Lab 7

Quadrature formulas (1)

Trapezium formula for double integral

Applying succesively trapezium formula with respect to y, and with respect to x, we have:

$$
\begin{aligned}
\int_a^b \int_c^d f(x,y)dydx \approx & \int_a^b \left(\frac{d-c}{4}\right)\left[f(x,c) + 2f\left(x, \frac{c+d}{2}\right) + f(x,d)\right] dx \\
&= \frac{b-a}{4} \cdot \frac{d-c}{4}\left[f(a,c) + 2f\left(a, \frac{c+d}{2}\right) + f(a,d)\right] \\
&+ \frac{b-a}{4} \cdot 2 \cdot \frac{d-c}{4}\left[f\left(\frac{a+b}{2}, c\right)\right. \\
&\left. +2f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \left(\frac{a+b}{2}, d\right)\right] \\
&+ \frac{b-a}{4} \cdot \frac{d-c}{4}\left[f(b,c) + 2f\left(b, \frac{c+d}{2}\right) + f(b,d)\right].
\end{aligned}
$$

We get

$$
\begin{aligned}
\int_a^b \int_c^d f(x,y)dydx \approx & \frac{(b-a)(d-c)}{16}\left[f(a,c) + f(a,d) + f(b,c) + f(b,d)\right. \\
&+ 2f\left(\frac{a+b}{2}, c\right) + 2f\left(\frac{a+b}{2}, d\right) + 2f\left(a, \frac{c+d}{2}\right) \\
&\left. + 2f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]
\end{aligned} \tag{1}
$$

The Romberg's iterative method for trapezium formula:

We have

$$Q_{T_0}(f) = \frac{h}{2}\left[f(a) + f(b)\right], \ h = b-a,$$

$Q_{T_0}(f)$ being the first element of the sequence.

We divide the interval $[a,b]$ in two equal parts, of length $\frac{h}{2}$ and applying to $[a, a+\frac{h}{2}]$ and $[a+\frac{h}{2}, b]$ the trapezium formula we get

$$Q_{T_1}(f) = \frac{h}{4}\left[f(a) + 2f\left(a + \frac{h}{2}\right) + f(b)\right]$$

or

$$Q_{T_1}(f) = \frac{1}{2}Q_{T_0}(f) + hf\left(a + \frac{h}{2}\right).$$

Dividing now each previous divisions $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ in two equal parts, we obtain a division of the initial interval in $4 = 2^2$ equal parts, each of length $\frac{h}{4}$. Applying the repeated trapezium formula, we get

$$\begin{aligned} Q_{T_2}(f) &= \frac{h}{8}\left[f(a) + 2\sum_{i=1}^{3} f\left(a + \frac{ih}{4}\right) + f(b)\right] \qquad (2) \\ &= \frac{1}{2}Q_{T_1}(f) + \frac{h}{2^2}\left[f\left(a + \frac{1}{2^2}h\right) + f\left(a + \frac{3}{2^2}h\right)\right]. \end{aligned}$$

Continuing in an analogous manner, we get

$$Q_{T_k}(f) = \frac{1}{2}Q_{T_{k-1}}(f) + \frac{h}{2^k}\sum_{j=1}^{2^{k-1}} f\left(a + \frac{2j-1}{2^k}h\right),\ k = 1, 2, ... \qquad (3)$$

We obtain the sequence

$$Q_{T_0}(f),\ Q_{T_1}(f), ..., Q_{T_k}(f), ... \qquad (4)$$

which converges to the value $I = \int_a^b f(x)dx$.

We approximate the error by $\left|Q_{T_n}(f) - Q_{T_{n-1}}(f)\right|$. If we want to approximate I with error less than ε, we compute successively the elements of (4) until the first index for which

$$\left|Q_{T_n}(f) - Q_{T_{n-1}}(f)\right| \le \varepsilon,$$

$Q_{T_n}(f)$ being the required value.

Problems

1. a) Approximate the integral

$$I = \int_0^1 f(x)dx, \qquad \text{for } f(x) = \frac{2}{1+x^2},$$

using trapezium formula.

b) Plot the graph of the function f and the graph of the trapezium with vertices $(0,0), (0, f(0)), (1, f(1))$ and $(1, 0)$.

c) Approximate the integral I using Simpson's formula.

2. Approximate the following double integral

$$\int_{1.4}^{2}\int_{1}^{1.5} \ln(x+2y)dydx$$

using trapezium formula for double integral, given in (1). (Exact value is: 0.4295545)

3. Evaluate the integral that arises in electrical field theory:

$$H(x,r)=\frac{60r}{r^2-x^2}\int_0^{2\pi}\left[1-\left(\frac{x}{r}\right)^2\sin\phi\right]^{1/2}d\phi,$$

for $r=110$, $x=75$, using the repeated trapezium formula. (*Answer:* 6.3131)

4. Evaluate the integral

$$\int_0^{\pi}\frac{dx}{4+\sin 20x}$$

using the repeated Simpson's formula for $n=10$ and 30. (*Answer:* 0.78;0.81)

5. Use Romberg's algorithm for trapezium formula to approximate the integral

$$\int_0^1\frac{2}{1+x^2}dx,$$

with precision $\varepsilon=10^{-5}$.