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Solutions to Exercise Sheet no.10

Analysis for CS

(G 27)

a) For all $(x,y,z) \in \mathbb{R}^3$ it holds $\frac{\partial f}{\partial x}(x,y,z) = 3x^2 - 3$, $\frac{\partial f}{\partial y}(x,y,z) = 2y$ and $\frac{\partial f}{\partial z}(x,y,z) = 2z$. Thus the stationary points of the function f are the solutions of the system

$$\begin{cases} 3x^2 - 3 = 0 \\ 2y = 0 \\ 2z = 0. \end{cases}$$

Therefore $(-1,0,0)$ and $(1,0,0)$ are the only stationary points of the function.

For all $(x,y,z) \in \mathbb{R}^3$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 6x, \quad \frac{\partial^2 f}{\partial y^2}(x,y,z) = 2, \quad \frac{\partial^2 f}{\partial z^2}(x,y,z) = 2,$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = 0, \quad \frac{\partial^2 f}{\partial x \partial z}(x,y,z) = 0, \quad \frac{\partial^2 f}{\partial y \partial z}(x,y,z) = 0,$$

thus

$$H_f(-1,0,0) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad H_f(1,0,0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The matrix $H_f(1,0,0)$ is positive definite, thus $(1,0,0)$ is a local minimum point of the function. The corresponding extreme value is $f(1,0,0) = -2$.

Since Sylvester's criterion cannot be applied to the matrix $H_f(-1,0,0)$, we will study its type by other means. The quadratic form associated to this matrix is $\Phi \colon \mathbb{R}^3 \to \mathbb{R}$, defined by

$$\Phi(h_1, h_2, h_3) = -6(h_1)^2 + 2(h_2)^2 + 2(h_3)^2.$$

Since $\Phi(1,0,0) = -6 < 0$ and $\Phi(0,1,0) = 2 > 0$ it follows that this quadratic form (and therefore its associated matrix) is indefinite. Thus $(-1,0,0)$ is not a local extremum of f.

b) For all $(x,y) \in \mathbb{R}^2$ it holds $\frac{\partial f}{\partial x}(x,y) = 4x^3 - 8(x-y)$ and $\frac{\partial f}{\partial y}(x,y) = 4y^3 + 8(x-y)$. The stationary points of the function f are the solutions of the system

$$\begin{cases} 4x^3 - 8(x-y) = 0 \\ 4y^3 + 8(x-y) = 0. \end{cases}$$

By adding the two equations we get $x^3 = -y^3$, thus $x = -y$. By replacing this in the first equation, we get $x^3 - 4x = 0$, thus $x \in \{-2, 0, 2\}$. Therefore the stationary points of the function are $(-2,2)$, $(0,0)$ and $(2,-2)$.

For all $(x,y) \in \mathbb{R}^2$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2 - 8, \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = 8, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 12y^2 - 8.$$

For the three stationary points we get

$$H_f(-2,2) = H_f(2,-2) = \begin{pmatrix} 40 & 8 \\ 8 & 40 \end{pmatrix} \quad \text{and} \quad H_f(0,0) = \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix}.$$

Since the matrix $H_f(-2,2) = H_f(2,-2)$ is positive definite, the points $(-2,2)$ and $(2,-2)$ are local minimum points. The corresponding extreme values are, respectively, $f(-2,2) = -32$ and $f(2,-2) = -32$.

Due to the fact that the determinant of the matrix $H_f(0,0)$ is zero (more exactly, by computing the quadratic form associated with this matrix, we get that it is negative semidefinite), the algorithm for determining the stationary points provides us with no useful information connected to the stationary point $(0,0)$. Nevertheless we notice that $f(0,0) = 0$, $f(x,x) = 2x^4$, $f(x,0) = x^2(x^2-4)$. It follows that in each neighborhood of $(0,0)$ the function f takes both positive and negative values (for example, for all $n \in \mathbb{N}^*$ it holds $f(\frac{1}{n}, \frac{1}{n}) > 0$ and $f(\frac{1}{n}, 0) < 0$). Therefore $(0,0)$ is not a local extremum.

c) For all $(x,y,z) \in \mathbb{R}^3$ it holds $\frac{\partial f}{\partial x}(x,y,z) = yz^2 + y$, $\frac{\partial f}{\partial y}(x,y,z) = xz^2 + x$ and $\frac{\partial f}{\partial z}(x,y,z) = 2z(1+xy)$. The stationary points of f are the solutions of the system

$$\begin{cases} y(z^2+1) = 0 \\ x(z^2+1) = 0 \\ 2z(1+xy) = 0. \end{cases}$$

From the first two equations it follows that $y = 0$ and $x = 0$, respectively. By replacing these values in the last equation, we reach the conclusion that $z = 0$. Hence $(0,0,0)$ is the only stationary point of the function.

