# Geometry[1]
# First Year, Computer science


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[1] These notes are not in a final form. They are continuously being improved

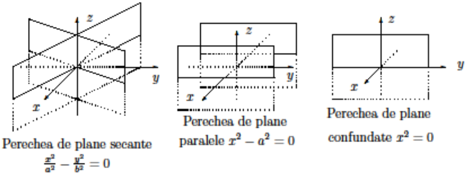
# Content

# Pairs of planes, the straight line, the point and the empty set

- *The pair of concurrent planes* is the quadric of equation

  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ along the anologous equations such as $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$

- *The pair of parallel planes* is the quadric of equation $x^2 - a^2 = 0$ along the anologous equations.
- *The pair of parallel double planes* is the quadric of equation $x^2 = 0$ along the anologous equations.
- *The line* is the quadric of equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$, along the anologous equations.




- *The point* is the quadric of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0. \tag{1.1}$$

- *The empty set* is the quadric of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1 \text{ or } \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1. \tag{1.2}$$

# Generated Surfaces

Consider the 3-dimensional Euclidean space $\mathcal{E}_3$, together with a Cartesian system of coordinates $Oxyz$. Generally, the set

$$S = \{M(x, y, z) : F(x, y, z) = 0\},$$

where $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a real function and $D$ is a domain, is called *surface* of implicit equation $F(x, y, z) = 0$ (the quadric surfaces, defined in the previous chapter for $F$ a polynomial of degree two, are such of surfaces). On the other hand, the set

$$S_1 = \{M(x, y, z) : x = x(u, v), y = y(u, v), z = z(u, v)\},$$

where $x, y, z : D_1 \subseteq \mathbb{R}^2 \to \mathbb{R}$, is a *parameterized surface*, of parametric equations

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \quad (u, v) \in D_1.$$

The intersection between two surfaces is a *curve* in 3-space (remember, for instance, that the intersection between a quadric surface and a plane is a conic section, hence the conics are plane curves). Then, the set

$$C = \{M(x, y, z) : F(x, y, z) = 0, G(x, y, z) = 0\},$$

where $F, G : D \subseteq \mathbb{R}^3 \to \mathbb{R}$, is the curve of *implicit* equations

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} .$$

As before, one can parameterize the curve. The set

$$C_1 = \{M(x, y, z) : x = x(t), y = y(t), z = z(t)\},$$

where $x, y, z : I \subseteq \mathbb{R} \to \mathbb{R}$ and $I$ is open, is called *parameterized curve* of parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} , \quad t \in I.$$

Let be given a family of curves, depending on one single parameter $\lambda$,

$$\mathcal{C}_\lambda : \begin{cases} F_1(x, y, z; \lambda) = 0 \\ F_2(x, y, z; \lambda) = 0 \end{cases} .$$

In general, the family $\mathcal{C}_\lambda$ does not cover the entire space. By eliminating the parameter $\lambda$ between the two equations of the family, one obtains the equation of the surface *generated* by the family of curves.
Suppose now that the family of curves depends on two parameters $\lambda$, $\mu$,

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} F_1(x, y, z; \lambda, \mu) = 0 \\ F_2(x, y, z; \lambda, \mu) = 0 \end{cases} ,$$

and that the parameters are related through $\varphi(\lambda, \mu) = 0$ If it can be obtained an equation which does not depend on the parameters (by eliminating the parameters between the three equations), then the set of all the points which verify it is called surface *generated* by the family (or the sub-family) of curves.

# Cylindrical Surfaces

## Definition 2.1

*The surface generated by a variable line (the generatrix), which remains parallel to a fixed line $d$ and intersects a given curve $\mathcal{C}$, is called cylindrical surface. The curve $\mathcal{C}$ is called the director curve of the cylindrical surface.*

## Theorem 2.2

*The cylindrical surface, with the generatrix parallel to the line $d$, where*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases},$$

