Geometry[1]
First Year, Computer science

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[1] These notes are not in a final form. They are continuously being improved

Content

The equation of the tangent plane and the normal line

Proposition 1.1 ([Pi, p. 256])

If $D \subseteq \mathbb{R}^3$ is an open set and $F : D \longrightarrow \mathbb{R}$ is a C^1-smooth function, then the gradient vector field $\mathrm{grad}F := (F_x, F_y, F_z)$ is normal to the level sets of F at every point $M_0(x_0, y_0, z_0) \in D$ where $(\mathrm{grad}F)(x_0, y_0, z_0)$ is nonzero. More precisely, $(\mathrm{grad}F)(x_0, y_0, z_0)$ is a normal vector of the plane $T_{M_0}(F^{-1}(F(x_0, y_0, z_0)))$ tangent to the level set $F^{-1}(F(x_0, y_0, z_0))$ at its point M_0.

Proposition 1.2

Let $D \subseteq \mathbb{R}^3$ be an open set and $F : D \longrightarrow \mathbb{R}$ be a C^1-smooth function. If $(\operatorname{grad} F)(x_0, y_0, z_0) \neq 0$ at some point $M_0(x_0, y_0, z_0) \in D$, then the equation of the plane $T_{M_0}(S)$-tangent to the implicit surface

$$S : F(x, y, z) = F(x_0, y_0, z_0)$$

at its point $M_0(x_0, y_0, z_0) \in S$ is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_y(x_0, y_0, z_0)(y - y_0) = 0.$$

The equations of the normal line to S at M_0 are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)}.$$

Exemple 1.3

The equations of the tangent plane an the normal line to the ellipsoid

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

at its point $M_0(x_0, y_0, z_0)$ are

$$T_{M_0}(\mathcal{E}) : \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$$

$$N_{M_0}(\mathcal{E}) : \frac{a^2}{x_0}(x - x_0) = \frac{b^2}{y_0}(y - y_0) = \frac{c^2}{z_0}(z - z_0).$$

Indeed, the gradient of the function $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$ is $(\mathrm{grad} F) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$ and

$\frac{1}{2}(\mathrm{grad} F)(x_0, y_0, z_0) = \left(\frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2}\right)$ is a normal vector to the tangent plane $T_{M_0}(\mathcal{E})$ to the ellipsoid $\mathcal{E} = F^{-1}(0)$ at $M_0(x_0, y_0, z_0)$.

Exemple 1.4

The equations of the tangent plane and the normal line to the hyperboloid of one sheet

$$\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

at its point $M_0(x_0, y_0, z_0)$ are

$$T_{M_0}(\mathcal{H}_1) : \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = 1$$

$$N_{M_0}(\mathcal{H}_1) : \frac{a^2}{x_0}(x - x_0) = \frac{b^2}{y_0}(y - y_0) = \frac{c^2}{z_0}(z_0 - z).$$

Exemple 1.5

The equations of the tangent plane and the normal line to the hyperboloid of two sheets

$$\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0,$$

at its point $M_0(x_0, y_0, z_0)$ are

$$T_{M_0}(\mathcal{H}_2) : \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = -1$$

$$N_{M_0}(\mathcal{H}_2) : \frac{a^2}{x_0}(x - x_0) = \frac{b^2}{y_0}(y - y_0) = \frac{c^2}{z_0}(z_0 - z).$$

The elliptic cone

The surface of equation

$$\mathcal{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{1.1}$$

is called *elliptic cone*.

- The coordinate planes are planes of symmetry for $\mathcal{C}$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{C}$;
- The intersections with the coordinates planes are

$$\begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \\ x = 0 \\ \text{two lines} \end{cases}, \begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \\ \text{two lines} \end{cases},$$

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \\ z = 0 \\ \text{the origin } O(0,0,0). \end{cases}$$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -\dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases} ; \quad \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = -\dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas.} \end{cases}$$

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{\lambda^2}{c^2} \\ z = \lambda \\ \text{ellipses} \end{cases}$$

The elliptic paraboloid

The surface of equation

$$\mathcal{P}_e : \frac{x^2}{p} + \frac{y^2}{q} = 2z, \qquad p, q \in \mathbb{R}_+^*, \tag{1.2}$$

is called *elliptic paraboloid*.