For all $(x,y,z) \in \mathbb{R}^3$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 0, \quad \frac{\partial^2 f}{\partial y^2}(x,y,z) = 0, \quad \frac{\partial^2 f}{\partial z^2}(x,y,z) = 2(1+xy),$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = z^2 + 1, \quad \frac{\partial^2 f}{\partial x \partial z}(x,y,z) = 2yz, \quad \frac{\partial^2 f}{\partial y \partial z}(x,y,z) = 2xz,$$

thus

$$H_f(0,0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since Sylvester's criterion cannot be applied to the matrix $H(f)(0,0,0)$, we study its nature by other means. Its associated quadratic form is the map $\Phi \colon \mathbb{R}^3 \to \mathbb{R}$, defined by

$$\Phi(h_1,h_2,h_3) = 2h_1h_2 + 2(h_3)^2.$$

Since $\Phi(-1,1,0) = -2 < 0$ and $\Phi(0,0,1) = 2 > 0$, it follows that this quadratic form (therefore its corresponding matrix) is indefinite. Thus $(0,0,0)$ is not a local extremum point of the function.

d) For all $(x,y) \in \mathbb{R}^2$ it holds $\frac{\partial f}{\partial x}(x,y) = 3x^2 + 3y^2 - 15$ and $\frac{\partial f}{\partial y}(x,y) = 6xy - 12$. The stationary points of f are the solutions of the system

$$\begin{cases} 3x^2 + 3y^2 - 15 = 0 \\ 6xy - 12 = 0. \end{cases}$$

It follows that $x^2+y^2=5$ and $xy=2$. As $x^2+y^2=(x+y)^2-2xy$ it follows that $(x+y)^2=9$, which means that $x+y\in\{-3,3\}$. Therefore x and y are the solutions of one of the equations $t^2+3t+2=0$ or $t^2-3t+2=0$. Thus the stationary point of the function are $(-2,-1)$, $(-1,-2)$, $(1,2)$ and $(2,1)$.

For all $(x,y)\in\mathbb{R}^2$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y)=6x,\quad \frac{\partial^2 f}{\partial x\partial y}(x,y)=6y,\quad \frac{\partial^2 f}{\partial y^2}(x,y)=6x.$$

For the four stationary points of the function we get

$$H_f(-2,-1)=6\begin{pmatrix}-2 & -1\\ -1 & -2\end{pmatrix},\quad H_f(-1,-2)=6\begin{pmatrix}-1 & -2\\ -2 & -1\end{pmatrix},$$

$$H_f(1,2)=6\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix},\quad H_f(2,1)=6\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}.$$

The matrix $H_f(-2,-1)$ is negative definite, $H_f(2,1)$ is positive definite, while $H_f(-1,-2)$ and $H_f(1,2)$ are indefinite. Thus $(-2,-1)$ is a local maximum point of f, and $(2,1)$ is a local minimum, while $(-1,-2)$ and $(1,2)$ are not local extrema. Moreover $f(-2,-1)=28$ and $f(2,1)=-28$.

HOMEWORK:

(H 25)

a) For all $(x,y,z)\in\mathbb{R}^3$ it holds $\frac{\partial f}{\partial x}(x,y,z)=4x-y+2z$, $\frac{\partial f}{\partial y}(x,y,z)=-x-1+3y^2$ and $\frac{\partial f}{\partial z}(x,y,z)=2x+2z$. The stationary points of f are the solutions of the system

$$\begin{cases} 4x-y+2z=0\\ -x-1+3y^2=0\\ 2x+2z=0.\end{cases}$$

From the last equation it follows that $x=-z$. By replacing this in the first equation, we get $y=2x$, equality which, together with the second equation, provides us with $12x^2-x-1=0$. Thus $x\in\{-\frac{1}{4},\frac{1}{3}\}$. It follows that $(-\frac{1}{4},-\frac{1}{2},\frac{1}{4})$ and $(\frac{1}{3},\frac{2}{3},-\frac{1}{3})$ are the stationary points of the function.

For all $(x,y,z)\in\mathbb{R}^3$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y,z)=4,\quad \frac{\partial^2 f}{\partial y^2}(x,y,z)=6y,\quad \frac{\partial^2 f}{\partial z^2}(x,y,z)=2,$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y,z)=-1,\quad \frac{\partial^2 f}{\partial x\partial z}(x,y,z)=2,\quad \frac{\partial^2 f}{\partial y\partial z}(x,y,z)=0,$$

thus

$$H_f\left(-\frac{1}{4},-\frac{1}{2},\frac{1}{4}\right)=\begin{pmatrix}4 & -1 & 2\\ -1 & -3 & 0\\ 2 & 0 & 2\end{pmatrix},\quad H_f\left(\frac{1}{3},\frac{2}{3},-\frac{1}{3}\right)=\begin{pmatrix}4 & -1 & 2\\ -1 & 4 & 0\\ 2 & 0 & 2\end{pmatrix}.$$

