Propositional logic

Exercise 1. Using the truth table method check if:

1) "$\uparrow$" connective is associative: $p \uparrow (q \uparrow r) \equiv (p \uparrow q) \uparrow r$.
2) "$\downarrow$" connective is associative: $p \downarrow (q \downarrow r) \equiv (p \downarrow q) \downarrow r$.
3) "$\uparrow$" connective is distributive over "$\downarrow$" connective: $p \uparrow (q \downarrow r) \equiv (p \uparrow q) \downarrow (p \uparrow r)$.
4) "$\downarrow$" connective is distributive over "$\uparrow$" connective: $p \downarrow (q \uparrow r) \equiv (p \downarrow q) \uparrow (p \downarrow r)$.
5) the properties of absorption: $p \downarrow (q \uparrow p) \equiv p$ and $p \uparrow (q \downarrow p) \equiv p$.
6) DeMorgan's laws for $\downarrow$ and $\uparrow$:

$$\neg(p \downarrow q) \equiv \neg p \uparrow \neg q; \qquad \neg(p \uparrow q) \equiv \neg p \downarrow \neg q;$$

Exercise 2.
Using the truth table method decide what kind of formula (consistent, inconsistent, tautology, contingent) is A. If A is consistent, write all the models of A.

1) A=$q \wedge \neg p \wedge r \to \neg p \vee \neg(q \wedge r)$
2) A=$\neg p \vee \neg(q \wedge r) \to q \wedge \neg p$
3) A=$\neg p \vee (\neg q \vee r) \to q \vee \neg p \vee r$
4) A=$\neg(\neg p \vee q) \vee r \to \neg p \vee (\neg q \vee r)$
5) A=$\neg p \vee (\neg q \vee \neg r) \to q \wedge \neg p$
6) $A = \neg p \vee (\neg q \wedge \neg r) \to q \wedge \neg p \wedge r$.

Exercise 3. Using the truth table method check if the following logical consequences hold:

1) $p \to q \models (p \to r) \to (p \to q \wedge r)$
2) $p \to q \models (q \to r) \to (p \to r)$
3) $p \to (q \to r) \models (p \to q) \to (p \to r)$
4) $p \to r \models (q \to r) \to ((p \vee q) \to r)$
5) $p \to q \models (\neg p \to q) \to q$
6) $p \to q \models (q \to r) \to (p \to q \wedge r)$;

Exercise 4.
Using the truth table method prove that the following formulas are tautologies.

1) $(p \to (q \wedge r)) \to ((p \to q) \wedge (p \to r))$ --- left-distribution of $\to$ over $\wedge$
2) $(p \to (q \to r)) \to (q \to (p \to r))$ --- permutation of the premises law.
3) $(p \to (q \to r)) \to (p \wedge q \to r)$ --- reunion of the premises law.
4) $(p \wedge q \to r) \to (p \to (q \to r))$ --- separation of the premises law.
5) $(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$ ---"cut" law.
6) $p \vee (q \to r) \to ((p \vee q) \to (p \vee r))$ --- left-distribution of $\vee$ over $\to$

Exercise 5. Transform the formula A into its equivalent CNF and DNF. Using one of these forms prove that A is a valid formula in propositional calculus.

1) A=$(p \to (q \to r)) \to (q \to (p \to r))$ --- permutation of the premises law.
2) A=$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$ ---"cut" law.
3) A=$(p \wedge q \to r) \to (p \to (q \to r))$ --- separation of the premises law.
4) A=$(p \to (q \to r)) \to (p \wedge q \to r)$ --- reunion of the premises law.
5) A=$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ --- axiom A2
6) A=$(p \to (q \wedge r)) \to ((p \to q) \wedge (p \to r))$ --- semidistributivity of $\to$ over $\wedge$

Exercise 6.
Using the appropriate normal form write all the models of the following formulas:
1) $(p \vee q \to r) \to (p \to r) \wedge q$;
2) $\neg(\neg p \vee q) \vee r \to \neg p \wedge \neg(q \wedge r)$;
3) $(p \wedge q \to r) \to (p \to r) \wedge q$;
4) $(p \vee q) \wedge \neg r \to p \wedge q \wedge r$;
5) $p \vee \neg(q \wedge \neg r) \to p \wedge q \wedge \neg r$;
6) $(p \vee q \to r) \to (q \to r) \wedge p$;
7) $(q \vee r \to p) \to (p \to r) \wedge q$;
8) $(q \wedge r \to p) \to (p \to r) \wedge q$.

