Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

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Chapter 1. Differential Equations. Forms and Solutions[1]

Forms. We will study *differential equations in the vectorial form*

$$x' = f(t,x) \tag{1}$$

where the function $f : D \to \mathbb{R}^n$ is continuous on the open subset $D \subset \mathbb{R} \times \mathbb{R}^n$. The natural number $n \geq 1$ is called *the dimension of the equation*. The unknown of the differential equation (1) is *a function* $x : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$. The variable of the function x is denoted by t. It is also said that t is *the independent variable* of (1), while x is *the dependent variable*. The symbol x' in (1) denotes *the first order derivative* of x with respect to t.

When $n = 1$ the equation is said to be *scalar*. Note that for $n \geq 2$ we can say that (1) is a system of n scalar differential equations with n scalar unknowns. More precisely, denoting the components of the vectorial functions x and f by $x_1, x_2, ..., x_n$ and $f_1, f_2, ..., f_n$, respectively (note that we consider x and f as column vectors), we can write equation (1) as

$$\begin{array}{rcl} x_1' &=& f_1(t, x_1, ..., x_n) \\ x_2' &=& f_2(t, x_1, ..., x_n) \\ ... && \\ x_n' &=& f_n(t, x_1, ..., x_n). \end{array}$$

When first presenting differential equations, one can say, roughly speaking, that a differential equation is a relation involving the derivatives of some unknown function up to a given order. This means that it is a scalar equation of the form

$$x^{(n)} = g(t, x, x', ..., x^{(n-1)}). \tag{2}$$

Here we consider $g : D \to \mathbb{R}$ a continuous function on the open subset $D \subset \mathbb{R} \times \mathbb{R}^n$. The natural number $n \geq 1$ is called *the order* of the differential equation (2). The unknown is a scalar *function* $x : I \to \mathbb{R}$ defined on $I \subset \mathbb{R}$ and whose variable is denoted by t. The symbol $x^{(k)}$ in (2) denotes *the k-th order derivative* of x with respect to t, for any $k = 1, ..., n$.

We will show in the sequel that an equation of the form (2) can be put into the form (1). First note that $x(t)$ in (2) is a scalar, while $x(t)$ in (1) is a vector. Hence we use a new notation, X, for a vectorial function, such that we have to arrive to the n-dimensional system

$$X' = f(t, X). \tag{3}$$

This also clarifies that we need to introduce $(n-1)$ scalar unknowns beside the scalar unknown x of our equation (2). These new unknowns will be the derivatives of x up to order $(n-1)$, that is

$$X_1 = x, \quad X_2 = x', \quad X_3 = x'', \quad ... \quad X_n = x^{(n-1)}. \tag{4}$$

From (2) and (4) we obtain that $X_1, ..., X_n$ satisfy

$$\begin{aligned} X_1' &= X_2 \\ X_2' &= X_3 \\ &\dots \\ X_{n-1}' &= X_n \\ X_n' &= g(t, X_1, X_2, ..., X_n). \end{aligned}$$

The final step in seeing that this system can be put into the vectorial form (3) is to identify the components of the vectorial function f. Of course, these are

$$f_1(t, X_1, ..., X_n) = X_2, \quad f_2(t, X_1, ..., X_n) = X_3, \quad ... \quad f_n(t, X_1, ..., X_n) = g(t, X_1, ..., X_n).$$

In the sequel we provide some examples.

1) $x' = 2t + \sin t, \quad x' = x, \quad x' = tx, \quad x' = \sin(t^2 x)$ are scalar first order differential equations. For each of them the unknown is the function x of variable t.

2) The same equations can be written using other notations for the variables. For example, when we denote the unknown function by u and let the independent variable be t, we have $\quad u' = 2t + \sin t, \quad u' = u, \quad u' = tu, \quad u' = \sin(t^2 u)$.

3) We write now the same equations as in 1) and 2) as $y' = 2x + \sin x$, $y' = y$, $y' = xy$, $y' = \sin(x^2 y)$. For each of them the unknown is the function y of variable x.

4) $x''' = t$, $x''' = 3\cos t + e^t - 5x' + 7x'x''$ are scalar third order differential equations. For each of them the unknown is the function x of variable t.

5) The following is a 2-dimensional differential system, or, in other words, a system of 2 (scalar) differential equations with two unknowns, the functions x_1, x_2 of variable t.

$$\begin{aligned} x_1' &= tx_1 + \sin x_2 \\ x_2' &= -\sin(2t)x_1. \end{aligned}$$

By denoting the unknowns as x, y of variable t, the same system can be written as

$$\begin{aligned} x' &= tx + \sin y \\ y' &= -\sin(2t)x. \end{aligned}$$

Solutions. Now we intend to present the precise notion of solution for a differential equation. We give also some examples.

Definition 1 *We say that a vectorial function $\varphi : I \to \mathbb{R}^n$ is a solution of the differential equation* (1) *if*

(i) $I \subset \mathbb{R}$ *is an open interval,* $\varphi \in C^1(I, \mathbb{R}^n)$,
(ii) $(t, \varphi(t)) \in D$, *for all* $t \in I$,
(iii) $\varphi'(t) = f(t, \varphi(t))$, *for all* $t \in I$.

In particular, for the nth order differential equation (2) the notion of solution is as follows.

Definition 2 *We say that scalar a function $\varphi : I \to \mathbb{R}$ is a solution of the nth order differential equation* (2) *if*

(i) $I \subset \mathbb{R}$ *is an open interval,* $\varphi \in C^n(I)$,
(ii) $(t, \varphi(t), \varphi'(t), ..., \varphi^{(n-1)}(t)) \in D$, *for all* $t \in I$,
(iii) $\varphi^{(n)}(t) = g(t, \varphi(t), \varphi'(t), ..., \varphi^{(n-1)}(t))$, *for all* $t \in I$.

We present now some examples in the form of exercises (this means that you have to check them).

1) The function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = t^2 - \cos t + 34$ is a solution of the scalar first order differential equation $x' = 2t + \sin t$.

2) The function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = -23\, e^t$ is a solution of the scalar first order differential equation $x' = x$.

3) The function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = 987\, e^{t^2/2}$ is a solution of the scalar first order differential equation $x' = tx$.

4) Let the functions $\varphi_1, \varphi_2, \varphi_3 : (-2, \infty) \to \mathbb{R}$ be given by the expressions $\varphi_1(t) = 1 + t$, $\varphi_2(t) = 1 + 2t$, $\varphi_3(t) = 1$. For the scalar first order differential equation

$$x' = \frac{x^2 - 1}{t^2 + 2t}$$

we have that φ_1 and φ_3 are solutions, while φ_2 is not a solution.

We remark that, in general, a differential equation has *many* solutions, where *many* does not mean 2 or 3, not even ten thousands. For example, for the scalar first order differential equation $x' = 2t + 1$ the function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = t^2 + t + c$ is a solution for an arbitrary constant $c \in \mathbb{R}$. Hence, this differential equation has as many solutions as real functions. In other words, the cardinal of the set of the solutions of $x' = 2t + 1$ is $\aleph$.

When talking about equations, one is used to say that he wants to "solve" it. To "solve" a differential equation means to find the whole family of solutions, which will be represented in a formula depending on one or more arbitrary constants. This formula is also called *the general solution* of the differential equation. For example, we say that $x' = 2t+1$ has the general solution $x = t^2+t+c$, for arbitrary $c \in \mathbb{R}$.

It is worth to say that one (human or computer) can not find the general solution of any differential equation. It is proved that the general solution of most of the differential equations can not be written as a finite combination of elementary functions.

The Initial Value Problem. When adding Initial Conditions to a differential equation, we say that an Initial Value Problem (IVP, for short) is formulated. These type of problems are also called Cauchy Problems, after the French mathematician Augustin-Louis Cauchy (1789-1857). More precisely, the IVP for (1) is

$$\begin{aligned} x' &= f(t, x) \\ x(t_0) &= \eta, \end{aligned}$$

where $f : D \to \mathbb{R}^n$ continuous on the open subset $D \subset \mathbb{R} \times \mathbb{R}^n$ and $(t_0, \eta) \in D$ are all given. Note that t_0 is called *the initial time* while η is called *the initial value* or *the initial position.* In the particular case $n = 2$ we have

$$\begin{aligned} x_1' &= f_1(t, x) \\ x_2' &= f_2(t, x) \\ x_1(t_0) &= \eta_1, \\ x_2(t_0) &= \eta_2. \end{aligned}$$

The IVP for (2) is

$$\begin{aligned} x^{(n)} &= g(t, x, x', ..., x^{(n-1)}) \\ x(t_0) &= \eta_1 \\ x'(t_0) &= \eta_2 \\ &\dots \\ x^{(n-1)}(t_0) &= \eta_n, \end{aligned}$$

where $g : D \to \mathbb{R}$ continuous on the open subset $D \subset \mathbb{R} \times \mathbb{R}^n$ and $(t_0, \eta_1, ..., \eta_n) \in D$ are all given. In the particular case $n = 2$ we have

$$\begin{aligned} x'' &= g(t, x, x') \\ x(t_0) &= \eta_1 \\ x'(t_0) &= \eta_2. \end{aligned}$$

In this case η_1 is called *the initial position* while η_2 is *the initial velocity.*

It is worth to say that, in general, an IVP has a unique solution.

Examples of problems which are not correctly-defined IVPs.

1) $x' = tx - 1, \quad x(0) = 0, \quad x'(0) = 2$. It is not correct since it has an extra condition, $x'(0) = 2$, while the scalar differential equation is of first order. A correctly-defined IVP is $x' = tx - 1, \quad x(0) = 0$.

2) $x'' = tx - 1, \quad x(2) = 5, \quad x'(0) = -6$. It is not correct because there are two different "initial times", $t_0 = 2$ and also $t_0 = 0$. Two correctly-defined IVPs are $x'' = tx - 1, \quad x(0) = 5, \quad x'(0) = -6$ and $x'' = tx - 1, \quad x(2) = 5, \quad x'(2) = -6$.

3) $x' = 2x + \sin t - 5t^2 y, \quad y' = xy - 3tx^2y^3, \quad x(0) = 1, \quad x'(0) = 2$. It is not correct because for this first order differential system appears a condition for the first order derivative of one of the unknowns, i.e. $x'(0) = 2$. A correctly-defined IVP is $\quad x' = 2x + \sin t - 5t^2 y, \quad y' = xy - 3tx^2y^3, \quad x(0) = 1, \quad y(0) = 2$.

