Seminar

INFORMATICĂ SECŢIA ENGLEZĂ (GEOMETRIE)

1. Consider the triangle ABC alongside its orthocenter H, its circumcenter O and the diametrically opposed point A' of A on the latter circle. Show that:
   (a) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$.
   (b) $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.
   (c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\,\overrightarrow{HO}$.
2. Consider the triangle ABC alongside its centroid G, its orthocenter H and its circumcenter O. Show that O, G, H are collinear and $3\,\overrightarrow{HG} = 2\,\overrightarrow{HO}$.
3. Consider two perpendicular chords AB and CD of a given circle and $\{M\} = AB \cap CD$. Show that
$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2\,\overrightarrow{OM}\,.$$
4. In a triangle ABC consider the points M, L on the side AB and N, T on the side AC such that $3\,\overrightarrow{AL} = 2\,\overrightarrow{AM} = \overrightarrow{AB}$ and $3\,\overrightarrow{AT} = 2\,\overrightarrow{AN} = \overrightarrow{AC}$. Show that $\overrightarrow{AB} + \overrightarrow{AC} = 5\,\overrightarrow{AS}$, where $\{S\} = MT \cap LN$.
5. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$, not necesarily in the same plane, alongside their centroids G_1, G_2. Show that $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\,\overrightarrow{G_1G_2}$.
6. Consider a tetrahedron $A_1A_2A_3A_4$ and the midpoints A_{ij} of the edges A_iA_j, $i \neq j$. Show that:
   (a) The lines $A_{12}A_{34}$, $A_{13}A_{24}$ and $A_{14}A_{23}$ are concurrent in a point G.
   (b) The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at G.
   (c) Determine the ratio in which the point G divides each median.
   (d) Show that $\overrightarrow{GA_1} + \overrightarrow{GA_2} + \overrightarrow{GA_3} + \overrightarrow{GA_4} = \vec{0}$.
   (e) If M is an arbitrary point, show that $\overrightarrow{MA_1} + \overrightarrow{MA_2} + \overrightarrow{MA_3} + \overrightarrow{MA_4} = 4\,\overrightarrow{MG}$.
7. Consider the points C' and B' on the sides AB and AC of the triangle ABC such that $\overrightarrow{AC'} = \lambda\,\overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\,\overrightarrow{CB'}$. The lines BB' and CC' meet at M. If $P \in \mathscr{P}$ is a given point and $\mathbf{r}_A = \overrightarrow{PA}$, $\mathbf{r}_B = \overrightarrow{PB}$, $\mathbf{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that
$$\mathbf{r}_M = \frac{\mathbf{r}_A - \lambda\mathbf{r}_B - \mu\mathbf{r}_C}{1-\lambda-\mu}. \tag{0.1}$$

   Soluţie. The equations of the lines BB' and CC' are:
$$BB' :\ \mathbf{r}_X = (1-t)\mathbf{r}_B + t\mathbf{r}_{B'},\ CC' :\ \mathbf{r}_Y = (1-s)\mathbf{r}_C + s\mathbf{r}_{C'}.$$
   In order to express $\mathbf{r}_{B'}$ in terms of $\mathbf{r}_A$ and $\mathbf{r}_C$ we observe that:
$$\overrightarrow{AB'} = \mu\,\overrightarrow{CB'} \Leftrightarrow \overrightarrow{PB'} - \overrightarrow{PA} = \mu\left(\overrightarrow{PB'} - \overrightarrow{PC}\right) \Leftrightarrow \mathbf{r}_{B'} = \frac{\mathbf{r}_A - \mu\mathbf{r}_C}{1-\mu}.$$
   One can similarly show that $\mathbf{r}_{C'} = \dfrac{\mathbf{r}_A - \lambda\mathbf{r}_B}{1-\lambda}$. Thus, the vector equations of the lines BB' and CC' become:
$$BB' :\ \mathbf{r}_X = \frac{t}{1-\mu}\mathbf{r}_A + (1-t)\mathbf{r}_B - \frac{t\mu}{1-\mu}\mathbf{r}_C$$
$$CC' :\ \mathbf{r}_Y = \frac{s}{1-\lambda}\mathbf{r}_A - \frac{s\lambda}{1-\lambda}\mathbf{r}_B + (1-s)\mathbf{r}_C.$$

Since $BB' \cap CC' = \{M\}$, it follows that

$$\mathbf{r}_M = \frac{s_0}{1-\lambda}\mathbf{r}_A - \frac{s_0\lambda}{1-\lambda}\mathbf{r}_B + (1-s_0)\mathbf{r}_C = \frac{t_0}{1-\mu}\mathbf{r}_A + (1-t_0)\mathbf{r}_B - \frac{t_0\mu}{1-\mu}\mathbf{r}_C,$$

for some $s_0, t_0 \in \mathbb{R}$. Taking into account that the system

$$\begin{cases} \dfrac{t}{1-\mu} &= \dfrac{s}{1-\lambda} \\ 1-t &= \dfrac{s\lambda}{\lambda-1} \\ \dfrac{t\mu}{\mu-1} &= 1-s \end{cases}$$

has the unique solution $s_0 = \dfrac{1-\lambda}{1-\lambda-\mu}$, $t_0 = \dfrac{1-\mu}{1-\lambda-\mu}$, it follows that

$$\begin{aligned} \mathbf{r}_M &= \frac{s_0}{1-\lambda}\mathbf{r}_A - \frac{s_0\lambda}{1-\lambda}\mathbf{r}_B + (1-s_0)\mathbf{r}_C \\ &= \frac{t_0}{1-\mu}\mathbf{r}_A + (1-t_0)\mathbf{r}_B - \frac{t_0\mu}{1-\mu}\mathbf{r}_C \\ &= \frac{\mathbf{r}_A - \lambda\mathbf{r}_B - \mu\mathbf{r}_C}{1-\lambda-\mu}. \end{aligned}$$

8. Consider the triangle ABC alongside its centroid G, its orthocenter H, its incenter I and its circumcenter O. If $P \in \mathscr{P}$ is a given point and $\mathbf{r}_A = \overrightarrow{PA}$, $\mathbf{r}_B = \overrightarrow{PB}$, $\mathbf{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that:
   (a) $\mathbf{r}_G := \overrightarrow{PG} = \dfrac{\mathbf{r}_A + \mathbf{r}_B + \mathbf{r}_C}{3}$.
   (b) $\mathbf{r}_I := \overrightarrow{PI} = \dfrac{a\mathbf{r}_A + b\mathbf{r}_B + c\mathbf{r}_C}{a+b+c}$.
   (c) $\mathbf{r}_H := \overrightarrow{PH} = \dfrac{(\tan A)\mathbf{r}_A + (\tan B)\mathbf{r}_B + (\tan C)\mathbf{r}_C}{\tan A + \tan B + \tan C}$.
   (d) $\mathbf{r}_O := \overrightarrow{PO} = \dfrac{(\sin 2A)\mathbf{r}_A + (\sin 2B)\mathbf{r}_B + (\sin 2C)\mathbf{r}_C}{\sin 2A + \sin 2B + \sin 2C}$.

   Soluţie. (8a) Taking into accout the property of the centroid to be the intersection point of the medians BB' and CC', $C' \in [AC]$, $B' \in [AB]$, it follows that

$$\overrightarrow{AC'} = -\overrightarrow{BC'},\ \overrightarrow{AB'} = -\overrightarrow{CB'},$$

   i.e. we may obtain $\mathbf{r}_G$ simply by taking $\lambda = -1 = \mu$ within the formula (0.1). By doing so we obtain

$$\mathbf{r}_G = \frac{\mathbf{r}_A + \mathbf{r}_B + \mathbf{r}_C}{3}.$$

   (8b) Recall that the incenter I is the intersection point of the angle bisectors BB' and CC'. In order to express $\mathbf{r}_I$ in terms $\mathbf{r}_A$, $\mathbf{r}_B$ and $\mathbf{r}_C$, we only need to find λ and μ with the properties $\overrightarrow{AC'} = \lambda\,\overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\,\overrightarrow{CB'}$. Since $C' \in]AC[$, $B' \in]AB[$ it follows that $\lambda, \mu < 0$. On the other hand the equalities $\overrightarrow{AC'} = \lambda\,\overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\,\overrightarrow{CB'}$ imply that

$$\|\overrightarrow{AC'}\| = |\lambda| \cdot \|\overrightarrow{BC'}\| = -\lambda \cdot \|\overrightarrow{BC'}\| \text{ and } \|\overrightarrow{AB'}\| = |\mu| \cdot \|\overrightarrow{CB'}\| = -\mu \cdot \|\overrightarrow{CB'}\|,$$

i.e.

$$\lambda = -\frac{\| \overrightarrow{AC'} \|}{\| \overrightarrow{BC'} \|} = -\frac{b}{a} \text{ and } \mu = -\frac{\| \overrightarrow{AB'} \|}{\| \overrightarrow{CB'} \|} = -\frac{c}{a}.$$

If we replace these values within the formula (0.1) we obtain

$$\mathbf{r}_I = \frac{a\mathbf{r}_A + b\mathbf{r}_B + c\mathbf{r}_C}{a+b+c}.$$

9. Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\mathbf{r}_M = m\frac{1-n}{1-mn}\mathbf{u} + n\frac{1-m}{1-mn}\mathbf{v} \tag{0.2}$$

and

$$\mathbf{r}_N = m\frac{n-1}{n-m}\mathbf{u} + n\frac{m-1}{m-n}\mathbf{v}, \tag{0.3}$$

where $\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$, $\mathbf{u} = \overrightarrow{OA}$, $\mathbf{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\,\overrightarrow{OA'}$, i.e.

