Exercises for predicate logic

Exercise 1: Using the definition of deduction in predicate logic prove:

1. $\neg p(a), (\forall x)(p(x) \vee q(x)) \vdash (\exists y)q(y)$;
2. $(\forall y)(p(y) \vee q(y)), \neg(\forall z)p(z) \vdash (\forall x)q(x)$;
3. $\neg q(a), (\forall x)(p(x) \to q(x)) \vdash (\exists x)p(x)$;
4. $(\forall y)(p(y) \to q(y)), (\forall z)p(z) \vdash (\forall x)q(x)$;
5. $p(a), (\forall x)(p(x) \to q(x)) \vdash (\exists y)q(y)$

Exercise 2: Transform the following statements from natural language into predicate formulas, choosing the appropriate constants, function symbols and predicate symbols:

1. For every positive integer *x*, if *x* is a square of an integer, then there exists an integer *y* such that $(y+1)*(y-1)=x-1$.
2. For every positive integer *x*, if *x* is not a prime, then there exists a prime *y* such that *y* divides *x* and *y* is smaller than *x*.
3. The sum of two even numbers is an even number and their product is divisible by 4.
4. In a plane there are lines parallel to a line d and there are lines perpedicular to d.
5. CS students like either algebra or logic, all of them like Java but only Bill likes history.
6. Anyone who owns a rabbit hates anything that chases any rabbit.
7. If Santa has some reindeer with a red nose, then every child loves Santa.
8. Every investor who bought something that falls is not happy.
9. Anyone who has any cats will not have any mice.
10. Caterpillars and snails are much smaller than birds, which are much smaller than foxes, which in turn are much smaller than wolves.
11. Caterpillars and snails like to eat some plants.
12. Every animal either likes to eat all plants or all animals much smaller than itself that like to eat some plants.

Exercise 3: Using the given interpretations evaluate the following formulas:

1. $U = (\exists x)A(x) \wedge (\exists x)B(x) \to (\forall x)(A(x) \vee B(x))$

   Interpretation I =<D,m>, where:
   D = the set of all straight lines of a plan P
   Fie $d \in P$, a constant straight line belonging to the interpretation domain.
   $m(A)$: $D \to \{T,F\}$, $m(A)(x)$: „ $x \perp d$ ”; $m(B)$: $D \to \{T,F\}$, $m(B)(x)$: „ $x \parallel d$ ”;

2. $U = (\exists x)(p(x) \wedge q(x)) \to (\exists x)p(x) \vee q(12)$

   Interpretation I =<D,m>, where:
   D =**N** (the set of natural numbers)
   $m(p)$: $\mathbf{N} \to \{T,F\}$, $m(p)(x)$: „$x \vdots 5$”; $m(q)$: $\mathbf{N} \to \{T,F\}$, $m(q)(x)$: „$x \vdots 7$”;

3. $U(z) = (\exists x)(\forall y)p(f(x,y),z)$

   Interpretation I =<D,m>, where: D =**Z** (the set of integer numbers),
   $m(f)$: $\mathbf{Z}^2 \to \mathbf{Z}$, $m(f)(x,y) = (x+y)^2$ and
   $m(p)$: $\mathbf{Z}^2 \to \{T,F\}$, $m(p)(x,y)$: „$x > y$”;

4. $U = (\forall x)(\exists y)p(x,y) \to (\exists y)(\forall x)p(x,y)$

   Interpretation I =<D,m>, where:
   D = the set of all triangles,
   $m(p)$: $D^2 \to \{T,F\}$, $m(p)(x,y)$: „$Area(x) \leq Area(y)$”;

5. $U = (\forall x)(\exists y)(\exists z)p(x, f(g(y), g(z)))$

Interpretation I =<D,m>, where:
D =**N** (the set of natural numbers)
$m(f)$: $\mathbf{N}^2 \to \mathbf{N}$, $m(f)(x, y) = x + y$;
$m(g)$: $\mathbf{N} \to \mathbf{N}$, $m(g)(x) = x^2$;
$m(p)$: $\mathbf{N}^2 \to \{T,F\}$, $m(p)(x, y)$: „$x = y$";

Exercise 4: Prove that the following formulas are not valid finding anti-models for them.