4) $x' = (x^2 - 1)/(t^2 + 2t), \quad x(-2) = 1$ is not correctly defined because the right-hand side of the differential equation is not defined for $t = -2$ (which is the initial time). More precisely, in the notations used here, consider $f(t, x) = (x^2 - 1)/(t^2 + 2t)$ and notice that it is not defined for $t \in \{-2, 0\}$. Hence, f it is defined only in $D = (-\infty, -2) \times \mathbb{R} \cup (-2, 0) \times \mathbb{R} \cup (0, \infty) \times \mathbb{R}$, but $(-2, 1) \notin D$ as it is required (see again the above definition where (t_0, η) must be in D).

Here are some exercises.

1) Knowing that the initial value problem $\quad x' = 1 - x^2, \quad x(0) = 1 \quad$ has a unique solution, find it among the following objects:

(a) *solution*; (b) the unit circle; (c) the constant function $x = 1$;
(d) the constant function $x = -1$; (e) the derivative.

2) Knowing that the initial value problem $\quad x' = 3x, \quad x(0) = 1 \quad$ has a unique solution, find it among the following functions:

(a) $x = e^t$; (b) $x = 1$; (c) $x = 1/3$; (d) $x = t$; (e) $x = e^{3t}$.

3) Knowing that the initial value problem $\quad x' = x - 3, \quad x(0) = 1 \quad$ has a unique solution, find it among the functions of the form $x = 3 + c\,e^t$, with $c \in \mathbb{R}$.

4) Knowing that the initial value problem $x' = x - e^t, \quad x(0) = 1$ has a unique solution, find it among the functions of the form $x = (at+b)\, e^t$, with $a, b \in \mathbb{R}$.

5) Check that, for any $c \geq 0$, the function $\varphi_c : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_c(t) = \begin{cases} 0 & , \quad t \leq c \\ \left[\frac{2}{3}(t-c)\right]^{3/2} & , \quad t > c \end{cases}$$

is a solution of the IVP $x' = x^{1/3}, \quad x(0) = 0$.

Chapter 2. Linear Differential Equations[2]

The form and the Existence and Uniqueness Theorem for the IVP. In the previous lecture we saw the general form of an nth order scalar differential equation. In this lecture we begin the study of a particular case of such equations, namely the class of nth order scalar linear differential equations which have the form

$$x^{(n)} + a_1(t)x^{(n-1)} + a_2(t)x^{(n-2)} + \cdots + a_{n-1}(t)x' + a_n(t)x = f(t), \tag{5}$$

where $a_1, ..., a_n, f \in C(I)$, $I \subset \mathbb{R}$ being a nonempty open interval.

A *solution* of (5) is a function $\varphi \in C^n(I)$ that satisfies (5) for all $t \in I$.

The functions $a_1, ..., a_n$ are called *the coefficients* and the function f is called *the nonhomogeneous part* or *the force* of equation (5). When $f \equiv 0$ we say that (5) is *linear homogeneous* or *unforced*, otherwise we say that (5) is *linear nonhomogeneous* or *forced*. When all the coefficients are constant functions, we say that (5) is *a linear differential equation with constant coefficients.*

Examples.

1) $x''' + x = 0$ is a third order linear homogeneous differential equation with constant coefficients.

2) $x'' + tx = 0$ is a second order linear homogeneous differential equation, but the coefficients are not all constant.

3) Let λ be a real parameter. The equation $x'' + \lambda x = 2\sin(3t) - t^2$ is a second order linear nonhomogeneous differential equation with constant coefficients. The nonhomogeneous part is $f(t) = 2\sin(3t) - t^2$.

4) The equation $x'' - 2x' + x^2 = 0$ is a second order *non-linear* differential equation. Indeed, it has one non-linear term, x^2.

In the conditions described above we have the following important result.

Theorem 1 *Let $t_0 \in I$ and $\eta_1, ..., \eta_n \in \mathbb{R}$ be given numbers. We have that the following IVP has a unique solution which is defined on the whole interval I.*

$$
\begin{aligned}
&x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = f(t)\\
&x(t_0) = \eta_1\\
&x'(t_0) = \eta_2\\
&...\\
&x^{(n-1)}(t_0) = \eta_n.
\end{aligned} \tag{6}
$$

For further reference we write below the form of a linear homogeneous differential equation.

$$x^{(n)} + a_1(t)x^{(n-1)} + a_2(t)x^{(n-2)} + \cdots + a_{n-1}(t)x' + a_n(t)x = 0. \tag{7}$$

When equation (5) is linear nonhomogeneous, we say (7) is *the linear homogeneous differential equation associated* to it.

The fundamental theorems for linear differential equations.[3] These theorems give the structure of the set of solutions of such equations. Their proofs relies on Linear Algebra. The key that opens the door of this theory is to associate a linear map to a linear differential equation. First we note that the set of continuous functions on an open interval, $C^n(I)$, has a linear structure when considering the usual operations of addition between functions and multiplication of a function with a real number. For each function $x \in C^n(I)$ we define a new function, denoted $\mathcal{L}x$, as

$$\mathcal{L}x(t) = x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \cdots + a_n(t)x(t), \quad \text{for all } t \in I.$$

It is not difficult to see that $\mathcal{L}x \in C(I)$. In this way we obtain a map between the linear spaces $C^n(I)$ and $C(I)$, i.e.

$$\mathcal{L} : C^n(I) \to C(I).$$

Proposition 1 (i) *The map $\mathcal{L}$ is linear, that is, for any $x, y \in C^n(I)$ and any $\alpha, \beta \in \mathbb{R}$ we have*

$$\mathcal{L}(\alpha x + \beta y) = \alpha\mathcal{L}x + \beta\mathcal{L}y.$$

(ii) *The linear homogeneous differential equation* (7) *can be written equivalently*

$$\mathcal{L}x = 0,$$

while the linear nonhomogeneous differential equation (5) *can be written equivalently*

$$\mathcal{L}x = f.$$

The fundamental theorem for linear homogeneous differential equations follows.

Theorem 2 *Let $x_1, ..., x_n$ be n linearly independent solutions of* (7)*. Then the general solution of* (7) *is*

$$x = c_1 x_1 + ... + c_n x_n, \quad c_1, ..., c_n \in \mathbb{R}.$$

Proof. Applying the previous Proposition, we obtain that the set of solutions of the linear homogeneous differential equation (7) is $\ker \mathcal{L}$, which, further, using Linear Algebra, is a linear subspace of $C^n(I)$. With all these in mind, note that, in order to complete the proof of our theorem it remains to prove that the linear space $\ker \mathcal{L}$ has dimension n. We know that the Euclidean space $\mathbb{R}^n$ has dimension n. We intend to find an isomorphism between $\mathbb{R}^n$ and $\ker \mathcal{L}$, because, as we know from Linear Algebra, an isomorphism between linear spaces preserves the dimension. Let us introduce a notation first. Let $t_0 \in I$ be fixed. For any $\eta \in \mathbb{R}^n$, whose components are denoted $\eta_1, ..., \eta_n$, we know by Theorem 1 that the IVP (6) has a unique solution. Denote this solution by $\phi(\cdot, \eta)$. It is not difficult to see that $\phi(\cdot, \eta) \in \ker \mathcal{L}$. In this way we defined the bijective map

$$\Phi : \mathbb{R}^n \to \ker \mathcal{L}, \quad \Phi(\eta) = \phi(\cdot, \eta).$$

We intend to show now that Φ is also a linear map. Let $\eta, \theta \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Denote $\varphi_1 = \Phi(\eta)$, $\varphi_2 = \Phi(\theta)$ and $\varphi_3 = \Phi(\alpha\eta + \beta\theta)$. Then, by the above definition of Φ, they satisfy

$$\mathcal{L}\varphi_1 = 0, \quad (\varphi_1(0), \varphi_1'(0), ..., \varphi_1^{(n-1)}(0)) = \eta, \tag{8}$$

$$\mathcal{L}\varphi_2 = 0, \quad (\varphi_2(0), \varphi_2'(0), ..., \varphi_2^{(n-1)}(0)) = \theta. \tag{9}$$

$$\mathcal{L}\varphi_3 = 0, \quad (\varphi_3(0), \varphi_3'(0), ..., \varphi_3^{(n-1)}(0)) = \alpha\eta + \beta\theta. \tag{10}$$

Denote now $\varphi_4 = \alpha\Phi(\eta) + \beta\Phi(\theta)$. Since $\varphi_4 = \alpha\varphi_1 + \beta\varphi_2$, by (8), (9), using the linearity of $\mathcal{L}$ and of the derivative, we deduce that

$$\mathcal{L}\varphi_4 = 0, \quad (\varphi_4(0), \varphi_4'(0), ..., \varphi_4^{(n-1)}(0)) = \alpha\eta + \beta\theta.$$

If we compare this last relation with (10) we see that both functions φ_3 and φ_4 are solutions of the same IVP, which, by Theorem 1, has a unique solution. Hence $\varphi_3 = \varphi_4$, that is

$$\Phi(\alpha\eta + \beta\theta) = \alpha\Phi(\eta) + \beta\Phi(\theta).$$

With this we finished to prove that Φ is a linear map.

As we discussed in the beginning of this proof, we can conclude now that the set of solutions of the linear homogeneous differential equation (7), which coincides with $\ker \mathcal{L}$, is a linear space of dimension n. The hypothesis of our theorem is that $x_1, ..., x_n$ are linearly independent solutions of (7), which, in other words, means that $\{x_1, ..., x_n\}$ is a basis of $\ker \mathcal{L}$. Then

$$\ker \mathcal{L} = \{c_1x_1 + ... + c_nx_n, \quad c_1, ..., c_n \in \mathbb{R}\},$$

which gives the conclusion of our theorem. □

The fundamental theorem for linear nonhomogeneous differential equations follows.

Theorem 3 *Let x_h be the general solution of the linear homogeneous differential equation associated to* (5) *and let x_p be some particular solution of* (5)*. Then the general solution of* (5) *is*

$$x = x_h + x_p.$$

Proof. The set of solutions of (5) coincides with the set of solutions of $\mathcal{L}x = f$. By Linear Algebra we know that the set of solutions of $\mathcal{L}x = f$ is $\ker \mathcal{L} + \{x_p\}$. With this the proof is finished. □

The linearity of the map $\mathcal{L}$ assures the validity of the following result, which is called *The superposition principle.*

Theorem 4 *Let $f_1, f_2 \in C(I)$ and $\alpha \in \mathbb{R}$. Suppose that x_{p1} is some particular solution of $\mathcal{L}x = f_1$ and x_{p2} is some particular solution of $\mathcal{L}x = f_2$.*

Then $x_p = x_{p1} + x_{p2}$ is a particular solution of $\mathcal{L}x = f_1 + f_2$ and $\tilde{x}_p = \alpha\, x_{p1}$ is a particular solution of $\mathcal{L}x = \alpha f_1$.