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\,\overrightarrow{OA} + n\frac{1-m}{1-mn}\,\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\,\overrightarrow{OA} + n\frac{m-1}{m-n}\,\overrightarrow{OA'}.$$

Solution. (0.2) The vector equations of the lines AB' and $A'B$ are:

$$AB' : \mathbf{r}_X = (1-\lambda)\mathbf{r}_A + \lambda\mathbf{r}_{B'},\ A'B : \mathbf{r}_Y = (1-\mu)\mathbf{r}_{A'} + \lambda\mathbf{r}_B,$$

or, equivalently $AB' : \mathbf{r}_X = (1-\lambda)\mathbf{u} + \lambda n\mathbf{v}$, $A'B : \mathbf{r}_Y = (1-\mu)\mathbf{v} + \lambda m\mathbf{u}$. Since $\{M\} =$

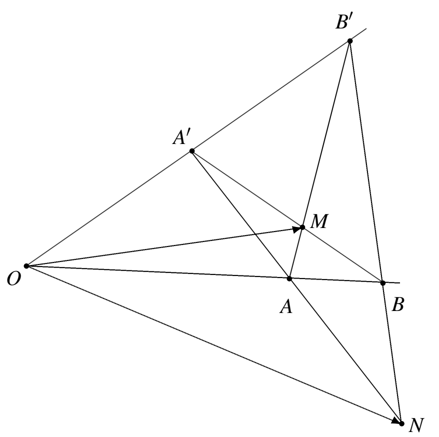

Fig. 0.1

$AB' \cap A'B$, it follows that $\mathbf{r}_M$ admits both a representation in the form $(1-\lambda)\mathbf{u} + \lambda n\mathbf{v}$ and a representation in the form $(1-\mu)\mathbf{v} + \mu m\mathbf{u}$, i.e.

$$\mathbf{r}_M = (1-\lambda)\mathbf{u} + \lambda n\mathbf{v} = (1-\mu)\mathbf{v} + \mu m\mathbf{u},\ \lambda, \mu \in \mathbb{R}.$$

The linear independence of the vectors $\mathbf{u} = \overrightarrow{OA}$, $\mathbf{v} = \overrightarrow{OA'}$ leads us to the compatible linear system

$$\begin{aligned} &1-\lambda = \mu m \\ &\lambda n = 1-\mu, \end{aligned}$$

whose solution is

$$\lambda = \frac{1-m}{1-mn}, \mu = \frac{1-n}{1-mn},$$

i.e.

$$1-\mu = n\frac{1-m}{1-mn}.$$

Thus,

$$\mathbf{r}_M = m\frac{1-n}{1-mn}\mathbf{u} + n\frac{1-m}{1-mn}\mathbf{v}.$$

(0.3) The vector equation of the straight lines AA' and BB' are:

$$AA' : \mathbf{r}_X = (1-\lambda)\mathbf{r}_A + \lambda \mathbf{r}_{A'},\ BB' : \mathbf{r}_Y = (1-\mu)\mathbf{r}_B + \lambda \mathbf{r}_{B'},$$

$AA' : \mathbf{r}_X = (1-\lambda)\mathbf{u} + \lambda\mathbf{v}, BB' : \mathbf{r}_Y = (1-\mu)m\mathbf{u} + \lambda n\mathbf{v}$. Since $\{N\} = AA' \cap BB'$, we deduce that $\mathbf{r}_N$ admits both a representation of the form $(1-\lambda)\mathbf{u}+\lambda\mathbf{v}$, and a representation of the form $(1-\mu)m\mathbf{u}+\mu n\mathbf{v}$, that is

$$\mathbf{r}_M = (1-\lambda)\mathbf{u} + \lambda\mathbf{v} = (1-\mu)m\mathbf{u} + \mu n\mathbf{v},\ \lambda,\mu \in \mathbb{R}.$$

The linear independence of the vectors $\mathbf{u} = \overrightarrow{OA}$, $\mathbf{v} = \overrightarrow{OA'}$ leads us to the compatible linear system

$$\begin{aligned} &1-\lambda = m(1-\mu) \\ &\lambda = \mu n, \end{aligned}$$

whose solution is

$$\lambda = n\frac{1-m}{n-m},\ \mu = \frac{1-m}{n-m}$$

and thus

$$1-\mu = n\frac{n-1}{n-m}.$$

Therefore,

$$\mathbf{r}_M = m\frac{n-1}{n-m}\mathbf{u} + n\frac{m-1}{m-n}\mathbf{v}.$$

10. The midpoints of the diagonals of a complet quadrilater are collinear (Newton's theorem).
Solution. Consider the convex quadrilater $OABC$ with pairwise unparallel opposite sides. Let us also consider $\{D\} = OC \cap AB$ and $\{E\} = OA \cap BC$. The figure $OABCDE$ is called *complete quadrilateral*, and its diagonals are OB, AC and DE. Denote by M, N and P the midpoints of the diagonals $[OB]$, $[AC]$ and $[DE]$ and observe that

$$\mathbf{r}_N = \frac{1}{2}(\mathbf{a}+\mathbf{c}) \text{ şi } \mathbf{r}_P = \frac{1}{2}(m\mathbf{a}+n\mathbf{c}),$$

where $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{c} = \overrightarrow{OC}$, $\overrightarrow{OE} = m\mathbf{a}$ and $\overrightarrow{OD} = n\mathbf{c}$. Using the relation (0.2), we conclude that

$$\mathbf{r}_B = m\frac{1-n}{1-mn}\mathbf{a} + n\frac{1-m}{1-mn}\mathbf{c},$$

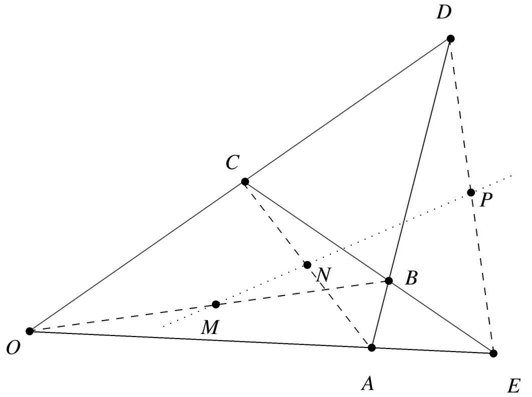

Fig. 0.2

i.e.

$$\mathbf{r}_M = \frac{1}{2}\left(m\frac{1-n}{1-mn}\mathbf{a} + n\frac{1-m}{1-mn}\mathbf{c}\right).$$

Therefore

$$\begin{aligned}\overrightarrow{MP} &= \mathbf{r}_P - \mathbf{r}_M = \frac{mn}{2(mn-1)}\left((m-1)\mathbf{a} + (n-1)\mathbf{c}\right)\\ \overrightarrow{NP} &= \mathbf{r}_P - \mathbf{r}_N = \frac{1}{2}\left((m-1)\mathbf{a} + (n-1)\mathbf{c}\right),\end{aligned}$$

which implies the equality $\overrightarrow{MP} = \dfrac{mn}{mn-1}\,\overrightarrow{NP}$ and shows the colinearity of M, N şi P.

