Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

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Solutions to Exercise Sheet no.13

Analysis for CS

(G 31)

The functions to be integrated in exercises a)–c) are continuous. Thus we can apply assertion 2° of **Th1** in lecture no. 13 to compute the corresponding multiple integrals, which we denote by I.

a) We have that $I = \int_1^2 dx \int_2^3 dy \int_0^2 \frac{2z}{(x+y)^2} dz = \int_1^2 dx \int_2^3 \frac{4}{(x+y)^2} dy$. Since for every $x \in [1, 2]$

$$\int_2^3 \frac{4}{(x+y)^2} dy = -\frac{4}{x+y}\Big|_2^3 = \frac{4}{x+2} - \frac{4}{x+3},$$

we finally obtain $I = \int_1^2 (\frac{4}{x+2} - \frac{4}{x+3}) dx = 4 \ln \frac{x+2}{x+3}\Big|_1^2 = 4 \ln \frac{16}{15}$.

b) We have that $I = \int_0^1 dy \int_0^{\sqrt{3}} \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}} dx$. Since for all $y \in [0, 1]$

$$\int_0^{\sqrt{3}} \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}} dx = -\frac{1}{(1+x^2+y^2)^{\frac{1}{2}}}\Bigg|_0^{\sqrt{3}} = \frac{1}{\sqrt{1+y^2}} - \frac{1}{\sqrt{4+y^2}},$$

we get $I = \int_0^1 \frac{1}{\sqrt{1+y^2}} dy - \int_0^1 \frac{1}{\sqrt{4+y^2}} dy = \ln \frac{y+\sqrt{1+y^2}}{y+\sqrt{4+y^2}}\Bigg|_0^1 = \ln \frac{2(1+\sqrt{2})}{1+\sqrt{5}}$.

c) We have that $I = \int_0^1 dx \int_0^1 dy \int_0^1 \frac{x^2 z^3}{1+y^2} dz = \frac{1}{4} \int_0^1 dx \int_0^1 \frac{x^2}{1+y^2} dy$. Since for all $x \in [0, 1]$

$$\int_0^1 \frac{x^2}{1+y^2} dy = x^2 \operatorname{arctg} y\big|_0^1 = \frac{\pi}{4} x^2,$$

we get $I = \frac{\pi}{16} \int_0^1 x^2 dx = \frac{\pi}{48}$.

(G 32)

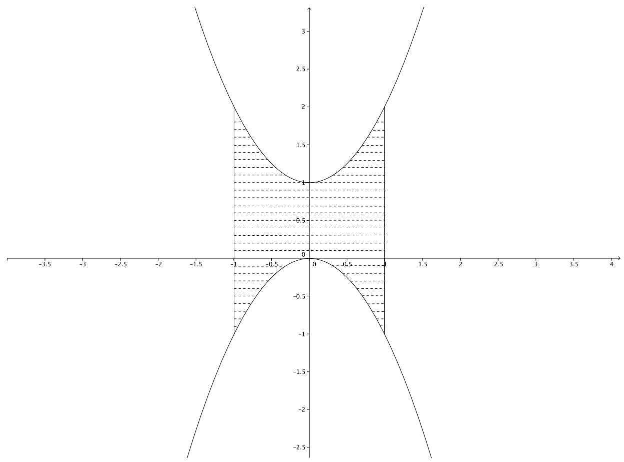
b) We denote by I the double integral to be computed. Since M is a normal domain with respect to the y-axis, we apply **Th1** in the exercise-class no. 13 to compute I. Hence $I = \int_{-1}^1 dx \int_{-x^2}^{1+x^2} (x^2 - 2y) dy$. For all $x \in [-1, 1]$

$$\int_{-x^2}^{1+x^2} (x^2 - 2y) dy = (x^2 y - y^2)\big|_{-x^2}^{1+x^2} = 2x^4 - x^2 - 1,$$

thus

$$I = \int_{-1}^1 (2x^4 - x^2 - 1) dx = \left(\frac{2}{5}x^5 - \frac{1}{3}x^3 - x\right)\Bigg|_{-1}^1 = -\frac{28}{15}.$$

a)



c) We see from the above figure that not every parallel line to the x-axis intersects M along a compact interval. Thus M is not normal with respect to the x-axis.

HOMEWORK:

(H 34)

The functions to be integrated in exercises a)–c) are continuous. Thus we can apply assertion $2°$ of **Th1** in lecture no. 13 to compute the corresponding multiple integrals, which we denote by I.

a) We have that $I = \int_0^1 dy \int_0^1 (xy + y^2)dx$. Since for every all $y \in [0, 1]$

$$\int_0^1 (xy + y^2)dx = \left(\frac{x^2 y}{2} + y^2 x\right)\Big|_0^1 = \frac{y}{2} + y^2,$$

we obtain $I = \int_0^1 (\frac{y}{2} + y^2)dy = \left(\frac{y^2}{4} + \frac{y^3}{3}\right)\Big|_0^1 = \frac{7}{12}$.

