Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică
Secţia: Informatică engleză, Curs: Dynamical Systems, An: 2015/2016

Seminar 1

1. Find all functions $x \in C^1(\mathbb{R})$ such that: a) $x' = 0$; b) $x' = 2t$; c) $x' = \sin t$; d) $x' = 2t + \sin t$; e) $x' = e^{2t} \cos t$; f) $x' = (t^2 - 5t + 7) \sin t$.

All these are simple examples of first order differential equations. Note that, in general, a first order differential equation has the form $x' = f(t, x)$, while all the above have the form $x' = f(t)$, i.e. the unknown function $x(t)$ does not appear in the right hand side. In turn, in the right hand side of the differential equation below, $x' = 3x$, appears the unknown function but does not appear the independent variable t.

2. Show that $x : \mathbb{R} \to \mathbb{R}$, given by the expression $x(t) = 2e^{3t}$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem $\quad x' = 3x, \quad x(0) = 2.$
Represent the corresponding integral curve* and describe its long term behavior**.

**A graphical representation of a solution of some differential equation is called an integral curve or a solution curve of this equation.*
***To describe the long term behavior of some function means to decide whether it is: periodic, oscillatory (i.e. it has "many many" zeros), bounded, increasing, and to describe how it behaves at $\pm\infty$.*

3. Show that $x : \mathbb{R} \to \mathbb{R}$, $x(t) = 2 \sin t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem $\quad x'' + x = 0, \quad x(0) = 0, \quad x'(0) = 2.$
Represent the corresponding integral curve and describe its long term behavior.

4. Show that $x(t) = e^{-2t} \cos t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem $\quad x'' + 4x' + 5x = 0, \quad x(0) = 1, \quad x'(0) = -2.$
Represent this integral curve and describe its long term behavior.

5. Decide whether $x : \mathbb{R} \to \mathbb{R}$, $x(t) = \cos t$ for all $t \in \mathbb{R}$, is a solution of the differential equations $x' + x = 0$ or $x'' - x = 0$ or $x''' + x' = 0$ or $x^{(4)} + x'' = 0$ or $x^{(5)} + x''' = 0$ or $x^{(6)} + x^{(4)} = 0$, ...

6. Find all constant solutions of the differential equations: a) $x' = x - x^3$; b) $x' = \sin x$; c) $x' = \dfrac{x+1}{2x^2+5}$; d) $x' = x^2 + x + 1$; e) $x' = x + 4x^3$; f) $x' = -1 + x + 4x^3$; g) $x' = 3 + x + 4x^3$; h) $x'' + 3x' + x^2 - 9 = 0$.

7. i) Let the functions $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}$ be defined by $x_1(1) = 1$, $x_2(t) = t$ and $x_3(t) = t^2$ for all $t \in \mathbb{R}$. Prove that they are linearly independent in the linear space $C(\mathbb{R})$ (over the filed $\mathbb{R}$ and with the usual operations).

(ii) Find all $a, b, c \in \mathbb{R}$ such that $x(t) = at^2 + bt + c$ to be a solution of $x' - 5x = 2t^2 + 3$ or $x'' = 0$ or $x''' = 0$. Write the solutions you found.

8. (i) Let the functions $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}$ be such that $x_1(1) = \cos t$, $x_2(t) = \sin t$ and $x_3(t) = e^t$ for all $t \in \mathbb{R}$. Prove that they are linearly independent in the linear space $C(\mathbb{R})$.

(ii) Find all $a, b, c \in \mathbb{R}$ such that $x(t) = a \sin t + b \cos t + c\, e^t$ to be a solution of $x' + x = -3 \sin t + 2\, e^t$ or $x'' + 4x = -3 \sin t$ or $x'' + x = -3 \sin t$ or $x'' + x = 0$ or $x''' - x'' + x' - x = 0$. Write the solutions you found.

9. Find $r \in \mathbb{R}$ such that $x(t) = e^{rt}$ to be a solution of $x'' - 5x' + 6x = 0$ or $x''' - 5x'' + 6x' = 0$ or $x^{(4)} - 5x''' + 6x'' = 0$ or $x'' + 9x = 0$ or $x'' + x' + x = 0$.

10. Find as many functions $x \in C^1(\mathbb{R})$ as you can such that: a) $x' = x$; b) $x' = 2x$; c) $x' = -x$; d) $x' = ax$, with $a \neq 0$ a real parameter.

11. *An integrating factor for some differential equation is a function $\mu = \mu(t, x)$ that helps us to integrate the differential equation.* For example, $\mu(x) = e^{-t}$ is an integrating factor of $x' = x$, since after multiplying the equation with μ, we can write the equation as $(xe^{-t})' = 0$. After integration, we obtain the general solution $x = ce^t$, where c is an arbitrary real constant. Integrate the following equations: a) $x' + x = 1 + t$; b) $x' + 2x = \sin t$; c) $tx' + x = 1$; d) $tx' + 2x = 1$; e) $xx' + t = 0$.

12. Represent the integral curves of the differential equations $x' = 0$, $x' = 2t$, $x' = x$ and $x' = 2x$. In each case, find the curve that passes through the point $(1, 3)$. Decide whether the following claim is true: "Through each point of the plane passes one and only one solution curve of the given differential equation".