Geometry[1]
First Year, Computer science

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[1] These notes are not in a final form. They are continuously being improved

Content

Definition 1.1

Given a fixed direction in the plane specified by a unit vector $v = (v_1, v_2)$, consider the lines d with direction v and the oriented distance d from the origin. The shear about the origin of factor r in the direction v is defined to be the transformation which maps a point $M(x, y)$ on d to the point $M' = M + rdv$. The equation of the line through M of direction v is $v_2X - v_1Y + (v_1y - v_2x) = 0$. The oriented distacnce from the origine to this line is $v_1y - v_2x$. Thus the action of the shear $Sh(v, r) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the origin of factor r in the direction v is

$$\begin{aligned} Sh(v, r)(x, y) &= (x, y) + rd(v_1, v_2) \\ &= (x, y) + (r(v_1y - v_2x)v_1, r(v_1y - v_2x)v_2) \\ &= (x, y) + \left(-rv_1v_2x + rv_1^2y, -rv_2^2x + rv_1v_2y\right) \\ &= \left((1 - rv_1v_2)x + rv_1^2y, -rv_2^2x + (1 + rv_1v_2)y\right) \end{aligned}$$

Thus

$$\begin{aligned}[Sh(v,r)^c]\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} (1-rv_1v_2)x + rv_1^2 y \\ -rv_2^2 x + (1+rv_1v_2)y \end{bmatrix} \\ &= \begin{bmatrix} 1-rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1+rv_1v_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},\end{aligned}$$

i.e. $[Sh(v,r)] = \begin{bmatrix} 1-rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1+rv_1v_2 \end{bmatrix}$.

Example 1.3

Consider a quadrilateral with vertices $A(1,1)$, $B(3,1)$, $C(2,2)$, *and* $D(1.5,3)$. *Find the image quadrilaterals through the translation* $T(1,2)$, *the scaling* $S(2,2.5)$, *the reflections about the* x *and* y*-axes, the clockwise rotation through the angle* $\pi/2$ *and the shear* $Sh\left((2/\sqrt{5}, 1/\sqrt{5}), 1.5\right)$.

Homogeneous coordinates

The affine transformation

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x, y) = (ax + by + c, dx + ey + f)$$

can be written by usin the matrix language and by equations:

1. 1.1 indentifying the vectors $(x, y) \in \mathbb{R}^2$ with the line matrices $[x\ y] \in \mathbb{R}^{1\times 2}$ and implicitely $\mathbb{R}^2$ with $\mathbb{R}^{1\times 2}$:

$$L[x\ y] = [x\ y]\begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c\ f].$$

   1.2 indentifying the vectors $(x, y) \in \mathbb{R}^2$ with the column matrices $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$ and implicitely $\mathbb{R}^2$ cu $\mathbb{R}^{2\times 1}$:

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}.$$

2. $\begin{cases} x' = ax + by + c \\ y' = dx + ey + f. \end{cases} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$

Observe that the representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

In this lesson we identify the points $(x, y) \in \mathbb{R}^2$ with the points $(x, y, 1) \in \mathbb{R}^3$ and even with the punctured lines of $\mathbb{R}^3$, (rx, ry, r), $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x, y) \in \mathbb{R}^2$ with the punctured lines of $\mathbb{R}^3$ represented in the form

$$\begin{bmatrix} rx \\ ry \\ r \end{bmatrix}, \ r \in \mathbb{R}^*,$$

and the latter ones we shall call *homogeneous coordinates* of the point $(x, y) \in \mathbb{R}^2$.

The set of homogeneous coordinates (x, y, w) will be denoted by $\mathbb{RP}^2$ and call it the real *projective plane*. The homogeneous coordinates $(x, y, w) \in \mathbb{RP}^2$, $w \neq 0$ şi $\left(\frac{x}{w}, \frac{y}{w}, 1\right)$ represent the same element of $\mathbb{RP}^2$.

Definition 1.4

A projective transformation *of the projective plane* $\mathbb{RP}^2$ *is a transformation* $L : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ *defined by*

$$L \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} ax + by + cw \\ dx + ey + fw \\ gx + hy + kw \end{bmatrix}, \quad (1.1)$$

unde $a, b, c, d, e, f, g, h, k \in \mathbb{R}$*. Note that*

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

is called the homogeneous transformation matrix*.*

If $g = h = 0$ and $k \neq 0$, then the projective transformation (1.1) is said to be *affine*. Observe that a projective transformation (1.1) is well defined since

$$L \begin{bmatrix} rx \\ ry \\ rw \end{bmatrix} = \begin{bmatrix} arx + bry + crw \\ drx + ery + frw \\ grx + hry + krw \end{bmatrix} = \begin{bmatrix} r(ax + by + cw) \\ r(dx + ey + fw) \\ r(gx + hy + kw) \end{bmatrix}.$$

Observation 1.5

The projective plane $\mathbb{RP}^2$ is actually the quotient set $(\mathbb{R}^3 \setminus \{0\}) / \sim$, where $' \sim '$ is one of the following equaivalence relation on $\mathbb{R}^3 \setminus \{0\}$:

$$(x, y, w) \sim (\alpha, \beta, \gamma) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.î. } (x, y, w) = r(\alpha, \beta, \gamma).$$

Observe that the equivalence classes of the equivalence relation $\sim'$ are right the punctured lines of $\mathbb{R}^3$ through the origine without the origine itself, i.e. the elements of the real projective plane $\mathbb{RP}^2$.

Observation 1.6. If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two

projective applications, then their product $L_1 \circ L_2$ is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of L_1 and L_2 .

Indeed, if

$$L_1 \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \text{ and } L_2 \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

then

$$(L_1 \circ L_2) \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \left(\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Observation 1.7

If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two affine applications, then their product $L_1 \circ L_2$ is also an affine transformation.

Transformations in homogeneous coordinates
Translations and scalings

In this section we shall identify an affine transformation of $\mathbb{RP}^2$ with its homogeneous transformation matrix

• The homogeneous transformation matrix of the translation $T(h,k)$ is

$$T(h,k) = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of the scaling $S(s_x, s_y)$ is

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Reflections

- The homogeneous transformation matrix of reflection S_x about the x-axis is

$$S_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- The homogeneous transformation matrix of reflection S_y about the y-axis is

$$S_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- The homogeneous transformation matrix of reflection S_l about the line $l: \; ax + by + c = 0$ is

$$S_l = \begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc}{a^2+b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Rotations

The homogeneous transformation matrix of the rotation R_θ about the origine through an angle θ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 1.8

The homogeneous transformation matrix of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ is

$$\begin{aligned} R_\theta(x_0, y_0) &= T(x_0, y_0) R_\theta T(-x_0, -y_0) \\ &= \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & -x_0\cos\theta + y_0\sin\theta + x_0 \\ \sin\theta & \cos\theta & -x_0\sin\theta - y_0\cos\theta + x_0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Thus the equations of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ are:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & -x_0\cos\theta + y_0\sin\theta + x_0 \\ \sin\theta & \cos\theta & -x_0\sin\theta - y_0\cos\theta + x_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

sau, echivalent

$$\begin{cases} x' & = x\cos\theta - y\sin\theta - x_0\cos\theta + y_0\sin\theta + x_0 \\ y' & = x\sin\theta + y\cos\theta - x_0\sin\theta - y_0\cos\theta + x_0. \end{cases}$$