11. Let d, d' be concurrent straight lines and $A, B, C \in d$, $A', B', C' \in d'$. If the following relations $AB' \nparallel A'B$, $AC' \nparallel A'C$, $BC' \nparallel B'C$ hold, show that the points $\{M\} := AB' \cap A'B$, $\{N\} := AC' \cap A'C$, $\{P\} := BC' \cap B'C$ are collinear (Pappus' theorem).
Solution. By using the relation (0.2), we conclude that

$$\begin{aligned}\mathbf{r}_M &= m\frac{1-m'}{1-mm'}\mathbf{u} + m'\frac{1-m}{1-mm'}\mathbf{v}\\ \mathbf{r}_N &= n\frac{1-n'}{1-nn'}\mathbf{u} + n'\frac{1-n}{1-nn'}\mathbf{v}\\ \mathbf{r}_P &= \frac{n}{m}\frac{1-\frac{n'}{m'}}{1-\frac{n}{m}\frac{n'}{m'}}\,\overrightarrow{OB} + \frac{n'}{m'}\frac{1-\frac{n}{m}}{1-\frac{n}{m}\frac{n'}{m'}}\,\overrightarrow{OB'}\\ &= mn\frac{m'-n'}{mm'-nn'}\mathbf{u} + m'n'\frac{m-n}{mm'-nn'}\mathbf{v},\end{aligned}$$

where $\mathbf{u} = \overrightarrow{OA}$, $\mathbf{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$, $\overrightarrow{OC} = n\,\overrightarrow{OA}$, $\overrightarrow{OB'} = m'\,\overrightarrow{OA'}$ and $\overrightarrow{OC'} = n'\,\overrightarrow{OA'}$. Thus $\overrightarrow{OC} = \dfrac{n}{m}\,\overrightarrow{OB}$ and $\overrightarrow{OC'} = \dfrac{n'}{m'}\,\overrightarrow{OB'}$. Consequently

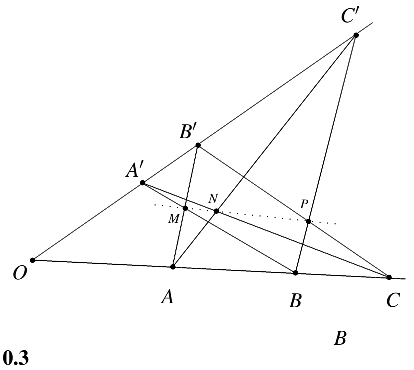

Fig. 0.3

$$
\begin{aligned}
(1-mm')\mathbf{r}_M &= m(1-m')\mathbf{u}+m'(1-m)\mathbf{v}\\
(1-nn')\mathbf{r}_N &= n(1-n')\mathbf{u}+n'(1-n)\mathbf{v}\\
(mm'-nn')\mathbf{r}_P &= mn(m'-n')\mathbf{u}+m'n'(m-n)\mathbf{v},
\end{aligned}
$$

or, echivalently

$$
\begin{aligned}
\left(\frac{1}{mm'}-1\right)\mathbf{r}_M &= \left(\frac{1}{m'}-1\right)\mathbf{u}+\left(\frac{1}{m}-1\right)\mathbf{v})\\
&= \frac{1}{m'}\mathbf{u}+\frac{1}{m}\mathbf{v}-(\mathbf{u}+\mathbf{v})\\
\left(\frac{1}{nn'}-1\right)\mathbf{r}_N &= \left(\frac{1}{n'}-1\right)\mathbf{u}+\left(\frac{1}{n}-1\right)\mathbf{v}\\
&= \frac{1}{n'}\mathbf{u}+\frac{1}{n}\mathbf{v}-(\mathbf{u}+\mathbf{v})\\
\left(\frac{1}{nn'}-\frac{1}{mm'}\right)\mathbf{r}_P &= \left(\frac{1}{n'}-\frac{1}{m'}\right)\mathbf{u}+\left(\frac{1}{n}-\frac{1}{m}\right)\mathbf{v}.
\end{aligned}
$$

Subtracting the first two relations we obtain

$$
\begin{aligned}
\left(\frac{1}{mm'}-1\right)\mathbf{r}_M+\left(1-\frac{1}{nn'}\right)\mathbf{r}_N &= \left(\frac{1}{m'}-\frac{1}{n'}\right)\mathbf{u}+\left(\frac{1}{m}-\frac{1}{n}\right)\mathbf{v}\\
&= \left(\frac{1}{mm'}-\frac{1}{nn'}\right)\mathbf{r}_P.
\end{aligned}
$$

Thus,

$$
\mathbf{r}_P = \underbrace{\frac{\frac{1}{mm'}-1}{\frac{1}{mm'}-\frac{1}{nn'}}}_{\lambda}\mathbf{r}_M+\underbrace{\frac{1-\frac{1}{nn'}}{\frac{1}{mm'}-\frac{1}{nn'}}}_{\mu}\mathbf{r}_N = \lambda\mathbf{r}_M+\mu\mathbf{r}_N,
$$

which shows the collinearity of the points M, N and P, as $\lambda+\mu=1$.

12. Let d, d' be two straight lines and $A,B,C \in d$, $A',B',C' \in d'$ three points on each line such that $AB' \| BA'$, $AC' \| CA'$. Show that $BC' \| CB'$ (the affine Pappus' theorem).

   Solution. Let us denote by $\mathbf{u}$ and $\mathbf{v}$ the vectors $\overrightarrow{OA}$ and $\overrightarrow{OA'}$ respectively. Observe that $\overrightarrow{OB}= m\mathbf{u}$ and $\overrightarrow{OC}= n\mathbf{u}$. and $\overrightarrow{A'B}= m\mathbf{u}-\mathbf{v}$ and $\overrightarrow{A'C}= n\mathbf{u}-\mathbf{v}$. Thus

$$\overrightarrow{OB'}= \alpha\mathbf{v} =\overrightarrow{OA} + \overrightarrow{AB'}= \mathbf{u}+\beta\ \overrightarrow{BA'}= \mathbf{u}+\beta(\mathbf{v}-m\mathbf{u}) = (1-\beta m)\mathbf{u}+\beta\mathbf{v}.$$

   which shows that $\alpha = \beta = \dfrac{1}{m}$ and $\overrightarrow{OB'}= \dfrac{\mathbf{v}}{m}$. One can similarly show that $\overrightarrow{OC'}= \dfrac{\mathbf{v}}{n}$. Thus

$$\overrightarrow{B'C}=\overrightarrow{OC} - \overrightarrow{OB'}= n\mathbf{u}-\frac{\mathbf{v}}{m} \ \text{ and } \ \overrightarrow{BC'}=\overrightarrow{OC'} - \overrightarrow{OB}= \frac{\mathbf{v}}{n}-m\mathbf{u}.$$

   Consequently

$$\overrightarrow{B'C}= -\frac{n}{m}\left(\frac{\mathbf{v}}{n}-m\mathbf{u}\right) = -\frac{n}{m}\ \overrightarrow{BC'},$$

   which shows that $B'C \| BC'$.

13. Let us consider two triangles ABC and $A'B'C'$ such that the lines AA', BB', CC' are concurrent at a point O and $AB \nparallel A'B'$, $BC \nparallel B'C'$ and $CA \nparallel C'A'$. Show that the points $\{M\} = AB \cap A'B'$, $\{N\} = BC \cap B'C'$ and $\{P\} = CA \cap C'A'$ are collinear (Desargues).

   Solution. By using the relation (0.3), we conclude that

$$\begin{aligned}
\mathbf{r}_M &= m\frac{n-1}{n-m}\mathbf{a}+n\frac{m-1}{m-n}\mathbf{b}\\
\mathbf{r}_N &= m\frac{p-1}{p-m}\mathbf{a}+p\frac{m-1}{m-p}\mathbf{c}\\
\mathbf{r}_P &= n\frac{p-1}{p-n}\mathbf{b}+p\frac{n-1}{n-p}\mathbf{c}
\end{aligned}$$

   where $\mathbf{a} =\overrightarrow{OA}$, $\mathbf{b} =\overrightarrow{OB}$, $\mathbf{c} =\overrightarrow{OC}$, $\overrightarrow{OA'}= m\ \overrightarrow{OA}$, $\overrightarrow{OB'}= n\ \overrightarrow{OB}$ and $\overrightarrow{OB'}= p\ \overrightarrow{OC'}$.

$$\begin{aligned}
M,\ N,\ P \text{ are collinear } &\Leftrightarrow \overrightarrow{NM}= t\ \overrightarrow{NP} \text{ for some } t \in \mathbb{R}\\
&\Leftrightarrow \mathbf{r}_M - \mathbf{r}_N = t(\mathbf{r}_P - \mathbf{r}_N) \text{ for } t \in \mathbb{R}
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \left(m\frac{n-1}{n-m}-m\frac{p-1}{p-m}\right)\mathbf{a}+n\frac{m-1}{m-n}\mathbf{b}-p\frac{m-1}{m-p}\mathbf{c}\\
&= t\left[-m\frac{p-1}{p-m}\mathbf{a}+n\frac{p-1}{p-n}\mathbf{b}+\left(p\frac{n-1}{n-p}-p\frac{m-1}{m-p}\right)\mathbf{c}\right]\\
&\Leftrightarrow m\left(\frac{n-1}{n-m}-\frac{p-1}{p-m}\right)\mathbf{a}+n\frac{m-1}{m-n}\mathbf{b}-p\frac{m-1}{m-p}\mathbf{c}\\
&= -tm\frac{p-1}{p-m}\mathbf{a}+tn\frac{p-1}{p-n}\mathbf{b}+tp\left(\frac{n-1}{n-p}-\frac{m-1}{m-p}\right)\mathbf{c}.
\end{aligned}$$

   Taking into account the relations

$$\begin{aligned}
&\frac{p-m}{p-1}\left(\frac{p-1}{p-m}-\frac{n-1}{n-m}\right) = \frac{m-1}{m-n}\cdot\frac{p-n}{p-1}\\
&= \frac{m-1}{m-p}\Bigg/\left(\frac{m-1}{m-p}-\frac{n-1}{n-p}\right),
\end{aligned}$$

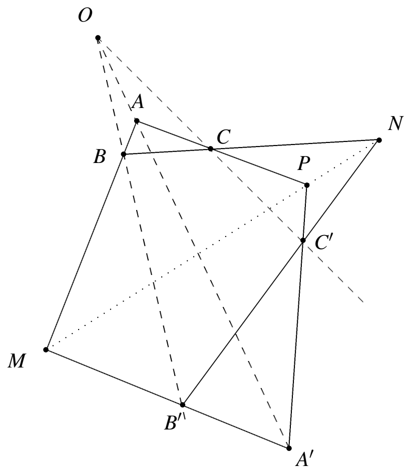

Fig. 0.4

observe that the equality $\overrightarrow{NM} = t\,\overrightarrow{NP}$ is satisfied for the common value t of the above expressions.

