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Solutions to Exercise Sheet no.11

Analysis for CS

(G 28)

In all cases we use the formula of Leibniz-Newton for improper integrals.

a) The function $F\colon (-1,1) \to \mathbb{R}$, defined by $F(x) = \arcsin x$, is an antiderivative of f. Since

$$\lim_{\substack{x\to -1\\ x>-1}} \arcsin x = -\frac{\pi}{2} \quad \text{and} \quad \lim_{\substack{x\to 1\\ x<1}} \arcsin x = \frac{\pi}{2},$$

it follows that f is improperly integrable on $(-1,1)$ and $\int_{-1+}^{1-} \frac{1}{\sqrt{1-x^2}} dx = \pi$.

b) We first determine an antiderivative F of the function f on $[1,\infty)$. Since

$$\int \frac{1}{x(1+x)} dx = \int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx = \ln x - \ln(1+x) + \mathcal{C},$$

it follows that the function $F\colon [1,\infty) \to \mathbb{R}$, defined by $F(x) = \ln \frac{x}{1+x}$, is an antiderivative of f. From

$$\lim_{x\to\infty} \ln \frac{x}{1+x} = 0,$$

it follows that f is improperly integrable on $[1,\infty)$ and $\int_1^{\infty} \frac{1}{x(1+x)} dx = \ln 2$.

c) We first determine an antiderivative F of the function f on $(0,1]$. Since

$$\int \ln x dx = \int (x)' \ln x dx = x\ln x - \int 1 dx = x\ln x - x + \mathcal{C},$$

it follows that the function $F\colon (0,1] \to \mathbb{R}$, defined by $F(x) = x\ln x - x$, is an antiderivative of f. Since

$$\lim_{\substack{x\to 0\\ x>0}} (x\ln x - x) = -\lim_{y\to\infty} \frac{\ln y}{y} = 0,$$

it follows that f is improperly integrable on $(0,1]$ and $\int_{0+}^{1} \ln x dx = -1$.

d) We first determine an antiderivative F of the function f on $[0,1)$. Since

$$\int \frac{\arcsin x}{\sqrt{1-x^2}} dx = \int \arcsin x (\arcsin x)' dx = \frac{1}{2}(\arcsin x)^2 + \mathcal{C},$$

it follows that the function $F\colon [0,1) \to \mathbb{R}$, defined by $F(x) = \frac{1}{2}(\arcsin x)^2$, is an antiderivative of f. From

$$\lim_{\substack{x\to 1\\ x<1}} (\arcsin x)^2 = \left(\frac{\pi}{2}\right)^2,$$

it follows that f is improperly integrable on $[0,1)$ and $\int_0^{1-} \frac{\arcsin x}{\sqrt{1-x^2}} dx = \frac{(\pi)^2}{8}$.

e) We first determine an antiderivative F of the function f on $(0,1]$. Since

$$\int \frac{\ln x}{\sqrt{x}} dx = 2\int (\sqrt{x})' \ln x dx = 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}\ln x - 4\sqrt{x} + \mathcal{C},$$

it follows that the function $F\colon (0,1] \to \mathbb{R}$, defined by $F(x) = 2\sqrt{x}\ln x - 4\sqrt{x}$, is an antiderivative of f. As

$$\lim_{\substack{x\to 0\\ x>0}} \left(2\sqrt{x}\ln x - 4\sqrt{x}\right) = -2\lim_{y\to\infty} \frac{\ln y}{\sqrt{y}} = 0,$$

it follows that f is improperly integrable on $(0,1]$ and $\int_{0+}^{1} \frac{\ln x}{\sqrt{x}} dx = -4$.

f) We first determine an antiderivative F of the function f on $[e,\infty)$. Since

$$\int \frac{1}{x(\ln x)^3} dx = \int \frac{(\ln x)'}{(\ln x)^3} dx = -\frac{1}{2(\ln x)^2} + \mathcal{C},$$

it follows that the function $F\colon [e,\infty[\to \mathbb{R}$, defined by $F(x) = -\frac{1}{2(\ln x)^2}$, is an antiderivative of f. As

$$\lim_{x\to\infty} -\frac{1}{2(\ln x)^2} = 0,$$

it follows that f is improperly integrable on $[e,\infty)$ and $\int_e^{\infty} \frac{1}{x(\ln x)^3} dx = \frac{1}{2}$.