*and having the director curve $\mathcal{C}$, where*

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

*(assume that $d$ and $\mathcal{C}$ are not coplanar), is characterized*

*by an equation of the form*

$$\varphi(\pi_1, \pi_2) = 0. \tag{2.1}$$

## Proof.

The equations of an arbitrary line, which is parallel to

$$d : \begin{cases} \pi_1(x, y, z) = 0 \\ \pi_2(x, y, z) = 0 \end{cases}, \text{ are } d_{\lambda,\mu} : \begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}.$$

Not every line from the family $d_{\lambda,\mu}$ intersects the curve $\mathcal{C}$. This happens only when the system of equations

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}$$

.

is compatible. By eliminating $x$, $y$ and $z$ between the four equations of the system, one obtains a *compatibility condition* $\varphi(\lambda, \mu) = 0$ for the parameters $\lambda$ and $\mu$. The equation of the surface can be determined now from the system

$$\begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \\ \varphi(\lambda, \mu) = 0 \end{cases},$$

and it is immediate that $\varphi(\pi_1, \pi_2) = 0$. □

*Remark*: Any equation of the form (2.1), where $\pi_1$ and $\pi_2$ are linear function of $x$, $y$ and $z$, represents a cylindrical surface, having the generatrices parallel to $d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$.

*Example* Let us find the equation of the cylindrical surface having the generatrices parallel to

$$d : \begin{cases} x + y = 0 \\ z = 0 \end{cases}$$

and the director curve given by

$$\mathcal{C} : \begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \end{cases} .$$

The equations of the generatrices $d$ are

$$d_{\lambda,\mu} : \begin{cases} x + y = \lambda \\ z = \mu \end{cases} .$$

They must intersect the curve $\mathcal{C}$, i.e. the system

$$\begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \\ x + y = \lambda \\ z = \mu \end{cases}$$

has to be compatible.

A solution of the system can be obtained using the three last equations

$$\begin{cases} x = 1 \\ y = \lambda - 1 \\ z = \mu \end{cases}$$

and, replacing in the first one, one obtains the compatibility condition

$$2(\lambda - 1)^2 + \mu - 1 = 0.$$

The equation of the surface is obtained by eliminating the parameters in

$$\begin{cases} x + y = \lambda \\ z = \mu \\ 2(\lambda - 1)^2 + \mu - 1 = 0 \end{cases}.$$

Then,

$$2(x + y - 1)^2 + x - 1 = 0.$$

# Conical Surfaces

## Definition 2.4

*The surface generated by a variable line, which passes through a fixed point $V$ and intersects a given curve $\mathcal{C}$, is called conical surface. The point $V$ is called the vertex of the surface and the curve $\mathcal{C}$ director curve.*

## Theorem 2.5

*The conical surface, of vertex $V(x_0, y_0, z_0)$ and director curve*

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$

*(suppose that $V$ and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form*

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0. \tag{2.2}$$

## Proof.

The equations of an arbitrary line through $V(x_0, y_0, z_0)$ are

$$d_{\lambda\mu} : \begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \end{cases} .$$

A generatrix has to intersect the curve $\mathcal{C}$, hence the system of equations

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

must be compatible. This happens for some values of the parameters $\lambda$ and $\mu$, which verify a *compatibility condition*

$$\varphi(\lambda, \mu),$$

obtained by eliminating $x$, $y$ and $z$ in the the previous system of equations. In these conditions, the surface is generated and its equation rises from the system

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ \varphi(\lambda, \mu) = 0 \end{cases} .$$

It follows that

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0.$$

□

*Remark*: If $\varphi$ is an algebraic function, then the equation (2.2) can be written in the form

$$\phi(x - x_0, y - y_0, z - z_0) = 0,$$

where $\phi$ is homogeneous with respect to $x - x_0$, $y - y_0$ and $z - z_0$.

If $\varphi$ is algebraic and $V$ is the origin of the system of coordinates, then the equation of the conical surface is $\phi(x, y, z) = 0$, with $\phi$ a homogeneous polynomial. Conversely, an algebraic homogeneous equation in $x$, $y$ and $z$ represents a conical surface with the vertex at the origin.