14. Write the equation of the line which passes through $A(1,-2,6)$ and is parallel to
    (a) The x-axis;
    (b) The line $(d_1)\ \dfrac{x-1}{2} = \dfrac{y+5}{-3} = \dfrac{z-1}{4}$.
    (c) The vector $\mathbf{v}(1,0,2)$.
15. Write the equation of the plane which contains the line
$$(d_1)\ \frac{x-3}{2} = \frac{y+4}{1} = \frac{z-2}{-3}$$
and is parallel to the line
$$(d_2)\ \frac{x+5}{2} = \frac{y-2}{2} = \frac{z-1}{2}.$$
16. Consider the points $A(\alpha,0,0)$, $B(0,\beta,0)$ and $C(0,0,\gamma)$ such that
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{a} \text{ where } a \text{ is a constatnt.}$$
Show that the plane (A,B,C) passes through a fixed point.
17. Write the equation of the line which passes through tht point $M(1,0,7)$, is parallel to the plane $(\pi)\ 3x - y + 2z - 15 = 0$ and intersects the line
$$(d)\ \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$
18. Write the equation of the plane which passes through $M_0(1,-2,3)$ and is parallel to the vectors $\mathbf{v}_1(1,-1,0)$ and $\mathbf{v}_2(-3,2,4)$.
19. Write the equation of the plane which passes through $M_0(1,-2,3)$ and cuts the positive coordinate axes through congruent segments.

20. Write the equation of the plane which passes through $A(1,2,1)$ and is parallel to the straight lines

$$(d_1)\ \begin{cases} x+2y-z+1=0 \\ x-y+z-1=0 \end{cases} \quad (d_2)\ \begin{cases} 2x-y+z=0 \\ x-y+z=0. \end{cases}$$

21. Write the equation of the plane determined by the line

$$(d)\ \begin{cases} x-2y+3z=0 \\ 2x+z-3=0 \end{cases}$$

and the point $A(-1,2,6)$.

22. Consider the rtiangle ABC and the midpoint A' of the side $[BC]$. Show that

$$4\,\overrightarrow{AA'}^2 - \overrightarrow{BC}^2 = 4\,\overrightarrow{AB}\cdot\overrightarrow{AC}.$$

23. Consider the rectangle $ABCD$ and the arbitrary point M witin the space. Show that
    (a) $\overrightarrow{MA}\cdot\overrightarrow{MC}=\overrightarrow{MB}\cdot\overrightarrow{MD}$.
    (b) $\overrightarrow{MA}^2+\overrightarrow{MC}^2=\overrightarrow{MB}^2+\overrightarrow{MD}^2$.
24. Consider the noncoplanar vectors $\overrightarrow{OA}\,(1,-1,-2)$, $\overrightarrow{OB}\,(1,0,-1)$, $\overrightarrow{OC}\,(2,2,-1)$ related to an orthonormal basis $\mathbf{i},\mathbf{j},\mathbf{k}$. Let H be the foot of the perpendicular through O on the plane ABC. Determine the components of the vectors $\overrightarrow{OH}$.
25. If two pairs of opposite edges of the tetrahedron $ABCD$ are perpendicular ($AB\perp CD$, $AD\perp BC$), show that
    (a) The third pair of opposite edges are perpendicular too ($AC\perp BD$).
    (b) $AB^2+CD^2=AC^2+BD^2=BC^2+AD^2$.
26. Two triangles ABC şi $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices A', B', C' on the sides BC, CA, AB are concurrent. Show that, in this case, the perpendicular lines from the vertices A, B, C on the sides $B'C'$, $C'A'$, $A'B'$ are concurrent too.
27. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be noncollinear vectors. Show that the necesssary and sufficient condition for the existence of a triangle ABC with the properties $\overrightarrow{BC}=\mathbf{a}$, $\overrightarrow{CA}=\mathbf{b}$, $\overrightarrow{AB}=\mathbf{c}$ is

$$\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}.$$

From the equalities of the norms deduce the low of sines.

28. Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.
29. Find the orthogonal projection
    (a) of the point $A(1,2,1)$ on the plane $\pi:\ x+y+3z+5=0$.
    (b) of the point $B(5,0,-2)$ on the straight line $(d)\ \dfrac{x-2}{3}=\dfrac{y-1}{2}=\dfrac{z-3}{4}$.
30. Compute the distance from the point $A(3,1,-1)$ to the plane $\pi: 22x+4y-20z-45=0$.
31. Find the equations of the bisector planes of the dihedral angles of the planes $(\pi_1)\ 2x+y-3z-5=0$, $(\pi_2)\ x+3y+2z+1=0$.
32. Find the angle between:
    (a) the straight lines

$$(d_1)\ \begin{cases} x+2y+z-1=0 \\ x-2y+z+1=0 \end{cases} \quad (d_2)\ \begin{cases} x-y-z-1=0 \\ x-y+2z+1=0. \end{cases}$$

(b) the planes $(\pi_1)\ x+3y+2z+1=0$, $(\pi_2)\ 3x+2y-z=6$.
(c) the coordinate plane xOy and the straight line M_1M_2, where $M_1(1,2,3)$, $M_2(-2,1,4)$.

33. Find the equations of the line through the point $P(4,3,0)$ which is perpendicular to the line
$$(d)\ \frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{5}.$$
as well as the symmetric point P' of P with respect to the line d.

34. Find the equation of the plane through the line M_1M_2 which is perpendicular to the plane $\pi:\ x-y+z=0$, where $M_1(2,1,-1)$ and $M_2(-3,0,2)$.

35. Show that
(a) $|(\mathbf{a},\mathbf{b},\mathbf{c})|\leq\|\mathbf{a}\|\cdot\|\mathbf{b}\|\cdot\|\mathbf{c}\|$;
(b) $(\mathbf{a}+\mathbf{b},\mathbf{b}+\mathbf{c},\mathbf{c}+\mathbf{a})=2(\mathbf{a},\mathbf{b},\mathbf{c})$.

36. Show the following equality:
$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=(\mathbf{a},\mathbf{c},\mathbf{d})\mathbf{b}-(\mathbf{b},\mathbf{c},\mathbf{d})\mathbf{a}=(\mathbf{a},\mathbf{b},\mathbf{d})\mathbf{c}-(\mathbf{a},\mathbf{b},\mathbf{c})\mathbf{d}.$$

Soluţie. By using the identity $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{v}$ for $\mathbf{u}=\mathbf{a}\times\mathbf{b}$, $\mathbf{v}=\mathbf{c}$ and $\mathbf{w}=\mathbf{d}$ we obtain
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= \mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{v}\\ &=[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}]\,\mathbf{c}-[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]\,\mathbf{d}\\ &=(\mathbf{a},\mathbf{b},\mathbf{d})\mathbf{c}-(\mathbf{a},\mathbf{b},\mathbf{c})\mathbf{d}.\end{aligned}$$

By using the identity $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}$ for $\mathbf{u}=\mathbf{a}$, $\mathbf{v}=\mathbf{b}$ and $\mathbf{w}=\mathbf{c}\times\mathbf{d}$ we obtain
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= (\mathbf{u}\times\mathbf{v})\times\mathbf{w}=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}\\ &=[\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{b}-[\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{a}\\ &=(\mathbf{a},\mathbf{c},\mathbf{d})\mathbf{b}-(\mathbf{b},\mathbf{c},\mathbf{d})\mathbf{a}.\end{aligned}$$

37. Show the following equality:
$$(\mathbf{u}\times\mathbf{v},\mathbf{v}\times\mathbf{w},\mathbf{w}\times\mathbf{u})=(\mathbf{u},\mathbf{v},\mathbf{w})^2.$$