Exercise 7.
Using the appropriate normal form prove that the following formulas are inconsistent:
1) $(U \to (V \to Z)) \wedge \neg((U \to V) \to (U \to Z))$;
2) $(\neg U \vee V) \wedge \neg(\neg V \to \neg U)$;
3) $(U \to V) \wedge (U \wedge V \to Z) \wedge (U \wedge \neg Z)$;
4) $(U \to (V \vee Z)) \wedge (\neg(U \to V) \wedge \neg(U \to Z))$;
5) $U \wedge (V \to Z) \wedge ((U \wedge V) \wedge \neg(U \wedge Z))$;
6) $(U \to (V \to Z)) \wedge (U \wedge V \wedge \neg Z)$;
7) $(U \to (V \to Z)) \wedge \neg(V \to (U \to Z))$;
8) $(U \wedge V \to Z) \wedge \neg(U \to (V \to Z))$;

Exercise 8.

Prove the following properties of the logical consequence relation where: R, S $\subseteq F_P$ and U, V, Z $\in F_P$.

1. *monotonicity*: if R $\models$U and R $\subseteq$ S then S $\models$U;
2. *cut:* if S $\models V_i$, $\forall i \in I$ and S$\cup\{V_i | i\in I\}\models$ U then S $\models$U;
3. *tranzitivity*: if S $\models$U and {U}$\models$V then S $\models$V;
4. *conjunction in conclusions* (right *„and”*):
   if S $\models$ U and S $\models$ V then S $\models$ U$\wedge$V;
5. *disjunction* in *premises* (left *„or”*):
   if S$\cup${U} $\models$ Z and S$\cup${V} $\models$ Z then S$\cup${U$\vee$V}$\models$Z;
6. *proof by cases*:
   if S$\cup${U} $\models$ V and S$\cup${$\neg$U } $\models$ V then S $\models$ V.

Remark: These properties are also properties of the syntactic consequence relation ” $\vdash$”.
Exercise 9.
Using the definition of deduction prove the following deductions:
1) $p \to q, r \to p, r \vdash q$;
2) $p \to r, p \vee r \to q, r \vdash q$;
3) $p \to q, q \to r, p \vdash r$;
4) $p \vee (q \to r), p \vee q, \neg p \vdash r$;

5) $p \to (q \to r), q, p \vdash r$;
6) $p \to (q \to r), p \wedge q, p \vdash r$;
7) $p \to (q \to r), p \to q, p \vdash r$;
8) $p \to q, r \to t, p \vee r, \neg q \vdash t$.

Exercise 10.
Using the theorem of deduction and its reverse prove that:
1) $\vdash (U \to (V \to Z) \to (U \wedge V \to Z)$ --- reunion of the premises law.
2) $\vdash (U \wedge V \to Z) \to (U \to (V \to Z))$ --- separation of the premises law
3) $\vdash (U \to (V \to Z)) \to (V \to (U \to Z))$ --- permutation of the premises law.
4) $\vdash (U \to (V \to Z)) \to ((U \to V) \to (U \to Z))$ --- second axiom of propositional logic
5) $\vdash (U \to V) \wedge (U \wedge V \to Z) \to (U \to Z)$ --- “cut” law.
6) $\vdash (U \to V) \to ((U \to Z) \to (U \to V \wedge Z))$

Exercise 11.
Using the theorem of deduction and its reverse prove that:
1) $\vdash (p \to q) \to ((r \to p) \to (r \to q))$
2) $\vdash (p \to r) \to ((q \to r) \to (p \vee q \to r))$
3) $\vdash (p \to q) \to ((r \to t) \to (p \wedge r \to q \wedge t))$
4) $\vdash (p \to r) \to ((p \wedge r \to q) \to (p \to q))$
5) $\vdash (q \to p) \to ((s \to q) \to (s \to p))$
6) $\vdash (p \to (q \to r)) \to (p \to (\neg r \to \neg q))$.

