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Solutions to Exercise Sheet no.9

Analysis for CS

(G 24)

a) Let $(x,y,z) \in \mathbb{R}^* \times \mathbb{R}^2$ be arbitrarily chosen. Then

$$\frac{\partial f}{\partial x}(x,y,z) = -\frac{z^2e^y}{x^2},\ \frac{\partial f}{\partial y}(x,y,z) = \frac{z^2e^y}{x} \text{ and } \frac{\partial f}{\partial z}(x,y,z) = \frac{2ze^y}{x}.$$

b) Let $(x,y,z) \in \mathbb{R}^* \times \mathbb{R}^2$ be arbitrarily chosen. Then

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 2\frac{z^2e^y}{x^3},\ \frac{\partial^2 f}{\partial y^2}(x,y,z) = \frac{z^2e^y}{x} \text{ and } \frac{\partial^2 f}{\partial z^2}(x,y,z) = \frac{2e^y}{x}.$$

The mixed second-order partial derivatives are

$$\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = -\frac{z^2e^y}{x^2} = \frac{\partial^2 f}{\partial x \partial y}(x,y,z),$$

$$\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = \frac{2ze^y}{x} = \frac{\partial^2 f}{\partial z \partial y}(x,y,z),$$

$$\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = -\frac{2ze^y}{x^2} = \frac{\partial^2 f}{\partial x \partial z}(x,y,z).$$

(G 25)

a) First we analyze the partial differentiability with respect to x at 0_2:

$$\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{\frac{x^4-0}{2(x^4)+0} - 0}{x} = \lim_{x\to 0} \frac{1}{2x}.$$

Since $\lim\limits_{\substack{x\to 0\\ x<0}} \frac{1}{2x} = -\infty$ and $\lim\limits_{\substack{x\to 0\\ x>0}} \frac{1}{2x} = +\infty$, we conclude that $\lim\limits_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0}$ does not exist. Hence f is not partially differentiable with respect to x at 0_2.

Then we analyze the partial differentiability with respect to y at 0_2:

$$\lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{\frac{0-y^4}{2(0+y^4)} - 0}{y} = \lim_{y\to 0} -\frac{1}{2y}.$$

Since $\lim\limits_{\substack{y\to 0\\ y<0}} -\frac{1}{2y} = +\infty$ and $\lim\limits_{\substack{y\to 0\\ y>0}} -\frac{1}{2y} = -\infty$ we conclude that $\lim\limits_{y\to 0} \frac{f(0,y)-f(0,0)}{x-0}$ does not exist. Hence f is not partially differentiable with respect to y at 0_2.

b) We will prove that f is not continuous at 0_2. Assume by contradiction that f is continuous at 0_2. Then, according to **Th3** in Lecture 9, for every sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^2$, with $\lim\limits_{k\to\infty} x^k = 0_2$, one should have that $\lim\limits_{k\to\infty} f(x^k) = f(0_2)$.

If we consider the sequence with the general term $a^k = (\frac{1}{k}, 0)$, then $\lim\limits_{k\to\infty} a^k = 0_2$ and $\lim\limits_{k\to\infty} f(a^k) = \lim\limits_{k\to\infty} \dfrac{\left(\frac{1}{k}\right)^4 - 0}{2\left(\left(\frac{1}{k}\right)^4 + 0\right)} = \lim\limits_{k\to\infty} \dfrac{1}{2} = \dfrac{1}{2} \neq 0 = f(0_2)$. Hence we have obtained a contradiction. Thus f is not continuous at 0_2.

(G 26)

Let $x \in \operatorname{int} M$. Then there exists $r > 0$ such that $B(x,r) \subseteq M$. Since $x \in B(x,r)$, we conclude that $x \in M$.

If $V \in \mathcal{V}(x)$ then there exists $r' > 0$ such that $B(x,r') \subseteq V$. Let $r_0 := \min\{r, r'\}$. Then

$$B(x, r_0) \setminus \{x\} \subseteq V \cap (M \setminus \{x\}),$$

thus $V \cap (M \setminus \{x\}) \neq \emptyset$, showing that $x \in M'$.

