Geometry[1]
First Year, Computer science

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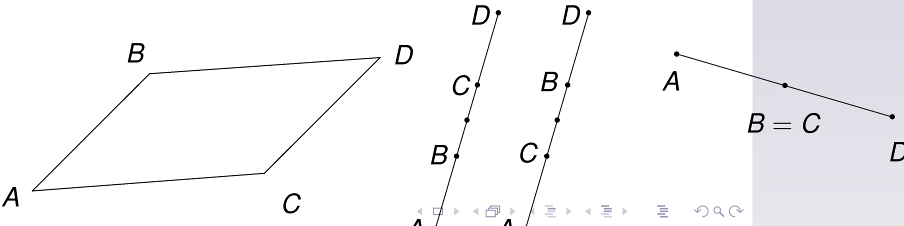
[1] These notes are not in a final form. They are continuously being improved

Cuprins

Let $\mathcal{P}$ be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If $(A, B) \in \mathcal{P} \times \mathcal{P}$ is an ordered pair, then A is called the *original point* or the *origin* and B is called the *terminal point* or the *extremity* of (A, B).

Definition 1.1

The ordered pairs (A, B), (C, D) are said to be equipollent, written $(A, B) \sim (C, D)$, if the segments $[AD]$ and $[BC]$ have the same midpoint.



Vectors

Remark 1.2

If A, B, C, D are not collinear points, then $(A, B) \sim (C, D)$ if and only if $ABDC$ is a parallelogram. In fact the length of the segments $[AB]$ and $[CD]$ is the same whenever $(A, B) \sim (C, D)$.

Proposition 1.3

If (A, B) is an ordered pair and $O \in \mathcal{P}$ is a given point, then there exists a unique point X such that $(A,B) \sim (O, X)$.

Proposition 1.4

The equipollence relation is an equivalence relation on $\mathcal{P} \times \mathcal{P}$.

Vectors

Definition 1.5

The equivalence classes with respect to the equipollence relation are called (free) vectors*.*

Denote by $\overrightarrow{AB}$ the equivalence class of the ordered pair (A, B), that is $\overrightarrow{AB} = \{(X, Y) \in \mathcal{P} \times \mathcal{P} \mid (X, Y) \sim (A, B)\}$ and let $\mathcal{V} = \mathcal{P} \times \mathcal{P} \big/ _{\sim} = \{\overrightarrow{AB} \mid (A, B) \in \mathcal{P} \times \mathcal{P}\}$ be the set of (free) vectors. The *length* or the *magnitude* of the vector $\overrightarrow{AB}$, denoted by $\| \overrightarrow{AB} \|$ or by $| \overrightarrow{AB} |$, is the length of the segment $[AB]$.

Remark 1.6

If two ordered pairs (A, B) and (C, D) are equipplent, i.e. the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.

Proposition 1.7

1. $\overrightarrow{AB}=\overrightarrow{CD} \Leftrightarrow \overrightarrow{AC}=\overrightarrow{BD}$.
2. $\forall A, B, O \in \mathcal{P}$, $\exists\, ! X \in \mathcal{P}$ *such that* $\overrightarrow{AB}=\overrightarrow{OX}$.
3. $\overrightarrow{AB}=\overrightarrow{A'B'}$, $\overrightarrow{BC}=\overrightarrow{B'C'} \Rightarrow \overrightarrow{AC}=\overrightarrow{A'C'}$.

Definition 1.8

If O, $M \in \mathcal{P}$, the the vector $\overrightarrow{OM}$ is denoted by $\vec{r}_M$ and is called the *position vector of* M *with respect to* O.

Corollary 1.9

The map $\varphi_O : \mathcal{P} \to \mathcal{V}$, $\varphi_O(M) = \vec{r}_M$ *is one-to-one and onto, i.e bijective.*

Operations with vectors

In this section we shall define and study the the addition operation of vectors alongside the multiplication of vectors with scalars.

The addition of vectors

Let $\vec{a}, \vec{b} \in \mathcal{V}$ and $O \in \mathcal{P}$ be such that $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{AB}$. The vector $\overrightarrow{OB}$ is called the *sum* of the vectors $\vec{a}$ and $\vec{b}$ and is written $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$.

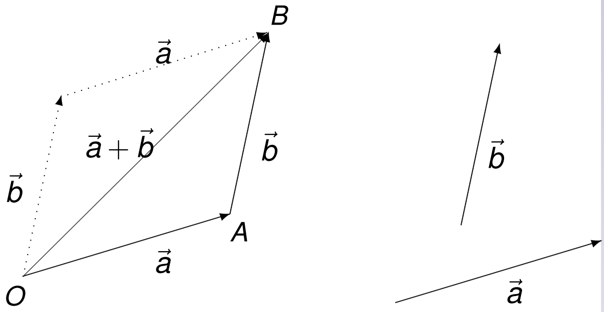


Figure: 1

The addition of vectors

Let O' be another point and $A', B' \in \mathcal{P}$ be such that $\overrightarrow{O'A'} = \vec{a}$, $\overrightarrow{A'B'} = \vec{b}$. Since $\overrightarrow{OA} = \overrightarrow{O'A'}$ and $\overrightarrow{AB} = \overrightarrow{A'B'}$ it follows, according to Proposition 1.4 (3), that $\overrightarrow{OB} = \overrightarrow{O'B'}$. Therefore the vector $\vec{a} + \vec{b}$ is independent on the choice of the point O.

