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Complements of Chapter 2: Linear Differential Equations

A real-world application: the Spring-Mass System. We have a mass sliding on an horizontal wall and attached to a vertical wall by an elastic spring. The position of the mass when the spring is relaxed is called *the equilibrium position* and, on a real axe Ox we identify it with the origin O. When it is not in the equilibrium position, we denote by $x(t)$ the distance from the equilibrium at time t. Using Newton's second law and Hooke's law one can find that the function $t \in \mathbb{R} \mapsto x(t) \in \mathbb{R}$ satisfies the differential equation

$$x'' + \frac{\nu}{m}x' + \frac{k}{m}x = f(t).$$

Here $m > 0$ is the mass, $\nu > 0$ is the friction (also called damping) coefficient, $k > 0$ is the elasticity constant (also called stiffness) of the spring, and $f(t)$ is the measure of some external force. As we know, this mathematical model is a second order linear differential equation with constant coefficients. We will study the following three cases:

Case 1. Undamped motion without external force:

$$x'' + \frac{k}{m}x = 0.$$

Case 2. Damped motion without external force:

$$x'' + \frac{\nu}{m}x' + \frac{k}{m}x = 0.$$

Case 3. Undamped motion with some external force:

$$x'' + \frac{k}{m}x = A\cos\omega t.$$

We consider now each case. We will find the general solution using mainly the characteristic equation method.

Case 1. Undamped motion without external force:

$$x'' + \frac{k}{m}x = 0. \tag{1}$$

The general solution of (1),

$$x = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \quad c_1, c_2 \in \mathbb{R},$$

where we used the notation

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

Each of these non-null functions is periodic with minimal period $T = 2\pi/\omega_0$. Moreover, note that for $A_0 > 0$ and $\varphi_0 \in [0, 2\pi)$ such that $c_1 = A_0 \cos \varphi_0$ and $c_2 = A_0 \sin \varphi_0$ then the general solution can be written equivalently as

$$x(t) = A_0 \cos(\omega_0 t - \varphi_0), \quad A_0 \in \mathbb{R},\ \varphi_0 \in [0, 2\pi).$$

Hence *the mass oscillates with constant amplitude* A_0 *around the equilibrium position.*

Case 2. Damped motion without external force:

$$x'' + \frac{\nu}{m}x' + \frac{k}{m}x = 0. \tag{2}$$

Using the characteristic equation method one can see that we have to consider three subcases, which correspond to the sign of the discriminant of the characteristic equation.

Case 2.1. The motion is underdamped: $\nu < \sqrt{4mk}$.
In this case the general solution is

$$x = e^{-\frac{\nu}{2m}t}(c_1 \cos \tilde{\omega}_0 t + c_2 \sin \tilde{\omega}_0 t), \quad c_1, c_2 \in \mathbb{R},$$

or, equivalently,

$$x(t) = A_0 e^{-\frac{\nu}{2m}t} \cos(\tilde{\omega}_0 t - \varphi_0), \quad A_0 \in \mathbb{R},\ \varphi_0 \in [0, 2\pi),$$

where we have used the notation

$$\tilde{\omega}_0 = \frac{\sqrt{-\nu^2 + 4mk}}{2m}.$$

Hence *the mass oscillates around the equilibrium position, but the amplitude decreases exponentially to* 0.

Case 2.2. The motion is critically damped: $\nu = \sqrt{4mk}$.
In this case the general solution is

$$x = e^{-\frac{\nu}{2m}t}(c_1 + c_2 t), \quad c_1, c_2 \in \mathbb{R}.$$

Hence *there are no oscillations and the mass goes rapidly to the equilibrium position.*

Case 2.3. The motion is overdamped: $\nu > \sqrt{4mk}$.
In this case the general solution is

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad c_1, c_2 \in \mathbb{R},$$

where

$$r_1 = \frac{-\nu + \sqrt{\nu^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-\nu - \sqrt{\nu^2 - 4mk}}{2m}.$$

Note that $r_1 < 0$ and $r_2 < 0$.
Hence *there are no oscillations and the mass goes rapidly to the equilibrium position.*

Case 3. Undamped motion with some external force:

$$x'' + \frac{k}{m}x = A\cos\omega t. \tag{3}$$

This is a linear nonhomogeneous differential equation whose homogeneous part is like in Case 1. Then

$$x_h = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t, \quad c_1, c_2 \in \mathbb{R},$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

In order to write a particular solution we need to consider two subcases.