Exercise 12.
Using the semantic tableaux method decide what kind of formula is A.
If A is consistent, write all its models.
1) A= $((p \wedge q) \vee (\neg p \wedge \neg r)) \to (q \leftrightarrow r)$.
2) A= $(p \wedge q \to r) \to (p \to r) \wedge q$
3) A= $(q \wedge r \to p) \to (p \to r) \wedge q$
4) A= $((r \wedge q) \vee (\neg p \wedge \neg r)) \to (p \leftrightarrow q)$.
5) A= $((p \wedge r) \vee (\neg p \wedge \neg r)) \to (q \leftrightarrow r)$
6) $A = (\neg (p \to r) \to \neg p) \to ((p \to q) \to (p \to r))$.

Exercise 13. Prove that the formula A is valid using the semantic tableaux method.
1) A= $(p \to (q \to r)) \to (q \to (p \to r))$ permutation of the premises
2) A= $(p \to q \vee r) \leftrightarrow (p \to q) \vee (p \to r)$ distribution of implication over disjunction
3) A= $(p \to q \wedge r) \leftrightarrow (p \to q) \wedge (p \to r)$ distribution of implication over disjunction
4) A= $(p \to q) \to ((r \to t) \to (p \wedge r \to q \wedge t))$
5) A= $(\neg(q \to r) \to p) \to ((p \to q) \to (p \to r))$
6) $A = (p \to (q \to r)) \leftrightarrow (p \wedge q \to r)$;

Exercise 14.
Using the semantic tableaux method check if the following logical consequences hold:
1) $p \to (\neg q \vee r \wedge s), p, \neg s \models \neg q$.
2) $\neg p \to (\neg q \to r), r \vee q \models (\neg p \to q) \vee r$
3) $p \to (q \vee r \wedge s), p, \neg r \models q$
4) $p \to q, r \to t, p \wedge r \models q \wedge t$
5) $p \wedge (q \to r), q \vee r \models p \to (q \to r)$
6) $p \to q \models (r \to t) \to (p \wedge r \to q \wedge t)$

Exercise 15. Using the sequent calculus check if the following sequent is true or not.
1. $p \vee q \to r, p \Rightarrow (p \vee r) \to q$.
2. $p \wedge q \to r \Rightarrow (p \to r) \wedge q$
3. $q \wedge r \to p, q \Rightarrow (p \to r) \wedge q$
4. $p, r \wedge q \vee \neg p \wedge \neg r \Rightarrow (p \to q) \wedge (q \to p)$.
5. $p \vee \neg r, \neg p \vee \neg q \Rightarrow q \to r, r \to q$
6. $p \wedge (q \to r), q \vee r \models p \to (q \to r)$

Exercise 16. Prove the validity of the formula A using the sequent calculus method:
1) A=$(p \to q \wedge r) \leftrightarrow (p \to q) \wedge (p \to r)$, distributivity of $\to$ over $\wedge$
2) A=$(p \to (q \to r)) \leftrightarrow (p \wedge q \to r)$, reunion of the premises
3) A=$(\neg(q \to r) \to p) \to ((p \to q) \to (p \to r))$
4) A=$(p \to q \vee r) \leftrightarrow (p \to q) \vee (p \to r)$, distributivity of $\to$ over $\vee$.
5) A= $(p \wedge q \to r) \leftrightarrow (p \to (q \to r))$, separation of the premises
6) $A = p \vee (q \to r) \leftrightarrow (p \vee q) \to (p \vee r)$

Exercise 17. Using the sequent calculus check if the following logical consequences hold:
1) $\neg p \to (\neg q \to r), r \vee q \models (\neg p \to q) \vee r$
2) $p \wedge (q \to r), q \vee r \models p \to (q \to r)$
3) $p \to q, r \to t, p \wedge r \models q \wedge t$
4) $p \to (\neg q \vee r \wedge s), p, \neg s \models \neg q \vee s$
5) $p \to (q \vee r \wedge s), p, \neg r \models q \wedge p$
6) $p \to (q \vee r \wedge s), p, \neg r \models q \vee r$

Exercise 18. Using general resolution prove that the following formulas are tautologies:
1) $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$;
2) $(B \to A) \wedge (C \to A) \to (B \wedge C \to A)$;
3) $(B \to A) \wedge (C \to A) \to (B \vee C \to A)$;
4) $(A \to C) \to ((\neg A \to B) \to (\neg B \to C))$;
5) $A \vee (B \to C) \to (A \vee B) \to (A \vee C)$;
6) $(A \to B) \to ((C \to A) \to (C \to B))$;
7) $(A \to B) \to ((\neg A \to C) \to (\neg B \to C))$;
8) $(A \to B \wedge C) \to (A \to B) \wedge (A \to C)$.

