Geometry[1]
First Year, Computer science

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Lecture 2, 05.03.2014

[1] These notes are not in a final form. They are continuously being improved

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Dependence an independence of vectors

Definition 1.1

1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *collinear* if the points O, A, B are collinear. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *noncollinear*.
2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}, \overrightarrow{OC}$ are said to be *coplanar* if the points O, A, B, C are coplanar. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are *noncoplanar*.

Remark 1.2

1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are linearly (in)dependent if and only if they are (non)collinear.
2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are linearly (in)dependent if and only if they are (non)coplanar.

Proposition 1.3

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ form a basis of $\mathcal{V}$ if and only if they are noncoplanar.

Corolar 1.4

The dimension of the vector space of free vectors $\mathcal{V}$ is three.

The directions of the straight lines and planes

Proposition 1.5

Let Δ be a straight line and let $A \in \Delta$ be a given point. The set $\vec{\Delta} = \{\overrightarrow{AM} \mid M \in \Delta\}$ is an one dimensional subspace of $\mathcal{V}$. It is independent on the choice of A on the line Δ and is called the director subspace *of Δ or the* direction *of Δ.*

Remark 1.6

The straight lines $\Delta,\ \Delta'$ are parallel if and only if $\vec{\Delta} = \vec{\Delta}'$

Definition 1.7

We call *director vector* of the straigh line Δ every nonzero vector $\{\vec{d}\} \in \vec{\Delta}$.

If $\vec{d} \in \mathcal{V}$ is a nonzero vector and $A \in \mathcal{P}$ is a given point, then there exits a unique straight line which passes through A and has the direction $\langle\vec{d}\rangle$. This stright line is $\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle\vec{d}\rangle\}$. Δ is called the straight line which passes through A and is parallel to the vector $\vec{d}$.

Proposition 1.8

Let π be a plane and let $A \in \pi$ be a given point. The set $\vec{\pi} = \{\overrightarrow{AM} \in \mathcal{V} \mid M \in \pi\}$ is a two dimensional subspace of $\mathcal{V}$. It is independent on the position of A inside π and is called the director subspace, the director plane *or the* direction *of the plane π.*

Remark 1.9

• The planes π, π' are parallel if and only if $\vec{\pi} = \vec{\pi}'$.
• If $\vec{d}_1$, $\vec{d}_2$ are two linearly independent vectors and $A \in \mathcal{P}$ is a fixed point, then there exists a unique plane $\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle\vec{d}_1, \vec{d}_2\rangle\}$ passing through A and having the direction $\langle\vec{d}_1, \vec{d}_2\rangle$. It is equally said to be parallel to the vectors $\vec{d}_1$ and $\vec{d}_2$.

The vector ecuation of the straight lines and planes

Let Δ be a straight line and let $A \in \Delta$ be a given point.

$$\vec{r}_M = \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \vec{r}_A + \overrightarrow{AM}\ .$$

Thus

$$\begin{aligned} \{\vec{r}_M \mid M \in \Delta\} &= \{\vec{r}_A + \overrightarrow{AM} \mid M \in \Delta\} \\ &= \vec{r}_A + \{\overrightarrow{AM} \mid M \in \Delta\} \\ &= \vec{r}_A + \vec{\Delta}. \end{aligned}$$

Similarly, for a plane π and $B \in \pi$ a given point, then

$$\{\vec{r}_M \mid M \in \pi\} = \vec{r}_B + \vec{\pi}.$$

Generally speaking, a subset X of a vector space is called *affine variety* if either $X = \emptyset$ or there exists $a \in V$ and a vector subspace U of V, such that $X = a + U$.

$$\dim(X) = \begin{cases} -1 & \text{dacă } X = \emptyset \\ \dim(U) & \text{dacă } X = a + U, \end{cases}$$

Proposition 1.10

The bijection φ_O transforms the straight lines and the planes of the space $\mathcal{P}$ into the one and two dimnensional affine varieties of the vector space $\mathcal{V}$.

