Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2015

1. (1p) Find the general solution of each of the following differential equations whose unknown is the function denoted $x(t)$.

(a) $x' + tx = 1$; (b) $x'' + 4x = 1$.

2. (2.5p) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$.

(a) Find the fixed points of f and study their stability using the linearization method.

(b) Represent the graph of f and find geometrically the fixed points of f.

(c) Find directly $\varphi(k, 0)$ (or, in other notation, $f^k(0)$) for any $k \geq 0$. Which is the orbit corresponding to the initial state 0? What remarkable property has this orbit? Depict this orbit using the stair-step diagram.

d) Let $\eta = 2$, and, respectively, $\eta = -1/4$. Using the stair-step diagram describe the long-term behavior of the orbit that starts at η (in other notation, of the sequence defined by $x_{k+1} = x_k^2 - 1,\ x_0 = \eta$).

3. (2p) Let $c \in [0, 1)$ be a parameter and consider the scalar dynamical system $\dot{x} = x(1 - x) - cx$.

a) Find its equilibria and study their stability using the linearization method.

b) Represent its phase portrait.

c) When $x(t) > 0$ is considered to be the number of fish in some lake, and $c \geq 0$ to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical result obtained at a) and b).

d) What will happen with the fish in the case that $c = 2$?

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1. (1.5p) We consider the linear planar system $\dot{x} = -x,\ \dot{y} = -y$.

a) Find its general solution and its flow.

b) Using the definition of the orbit, find two of its orbits: the ones corresponding to the initial states $\eta = (1, 2)$, and, respectively, $\eta = (-1, -2)$.

c) Find its isocline for the slope $m = 2$. Find its isocline for the slope $m \in \mathbb{R}$. Represent few isoclines and find the shape of the orbits.

d) Represent its phase portrait.

2. (0.5p) The following proposition is true or false? Justify.

"The isoclines of a linear planar system are straight lines that pass through the origin".

3. (1.5) Find the general solution of the differential equations $x' + tx = 2t$ and $x'' + \omega^2 x = 1$ (the unknown denoted $x(t)$ and the parameter $\omega > 0$) and of the difference equation $x_{k+1} = 3\,x_k - 4$.

4. (2p) Using the stair-step diagram, estimate the basin of attraction for each of the fixed points (if there is any which is an attractor) of the map

$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x^2 + 5}{2x} \ .$$

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1. (1.5p) Represent the phase portrait of the scalar dynamical system $\dot{x} = x(1 - x^2)$. Find $\varphi(t, 1)$ and justify. Specify the monotony of $\varphi(t, 2)$ and, respectively, $\varphi(t, 0.5)$.

2. (0.25p) Find the polar coordinates of the points whose cartesian coordinates are: $(1, 0)$, $(0, 1)$, $(-2, 0)$ and $(0, -0.5)$, respectively.

3. (2.5p) We consider the planar system
$\dot{x} = -y + x(1 - x^2 - y^2),\ \dot{y} = x + y(1 - x^2 - y^2)$.
a) Study the type and stability of the equilibrium point $(0, 0)$ using the linearization method. There are other equilibria?
b) Transform the given system to polar coordinates.
c) What is the shape of the orbit corresponding to: $\varphi(t, 1, 0)$, $\varphi(t, 0, 1)$, $\varphi(t, -2, 0)$ and $\varphi(t, 0, -0.5)$, respectively? Justify.
d) What remarkable property has the function $\varphi(t, 1, 0)$?

4. (1.25p) Find all the solutions of each of the following difference equations and which also satisfies the given conditions: a) $x_{k+2} - 5\,x_{k+1} + 6x_k = 12$; b) $x_{k+1} = 1 - x_k^2,\ x_0 = 0$; c) $x_{k+2} + x_{k+1} + x_k = 0,\ x_0 = 0$.

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Exam on Dynamical Systems
June, 2015

1. (1.5p) Find the linear homogeneous differential equation of minimal order that has as solutions:

a) $t\,e^{2t}$ and e^{-t};

b) $\cos(\omega t)$ and $3\sin(\omega t)$ (here $\omega > 0$).

Find also the general solution of each of these two equations.

2. (2p) We consider the planar Lotka-Volterra system

$\dot{x} = x(1-y), \quad \dot{y} = y(2-x).$

a) Find its equilibria and study their stability using the linearization method.

b) Find a first integral in $(0,\infty)\times(0,\infty)$.

