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Solutions to Exercise Sheet no.3

Analysis for CS

(G 9)

a) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{-n^3+3n}\right)^{n^2-n^3} = \lim\limits_{n\to\infty}\left(\left(1+\dfrac{1}{-n^3+3n}\right)^{-n^3+3n}\right)^{\frac{n^2-n^3}{-n^3+3n}} = e.$

b) $\lim\limits_{n\to\infty}(3n^2+5)\ln\left(1+\dfrac{1}{n^2}\right) = \lim\limits_{n\to\infty}\ln\left(1+\dfrac{1}{n^2}\right)^{3n^2+5} = \ln\left(\left(1+\dfrac{1}{n^2}\right)^{n^2}\right)^{\frac{3n^2+5}{n^2}} = \ln e^3 = 3.$

c) For $n \geq 1$ let $x_n = n^n$ and $y_n = 1 + 2^2 + 3^3 + \cdots + n^n$. The sequence $(y_n)_{n\in\mathbb{N}^*}$ is strictly increasing and has limit ∞. Since

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \lim_{n\to\infty}\frac{(n+1)^{n+1}-n^n}{(n+1)^{n+1}} = 1 - \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{n+1} = 1,$$

the Theorem of Stolz-Cesàro yields that $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n} = 1$.

d) For $n \geq 1$ let $a_n = x_1 + 2x_2 + \cdots + nx_n$ and $b_n = n^2$. The sequence $(b_n)_{n\in\mathbb{N}^*}$ is strictly increasing and has limit ∞. Since

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{(n+1)x_{n+1}}{2n+1} = \frac{x}{2},$$

the Theorem of Stolz-Cesàro yields that $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{x}{2}$.

(G 10)

(i) The proposition we prove, for $n \in \mathbb{N}$, through mathematical induction is

$$P(n) : x_n < \frac{1}{a}.$$

I. $P(0)$ is true from the hypothesis.

II. We assume that $P(k)$ is true and prove that $P(k+1)$ is also true, for some $k \in \mathbb{N}$. Thus we know that $x_k < \frac{1}{a}$. Since $a > 0$, let us further notice the following chain of equivalences

$$x_{k+1} < \frac{1}{a} \iff 2x_k - ax_k^2 < \frac{1}{a} \iff 2ax_k - a^2x_k^2 < 1 \iff (ax_k-1)^2 > 0.$$

Thus $P(k+1)$ is true, since $ax_k - 1 \neq 0$, fact known from $P(k)$.

(ii) The proposition we prove, for $n \in \mathbb{N}$, through mathematical induction is

$$Q(n) : x_n > 0.$$

I. $Q(0)$ is true from the hypothesis.

II. We assume that $Q(k)$ is true and prove that $Q(k+1)$ is also true, for some $k \in \mathbb{N}$. Thus we know that $x_k > 0$. Let us further notice the following chain of equivalences

$$x_{k+1} > 0 \Longleftrightarrow 2x_k - ax_k^2 > 0 \Longleftrightarrow 2 - ax_k > 0 \Longleftrightarrow \frac{2}{a} > x_k.$$

Thus $P(k+1)$ is true, since $\frac{2}{a} > \frac{1}{a} > x_k$, fact known from $P(k)$.

(iii) Let $n \in \mathbb{N}$ be arbitrarily chosen. Then, using (i) and (ii), we get

$$x_{n+1} - x_n = x_n - ax_n^2 = x_n(1 - ax_n) > 0.$$

Thus $(x_n)_n \in \mathbb{N}$ is a strictly increasing sequence.

(iv) From (i), (ii) and (iii) we have that the sequence $(x_n)_{n\in\mathbb{N}}$ is both strictly increasing and bounded. Thus, it is convergent, so there exists $l = \lim\limits_{n\to\infty} x_n \in \mathbb{R}$. Hence, we may pass to limit in the recurrence relation

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} (2x_n - ax_n^2) \Longleftrightarrow l = 2l - al^2 \Longleftrightarrow l(1 - al) = 0 \Longleftrightarrow l = 0 \text{ or } l = \frac{1}{a}.$$

As $(x_n)_{n\in\mathbb{N}}$ is strictly increasing with positive terms, we get the conclusion that $l = \frac{1}{a}$.