Case 3.1: $\omega \neq \omega_0$. A particular solution is

$$x_p = -\frac{A}{\omega^2 - \omega_0^2}\cos\omega t,$$

then the general solution is

$$x = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t - \frac{A}{\omega^2 - \omega_0^2}\cos\omega t, \quad c_1, c_2 \in \mathbb{R}.$$

In this case *oscillations occur with bounded amplitude, but the motion is periodic if and only if* $\omega/\omega_0 \in \mathbb{Q}$.

Case 3.2: $\omega = \omega_0$. A particular solution is

$$x_p = \frac{1}{2\omega_0} t \sin \omega_0 t,$$

then the general solution is

$$x = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{1}{2\omega_0} t \sin \omega_0 t, \quad c_1, c_2 \in \mathbb{R}.$$

Any function of the above form is unbounded. In this case *oscillations occur with an amplitude that increases to* ∞. This phenomenon is called *resonance*.

The exponential function: scalar, complex, matrix.

Proposition 1 *Let $a \in \mathbb{R}$. The series of real numbers*

$$1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots + \frac{1}{k!}a^k + \dots \tag{4}$$

is convergent in $\mathbb{R}$.

Definition 1 *The sum of series (4) is denoted by $exp(a)$ or e^a. The function $exp : \mathbb{R} \to \mathbb{R}$ is called the scalar exponential function.*

The properties of the exponential function follow from this definition. Some of them are listed below:

$$e^0 = 1, \quad e^a \cdot e^b = e^{a+b}, \quad \frac{d}{dt}(e^{at}) = ae^{at}.$$

Proposition 2 *Let $z \in \mathbb{C}$. The series of complex numbers*

$$1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots + \frac{1}{k!}z^k + \dots \tag{5}$$

is convergent in $\mathbb{C}$.

Definition 2 *The sum of series (5) is denoted by $exp(z)$ or e^z. The function $exp : \mathbb{C} \to \mathbb{C}$ is called the complex exponential function.*

Proposition 3 (Euler's formula) *For any $z = \alpha + i\beta \in \mathbb{C}$ we have the following formula for the complex number e^z*

$$e^{\alpha+i\beta} = e^{\alpha}\cos\beta + i\, e^{\alpha}\sin\beta.$$

The properties of the complex exponential function follow from the above formula. Some of them are listed below:

$$e^z \cdot e^w = e^{z+w}, \quad \frac{d}{dt}(e^{zt}) = ze^{zt}.$$

Proposition 4 *Let $A \in \mathcal{M}_n(\mathbb{R})$. The series of matrices*

$$I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{k!}A^k + \dots \tag{6}$$

is convergent in $\mathcal{M}_n(\mathbb{R})$.

Definition 3 *The sum of series* (6) *is denoted by* $exp(A)$ *or* e^A. *The function* $exp : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ *is called the matrix exponential function.*

The properties of the matrix exponential function follow from the above definition. Some of them are listed below:

$e^{O_n} = I_n$, $\quad e^A \cdot e^B = e^{A+B}$ if and only if $AB = BA$, $\quad e^A$ is nonsingular and $(e^A)^{-1} = e^{-A}$,

$$\frac{d}{dt}(e^{tA}) = Ae^{tA}, \tag{7}$$

the components of the matrix e^{tA} *(where* $t \in \mathbb{R}$*) are finite sums of terms of the form*

$$p(t)e^{\alpha t}\cos\beta t \quad \textit{and} \quad p(t)e^{\alpha t}\sin\beta t$$

where α *and* β *are real numbers such that* $\alpha + i\beta$ *is an eigenvalue of* A, *and* $p(t)$ *is a polynomial of degree at most* $n - 1$.