g) We notice that the roots of the equation $2x^2 - 2x - 1 = 0$ are $x_1 = \frac{1-\sqrt{3}}{2}$ and $x_2 = \frac{1+\sqrt{3}}{2}$. We first determine an antiderivative F of the function f on $(x_2, 2]$. Since

$$\int \frac{1}{x\sqrt{2x^2-2x-1}} dx = \int \frac{1}{x^2\sqrt{2-\frac{2}{x}-\frac{1}{x^2}}} dx = -\int \frac{\left(1+\frac{1}{x}\right)'}{\sqrt{3-\left(1+\frac{1}{x}\right)^2}} dx = -\arcsin\frac{1+\frac{1}{x}}{\sqrt{3}} + \mathcal{C},$$

it follows that the function $F\colon]x_2, 2] \to \mathbb{R}$, defined by $F(x) = -\arcsin\frac{1+\frac{1}{x}}{\sqrt{3}}$, is an antiderivative of f. As

$$\lim_{\substack{x\to x_2\\ x>x_2}} \arcsin\frac{1+\frac{1}{x}}{\sqrt{3}} = \arcsin 1 = \frac{\pi}{2} \quad \text{and} \quad \arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3},$$

it follows that f is improperly integrable $(x_2, 2]$ and $\int_{x_2+}^{2} \frac{1}{x\sqrt{2x^2-2x-1}} dx = \frac{\pi}{6}$.

h) We first determine an antiderivative F of the function f on $[0,\infty)$. Since

$$\int \left(\frac{\pi}{2} - \operatorname{arctg} x\right) dx = \frac{\pi}{2}x - \int (x)' \operatorname{arctg} x dx = \frac{\pi}{2}x - x\operatorname{arctg} x + \int \frac{x}{1+x^2} dx =$$
$$= x\left(\frac{\pi}{2} - \operatorname{arctg} x\right) + \frac{1}{2}\ln(1+x^2) + \mathcal{C} = x\left(\frac{\pi}{2} - \operatorname{arctg} x\right) + \ln(\sqrt{1+x^2}) + \mathcal{C},$$

it follows that the function the function $F\colon [0,\infty) \to \mathbb{R}$, defined by $F(x) = x\left(\frac{\pi}{2} - \operatorname{arctg} x\right) + \ln(\sqrt{1+x^2})$, is an antiderivative of f. As

$$\lim_{x\to\infty} x\left(\frac{\pi}{2} - \operatorname{arctg} x\right) = \lim_{\substack{y\to 0\\ y>0}} \frac{\frac{\pi}{2} - \operatorname{arctg}\frac{1}{y}}{y} = \lim_{\substack{y\to 0\\ y>0}} \frac{1}{y^2+1} = 1$$

(when computing the limit we used L'Hospital's rules). It follows that

$$\lim_{x\to\infty} \left(x\left(\frac{\pi}{2} - \operatorname{arctg} x\right) + \ln(\sqrt{1+x^2}\right) = \infty,$$

thus f is not improperly integrable on $[0,\infty)$.

HOMEWORK:

(H 28)

a) We first determine an antiderivative F of the function f on $(0,\infty)$. Since

$$\int \frac{1}{4\sqrt{x}+\sqrt{x^3}}dx = \int \frac{1}{\sqrt{x}(4+x)}dx = \int \frac{1}{\sqrt{x}(4+(\sqrt{x})^2)}dx = 2\int \frac{(\sqrt{x})'}{4+(\sqrt{x})^2}dx = \operatorname{arctg}\frac{\sqrt{x}}{2} + \mathcal{C},$$

it follows that the function $F\colon (0,\infty)\to\mathbb{R}$, defined by $F(x)=\operatorname{arctg}\frac{\sqrt{x}}{2}$, is an antiderivative of f. As

$$\lim_{\substack{x\to 0\\ x>0}} \operatorname{arctg}\frac{\sqrt{x}}{2} = 0 \quad \text{and} \quad \lim_{x\to\infty} \operatorname{arctg}\frac{\sqrt{x}}{2} = \frac{\pi}{2},$$

it follows that f is improperly integrable on $(0,\infty)$ and $\int_{0+}^{\infty} \frac{1}{4\sqrt{x}+\sqrt{x^3}}dx = \frac{\pi}{2}$.