(G 11)

Th6 (Limits and boundedness properties)

For a sequence $(x_n)_{n\in\mathbb{N}}$, the following assertions hold:

1) If $(x_n)_{n\in\mathbb{N}}$ is convergent, then $(x_n)_{n\in\mathbb{N}}$ is bounded.
2) If $\lim\limits_{n\to\infty} x_n = \infty$ then $(x_n)_{n\in\mathbb{N}}$ is unbounded above.
3) If $\lim\limits_{n\to\infty} x_n = -\infty$ then $(x_n)_{n\in\mathbb{N}}$ is unbounded below.

Proof: 1) We have $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$. This means, from the definition, that

$$\forall \varepsilon > 0, \exists n(\varepsilon) \in \mathbb{N} \text{ such that } |x_n - x| < \varepsilon, \forall n \geq n(\varepsilon).$$

For $\varepsilon = 1, \exists\ n(1) \in \mathbb{N}^*$ such that $|x_n - x| < 1, \forall n \geq n(1)$, which can be rewritten as

$$-1 < x_n - x < 1 \Longleftrightarrow x - 1 < x_n < x + 1 \Longleftrightarrow |x_n| < \max\{|x+1|, |x-1|\}, \forall n \geq n(1)$$
$$\Longleftrightarrow |x_n| < \max\{|x+1|, |x-1|\}, \forall n \geq n(1).$$

Thus we know that all the terms of the sequence $(x_n)_{n\in\mathbb{N}}$, which have an index $\geq n(1)$, are bounded. As well, the other elements, i.e., $x_0, x_1, ..., x_{n(1)-1}$ are all bounded by their maximum, maximum that exists as it is taken from a set of finite elements. Hence we have that

$$|x_n| < \max\{|x+1|, |x-1|, |x_0|, |x_1|, ..., |x_{n(1)-1}|\}, \forall\ n \in \mathbb{N},$$

showing that $(x_n)_{n\in\mathbb{N}}$ is bounded.

2) Since $\lim\limits_{n\to\infty} x_n = \infty$, we have that, for every $t \in \mathbb{R}$, there exists an index $n(t)$ such that $x_n > t$. This fact shows that $(x_n)_{n\in\mathbb{N}}$ is unbounded above.

3) Since $\lim\limits_{n\to\infty} x_n = -\infty$, we have that, for every $t \in \mathbb{R}$, there exists an index $n(t)$ such that $x_n < t$. This fact shows that $(x_n)_{n\in\mathbb{N}}$ is unbounded below.

HOMEWORK:

(H 11)

1) a) For $n \geq 1$ let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ and $y_n = n$. The sequence $(y_n)_{n\in\mathbb{N}^*}$ is strictly increasing and has limit ∞. Since

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

the Theorem of Stolz-Cesàro yields that $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = 0$.

b) For $n \geq 1$ let $x_n = \sqrt{1+2^2} + \sqrt{1+3^2} + \cdots + \sqrt{1+n^2}$ and $y_n = 1 + n^2$. The sequence $(y_n)_{n\in\mathbb{N}^*}$ is strictly increasing and has limit ∞. Since

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \lim_{n\to\infty} \frac{\sqrt{1+(n+1)^2}}{2n+1} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{2}{n}+\frac{2}{n^2}}}{2+\frac{1}{n}} = \frac{1}{2},$$

the Theorem of Stolz-Cesàro yields that $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{1}{2}$.

c) For $n \in \mathbb{N}$ let $a_n = x_0 + 2^1x_1 + 2^2x_2 + \cdots + 2^nx_n$ and $b_n = 2^{n+1}$. The sequence $(b_n)_{n\in\mathbb{N}}$ is strictly increasing and has limit ∞. Since

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{2^{n+1}x_{n+1}}{2^{n+2} - 2^{n+1}} = \lim_{n\to\infty} x_{n+1} = x$$

the Theorem of Stolz-Cesàro yields that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = x$.