Exercises. 1) Let $\lambda_1, \lambda_2 \in \mathbb{R}$. Prove that

$$\exp\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \exp(\lambda_1) & 0 \\ 0 & \exp(\lambda_2) \end{pmatrix}.$$

2) Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Prove that $\quad e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for any $t \in \mathbb{R}$.

Linear homogeneous systems with constant coefficients. Let $n \geq 1$. We consider the system of n first order differential equations with n unknowns, the functions $x_1, x_2, ..., x_n$ of independent variable t.

$$
(8) \qquad \begin{aligned}
&x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
&x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\cdots \\
&x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n.
\end{aligned}
$$

Associated to system (8) we consider the matrix of its coefficients, or, simply, the matrix of (8)

$$A = (a_{ij})_{1 \leq i,j \leq n}.$$

Denoting X a vector valued function whose components are the unknowns of system (8),

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$$

we write system (8) in the vectorial form

$$(9) \qquad X' = AX.$$

We remark that any nth order linear homogeneous differential equation with constant coefficients can be written in the form (8). Indeed, consider

$$(10) \qquad x^{(n)} + \alpha_1 x^{(n-1)} + \cdots + \alpha_{n-1}x' + \alpha_n x = 0,$$

where the coefficients are real constants. The n scalar unknowns are

$$x_1 = x, \quad x_2 = x', \quad ... \quad , \quad x_n = x^{(n-1)},$$

such that they satisfy the following system

$$
(11) \qquad \begin{aligned}
&x_1' = x_2 \\
&x_2' = x_3 \\
&\cdots \\
&x_n' = -\alpha_n x_1 - \alpha_{n-1}x_2 + \cdots - \alpha_1 x_n,
\end{aligned}
$$

which, of course, has the form (8).

Definition 4 *A solution of system* (9) *is a vector-valued function* $\varphi \in C^1(\mathbb{R}, \mathbb{R}^n)$ *that satisfies* $\varphi'(t) = A\varphi(t)$ *for all* $t \in \mathbb{R}$.

A matrix solution of system (9) *is a matrix-valued function* $\Phi \in C^1(\mathbb{R}, \mathcal{M}_n(\mathbb{R}))$ *that satisfies* $\Phi'(t) = A\Phi(t)$ *for all* $t \in \mathbb{R}$.

A matrix solution whose columns are linearly independent functions is called a fundamental matrix solution.

A matrix solution $\Phi(t)$ *such that* $\Phi(0) = I_n$ *is called the principal matrix solution.*

We give without proof the following Existence and Uniqueness Theorem

Theorem 1 *Let* $\eta \in \mathbb{R}^n$ *and* $M \in \mathcal{M}_n(\mathbb{R})$. *We have that the IVP*

$$x' = Ax, \quad x(0) = \eta \tag{12}$$

has a unique solution. Moreover, also the IVP

$$X' = AX, \quad x(0) = M \tag{13}$$

has a unique matrix solution.

An important consequence of the above theorem and of the linearity of the map

$$\mathcal{L} : C^1(\mathbb{R}, \mathcal{M}_n(\mathbb{R})) \to C(\mathbb{R}, \mathcal{M}_n(\mathbb{R})), \quad \mathcal{L}(X) = X' - AX$$

is the following fundamental theorem for linear homogeneous systems

Theorem 2 *The set of solutions of the linear system* $X' = AX$ *is a linear space of dimension* n.

