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Solutions to Exercise Sheet no.2

Analysis for CS

(G 6)

a) $\lim\limits_{n\to\infty} \dfrac{5^n+1}{7^n+1} = \lim\limits_{n\to\infty} \left(\left(\dfrac{5}{7}\right)^n \cdot \dfrac{1+\frac{1}{5^n}}{1+\frac{1}{7^n}} \right) = 0.$

b) $\lim\limits_{n\to\infty} \dfrac{4^n+(-2)^n}{4^{n-1}+2} = \lim\limits_{n\to\infty} \dfrac{4^n(1+\left(-\frac{1}{2}\right)^n)}{4^{n-1}\left(1+\frac{2}{4^{n-1}}\right)} = 4.$

c) $\lim\limits_{n\to\infty} \left(\sin\dfrac{\pi}{10}\right)^n = 0$, since $-1 < \sin\frac{\pi}{10} < 1$.

d) We have that

$$\lim_{n\to\infty} \sqrt{9n^2+2n+1} - 3n = \lim_{n\to\infty} \frac{2n+1}{\sqrt{9n^2+2n+1}+3n} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{9+\frac{2}{n}+\frac{1}{n^2}}+3} = \frac{1}{3}.$$

e) $\lim\limits_{n\to\infty} \left(5 + \dfrac{1-2n^3}{3n^4+2}\right)^2 = 25.$

f) $\lim\limits_{n\to\infty} \sqrt{n^2+3} - \sqrt{n^3+1} = \lim\limits_{n\to\infty} \sqrt{n^3+1}\left(\sqrt{\dfrac{n^2+3}{n^3+1}} - 1\right) = -\infty.$

g) $\lim\limits_{n\to\infty} \left(\dfrac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}} = \infty^{-\frac{5}{6}} = 0.$

h) Since

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{n}\right) = \frac{1}{n},$$

we get $\lim\limits_{n\to\infty} \left(1-\dfrac{1}{2}\right)\left(1-\dfrac{1}{3}\right)\dots\left(1-\dfrac{1}{n}\right) = 0.$

(G 7)

a) We first note that, if $q \in (0,1)$, then $q^n > q^{n+1}$, for every $n \in \mathbb{N}$. Thus the sequence $(q^n)_{n\in\mathbb{N}}$ is strictly decreasing. In our case $x_n = \left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n$, for $n \in \mathbb{N}^*$. By the previous observation, we conclude that the sequence $(x_n)_{n\in\mathbb{N}^*}$ is strictly decreasing, hence it is bounded above. Since $x_n > 0$, for every $n \in \mathbb{N}^*$, the sequence is also bounded below. The sequence converges to 0.

b) We notice that for $n = 2k$, $x_{2k} = \frac{1}{k}$, while for $n = 2k-1$, $x_{2k-1} = -\frac{1}{2k-1}$, for $k \in \mathbb{N}^*$. Since $x_1 < x_2$ and $x_2 > x_3$, the sequence $(x_n)_{n\in\mathbb{N}^*}$ is not monotonic. From $|x_n| = \frac{1}{n} \in [-1,1]$, for every $n \in \mathbb{N}^*$, we conclude that $(x_n)_{n\in\mathbb{N}^*}$ is bounded. Since $\lim\limits_{n\to\infty} |x_n| = 0$, the sequence $(x_n)_{n\in\mathbb{N}^*}$ converges to 0.

c) We notice that

$$\frac{x_{n+1}}{x_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \leq 1, \forall n \in \mathbb{N}^*.$$

Thus $(x_n)_{n\in\mathbb{N}^*}$ is a decreasing sequence. Moreover $x_n \leq x_1 = 2$ for all $n \in \mathbb{N}^*$. It is obvious that $x_n > 0$ for all $n \in \mathbb{N}^*$, implying thus that $(x_n)_{n\in\mathbb{N}^*}$ is bounded. But then $(x_n)_{n\in\mathbb{N}^*}$ converges to a real number ℓ. Assuming, by contradiction, that $\ell \neq 0$, we get that

$$1 = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{2}{n+1} = 0,$$

a contradiction. Thus $\ell = 0$.

d) We notice that

$$x_{n+1} - x_n = \frac{n+1}{n^2+2n+2} - \frac{n}{n^2+1} = \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0 \text{ for all } n \in \mathbb{N}^*.$$

Thus $(x_n)_{n\in\mathbb{N}^*}$ is a strictly decreasing sequence. Moreover $x_n \leq x_1 = \frac{1}{2}$ for all $n \in \mathbb{N}^*$. It is obvious that $x_n > 0$ for all $n \in \mathbb{N}^*$, implying thus that $(x_n)_{n\in\mathbb{N}^*}$ is bounded. The sequence $(x_n)_{n\in\mathbb{N}^*}$ converges to 0.

(G 8)

Case 3 $x \in \mathbb{R}$ and $y = -\infty$.

From $\lim\limits_{n\to\infty} x_n = x$ we have that $\exists n(1) \in \mathbb{N}$ such that $|x_n - x| < 1$, $\forall n \geq n(1)$.

From $\lim\limits_{n\to\infty} x_n = -\infty$ we have that $\exists n(x-1) \in \mathbb{N}$ such that $x_n < x-1$, $\forall n \geq n(x-1)$.