Exercise 19.
Using lock resolution check the inconsistency of the following sets of clauses.

Choose two different indexings:

1) $\{p \vee q, p \vee \neg q \vee r, p \vee \neg q \vee \neg r, \neg p \vee r, \neg p \vee \neg r\}$;
2) $\{\neg p \vee \neg q, \neg p \vee q \vee \neg r, p \vee \neg r, \neg p \vee r, p \vee r\}$;
3) $\{p \vee q, p \vee \neg q \vee \neg r, \neg p \vee \neg r, r, \neg p \vee r\}$;
4) $\{p \vee q, \neg p \vee q \vee \neg r, \neg p \vee q \vee r, \neg q \vee \neg r, \neg q \vee r\}$;
5) $\{p \vee \neg q, \neg p \vee \neg q \vee r, \neg p \vee q \vee r, p \vee q, \neg r\}$;
6) $\{p \vee q, \neg p \vee q \vee \neg r, \neg p \vee \neg q \vee \neg r, p \vee \neg q, r\}$;
7) $\{p \vee \neg q, \neg p \vee \neg q \vee r, \neg p \vee \neg q \vee \neg r, r \vee q, \neg r \vee q\}$;
8) $\{p \vee r, p \vee q \vee \neg r, \neg p \vee \neg q \vee r, \neg p \vee q \vee r, \neg r\}$.

Exercise 20. Build a linear refutation from the following set of clauses:

1) $S = \{p \vee q \vee r, \neg q \vee r, \neg r, \neg p \vee r\}$;
2) $S = \{p \vee \neg r, q \vee r, \neg q \vee r, \neg p \vee \neg r\}$;
3) $S = \{q \vee r, \neg p, \neg q \vee r, p \vee \neg r\}$;
4) $S = \{\neg p \vee q, p \vee \neg q \vee r, \neg r, p \vee q \vee r, \neg p \vee \neg q\}$;
5) $S = \{p \vee r, \neg q, p \vee q \vee \neg r, \neg p \vee \neg r, q \vee r\}$;
6) $S = \{p \vee q, \neg p \vee q, \neg p \vee \neg q, p \vee \neg q\}$;
7) $S = \{p, q \vee r, \neg p \vee q \vee \neg r, \neg p \vee \neg q\}$;
8) $S = \{p \vee \neg q \vee r, q, \neg p \vee \neg q \vee r, \neg p \vee \neg q \vee \neg r, p \vee \neg r\}$.

Exercise 21.

Using the set-of-support strategy prove the following deductions:

1) $\neg(p \vee q) \rightarrow r, \neg p \vee q \vee r, \neg r \vdash q \wedge \neg r$;
2) $p \vee \neg r, \neg q \rightarrow r, \neg q \vdash \neg(p \rightarrow q)$;
3) $q \wedge r \rightarrow p, p \vee q, q \rightarrow r \vdash p$;
4) $r \rightarrow p \vee q, \neg p \rightarrow r, \neg q \vdash p \wedge \neg q$;
5) $\neg p \rightarrow q, (q \rightarrow r) \wedge \neg r \vdash p \wedge \neg r$;
6) $q \rightarrow p, q \vee r, p \rightarrow r \vdash r$;
7) $\neg p \rightarrow q \vee r, \neg q, p \rightarrow q \vdash \neg(p \vee q) \wedge r$;
8) $r \rightarrow p, \neg p, q \rightarrow p \vee r \vdash \neg(\neg p \rightarrow q \vee r)$.

Exercise 22.

Prove the syllogisme rule: $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$ using:

1) a syntactic method;
2) a semantic method;
3) a direct method;
4) a refutation method;
5) a semantic and direct method;
6) a semantic and refutation method;
7) a syntactic and refutation method.