{x}) = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{(\sqrt{x+1}+\sqrt{x})} =$ $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{x}}+1} = \dfrac{1}{2}.$

(16) Since

$$\left|\frac{(-1)^{[x]}}{x}\right| = \frac{1}{x}, \text{ for all } x>0,$$

we get $\lim\limits_{x\to\infty}\dfrac{(-1)^{[x]}}{x}=0$.

(17) $\lim\limits_{x\to-\infty} e^{\frac{|x|+1}{x-1}} = \lim\limits_{x\to-\infty} e^{\frac{-x+1}{x-1}} = \dfrac{1}{e}$.

(18) We get

$$\lim_{x\to-\infty}\left(\frac{x^2+x+1}{x^2-x+1}\right)^{\sqrt{-x}} = \lim_{x\to-\infty}\left(1+\frac{2x}{x^2-x+1}\right)^{\sqrt{-x}} = \lim_{x\to-\infty}\left(\left(1+\frac{2x}{x^2-x+1}\right)^{\frac{x^2-x+1}{2x}}\right)^{\frac{2x\sqrt{-x}}{x^2-x+1}}$$

$$= e^0 = 1.$$

(19) We get

$$\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1}{x} = \lim_{x\to 0}\frac{(\sqrt[3]{1+x}-1)(\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1)}{x(\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1)} =$$

$$= \lim_{x\to 0}\frac{1}{\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1} = \frac{1}{3}.$$