Making a summary of the fundamental theorems we can describe the main steps of a *method for finding the general solution* of a linear nonhomogeneous equation of the form (5), i.e.

$$x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = f(t). \tag{1}$$

Step 1. Write the linear homogeneous differential equation associated $x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = 0$ and find its general solution. Denote it by x_h. For this it is sufficient to find n linearly independent solutions, denote them by $x_1, ..., x_n$. Hence,

$$x_h = c_1x_1 + ... + c_nx_n, \quad c_1, ..., c_n \in \mathbb{R}.$$

Step 2. Find a particular solution of the linear nonhomogeneous equation (5). Denote it by x_p.

Step 3. Write the general solution of (5) as

$$x = x_h + x_p.$$

Example-exercise. Find the general solution of

$$x' - x = -5.$$

First we notice that this is a first order linear nonhomogeneous differential equation. We follow the steps of the method presented above.

Step 1. The linear homogeneous differential equation associated is

$$x' - x = 0.$$

In order to find its general solution it is sufficient if we find a non-null solution. We notice that $x_1 = e^t$ verifies $x' = x$, hence it is a nonnull solution. Then

$$x_h = c\,e^t, \quad c \in \mathbb{R}.$$

Step 2. We notice that $x_p = 5$ verifies $x' - x = -5$.

Step 3. The general solution of $x' - x = -5$ is

$$x = c\,e^t + 5, \quad c \in \mathbb{R}.$$

[4] **The general solution of a first order linear differential equation**. Take

$$x' + a(t)x = f(t) \tag{11}$$

where $a, f \in C(I)$ and write also the linear homogeneous equation associated,

$$x' + a(t)x = 0. \tag{12}$$

Let $t_0 \in I$ be fixed and denote by A a primitive of a, that is

$$A(t) = \int_{t_0}^{t} a(s)ds.$$

It is not difficult to check the following result on (12).

Proposition 2 (i) *We have that* $x_1 = e^{-A(t)}$ *is a solution of* (12)*. Hence, the general solution of this differential equation is* $x = ce^{-A(t)}$, $c \in \mathbb{R}$.

(ii) *In particular, when* a *is a constant function, that is* $a(t) = \lambda$ *for all* $t \in I$ *and for some* $\lambda \in \mathbb{R}$*, then* $x_1 = e^{-\lambda t}$ *is a solution of* $x' + \lambda x = 0$*. Hence, the general solution of this differential equation is* $x = ce^{-\lambda t}$, $c \in \mathbb{R}$.

Let us now deduce *qualitative* properties of the solutions of (12).

Proposition 3 (i) *Let* $\varphi : I \to \mathbb{R}$ *be a solution of* (12)*. Then either* $\varphi(t) = 0$ *for all* $t \in I$*, or* $\varphi(t) \neq 0$ *for all* $t \in I$.

(ii) *Assume that* $a(t) \neq 0$ *for all* $t \in I$ *and let* $\varphi : I \to \mathbb{R}$ *be a non-null solution of* (12)*. Then* φ *is strictly monotone.*

Proof. (i) In this situation the most handful way to prove this, is to use that $x = ce^{-A(t)}$, $c \in \mathbb{R}$ is the general solution of (12). Indeed, we deduce that there exists some $\tilde{c} \in \mathbb{R}$ such that $\varphi(t) = \tilde{c}\, e^{-A(t)}$ for all $t \in I$. Then either $\tilde{c} = 0$ or $\tilde{c} \neq 0$. Using that the exponential function is always positive, we obtain the conclusion.

We comment that there is another proof that uses the Existence and Uniqueness Theorem. Indeed, let φ be a solution of (12) such that $\varphi(t_0) = 0$ for some $t_0 \in I$. Then φ is a solution of the IVP

$$x' + a(t)x = 0$$
$$x(t_0) = 0.$$

As one can easily see the null function is also a solution of this IVP, which, by Theorem 1, has a unique solution. Hence $\varphi \equiv 0$.

(ii) Since a is a continuous function on I, the hypothesis $a(t) \neq 0$ for all $t \in I$ assures that, either $a(t) > 0$ for all $t \in I$ or $a(t) < 0$ for all $t \in I$. Applying (i) we deduce that a similar result holds for φ. Hence

$$\text{either } a(t)\varphi(t) > 0 \text{ for all } t \in I \text{ or } a(t)\varphi(t) < 0 \text{ for all } t \in I. \tag{13}$$

We are interested to study the sign of φ'. Since φ is a solution of (12), we have that $\varphi'(t) = -a(t)\varphi(t)$ for all $t \in I$. Using (13) we obtain that also φ' has a definite sign on I, hence φ is strictly monotone on I. □

An alternative method to find the general solution of (12) is *the separation of variables method*, which we present below.

Step 1. We notice that $x = 0$ is always a solution of (12).

Step 2. We look now for the non-null solutions of (12) which we write as $x'(t) = -a(t)x(t)$. We write this equation in the form (we "separate" the dependent variable x from the independent variable t)

$$\frac{x'(t)}{x(t)} = -a(t).$$

Step 3. We integrate the above equation, that is we look for primitives of each side of the equation. Note that a primitive for the left-hand side is $\ln|x(t)|$, while a primitive for the right-hand side is $-A(t)$. Hence we obtain

$$\ln|x(t)| = -A(t) + c, \quad c \in \mathbb{R}.$$

Step 4. We write the solution explicitly. We have $|x(t)| = e^{-A(t)+c}$, hence $x(t) = \pm e^c\, e^{-A(t)}$ for an arbitrary constant $c \in \mathbb{R}$. Now we note that $\{\pm e^c \ : \ c \in \mathbb{R}\} = \mathbb{R}^*$. Then we can write equivalently $x(t) = c\, e^{-A(t)}$ for an arbitrary constant $c \in \mathbb{R}^*$.

Step 5. The solution $x = 0$ found at *Step 1* and the family of solutions $x(t) = c\, e^{-A(t)}$, $c \in \mathbb{R}^*$ found at *Step 4* can be written together into the formula

$$x(t) = c\, e^{-A(t)}. \quad c \in \mathbb{R}.$$

Now we present *the Lagrange method*, also called *the variation of the constant method* used to find a particular solution of the first order linear nonhomogeneous differential equation (11). This consists in looking for some function $\varphi \in C^1(I)$ with the property that

$$x_p = \varphi(t)\, e^{-A(t)}$$

is a solution of (11). After replacing this form in (11) we obtain $\varphi'(t) = e^{A(t)} f(t)$. Hence, some function φ can be written as $\varphi(t) = \int_{t_0}^t e^{A(s)} f(s)ds$. Consequently we found a particular solution of (11)

$$x_p = \int_{t_0}^t e^{-A(t)+A(s)} f(s)ds.$$

The next result follows now applying the Fundamental Theorem for linear nonhomogeneous differential equations.

Proposition 4 *The general solution of the first order linear nonhomogeneous differential equation* (11) *is*

$$x(t) = c\, e^{-A(t)} + \int_{t_0}^t e^{-A(t)+A(s)} f(s)ds, \quad c \in \mathbb{R}.$$

We mention that, in practice, the separation of variables method and, respectively, the Lagrange method are widely used. An alternative way to solve both (12) and (11) is using the

Property 1 *The function $\mu(t) = e^{A(t)}$ is an integrating factor for* (11).

Proof. We will show that, after multiplying (11) with the function $\mu(t)$ given in the statement, it is possible to integrate it, thus finding its general solution. Indeed, after multiplying (11) with $e^{A(t)}$ we obtain

$$x'(t)e^{A(t)} + x(t)\, a(t)e^{A(t)} = f(t)e^{A(t)},$$

that, further can be written as $(x(t)\, e^{A(t)})' = f(t)e^{A(t)}$. Of course, a primitive of the left-hand side is $x(t)\, e^{A(t)}$, and a primitive of the right-hand side is $\int_{t_0}^t e^{A(s)} f(s)ds$. We thus obtain

$$x(t)\, e^{A(t)} = \int_{t_0}^t e^{A(s)} f(s)ds + c, \quad c \in \mathbb{R}.$$

Writing explicitly the unknown $x(t)$ we obtain the same expression of the general solution as in Proposition 4. □

[5] **Linear differential equations with constant coefficients.** In this special case there is a method, called *the characteristic equation method* to find the n linearly independent solutions of an nth order linear homogeneous equation. We will also present here *the undetermined coefficients method* to find a particular solution for such kind of equations when the nonhomogeneous part has some special forms.

We write now a linear homogeneous differential equation with constant coefficients denoted $a_1, ..., a_n \in \mathbb{R}$.

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1} x' + a_n x = 0, \tag{14}$$

and consider again the linear map $\mathcal{L}$ (defined in the beginning) corresponding to (14).

We start by noticing that, when looking for solutions of (14) of the form

$$x = e^{rt}$$

(with $r \in \mathbb{R}$ that has to be found), we obtain that r must be a root of the nth degree algebraic equation

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0. \tag{15}$$

More precisely, we have that

$$\mathcal{L}(e^{rt}) = e^{rt}\, l(r),$$

where

$$l(r) = r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n.$$

Then every real root of (15) provides a solution of (14). But we know that, in general, not all the roots of an algebraic equation are real. However, we will show how the roots of the algebraic equation (15) provide all the n linearly independent solutions of (14) needed to obtain its general solution.

For our purpose we need to see that the concept of real-valued solution for (14) can be extended to that of complex-valued solution. Denoting a complex-valued function by $\gamma : \mathbb{R} \to \mathbb{C}$, its real part by $u : \mathbb{R} \to \mathbb{R}$ and its imaginary part by $v : \mathbb{R} \to \mathbb{R}$ we have $\gamma(t) = u(t) + i\, v(t)$, for all $t \in \mathbb{R}$. The function γ can be identified with a vectorial function of one real variable t and with two real components u and

v. Hence properties of u and v (as, for example, continuity or differentiability) are transferred to γ and viceversa.

Since we defined a solution to be a real-valued function, we will only say that a complex-valued function *verifies* or not a differential equation. With respect to the linear homogeneous differential equation with constant real coefficients (14) we have the following result.

