

Seminar Nr.4, Discrete Random Variables and Discrete Random Vectors

Theory Review

Bernoulli Distribution with parameter $p \in (0, 1)$: $X \left(\begin{matrix} 0 & 1 \\ 1-p & p \end{matrix} \right)$

Binomial Distribution with parameters $n \in \mathbb{N}, p \in (0, 1)$: $X \left(\begin{matrix} k \\ C_n^k p^k q^{n-k} \end{matrix} \right)_{k=\overline{0,n}}$

Hypergeometric Distribution with parameters $N, n_1, n \in \mathbb{N}, n, n_1 \leq N$: $X \left(\begin{matrix} k \\ p_k \end{matrix} \right)_{k=\overline{0,n}}$,

where $p_k = \frac{C_{n_1}^k C_{N-n_1}^{n-k}}{C_N^n}$

Poisson Distribution with parameter $\lambda > 0$: $X \left(\begin{matrix} k \\ p_k \end{matrix} \right)_{k \in \mathbb{N}}$, where $p_k = \frac{\lambda^k}{k!} e^{-\lambda}$

Pascal Distribution with parameters $n \in \mathbb{N}, p \in (0, 1)$: $X \left(\begin{matrix} k \\ C_{n+k-1}^k p^n q^k \end{matrix} \right)_{k \in \mathbb{N}}$

Geometric Distribution with parameter $p \in (0, 1)$: $X \left(\begin{matrix} k \\ pq^k \end{matrix} \right)_{k \in \mathbb{N}}$

Discrete Uniform Distribution with parameter $m \in \mathbb{N}$: $X \left(\begin{matrix} k \\ \frac{1}{m} \end{matrix} \right)_{k=\overline{1,m}}$

Cumulative Distribution Function $F_X : \mathbb{R} \rightarrow \mathbb{R}, F_X(x) = P(X < x)$

$(X, Y) : S \rightarrow \mathbb{R}^2$ discrete random vector with pdf $p_{ij} = P(X = x_i, Y = y_j), (i, j) \in I \times J$ and cdf $F = F_{(X,Y)} : \mathbb{R}^2 \rightarrow \mathbb{R}, F(x, y) = P(X < x, Y < y) = \sum_{x_i < x} \sum_{y_j < y} p_{ij}, \forall (x, y) \in \mathbb{R}^2$.

$$p_i = P(X = x_i) = \sum_{j \in J} p_{ij}, \forall i \in I, q_j = P(Y = y_j) = \sum_{i \in I} p_{ij}, \forall j \in J \text{ (marginal dens.)}$$

Let $X \left(\begin{matrix} x_i \\ p_i \end{matrix} \right)_{i \in I}, Y \left(\begin{matrix} y_j \\ q_j \end{matrix} \right)_{j \in J}$ be discrete random variables. Then

X and Y are **independent** $\Leftrightarrow p_{ij} = P(X = x_i, Y = y_j) = P(X = x_i) P(Y = y_j) = p_i q_j$.

$$X+Y \left(\begin{matrix} x_i + y_j \\ p_{ij} \end{matrix} \right)_{(i,j) \in I \times J}, \alpha X \left(\begin{matrix} \alpha x_i \\ p_i \end{matrix} \right)_{i \in I}, XY \left(\begin{matrix} x_i y_j \\ p_{ij} \end{matrix} \right)_{(i,j) \in I \times J}, X/Y \left(\begin{matrix} x_i / y_j \\ p_{ij} \end{matrix} \right)_{(i,j) \in I \times J} \\ (y_j \neq 0)$$

1. A coin is flipped 3 times. Let X denote the number of heads that appear.

a) Find the probability distribution function (pdf) of X . What type of distribution does X have?

b) Find the cumulative distribution function (cdf) of X, F_X .

c) Find $P(X \leq 2)$ and $P(X < 2)$.

2. It was found that the probability to log on to a computer from a remote terminal is 0.7. Let X denote the number of attempts that must be made to gain access to the

computer:

- a) Find the pdf of X . What type of distribution does X have?
- b) Find the probability (express it in terms of the cdf F_X) that at most 4 attempts must be made to gain access to the computer.
- c) Find the probability (express it in terms of the cdf F_X) that at least 3 attempts must be made to gain access to the computer.

3. A number is picked randomly out of 1, 2, 3, 4 and 5. Let X denote the number picked. Let Y be 1 if the number picked was 1, 2 if the number was prime and 3, otherwise.

- a) Find the pdf of X, Y .
- b) Find the pdf's of $X + Y, XY$.

4. Same problem with 2 numbers being picked randomly. Variable X refers to the 1st number, variable Y to the 2nd. Is there a difference in the answers, from the previous problem?

5. Eight letters are randomly distributed into 3 mailboxes. Let X be the number of letters in the 1st mailbox. Find the pdf of X .

6. In an automobile factory two tasks are performed by robots: welding two joints and tightening three bolts. Let X denote the number of defective welds and Y the number of improperly tightened bolts produced per car. The joint pdf for (X, Y) is given in the following table:

$X \backslash Y$	0	1	2	3
0	.840	.030	.020	.010
1	.060	.010	.008	.002
2	.010	.005	.004	.001

Find

- a) The marginal densities f_X, f_Y ;
- b) The probability that exactly two defective welds and one improperly tightened bolt will be produced by the robots (event A);
- c) The probability that at least one defective weld and at least one improperly tightened bolt will be produced (event B);
- d) The probability that at most one defective weld will be produced (event C).

7. Let X_1, \dots, X_n be independent and identically distributed, with a Bernoulli distribution with parameter p . Find the pdf of $Y = \sum_{i=1}^n X_i$. What type of distribution does Y have?

Bonus Problems:

9. The independent variables X, Y have binomial distributions with parameters m, p and n, p , respectively. Find the pdf of $X + Y$.

10. A die is thrown 3 times. Let X_1, X_2 and X_3 be the number that shows on the 1st, 2nd and 3rd throw, respectively.

- a) Find $P(X_1 + X_2 = X_3)$ and $P(X_1 + X_2 = 7 - X_3)$.
- b) Consider the equation $ax + by - c = 0$, where a, b, c are determined by throwing a die 3 times. Find the probability that the straight line defined by the equation passes through the point $(1, 1)$.