1. $U = ((\exists x)p(x) \to (\exists x)q(x)) \to (\forall x)(p(x) \to q(x))$
2. $U = (\exists x)(p(x) \to q(x)) \to ((\exists x)p(x) \to (\exists x)q(x))$;
3. $U = ((\forall x)p(x) \to (\forall x)q(x)) \to (\forall x)(p(x) \to q(x))$;
4. $U = (\forall x)(p(x) \vee q(x)) \to (\forall x)p(x) \vee (\forall x)q(x)$;
5. $U = ((\forall x)p(x) \to (\exists x)q(x)) \to (\forall x)(p(x) \to q(x))$;
6. $U = ((\exists x)p(x) \to (\exists x)q(x)) \to (\forall x)(p(x) \to q(x))$;
7. $U = (\exists x)p(x) \wedge (\exists x)q(x) \to (\forall x)(p(x) \wedge q(x))$.

Exercise 5:
Choose two arbitrary interpretations (one with a finite domain and the other with an infinite domain) for the formula *U* and prove that they are models of *U*.

1. $U = (\forall x)(A(x) \leftrightarrow B(x)) \to ((\exists x)A(x) \leftrightarrow (\exists x)B(x))$;
2. $U = (\exists x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\exists x)B(x))$;
3. $U = ((\exists x)A(x) \to (\forall x)B(x)) \to (\forall x)(A(x) \to B(x))$;
4. $U = (\forall x)(A(x) \vee B(x)) \to ((\forall x)A(x) \vee (\exists x)B(x))$;
5. $U = (\forall x)(A(x) \to B(x)) \to ((\exists x)A(x) \to (\exists x)B(x))$;
6. $U = (\forall x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\exists x)B(x))$.
7. $U = (\forall x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\forall x)B(x))$;
8. $U = (\forall x)(A(x) \leftrightarrow B(x)) \to ((\forall x)A(x) \leftrightarrow (\forall x)B(x))$;

Remark: All the formulas from Exercise 5 are tautologies.

Exercise 6. Transform the following formulas into prenex, Skolem and clausal normal forms.

1. $(\exists x)\ (\neg(\exists y)p(y) \to (\forall y)(q(y) \to r(x)))$;
2. $(\exists x)\ ((\exists y)p(y) \to \neg(\forall y)(q(y) \to r(x)))$;
3. $(\forall x)\ (\neg(\exists y)p(y) \to (\forall y)(q(y) \to r(x)))$;
4. $(\forall x)\ ((\exists y)p(y) \to \neg(\forall y)(q(y) \to r(x)))$;
5. $(\exists x)\ ((\forall y)p(y) \to \neg(\exists y)(q(y) \to r(x)))$;
6. $(\forall x)\ (\neg(\forall y)p(y) \to (\exists y)(q(y) \to r(x)))$;
7. $(\forall x)\ ((\forall y)p(y) \to \neg(\exists y)(q(y) \to r(x)))$;
8. $(\exists x)\ (\neg(\forall y)p(y) \to (\exists y)(q(y) \to r(x)))$.

Exercise 7. Transform the following formulas into prenex, Skolem and clausal normal forms.