Proposition 5 *Assume that the complex-valued function* $\gamma \in C^n(\mathbb{R}, \mathbb{C})$ *verifies* (14). *Then, both its real part* u *and its imaginary part* v *are solutions of* (14).

Proof. In order to shorten the presentation, we use again the notation of the linear map $\mathcal{L}$ as presented in the beginning of this lecture. Thus equation (14) can be written equivalently as $\mathcal{L}x = 0$ It is not difficult to see that $\mathcal{L}(\gamma) = \mathcal{L}(u + iv) = \mathcal{L}u + i\mathcal{L}v$, where, of course, $\mathcal{L}u$ and $\mathcal{L}v$ are real-valued functions. By hypothesis we have that $\mathcal{L}(\gamma) = 0$. Thus $\mathcal{L}u = 0$ and $\mathcal{L}v = 0$, which give the conclusion. □

We need to work with the complex-valued function of real variable

$$\gamma(t) = e^{(\alpha+i\beta)t}, \quad t \in \mathbb{R},$$

where $\alpha, \beta \in \mathbb{R}$ are fixed real numbers. Using Euler's formula we know that its real and, respectively, imaginary parts are

$$u(t) = e^{\alpha t} \cos \beta t, \quad v(t) = e^{\alpha t} \sin \beta t.$$

Using that $\gamma'(t) = u'(t) + iv'(t)$, one can check that

$$\gamma'(t) = (\alpha + i\beta)e^{(\alpha+i\beta)t}, \quad t \in \mathbb{R}.$$

This last formula tells us that the derivatives of the function e^{rt}, where $r \in \mathbb{C}$ is fixed, are computed using the same rules as when $r \in \mathbb{R}$. Hence in the case that $r = \alpha + i\beta$ is a root of (15), the complex-valued function e^{rt} verifies (14). We thus have

Proposition 6 *If* $r = \alpha + i\beta$ *with* $\beta \neq 0$, *is a root of* (15), *then* $e^{\alpha t} \cos \beta t$ *and* $e^{\alpha t} \sin \beta t$ *are solutions of* (14).

We notice that, since the polynomial $l(r)$ has real coefficients, in the case that $r = \alpha + i\beta$ with $\beta \neq 0$ is a root of l, we have that its conjugate, $r = \alpha - i\beta$ is a root, too. According to the previous proposition, this gives that $e^{\alpha\, t}\cos\beta\, t$ and $-e^{\alpha\, t}\sin\beta\, t$ are solutions of (14). But this is no new information. In fact, it is usually said that the two solutions indicated in the proposition comes from the two roots $\alpha \pm i\beta$.

Hence we have seen that any complex root provides a solution of (14). But still there is the possibility that the solutions obtained are not enough, since we know by the Fundamental Theorem of Algebra that a polynomial of degree n has indeed n roots, but counted with their multiplicity. We will show that

Proposition 7 *If $r \in \mathbb{C}$ is a root of multiplicity m of the polynomial l, then $t^k\, e^{rt}$ verifies* (14) *for any $k \in \{0, 1, 2, ..., m-1\}$.*

Proof. We remind first that $r \in \mathbb{C}$ is a root of multiplicity m of the polynomial l if and only if

$$l(r) = l'(r) = ... = l^{(m-1)}(r) = 0.$$

By direct calculations we obtain for each $k \in \{0, 1, 2, ..., m-1\}$ that

$$\mathcal{L}(t^k\, e^{rt}) = e^{rt} \sum_{j=0}^{k} C_k^j\, t^{k-j}\, l^{(j)}(r),$$

which, in the case that $r \in \mathbb{C}$ is a root of multiplicity m gives that $\mathcal{L}(t^k\, e^{rt}) = 0$. $\square$

We describe now **The characteristic equation method** for the linear homogeneous differential equation with constant coefficients (14).

Step 1. Write the *characteristic equation* (15). Note that it is an algebraic equation of degree n (equal to the order of the differential equation) and with the same coefficients as the differential equation.

Step 2. Find all the n roots in $\mathbb{C}$ of (15), counted with their multiplicity.

Step 3. Associate n functions obeying the following rules.

For $r = \alpha$ a real root of order m we take m functions:

$$e^{\alpha t},\ te^{\alpha t}, \ldots,\ t^{m-1}e^{\alpha t}.$$

For $r = \alpha + i\beta$ and $r = \alpha - i\beta$ roots of order m we take $2m$ functions

$$e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t, \ldots,\ t^{m-1}e^{\alpha t}\cos\beta t,\ t^{m-1}e^{\alpha t}\sin\beta t.$$

The following useful result holds true.

Theorem 5 *The n functions found by applying the characteristic equation method are n linearly independent solutions of* (14).

In the discussion before the presentation of this method we proved that the n functions are solutions of (14). The proof of the above theorem would be completed by showing that they are linearly independent. But this is beyond the aim of these lectures.

We present now *the undetermined coefficients method* to find a particular solution for a linear nonhomogeneous differential equation

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1}x' + a_n x = f(t), \tag{16}$$

with constant coefficients $a_1, ..., a_n$ and with $f \in C(\mathbb{R})$ of special form. Denote again the characteristic polynomial $l(r) = r^n + a_1\, r^{(n-1)} + ... + a_n$.

We consider those functions f which can be solutions to some linear homogeneous differential equation with constant coefficients. More exactly, the function f can be either of the form $P_k(t)e^{\alpha t}$ or $P_k(t)e^{\alpha t}\cos\beta t + \tilde{P}_k(t)e^{\alpha t}\sin\beta t$, where $P_k(t)$ and $\tilde{P}_k(t)$ denote some polynomials in t of degree at most k. Roughly speaking, the idea behind this method is that (16) has a particular solution of the same form as $f(t)$. The following rules apply.

Assume that $f(t) = P_k(t)e^{\alpha t}$.

In the case that $r = \alpha$ is not a root of the characteristic polynomial $l(r)$, then $x_p = Q_k(t)e^{\alpha t}$ for some polynomial $Q_k(t)$ of degree at most k whose coefficients have to be determined.

In the case that $r = \alpha$ is a root of multiplicity m of the characteristic polynomial $l(r)$, then $x_p = t^m Q_k(t)e^{\alpha t}$ for some polynomial $Q_k(t)$ of degree at most k whose coefficients have to be determined.

Assume now that $f(t) = P_k(t)e^{\alpha t}\cos\beta t + \tilde{P}_k(t)e^{\alpha t}\sin\beta t$.

In the case that $r = \alpha + i\beta$ is not a root of the characteristic polynomial $l(r)$, then $x_p = Q_k(t)e^{\alpha t}\cos\beta t + \tilde{Q}_k(t)e^{\alpha t}\sin\beta t$ for some polynomials $Q_k(t)$ and $\tilde{Q}_k(t)$ of degree at most k whose coefficients have to be determined.

In the case that $r = \alpha + i\beta$ is a root of multiplicity m of the characteristic polynomial $l(r)$, then $x_p = t^m[Q_k(t)e^{\alpha t}\cos\beta t + \tilde{Q}_k(t)e^{\alpha t}\sin\beta t]$ for some polynomials $Q_k(t)$ and $\tilde{Q}_k(t)$ of degree at most k whose coefficients have to be determined.

We present now some examples to understand the rules of the *undetermined coefficients method*. For simplicity, we take equations with the same homogeneous part. This will be $\mathcal{L}x = x'' - 4x$, whose characteristic polynomial $l(r) = r^2 - 4$ has the real simple roots $r_1 = -2$ and $r_2 = 2$.

1) For $x'' - 4x = 1$ we have $f(t) = 1$, which is a polynomial of degree 0. We have to check whether $r = 0$ is a root of $l(r)$. Of course, it is not a root. Then we look for $x_p = a$, where $a \in \mathbb{R}$ has to be determined.

2) For $x'' - 4x = 2t$ we have $f(t) = 2t^2$, which is a polynomial of degree 2. We have to check whether $r = 0$ is a root of $l(r)$. Of course, it is not a root. Then we look for $x_p = at^2 + bt + c$, where $a, b, c \in \mathbb{R}$ have to be determined.

3) For $x'' - 4x = -5e^{3t}$ we have $f(t) = -5e^{3t}$. We have to check whether $r = 3$ is a root of $l(r)$. Of course, it is not a root. Then we look for $x_p = ae^{3t}$, where $a \in \mathbb{R}$ has to be determined.

4) For $x'' - 4x = -5te^{3t}$ we have $f(t) = -5te^{3t}$. We have to check whether $r = 3$ is a root of $l(r)$. Of course, it is not a root. Then we look for $x_p = (at + b)e^{3t}$, where $a, b \in \mathbb{R}$ have to be determined.

5) For $x'' - 4x = -5e^{2t}$ we have $f(t) = -5e^{2t}$. We have to check whether $r = 2$ is a root of $l(r)$. It is a simple root. Then we look for $x_p = ate^{2t}$, where $a \in \mathbb{R}$ has to be determined.

6) For $x'' - 4x = -5 \sin 2t$ we have $f(t) = -5 \sin 2t$. We have to check whether $r = 2i$ is a root of $l(r)$. Of course, it is not a root. Then we look for $x_p = a \sin 2t + b \cos 2t$, where $a, b \in \mathbb{R}$ have to be determined.

Chapter 3
The dynamical system generated by a differential equation[6]

We consider differential equations of the form

$$\dot{x} = f(x) \tag{17}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given C^1 function, the unknown is a function x of variable t (from time), $\dot{x}$ is the Newton's notation for the derivative with respect to time. Equation (17) is said to be *autonomous* because the function f does not depend on t. In this lecture we define important concepts that, all together, define what is called *the dynamical system generated by* (17), such as: *the state space, the flow, the orbits, the phase portrait.*

A very important result is the following *existence and uniqueness theorem.*

Theorem 6 *Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\eta \in \mathbb{R}^n$. Then the Initial Value Problem*

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= \eta \end{aligned} \tag{18}$$

has a unique solution defined on an open (maximal) interval $I_\eta = (\alpha_\eta, \omega_\eta) \subset \mathbb{R}$, which, of course, is such that $0 \in I_\eta$. Denote this solution by $\varphi(\cdot, \eta)$.

If $\varphi(\cdot, \eta)$ is bounded then $I_\eta = \mathbb{R}$.

If $\varphi(\cdot, \eta)$ is bounded to the right then $\omega_\eta = \infty$.

If $\varphi(\cdot, \eta)$ is bounded to the left then $\alpha_\eta = -\infty$.

