Geometry[1]
First Year, Computer science

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[1] These notes are not in a final form. They are continuously being improved

Content

Definition 1.1

The basis $[\vec{a}, \vec{b}, \vec{c}]$ of the space $\mathcal{V}$ is said to be *directe* if $(\vec{a}, \vec{b}, \vec{c}) > 0$. If, on the contrary, $(\vec{a}, \vec{b}, \vec{c}) < 0$, we say that the basis $[\vec{a}, \vec{b}, \vec{c}]$ is *inverse*

Definition 1.2

The *oriented volume* of the parallelipiped constructed on the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is $\varepsilon \cdot V$, where V is the volume of this parallelepiped and $\varepsilon = +1$ or -1 insomuch as the basis $[\vec{a}, \vec{b}, \vec{c}]$ is directe or inverse respectively.

Propoziţia 1.3

The triple scalar product $(\vec{a}, \vec{b}, \vec{c})$ of the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ equals the oriented volume of the parallelepiped constructed on these vectors.

Applications of the vector products
The area of the triangle ABC

$S_{ABC} = \frac{1}{2} || \overrightarrow{AB} || \cdot || \overrightarrow{AC} || \sin \widehat{BAC} = \frac{1}{2} || \overrightarrow{AB} \times \overrightarrow{AC} ||$. Since the coordinates of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are $(x_B - x_A, y_B - x_A, z_B - z_A)$ and $(x_C - x_A, y_C - x_A, z_C - z_A)$ respectively, we deduce that

$$S_{ABC} = \frac{1}{2} \left|\left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_B - x_A & y_B - x_A & z_B - z_A \\ x_C - x_A & y_C - x_A & z_C - z_A \end{vmatrix} \right|\right|,$$

or, equivalently

$$4S_{ABC}^2 = \begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2.$$

The distance from one point to a straight line

a) *The distance* $\delta(A, BC)$ *from the point* $A(x_A, y_A, z_A)$ *to the straight line* BC, where $B(x_B, y_B, z_B)$ şi $C(x_C, y_C, z_C)$. Since

$$S_{ABC} = \frac{\| \overrightarrow{BC} \| \cdot \delta(A, BC)}{2}$$

rezultă că

$$\delta^2(A, BC) = \frac{4S_{ABC}^2}{\| \overrightarrow{BC} \|^2}.$$

Thus, we obtain

$$\delta^2(A, BC) = \frac{\begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2}{(x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2}.$$

(*b*) The distance from $\delta(A, d)$ from one point $A(A_A, y_A, z_A)$ to the straight line d : $\frac{x-x_0}{p} + \frac{y-y_0}{p} + \frac{z-z_0}{p}$.

$$\delta(A, d) = \frac{\|\vec{d} \times \overrightarrow{A_0 A}\|}{\vec{d}}, \quad \text{where } A_0(x_0, y_0, z_0) \in \delta.$$

Since

$$\begin{aligned} \vec{d} \times \overrightarrow{A_0 A} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ x_A - x_0 & y_A - y_0 & z_A - z_0 \end{vmatrix} \\ &= \begin{vmatrix} q & r \\ y_A - y_0 & z_A - z_0 \end{vmatrix} \vec{i} + \begin{vmatrix} r & p \\ z_A - z_0 & x_A - x_0 \end{vmatrix} \vec{j} + \begin{vmatrix} p & q \\ x_A - x_0 & y_A - y_0 \end{vmatrix} \vec{k} \end{aligned}$$

it follows that

$$\delta(A, d) = \frac{\sqrt{\begin{vmatrix} q & r \\ y_A - y_0 & z_A - z_0 \end{vmatrix}^2 + \begin{vmatrix} r & p \\ z_A - z_0 & x_A - x_0 \end{vmatrix}^2 + \begin{vmatrix} p & q \\ x_A - x_0 & y_A - y_0 \end{vmatrix}^2}}{\sqrt{p^2 + q^2 + r^2}}.$$

The distance between two straight lines

If $d_1,\ d_2$ are two straight lines, then the distance between them, denoted by $\delta(d_1, d_2)$, is being defined as

$$\min\{\|\ \overrightarrow{M_1M_2}\ \|\ |\ M_1 \in d_1,\ M_2 \in d_2\}.$$

1. If $d_1 \cap d_2 \neq \emptyset$, then $\delta(d_1, d_2) = 0$.
2. If $d_1 || d_2$, then $\delta(d_1, d_2) = \|\ \overrightarrow{MN}\ \|$ where $\{M\} = d \cap d_1,\ \{N\} = d \cap d_2$ and d is a straight line perpendicular to the lines d_1 and d_2. Obviously $\|\ \overrightarrow{MN}\ \|$ is independent on the choice of the line d.
3. We now assume that the straight lines $d_1,\ d_2$ are noncoplanar (skew lines). In this case there exits a unique straight line d such that $d \perp d_1, d_2$ and $d \cap d_1 = \{M_1\},\ d \cap d_2 = \{M_2\}$. The straight line d is called the *common perpendicular* of the lines $d_1,\ d_2$ and obviously $\delta(d_1, d_2) = \|\ \overrightarrow{M_1M_2}\ \|$.

Assume that the straight lines d_1, d_2 are given by their points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$ and their vectors şi au vectorii directori $\vec{d}_1(p_1, q_1, r_1)$ $\vec{d}_2(p_2, q_2, r_2)$, that is, thei equations are

$$d_1 : \frac{x - x_1}{p_1} = \frac{y - y_1}{q_1} = \frac{z - z_1}{r_1}$$

$$d_2 : \frac{x - x_2}{p_2} = \frac{y - y_2}{q_2} = \frac{z - z_2}{r_2}.$$

The common perpendicular of the lines d_1, d_2 is the intersection line between the plane containing the line d_1 which is parallel to the vector $\vec{d}_1 \times \vec{d}_2$, and the plane containing the line d_2 which is parallel to $\vec{d}_1 \times \vec{d}_2$. Since

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} \vec{i} + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} \vec{j} + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \vec{k}$$

it follows that the equations of the common perpendicular are

$$\begin{cases} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ p_1 & q_1 & r_1 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0 \\ \\ \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ p_2 & q_2 & r_2 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0. \end{cases} \tag{2.1}$$

The distance between the straight lines d_1, d_2 can be also regarded the altitude of the parallelogram constructed on the vectors $\vec{d}_1, \vec{d}_2, \vec{d}_1 \times \vec{d}_2$. Thus

$$\delta(d_1, d_2) = \frac{|(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2)|}{||\vec{d}_1 \times \vec{d}_2||}. \tag{2.2}$$

Therefore we obtain

$$\delta(d_1, d_2) = \frac{\left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} \right|}{\sqrt{\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}^2 + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}^2 + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}^2}} \tag{2.3}$$