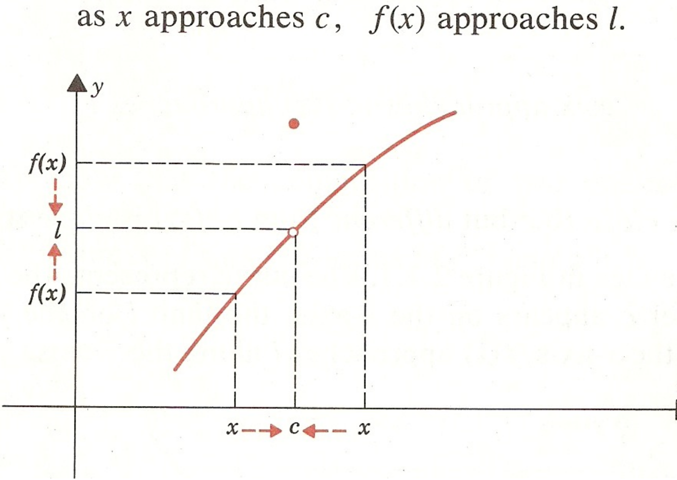
Analysis for CS, Winter semester 2013-2014

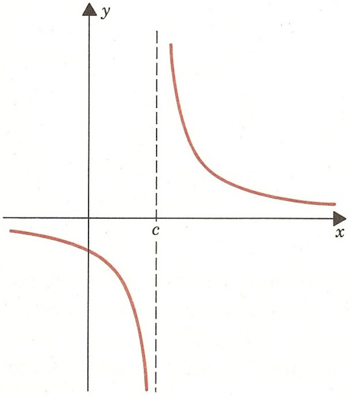
Course 8:

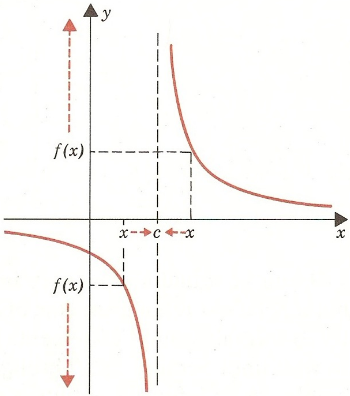
Overview: Properties of real-valued functions of one variable

Limit of a function at a point



Limit of a function at a point



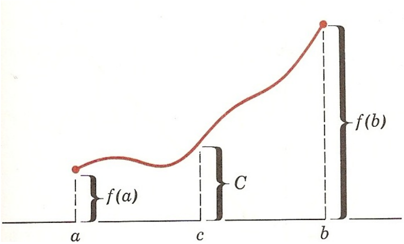


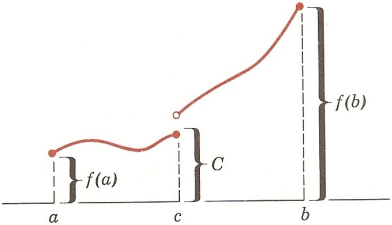
In ordinary language, to say that a certain process is continuous is to say that it goes on without interruption.
In mathematics the word continuous has much the same meaning.

Describing continuity

A function which is continuous on an interval does not skip any values. $\Longrightarrow$ Its graph is unbroken: there are no holes in it and no gaps.

Continuity of functions





The intermediate value theorem (Darboux)

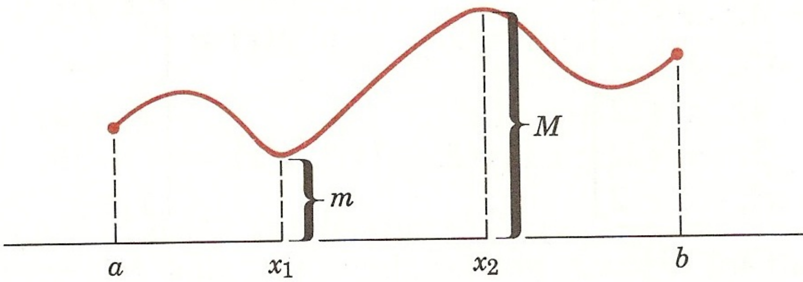
If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and C is a number between $f(a)$ and $f(b)$, then there exists at least one point c between a and b for which $f(c) = C$.

