Geometry[1] First Year, Computer science

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[1] These notes are not in a final form. They are continuously being improved

Content

The dot product

Definition 1.1

The real number

$$\vec{a} \cdot \vec{b} = \begin{cases} 0 \text{ if } \vec{a} = 0 \text{ or } \vec{b} = 0 \\ ||\vec{a}|| \cdot ||\vec{b}|| cos(\widehat{\vec{a}, \vec{b}}) \text{ if } \vec{a} \neq 0 \text{ and } \vec{b} \neq 0 \end{cases} \quad (1.1)$$

is called the *dot product* of the vectors $\vec{a}$, $\vec{b}$.

Remark 1.2

$\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$

Proposition 1.3

The dot product has the following properties:

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}, \ \forall \vec{a}, \vec{b} \in \mathcal{V}$.
2. $\vec{a} \cdot (\lambda \vec{b}) = \lambda (\vec{a} \cdot \vec{b}), \ \forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}$.
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}, \ \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.
4. $\vec{a} \cdot \vec{a} \geq 0, \ \forall \vec{a} \in \mathcal{V}$.
5. $\vec{a} \cdot \vec{a} = 0 \Leftrightarrow \vec{a} = \vec{0}$.

Definition 1.4

A basis of the vector space $\mathcal{V}$ is said to be *orthonormal*, if $||\vec{i}|| = ||\vec{j}|| = ||\vec{k}|| = 1, \ \vec{i} \perp \vec{j}, \vec{j} \perp \vec{k}, \vec{k} \perp \vec{i}$ $(\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1, \ \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0)$. A cartesian reference system $R = (O, \vec{i}, \vec{j}, \vec{k})$ is said to be *orthonormal* if the basis $[\vec{i}, \vec{j}, \vec{k}]$ is orthonormal.

Proposition 1.5

Let $[\vec{i}, \vec{j}, \vec{k}]$ be an orthonormal basis and $\vec{a}, \vec{b} \in \mathcal{V}$. If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad (1.2)$$

Thus, $\vec{a} \cdot \vec{a} = ||\vec{a}|| \cdot ||\vec{a}|| \cos 0 = ||\vec{a}||^2$. On the other hand $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$ and we conclude that

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Consequently

$$\begin{aligned}
\cos(\widehat{\vec{a},\vec{b}}) &= \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||\cdot||\vec{b}||} \\
&= \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}.
\end{aligned} \tag{1.3}$$

In particular

$$\cos(\widehat{\vec{a},\vec{i}}) = \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}$$
$$\cos(\widehat{\vec{a},\vec{j}}) = \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}$$
$$\cos(\widehat{\vec{a},\vec{k}}) = \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}$$

Remark 1.6

$\vec{a}\perp\vec{b} \Leftrightarrow a_1b_1+a_2b_2+a_3b_3=0$

Aplications of the dot product

• **The distance between two points.** Consider two points $A(x_A, y_A, z_A),\ B(x_B, y_B, z_B) \in \mathcal{P}$. The norm of the vector $\overrightarrow{AB}\,(x_B - x_A, y_B - y_A, z_B - z_A)$ is

$$|| \overrightarrow{AB} || = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

• **The normal vector of a plane.** Consider the plane $\pi\ :\ Ax + By + Cz + D = 0$ and the point $P(x_0, y_0, z_0) \in \pi$. The equation of π becomes

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{1.4}$$

If $M(x, y, z) \in \pi$, the coordinates of $\overrightarrow{PM}$ are $(x - x_0, y - y_0, z - z_0)$ and the equation (1.4) tells us that $\vec{n}\cdot \overrightarrow{PM} = 0$, for every $M \in \pi$, that is $\vec{n} \perp \overrightarrow{PM} = 0$, for every $M \in \pi$, which is equivalent to $\vec{n} \perp \vec{\pi}$, where $\vec{n}(A, B, C)$. This is the reason to call $\vec{n}(A, B, C)$ *the normal vector* of the plane π.

• **The distance from a point to a plane.** Consider the plane $\pi : Ax + By + Cz + D = 0$, a point $P(x_P, y_P, z_P) \in \mathcal{P}$ and M the orthogonal projection of P on π.