Let Δ be a straight line, let π be a plane, $\{\vec{d}\}$ be a basis of $\vec{\Delta}$ and let $[\vec{d}_1, \vec{d}_2]$ be a casis of $\vec{\pi}$. Then for $A \in \Delta$, we obtain the equivalence $M \in \Delta$ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\vec{r}_M = \vec{r}_A + \lambda \vec{d}. \tag{1.1}$$

The relation (1.1) is called *the vector equation* of the straight line Δ. Similalrly, for $B \in \pi$, we obtain the equivalence $M \in \pi$ if and only if there exists $\lambda_2, \lambda_2 \in \mathbb{R}$ such that

$$\vec{r}_M = \vec{r}_A + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2. \tag{1.2}$$

The relation (1.2) is called the *vector equation* of the plane π.

Proposition 1.11

If A, B are different points of a straight line Δ, then its vector equation can be put in the form

$$\vec{r}_M = (1-\lambda)\vec{r}_A + \lambda\vec{r}_B,\ \lambda \in \mathbb{R}. \tag{1.3}$$

Proposition 1.12

If A, B, C are three noncolinear points within the plane π, then the vector equation of the plane π can be put in the form

$$\vec{r}_M = (1-\lambda_1-\lambda_2)\vec{r}_A + \lambda_1\vec{r}_B + \lambda_2\vec{r}_C,\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{1.4}$$

Exemple 1.13

Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. We shall show that

$$\vec{r}_M = m\frac{1-n}{1-mn}\vec{u} + n\frac{1-m}{1-mn}\vec{v},$$

where $\{M\} = AB' \cap A'B$, $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\,\overrightarrow{OA'}$, i.e.

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\,\overrightarrow{OA} + n\frac{1-m}{1-mn}\,\overrightarrow{OA'}\,.$$

The vector equations of the lines AB' and $A'B$ are:

$$AB' : \vec{r}_X = (1-\lambda)\vec{r}_A + \lambda\vec{r}_{B'},\ A'B : \vec{r}_Y = (1-\mu)\vec{r}_{A'} + \lambda\vec{r}_B,$$

or, equivalently

$$AB' : \vec{r}_X = (1-\lambda)\vec{u} + \lambda n\vec{v},\ A'B : \vec{r}_Y = (1-\mu)\vec{v} + \mu m\vec{u}.$$

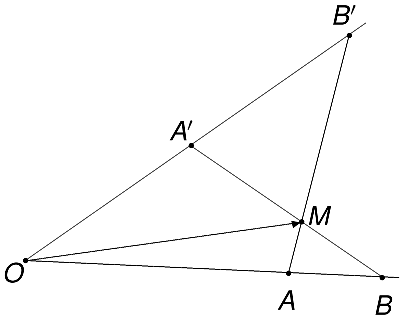

Figure: 1

Since $\{M\} = AB' \cap A'B$, it follows that $\vec{r}_M$ both a representation in the form $(1-\lambda)\vec{u} + \lambda n\vec{v}$ and a representation in the form $(1-\mu)\vec{v} + \mu m\vec{u}$, i.e.

$$\vec{r}_M = (1-\lambda)\vec{u} + \lambda n\vec{v} = (1-\mu)\vec{v} + \mu m\vec{u},\ \ \lambda, \mu \in \mathbb{R}.$$

The linear independence of the vectors $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OA'}$ leads us to the compatible linear system

$$\begin{aligned} 1 - \lambda &= \mu m \\ \lambda n &= 1 - \mu, \end{aligned}$$

whose solution is

$$\lambda = \frac{1-m}{1-mn}, \mu = \frac{1-n}{1-mn},$$

i.e.

$$1 - \mu = n\frac{1-m}{1-mn}.$$

Thus,

$$\vec{r}_M = m\frac{1-n}{1-mn}\vec{u} + n\frac{1-m}{1-mn}\vec{v}.$$

Exemple 1.14

(Newton's theorem) The midpoints of a the diagonals of a complet quadrilater are collinear.

Consider the convex quadrilater $OABC$ with pairwise unparallel opposite sides. Let us also consider $\{D\} = OC \cap AB$ and $\{E\} = OA \cap BC$. The figure $OABCDE$ is called *complete quadrilateral*, and its diagonals are OB, AC and DE.