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Exercise Sheet no.8

Analysis for CS

GROUPWORK:

(G 21)

Let $u,\, v \in \mathbb{R}^n$. Denote by $\alpha = \langle u, v \rangle$, $\beta = ||u||$ and $\gamma = ||v||$.

a) Using the properties of the scalar product and the definition of the Euclidean norm, determine in terms of α, β and γ the numbers $\langle u+v, v\rangle$, $\langle u, 2u-3v\rangle$ and $||u-v||$.

b) If $n = 3$, $u = (-1, 2, 3)$ and $v = (-2, 1, -3)$,

b1) compute α, β and γ,

b2) determine all reals $r > 0$ with the property that the open ball $B(u, r)$ doesn't contain the point v,

b3) determine all reals t with the property that the closed ball $\overline{B}(u, 5)$ contains the vector $(1, -1, t)$.

(G 22)

Decide whether the following sequences $(x^k)_{k\in\mathbb{N}^*}$ in $\mathbb{R}^n$ are convergent or not, and, in case they are convergent, determine their limit.

a) $n = 2$ and $x^k = \left(\left(-\frac{1}{2}\right)^k, (-1)^k\right)$, b) $n = 3$ and $x^k = \left(\frac{2^k}{k!}, \frac{1-4k^7}{k^7+12k}, \frac{\sqrt{k}}{e^{3k}}\right)$,

c) $n = 2$ and $x^k = \left(\frac{\sin k}{k}, -k^3 + k\right)$, d) $n = 4$ and $x^k = \left(\frac{2^{2k}}{\left(2+\frac{1}{k}\right)^{2k}}, \frac{1}{\sqrt[k]{k!}}, (e^k + k)^{\frac{1}{k}}, \frac{\alpha^k}{k}\right)$, where $\alpha \in \mathbb{R}_+$ is fixed.

(G 23) (Train your brain)

Prove **TH2** in lecture no. 8 concerning the uniqueness of the limit of a convergent sequence in $\mathbb{R}^n$.

Homework:

(H 20) (To be delivered in the next exercise-class)

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{2x} \sin x$. Write down:

a) Taylor's polynomial $T_2(x, 0)$,

b) the remainder term $R_2(x, 0)$, for $x \in \mathbb{R} \setminus \{0\}$, according to Taylor's formula,

c) $(e^{2x})^{(n)}$, for $n \in \mathbb{N}$, $x \in \mathbb{R}$,

d) $f^{(n)}(x)$, for $n \in \mathbb{N}$, $x \in \mathbb{R}$, using also the formula for $\sin^{(n)}(x)$ given in exercise (H 18).

(H 21)

Let $f\colon (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$, and let $n \in \mathbb{N}$. Write down:

a) $f^{(n)}(x)$, for $x > 0$,

b) Taylor's polynomial $T_n(x, 1)$,

c) the remainder term $R_n(x, 1)$, for $x \in (0, \infty) \setminus \{1\}$, according to Taylor's formula.

(H 22)

Let x, $y \in \mathbb{R}^n$. Using the definition of the Euclidean norm and the properties of the scalar product, prove the following equality, known as the *parallelogram identity*

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2).$$

Remark. A particular case (if $n = 2$) of the above identity is a result that belongs to elementary geometry. It states that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.