Applications

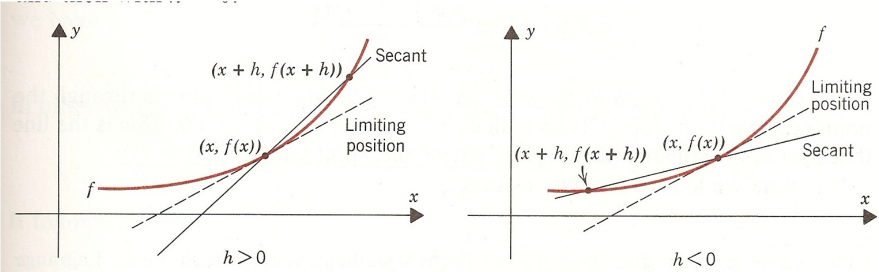
Numerical methods to compute zeros of functions: If $f(a)f(b) < 0$ (thus take $C = 0$) $\Longrightarrow \exists\ c$ between a and b for which $f(c) = 0$.

The maximum-minimum theorem (Weierstrass)

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f takes on both a maximum value M and a minimum value m on $[a, b]$.



Geometric interpretation of the derivative: the tangent to the graph of functions



Differentiability and one-sided differentiability

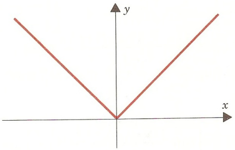
Let $f\colon M \to \mathbb{R}$, $\alpha \in M$ such that $\alpha \in (M_\ell)' \cap (M_r)'$, where $M_\ell := M \cap (-\infty, \alpha)$ and $M_r := M \cap (\alpha, \infty)$. Then:

- If f is differentiable at α, then f is both left-hand and right-hand differentiable at α and $f'(\alpha) = f'_\ell(\alpha) = f'_r(\alpha)$.
- If f is both left-hand and right-hand differentiable at α and if $f'_\ell(\alpha) = f'_r(\alpha)$, then f is differentiable at α and $f'(\alpha) = f'_\ell(\alpha)$.

Differentiability and continuity

Let $f : M \to \mathbb{R}$ and $\alpha \in M \cap M'$. If f is differentiable at α, then f is continuous at α.

Continuity $\not\Longrightarrow$ differentiability



no derivative at 0

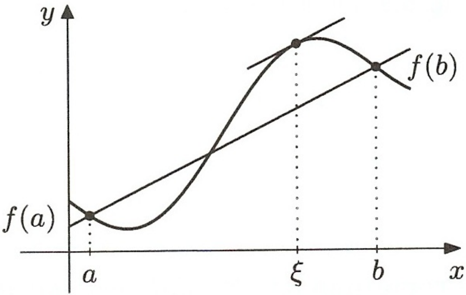
An amazing fact

There exist continuous nowhere differentiable function.
$\hookrightarrow$ The first example was constructed by K. Weierstrass in 1885.
The graph of such a function is a fractal.

The mean value theorem (Lagrange)

Let a, $b \in \mathbb{R}$, $a < b$, and let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one point $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$



Applications of the derivative

- the study of local and global properties of functions, such as
  - monotonicity,
  - maxima and minima,
- gives information concerning the features of graphs,
- play an important role in computing zeros of functions, approximations, modeling etc.

L'Hospital's rules

Let $a,\, b \in \overline{\mathbb{R}}$, $a < b$, and let $f,\, g\colon (a,b) \to \mathbb{R}$ be differentiable functions satisfying the following conditions

(1) $g'(x) \neq 0,\ \forall\ x \in (a,b)$,

(2) $\lim\limits_{\substack{x\to a\\ x>a}} f(x) = \lim\limits_{\substack{x\to a\\ x>a}} g(x) = \ell$, where $\ell \in \{-\infty,\, 0,\, \infty\}$.