Soluţie. We have successively:
$$\begin{aligned}(\mathbf{u}\times\mathbf{v},\mathbf{v}\times\mathbf{w},\mathbf{w}\times\mathbf{u}) &= [(\mathbf{u}\times\mathbf{v})\times(\mathbf{v}\times\mathbf{w})]\cdot(\mathbf{w}\times\mathbf{u})\\ &=[(\mathbf{u},\mathbf{v},\mathbf{w})\mathbf{v}-(\mathbf{u},\mathbf{v},\mathbf{v})\mathbf{w}]\cdot(\mathbf{w}\times\mathbf{u})\\ &=(\mathbf{u},\mathbf{v},\mathbf{w})\,[\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})]=(\mathbf{u},\mathbf{v},\mathbf{w})(\mathbf{v},\mathbf{w},\mathbf{u})=(\mathbf{u},\mathbf{v},\mathbf{w})^2.\end{aligned}$$

38. The *reciprocal vectors* of the noncoplanar vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ are defined by
$$\mathbf{u}'=\frac{\mathbf{v}\times\mathbf{w}}{(\mathbf{u},\mathbf{v},\mathbf{w})},\ \mathbf{v}'=\frac{\mathbf{w}\times\mathbf{u}}{(\mathbf{u},\mathbf{v},\mathbf{w})},\ \mathbf{w}'=\frac{\mathbf{u}\times\mathbf{v}}{(\mathbf{u},\mathbf{v},\mathbf{w})}.$$

Show that:
(a) $\mathbf{a}=(\mathbf{a}\cdot\mathbf{u}')\mathbf{u}+(\mathbf{a}\cdot\mathbf{v}')\mathbf{v}+(\mathbf{a}\cdot\mathbf{w}')\mathbf{w}=\dfrac{(\mathbf{a},\mathbf{v},\mathbf{w})}{(\mathbf{u},\mathbf{v},\mathbf{w})}\mathbf{u}+\dfrac{(\mathbf{u},\mathbf{a},\mathbf{w})}{(\mathbf{u},\mathbf{v},\mathbf{w})}\mathbf{v}+\dfrac{(\mathbf{u},\mathbf{v},\mathbf{a})}{(\mathbf{u},\mathbf{v},\mathbf{w})}\mathbf{w}$.
(b) the reciprocal vectors of $\mathbf{u}',\mathbf{v}',\mathbf{w}'$ are the vectors $\mathbf{u},\mathbf{v},\mathbf{w}$.

Soluţie. (38a) Obviously $\mathbf{a}=\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{c}$, as $\mathbf{u},\mathbf{v},\mathbf{w}$ are three linearly independent vectors of the three dimensional vector space $\mathscr{V}$, i.e. $\mathbf{u},\mathbf{v},\mathbf{w}$ form a basis of $\mathscr{V}$. Moreover we have

$$\begin{aligned}\mathbf{a}\cdot\mathbf{u}' &= \frac{\mathbf{a}\cdot(\mathbf{v}\times\mathbf{w})}{(\mathbf{u},\mathbf{v},\mathbf{w})}=\frac{(\mathbf{a},\mathbf{v},\mathbf{w})}{(\mathbf{u},\mathbf{v},\mathbf{w})}=\frac{(\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{c},\mathbf{v},\mathbf{w})}{(\mathbf{u},\mathbf{v},\mathbf{w})}\\ &= \frac{\alpha(\mathbf{u},\mathbf{v},\mathbf{w})+\beta(\mathbf{v},\mathbf{v},\mathbf{w})+\gamma(\mathbf{w},\mathbf{v},\mathbf{w})}{(\mathbf{u},\mathbf{v},\mathbf{w})}=\alpha.\end{aligned}$$

One can similalry show that

$$\mathbf{a}\cdot\mathbf{v}'=\frac{(\mathbf{u},\mathbf{a},\mathbf{w})}{(\mathbf{u},\mathbf{v},\mathbf{w})}=\beta \text{ and } \mathbf{a}\cdot\mathbf{w}'=\frac{(\mathbf{u},\mathbf{v},\mathbf{a})}{(\mathbf{u},\mathbf{v},\mathbf{w})}=\gamma.$$

(38b) Let us first observe that

$$(\mathbf{u}',\mathbf{v}',\mathbf{w}')=(\mathbf{w}',\mathbf{u}',\mathbf{v}')=\frac{(\mathbf{u}\times\mathbf{v},\mathbf{v}\times\mathbf{w},\mathbf{w}\times\mathbf{u})}{(\mathbf{u},\mathbf{v},\mathbf{w})^3}=\frac{(\mathbf{u},\mathbf{v},\mathbf{w})^2}{(\mathbf{u},\mathbf{v},\mathbf{w})^3}=\frac{1}{(\mathbf{u},\mathbf{v},\mathbf{w})}.$$

On the other hand we have:

$$\frac{\mathbf{v}'\times\mathbf{w}'}{(\mathbf{u}',\mathbf{v}',\mathbf{w}')}=(\mathbf{u},\mathbf{v},\mathbf{w})(\mathbf{v}'\times\mathbf{w}')=(\mathbf{u},\mathbf{v},\mathbf{w})\frac{(\mathbf{w}\times\mathbf{u})\times(\mathbf{u}\times\mathbf{v})}{(\mathbf{u},\mathbf{v},\mathbf{w})^2}=\frac{(\mathbf{w},\mathbf{u},\mathbf{v})\mathbf{u}-(\mathbf{w},\mathbf{u},\mathbf{u})\mathbf{v}}{(\mathbf{u},\mathbf{v},\mathbf{w})}=\mathbf{u}.$$

One can similarly show that

$$\frac{\mathbf{w}'\times\mathbf{u}'}{(\mathbf{u}',\mathbf{v}',\mathbf{w}')}=\mathbf{v} \text{ and } \frac{\mathbf{u}'\times\mathbf{v}'}{(\mathbf{u}',\mathbf{v}',\mathbf{w}')}=\mathbf{w}.$$

39. Let d_1,d_2,d_3,d_4 be pairwise skew straight lines. Assuming that $d_{12}\perp d_{34}$ and $d_{13}\perp d_{24}$, show that $d_{14}\perp d_{23}$, where d_{ik} is the common perpendicular of the lines d_i and d_k.
Solution. A director vector of the common perpendicular d_{ij} is $\mathbf{d}_i\times\mathbf{d}_j$, where $\mathbf{d}_r$ stands for a director vector of d_r. Therefore we have successively:

$$\begin{aligned} d_{12}\perp d_{34} &\Leftrightarrow \mathbf{d}_1\times\mathbf{d}_2\perp\mathbf{d}_3\times\mathbf{d}_4\Leftrightarrow(\mathbf{d}_1\times\mathbf{d}_2)\cdot(\mathbf{d}_3\times\mathbf{d}_4)=0\\ &\Leftrightarrow \begin{vmatrix}\mathbf{d}_1\cdot\mathbf{d}_3 & \mathbf{d}_1\cdot\mathbf{d}_4\\ \mathbf{d}_2\cdot\mathbf{d}_3 & \mathbf{d}_2\cdot\mathbf{d}_4\end{vmatrix}=0\Leftrightarrow(\mathbf{d}_1\cdot\mathbf{d}_3)(\mathbf{d}_2\cdot\mathbf{d}_4)=(\mathbf{d}_1\cdot\mathbf{d}_4)(\mathbf{d}_2\cdot\mathbf{d}_3).\end{aligned}$$

Similalry

$$\begin{aligned} d_{13}\perp d_{24} &\Leftrightarrow \mathbf{d}_1\times\mathbf{d}_3\perp\mathbf{d}_2\times\mathbf{d}_4\Leftrightarrow(\mathbf{d}_1\times\mathbf{d}_3)\cdot(\mathbf{d}_2\times\mathbf{d}_4)=0\\ &\Leftrightarrow \begin{vmatrix}\mathbf{d}_1\cdot\mathbf{d}_2 & \mathbf{d}_1\cdot\mathbf{d}_4\\ \mathbf{d}_3\cdot\mathbf{d}_2 & \mathbf{d}_3\cdot\mathbf{d}_4\end{vmatrix}=0\Leftrightarrow(\mathbf{d}_1\cdot\mathbf{d}_2)(\mathbf{d}_3\cdot\mathbf{d}_4)=(\mathbf{d}_1\cdot\mathbf{d}_4)(\mathbf{d}_3\cdot\mathbf{d}_2).\end{aligned}$$

Therefore we have

$$(\mathbf{d}_1\cdot\mathbf{d}_3)(\mathbf{d}_2\cdot\mathbf{d}_4)=(\mathbf{d}_1\cdot\mathbf{d}_4)(\mathbf{d}_2\cdot\mathbf{d}_3)=(\mathbf{d}_1\cdot\mathbf{d}_2)(\mathbf{d}_3\cdot\mathbf{d}_4),$$

which shows that

$$(\mathbf{d}_1\cdot\mathbf{d}_3)(\mathbf{d}_2\cdot\mathbf{d}_4)-(\mathbf{d}_1\cdot\mathbf{d}_2)(\mathbf{d}_3\cdot\mathbf{d}_4)=0\Leftrightarrow\begin{vmatrix}\mathbf{d}_1\cdot\mathbf{d}_2 & \mathbf{d}_1\cdot\mathbf{d}_3\\ \mathbf{d}_4\cdot\mathbf{d}_2 & \mathbf{d}_4\cdot\mathbf{d}_3\end{vmatrix}=0\Leftrightarrow d_{14}\perp d_{23}.$$