The map φ of two variables t and η defined in the previous theorem is called *the flow of the dynamical system generated by equation* (17). Some important properties of this map are

(i) $\varphi(0, \eta) = \eta$;

(ii) $\varphi(t + s, \eta) = \varphi(t, \varphi(s, \eta))$ for each t and s when the map on either side is defined;

(iii) φ is continuous with respect to η.

It is easy to see that (i) holds true. In order to prove (ii) let us consider s and η be fixed and x_1, x_2 be two functions given by

$$x_1(t) = \varphi(t+s,\eta) \quad \text{and} \quad x_2(t) = \varphi(t, \varphi(s,\eta)).$$

By the definition of the flow, x_2 is a solution of the IVP

$$\begin{aligned} &\dot{x} = f(x) \\ &x(0) = \varphi(s,\eta). \end{aligned} \tag{19}$$

In the same time we have that $x_1(0) = \varphi(s,\eta)$ and $\dot{x}_1(t) = \frac{d}{dt}(\varphi(t+s,\eta)) = \dot{\varphi}(t+s,\eta) = f(\varphi(t+s,\eta)) = f(x_1(t))$. Hence, x_1 is also a solution of the IVP (19). As a consequence of Theorem 6, this IVP has a unique solution, thus the two functions x_1 and x_2 must be equal. The proof of (iii) is beyond the aim of these lectures.

When working with the flow, η it is said to be *the initial state* of the dynamical system generated by equation (17), while $\varphi(t,\eta)$ is said to be the *state at time t*. According to these, the space $\mathbb{R}^n$ to which belong the states it is called *the state space* of the dynamical system generated by (17). It is also called *the phase space*.

We say that $\eta^* \in \mathbb{R}^n$ is an *equilibrium state*/point (or critical point, or stationary point or steady-state solution) of the dynamical system generated by (17) when

$$\varphi(t,\eta^*) = \eta^* \quad \text{for any} \quad t \in \mathbb{R}.$$

It is important to notice that the equilibria of (17) can be found solving in $\mathbb{R}^n$ the equation

$$f(x) = 0.$$

The *orbit* of the initial state η is

$$\gamma(\eta) = \{\ \varphi(t,\eta)\ :\ t \in I_\eta\ \}.$$

The *positive orbit* of the initial state η is

$$\gamma^+(\eta) = \{\ \varphi(t,\eta)\ :\ t \in I_\eta,\ t > 0\ \}.$$

The *negative orbit* of the initial state η is

$$\gamma^-(\eta) = \{\ \varphi(t,\eta)\ :\ t \in I_\eta,\ t < 0\ \}.$$

Note that an orbit is a curve in the state space $\mathbb{R}^n$ parameterized by the time t. Also, note that the orbit of an equilibrium point is formed only by this point, that is, $\gamma(\eta^*) = \{\,\eta^*\,\}$ when η^* is an equilibrium.

The phase portrait of the dynamical system (17) is the representation in the state space $\mathbb{R}^n$ of all its orbits, together with some arrows that indicate the evolution in time.

The particular case $n = 1$. In this case the state space is $\mathbb{R}$ and we say that equation (17) is scalar. We start with an example.

Example 1. Study the dynamical system generated by the scalar equation

$$\dot{x} = -x.$$

The state space is $\mathbb{R}$. The flow is

$$\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \varphi(t,\eta) = \eta e^{-t}.$$

There is a unique equilibrium point, $\eta^* = 0$ whose orbit is

$$\gamma(0) = \{0\}.$$

For any $\eta > 0$ we have $\quad \gamma(\eta) = \{\,\eta e^{-t}\ :\ t \in \mathbb{R}\,\} = (0,\infty)$,
$\gamma^+(\eta) = \{\,\eta e^{-t}\ :\ t > 0\,\} = (0,\eta), \quad \gamma^-(\eta) = \{\,\eta e^{-t}\ :\ t < 0\,\} = (\eta,\infty)$.
For any $\eta < 0$ we have $\quad \gamma(\eta) = (-\infty,0), \quad \gamma^+(\eta) = (\eta,0), \quad \gamma^-(\eta) = (-\infty,\eta)$.
Hence, the orbits are: $(-\infty,0), \quad \{0\}, \quad (0,\infty)$. The arrows must indicate that the states of the dynamical system evolves toward the equilibrium point 0. $\diamond$

We have the following result.

Lemma 1 *Let $f \in C^1(\mathbb{R})$ and φ be the flow of $\dot{x} = f(x)$. Let $\eta, \xi \in \mathbb{R}$ be fixed. Then*

(i) $\gamma(\eta) \subset \mathbb{R}$ is an open interval and $\varphi(\cdot,\eta)$ is a strictly monotone function for each η which is not an equilibrium;

(ii) $\varphi(t,\eta) < \varphi(t,\xi)$ *for all* t*, if* $\eta < \xi$*;*

(iii) if $\gamma^+(\eta)$ *is bounded, then* $\lim\limits_{t\to\infty} \varphi(t,\eta) = \eta^*$*, where* η^* *is an equilibrium point;*

(iv) if $\gamma^-(\eta)$ *is bounded, then* $\lim\limits_{t\to-\infty} \varphi(t,\eta) = \eta^*$*, where* η^* *is an equilibrium point.*

Proof. (i) We prove first the second statement. Since η is not an equilibrium point we have that either $f(\eta) > 0$ or $f(\eta) < 0$. We consider the case when $f(\eta) > 0$ (the other case is similar).

Then we have that $\frac{d}{dt}\varphi(0,\eta) = f(\eta) > 0$. We assume by contradiction that there exists t_1 such that $\frac{d}{dt}\varphi(t_1,\eta) \leq 0$. We denote $\eta_1 = \varphi(t_1,\eta)$. Since $f(\eta) > 0$ and $f(\eta_1) \leq 0$, it follows that there exists η^* between η and η_1, such that $f(\eta^*) = 0$. But the function $\varphi(\cdot,\eta)$ is continuous on the open interval $(0,t_1)$, or $(t_1,0)$. Hence, it takes all the values between η and η_1. This means that there exists t_2 such that $\varphi(t_2,\eta) = \eta^*$. We consider now the IVP

$$\begin{aligned} \dot{x} &= f(x) \\ x(t_2) &= \eta^* \end{aligned}$$

and see that it has two solutions: $\varphi(\cdot,\eta)$ and the constant function η^*. This fact contradicts the unicity property.

The first statement follows by the fact that $\gamma(\eta)$ is the image of the continuous and strictly monotone function $\varphi(\cdot,\eta)$ which, by Theorem 6, is defined on an open interval.

(ii) In these hypotheses we have that $\varphi(0,\eta) - \varphi(0,\xi) < 0$. Assume by contradiction that there exists t_1 such that $\varphi(t_1,\eta) - \varphi(t_1,\xi) \geq 0$. From here, using the continuity of the function $\varphi(t,\eta) - \varphi(t,\xi)$, we deduce that there exists t_2 such that $\varphi(t_2,\eta) - \varphi(t_2,\xi) = 0$. We consider now the IVP

$$\begin{aligned} \dot{x} &= f(x) \\ x(t_2) &= \varphi(t_2,\eta) \end{aligned}$$

and see that it has two different solutions: $\varphi(t,\eta)$ and $\varphi(t,\xi)$. This fact contradicts the unicity property.

(iii) The function $\varphi(t,\eta)$ is a solution of $\dot{x} = f(x)$, hence

$$\frac{d\varphi}{dt}(t,\eta) = f(\varphi(t,\eta)). \tag{20}$$

Since, in addition, the C^1 function $\varphi(t, \eta)$ is monotone and bounded as t goes to ∞, we deduce that there exists some $\eta^* \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \varphi(t,\eta) = \eta^* \quad \text{and} \quad \lim_{t\to\infty} \frac{d\varphi}{dt}(t,\eta) = 0. \tag{21}$$

Passing to the limit as $t \to \infty$ in (20) and taking into account equations (21), we obtain that

$$0 = f(\eta^*),$$

which means that η^* must be an equilibrium point. The proof of (iv) is similar. $\square$

As a consequence of the above result we give the following procedure useful to represent the phase portrait of any scalar dynamical system $\dot{x} = f(x)$.

Step 1. Find all the equilibria, i.e. solve $f(x) = 0$.

Step 2. Represent the equilibria on the state space, $\mathbb{R}$. The orbits are the ones corresponding to the equilibria and the open intervals of $\mathbb{R}$ delimited by the equilibria.

Step 3. Determine the sign of f on each orbit. According to this sign, insert an arrow on each orbit. If the sign is $+$, the arrow must indicate that x increases, while if the sign is $-$, the arrow must indicate that x decreases.

Example 2. Consider the differential equation $\dot{x} = x - x^3$.

The state space is $\mathbb{R}$.

The equilibrium points are -1, 0, 1.

The orbits are $(-\infty, -1)$, $\{-1\}$, $(-1, 0)$, $\{0\}$, $(0, 1)$, $\{1\}$, $(1, \infty)$.

The function $f(x) = x - x^3$ is positive on $(-\infty, -1)$, negative on $(-1, 0)$, positive on (0,1) and negative on $(1, \infty)$. $\diamond$

Example 3. How to read a phase portrait? Assume that we see a phase portrait of some scalar differential equation $\dot{x} = f(x)$ and note that, for example, the open bounded interval (a, b) is an orbit such that the arrow on it indicates to the right. Only with this information we can deduce some important properties of the flow of the differential equation having this phase portrait.

Let $\eta \in (a, b)$ be a fixed initial state. Then $\gamma(\eta) = (a, b)$, which means that *the image* of the function $\varphi(\cdot, \eta)$ *is the open bounded interval* (a, b) (we used only the definition of the orbit). By Theorem 6, since $\varphi(\cdot, \eta)$ is bounded, we must have that its interval of definition is $\mathbb{R}$. The fact that the arrow indicates to the right provides the information that the function $\varphi(\cdot,\eta)$ *is strictly increasing*. We know that a continuous increasing function defined on the interval $(-\infty,\infty)$ whose image is the interval (a,b) must have the limit as $t\to-\infty$ equal to a, and limit as $t\to\infty$ equal to b, hence $\lim\limits_{t\to-\infty}\varphi(t,\eta)=a$ and $\lim\limits_{t\to\infty}\varphi(t,\eta)=b$. By Lemma 1 we deduce that a and b must be *equilibria*.

Stability of the equilibria of dynamical systems

The notion of stability is of considerable theoretical and practical importance. Roughly speaking, an equilibrium point η^* is stable if all solutions starting near η^* stay nearby. If, in addition, nearby solutions tend to η^* as $t\to\infty$, then η^* is asymptotically stable. Precise definitions were given by the Russian mathematician Aleksandr Lyapunov in 1892.