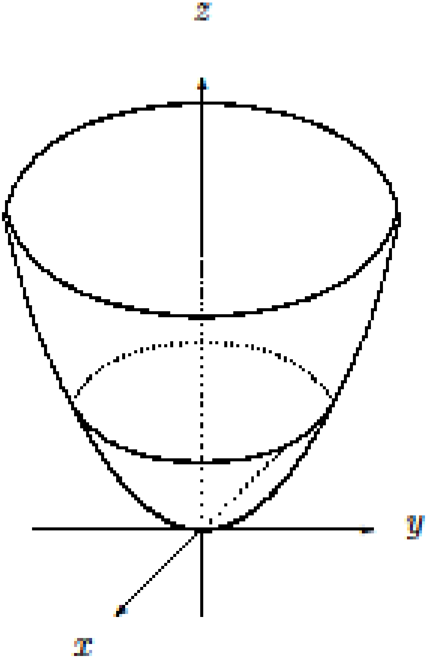
- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are

$$\left\{ \begin{array}{c} \dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{array} \right., \left\{ \begin{array}{c} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{array} \right., \left\{ \begin{array}{c} \dfrac{x^2}{p} + \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{the origin } O(0,0,0). \end{array} \right.$$

- The intersection with the planes parallel to the coordinate planes are $\begin{cases} \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \end{cases}$,
  - If $\lambda > 0$, the section is an ellipse;
  - If $\lambda = 0$, the intersection reduces to the origin;
  - If $\lambda < 0$, one has the empty set;

  and

$$\begin{cases} \dfrac{y^2}{q} = 2z - \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{cases}; \qquad \begin{cases} \dfrac{x^2}{p} = 2z - \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas} \end{cases};$$


z
y
x

The hyperbolic paraboloid

The *hyperbolic paraboloid* is the surface given by the equation

$$\mathcal{P}_h : \frac{x^2}{p} - \frac{y^2}{q} = 2z, \qquad p, q \in \mathbb{R}^*_+. \tag{1.3}$$

- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are, respectively,

$$-\begin{cases} \dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{cases}; \quad \begin{cases} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{cases}; \quad \begin{cases} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{two lines.} \end{cases};$$

- The intersection with the planes parallel to the coordinate planes are

$$\begin{cases} -\dfrac{y^2}{q} = 2z + \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{cases} ; \quad \begin{cases} \dfrac{x^2}{p} = 2z - \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas.} \end{cases}$$

$$\begin{cases} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \\ \text{hyperbolas} \end{cases}$$

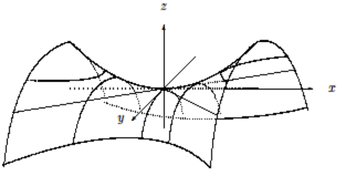
Remark: The hyperbolic paraboloid admits two families of rectilinear generatrices. Since

$$\left(\frac{x}{\sqrt{p}}+\frac{y}{\sqrt{q}}\right)\left(\frac{x}{\sqrt{p}}-\frac{y}{\sqrt{q}}\right)=2z,$$

then the two families are, respectively, of equations

$$d_\lambda:\begin{cases}\dfrac{x}{\sqrt{p}}-\dfrac{y}{\sqrt{q}}=\lambda\\ \lambda\left(\dfrac{x}{\sqrt{p}}+\dfrac{y}{\sqrt{q}}\right)=2z\end{cases},\lambda\in\mathbb{R}^*\text{ and}$$

$$d'_\mu:\begin{cases}\dfrac{x}{\sqrt{p}}+\dfrac{y}{\sqrt{q}}=\mu\\ \mu\left(\dfrac{x}{\sqrt{p}}-\dfrac{y}{\sqrt{q}}\right)=2z\end{cases},\mu\in\mathbb{R}^*.$$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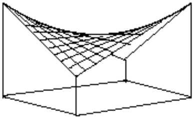


Singular Quadrics: Elliptic Cylinder, Hyperbolic Cylinder, Parabolic Cylinder

- The *elliptic cylinder* is the surface of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0. \tag{1.4}$$

or

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0, \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

- The *hyperbolic cylinder* is the surface of equation

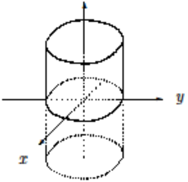
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0, \tag{1.5}$$

or

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0, \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0.$$

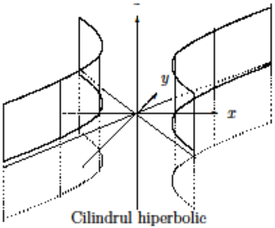
- The *parabolic cylinder* is the surface of equation

$$y^2 = 2px, \quad p > 0, \quad \text{(or an alternative equation)}. \tag{1.6}$$



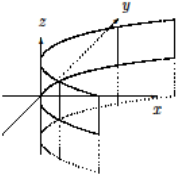
Cilindrul eliptic

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



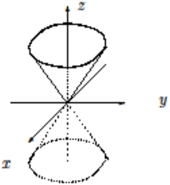
Cilindrul hiperbolic

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Cilindrul parabolic

$y^2 = 2px$



Conul $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

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