1. $(\forall x)(\forall y)((\exists z)p(z) \wedge (\exists u)(q(x,u) \to (\exists z)q(y,z)))$;
2. $(\exists x)(\forall y)((\exists z)p(z) \wedge (\exists u)(q(x,u) \to (\exists z)q(y,z)))$;
3. $(\forall x)(\exists y)((\exists z)p(z) \wedge (\exists u)(q(x,u) \to (\exists z)q(y,z)))$;
4. $(\exists x)(\exists y)((\exists z)p(z) \wedge (\forall u)(q(x,u) \to (\exists z)q(y,z)))$;
5. $(\forall x)(\exists y)((\exists z)p(z) \wedge (\forall u)(q(x,u) \to (\exists z)q(y,z)))$;
6. $(\forall x)(\forall y)((\exists z)p(z) \wedge (\forall u)(q(x,u) \to (\exists z)q(y,z)))$;

7. $(\forall x)(\forall y)((\exists z)p(z) \wedge (\exists u)(q(x,u) \rightarrow (\forall z)q(y,z)))$;
8. $(\exists x)(\forall y)((\exists z)p(z) \wedge (\forall u)(q(x,u) \rightarrow (\exists z)q(y,z)))$.

Exercise 8.
Using the semantic tableaux method prove the following properties:

1. **$\exists$ is semi-distributive over $\wedge$:**

$\models (\exists x)(A(x) \wedge B(x)) \rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$ and

$$\not\models (\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)(A(x) \wedge B(x))$$

2. **$\forall$ is semi-distributive over $\vee$:**

$\models (\forall x)A(x) \vee (\forall x)B(x) \rightarrow (\forall x)(A(x) \vee B(x))$ and

$$\not\models (\forall x)(A(x) \vee B(x)) \rightarrow (\forall x)A(x) \vee (\forall x)B(x)$$

3. $\exists$ is semi-distributive over $\rightarrow$:

$\models ((\exists x)A(x) \rightarrow (\exists x)B(x)) \rightarrow (\exists x)(A(x) \rightarrow B(x))$ and

$$\not\models (\exists x)(A(x) \rightarrow B(x)) \rightarrow ((\exists x)A(x) \rightarrow (\exists x)B(x))$$

4. $\forall$ is semi-distributive over $\rightarrow$:

$\models (\forall x)(A(x) \rightarrow B(x)) \rightarrow ((\forall x)A(x) \rightarrow (\forall x)B(x))$ and

$$\not\models ((\forall x)A(x) \rightarrow (\forall x)B(x)) \rightarrow (\forall x)(A(x) \rightarrow B(x))$$

5. $\models (\exists x)(A(x) \rightarrow B(x)) \rightarrow ((\forall x)A(x) \rightarrow (\exists x)B(x))$ and

$$\not\models ((\exists x)A(x) \rightarrow (\exists x)B(x)) \rightarrow (\forall x)(A(x) \rightarrow B(x))$$

6. **$\exists$ is distributive over $\vee$**

$\models (\exists x)(A(x) \vee B(x)) \leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$

7. **$\forall$ is distributive over $\wedge$**

$\models (\forall x)(A(x) \wedge B(x)) \leftrightarrow (\forall x)A(x) \wedge (\forall x)B(x)$

Exercise 9: Using the semantic tableaux method check the validity of the following formulas:

1. $(\forall x)(\forall y)p(x,y) \leftrightarrow (\exists x)(\forall y)p(x,y)$;
2. $(\exists x)(\forall y)p(x,y) \leftrightarrow (\forall y)(\exists x)p(x,y)$;
3. $(\forall y)(\exists x)p(x,y) \leftrightarrow (\exists y)(\exists x)p(x,y)$;
4. $(\forall x)(\forall y)p(x,y) \leftrightarrow (\forall y)(\forall x)p(x,y)$;
5. $(\forall y)(\forall x)p(x,y) \leftrightarrow (\forall x)(\exists y)p(x,y)$;
6. $(\exists y)(\exists x)p(x,y) \leftrightarrow (\exists x)(\forall y)p(x,y)$;
7. $(\exists y)(\exists x)p(x,y) \leftrightarrow (\forall x)(\exists y)p(x,y)$;
8. $(\exists y)(\exists x)p(x,y) \leftrightarrow (\exists x)(\exists y)p(x,y)$.

Exercise 10. Using the semantic tableaux method check if the following logical consequences hold.