40. Find the points of the z-axis which is equidistant with respect to the planes

$$(\pi_1)\ 12x+9y-20z-19=0 \text{ and } (\pi_2)\ 16x+12y+15z-9=0.$$

Solution. Assume that $M(0,0,z) \in Oz$ satisfies the requirement $\delta(M,\pi_1)=\delta(M,\pi_2)$, i.e. we have ssuccessively

$$\begin{aligned}\frac{|-20z-19|}{\sqrt{12^2+9^2+(-20)^2}}=\frac{|15zz-9|}{\sqrt{16^2+12^2+15^2}} &\Leftrightarrow \frac{|-20z-19|}{\sqrt{625}}=\frac{|15zz-9|}{\sqrt{625}}\\ &\Leftrightarrow |-20z-19|=|15zz-9|\\ &\Leftrightarrow -20z-19=\pm(15zz-9)\\ &\Leftrightarrow -20z-19=15z-9\\ &\quad \text{or } -20z-19=-15z+9\\ &\Leftrightarrow -20z-19=15z-9\\ &\quad \text{or } -20z-19=-15z+9\\ &\Leftrightarrow z=-\frac{2}{7} \text{ or } z=-\frac{28}{5}.\end{aligned}$$

Thus, the equidistant points of the z-axis w.r.t. the planes π_1 and π_2 are $M_1\left(0,0,-\frac{2}{7}\right)$ and $M_2\left(0,0,-\frac{28}{5}\right)$.

41. Find the distance from the point $P(1,2,-1)$ to the straight line $(d)\ x=y=z$.
Solution. According to the formula for the distance from one point to a given line we have

$$\delta(P,d)=\frac{\|\overrightarrow{OP}\times\mathbf{d}\|}{\|\mathbf{d}\|},$$

where $O(0,0,0)\in d$ and $\mathbf{d}(1,1,1)$ is a director vector of d. On the other hand $\|\mathbf{d}\|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$ and

$$\overrightarrow{OP}\times\mathbf{d}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & -1\\ 1 & 1 & 1\end{vmatrix}=3\mathbf{i}-2\mathbf{j}-\mathbf{k}\Longrightarrow\|\overrightarrow{OP}\times\mathbf{d}\|=\sqrt{14}.$$

Thus

$$\delta(P,d)=\frac{\|\overrightarrow{OP}\times\mathbf{d}\|}{\|\mathbf{d}\|}=\sqrt{\frac{14}{3}}.$$

42. Find the distance between the straight lines

$$(d_1)\ \frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{1},\ (d_2)\ \frac{x+1}{3}=\frac{y}{4}=\frac{z-1}{3}$$

as well as the equations of the common perpendicular.
Solution. According to the formula for the distance between two lines we have

$$\delta(d_1,d_2)=\frac{|(\overrightarrow{A_1A_2},\mathbf{d}_1,\mathbf{d}_2)|}{\|\mathbf{d}_1\times\mathbf{d}_2\|},$$

where $A_1(1,-1,0)\in d_1, A_2(-1,0,1)\in d_2$ and $\mathbf{d}_1(2,3,1), \mathbf{d}_2(3,4,3)$ are director vectors of d_1 and d_2 respectively. On the other hand $\overrightarrow{A_1A_2}\,(-2,1,1)$ and

$$(\overrightarrow{A_1A_2},\mathbf{d}_1,\mathbf{d}_2)=\begin{vmatrix}-2 & 1 & 1\\ 2 & 3 & 1\\ 3 & 4 & 3\end{vmatrix}=-18+8+3-9-6+8=-14$$

$$\mathbf{d}_1\times\mathbf{d}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&1\\3&4&3\end{vmatrix}=\begin{vmatrix}3&1\\4&3\end{vmatrix}\mathbf{i}+\begin{vmatrix}1&2\\3&3\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&3\\3&4\end{vmatrix}\mathbf{k}=5\mathbf{i}-3\mathbf{j}-\mathbf{k}\Longrightarrow\|\mathbf{d}_1\times\mathbf{d}_2\|=\sqrt{35}.$$

Thus

$$\delta(d_1,d_2)=\frac{|(\overrightarrow{A_1A_2},\mathbf{d}_1,\mathbf{d}_2)|}{\|\mathbf{d}_1\times\mathbf{d}_2\|}=\frac{14}{\sqrt{35}}.$$

The equations of the common perpendicular are

$$\begin{cases}\begin{vmatrix}x-1&y+1&z-0\\2&3&1\\\begin{vmatrix}3&1\\4&3\end{vmatrix}&\begin{vmatrix}1&2\\3&3\end{vmatrix}&\begin{vmatrix}2&3\\3&4\end{vmatrix}\end{vmatrix}=0\\ \\ \begin{vmatrix}x+1&y-0&z-1\\3&4&3\\\begin{vmatrix}3&1\\4&3\end{vmatrix}&\begin{vmatrix}1&2\\3&3\end{vmatrix}&\begin{vmatrix}2&3\\3&4\end{vmatrix}\end{vmatrix}=0\end{cases}\Longleftrightarrow\begin{cases}\begin{vmatrix}x-1&y+1&z\\2&3&1\\5&-3&-1\end{vmatrix}=0\\ \\ \begin{vmatrix}x+1&y&z-1\\3&4&3\\5&-3&-1\end{vmatrix}=0\end{cases}$$

$$\Longleftrightarrow\begin{cases}7(y+1)-21z=0\\5(x+1)-18y-29(z-1)=0\end{cases}$$
$$\Longleftrightarrow\begin{cases}y-3z+1=0\\5x-18y-29z+29=0.\end{cases}$$

43. Find the distance between the straight lines M_1M_2 and d, where $M_1(-1,0,1)$, $M_2(-2,1,0)$ and

$$(d)\ \begin{cases}x+y+z=1\\2x-y-5z=0.\end{cases}$$

as well as the equations of the common perpendicular.
Solution. According to the formula for the distance between two lines we have

$$\delta(M_1M_2,d)=\frac{|(\overrightarrow{M_1A},\overrightarrow{M_1M_2},\mathbf{d})|}{\|\overrightarrow{M_1M_2}\times\mathbf{d}\|},$$

where $A(3,-4,2)\in d$ and $\mathbf{d}(-4,7,-3)$ is a director vector of d. On the other hand $\overrightarrow{M_1M_2}\,(-1,1,-1)$, $\overrightarrow{M_1A}\,(4,-4,1)$ and

$$(\overrightarrow{M_1A},\overrightarrow{M_1M_2},\mathbf{d})=\begin{vmatrix}4&-4&1\\-1&1&-1\\-4&7&-3\end{vmatrix}=-12-7-16+4+28+12=9$$

$$\overrightarrow{M_1M_2}\times\mathbf{d}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&1&-1\\-4&7&-3\end{vmatrix}=\begin{vmatrix}1&-1\\7&-3\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1&-1\\-4&-3\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1&1\\-4&7\end{vmatrix}\mathbf{k}=4\mathbf{i}+\mathbf{j}-3\mathbf{k}\Longrightarrow\|\overrightarrow{M_1M_2}\times\mathbf{d}\|=\sqrt{26}.$$

Thus

$$\delta(M_1M_2,d)=\frac{|(\overrightarrow{M_1A},\overrightarrow{M_1M_2},\mathbf{d})|}{\|\overrightarrow{M_1M_2}\times\mathbf{d}\|}=\frac{9}{\sqrt{26}}.$$

The equations of the common perpendicular are

$$\left\{\begin{array}{l}\begin{vmatrix} x+1 & y-0 & z-1 \\ -1 & 1 & -1 \\ \begin{vmatrix} 1 & -1 \\ 7 & -3 \end{vmatrix} & \begin{vmatrix} -1 & -1 \\ -3 & -4 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ -4 & 7 \end{vmatrix} \end{vmatrix} = 0 \\ \\ \begin{vmatrix} x-3 & y+4 & z-2 \\ -4 & 7 & -3 \\ \begin{vmatrix} 1 & -1 \\ 7 & -3 \end{vmatrix} & \begin{vmatrix} -1 & -1 \\ -3 & -4 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ -4 & 7 \end{vmatrix} \end{vmatrix} = 0 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l}\begin{vmatrix} x+1 & y & z-1 \\ -1 & 1 & -1 \\ 4 & 1 & -3 \end{vmatrix} = 0 \\ \\ \begin{vmatrix} x-3 & y+4 & z-2 \\ -4 & 7 & -3 \\ 4 & 1 & -3 \end{vmatrix} = 0 \end{array}\right.$$

$$\Longleftrightarrow \begin{cases} -2(x+1)-7y-5(z-1)=0 \\ -18(x-3)-24(y+4)-32(z-2)=0 \end{cases}$$
$$\Longleftrightarrow \begin{cases} 2x+7y+5z-3=0 \\ 9x+12y+16z-11=0. \end{cases}$$

44. The points $A(1,2\alpha,\alpha)$, $B(3,2,1)$, $C(-\alpha,0,\alpha)$ and $D(-1,3,-3)$ are being considered with respect to some orthonormal cartezian system. Find the value of the parameter α for which the pencil of planes through the straight line AB has a common plane with the pencil of planes through the straight line CD.

