Dynamical systems. Final exam 20-06-2007

1. Find the general solution of $\ddot{\theta} + \dot{\theta} + \theta = 0$. Prove that $\lim_{t\to\infty} \theta(t) = 0$ for any solution θ of this differential equation.

2. Prove that $\lim_{t\to\infty} \theta(t) = 0$ for any solution θ of the differential equation $\ddot{\theta} + \dot{\theta} + \sin\theta = 0$ with $|\theta(0)|$ sufficiently small.

3. Find the general solution of the differential equation

$$y' = \frac{3x - y}{x + 3y}.$$

(Hint: write it in symmetrical form)

4. Specify the type and study the stability of the equilibrium $(0, 0)$ of the planar system $\dot{x} = x + 3y\,, \quad \dot{y} = 3x - y\,.$ Find also a first integral for this system.

5. Define the notion of first integral for a planar autonomous system.

6. Write the statements of the Existence Theorem of Peano and of the Local Existence and Uniqueness Theorem for a first order scalar Initial Value Problem.

7. Prove that the Initial Value Problem

$$y' = \frac{y}{x^2 - 2x + 1}\,, \quad y(0) = 2$$

has a unique maximal solution and than find it.