1. $p(a), (\forall x)(p(x) \rightarrow p(f(x))) \models (\forall x)p(x)$;
2. $(\forall x)(p(x) \rightarrow q(x)), (\forall x)p(x) \models (\forall x)q(x)$;

1. $(\forall x)(\forall y)(q(x,y) \rightarrow p(x,y)), (\forall z)q(z,z) \models (\forall x)p(x,x)$;
2. $(\exists x)(\forall y)(p(x,y) \rightarrow r(x)), (\forall x)(\forall y)p(x,y) \models (\exists z)r(z)$;
3. $(\forall x)(p(x) \rightarrow q(x)), (\exists x)p(x) \models (\exists x)q(x)$;
4. $(\exists x)(\forall y)(q(x,y) \rightarrow p(x,y)), (\forall z)q(z,z) \models (\exists x)p(x,x)$;
5. $(\forall x)(\forall y)(p(x,y) \rightarrow r(x)), (\exists x)(\exists y)p(x,y) \models (\exists z)r(z)$;
6. $(\forall x)(\forall y)((p(x,y) \rightarrow p(y,x)) \models (\forall x)p(x,x)$.

Exercise 11. Are the atoms from the following pairs unifiable? If yes, write the most general unifier.

1. $P(a,x,g(g(y)))$ and $P(y,f(z),f(z))$;
$P(x,g(f(a)),f(x))$ and $P(f(y),z,y)$;
$P(a,x,g(g(y)))$ and $P(z,h(z,u),g(u),z)$;
2. $P(a,x,f(g(y)))$ and $P(y,f(z),f(z))$;
$P(x,g(f(a)),f(b))$ and $P(f(y),z,z)$;
$P(a,x,f(g(y)))$ and $P(z,h(z,u),f(b),z)$;
6. $P(a,f(x),g(h(y)))$ and $P(y,f(z),g(z))$;
$P(x,g(f(a)),h(x,y))$ and $P(f(z),g(z),y)$;
$P(g(y),x,f(g(y)))$ and $P(z,h(z,u),f(u))$;
7. $P(a,g(x),f(g(y)))$ and $P(y,z,f(z))$;
$P(b,g(f(a)),z)$ and $P(f(y),z,g(y))$;
$P(a,h(x,b),f(g(y)))$ and $P(z,h(z,u),f(u))$;
8. $P(a,x,g(f(y)))$ and $P(f(z),z,g(x))$;
$P(a,x,g(f(y)))$ and $P(x,y,g(f(b)))$;
$P(a,h(x,u),g(f(z)))$ and $P(y,h(y,f(z)),g(x))$;
9. $P(a,y,g(f(z)))$ and $P(z,f(z),x)$;
$P(y,f(x),z)$ and $P(y,f(y),f(y))$;
$P(h(x,y),x,y)$ and $P(h(y,x),f(z),z)$;
10. $P(a,x,g(f(y)))$ and $P(f(y),z,x)$;
$P(x,a,g(b))$ and $P(f(y),f(y),g(x))$;
$P(h(x,a),f(z),z)$ and $P(h(f(y),x),f(x),a)$;
11. $P(a,x,g(f(y)))$ and $P(f(y),f(z),g(z))$;
$P(x,g(f(a)),x)$ and $P(f(y),z,h(y,f(y)))$;
$P(a,h(x,u),f(g(y)))$ and $P(z,h(z,u),g(u))$.

Exercise 12.

Prove the inconsistency of the following set of clauses using lock resolution. Try two different indexings for the literals.