The matrix $H_f\left(\frac{1}{3},\frac{2}{3},-\frac{1}{3}\right)$ is positive definite, thus $(\frac{1}{3},\frac{2}{3},-\frac{1}{3})$ is a local minimum point. Moreover

$$f\left(\frac{1}{3},\frac{2}{3},-\frac{1}{3}\right)=-\frac{13}{27}.$$

Since Sylvester's criterion cannot be applied to the matrix $H_f\left(-\frac{1}{4},-\frac{1}{2},\frac{1}{4}\right)$, we will study its nature by other means. The quadratic from associated to this matrix is the map $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$, defined by

$$\Phi(h_1,h_2,h_3) = 4(h_1)^2 - 2h_1h_2 + 4h_1h_3 - 3(h_2)^2 + 2(h_3)^2.$$

Since $\Phi(0,1,0) = -3 < 0$ and $\Phi(0,0,1) = 2 > 0$, it follows that this quadratic form (and its corresponding matrix) is indefinite. Therefore, $(-\frac{1}{4},-\frac{1}{2},\frac{1}{4})$ is not a local extremum.

b) For all $(x,y) \in \mathbb{R}^2$ it holds $\frac{\partial f}{\partial x}(x,y) = 2x - 3y^2(1-x)^2$ and $\frac{\partial f}{\partial y}(x,y) = 2y(1-x)^3$. The stationary points of f are the solutions of the system

$$\begin{cases} 2x - 3y^2(1-x)^2 = 0 \\ 2y(1-x)^3 = 0. \end{cases}$$

From the last equation we get $y=0$ or $x=1$. If $y=0$, then from the first equation of the system we get $x=0$. We notice that $x=1$ does not obey the first equation of the system, thus it is excluded. In conclusion $(0,0)$ is the only stationary point of the function.

For all $(x,y) \in \mathbb{R}^2$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2 + 6y^2(1-x), \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = -6y(1-x)^2, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 2(1-x)^3,$$

thus

$$H_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

It follows that $H(f)(0,0)$ is positive definite, thus $(0,0)$ is a local minimum point. The corresponding extreme value is $f(0,0) = 0$.

(H 26)

a) Recall from the solution to (H 23) that

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{y^3(y^2-x^2)}{(x^2+y^2)^2}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2 \end{cases} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{xy^2(3x^2+y^2)}{(x^2+y^2)^2}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2. \end{cases}$$

We study the second-order partial differentiability of f with respect to (x,y) at 0_2:

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y} = \lim_{y \to 0} \frac{\frac{y^5}{y^4}}{y} = 1.$$

We study the second-order partial differentiability of f with respect to (y,x) at 0_2:

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x\to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} = 0.$$

We get that

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \begin{cases} \frac{y^2(5y^2-3x^2)(x^2+y^2)-4y^4(y^2-x^2)}{(x^2+y^2)^3}, & (x,y) \neq 0_2 \\ 1, & (x,y) = 0_2 \end{cases}$$

and

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \begin{cases} \frac{y^2(9x^2+y^2)(x^2+y^2)-4x^2y^2(3x^2+y^2)}{(x^2+y^2)^3}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2. \end{cases}$$

b) Since

$$\lim_{n\to\infty} \frac{\partial^2 f}{\partial y \partial x}\left(\frac{1}{n}, 0\right) = 0 \neq \frac{\partial^2 f}{\partial y \partial x}(0,0) \text{ and } \lim_{n\to\infty} \frac{\partial^2 f}{\partial x \partial y}\left(0, \frac{1}{n}\right) = 1 \neq \frac{\partial^2 f}{\partial x \partial y}(0,0),$$

Th3 in lecture no. 9 implies that the functions $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are not continuous at 0_2.

(H 27)

Let $(x, y) \in M$ be arbitrarily chosen. Then

$$\frac{\partial f}{\partial x}(x,y) = \frac{2x(x+y) - (x^2+y^2)}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

and

$$\frac{\partial f}{\partial y}(x,y) = \frac{2y(x+y) - (x^2+y^2)}{(x+y)^2} = \frac{y^2 + 2xy - x^2}{(x+y)^2}.$$

Therefore we get

$$\begin{aligned} x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) &= x\frac{x^2 + 2xy - y^2}{(x+y)^2} + y\frac{y^2 + 2xy - x^2}{(x+y)^2} \\ &= \frac{x^3 + 2x^2y - xy^2 + y^3 + 2xy^2 - x^2y}{(x+y)^2} \\ &= \frac{x^3 + x^2y + y^3 + xy^2}{(x+y)^2} = f(x,y). \end{aligned}$$