We remind that we study differential equations of the form

$$\dot{x}=f(x) \tag{17}$$

where $f:\mathbb{R}^n\to\mathbb{R}^n$ is a given C^1 function.

Definition 3 *An equilibrium point η^* of equation (17) is said to be stable if, for any given $\varepsilon>0$, there is a $\delta>0$ such that, for every η for which $||\eta-\eta^*||<\delta$ we have that $||\varphi(t,\eta)-\eta^*||<\varepsilon$ for all $t\geq 0$.*

The equilibrium point η^* is said to be unstable if it is not stable.

An equilibrium point η^* is said to be asymptotically stable if it is stable and, in addition, there is an $r>0$ such that $||\varphi(t,\eta)-\eta^*||\to 0$ as $t\to\infty$ for all η satisfying $||\eta-\eta^*||<r$.

Stability of linear dynamical systems. We consider

$$\dot{x}=Ax \tag{22}$$

where the matrix $A\in\mathcal{M}_n(\mathbb{R})$ is called *the matrix of the coefficients* of the linear system (22). We assume that

$$\det A\neq 0$$

such that the only equilibrium point of (22) is $\eta^*=0$ (here 0 denotes the null vector from $\mathbb{R}^n$).

Definition 4 *We say that the linear system* (22) *is stable / asymptotically stable / unstable when its equilibrium point at the origin has this quality.*

We have the following important result. Its proof is beyond the aim of these lectures. It is given in terms of the *eigenvalues* of the matrix A. Remember that the eigenvalues of A have the property that they are the roots of the algebraic equation

$$\det(A - \lambda I_n) = 0.$$

Denote by $\sigma(A)$ the set of all eigenvalues $\lambda \in \mathbb{C}$ of the matrix A.

The notation $\Re(\lambda)$ for $\lambda \in \mathbb{C}$ means the real part of λ.

Theorem 7 *If* $\Re(\lambda) < 0$ *for any* $\lambda \in \sigma(A)$ *then the linear system* $\dot{x} = Ax$ *is asymptotically stable.*

If there exists some $\lambda \in \sigma(A)$ *such that* $\Re(\lambda) > 0$ *then the linear system* $\dot{x} = Ax$ *is unstable.*

The linearization method to study the stability of an equilibrium point of a nonlinear system. An equilibrium point η^* of (17) is said to be *hyperbolic* when $\Re(\lambda) \neq 0$ for any eigenvalue λ of the Jacobian matrix $Jf(\eta^*)$.

Theorem 8 *Let* η^* *be a hyperbolic equilibrium point of* (17). *We have that* η^* *is asymptotically stable / unstable if and only if the linear system*

$$\dot{x} = Jf(\eta^*)x$$

has the same quality.

Corollary 1 *Let* $n = 1$ *and* η^* *be an equilibrium point of* $\dot{x} = f(x)$.

If $f'(\eta^*) < 0$ *then* η^* *is asymptotically stable.*

If $f'(\eta^*) > 0$ *then* η^* *is unstable.*

Exercise 1. Study the stability of the equilibria of the damped pendulum equation

$$\ddot{\theta} + \frac{\nu}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0,$$

where $\nu > 0$ is the damping coefficient, m is the mass of the bob, $L > 0$ is the length of the rod and $g > 0$ is the gravity constant. What happen when $\nu = 0$?

Phase portraits of planar systems

Phase portraits of linear planar systems. We consider $\dot{x} = Ax$ where $A \in \mathcal{M}_2(\mathbb{R})$ with $\det A \neq 0$. In this case the state space is $\mathbb{R}^2$ and the orbits are curves. Denote by $\lambda_1, \lambda_2 \in \mathbb{C}$ the two eigenvalues of A. In the next definition the equilibrium point at the origin is classified as *node, focus, center, saddle*, depending on the eigenvalues of A.

Definition 5 *The equilibrium point $\eta^* = 0$ of the linear planar system $\dot{x} = Ax$ is a*

(i) **node** *if $\lambda_1 \leq \lambda_2 < 0$ or $0 < \lambda_1 \leq \lambda_2$. A node can be either asymptotically stable (when $\lambda_1 \leq \lambda_2 < 0$) or unstable (when $0 < \lambda_1 \leq \lambda_2$).*

(ii) **saddle** *if $\lambda_1 < 0 < \lambda_2$. Any saddle is unstable.*

(iii) **focus** *if $\lambda_{1,2} = \alpha \pm i\beta$ with $\beta \neq 0$ and $\alpha \neq 0$. A focus can be either asymptotically stable (when $\alpha < 0$) or unstable (when $\alpha > 0$).*

(iv) **center** *if $\lambda_{1,2} = \pm i\beta$ with $\beta \neq 0$. Any center is stable, but not asymptotically stable.*

In the sequel we provide some examples of linear planar systems having the equilibrium point at the origin of each of the types presented in the above definition. For each system given as an example our purpose is to study its type and stability, then to find the flow, the orbits and represent the phase portrait.

Example 1. $\dot{x} = -x, \quad \dot{y} = -2y.$

The matrix of the system is $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, which have the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence the equilibrium point at the origin is an *asymptotically stable node*.

In order to find the flow we have to consider the IVP

$$\dot{x} = -x, \quad \dot{y} = -2y, \quad x(0) = \eta_1, \quad y(0) = \eta_2$$

for each fixed $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. Calculations yields that the flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has the expression

$$\varphi(t, \eta_1, \eta_2) = (\eta_1 e^{-t},\ \eta_2 e^{-2t}).$$

The orbit corresponding to a fixed initial state $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ is

$$\gamma(\eta) = \{(\eta_1 e^{-t},\ \eta_2 e^{-2t}) \quad : \quad t \in \mathbb{R}\}.$$

In other words, the orbit is the curve in the plane xOy of parametric equations

$$x = \eta_1 e^{-t}, \;\; y = \eta_2 e^{-2t}, \;\; t \in \mathbb{R}.$$

Note that the parameter t can be eliminated and thus obtain the cartesian equation

$$\eta_1^2 \, y = \eta_2 \, x^2,$$

which, in general, is an equation of a parabola with the vertex in the origin. In the special case $\eta_1 = 0$ this is the equation $x = 0$, that is the Oy axis, while in the special case $\eta_2 = 0$ this is the equation $y = 0$, that is the Ox axis. Note that each orbit lie on one of these planar curves, but it is not the whole parabola or the whole line. More precisely, we have

$$\begin{array}{rcl}
\gamma(\eta) & = & \{(x,y) \in \mathbb{R}^2 \quad : \quad \eta_1^2 \, y = \eta_2 \, x^2, \quad \eta_1 x > 0, \quad \eta_2 y > 0\} \quad \text{when} \quad \eta_1\eta_2 \neq 0, \\
\gamma(\eta) & = & \{(0,y) \in \mathbb{R}^2 \quad : \quad \eta_2 y > 0\} \quad \text{when} \quad \eta_1 = 0, \; \eta_2 \neq 0, \\
\gamma(\eta) & = & \{(x,0) \in \mathbb{R}^2 \quad : \quad \eta_1 x > 0\} \quad \text{when} \quad \eta_1 \neq 0, \; \eta_2 = 0, \\
\gamma(0) & = & \{0\}.
\end{array}$$

On each orbit the arrows must point toward the origin.

Example 2. $\quad \dot{x} = x, \quad \dot{y} = -y.$

The matrix of the system is $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which have the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. Hence the equilibrium point at the origin is a *saddle*, which is *unstable*. In order to find the flow we have to consider the IVP

$$\dot{x} = x, \quad \dot{y} = -y, \quad x(0) = \eta_1, \quad y(0) = \eta_2$$

for each fixed $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. Calculations yields that the flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has the expression

$$\varphi(t, \eta_1, \eta_2) = (\eta_1 e^{t}, \, \eta_2 e^{-t}).$$

The orbit corresponding to a fixed initial state $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ is

$$\gamma(\eta) = \{(\eta_1 e^{t}, \, \eta_2 e^{-t}) \quad : \quad t \in \mathbb{R}\}.$$

In other words, the orbit is the curve in the plane xOy of parametric equations

$$x = \eta_1 e^{t}, \;\; y = \eta_2 e^{-t}, \;\; t \in \mathbb{R}.$$

Note that the parameter t can be eliminated and thus obtain the cartesian equation

$$xy = \eta_1\eta_2,$$

which, in general, is an equation of a hyperbola. More precisely, we have

$$\begin{aligned}
\gamma(\eta) &= \{(x,y)\in\mathbb{R}^2 \quad : \quad xy=\eta_1\eta_2, \quad \eta_1 x>0, \quad \eta_2 y>0\} \quad \text{when} \quad \eta_1\eta_2\neq 0,\\
\gamma(\eta) &= \{(0,y)\in\mathbb{R}^2 \quad : \quad \eta_2 y>0\} \quad \text{when} \quad \eta_1=0,\ \eta_2\neq 0,\\
\gamma(\eta) &= \{(x,0)\in\mathbb{R}^2 \quad : \quad \eta_1 x>0\} \quad \text{when} \quad \eta_1\neq 0,\ \eta_2=0,\\
\gamma(0) &= \{0\}.
\end{aligned}$$

On each orbit the arrows must point such that x moves away from 0, while y moves toward 0.

Example 3. $\dot{x} = -y, \quad \dot{y} = x.$

The matrix of the system is $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which have the eigenvalues $\lambda_{1,2} = \pm i$. Hence the equilibrium point at the origin is a *center*, which is *stable* but not asymptotically stable.