(3) $\exists \lim\limits_{\substack{x\to a\\ x>a}} \dfrac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$.

$$\Longrightarrow \exists \lim_{\substack{x\to a\\ x>a}} \frac{f(x)}{g(x)} \text{ and } \lim_{\substack{x\to a\\ x>a}} \frac{f(x)}{g(x)} = \lim_{\substack{x\to a\\ x>a}} \frac{f'(x)}{g'(x)}.$$

Remark

A similar result holds for $\lim\limits_{\substack{x\to b\\ x<b}} \dfrac{f(x)}{g(x)}$.

How to use L'Hospital's rules to compute limits of sequences

- not directly: sequences may not be differentiated; keep in mind: $(\mathbb{N})' = \{\infty\} \Longrightarrow$ we cannot compute derivatives for functions defined on $\mathbb{N}$ (thus for sequences),
- by considering suitable functions, then applying L'Hospital's rules for them and using finally the passage from limits of functions to limits of sequences.

Example

Compute $\lim\limits_{n\to\infty} \dfrac{n}{e^n}$. Consider $\lim\limits_{x\to\infty} \dfrac{x}{e^x}$. Since

$$\lim_{x\to\infty} \frac{x'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = 0,$$

we get that $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0 \Longrightarrow \lim\limits_{n\to\infty} \dfrac{n}{e^n} = 0.$

Taylor's formula

If $f : [a, b] \to \mathbb{R}$ is n-times continuously differentiable on $[a, b]$ and $(n+1)$-times differentiable on (a, b), then, for all $x, x_0 \in [a, b]$ with $x \neq x_0$, there exists c strictly between x and x_0 such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

The case $x = x_0$

If f is $(n+1)$-times differentiable at x_0, then the above formula holds true, considering $c = x_0$.

A particular case: $n = 0$, $x_0 = a$, $x = b$

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) = f(a) + f'(c)(b - a)$.
$\hookrightarrow$ the mean value theorem of Lagrange

Approximating the exponential function

Problem: Determine the minimal degree of the Taylor polynomial $T_n(x, 0)$ which approximates e^x in $[0, 1]$ correct to five digits.

Solution: Let $x \in [0, 1]$. According to Taylor's formula $\exists c \in [0, 1]$ such that

$$|e^x - T_n(x, 0)| = |R_n(x, 0)| = R_n(x, 0) = \frac{e^c}{(n+1)!}x^{n+1} \leq 10^{-5}.$$

The remainder is maximal for $x = c = 1 \implies$

$$\frac{e}{(n+1)!} \leq 10^{-5}.$$

Since $e < 3$, we get $n \geq 8$. Thus

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{8!}x^8 \pm 10^{-5}, \forall\ x \in [0, 1].$$

Problem

Estimate $\sin(0.5)$ with an error $< 10^{-3}$.

Solution

There exists $c \in \left(0, \frac{1}{2}\right)$ such that

$$|R_n(0.5, 0)| = \left| \frac{\sin^{(n+1)}(c)}{(n+1)!}(0.5)^{n+1} \right| \leq \frac{1}{(n+1)!}(0.5)^{n+1}.$$

Since $\frac{(0.5)^5}{5!} = \frac{1}{(2^5)(5!)} = \frac{1}{3840} < 10^{-3}$,
we can be sure that

$$T_4(0.5, 0) = T_3(0.5, 0) = 0.5 - \frac{(0.5)^3}{3!} = \frac{23}{48}$$

approximates $\sin(0.5)$ with an error $< 10^{-3}$.

The sine function

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \forall\ x \in \mathbb{R}.$$

- There are only odd powers.
- sin is an odd function, i.e., $\sin(-x) = -\sin x,\ \forall\ x \in \mathbb{R}$.

The cosine function

$$\cos x = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \forall\ x \in \mathbb{R}.$$

- There are only even powers.
- cos is an even function, i.e., $\cos(-x) = \cos x,\ \forall\ x \in \mathbb{R}$.

Approximating sin and cos by Taylor polynomials