The real number δ given by $\overrightarrow{MP} = \delta \cdot \vec{n}_0$ is called the *oriented distance* from P to the plane π, where $\vec{n}_0 = \frac{1}{||\vec{n}||}\vec{n}$ is the versor of $\vec{n}$. Since $\overrightarrow{MP} = \delta \cdot \vec{n}_0$, it follows that $\delta(P, M) = || \overrightarrow{MP} || = |\delta|$, where $\delta(P, M)$ stands for the distance from P to π.

On the other hand we shall show that

$$\delta = \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2 + B^2 + C^2}}.$$

Indeed, since $\overrightarrow{MP} = \delta \cdot \vec{n}_0$, we get successively:

$$\begin{aligned}
\delta &= \vec{n}_0 \cdot \overrightarrow{MP} = \left(\frac{1}{||\vec{n}||}\vec{n}\right) \cdot \overrightarrow{MP} = \frac{\vec{n} \cdot \overrightarrow{MP}}{||\vec{n}||} \\
&= \frac{A(x_P - x_M) + B(y_P - y_M) + C(z_P - z_M)}{\sqrt{A^2 + B^2 + C^2}} \\
&= \frac{Ax_P + By_P + Cz_P - (Ax_M + By_M + Cz_M)}{\sqrt{A^2 + B^2 + C^2}} \\
&= \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2 + B^2 + C^2}}.
\end{aligned}$$

Consequently

$$\delta(P, M) = ||\ \overrightarrow{MP}\ || = |\delta| = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

The vector product

Definition 1.7

The *vector product* or the *cross product* of the vectors $\vec{a}, \vec{b} \in \mathcal{V}$ is a vector, denoted by $\vec{a} \times \vec{b}$, which is defined to be zero if $\vec{a}, \vec{b}$ are linearly dependent (collinear), and if $\vec{a}, \vec{b}$ are linearly independent (noncollinear), then it is defined by the following data:

1. $\vec{a} \times \vec{b}$ is a vector orthogonal on the two-dimensional subspace $\langle \vec{a}, \vec{b} \rangle$ of $\mathcal{V}$;
2. if $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, then the sense of $\vec{a} \times \vec{b}$ is the one in which a right-handed screw, placed along the line passing through O orthogonal to the vectors $\vec{a}$ and $\vec{b}$, advances when it is being rotated simultaneously with the vector $\vec{a}$ from $\vec{a}$ towards $\vec{b}$ within the vector subspace $\langle \vec{a}, \vec{b} \rangle$ and the support half line of $\vec{a}$ sweeps the interior of the angle $\widehat{AOB}$ (Screw rule).
3. the *norm* (*magnitude* or *length*) of $\vec{a} \times \vec{b}$ is defined by

$$||\vec{a} \times \vec{b}|| = ||\vec{a}|| \cdot ||\vec{b}|| \sin(\widehat{\vec{a}, \vec{b}}).$$

Remarks 1.8

1. The *norm* (*magnitude* or *length*) of the vector $\vec{a} \times \vec{b}$ is actually the area of the parallelogram constructed on the vectors $\vec{a}$, $\vec{b}$.
2. The vectors $\vec{a}, \vec{b} \in \mathcal{V}$ are linearly dependent (collinear) if and only if $\vec{a} \times \vec{b} = \vec{0}$.

Proposition 1.9

The vector product has the following properties:

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}, \forall\, \vec{a}, \vec{b} \in \mathcal{V}$;
2. $(\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b}) = \lambda(\vec{a} \times \vec{b}), \forall\, \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}$;
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}, \forall\, \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.

If $[\vec{i}, \vec{j}, \vec{k}]$ is an orthonormal basis, observe that $\vec{i} \times \vec{j} \in \{-\vec{k}, \vec{k}\}$. We say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *direct* if $\vec{i} \times \vec{j} = \vec{k}$. If, on the contrary, $\vec{i} \times \vec{j} = -\vec{k}$, we say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *inverse*. Therefore, if $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis, then $\vec{i} \times \vec{j} = \vec{k},\ \vec{j} \times \vec{k} = \vec{i},\ \vec{k} \times \vec{i} = \vec{j}$ and obviously $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

Proposition 1.10

If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k},\ \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}, \tag{1.5}$$

or, equivalently,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k} \tag{1.6}$$

One can rewrite formula (1.5) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.7}$$

the right hand side determinant being understood in the sense of its cofactor expansion along the first line.