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Warming up Exercises

Analysis for CS

GROUPWORK:

(G 1) (Bernoulli-type inequalities with $\geq$)

a) Let $n \in \mathbb{N}^*$ and let $x_1, \ldots, x_n \in \mathbb{R}$ be real numbers satisfying the following properties:

(1) $x_i \geq -1$, for all $i \in \{1, \ldots, n\}$.

(2) $x_i x_j \geq 0$, for all $i, j \in \{1, \ldots, n\}$.

Prove that the **generalized Bernoulli-inequality**

$$(1 + x_1) \ldots (1 + x_n) \geq 1 + x_1 + \cdots + x_n$$

does hold.

b) Prove that for every $n \in \mathbb{N}^*$ and every real number $x \geq -1$ the **Bernoulli-inequality**

$$(1 + x)^n \geq 1 + nx$$

does hold.

(G 2) (AM-GM-HM inequalities)

Let $n \in \mathbb{N}^*$ and let $x_1, \ldots, x_n \in \mathbb{R}_+^*$. Prove the following sequence of inequalities

$$\min\{x_1, \ldots, x_n\} \leq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n} \leq \max\{x_1, \ldots, x_n\}.$$

Remarks. 1) The expression $\frac{x_1+\cdots+x_n}{n}$ is the *arithmetic mean* (AM, for short), $\sqrt[n]{x_1 \cdots x_n}$ is the *geometric mean* (GM, for short), and $\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}$ is the *harmonic mean* (HM, for short) of the positive reals $x_1, \ldots, x_n$.

2) It can be shown that equality holds in each of the inequalities of the above sequence if and only if $x_1 = \cdots = x_n$.

(G 3)

Let $n \in \mathbb{N}^*$.

a) Show that if the positive reals $x_1, \ldots, x_n > 0$ are so that their product $x_1 \cdots x_n = 1$, then $x_1 + \cdots + x_n \geq n$.

b) If $n \geq 2$, prove that $n! < \left(\frac{n+1}{2}\right)^n$. (We recall that $n!$, called the *factorial* of n, denotes the product $1 \cdot 2 \cdot \ldots \cdot n$.)

HOMEWORK:

(H 1) (To be delivered in the next exercise-class)

Let $n \in \mathbb{N}^*$. Compute the following sums in a direct way and prove afterwards (using mathematical induction) that the formula you have got does hold for every $n \in \mathbb{N}^*$.

(a) $1^2 + 2^2 + \cdots + n^2$,

(b) $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$.

(H 2) (Bernoulli-type inequalities with $>$)

a) Let $n \in \mathbb{N}$ with $n \geq 2$ and let $x_1, \ldots, x_n \in \mathbb{R}$ be real numbers satisfying the following properties:

(1) $x_i \geq -1$, for all $i \in \{1, \ldots, n\}$.

(2) $x_i x_j > 0$, for all $i, j \in \{1, \ldots, n\}$.

Prove that

$$(1 + x_1) \ldots (1 + x_n) > 1 + x_1 + \cdots + x_n.$$

b) Prove that for every $n \in \mathbb{N}$ with $n \geq 2$ and every nonzero real number $x \geq -1$

$$(1 + x)^n > 1 + nx.$$

(H 3)

Prove the following inequalities

a) $\left(1 + \frac{1}{n-1}\right)^n > \left(1 + \frac{1}{n}\right)^{n+1}$, $\forall n \in \mathbb{N}$ with $n \geq 2$,

b) $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$, $\forall n \in \mathbb{N}^*$.

(H 4) (The geometric interpretation of the AM–GM inequality)

Explain why for $n = 2$ the AM–GM inequality states that the square has the smallest perimeter amongst all rectangles of equal area.