Proof. It is easy to see that the map $\mathcal{L}$ defined above is linear and its kernel is the set of solutions of the linear system $X' = AX$, hence it is a linear space. The fact that it has dimension n will be proved by finding an isomorphism of linear spaces between $\ker \mathcal{L}$ and $\mathbb{R}^n$. Let

$$f : \ker \mathcal{L} \to \mathbb{R}^n, \quad f(X) = X(0) \text{ for all } X \in \ker \mathcal{L}.$$

It is not difficult to see that f is linear. The property that f is bijective follows from the Existence and Uniqueness Theorem. Hence f is an isomorphism between the linear spaces $\ker \mathcal{L}$ and $\mathbb{R}^n$. As we know, $\mathbb{R}^n$ has dimension n and the dimension is preserved between isomorphic linear spaces. □

Relation (7) given above expresses the very important property for linear systems

Proposition 5 (i) *The matrix-valued function $\Phi(t) = e^{tA}$ is the principal fundamental matrix solution of system $X' = AX$.*

(ii) *The unique solution of the IVP* (12) *is $\varphi(t) = e^{tA}\eta$, while the unique matrix solution of the IVP* (13) *is $\Phi(t) = e^{tA}M$.*

(iii) *The general solution of system $X' = AX$ is*

$$X = e^{tA}C, \quad C \in \mathbb{R}^n.$$

Proof. (i) From (7) we see that $\Phi(t) = e^{tA}$ is a matrix solution of $X' = AX$. Also, we know that $\Phi(0) = I_n$. It remained to prove that it is fundamental, that is to prove that its columns are linearly independent functions. Assume by contradiction that they are linearly dependent. Then $\det e^{tA} = 0$ for all $t \in \mathbb{R}$. But we know that for $t = 0$ the matrix e^{tA} takes the value I_n whose determinant is 1. In fact the matrix e^{tA} is nonsingular for any $t \in \mathbb{R}$. We arrive to a contradiction. Then, indeed, e^{tA} is a fundamental matrix solution.

(ii) It is easy to check.

(iii) Again it is easy to check that for each $C \in \mathbb{R}^n$ the function $e^{tA}C$ is a solution. The fact that any solution has this form will be justified in two ways. The first way is using Theorem 2. Note that the set $\{e^{tA}C, \quad C \in \mathbb{R}^n\}$ is a linear space of dimension n, as the set of all solutions of the system. Hence there are no other solutions. The other way is to use the properties of e^{tA} and is similar to the integrating factor method to integrate the scalar equation $x' = ax$. We note that, using the definition of the matrix exponential it can be proved that the matrices A and e^{-tA} commute. Note also that the rule of finding the derivative of the product between a matrix-valued function and a vector-valued function is the same as for the derivative of a product between two scalar functions. Thus we have

$$e^{-tA}(X' - AX) = e^{-tA}X' - Ae^{-tA}X = (e^{-tA}X)',$$

such that we can say that X is a solution of $X' = AX$ if and only if $(e^{-tA}X)' = 0$ which is equivalent to $e^{-tA}X = C$ for some $C \in \mathbb{R}^n$. To conclude we just notice that $(e^{-tA})^{-1} = e^{tA}$. $\square$

Until now, theoretically, or up to the matrix e^{tA}, we know everything about the linear homogeneous systems with constant coefficients. But for most of the matrices A it is a formidable task to find the components of e^{tA} using the definition. That is why we present in the sequel some results which are important in practice. First we remind some notions from Linear Algebra. We say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of the matrix A if there exists $v \in \mathbb{C}^n$, $v \neq 0$ such that $Av = \lambda v$. The vector v is said to be an *eigenvector* corresponding to the eigenvalue λ.

Note that $\lambda \in \mathbb{C}$ is an eigenvalue of the matrix A if and only if it satisfies the algebraic equation of degree n

$$\det(A - \lambda I_n) = 0, \tag{14}$$

which is also called *the characteristic equation* of the linear system $X' = AX$.

