Geometry[1]
First Year, Computer science

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[1] These notes are not in a final form. They are continuously being improved

Content

Definiţia 1.1

The parabola is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line d is equal to its distance to a fixed point F.

The line d is the *director line* and the point F is the *focus*. The distance between the focus and the director line is denoted by p and represents the *parameter* of the parabola.
Consider a Cartesian system of coordinates xOy, in which $F\left(\frac{p}{2}, 0\right)$ and $d : x = -\frac{p}{2}$. If $M(x, y)$ is an arbitrary point of the parabola, then it verifies

$$|MN| = |MF|,$$

where N is the orthogonal projection of M on Oy.

Thus, the coordinates of a point of the parabola verify

$$\sqrt{\left(x+\frac{p}{2}\right)^2+0}=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2} \Leftrightarrow$$

$$\Leftrightarrow \left(x+\frac{p}{2}\right)^2=\left(x-\frac{p}{2}\right)^2=y^2 \Leftrightarrow$$

$$\Leftrightarrow x^2+px+\frac{p^2}{4}=x^2-px+\frac{p^2}{4}+y^2,$$

and the equation of the parabola is

$$y^2=2px. \tag{1.1}$$

Remark 1.2

The equation (1.1) is equivalent to $y=\pm\sqrt{2px}$, so that the parabola is symmetric with respect to the x-axis.

Representing the graph of the function $f : [0, \infty) \to [0, \infty)$ and using the symmetry of the curve with respect to he x-axis, one obtains the graph of the parabola.
One has

$$f'(x) = \frac{p}{\sqrt{2px_0}}; \qquad f''(x) = -\frac{p}{2x\sqrt{2x}}.$$

x	0				∞
$f'(x)$	$\vert$	$+$	$+$	$+$	$+$
$f(x)$	0		$\nearrow$		∞
$f''(x)$	$-$	$-$	$-$	$-$	$-$

Finaly, one can easily show that the equation of tangent to the parabola $\mathcal{P} : y^2 = 2px$ at one of its point $M_0(x_0, y_0)$ is $T_{M_0}(\mathcal{P}) : yy_0 = p(x + x_0)$.

The ellipsoid

The *ellipsoid* is the quadric surface given by the equation

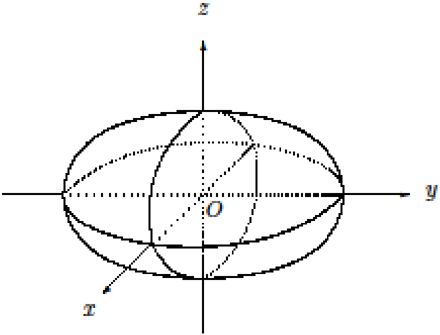
$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{2.1}$$

- The coordinate planes are all planes of symmetry of $\mathcal{E}$ since, for an arbitrary point $M(x, y, z) \in \mathcal{E}$, its symmetric points with respect to these planes, $M_1(-x, y, z)$, $M_2(x, -y, z)$ and $M_3(x, y, -z)$ belong to $\mathcal{E}$; therefore, the coordinate axes are axes of symmetry for $\mathcal{E}$ and the origin O is the center of symmetry of the ellipsoid (2.1);
- The traces in the coordinates planes are ellipses of equations

$$\begin{cases} \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ z = 0. \end{cases}$$

- The sections with planes parallel to *xOy* are given by setting $z = \lambda$ in (2.1). Then, a section is of equations
$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 - \dfrac{\lambda^2}{c^2} \\ z = \lambda \end{cases}.$$
- If $|\lambda| < c$, the section is an ellipse
$$\begin{cases} \dfrac{x^2}{\left(a\sqrt{1 - \dfrac{\lambda^2}{c^2}}\right)^2} + \dfrac{y^2}{\left(b\sqrt{1 - \dfrac{\lambda^2}{c^2}}\right)^2} = 1 \\ z = \lambda \end{cases};$$
- If $|\lambda| = c$, the intersection is reduced to one (tangency) point $(0, 0, \lambda)$;
- If $|\lambda| > c$, the plane $z = \lambda$ does not intersect the ellipsoid $\mathcal{E}$.