1. $S = \{\neg p(x) \vee q(x),\ p(a),\ \neg q(x) \vee \neg r(x),\ \neg w(a),\ r(y) \vee w(y)\}$;
2. $S = \{\ p(x) \vee \neg q(x),\ \neg p(a) \vee r(x),\ q(x),\ w(z),\ \neg r(y) \vee \neg w(y)\}$;
3. $S = \{\ p(x) \vee q(x) \vee r(x),\ \neg p(a),\ \neg q(x),\ \neg w(a),\ \neg r(y) \vee w(y)\}$;
4. $S = \{\ p(x) \vee q(x),\ \neg p(x) \vee r(x),\ \neg q(y) \vee r(y),\ \neg r(x) \vee w(x),\ \neg w(f(z))\}$
5. $S = \{\ p(x) \vee q(x),\ \neg p(a) \vee w(x),\ \neg q(y) \vee r(y),\ \neg r(x) \vee w(x),\ \neg w(a)\}$;

6. $S: \{\neg p(x) \vee \neg q(x), p(z) \vee w(x), q(y) \vee w(y) \vee \neg r(y),$

$\neg r(x) \vee \neg w(x), r(g(a,b))\}$;

7. $S = \{ p(x) \vee q(x), \neg p(x), \neg q(f(a)) \vee r(z), \neg w(z), \neg r(y) \vee w(y) \}$

8. EMBED Equation.3

$S = \{\neg p(x) \vee q(x) \vee \neg r(x), p(f(b)), \neg q(x), \neg w(y), r(y) \vee w(y) \}$.

Exercise 13.

Using general resolution check if the following formulas are theorems or not.

1. $(\forall x)(\forall y)((p(x,y) \to p(y,x)) \to (\forall x)p(x,x)$;
2. $((\exists x)p(x) \to (\exists x)q(x)) \to (\exists x)(p(x) \to q(x))$;
3. $(\forall x)(p(x) \to q(x)) \to ((\exists x)p(x) \to (\exists x)q(x))$;
4. $p(a) \wedge (\forall x)(p(x) \to p(f(x))) \to (\forall x)p(x)$;
5. $(\forall x)(\forall y)(q(x,y) \to p(x,y)) \to ((\forall z)q(z,z) \to (\forall x)p(x,x))$;
6. $((\forall x)p(x) \to (\forall x)q(x)) \to (\forall x)(p(x) \to q(x))$;
7. $(\exists x)(\forall y)(p(x,y) \to r(x)) \to ((\forall x)(\forall y)p(x,y) \to (\exists z)r(z))$.

Exercise 14. Prove the following deductions using linear resolution

1. $(\forall x)(\forall y)(p(y,x) \wedge q(x) \to q(y)), (\forall x)(\forall y)(r(y,x) \to q(y)),$
   $r(b,a), p(c,b) \vdash (\exists z)q(z)$;
2. $(\forall x)(p(x) \to r(x)), (\forall y)(r(y) \to q(y)), p(a), p(b) \vdash (\exists z)q(z)$;
3. $(\forall x)(\neg p(x) \wedge \neg q(x) \to r(x)), (\forall y)(r(y) \to w(y)), (\forall x)(w(x) \to p(x)),$
   $\neg p(a), \neg p(b), \neg w(c) \vdash (\exists z)q(z)$;
4. $(\forall x)(p(x) \to r(x)), (\forall y)(r(y) \to q(y)), r(a), r(b), \neg r(c) \vdash (\exists z)q(z)$;
5. $(\forall x)(\neg p(x) \wedge \neg q(x) \to r(x)), (\forall y)(r(y) \to w(y)), (\forall x)(w(x) \to p(x)),$
   $\neg p(a), \neg w(c) \vdash (\exists z)q(z)$;
6. $(\forall x)(\forall y)(\neg p(y,x) \to q(y)), (\forall x)(\forall y)(r(y,x) \wedge q(x) \to q(y)),$
   $r(b,a), \neg p(a,b) \vdash (\exists z)q(z)$;
7. $(\forall x)(p(x) \to r(x)), (\forall y)(p(y) \to q(y)), p(a), \neg r(c) \vdash (\exists z)q(z)$;
8. $(\forall x)(p(x) \to r(x)), (\forall y)(p(y) \to q(y)), p(a), p(b), \neg p(c) \vdash (\exists z)q(z)$.