Proposition 2.1

The set $\mathcal{V}$ endowed to the binary operation $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, $(\vec{a}, \vec{b}) \mapsto \vec{a} + \vec{b}$, is an abelian group whose zero element is the vector $\overrightarrow{AA} = \overrightarrow{BB} = \vec{0}$ and the opposite of $\overrightarrow{AB}$, denoted by $-\overrightarrow{AB}$, is the vector $\overrightarrow{BA}$.

In particular, then addition operation is associative, the vector $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ is usually denoted by $\vec{a} + \vec{b} + \vec{c}$. Moreover the expression

$$((\cdots(\vec{a}_1 + \vec{a}_2) + \vec{a}_3 + \cdots + \vec{a}_n)\cdots), \tag{1.1}$$

The addition of vectors

is independent on the distribution of paranthesis and it is usually denoted by $\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n$.

Example 2.2

If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{P}$ are some given points, then $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n}$. This shows that $\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1} = \vec{0}$, namely the sum of vectors constructed on the edges of a closed broken line is zero.

Example 2.3

Consider the parallelograms in $\mathcal{P}$, $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, and M_1, M_2, M_3, M_4 the midpoints of the segments $[A_1B_1], [A_2B_2], [A_3B_3], [A_4B_4]$ respectively. Show that:

- $2\,\overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}$ *and* $2\,\overrightarrow{M_3M_4} = \overrightarrow{A_3A_4} + \overrightarrow{B_3B_4}$.
- M_1, M_2, M_3, M_4 *are the vertices of a parallelogram.*

The addition of vectors

Corolarul 2.4

If $\vec{a} = \overrightarrow{OA}, \vec{b} = \overrightarrow{OB}$ are given vectors, there exists a unique vector $\vec{x} \in \mathcal{V}$ such that $\vec{a} + \vec{x} = \vec{b}$. In fact $\vec{x} = \vec{b} + (-\vec{a}) = \overrightarrow{AB}$ and is denoted by $\vec{b} - \vec{a}$.

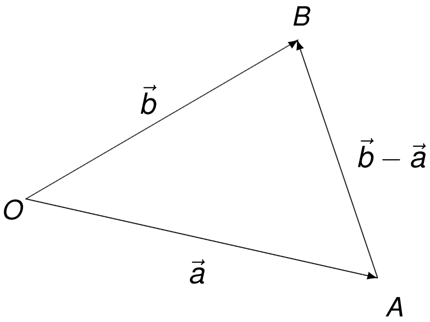


Figure: 1

The multiplication of vectors with scalars

Let $\alpha \in \mathbb{R}$ be a scalar and $\vec{a} = \overrightarrow{OA} \in \mathcal{V}$ be a vector. We define the vector $\alpha \cdot \vec{a}$ as follows: $\alpha \cdot \vec{a} = \vec{0}$ if $\alpha = 0$ or $\vec{a} = \vec{0}$; if $\vec{a} \neq \vec{0}$ and $\alpha > 0$, there exists a unique point on the half line $]OA$ such that $||OB|| = \alpha \cdot ||OA||$ and define $\alpha \cdot \vec{a} = \overrightarrow{OB}$; if $\alpha < 0$ we define $\alpha \cdot \vec{a} = -(|\alpha| \cdot \vec{a})$. The external binary operation

$$\mathbb{R} \times \mathcal{V} \to \mathcal{V}, \ (\alpha, \vec{a}) \mapsto \alpha \cdot \vec{a}$$

is called the *multiplication of vectors with scalars*.

Proposition 2.5

The following properties hold:

(v1) $(\alpha + \beta) \cdot \vec{a} = \alpha \cdot \vec{a} + \beta \cdot \vec{a}, \ \forall \, \alpha, \beta \in \mathbb{R}, \ \vec{a} \in \mathcal{V}.$

(v2) $\alpha \cdot (\vec{a} + \vec{b}) = \alpha \cdot \vec{a} + \alpha \cdot \vec{b}, \ \forall \, \alpha \in \mathbb{R}, \ \vec{a}, \vec{b} \in \mathcal{V}.$

(v3) $\alpha \cdot (\beta \cdot \vec{a}) = (\alpha\beta) \cdot \vec{a}, \forall \, \alpha, \beta \in \mathbb{R}.$

(v4) $1 \cdot \vec{a} = \vec{a}, \ \forall \, \vec{a} \in \mathcal{V}.$

The vector structure on the set of vectors

Theorem 2.6

The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.

Example 2.7

If A' is the midpoint of the egde $[BC]$ of the triangle ABC, then $\overrightarrow{AA'} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$.

Example 2.8

If O is a fixed point in $\mathcal{P}$, then the position vector $\vec{r}_G = \overrightarrow{OG}$ of the centroid G with respect to the point O is $\vec{r}_G = \frac{1}{3}\left(\vec{r}_A + \vec{r}_B + \vec{r}_C\right)$.