3. (2p) Let $f:\mathbb{R}\to\mathbb{R}$, $f(x) = 2x(1-x)$.

a) Find its fixed points and study their stability.

b) Let $I_1 = (-\infty, 0)$, $I_2 = (0,1)$ and $I_3 = (1,\infty)$. Find $f(I_1)$, $f(I_2)$ and $f(I_3)$.

c) Find the orbits corresponding to the initial states $\eta = 0$ and, respectively, $\eta = 1$.

d) Using the stair-step diagram, describe the long-term behavior of the orbits corresponding to the initial states: $\eta = 1/8$, $\eta = 7/8$, $\eta = -1/8$ and, respectively, $\eta = 9/8$.

e) Estimate the basin of attraction of the stable fixed point of f.

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Exam on Dynamical Systems
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1. (1p)

a) Write the statement of the Fundamental Theorem for linear homogeneous second order differential equations.

b) The following proposition is true or false? Justify. We remind you that $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$.

"The general solution of the differential equation $x'' - x = 0$ is
$x(t) = c_1 \cosh t + c_2 \sinh t$, where c_1, c_2 are arbitrary real constants."

2. (1p) Find a first integral in $\mathbb{R}^2$ of the pendulum equation

$$\ddot{\theta} + \omega^2 \sin\theta = 0.$$

Hint: Write first the planar system equivalent to this equation.

3. (0.75p) Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k,\ x_0 = 0,\ x_1 = 1.$$

4. (2p) We consider the following nonlinear planar systems
$\dot{x} = -x + xy, \quad \dot{y} = -2y + 3y^2.$

a) Find its equilibria and study their stability using the linearization method.

b) Find $\varphi(t, 0, 2/3)$, $\varphi(t, 4, 0)$ and $\varphi(t, 1, 2/3)$.

5. (0.75p) We consider the IVP $y' = 1 + xy^2$, $y(0) = 0$. Write the Euler numerical formula on the interval $[0, 1]$ with step-size $h = 0.02$. Specify the initial values and the number of steps necessary to find the approximate value of $\varphi(0.5)$ and, respectively, of $\varphi(1)$. Here with φ is denoted the exact solution of the given IVP.

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Exam on Dynamical Systems
June, 2015

1. (1.25p) For each $k > 0$ we consider the differential equation $\dot{x} = -k\,(x - 21)$, which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time t.

a) Find its flow.

b) An experiment revealed the following fact. A cup of tea with initial temperature of $49°C$ has a temperature of $37°C$ after 10 minutes. Find the initial temperature of a cup of tea such that after 20 minutes the tea has $37°C$.

2. (2.5p) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$.

(a) Find the fixed points of f and study their stability using the linearization method.

(b) Represent the graph of f and find geometrically the fixed points of f.

(c) Find the orbit corresponding to the initial state 0? What remarkable property has this orbit? Depict this orbit using the stair-step diagram.

d) Let $\eta = 2$, and, respectively, $\eta = -1/4$. Find $f(\eta)$ and $f^2(\eta)$ (here f^2 denotes the second iterate of f). Using the stair-step diagram describe the long-term behavior of the orbit corresponding to the initial state η .

3. (1.75p) For what values of the real parameter a the system

$\dot{x} = ax - 5y,\ \dot{y} = x - 2y$ has a center at the origin?

For $a = 0$ find the general solution of this system and specify its type and stability.

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Exam on Dynamical Systems
July, 2015

1. (1.5p) We consider the differential equation

$$x'' + 9x = \cos 3t.$$

a) Find a solution of the form $x_p = t(a\cos 3t + b\sin 3t)$, with $a, b \in \mathbb{R}$.
b) Find its general solution.
c) Describe the motion of a spring-mass system governed by this equation.

2. (1p) Find the solution of

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k, \quad x_0 = 0, \quad x_1 = 0.$$

Hint: look for $a, b \in \mathbb{R}$ such that $(x_k)_p = ak + b$ is a particular solution of the difference equation.

3. (1.5p) We consider the IVP $x' = -200x, \quad x(0) = 1.$
a) Find the solution and its limit as $t \to \infty$.
b) Write the Euler's numerical formula with constant step-size h.
c) Find a range of values for the step-size h such that the solution $(x_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} x_k = 0$.

4. (1.5p) Find the fixed points and the 2–periodic points of the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - 2x^2$. Study the stability of the fixed points.

ST1. (1p) Find the general solution of the scalar differential equation $x' - ax = at - 1$, where the unknown is the function x of variable t and $a \in \mathbb{R}^*$ is a fixed parameter.

ST2. (1p) We consider the scalar differential equation

$$(*) \quad \dot{x} = 2x(2 - x),$$

whose unknown is the function x of variable t. We denote by $\varphi(t; \eta)$ the flow of $(*)$.

a) For $(*)$, find its equilibria, its orbits, depict its phase portrait, and study the stability of its equilibria.

b) Find $\lim\limits_{t \to \infty} \varphi(t; 1)$.

c) There exists some $\eta \in \mathbb{R}$ such that $\lim\limits_{t \to \infty} \varphi(t; \eta) = 3$?