Dynamical systems. Final exam 25-01-2005

1. Find a first integral of $(2x+1)dx + 2ydy = 0$.

2. Represent the phase portrait of $x' = \lambda - x^2$, where $\lambda \in \mathbb{R}$ is a parameter. Study the stability of the equilibrium points.

3. Find the maximal solution of the Initial Value Problem:
$x^2y'' - 2xy' + 2y = x^3 , \quad y(1) = 1 , \quad y'(1) = 1 .$

4. We consider the system $\dot{x} = 3x + 2y , \quad \dot{y} = -x + y .$
a) Write its general solution.
b) Specify the type and study the stability of the equilibrium point.

5. Write the Euler's numerical formula to find the approximate solution of the Initial Value Problem: $y' = y + \sin y , \quad y(0) = 2$ on the interval $[0,\, 1.5]$.

6. We consider the Initial Value Problem: $y' = f(x, y) , \quad y(x_0) = y_0$, where $f : [x_0 - a,\, x_0 + a] \times \mathbb{R} \to \mathbb{R}$ and $a > 0, \quad y_0 \in \mathbb{R}$.

a) Write the definition and state sufficient conditions for the function f to be Lipschitz with respect to y .

b) Write the statement of the Global Existence and Uniqueness Theorem.

c) Prove the convergence of the sequence of functions $\varphi_n \in C[-1, 1]$, for all $n \geq 0$ given by the recurrence:

$$\varphi_{n+1}(x) = 1 + 2\int_0^x s\varphi_n(s)ds \, , \; n \geq 0 \, , \; \varphi_0(x) = 1 \text{ for all } x \in [-1, 1] \, .$$

(Hint: use b) and the formula $\left(\int_0^x u(s)ds\right)' = u(x)$.)