b) We have that $I = \int_0^1 dx \int_0^2 \min\{x, y\}dy$. Note that for all $x \in [0, 1]$

$$\min\{x, y\} = \begin{cases} y, & 0 \le y \le x \\ x, & x \le y \le 2, \end{cases}$$

thus $\int_0^2 \min\{x, y\}dy = \int_0^x ydy + \int_x^2 xdy = -\frac{x^2}{2} + 2x$. We finally get $I = \int_0^1 (-\frac{x^2}{2} + 2x)dx = \frac{5}{6}$.

c) We have that $I = \int_1^2 dx \int_1^2 dy \int_1^2 \frac{1}{(x+y+z)^3} dz$. For every $(x, y) \in [1, 2] \times [1, 2]$ we compute

$$\int_1^2 \frac{1}{(x+y+z)^3} dz = -\frac{1}{2(x+y+z)^2}\Big|_1^2 = -\frac{1}{2}\left(\frac{1}{(x+y+2)^2} - \frac{1}{(x+y+1)^2}\right).$$

For $x \in [1, 2]$ we then get

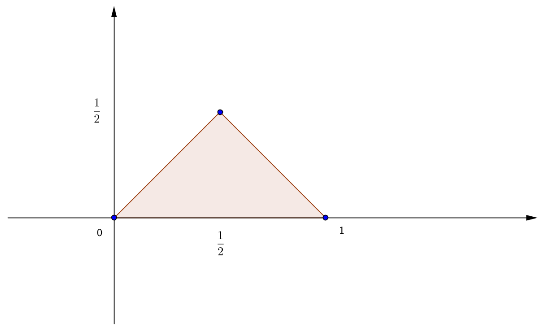
$$-\frac{1}{2}\int_1^2 \left(\frac{1}{(x+y+2)^2} - \frac{1}{(x+y+1)^2}\right) dy = \frac{1}{2}\left(\frac{1}{x+y+2} - \frac{1}{x+y+1}\right)\Big|_1^2.$$

Thus

$$I = \frac{1}{2}\int_1^2 \left(\frac{1}{x+4} - \frac{2}{x+3} + \frac{1}{x+2}\right) dx = \frac{1}{2}\ln\frac{(x+4)(x+2}{(x+3)^2}\Big|_1^2 = \frac{1}{2}\ln\frac{128}{125}.$$

(H 35)

a)



b) We see from the representation of M performed at a) that this set is a normal domain with respect to both axes. When viewed as a normal domain with respect to the x-axis, then the boundary functions $\psi_1, \psi_2 \colon [0, \frac{1}{2}] \to \mathbb{R}$ of M are, respectively, given by

$$\psi_1(y) = y, \quad \psi_2(y) = 1 - y.$$

Thus

$$M = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \le y \le \frac{1}{2},\ y \le x \le 1 - y \right\}.$$

c) We denote by I the integral to be computed. Using **Th2** in the exercise-class no. 13, we get that $I = \int_1^{\frac{1}{2}} dy \int_y^{1-y} (x^2 + y^2) dx$. Since for $y \in [0, \frac{1}{2}]$ we have that

$$\int_y^{1-y} (x^2 + y^2) dx = \left(\frac{1}{3} x^3 + x y^2 \right) \Big|_y^{1-y} = \frac{1}{3}(1 - y)^3 + y^2 - \frac{7}{3} y^3,$$

we finally obtain that $I = \int_0^{\frac{1}{2}} \left(\frac{1}{3}(1 - y)^3 + y^2 - \frac{7}{3} y^3 \right) dy = \frac{1}{12}$.

(H 36)

Let $(\tilde{\Delta}_k)_{k \in \mathbb{N}^*}$ be a partition of the interval $[0, 1]$ such that $\lim\limits_{k \to \infty} ||\tilde{\Delta}_k|| = 0$. (Take for instance, for every $k \in \mathbb{N}^*$, $\tilde{\Delta}_k$ to be the equidistant partition determined by the points $0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1$. In this case $||\tilde{\Delta}_k|| = \frac{1}{k}$, for every $k \in \mathbb{N}^*$.) For every $k \in \mathbb{N}^*$ denote by Δ_k the partition of the square $[0, 1] \times [0, 1]$ determined by $\tilde{\Delta}_k$ and $\tilde{\Delta}_k$. Then clearly $\lim\limits_{k \to \infty} ||\Delta_k|| = 0$. For every $k \in \mathbb{N}^*$ let $\Delta_k := \{D_1^k, \ldots, D_{m_k}^k\}$. For $k \in \mathbb{N}^*$ consider now, for every $j \in \{1, \ldots, m_k\}$, a point $u_j^k \in D_j^k \cap \mathbb{Q}^2$ and a point $v^k \in D_j^k \setminus \mathbb{Q}^2$. Denote by $s_k := (u_1^k, \ldots, u_{m_k}^k)$ and by $s'_k := (v_1^k, \ldots, v_{m_k}^k)$. Then $s_k, s'_k \in S_{\Delta_k}$, for every $k \in \mathbb{N}^*$. We thus obtain the following values for the Riemann sums

$$S(f, \Delta_k, s_k) = 1, \quad S(f, \Delta_k, s'_k) = 0, \forall\ k \in \mathbb{N}^*.$$

Thus $\lim\limits_{k \to \infty} S(f, \Delta_k, s_k) \neq \lim\limits_{k \to \infty} S(f, \Delta_k, s'_k)$, so f is not Riemann integrable on $[0, 1] \times [0, 1]$.