In order to find the flow we have to consider the IVP

$$\dot{x} = -y, \quad \dot{y} = x, \quad x(0)=\eta_1, \quad y(0)=\eta_2$$

for each fixed $\eta = (\eta_1,\eta_2)\in\mathbb{R}^2$. Calculations yields that the flow $\varphi : \mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$ has the expression

$$\varphi(t,\eta_1,\eta_2) = (\eta_1\cos t - \eta_2\sin t,\ \eta_1\sin t + \eta_2\cos t).$$

The orbit corresponding to a fixed initial state $\eta = (\eta_1,\eta_2)\in\mathbb{R}^2$ is

$$\gamma(\eta) = \{(\eta_1\cos t - \eta_2\sin t,\ \eta_1\sin t + \eta_2\cos t) \quad : \quad t\in\mathbb{R}\}.$$

In other words, the orbit is the curve in the plane xOy of parametric equations

$$x = \eta_1\cos t - \eta_2\sin t,\ \ y = \eta_1\sin t + \eta_2\cos t,\ \ t\in\mathbb{R}.$$

Note that the parameter t can be eliminated and thus obtain the cartesian equation

$$x^2 + y^2 = \eta_1^2 + \eta_2^2,$$

which, in general, is an equation of a circle with the center at the origin and radius $\sqrt{\eta_1^2+\eta_2^2}$. More precisely, we have

$$\begin{aligned}\gamma(\eta) &= \{(x,y)\in\mathbb{R}^2 \quad : \quad x^2+y^2=\eta_1^2+\eta_2^2\} \quad \text{when} \quad \eta_1^2+\eta_2^2\neq 0,\\ \gamma(0) &= \{0\}.\end{aligned}$$

On each orbit the arrows must point in the trigonometric sense.

Example 4. $\dot{x}=x-y, \quad \dot{y}=x+y.$

The matrix of the system is $A=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$, which have the eigenvalues $\lambda_{1,2}=1\pm i$. Hence the equilibrium point at the origin is an *unstable focus.*

In order to find the flow we have to consider the IVP

$$\dot{x}=x-y, \quad \dot{y}=x+y, \quad x(0)=\eta_1, \quad y(0)=\eta_2$$

for each fixed $\eta=(\eta_1,\eta_2)\in\mathbb{R}^2$. Calculations yields that the flow $\varphi:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$ has the expression

$$\varphi(t,\eta_1,\eta_2)=(\eta_1e^t\cos t-\eta_2e^t\sin t,\, \eta_2e^t\cos t+\eta_1e^t\sin t).$$

In order to find the shape of the orbits it is more convenient to pass to polar coordinates, that is, instead of the unknowns $x(t)$ and $y(t)$, to consider new unknowns $\rho(t)$ and $\theta(t)$ related by

$$x(t)=\rho(t)\cos\theta(t), \quad y(t)=\rho(t)\sin\theta(t), \tag{23}$$

where

$$\rho(t)>0 \quad \text{for any} \quad t\in\mathbb{R}.$$

We can write equivalently

$$\rho(t)^2=x(t)^2+y(t)^2, \quad \tan\theta(t)=\frac{y(t)}{x(t)}\,. \tag{24}$$

Our aim is to find a system satisfied by the new unknowns ρ and θ. In this system their derivatives will be involved. We will show two ways to find this new system.

Method 1. We take the derivatives in the equalities (23) and obtain

$$\dot{x}=\dot{\rho}\cos\theta-\rho\dot{\theta}\sin\theta, \quad \dot{y}=\dot{\rho}\sin\theta+\rho\dot{\theta}\cos\theta.$$

After we replace in our system, $\quad \dot{x} = x - y, \quad \dot{y} = x + y$, we obtain

$$\dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta = \rho\cos\theta - \rho\sin\theta, \quad \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta = \rho\cos\theta + \rho\sin\theta.$$

Calculations yields the system

$$\dot{\rho} = \rho, \quad \dot{\theta} = 1,$$

whose solution for a given initial state (ρ_0, θ_0) is given by

$$\rho(t) = \rho_0 e^t, \quad \theta(t) = \theta_0 + t,$$

which defines a logarithmic spiral in the (x, y) plane.

Since $\rho(t)$ is strictly increasing, the arrow on each orbit must point toward the infinity.

Method 2. We show that we arrive to the same system by taking the derivatives in the equalities (24). We obtain

$$2\rho\dot{\rho} = 2x\dot{x} + 2y\dot{y}, \quad \frac{\dot{\theta}}{\cos^2\theta} = \frac{\dot{y}x - y\dot{x}}{x^2}.$$

After we replace $\quad \dot{x} = x - y, \quad \dot{y} = x + y$, we obtain

$$\rho\dot{\rho} = x^2 - xy + xy + y^2, \quad \dot{\theta} = (x^2 + xy - xy + y^2)\frac{\cos^2\theta}{x^2},$$

which further can be written

$$\rho\dot{\rho} = \rho^2, \quad \dot{\theta} = \rho^2 \frac{\cos^2\theta}{\rho^2\cos^2\theta}.$$

It is not difficult to see that we arrive to the same system.

When studying nonlinear systems, the linerization method gives also information about the behavior of the orbits in a neighborhood of an equilibrium point. More precisely, we have the following result for planar systems.

Theorem 9 *Let $n = 2$ and η^* be a hyperbolic equilibrium point of $\dot{x} = f(x)$. Then η^* is a node / saddle / focus if and only if for the linear system $\dot{x} = Jf(\eta^*)x$, the origin has the same type.*

First integrals. The cartesian differential equation of the orbits of a planar system. We consider the planar autonomous system

$$\begin{aligned} \dot{x} &= f_1(x, y) \\ \dot{y} &= f_2(x, y) \end{aligned} \tag{25}$$

where $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a given C^1 function.

Definition 6 *Let $U \subset \mathbb{R}^2$ be an open nonempty set. We say that $H : U \to \mathbb{R}$ is a first integral in U of* (25) *if it is a non-constant C^1 function and the orbits of* (25) *lie on the level curves of H.*

Example 1. We saw that the orbits of the linear system with a center at the origin $\dot{x} = -y,\ \dot{y} = x$ are the circles of cartesian equation $x^2 + y^2 = c$, for any real constant $c \geq 0$. Hence, taking into account the definition of a first integral we can say that the function $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = x^2 + y^2$ is a first integral in $\mathbb{R}^2$ of this system. $\diamond$

Example 2. For the linear system with a saddle at the origin $\dot{x} = x,\ \dot{y} = -y$ the function $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = xy$ is a first integral in $\mathbb{R}^2$. $\diamond$

Example 3. There are systems without a first integral in $\mathbb{R}^2$. For the linear system with a node at the origin $\dot{x} - y,\ \dot{y} = x$ the function $H : \mathbb{R}^2 \setminus \{(0, y)\ :\ y \in \mathbb{R}\} \to \mathbb{R}$, $H(x, y) = \frac{y}{x^2}$ is a first integral in $\mathbb{R}^2 \setminus \{(0, y)\ :\ y \in \mathbb{R}\}$ of this system. In fact, this system does not have a first integral defined in a neighborhood of the origin. Only centers and saddles have this property. $\diamond$

In each of the previous examples a first integral was found after long calculations: first we found the flow, after the parametric equations of the orbits, and after the cartesian equation of the orbits. We used only the definitions of an orbit and, respectively, of a first integral and we had the advantage that the systems were simple enough to find explicitly their solutions. On the other hand, note that the a priori knowledge of a first integral is very helpful to draw the phase portrait.

Example 4. Knowing that $H(x, y) = y^2 + 2x^2$ is a first integral in $\mathbb{R}^2$ of the system $\dot{x} = y,\ \dot{y} = -2x$, represent its phase portrait.

First note that the level curves of H are ellipses that encircle the origin. Hence, these are the orbits of our system. The arrows on each orbit must point in the clockwise direction. $\diamond$

New questions arise: *How to check that a given function is a first integral? How to find a first integral?* The answer to the first question is given by the following result.

Proposition 8 *A nonconstant C^1 function $H : U \to \mathbb{R}$ is a first integral in U of* (25) *if and only if it satisfies the first order linear partial differential equation*

$$f_1(x,y)\frac{\partial H}{\partial x}(x,y) + f_2(x,y)\frac{\partial H}{\partial y}(x,y) = 0, \text{ for any } (x,y) \in U. \tag{26}$$

Example 5. We want to check that $H(x,y) = y^2 + 2x^2$ is a first integral in $\mathbb{R}^2$ of the system $\dot{x} = y,\ \dot{y} = -2x$. In the case of this system equation (26) becomes

$$y\frac{\partial H}{\partial x}(x,y) - 2x\frac{\partial H}{\partial y}(x,y) = 0.$$

It is not difficult to check that this equation is identically satisfied in $\mathbb{R}^2$ by the function $H(x,y) = y^2 + 2x^2$. $\diamond$

Of course, Proposition 8 gives also the answer to the second question, *How to find a first integral?*, only that we do not know how to solve a first order linear partial differential equation. It is not the aim of this course to explain all these in detail, but we will give the following helpful practical result.
A first integral of the planar system (25) (or, equivalently, a solution of the linear partial differential equation (26)) can be found by integrating the equation

$$\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}\,, \tag{27}$$

which is called *the cartesian differential equation of the orbits of* (25). After the integration of (27) we look for a function of two variables H such that we can write the general solution of (27) as $H(x,y) = c$, $c \in \mathbb{R}$. This H is a first integral of (25).

Example 5. We come back to the system $\dot{x} = y,\ \dot{y} = -2x$. This time we want to find a first integral. The previous statement says that we need to integrate the equation

$$\frac{dy}{dx} = \frac{-2x}{y}.$$

This is separable, and it can be written as $ydy = -2xdx$. After integration we obtain $y^2/2 = -x^2 + c$, $c \in \mathbb{R}$. Hence $H(x,y) = y^2/2 + x^2$ is a first integral in $\mathbb{R}^2$. $\diamond$

With the previous example, note that the first integral is not unique. Having one first integral, we can find many more, for example by multiplying it with any non null constant.

Exercise 1. Find a first integral in $\mathbb{R}^2$ of the undamped pendulum system

$$\dot{x} = y,\ \dot{y} = -\omega^2 \sin x,$$

where $\omega > 0$ is a real parameter. Show that there exists a region U in the state space $\mathbb{R}^2$ where the orbits are closed curves that encircle the origin, thus the origin is an equilibrium point of center type and it is stable. Note that the equilibrium point at the origin is not hyperbolic, which implies that the linearization method fails.

Similarly show that there exists a region U_k in the state space $\mathbb{R}^2$ where the orbits are closed curves that encircle the equilibrium point $(2k\pi, 0)$ for any $k \in \mathbb{Z}$.

Apply the linearization method to study the behavior of the orbits around the equilibrium point $(\pi, 0)$ and similarly around $(2k\pi + \pi, 0)$ for any $k \in \mathbb{Z}$.

Represent the phase portrait in $\mathbb{R}^2$.

