Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică
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Seminars 4 and 5. [1]

Consider

$$x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \tag{1}$$

whose matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Our aim is to find explicitly the general solution of this system.

First we distinguish two classes of such systems: the *uncoupled* systems

$$x_1' = a_{11}x_1 \\ x_2' = a_{22}x_2 \tag{2}$$

and, respectively, the *coupled* systems, which are the systems which are not uncoupled, that is either $a_{12} \neq 0$ or $a_{21} \neq 0$.

It is very easy to see that the uncoupled system (2) has the general solution

$$x_1 = c_1 e^{a_{11}t}, \quad x_2 = c_2 e^{a_{22}t}, \quad c_1, c_2 \in \mathbb{R}.$$

From now on we will study only coupled systems.

In the sequel we will see that for a coupled system with $a_{12} \neq 0$ the variable x_1 can be found as the solution of a second-order linear homogeneous differential equation. Similar property holds for the variable x_2 if $a_{22} \neq 0$.

Consider, for example, that $a_{12} \neq 0$. We use the first equation in (1) to write explicitly x_2 in function of x_1' and x_1,

$$x_2 = \frac{x_1' - a_{11}x_1}{a_{12}}, \tag{3}$$

and also to compute x_1'',

$$x_1'' = a_{11}x_1' + a_{12}x_2'.$$

Here we use the second equation in (1) to replace x_2' by $a_{21}x_1 + a_{22}x_2$. Now we use (3) to obtain x_1'' only in function of x_1' and x_1,

$$x_1'' = a_{11}x_1' + a_{12}a_{21}x_1 + a_{22}(x_1' - a_{11}x_1).$$

This last relation is the second order linear homogeneous equation

$$x_1'' - (a_{11} + a_{22})x_1' + (a_{11}a_{22} - a_{12}a_{21})x_1 = 0. \tag{4}$$

The general solution of system (1) can be found now in two steps. First find x_1 as the general solution of (4), then find x_2 using (3). This method is called *the method of reduction of the coupled system* (1) *to a second order differential equation.*

1. Prove that the roots of the characteristic equation of (4) coincide with the eigenvalues of the matrix A of system (1). To deduce this, it is sufficient if you show that both the equation of the eigenvalues of A and the characteristic equation of (4) are

$$\lambda^2 - (\mathrm{tr}A)\,\lambda + \det A = 0,$$

where $\mathrm{tr}A$ denotes the trace of the matrix A, while $\det A$ denotes its determinant.

Sketch the phase portrait of each of the following scalar differential equations. Indicate the stability type of their equilibria using the linearization method. Given an equilibrium point η^*, reading the phase portrait, indicate the maximal range for the initial state η such that either $\lim\limits_{t\to+\infty} \varphi(t,\eta) = \eta^*$ or $\lim\limits_{t\to-\infty} \varphi(t,\eta) = \eta^*$. For each $\eta \in \mathbb{R}$ establish the monotonicity properties of the function $\varphi(\cdot,\eta)$. As usual, $\varphi(t,\eta)$ denotes the flow of the given scalar differential equation.

2. a) $\dot{x} = -2x$ b) $\dot{x} = 1 + x$ c) $\dot{x} = 1 - x^2$ d) $\dot{x} = -4 + x^2$ e) $\dot{x} = 8 - x^3$
3. (*The logistic equation*) $\dot{x} = x(N - x)$ where $N > 0$ is a parameter
4. $\dot{x} = x(1 - x) - c$ 5. $\dot{x} = x(1 - x) - cx$ where $c > 0$ is a parameter
6. $\dot{x} = -x - x^3 + 1$ 7. $\dot{x} = -x - x^3 + \lambda$ where $\lambda \in \mathbb{R}$ is a parameter
8. $\dot{x} = \sin x$ 9. $\dot{x} = 2\sin x$ 10. $\dot{x} = 1 - 2\sin x$ 11. $\dot{x} = 2 - \sin x$
12. $\dot{x} = \lambda - x^2$ where $\lambda \in \mathbb{R}$ is a parameter.

For each of the following linear systems study the type and stability of its equilibrium point $(0,0)$. Then find its general solution.

13. $x' = -2x, \quad y' = 3y$
14. $x' = x, \quad y' = -x + 2y$
15. $x' = x + y, \quad y' = -2x + 4y$
16. $x' = x + y, \quad y' = x - 4y$
17. $x' = 4x - 5y, \quad y' = x - 2y$

18. For what values of the real parameter a the system $\dot{x} = ax - 5y, \quad \dot{y} = x - 2y$ has a center at the origin? In that cases find the general solution of the system.

19. a) Give an example of a coupled linear planar system which has a node at the origin.
b) Give an example of a coupled linear planar system which has a saddle at the origin.
c) There exist uncoupled linear planar systems with either a center or a focus at the origin?

For each of the following linear systems study the type and stability of its equilibrium point $(0,0)$. For those systems for which $(0,0)$ is not a focus, represent their phase portrait in two ways: by using the definition of an orbit and then by using the differential equation of the orbits. For those systems for which $(0,0)$ is a focus, represent their phase portrait by passing to polar coordinates.
20. $x' = -3x, \quad y' = -3y$ 21. $x' = -x, \quad y' = -2y$
22. $x' = 3x, \quad y' = 3y$ 23. $x' = -2x, \quad y' = 2y$
24. $x' = -y, \quad y' = \omega^2 x$ where $\omega > 0$ is a parameter
25. $\dot{x} = -x - y, \quad \dot{y} = x - y$ 26. $\dot{x} = -x + y, \quad \dot{y} = -x - y.$

Find the equilibria and study their stability for the following nonlinear planar systems.
27. $\dot{x} = -x + xy, \quad \dot{y} = -2y + 3y^2,$ 28. $\dot{x} = 2x - x^2 - xy, \quad \dot{y} = -y + xy,$
29. $\dot{x} = x - 2xy, \quad \dot{y} = x^2/2 - y,$ 30. $\dot{x} = 1 - xy, \quad \dot{y} = x - y^2,$
31. (*The Van der Pol equation*) $\ddot{y} + y + \lambda\left(\frac{1}{3}\dot{y}^3 + y\right) = 0$ where $\lambda \in \mathbb{R}$ is a parameter;
32. $\ddot{y} + \dot{y} + y^3 = 0.$

33. We consider the Lotka-Volterra system (also called the predator-prey system)

$$\dot{x} = x - xy, \quad \dot{y} = -0.3y + 0.3xy.$$

a) Notice that $(1,1)$ is an equilibrium point and show that it is non-hyperbolic.
b) Write the differential equation of the orbits of this system and notice that it is separable. Find its general solution.
c) Notice that the general solution found at b) can be written as $H(x,y) = c, \quad c \in \mathbb{R}$, with the function $H : (0,\infty) \times (0,\infty) \to \mathbb{R}$,

$$H(x,y) = y - \ln y + 0.3(x - \ln x).$$

2