45. Find the value of the parameter λ for which the straight lines

$$(d_1)\ \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1},\ (d_2)\ \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

are coplanar. Find the coordinates of their intersection point in that case.
Solution. The coplanarity condition is

$$(\overrightarrow{A_1A_2}, \mathbf{d}_1, \mathbf{d}_2) = 0,$$

where $A_1(1,-2,0) \in d_1$, $A_2(-1,3,0) \in d_2$ and $\mathbf{d}_1(3,-2,1)$, $\mathbf{d}_2(4,1,\lambda)$ are the director vectors of d_1 and d_2 respectively. On the other hand $\overrightarrow{A_1A_2}\ (-2,5,0)$ and

$$(\overrightarrow{A_1A_2}, \mathbf{d}_1, \mathbf{d}_2) = \begin{vmatrix} -2 & 5 & 0 \\ 3 & -2 & 1 \\ 4 & 1 & \lambda \end{vmatrix} = 4\lambda + 0 + 20 - 0 + 2 - 15\lambda = -11\lambda + 22.$$

Thus $(\overrightarrow{A_1A_2}, \mathbf{d}_1, \mathbf{d}_2) = 0 \Longleftrightarrow \lambda = 2$.
In order to find the intersection point of the lines d_1 and d_2, let us first find the intersection point of the line d_1 with onhe of the planes in the pencil through d_2, i.e. $(\pi)\ x-4y+13=0$. The parametric equations of the line d_1 are

$$\begin{cases} x = 1+3t \\ y = -2-2t \\ z = t \end{cases},\ t \in \mathbb{R}.$$

If we require the point of coordinates $(1+3t, -2-2t, t)$ to verify the equation $x-4y+13=0$, we obtain $1+3t+8+8t+13=0$, i.e. $t=-2$ and the coordinates of the intersection point are $(-5, 2-2)$.

46. Consider the planes $(\pi_1)\ 2x+y-3z-5=0$, $(\pi_2)\ x+3y+2z+1=0$. Find the equations of the bisector planes of the dihedral angles formed by the planes π_1 and π_2 and select the one contained into the acute regions of the dihedral angles formed by the two planes.

47. Let a, b two numbers such that $a^2 \neq b^2$. We also consider the planes:

$$(\alpha_1) ax + by - (a+b)z = 0$$

$$(\alpha_2) ax - by - (a-b)z = 0$$

and the quadric

$$(\mathscr{C}) : a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz - a^2b^2 = 0.$$

If $a^2 < b^2$, show that the quadric $\mathscr{C}$ is contained in the acute regions of the dihedral angles formed by the two planes. If, on the contrary, $a^2 > b^2$, show that the quadric $\mathscr{C}$ is contained in the obtuse regions of the dihedral angles formed by the two planes.
Solution. Fie $M(x,y,z) \in \mathscr{C}$. Cu notaţiile

$$\alpha_1(x,y,z) = ax + by - (a+b)z = 0,$$

$$\alpha_2(x,y,z) = ax - by - (a-b)z = 0,$$

$$\overrightarrow{n}_{\alpha_1}(a, b, -(a+b)),$$

$$\overrightarrow{n}_{\alpha_2}(a, -b, -(a-b)),$$

avem:

$$\alpha_1(x,y,z)\alpha_2(x,y,z)(\overrightarrow{n}_{\alpha_1} \cdot \overrightarrow{n}_{\alpha_2}) =$$

$$= (ax + by - (a+b)z)(ax - by - (a-b)z) \cdot (a^2 - b^2 + a^2 - b^2) =$$

$$= 2(a^2 - b^2)[a(x-z) + b(y-z)][a(x-z) - b(y-z)] =$$

$$= 2(a^2 - b^2)[a^2(x-z)^2 + b^2(y-z)^2] =$$

$$= 2(a^2 - b^2)(a^2x^2 - 2a^2xz + a^2z^2 - b^2y^2 + 2b^2yz - b^2z^2) =$$

$$= 2(a^2 - b^2)(a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz) =$$

$$= 2(a^2 - b^2)a^2b^2.$$

Deci dacă $a^2 < b^2$ atunci $\alpha_1(x,y,z) \cdot \alpha_1(x,y,z)(\overrightarrow{n}_{\alpha_1} \cdot \overrightarrow{n}_{\alpha_2}) < 0$, iar dacă $a^2 > b^2$ atunci $\alpha_1(x,y,z) \cdot \alpha_1(x,y,z)(\overrightarrow{n}_{\alpha_1} \cdot \overrightarrow{n}_{\alpha_2}) > 0$ şi astfel folosind punctul 1, punctul 2 rezultă imediat.

48. Determine the coordinates of the foci of the ellipse $(E)\ 9x^2 + 25y^2 - 225 = 0$.
49. Sketch the graph of $y = -\dfrac{3}{4}\sqrt{16 - x^2}$.
50. Find the intersection points between the line $(d)\ x + 2y - 7 = 0$ and the ellipse $(E)\ x^2 + 3y^2 - 25 = 0$.
51. Find the equations of the tangent lines to the ellipse $\mathscr{E}:\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ having a given angular coefficient $m \in \mathbb{R}$. (see [1, p. 110]).
52. Find the equations of the tangent lines to the ellipse $\mathscr{E}:\ x^2 + 4y^2 - 20 = 0$ which are orthogonal to the line $d:\ 2x - 2y - 13 = 0$.

53. Find the equations of the tangent lines to the ellipse $\mathscr{E}: \dfrac{x^2}{25}+\dfrac{y^2}{16}-1=0$, passing through $P_0(10,-8)$.
54. Determine the coordinates of the foci of the hyperbola $\mathscr{H}: \dfrac{x^2}{9}-\dfrac{y^2}{4}-1=0$.
55. Find the intersection points between the line $(d)\ 2x-y-10=0$ and the hyperbola $\mathscr{H}: \dfrac{x^2}{20}-\dfrac{y^2}{5}-1=0$.
56. Find the equations of the tangent lines to the hyperbola $\mathscr{H}: \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-1=0$ having a given angular coefficient $m\in\mathbb{R}$. (see [1, p. 115]).
57. Find the equations of the tangent lines to the hyperbola $\mathscr{H}: \dfrac{x^2}{20}-\dfrac{y^2}{5}-1=0$ which are orthogonal to the line $d: 4x+3y-7=0$.
58. Find the focus and the director line of the parabola $\mathscr{P}: y^2=2px$.
59. Find the equation of the parabola having the focus $F(-7,0)$ and the director line $d: x-7=0$.
60. Find the equations of the tangent lines to the parabola $\mathscr{P}: y^2=2px$ having a given angular coefficient $m\in\mathbb{R}$. (see [1, p. 119]).
61. Find the equation of the tangent line to the parabola $\mathscr{P}: y^2-8x=0$, parallel to $d: 2x+2y-3=0$.
62. Find the equation of the tangent line to the parabola $\mathscr{P}: y^2-36x=0$, passing through $P(2,9)$.
63. Show that the sum of the distances from any point inside the ellipse to its foci is less than the length of the major axis.
64. Show that a ray of light through a focus of an ellipse reflects to a ray that passes through the other focus (optical property of the ellipse).

Solution. Let $F_1(-c,0)$, $F_2(c,0)$ be the foci of the ellipse $\mathscr{E}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Recall that the gradient $\mathrm{grad}(f)(x_0,y_0)=(f_x(x_0,y_0),f_y(x_0,y_0))$ is a normal vector of the ellipse $\mathscr{E}$ to its point $M_0(x_0,y_0)$, where

$$f(x,y)=\delta(F_1,M)+\delta(F_2,M)=\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}$$

and $M(x,y)$. Note that

$$f_x(x_0,y_0)=\frac{x_0-c}{\delta(F_1,M_0)}+\frac{x_0+c}{\delta(F_2,M_0)} \text{ and } f_y(x_0,y_0)=\frac{y}{\delta(F_1,M_0)}+\frac{y}{\delta(F_2,M_0)},$$

and shows that

$$\begin{aligned}\mathrm{grad}(f) &= (f_x(x_0,y_0),f_y(x_0,y_0))=\left(\frac{x_0-c}{\delta(F_1,M_0)}+\frac{x_0+c}{\delta(F_2,M_0)},\frac{y_0}{\delta(F_1,M_0)}+\frac{y_0}{\delta(F_2,M_0)}\right)\\ &=\frac{(x_0-c,y)}{\delta(F_1,M_0)}+\frac{(x_0+c,y)}{\delta(F_2,M_0)}=\frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}+\frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}.\end{aligned}$$

The versors $\dfrac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}$ and $\dfrac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}$ point towards the exterior of the ellipse $\mathscr{E}$ and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{F_1M_0}$ and $\overrightarrow{F_2M_0}$ and (the sum) is also orthogonal to the tangent $T_{M_0}(\mathscr{E})$ of the ellipse at $M_0(x_0,y_0)$.