Exercise 2. Find a first integral in the first quadrant $(0, \infty) \times (0, \infty)$ of the Lotka-Volterra system (also called the prey-predator system)

$$\dot{x} = N_1 x - xy,\ \dot{y} = -N_2 y + xy,$$

where $N_1, N_2 > 0$ are real parameters. [7]

Chapter 4. Numerical methods for differential equations[8]

We consider the IVP for a scalar first order differential equation

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{28}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is C^1 and $(x_0, y_0) \in \mathbb{R}^2$ is fixed. Here the unknown is the function y of variable x, and y' denote its derivative with respect to x. We have the following result

The IVP (28) *has a unique solution denoted* φ*, defined at least on some interval* $[x_0, x^*]$ *for* $x^* > x_0$.

As we already know, not always it is possible to find the exact expression of the solution φ. Because of this, a theory on how to find *good* approximations of φ had been developed. The *numerical methods* are part of this theory. Their aim is to find approximations for the values of the solution on some given points in the interval $[x_0, x^*]$. More precisely, if we consider a partition of the interval $[x_0, x^*]$,

$$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = x^*,$$

the purpose is to find some values denoted y_k as good approximations of $\varphi(x_k)$, for any $k = \overline{1, n}$. Then an approximate solution (i.e. function) can be found using interpolation methods (these type of methods are able to find a smooth function whose graph passes through the points (x_k, y_k) , for any $k = \overline{1, n}$).

The approximate values y_k are usually computed using a recurrence formula. There are now many such formulas, many of them adapted to particular classes of equations or systems. We will present here only the basic ones: the Euler method (discovered by the Swiss mathematician Leonhard Euler around 1765) and the Runge-Kutta method of order 2 (discovered by the German mathematicians Carl Runge and Martin Kutta around 1900).

For simplicity we work only with partitions of the interval where the points are at equal distance $h > 0$, i.e. they satisfy for any $k = \overline{0, n-1}$,

$$x_{k+1} = x_k + h.$$

One can deduce that $x_k = x_0 + k\,h$ for any $k = \overline{1, n}$. The number n is called *the number of steps* to reach the end of the interval. When given a step size $h > 0$ and an interval $[x_0, x^*]$ the number of steps to be as close to x^* as possible is

$$n = \left[\frac{x^* - x_0}{h}\right]$$

where $[\cdot]$ denotes the entire part.

When given the number of steps $n \in \mathbb{N}^*$ and an interval $[x_0, x^*]$, the step size is

$$h = \frac{x^* - x_0}{n}.$$

The Euler method formula for the IVP (28) *with constant step size h is*

$$y_{k+1} = y_k + h\, f(x_k, y_k), \quad k = \overline{0, n-1}.$$

The Runge-Kutta method formula for the IVP (28) *with constant step size h is*

$$y_{k+1} = y_k + \frac{h}{2}\, f(x_k, y_k) + \frac{h}{2}\, f(x_{k+1}, y_k + h\, f(x_k, y_k)), \quad k = \overline{0, n-1}.$$

Note that the starting point y_0 is the one the appears in the initial condition in (28). Hence $y_0 = \varphi(x_0)$ is an exact value. In fact, only in theory y_0 is an exact value, since practically, when y_0 has too many decimals (for example is an irrational number) a human or even a computer use only a truncation of it. When applying a numerical method, the errors are due to the formula itself and to the truncations made. Moreover, the errors accumulate at each step, thus, in general, *the errors are larger as the interval* $[x_0, x^*]$ *is larger.*

When the step size h is smaller, the partition of the interval $[x_0, x^*]$ is finer. In general *the errors are smaller as the step size h is smaller.*

Exercise 1. We consider the IVP $y' = y,\ y(0) = 1$ whose solution we know that it is $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = e^x$. Apply the Euler numerical method with a constant step size $h > 0$ on the interval $[0, x^*]$ where $x^* > 0$ is fixed. Prove that

$$y_k = (1+h)^k, \quad k = \overline{0, n} \quad \text{where } h = \frac{x^*}{n}.$$

Prove that $y_n \to \varphi(x^*) = e^{x^*}$ as $n \to \infty$. $\diamond$

Exercise 2. We consider the IVP $y' = 1 + xy^2,\ y(0) = 0$ whose unique solution is denoted by φ. Write the two numerical formulas with constant step size $h > 0$ for this IVP. Now take $h = 0.1$. Find the number of steps to reach $x^* = 1$. For each of the two formulas, compute approximate values for $\varphi(0.1)$, $\varphi(0.2)$ and $\varphi(0.3)$. $\diamond$

In the rest of the lecture we present two ideas on how the Euler numerical formula can be derived.

The first idea uses the notion of *Taylor polynomial.* We know that the Taylor polynomial around a point a of some function φ is a good approximation of it at least in a small neighborhood of a. The highest the degree of the Taylor polynomial, the better the approximation. But we consider only the Taylor polynomial of degree 1, that is

$$\varphi(a) + (x - a)\varphi'(a).$$

With this we approximate $\varphi(x)$ for x sufficiently close to a. Now consider that φ is the exact solution of the IVP (28). Remind that this implies

$$\varphi'(x) = f(x, \varphi(x)).$$

Instead of a we take a point x_k from the partition of the interval $[x_0, x^*]$ and instead of x we take x_{k+1} which must be close to x_k. Denote, as before, an approximation of $\varphi(x_k)$ by y_k. Then

$$\varphi(x_k) + (x_{k+1} - x_k)f(x_k, \varphi(x_k))$$

is an approximation for $\varphi(x_{k+1})$. But this formula is not practical since it uses the exact value $\varphi(x_k)$ which is not known. That is why it is replaced by an approximation y_k. After this we obtain

$$y_{k+1} = y_k + (x_{k+1} - x_k)f(x_k, y_k).$$

The second idea uses the geometrical interpretation of a differential equation, more exactly the notion of *direction field*. We will see that these directions are tangent to the solution curves of the differential equations and that, an approximate solution is constructed "following" these directions as close as possible. Since the direction field is an important tool also in the qualitative methods, we will present this notion together with some examples.

The *direction field, also called slope field* in $\mathbb{R}^2$ of the scalar differential equation $y' = f(x,y)$ (with $f : \mathbb{R}^2 \to \mathbb{R}$ a continuous function) is a collection of vectors. For an arbitrary given point $(x,y) \in \mathbb{R}^2$, such a vector is based in (x,y) and have the slope $m = f(x,y)$. This number $m = f(x,y)$ it is said to be the slope of the direction field in the point (x,y).

For example, considering the differential equation

$$y' = 1 - \frac{x}{y^2} \tag{29}$$

the slope of its direction field in the point $(0,1)$ is 1, that means that the corresponding vector in $(0,1)$ is parallel to the first bisectrix. Also, the slope of its direction field in the point $(1,1)$ is 0, that means that the corresponding vector in $(1,1)$ is parallel to the Ox-axis. Although the right hand side of the equation is not defined in the point $(1,0)$, we say that the slope in $(1,0)$ is ∞, that means that the corresponding vector in $(1,0)$ is parallel to the Oy-axis.

In order to have a clearer picture of the direction field it is useful to "organize" the vectors finding some isoclines. The *isocline for the slope* m is the curve

$$I_m = \{(x,y) \,:\, f(x,y) = m\}.$$

For example, for (29), the isocline for the slope 1 is the curve of equation

$$1 - \frac{x}{y^2} = 1,$$

that after simplification gives the line

$$x = 0.$$

Also, the isocline for the slope 0 is the parabola

$$y^2 = x.$$

The usefulness of the direction field comes after the following property. *The slope of the direction field in some given point is the slope of the solution curve that passes through that point.* More precisely, let the point (x_1, y_1) be given and let a solution $\varphi(x)$ of $y' = f(x, y)$, whose graph passes through this point. We know that the slope of the direction field is $f(x_1, y_1)$ and that the slope of the solution curve is $\varphi'(x_1)$. We have to prove that

$$\varphi'(x_1) = f(x_1, y_1).$$

Indeed, since the graph of φ passes through (x_1, y_1) we have that $\varphi(x_1) = y_1$, and since φ is a solution of $y' = f(x, y)$ we have that $\varphi'(x_1) = f(x_1, \varphi(x_1))$. The proof is done.

Now we come back to the Euler numerical method to find an approximate solution of the IVP $y' = f(x, y),\ y(x_0) = y_0$. The geometrical idea behind it is the following. We start in (x_0, y_0) and follow the vector of slope $f(x_0, y_0)$ until it intersects the vertical line $x = x_1$ in a point (x_1, y_1). Remind that x_0, y_0, x_1 are given, and deduce that y_1 satisfies

$$y_1 - y_0 = f(x_0, y_0)(x_1 - x_0).$$

Thus

$$y_1 = y_0 + (x - x_0)f(x_0, y_0).$$

Once we are in (x_1, y_1) we follow the vector of slope $f(x_1, y_1)$ until it intersects the vertical line $x = x_2$ in a point (x_2, y_2) with

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1).$$

We proceed in the same way until the end of the interval, x^*, obtaining

$$y_{k+1} = y_k + f(x_k, y_k)(x_{k+1} - x_k).$$

Now we continue our study of the direction field with a second example, where we consider the differential equation

$$y' = -\frac{x}{y}.$$

We will find the shape of the solution curves using the direction field. First we notice that, given m, the isocline for the slope m is the line

$$y = -\frac{1}{m}x.$$

We notice that the vectors of the directions field are orthogonal to the corresponding isocline. Hence, any solution curve is orthogonal to all the lines that passes through the origin of coordinates. We deduce that a solution curve must be a circle centered in the origin.

The *direction field* in the phase space $\mathbb{R}^2$ of a planar dynamical system is defined in a similar way and we will see that it is tangent to its orbits. More precisely, let $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions and let

$$\dot{x} = f_1(x, y), \quad \dot{y} = f_2(x, y).$$

By definition, the slope of the direction field in the point (x, y) is

$$m = \frac{f_2(x, y)}{f_1(x, y)}.$$

We have the following useful property. *The slope of the direction field in some given point is the slope of the orbit that passes through that point.* Indeed, let the point (x_1, y_1) be given and let a solution $(\varphi_1(t), \varphi_2(t))$ of the system whose orbit passes through this point. We know that the vector $(\varphi_1'(t), \varphi_2'(t))$ is tangent to the orbit for any t. Take now t_1 such that $(\varphi_1(t_1), \varphi_2(t_1)) = (x_1, y_1)$. Note that $(\varphi_1'(t_1), \varphi_2'(t_1)) = (f_1(x_1, y_1), f_2(x_1, y_1))$ and that this vector is tangent to the orbit in (x_1, y_1). Hence the slope of the orbit in (x_1, y_1) is $f_2(x_1, y_1)/f_1(x_1, y_1)$. The proof is done.