b) We first determine an antiderivative F of the function f on $[e,\infty)$. Since

$$\int \frac{1}{x(\ln x)^3}dx = \int \frac{(\ln x)'}{(\ln x)^3}dx = -\frac{1}{2(\ln x)^2} + \mathcal{C},$$

it follows that the function $F\colon [e,\infty[\to\mathbb{R}$, defined by $F(x) = -\frac{1}{2(\ln x)^2}$, is an antiderivative of f. As

$$\lim_{x\to\infty} -\frac{1}{2(\ln x)^2} = 0,$$

it follows that f is improperly integrable on $[e,\infty)$ and $\int_{e}^{\infty} \frac{1}{x(\ln x)^3}dx = \frac{1}{2}$.

(H 29)

Let $(x,y)\in(0,\pi)\times(0,\pi)$ be arbitrarily chosen. Then

$$\frac{\partial f}{\partial x}(x,y) = \cos x + \cos(x+y) \text{ and } \frac{\partial f}{\partial y}(x,y) = \cos y + \cos(x+y).$$

The stationary points of f are the solutions of the system

$$\begin{cases} \cos x + \cos(x+y) = 0 \\ \cos y + \cos(x+y) = 0. \end{cases}$$

By subtracting the two equation we get $\cos x = \cos y$, and taking into account the domain of f, we obtain that $x=y$. By replacing this into the first equation we obtain

$$\cos x + \cos 2x = 0 \iff \cos x + 2(\cos x)^2 - 1 = 0$$

We make the notation $\cos x = t$. Then we get the equation $2t^2+t-1=0$ which has the solutions $t_1=-1$ and $t_2=\frac{1}{2}$. Since $x\in(0,\pi)$, we have that $\cos x\neq -1$, thus $\cos x=\frac{1}{2}$. We conclude that $(\frac{\pi}{3},\frac{\pi}{3})$ is the only stationary point of the function f. We compute now the second-order partial derivatives. For all $(x,y)\in(0,\pi)\times(0,\pi)$ we get

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\sin x - \sin(x+y), \frac{\partial^2 f}{\partial y^2}(x,y) = -\sin y - \sin(x+y),$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y) = -\sin(x+y) = \frac{\partial^2 f}{\partial x\partial y}(x,y),$$

thus

$$H_f(x,y) = \begin{pmatrix} -\sin x - \sin(x+y) & -\sin(x+y) \\ -\sin(x+y) & -\sin x - \sin(x+y) \end{pmatrix} \text{ and}$$

$$H_f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \begin{pmatrix} -\sqrt{3} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \sqrt{3} \end{pmatrix}.$$

Since $\Delta_1 = -\sqrt{3} < 0$ and $\Delta_2 = \frac{3}{2} > 0$, it follows (from **P2** or **Th3** in Lecture 10) that $H_f(\frac{\pi}{3},\frac{\pi}{3})$ is negative definite, thus $(\frac{\pi}{3},\frac{\pi}{3})$ is a local maximum point of f. The corresponding extreme value is

$$f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}.$$

(H 30)

1° According to the definition, f is improperly integrable on $[a,b)$ if and only if

$$\exists\ \ell := \lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)dx \in \mathbb{R}.$$

Let $t \in (a,b)$. Then $t \in \mathbb{R}$, f is continuous on $[a,t]$, F is an antiderivative of f on $[a,t]$, and thus, according to the formula of Leibniz-Newton for definite integrals

$$\int_a^t f(x)dx = F(x)\big|_a^t = F(t) - F(a). \tag{1}$$

Thus f is improperly integrable on $[a,b)$ if and only if F has finite (left-hand) limit at b.

2° Using (1) and the definition of the improper integral, we obtain that the improper integral I of f on $[a,b)$ is

$$I = \lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)dx = \lim_{\substack{t\to b\\ t<b}} F(t) - F(a).$$