Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.5

Analysis for CS

(G 13)

a) The inequality $\frac{3^n}{4^n+5^n} < \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$, for all $n \geq 0$, and the convergence of the geometric series $\sum\limits_{n\geq 0}\left(\dfrac{3}{4}\right)^n$ imply, according to the first comparison criterion, the convergence of the given series.

b) Since $\sum\limits_{n\geq 1}\dfrac{1}{(2n)^\alpha} = \dfrac{1}{2^\alpha}\sum\limits_{n\geq 1}\dfrac{1}{n^\alpha}$, we conclude that the series is convergent if $\alpha > 1$ and divergent if $\alpha \leq 1$.

c) Since

$$\lim_{n\to\infty}\frac{n^{\frac{5}{4}}(\sqrt{n+1}-\sqrt{n})}{n^{\frac{3}{4}}} = \lim_{n\to\infty}\frac{n^{\frac{5}{4}}}{(\sqrt{n+1}+\sqrt{n})n^{\frac{3}{4}}} = \lim_{n\to\infty}\frac{n^{\frac{5}{4}}}{\left(\sqrt{1+\frac{1}{n}}+1\right)n^{\frac{5}{4}}} = \frac{1}{2},$$

the second comparison criterion yields that the given series is equivalent to the series $\sum\limits_{n\geq 1}\dfrac{1}{n^{\frac{5}{4}}}$, hence it is convergent.

d) The relations

$$\lim \sqrt[n]{x_n} = \lim\left(\frac{n}{n+1}\right)^n = \lim\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

imply, according to the root criterion, the convergence of the series.

e) We put $x_n := \frac{x^n}{n^p}$, for $n \geq 1$ and apply the 2-steps-algorithm. We compute

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty} x\left(\frac{n}{n+1}\right)^p = x.$$

Thus, if $x < 1$, the series is convergent, and, if $x > 1$, it is divergent. If $x = 1$ the series becomes the generalized harmonic series $\sum\limits_{n\geq 1}\dfrac{1}{n^p}$, which is convergent if $p > 1$, and divergent if $p \leq 1$.

f) According to the hint,

$$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1.$$

Hence, by the second comparison criterion, the given series is equivalent to the harmonic series $\sum\limits_{n\geq 1}\dfrac{1}{n}$, so it is divergent.

(G 14)

a) Let $x_n := e - \left(1+\frac{1}{n}\right)^n$, $n \geq 1$. The sequence $(x_n)_{n\in\mathbb{N}^*}$ is a strictly decreasing sequence converging to 0 (since, according to the third lecture, the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\geq 1}$ is a strictly increasing sequence converging to e). The criterion of Leibniz assures now the convergence of the given series. The equality $(*)$ given in the hint implies, according to the second comparison criterion, that the series $\sum_{n\geq 1}\left(e-\left(1+\dfrac{1}{n}\right)^n\right)$ is equivalent to the harmonic series. This shows that the given series is not absolutely convergent.

b) It is obvious that the series is absolutely convergent (hence also convergent) if $x=0$. Assume now that $x>0$. Since $\lim_{n\to\infty}\dfrac{x}{n}=0$, there exists an index $n_0\in\mathbb{N}^*$ such that $\frac{x}{n}\in(0,\pi)$, for all $n\geq n_0$. This means that the terms of the given series are positive up to the index n_0. Since, when studying the convergence type of a series it is not important where the summation starts, we may assume without any loss of generality that the summation starts at n_0 (in order to assure that all terms of the series are positive). Using the hint, we have

$$\lim_{n\to\infty}\frac{\sin\frac{x}{n}}{\frac{1}{n}}=x>0.$$

By the second comparison criterion the given series is equivalent to the harmonic series, so it is divergent. It follows that the series is not absolutely convergent.

If $x<0$ then, taking into account that $\sin\frac{x}{n}=-\sin\frac{-x}{n}$, for all $n\geq 1$, we can apply the previously obtained result. Thus the series is divergent in this case, too. It follows that it is not absolutely convergent.

HOMEWORK:

(H 15)

1) a) In view of $\lim_{n\to\infty}\dfrac{n^\alpha}{(2n+1)^\alpha}=\left(\dfrac{1}{2}\right)^\alpha$, using the second comparison criterion, the given series is equivalent to the harmonic series $\sum_{n\geq 1}\dfrac{1}{n^\alpha}$, hence it is convergent if $\alpha>1$, and divergent if $\alpha\leq 1$.

b) Since $\lim_{n\to\infty}\dfrac{n}{\sqrt{n(n+1)}}=1$, the second comparison criterion yields that the given series is equivalent to the series $\sum_{n\geq 1}\dfrac{1}{n}$, hence it is divergent.

c) Since

$$\lim_{n\to\infty}\frac{\sin^3\frac{1}{n}}{\frac{1}{n^3}}=1,$$

the second comparison criterion yields that the given series is equivalent to the series $\sum_{n\geq 1}\dfrac{1}{n^3}$, hence it is convergent.

d) The relations $\lim_{n\to\infty}\sqrt[n]{x_n}=\lim_{n\to\infty}\dfrac{\sqrt[n]{n}}{\ln n}=0<1$ imply, by the root criterion, the convergence of the given series.

e) The relations

$$\lim_{n\to\infty}D_n=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{(n+1)^4}{n^4e^{2n+1}}=0<1$$

imply, in view of the quotient criterion, the convergence of the given series.

2) Denote by x_n the general term of the given series. If $\alpha < 0$, then $\lim\limits_{n\to\infty} |x_n| = \infty \neq 0$, hence the series $\sum\limits_{n\geq 1} \dfrac{(-1)^{n+1}}{n^\alpha}$ is divergent in this case. This series is divergent in the case $\alpha = 0$, too, since then $\lim\limits_{n\to\infty} |x_n| = 1 \neq 0$. If $\alpha > 0$, then $\left(\frac{1}{n^\alpha}\right)$ is a decreasing sequence which converges to 0. Using the criterion of Leibniz, we conclude that $\sum\limits_{n\geq 1} \dfrac{(-1)^{n+1}}{n^\alpha}$ is convergent in this case.

Knowing the type of the generalized harmonic series, we conclude that $\sum\limits_{n\geq 1} \dfrac{(-1)^{n+1}}{n^\alpha}$ is absolutely convergent if $\alpha > 1$, and not absolutely convergent if $\alpha \leq 1$.

(H 16)

We have that

$$D_n = \frac{x_{n+1}}{x_n} = x\frac{n^n}{(n+1)^n} = x\frac{1}{\left(1+\frac{1}{n}\right)^n},$$

hence $\lim\limits_{n\to\infty} D_n = \dfrac{x}{e}$. According to the first step of the 2-steps-algorithm, we get that, if $x < e$, the given series is convergent, and, if $x > e$, it is divergent. If $x = e$ we continue the algorithm. We have that

$$R_n = n\left(\frac{1}{D_n} - 1\right) = \frac{n}{e}\left(\left(1+\frac{1}{n}\right)^n - e\right).$$

From $(*)$ given in the hint to (G 14) we obtain that

$$\lim_{n\to\infty} n\left(e - \left(1+\frac{1}{n}\right)^n\right) = \frac{e}{2},$$

thus $\lim\limits_{n\to\infty} R_n = -\dfrac{1}{2} < 1$, which yields that the series is divergent.

In conclusion, the given series is convergent if and only if $x \in (0, 1)$.