Geometry[1]
First Year, Computer science

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[1] These notes are not in a final form. They are continuously being improved

Content

The vector product

Proposition 1.1

If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k},\ \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}, \tag{1.1}$$

or, equivalently,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k} \tag{1.2}$$

One can rewrite formula (1.1) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.3}$$

the right hand side determinant being understood in the sense of its cofactor expansion along the first line.
For the proof of Proposition 1.1 we rely on some equalities related to a direct orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$.
More precisely $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and obviously $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

The double vector (cross) product

The *double vector (cross) product* of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the vector $\vec{a} \times (\vec{b} \times \vec{c})$

Proposition 1.2

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \end{vmatrix}, \quad \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}.$$

Corollary 1.3

(*i*) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{a} \end{vmatrix}$

(*ii*) $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}\ \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$ (*Jacobi's identity*).

PROOF.

(*i*) $(\vec{a}\times\vec{b})\times\vec{c} = -\vec{c}\cdot(\vec{a}\times\vec{b}) = -[(\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}] =$
$(\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{a} \end{vmatrix}.$
(*ii*) $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) =$
$(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b} = \vec{0}.\square$

The triple scalar product

The *triple scalar product* $(\vec{a}, \vec{b}, \vec{c})$ of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the real number $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

Proposition 1.4

If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k},\ \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ şi $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ then

$$(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{1.4}$$

Proof.

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}.$$

Thus

$$\begin{aligned}(\vec{a},\vec{b},\vec{c}) &= (\vec{a}\times\vec{b})\cdot\vec{c}=\vec{c}\cdot(\vec{a}\times\vec{b})\\ &= c_1\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-c_2\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+c_3\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.\end{aligned}$$

$\square$

Corollary 1.5

1. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent (collinear) if and only if $(\vec{a}, \vec{b}, \vec{c}) = 0$*
2. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent (noncollinear) if and only if $(\vec{a}, \vec{b}, \vec{c}) \neq 0$*
3. *The free vectors $\vec{a}, \vec{b}, \vec{c}$ form a basis of the space $\mathcal{V}$ if and only if $(\vec{a}, \vec{b}, \vec{c}) \neq 0$.*
4. *The correspondence $F : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R},\ F(\vec{a}, \vec{b}, \vec{c}) = (\vec{a}, \vec{b}, \vec{c})$ is a skew-symmetric, i..e*

$$\begin{aligned}
(\alpha\vec{a} + \alpha'\vec{a}\,', \vec{b}, \vec{c}) &= \alpha(\vec{a}, \vec{b}, \vec{c}) + \alpha'(\vec{a}\,', \vec{b}, \vec{c})\\
(\vec{a}, \beta\vec{b} + \beta'\vec{b}\,', \vec{c}) &= \beta(\vec{a}, \vec{b}, \vec{c}) + \beta'(\vec{a}, \vec{b}', \vec{c})\\
(\vec{a}, \vec{b}, \gamma\vec{c} + \gamma'\vec{c}\,') &= \gamma(\vec{a}, \vec{b}, \vec{c}) + \gamma'(\vec{a}, \vec{b}, \vec{c}\,')
\end{aligned} \tag{1.5}$$

$\forall\, \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{R},\ \forall\, \vec{a}, \vec{b}, \vec{c}, \vec{a}\,', \vec{b}\,', \vec{c}\,' \in \mathcal{V}$ *şi*

$$(\vec{a}_1, \vec{a}_2, \vec{a}_3) = sgn(\sigma)(\vec{a}_{\sigma(1)}, \vec{a}_{\sigma(2)}, \vec{a}_{\sigma(3)}),\ \ \forall\, \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V} \text{ şi } \forall\, \sigma \in S_3 \tag{1.6}$$

Remark 1.6

One can rewrite the relations (1.6) as follows:

$$\begin{aligned}(\vec{a}_1, \vec{a}_2, \vec{a}_3) &= (\vec{a}_2, \vec{a}_3, \vec{a}_1) = (\vec{a}_3, \vec{a}_1, \vec{a}_2) \\ &= -(\vec{a}_2, \vec{a}_1, \vec{a}_3) = (\vec{a}_1, \vec{a}_3, \vec{a}_2) = -(\vec{a}_3, \vec{a}_2, \vec{a}_1),\end{aligned}$$

$\forall\, \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V}$

Corollary 1.7

1. $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) \forall\, \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.
2. *For every* $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathcal{V}$ *the* Laplace formula

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

holds.

Proof.

(i) $(\vec{a}\times\vec{b})\cdot\vec{c}=(\vec{a},\vec{b},\vec{c})=(\vec{b},\vec{c},\vec{a})=(\vec{b}\times\vec{c})\vec{a}=\vec{a}\cdot(\vec{b}\times\vec{c})$.
(ii) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=(\vec{a},\vec{b},\vec{c}\times\vec{d})=(\vec{c}\times\vec{d},\vec{a},\vec{b})=[(\vec{c}\times\vec{d})\times\vec{a}]\cdot\vec{b}=(\vec{a}\cdot\vec{c})\vec{d}-(\vec{a}\cdot\vec{d})\vec{c}]\cdot\vec{b}=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}]=$

$$=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}.$$

$\square$