Fix a natural number $n \geq \max\{n(1), n(x-1)\}$, then we have

$$|x_n - x| < 1 \text{ and } x_n < x - 1.$$

As $x_n < x - 1$ we have that $|x_n - x| = x - x_n$. Hence we get

$$x - x_n < 1 \text{ and } 1 < x - x_n$$

which is a contradiction.

Case 3 $x = -\infty$ and $y = \infty$.

From $\lim\limits_{n\to\infty} x_n = -\infty$ we have that $\exists n(1) \in \mathbb{N}$ such that $x_n < 1$, $\forall n \geq n(1)$.

From $\lim\limits_{n\to\infty} x_n = \infty$ we have that $\exists m(1) \in \mathbb{N}$ such that $x_n > 1$, $\forall n \geq m(1)$.

Fix a natural number $n \geq \max\{n(1), m(1)\}$, then we have

$$x_n < 1 \text{ and } x_n > 1,$$

which is a contradiction.

HOMEWORK:

(H 8)

1) a) $\lim\limits_{n\to\infty} \dfrac{3^n}{4^n} = \lim\limits_{n\to\infty} \left(\dfrac{3}{4}\right)^n = 0$, since $-1 < \frac{3}{4} < 1$.

b) $\lim\limits_{n\to\infty} \dfrac{2^n + (-2)^n}{3^n} = \lim\limits_{n\to\infty} \left(\left(\dfrac{2}{3}\right)^n + \left(-\dfrac{2}{3}\right)^n\right) = 0.$

c) $\lim_{n\to\infty} \frac{5-n^3}{n^2+1} = -\infty.$

d) $\lim_{n\to\infty} \left(2 + \frac{4^n + (-5)^n}{7^n + 1}\right)^{2n^3-n^2} = 2^\infty = \infty.$

e) Since $1+2+\cdots+n = \frac{n(n+1)}{2}$, we get

$$\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \lim_{n\to\infty} \frac{n^2+n}{2n^2} = \frac{1}{2}.$$

f) $\lim_{n\to\infty} \left(\frac{n^3+4n+1}{2n^3+5}\right)^{\frac{-2n^4+1}{n^4+3n+1}} = \left(\frac{1}{2}\right)^{-2} = 4.$

g) $\lim_{n\to\infty} (\cos(-2013))^n = 0$, since $-1 < \cos(-2013) < 1$.

h) $\lim_{n\to\infty} \left(\frac{n^5+3n+1}{2n^5-n^4+3}\right)^{\frac{3n-n^4}{n^3+1}} = \left(\frac{1}{2}\right)^{-\infty} = \infty.$

2) Note that $x_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, for every $n \in \mathbb{N}$. The sequence $(\sqrt{n})_{n\in\mathbb{N}}$ being strictly increasing, we conclude that $(x_n)_{n\in\mathbb{N}}$ is strictly decreasing, hence bounded above. Since $x_n > 0$, for every $n \in \mathbb{N}$, the sequence is also bounded below. We have that $\lim_{n\to\infty} x_n = 0$, thus the sequence converges to 0.

(H 9)

For every $n \geq 2$ we have that

$$(2^2-1)\cdot\cdots\cdot(n^2-1) = \prod_{k=2}^{n}(k^2-1) = \prod_{k=2}^{n}(k-1)(k+1) = (n-1)!\frac{(n+1)!}{2}.$$

We obtain that

$$a_n = \frac{(n-1)!(n+1)!}{2n!\,n!} = \frac{n+1}{2n},$$

thus $\lim_{n\to\infty} a_n = \frac{1}{2}.$

(H 10)

Let $x \in \mathbb{R}$ be arbitrary. We prove that x is the limit of a decreasing sequence of real numbers. By the density property of the set $\mathbb{Q}$, there exists a rational number $x_1 \in (x, x+1)$. Using once again this property, there exists a rational number $x_2 \in (x, \min\{x_1, x+\frac{1}{2}\})$. Thus $x < x_2 < x_1$ and $x_2 < x + \frac{1}{2}$. We continue inductively this procedure: Assuming that $x_n \in \mathbb{Q}$, $n \geq 2$, has been chosen such that $x < x_n < x_{n-1}$ and $x_n < x + \frac{1}{n}$, there exists (according to the density property of the set $\mathbb{Q}$) a rational number x_{n+1} such that $x_{n+1} \in (x, \min\{x_n, x + \frac{1}{n+1}\})$. Hence $x < x_{n+1} < x_n$ and $x_{n+1} < x + \frac{1}{n+1}$. This way we obtain the strictly decreasing sequence $(x_n)_{n\in\mathbb{N}^*}$ of rational numbers with the property that

$$x < x_n < x + \frac{1}{n}, \forall\, n \in \mathbb{N}^*.$$

Applying the Sandwich-Theorem, we conclude that $\lim_{n\to\infty} x_n = x$. Hence every real number is the limit of a decreasing sequence of rational numbers.

In order to prove that every real number is the limit of an increasing sequence of rationals, consider an arbitrary $x \in \mathbb{R}$. We already know that $-x$ is the limit of a decreasing sequence $(x_n)_{n\in\mathbb{N}^*}$ of rationals. Then $(-x_n)_{n\in\mathbb{N}^*}$ is an increasing sequence of rationals converging to x.