Proposition 6 *Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and $v \in \mathbb{C}^n$ be a corresponding eigenvector. Then*

(i) *the function $\varphi(t) = e^{\lambda t} v$ satisfies $X' = AX$.*

(ii) *when $\lambda \in \mathbb{R}$ we have that $v \in \mathbb{R}^n$ and the function $\varphi(t) = e^{\lambda t} v$ is a solution of $X' = AX$.*

(iii) *when $\lambda = \alpha + i\beta$ and $v = u + iw$ then*

$$\varphi_1(t) = e^{\alpha t} \cos(\beta t)\, u - e^{\alpha t} \sin(\beta t)\, w, \quad \varphi_2(t) = e^{\alpha t} \sin(\beta t)\, u - e^{\alpha t} \cos(\beta t)\, w$$

are solutions of $X' = AX$. Moreover, when $w \neq 0$ they are linearly independent.

(iv) *when $\lambda \in \mathbb{R}$ is an eigenvalue with both algebraic multiplicity and geometric multiplicity equal to 2 and v_1 and v_2 are two linearly independent eigenvectors corresponding to λ then*

$$\varphi_1(t) = e^{\lambda t} v_1, \quad \text{and} \quad \varphi_2(t) = e^{\lambda t} v_2$$

are two linearly independent solutions of $X' = AX$.

(v) *when $\lambda \in \mathbb{R}$ is an eigenvalue with algebraic multiplicity 2 and geometric multiplicity 1, v_1 is an eigenvector and v_2 is a generalized eigenvector corresponding to λ (i.e. $(A - \lambda I_n)v_2 = v_1$) then*

$$\varphi_1(t) = e^{\lambda t} v_1, \quad \text{and} \quad \varphi_2(t) = e^{\lambda t}(t v_1 + v_2)$$

are two linearly independent solutions of $X' = AX$.

Proof. (i) By direct computations and using that $Av = \lambda v$ we obtain $\varphi'(t) = e^{\lambda t} \lambda v = e^{\lambda t} A v = A\varphi(t)$ for all $t \in \mathbb{R}$.

(ii) By (i) this function satisfies the system and, since it is real-valued it is a solution in the sense of the definition given above.

(iii) By (i) the complex-valued function $\varphi(t) = e^{\lambda t}v$ satisfies the system. The Euler's formula helps us to find the real and, respectively, the imaginary part of this function,

$$\varphi(t) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(u + iw) = \varphi_1(t) + i\varphi_2(t).$$

Then $(\varphi_1 + i\varphi_2)' = A(\varphi_1 + i\varphi_2)$, which further gives $(\varphi_1' - A\varphi_1) + i(\varphi_2' - A\varphi_2) = 0$. Since both $(\varphi_1' - A\varphi_1)$ and $(\varphi_2' - A\varphi_2)$ are real, we deduce that they must be null. Hence φ_1 and φ_2 are solutions of the system $X' = AX$. To show that they are linearly independent suppose that some linear combination with coefficients c_1 and c_2 is identically zero. Evaluation at $t = 0$ and $t = \pi/(2\beta)$ yields the equations

$$c_1u + c_2w = 0, \quad c_2u - c_1w = 0.$$

Assume by contradiction that either c_1 or c_2 are not null. Fix, for example, that $c_1 \neq 0$. Then $u = -\frac{c_2}{c_1}w$ and, further, $-\frac{c_2^2}{c_1}w - c_1w = 0$. So $(c_1^2 + c_2^2)w = 0$. Since $w \neq 0$ we must have $c_1 = c_2 = 0$ which contradicts our assumption. $\square$

The particular case $n = 2$ will be treated in detail. Consider

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 \\ x_2' &= a_{21}x_1 + a_{22}x_2 \end{aligned} \tag{15}$$

whose matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Our aim is to find explicitly the general solution of this system. We will present two methods and, for a better understanding, we will discuss also on the similarities between them.

First we distinguish two classes of such systems: the *uncoupled* systems

$$\begin{aligned} x_1' &= a_{11}x_1 \\ x_2' &= a_{22}x_2 \end{aligned} \tag{16}$$

and, respectively, the *coupled* systems, which are the systems which are not uncoupled, that is either $a_{12} \neq 0$ or $a_{21} \neq 0$.