Exercise 15. Using linear resolution prove:

1. semidistributivity of „ $\forall$” over „ $\vee$ ”:
   $\vdash (\forall x)p(x) \vee (\forall x)q(x) \to (\forall x)(p(x) \vee q(x))$ and
   $$\nvdash (\forall x)(p(x) \vee q(x)) \to (\forall x)p(x) \vee (\forall x)q(x)$$
2. semidistributivity of „ $\exists$” over „ $\to$ ”:
   $\vdash ((\exists x)p(x) \to (\exists x)q(x)) \to (\exists x)(p(x) \to q(x))$ and
   $$\nvdash (\exists x)(p(x) \to q(x)) \to ((\exists x)p(x) \to (\exists x)q(x))$$
3. semidistributivity of „ $\forall$” over „ $\to$ ”:
   $\vdash (\forall x)(p(x) \to q(x)) \to ((\forall x)p(x) \to (\forall x)q(x))$ and

$$\not\models ((\forall x)p(x) \to (\forall x)q(x)) \to (\forall x)(p(x) \to q(x))$$

4. $\models (\exists x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\exists x)B(x))$ and

$$\not\models ((\exists x)A(x) \to (\exists x)B(x)) \to (\forall x)(A(x) \to B(x))$$

5.semidistributivity of „ $\exists$” over „ $\wedge$ ”:

$\vdash (\exists x)(p(x) \wedge q(x)) \to (\exists x)p(x) \wedge (\exists x)q(x)$ and

$$\not\models (\exists x)p(x) \wedge (\exists x)q(x) \to (\exists x)(p(x) \wedge q(x))$$

6. distributivity of „ $\forall$” over „ $\wedge$ ”:

$\vdash (\forall x)p(x) \wedge (\forall x)q(x) \leftrightarrow (\forall x)(p(x) \wedge q(x))$.

7. distributivity of „ $\exists$” over „ $\vee$ ”:

$\vdash (\exists x)(p(x) \vee q(x)) \leftrightarrow (\exists x)p(x) \vee (\exists x)q(x)$;

Exercise 16.

Check if the following formulas are tautologies using lock resolution.

1. $(\forall x)(\forall y)p(x,y) \leftrightarrow (\forall y)(\forall x)p(x,y)$;
2. $(\exists y)(\exists x)p(x,y) \leftrightarrow (\exists x)(\exists y)p(x,y)$;
3. $(\forall x)(\forall y)p(x,y) \leftrightarrow (\exists x)(\forall y)p(x,y)$;
4. $(\exists x)(\forall y)p(x,y) \leftrightarrow (\forall y)(\exists x)p(x,y)$;
5. $(\exists y)(\exists x)p(x,y) \leftrightarrow (\forall x)(\exists y)p(x,y)$;
6. $(\forall y)(\forall x)p(x,y) \leftrightarrow (\forall x)(\exists y)p(x,y)$;
7. $(\exists y)(\exists x)p(x,y) \leftrightarrow (\exists x)(\forall y)p(x,y)$;
8. $(\forall y)(\exists x)p(x,y) \leftrightarrow (\exists y)(\exists x)p(x,y)$.

Exercisea 17.

Check if the following formulas are theorems using resolution.

1. $(\forall x)(\exists y)\neg(p(y,x) \leftrightarrow \neg p(y,y))$;
2. $(\forall x)(\exists y)\neg(p(x,y) \leftrightarrow \neg p(y,y))$;
3. $(\forall x)(\exists y)\neg(p(y,y) \leftrightarrow \neg p(x,y))$;
4. $(\forall x)(\exists y)\neg(p(y,y) \leftrightarrow \neg p(y,x))$;
5. $(\exists y)(\exists x)p(x,y) \leftrightarrow (\exists x)(\forall y)p(x,y)$;
6. $(\exists y)(\exists x)p(x,y) \leftrightarrow (\forall x)(\exists y)p(x,y)$;
7. $(\forall y)(\forall x)p(x,y) \leftrightarrow (\forall x)(\exists y)p(x,y)$;
8. $(\forall y)(\exists x)\neg(p(x,y) \leftrightarrow \neg p(x,x))$.