Seminar Nr.1, Euler's Functions

Euler's Gamma Function

Definition: $\Gamma:(0,\infty)\to(0,\infty)$

$$\Gamma(a) = \int_{0}^{\infty} x^{a-1} e^{-x} dx.$$

Properties:

- 1. $\Gamma(1) = 1$;
- 2. $\Gamma(a+1) = a\Gamma(a), \forall a > 0;$
- 3. $\Gamma(n+1) = n!$, $\forall n \in \mathbb{N}$;

4.
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{2} \int_{0}^{\infty} e^{-\frac{t^2}{2}} dt = \int_{\mathbb{R}} e^{-t^2} dt = \sqrt{\pi}.$$

Euler's Beta Function

Definition: $\beta:(0,\infty)\times(0,\infty)\to(0,\infty)$

$$\beta(a,b) = \int_{0}^{1} x^{a-1} (1-x)^{b-1} dx.$$

Properties:

- 1. $\beta(a,1) = \frac{1}{a}, \forall a > 0;$
- 2. $\beta(a,b) = \beta(b,a), \forall a,b > 0;$
- 3. $\beta(a,b) = \frac{a-1}{b}\beta(a-1,b+1), \forall a > 1, b > 0;$
- 4. $\beta(a,b) = \frac{b-1}{a+b-1}\beta(a,b-1) = \frac{a-1}{a+b-1}\beta(a-1,b), \forall a > 1, b > 1;$
- 5. $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \forall a > 0, b > 0.$