Geometry[1]
First Year, Computer science

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Lecture 12, 28.05.2014

[1] These notes are not in a final form. They are continuously being improved

Content

Definition 1.1

An affine transformation of the plane is a mapping

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x, y) = (ax + by + c, dx + ey + f), \quad (1.1)$$

for some constant real numbers a, b, c, d, e, f.

By using the matrix language, the action of the map L can be written in the form

$$L(x, y) = [x\ y] \begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c\ f].$$

The affine transformation L can be also identified with the map $L^c : \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}^{2\times 1}$ given by

$$\begin{aligned} L^c\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} ax + by + c \\ dx + ey + f \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix} \\ &= [L] \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b \\ d & e \end{bmatrix}. \end{aligned}$$

Lemma 1.1

If $(aB - bA)^2 + (dB - eA)^2 > 0$*, then the affine transformation (1.1) maps the line* $(d)\ Ax + By + C = 0$ *to the line*

$$(eA - dB)x + (aB - bA)y + (bf - ce)A - (af - cd)B + (ae - bd)C = 0.$$

If $aB - bA = dB - eA = 0$*, then* $ae - bd = 0$ *and L is the constant map* $\left(\frac{cB - bC}{B}, \frac{fB - eC}{B}\right)$.

Definition 1.3

An affine transformation (1.1) is said to be singular *if*

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0 \text{ i.e. } ae - bd = 0.$$

and non-singular otherwise.

Note that the affine transformation L is nonsingular if and only if it is invertible. In such a case the inverse L^{-1} is a non-singular affine transformation and $[L^{-1}] = [L]^{-1}$.

Definition 1.4

The translation *of vector* $(h, k) \in \mathbb{R}^2$ *is the affine transformation*

$$T(h,k) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ [T(h,k)](x,y) = (x+h, y+k).$$

Thus

$$[T(h,k)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+h \\ y+k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix},$$

i.e.

$$[T(h,k)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the translation $T(h,k)$ is non-singular (invertible) and $(T(h,k))^{-1} = T(-h,-k)$.

Definition 1.5

The scaling about the origine *by non-zero scaling factors* $(s_x, s_y) \in \mathbb{R}^2$ *is the affine transformation*

$$S(s_x, s_y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \; [S(s_x, s_y)](x, y) = (s_x \cdot x, s_y \cdot y).$$

Thus

$$[S(s_x, s_y)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[S(s_x, s_y)] = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}.$$

Note that the scaling about the origine by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is non-singular (invertible) and $(S(s_x, s_y))^{-1} = S(s_x^{-1}, s_y^{-1})$.

Definition 1.6

The reflections about the x-axis *and the* y*-axis respectively are the affine transformation*

$$S_x, S_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ S_x(x,y) = (x,-y),\ S_y = (-x,y).$$

Thus

$$[S_x^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[S_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \textit{ Similarly } [S_y] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $S_x = S(-1,1)$ and $S_y = S(1,-1)$. Thus the two reflections are non-singular (invertible) and $S_x^{-1} = S_x$, $S_y^{-1} = S_y$.

Definition 1.7

The reflection $S_l : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the line l *maps a given point M to the point M' defined by the property that l is the perpendicular bisector of the segment MM'. One can show that the action of the reflection about the line $l : ax + by + c = 0$ is*

$$S_l(x,y)=\left(\frac{b^2-a^2}{a^2+b^2}x-\frac{2ab}{a^2+b^2}y-\frac{2ac}{a^2+b^2},-\frac{2ab}{a^2+b^2}x+\frac{a^2-b^2}{a^2+b^2}y-\frac{2bc}{a^2+b^2}\right)$$

$$\text{Thus } [S_l^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2}x-\frac{2ab}{a^2+b^2}y-\frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2}x+\frac{a^2-b^2}{a^2+b^2}y-\frac{2bc}{a^2+b^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{2ac}{a^2+b^2} \\ \frac{2ac}{a^2+b^2} \end{bmatrix},$$

i.e. $[S_l] = \frac{1}{a^2+b^2}\begin{bmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}$. *Note that the reflection S_l is non-singular (invertible) and $S_l^{-1} = S_l$.*

Definition 1.8

The rotation $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *about the origine through an angle* θ *maps a point* $M(x, y)$ *into a point* $M'(x', y')$ *with the properties that the segments* $[OM]$ *and* $[OM']$ *are congruent and the* $m(\widehat{MOM'}) = \theta$. *If* $\theta > 0$ *the rotation is supposed to be* anticlockwise *and for* $\theta < 0$ *the rotation is* clockwise. *If* $(x, y) = (r\cos\varphi, r\sin\varphi)$, *then the coordinates of the rotated point are* $\big(r\cos(\theta+\varphi), r\sin(\theta+\varphi)\big) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, *i.e.* $R_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

$$\textit{Thus}\ \left[R_\theta^c\right]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

i.e. $[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. *Note that the rotation* R_θ *is non-singular (invertible) and* $R_\theta^{-1} = R_{-\theta}$.