*Example*: Let us determine the equation of the conical surface, having the vertex $V(1, 1, 1)$ and the director curve

$$\mathcal{C} : \begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \end{cases} .$$

The family of lines passing through $V$ has the equations

$$d_{\lambda\mu} : \begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases} .$$

The system of equations

$$\begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \\ x-1 = \lambda(z-1) \\ y-1 = \mu(z-1) \end{cases}$$

must be compatible. A solution is

$$\begin{cases} x = 1-\lambda \\ y = 1-\mu \\ z = 0 \end{cases},$$

and, replaced in the first equation of the system, gives the compatibility condition

$$[(1-\lambda)^2 + (1-\mu)^2]^2 - (1-\lambda)(1-\mu) = 0.$$

The equation of the conical surface is obtained by eliminating the parameters $\lambda$ and $\mu$ in

$$\begin{cases} x-1 = \lambda(z-1) \\ y-1 = \mu(z-1) \\ ((1-\lambda)^2 + (1-\mu)^2)^2 - (1-\lambda)(1-\mu) = 0 \end{cases}.$$

# Conoidal Surfaces





## Definition 2.6

*The surface generated by a variable line, which intersects a given line $d$ and a given curve $\mathcal{C}$, and remains parallel to a given plane $\pi$, is called* conoidal surface*. The curve $\mathcal{C}$ is the* director curve *and the plane $\pi$ is the* director plane *of the conoidal surface.*

## Theorem 2.7

*The conoidal surface whose generatrix intersects the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$$

*and the curve*

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

*and has the director plane $\pi = 0$, (suppose that $\pi$ is not parallel to d and that $\mathcal{C}$ is not contained into $\pi$), is characterized by an equation of the form*

$$\varphi\left(\pi, \frac{\pi_1}{\pi_2}\right) = 0. \tag{2.3}$$

## Proof.

An arbitrary generatrix of the conoidal surface is contained into a plane parallel to $\pi$ and, on the other hand, comes from the bundle of planes containing $d$. Then, the equations of a generatrix are

$$d_{\lambda\mu} : \begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \end{cases} .$$

□

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Again, the generatrix must intersect the director curve, hence the system of equations

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

has to be compatible. This leads to a compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

and the equation of the conoidal surface is obtained from

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ \varphi(\lambda, \mu) = 0 \end{cases}.$$

By expressing $\lambda$ and $\mu$, one obtains (2.3). $\square$

## Example 2.9

*Let us find the equation of the conoidal surface, whose generatrices are parallel to $xOy$ and intersect $Oz$ and the curve*

$$\begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases} .$$

The equations of $xOy$ and $Oz$ are, respectively,

$$xOy : z = 0, \quad \text{and} \quad Oz : \begin{cases} x = 0 \\ z = 0 \end{cases} ,$$

so that the equations of the generatrix are

$$d_{\lambda,\mu} : \begin{cases} x = \lambda y \\ z = \mu \end{cases} .$$

From the compatibility of the system of equations

$$\begin{cases} x = \lambda y \\ z = \mu \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases},$$

one obtains the compatibility condition

$$2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0,$$

and, replacing $\lambda = \dfrac{y}{x}$ and $\mu = z$, the equation of the conoidal surface is

$$2x^2z - 2y^2z - 2x^2 + y^2 = 0.$$

# Revolution Surfaces

## Definition 2.10

*The surface generated after the rotation of a given curve $\mathcal{C}$ around a given line $d$ is said to be a* revolution surface.

## Theorem 2.11

*The equation of the revolution surface generated by the curve*

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

*in its rotation around the line*

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r},$$

*is of the form*

$$\varphi((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, px + qy + rz) = 0. \quad (2.4)$$

## Proof.

An arbitrary point on the curve $\mathcal{C}$ will describe, in its rotation around $d$, a circle situated into a plane orthogonal on $d$ and having the center on the line $d$. This circle can be seen as the intersection between a sphere, having the center on $d$ and of variable radius, and a plane, orthogonal on $d$, so that its equations are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases} .$$

The circle has to intersect the curve $\mathcal{C}$, therefore the system

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}$$

must be compatible. One obtains the compatibility

$$\varphi(\lambda, \mu) = 0,$$

which, after replacing the parameters, gives the equation of the surface (2.4). $\square$

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