2) Case 1: The set X is unbounded below. In this case $\inf X = -\infty$. Let $t \in \mathbb{R}$ be arbitrary. Since X is unbounded below, there exists an index $n(t) \in \mathbb{N}$ such that $x_{n(t)} < t$. The sequence being decreasing, we get that $x_n \leq x_{n(t)} < t$, for every $n \geq n(t)$. Thus $\lim\limits_{n\to\infty} x_n = -\infty$.

Case 2: The set X is bounded below. In this case $x := \inf X \in \mathbb{R}$. Let $\varepsilon > 0$. Then $x + \varepsilon > x$, hence $x + \varepsilon$ cannot be a lower bound of X. Thus there exists $n(\varepsilon) \in \mathbb{N}$ such that $x_{n(\varepsilon)} < x + \varepsilon$. The sequence being decreasing, we get that $x_n \leq x_{n(\varepsilon)} < x + \varepsilon$, for every $n \geq n(\varepsilon)$. Hence $x \leq x_n < x + \varepsilon$, for every $n \geq n(\varepsilon)$. We conclude that $|x_n - x| = x_n - x < \varepsilon$, for every $n \geq n(\varepsilon)$, showing that $\lim\limits_{n\to\infty} x_n = x$.

(H 12)

a) The equalities follow by a direct computation.

b) We have

$$\left(1+\frac{1}{1}\right)^1 < e < \left(1+\frac{1}{1}\right)^2,$$

$$\left(1+\frac{1}{2}\right)^2 < e < \left(1+\frac{1}{2}\right)^3,$$

$$\cdots$$

$$\left(1+\frac{1}{n-1}\right)^{n-1} < e < \left(1+\frac{1}{n-1}\right)^n.$$

All the terms involved in the equalities written above are positive, therefore we may multiply them and keep the inequalities. Hence we get

$$\left(1+\frac{1}{1}\right)^{1}\cdot\left(1+\frac{1}{2}\right)^{2}\cdot\ldots\cdot\left(1+\frac{1}{n-1}\right)^{n-1} < e^{n-1} < \left(1+\frac{1}{1}\right)^{2}\cdot\left(1+\frac{1}{2}\right)^{3}\cdot\ldots\cdot\left(1+\frac{1}{n-1}\right)^{n}.$$

Applying now a) we obtain the following

$$\frac{n^n}{n!} < e^{n-1} < \frac{n^n}{(n-1)!} \Longleftrightarrow e\frac{n^n}{n!} < e^n < e\frac{n^n}{(n-1)!} \Longleftrightarrow e\left(\frac{n}{e}\right)^n \le n! \le en\left(\frac{n}{e}\right)^n, \ \forall n\in\mathbb{N}^*.$$

c) From the first inequality in the above chain of inequalities we get that

$$\frac{n^n}{n!} < e^{n-1} \text{ and } e^{n-1} < n\frac{n^n}{n!}, \ \forall n\in\mathbb{N}^*,$$

thus

$$\frac{e^{n-1}}{n} < \frac{n^n}{n!} < e^{n-1}, \forall n\in\mathbb{N}^*.$$

Since all the terms in the above inequality are positive, by taking the n-th root we do not affect them, hence

$$\frac{e^{\frac{n-1}{n}}}{\sqrt[n]{n}} < \frac{n}{\sqrt[n]{n!}} < e^{\frac{n-1}{n}}, \forall n\in\mathbb{N}^*.$$

By passing to limit, knowing from a previous seminar that $\lim\limits_{n\to\infty}\sqrt[n]{n} = 1$, and computing $\lim\limits_{n\to\infty} e^{\frac{n-1}{n}} = e$, we obtain, using the Sandwich-Theorem, the desired conclusion, i.e.,

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = e.$$