It is very easy to see that the uncoupled system (16) has the general solution

$$x_1 = c_1 e^{a_{11}t}, \quad x_2 = c_2 e^{a_{22}t}, \quad c_1, c_2 \in \mathbb{R}.$$

From now on we will study only coupled systems.

In the sequel we will see that for a coupled system with $a_{12} \neq 0$ the variable x_1 can be found as the solution of a second-order linear homogeneous differential equation. Similar property holds for the variable x_2 if $a_{22} \neq 0$.

Consider, for example, that $a_{12} \neq 0$. We use the first equation in (15) to write explicitly x_2 in function of x_1' and x_1,

$$x_2 = \frac{x_1' - a_{11}x_1}{a_{12}}, \tag{17}$$

and also to compute x_1'',

$$x_1'' = a_{11}x_1' + a_{12}x_2'.$$

Here we use the second equation in (15) to replace x_2' by $a_{21}x_1 + a_{22}x_2$. Now we use (17) to obtain x_1'' only in function of x_1' and x_1,

$$x_1'' = a_{11}x_1' + a_{12}a_{21}x_1 + a_{22}(x_1' - a_{11}x_1).$$

This last relation is the second order linear homogeneous equation

$$x_1'' - (a_{11} + a_{22})x_1' + (a_{11}a_{22} - a_{12}a_{21})x_1 = 0. \tag{18}$$

The general solution of system (15) can be found now in two steps. First find x_1 as the general solution of (18), then find x_2 using (17). This method is called *the method of reduction of the coupled system* (15) *to a second order differential equation.*

The next method is based on the Fundamental Theorem 2 and Proposition 6 and is called *the characteristic equation method*. We will present it like an algorithm.

Step 1. We write the characteristic equation (14), $\det(A - \lambda I_2) = 0$ which can be written explicitly as

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{19}$$

Step 2. We find the roots λ_1 and λ_2 of the characteristic equation. Remind that these are eigenvalues of the matrix A. We have to find also corresponding eigenvalues.

According to the nature of the eigenvalues we attach two vector-valued functions following the rules.

When the roots are real and distinct consider

$$\varphi_1(t) = e^{\lambda_1 t} v_1 \quad \text{and} \quad \varphi_2(t) = e^{\lambda_2 t} v_2,$$

where $(A - \lambda_1 I_2)v_1 = 0$ and $(A - \lambda_2 I_2)v_2 = 0$.

When λ_1 is a (real) double root consider

$$\varphi_1(t) = e^{\lambda_1 t} v_1 \quad \text{and} \quad \varphi_2(t) = e^{\lambda_1 t}(t v_1 + v_2),$$

where $(A - \lambda_1 I_2)v_1 = 0$ and $(A - \lambda_1 I_2)v_2 = v_1$.

When $\lambda_{1,2} = \alpha \pm i\beta$ with $\beta \neq 0$ consider

$$\varphi_1(t) = e^{\alpha t}(\cos\beta t)\, v_1 - e^{\alpha t}(\sin\beta t)\, v_2 \quad \text{and} \quad \varphi_2(t) = e^{\alpha t}(\sin\beta t)\, v_1 - e^{\alpha t}(\cos\beta t)\, v_2,$$

where $(A - (\alpha + i\beta)I_2)(v_1 + i v_2) = 0$.

Step 3. The general solution of the coupled system (15) is

$$X = c_1\varphi_1(t) + c_2\varphi_2(t), \quad c_1, c_2 \in \mathbb{R}.$$

Note that the characteristic equation (19) of the coupled system (15) is also the characteristic equation of the second order differential equation (18). Moreover, note that $\mathrm{tr}A = a_{11} + a_{22}$ and $\det A = a_{11}a_{22} - a_{12}a_{21}$ such that the characteristic equation can be written

$$\lambda^2 - \mathrm{tr}A\,\lambda + \det A = 0.$$