This shows that the angle between the ray F_1M and the tangent $T_{M_0}(\mathscr{E})$ equals the angle between the ray F_2M and the tangent $T_{M_0}(\mathscr{E})$.

65. Show that a ray of light through a focus of a hyperbola reflects to a ray that passes through the other focus (optical property of the hyperbola).

Solution. Let $F_1(-c,0)$, $F_2(c,0)$ be the foci of the hyperbola $\mathscr{E}: \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Recall that the gradient $\mathrm{grad}(f)(x_0,y_0)=(f_x(x_0,y_0),f_y(x_0,y_0))$ is a normal vector of the hyperbola $\mathscr{H}$ to its point $M_0(x_0,y_0)$, where

$$f(x,y)=\delta(F_2,M)-\delta(F_1,M)=\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2} \tag{0.4}$$

on the left hand side branch of $\mathscr{H}$ and

$$f(x,y)=\delta(F_1,M)-\delta(F_2,M)=\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2} \tag{0.5}$$

on the right hand side branch of $\mathscr{H}$ and $M(x,y)$. We shall only use the version (0.4) of f, as judgement for the version (0.5) works in a similar way. Note that

$$f_x(x_0,y_0)=\frac{x_0-c}{\delta(F_1,M_0)}-\frac{x_0+c}{\delta(F_2,M_0)} \text{ and } f_y(x_0,y_0)=\frac{y}{\delta(F_1,M_0)}-\frac{y}{\delta(F_2,M_0)},$$

and shows that

$$\begin{aligned}\mathrm{grad}(f) &= (f_x(x_0,y_0),f_y(x_0,y_0))=\left(\frac{x_0-c}{\delta(F_1,M_0)}-\frac{x_0+c}{\delta(F_2,M_0)},\frac{y_0}{\delta(F_1,M_0)}-\frac{y_0}{\delta(F_2,M_0)}\right)\\ &=\frac{(x_0-c,y)}{\delta(F_1,M_0)}-\frac{(x_0+c,y)}{\delta(F_2,M_0)}=\frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}-\frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}.\end{aligned}$$

The versors $\frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}$ and $-\frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}$ point towards the 'exterior' of the hyperbola $\mathscr{H}$[1] and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{F_1M_0}$ and $\overrightarrow{F_2M_0}$ and (the sum) is also orthogonal to the tangent $T_{M_0}(\mathscr{H})$ of the hyperbola at $M_0(x_0,y_0)$. This shows that the angle between the ray F_1M and the tangent $T_{M_0}(\mathscr{H})$ equals the angle between the ray F_2M and the tangent $T_{M_0}(\mathscr{H})$.

66. Show that a ray of light through a focus of a parabola reflects to a ray parallel to the axis of the parabola (optical property of the parabola).

Solution. Let $F(\frac{p}{2},0)$ be the focus of the parabola $\mathscr{P}:\ y^2=2px$ and $d: x=-\frac{p}{2}$ be its director line. Recall that the gradient $\mathrm{grad}(f)(x_0,y_0)=(f_x(x_0,y_0),f_y(x_0,y_0))$ is a normal vector of parabola $\mathscr{P}$ to its point $M_0(x_0,y_0)$, where

$$f(x,y)=\delta(F,M)-\delta(M,d)=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}-\left(x+\frac{p}{2}\right)$$

and $M(x,y)$. Note that

$$f_x(x_0,y_0)=\frac{x_0-\frac{p}{2}}{\delta(F,M_0)}-1 \text{ and } f_y(x_0,y_0)=\frac{y_0}{\delta(F,M_0)},$$

[1] The exterior of a hyperbola is the nonconvex component of its complement

and shows that

$$\operatorname{grad}(f) = (f_x(x_0,y_0), f_y(x_0,y_0)) = \left(\frac{x_0 - \frac{p}{2}}{\delta(F,M_0)} - 1, \frac{y_0}{\delta(F,M_0)} \right)$$
$$= \left(\frac{x_0 - \frac{p}{2}}{\delta(F,M_0)}, \frac{y_0}{\delta(F,M_0)} \right) - (1,0) = \frac{\overrightarrow{FM_0}}{\delta(F,M_0)} - \mathbf{i}.$$

The versors $\dfrac{\overrightarrow{FM_0}}{\delta(F,M_0)}$ and $-\mathbf{i}$ point towards the 'exterior' of the parabola $\mathscr{P}$[2] and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{FM_0}$ and $\mathbf{i}$ and (the sum) is also orthogonal to the tangent line $T_{M_0}(\mathscr{P})$ of the parabola at $M_0(x_0,y_0)$. This shows that the angle between the ray FM and the tangent line $T_{M_0}(\mathscr{P})$ equals the angle between Ox and the tangent $T_{M_0}(\mathscr{E})$.

67. Find the intersection points of the ellipsoid

$$(\mathscr{E})\ \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1 \text{ with the line } (d)\ \frac{x-4}{2} = \frac{y+6}{-3} = \frac{z+2}{-2}.$$

Write the equations of the tangent plane and the normal line to the ellipsoid $(\mathscr{E})$ at those intersection points $\mathscr{E} \cap d$.

68. Find the rectilinea generatrices of the hyperboloid of one sheet

$$(\mathscr{H}_1)\ \frac{x^2}{36} + \frac{y^2}{9} - \frac{z^2}{4} = 1$$

which are parallel to the plane $(\pi)\ x+y+z=0$.

69. Find the locus of points on the hyperbolic paraboloid $(\mathscr{P}_h)\ y^2 - z^2 = 2x$ through wich the rectilinear generatrices are perpendicular.
70. Find the locus of points in the space equidistant to two given straight lines.
71. Find the equation of the cylindrical surface whose director curve is the planar curve

$$(C)\ \begin{cases} y^2 + z^2 = x \\ x = 2z \end{cases}$$

and the generatrix is perpendicular to the plane of the director curve.

72. Find the equation of the cylindrical surface generated by a variable straight line parallel to the line

$$(\Delta)\ \begin{cases} y - 3z = 0 \\ y + 2z = 0 \end{cases} \text{ which is tangent to the surface } (\mathscr{E})\ 4x^2 + 3y^2 + 2z^2 = 1.$$

73. Find the equation of the conical surface with the vertex $V(1,1,1)$ whose directrix is the curve

$$(C)\ \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0. \end{cases}$$

74. Find the equation of the conoidal surface generated by one variable line which intersects a linne (d) and acircle (C), whose plane is parallel to (d), and remains parallel to a plane orthogonal to (d) (The Willis conoid).
75. Find the equation of the revolution surface generated by the rotation of a variable line through a fixed line.

[2] The exterior of a parabola is the nonconvex component of its complement

76. The *torus* is the revolution surface obtained by the rotation of a circle C about a fixed line (d) within the plane of the circle such that $d \cap C = \emptyset$. Find the equation of the torus.
77. Consider a quadrilateral with vertices $A(1,1)$, $B(3,1)$, $C(2,2)$, and $D(1.5,3)$. Find the image quadrilaterals through the translation $T(1,2)$, the scaling $S(2,2.5)$, the reflections about the x and y-axes, the clockwise and anticlockwise rotations through the angle $\pi/2$ and the shear $Sh\left(\left(2/\sqrt{5},1/\sqrt{5}\right),1.5\right)$.
78. Find the concatenation (product) of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$ followed by a scaling by a factor of 3 units in the x-direction and 2 units in the y-direction. (Hint: $S(3,2)R_{3\pi/2}$)
79. Show that the concatenation (product) of two rotations, the first through an angle θ about a point $P(x_0,y_0)$ and the second about a point $Q(x_1,y_1)$ (distinct from P) through an angle $-\theta$ is a translation.
80. Consider a line (d) $ax+by+c$ whose slope is $\mathrm{tg}\theta = -\frac{a}{b}$. By using the observation that the reflection S_d in the line d is the following concatenation (product)

$$T(0,c/b)\circ R_\theta \circ S_x \circ R_{-\theta}\circ T(0,-c/b),$$

show that

$$S_d = \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc}{a^2+b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

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