

Seminar Nr.7, Inequalities; Sequences of Random Variables

Theory Review

Hölder's Inequality: $E(|XY|) \leq (E(|X|^p))^{\frac{1}{p}} \cdot (E(|Y|^q))^{\frac{1}{q}}, \forall p, q > 1, \frac{1}{p} + \frac{1}{q} = 1.$

Markov's Inequality: $P(|X| \geq a) \leq \frac{1}{a} E(|X|), \forall a > 0.$

Chebyshev's Inequality: $P(|X - E(X)| \geq \epsilon) \leq \frac{V(X)}{\epsilon^2}, \forall \epsilon > 0.$

Convergence:

- 1) **in probability** $X_n \xrightarrow{p} X$, if $\lim_{n \rightarrow \infty} P(|X_n - X| < \epsilon) = 1, \forall \epsilon > 0;$
- 2) **almost surely** $X_n \xrightarrow{a.s.} X$, if $P(\lim_{n \rightarrow \infty} X_n = X) = 1;$
- 3) **in distribution** $X_n \xrightarrow{d} X$, if $\lim_{n \rightarrow \infty} F_n(x) = F(x), \forall x \in \mathbb{R}$ continuity point for F ;
- 4) **in mean of order r** , $0 < r < \infty$ $X_n \xrightarrow{L^r} X$, if $\lim_{n \rightarrow \infty} E(|X_n - X|^r) = 0.$

Properties

1. $2) \Rightarrow 1) \Rightarrow 3)$
 2. $4) \Rightarrow 1) \Rightarrow 3)$
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1. (The 3σ Rule). For any random variable X , most of the values of X lie within 3 standard deviations away from the mean.

2. True or False: There is at least a 90% chance of the following happening: when flipping a coin 1000 times, the number of "heads" that appear is between 450 and 550.

3. Let $S = [0, 1], \mathcal{K} = \mathcal{B}[0, 1]$ (the set of all open subsets of $[0, 1]$) and let $P = \mu$ be the Lebesgue measure on $[0, 1]$ (length, distance). In the probability space (S, \mathcal{K}, P) , consider the sequence of random variables given by

$$X_n(e) = \begin{cases} 2^n, & \text{if } 0 \leq e \leq \frac{1}{n} \\ 1, & \text{if } e > \frac{1}{n}. \end{cases}$$

Study the various types of convergence of X_n to $X = 1$.

4. Consider a sequence $\{X_n\}_{n \in \mathbb{N}}$ of random variables such that each X_n is uniformly distributed on $[-n, n]$. Does $\{X_n\}_{n \in \mathbb{N}}$ converge in distribution?

5. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables and let X be a random variable such that $X_n \xrightarrow{L^r} X$. Prove that $X_n \xrightarrow{L^s} X$ for each $s \leq r$.

6. Let $X_n \in N(\mu_n, \sigma_n)$ be a sequence of random variables such that $X_n \xrightarrow{L^r} 0, r \geq 2$. Show that $\lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} \sigma_n = 0$.