The sections with planes parallel to *xOz* or *yOz* are obtained in a similar way.


z
y
O
x

Hyperboloids of One Sheet

The surface of equation

$$\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*, \qquad (2.2)$$

is called *hyperboloid of one sheet*.

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$; hence, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively, of equations

$$\left\{\begin{array}{c} \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \\ \text{a hyperbola} \end{array}\right. ; \left\{\begin{array}{c} \frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{array}\right. ; \left\{\begin{array}{c} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ z = 0 \\ \text{an ellipse} \end{array}\right. ;$$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases} ; \quad \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{cases} ;$$

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{\lambda^2}{c^2} \\ z = \lambda \\ \text{ellipses} \end{cases} ;$$

Remark: The surface $\mathcal{H}_1$ contains two families of lines, as

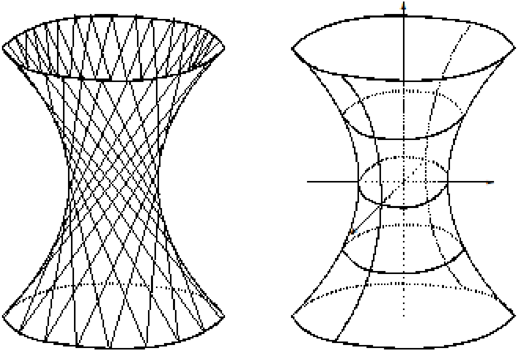
$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} \Leftrightarrow \left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right).$$

The equations of the two families of lines are:

$$d_\lambda : \begin{cases} \lambda\left(\dfrac{x}{a}+\dfrac{z}{c}\right)=1+\dfrac{y}{b} \\ \dfrac{x}{a}-\dfrac{z}{c}=\lambda\left(1-\dfrac{y}{b}\right) \end{cases}, \lambda \in \mathbb{R},$$

$$d'_\mu : \begin{cases} \mu\left(\dfrac{x}{a}+\dfrac{z}{c}\right)=1-\dfrac{y}{b} \\ \dfrac{x}{a}-\dfrac{z}{c}=\mu\left(1+\dfrac{y}{b}\right) \end{cases}, \mu \in \mathbb{R}.$$

Through any point on $\mathcal{H}_1$ pass two lines, one line from each family.



Hiperboloidul cu o pânză H_1

Hyperboloids of Two Sheets

The *hyperboloid of two sheets* is the surface of equation

$$\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \qquad (2.3)$$

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively,

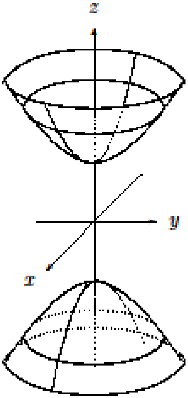
$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} + 1 = 0 \\ x = 0 \\ \text{a hyperbola;} \end{cases}, \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} + 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{cases}, \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + 1 = 0 \\ z = 0 \\ \text{the empty set} \end{cases};$$

- The intersections with planes parallel to the coordinate planes are

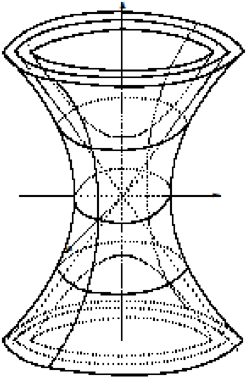
$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1 - \dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases}, \quad \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = -1 - \dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{cases}$$

and $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1 + \dfrac{\lambda^2}{c^2} \\ z = \lambda \end{cases}$.

  - If $|\lambda| > c$, the section is an ellipse;
  - If $|\lambda| = c$, the intersection reduces to a point $(0, 0, \lambda)$;
  - If $|\lambda| < c$, one obtains the empty set.



hiperboloidul
cu două pânze H_2



hiperboloizii H_1, H_2
şi conul asimptot comun

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