Hence $\operatorname{int} M \subseteq M$ and $\operatorname{int} M \subseteq M'$

HOMEWORK:

(H 23)

a) First we analyze the partial differentiability with respect to x at 0_2:

$$\lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x\to 0} \frac{0-0}{x} = \lim_{x\to 0} 0 = 0.$$

This means that f is partially differentiable with respect to x at 0_2, and that $\frac{\partial f}{\partial x}(0,0) = 0$.

Then we analyze the partial differentiability with respect to y at 0_2:

$$\lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{0-0}{y} = \lim_{y\to 0} 0 = 0.$$

This means that f is partially differentiable with respect to y at 0_2, and that $\frac{\partial f}{\partial y}(0,0) = 0$.

Second we analyze the case when $(x,y) \in \mathbb{R}^2 \setminus \{0_2\}$.

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^3(x^2+y^2) - xy^3(2x)}{(x^2+y^2)^2} = \frac{y^3(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{3xy^2(x^2+y^2) - xy^3(2y)}{(x^2+y^2)^2} = \frac{xy^2(3x^2+y^2)}{(x^2+y^2)^2}.$$

Thus

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{y^3(y^2-x^2)}{(x^2+y^2)^2}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2 \end{cases} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{xy^2(3x^2+y^2)}{(x^2+y^2)^2}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2. \end{cases}$$

b) Let (x,y,z) be arbitrarily chosen in $\mathbb{R}^3$. Then

$$\frac{\partial f}{\partial x}(x,y,z) = z\cos(x-y), \quad \frac{\partial f}{\partial y}(x,y,z) = -z\cos(x-y) \text{ and } \frac{\partial f}{\partial z}(x,y,z) = \sin(x-y).$$

Moreover

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = -z\sin(x-y),\ \frac{\partial^2 f}{\partial y^2}(x,y,z) = -z\sin(x-y) \text{ and } \frac{\partial^2 f}{\partial z^2}(x,y,z) = 0$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = z\sin(x-y) = \frac{\partial^2 f}{\partial x \partial y}(x,y,z),$$

$$\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = \cos(x-y) = \frac{\partial^2 f}{\partial x \partial y}(x,y,z),$$

$$\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = -\cos(x-y) = \frac{\partial^2 f}{\partial z \partial y}(x,y,z).$$

(H 24)

a) Let $(x,y) \in \mathbb{R}^2$ be arbitrarily chosen. Then $\frac{\partial f}{\partial x}(x,y) = -e^{-x}\sin(x+2y) + e^{-x}\cos(x+2y)$ and $\frac{\partial f}{\partial y}(x,y) = 2e^{-x}\cos(x+2y)$. Thus

$$\begin{aligned}\nabla f\left(0,\frac{\pi}{4}\right) &= \left(\frac{\partial f}{\partial x}\left(0,\frac{\pi}{4}\right), \frac{\partial f}{\partial y}\left(0,\frac{\pi}{4}\right)\right) \\ &= \left(-e^{-0}\sin\left(0+\frac{\pi}{2}\right) + e^{-0}\cos\left(0+\frac{\pi}{2}\right), 2e^{-0}\cos\left(0+\frac{\pi}{2}\right)\right) = (-1,0).\end{aligned}$$

b) Let $(x,y,z) \in \mathbb{R}^3$ be arbitrarily chosen. Then $\frac{\partial f}{\partial x}(x,y,z) = \cos\pi z$, $\frac{\partial f}{\partial x}(x,y,z) = -\cos\pi z$ and $\frac{\partial f}{\partial x}(x,y,z) = -(x-y)\pi\sin\pi z$. Thus

$$\nabla f\left(1,0,\frac{1}{2}\right) = \left(\cos\frac{\pi}{2}, -\cos\frac{\pi}{2}, -(1-0)\pi\sin\frac{\pi}{2}\right) = (0,0,-\pi).$$