Geometry[1]
First Year, Computer science

Assoc. Prof. Cornel-Sebastian PINTEA

"Babeş-Bolyai" University
Faculty of Mathematics and Computer Sciences
Cluj-Napoca, Romania

Lecture 8, 30.04.2014

[1] These notes are not in a final form. They are continuously being improved

Content

Conics (This chapter is done following [AnTo]) The Ellipse

Definition 1.1

An ellipse is the locus of points in a plane, the sum of whose distances from two fixed points, say F and F', called foci *is constant.*

The distance between the two fixed points is called the *focal distance*
Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$. If M is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

One may chose a Cartesian system of coordinates centered at the midpoint of the segment $[F'F]$, so that $F(c, 0)$ and $F'(-c, 0)$.

Remark 1.2

In $\Delta MFF'$ the following inequality $|MF| + |MF'| > |FF'|$ holds. Hence $2a > 2c$. Thus, the constants a and c must verify $a > c$.

The equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \tag{1.1}$$

where $b^2 = a^2 - c^2$.

Remark 1.3

The equation (1.1) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}; \qquad x = \pm\frac{a}{b}\sqrt{b^2 - y^2},$$

which means that the ellipse is symmetric with respect to both the x and the y axes. In fact, the line FF', determined by the foci of the ellipse, and the perpendicular line on the midpoint of the segment $[FF']$ are axes of symmetry for the ellipse. Their intersection point, which is the midpoint of $[FF']$, is the center of symmetry of the ellipse, or, simply, its center.

Remark 1.4

In order to sketch the graph of the ellipse, observe that it is enough to represent the function

$$f : [-a, a] \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{a^2 - x^2},$$

and to complete the ellipse by symmetry with respect to the x-axis.

One has

$$f'(x) = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}, \qquad f''(x) = -\frac{ab}{(a^2 - x^2)\sqrt{a^2 - x^2}}.$$

x	$-a$		0		a
$f'(x)$	$\vert$	$+\ +\ +$	0	$-\ -\ -$	$\vert$
$f(x)$	0	$\nearrow$	b	$\searrow$	0
$f''(x)$	$\vert$	$-\ -\ -$	$-$	$-\ -\ -$	$\vert$

The Hyperbola

Definiţia 1.5

The hyperbola is defined as the geometric locus of the points in the plane, whose absolute value of the difference of their distances to two fixed points, say F and F', have a constant difference.

The two fixed points are called the *foci* of the hyperbola, and the distance $|FF'| = 2c$ between the foci is the *focal distance*.
Suppose that the constant in the definition is $2a$. If $M(x, y)$ is an arbitrary point of the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

Choose a Cartesian system of coordinates, having the origine at the midpoint of the segment $[FF']$ and such that $F(c, 0)$, $F'(-c, 0)$.

Remark 1.6

In the triangle $\Delta MFF'$, $||MF| - |MF'|| < |FF'|$, so that $a < c$.

Let us determine the equation of a hyperbola. By using the definition we get $|MF| - |MF'| = \pm 2a$, namely

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a,$$

or, equivalently

$$\sqrt{(x-c)^2+y^2} = \pm 2a + \sqrt{(x+c)^2+y^2}.$$

We therefore have successively

$$x^2-2cx+c^2+y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2+y^2}+x^2+2cx+c^2+y^2 \iff$$

$$\iff cx + a^2 = \pm a\sqrt{(x+c)^2+y^2} \iff$$

$$\Longleftrightarrow c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 \Longleftrightarrow$$

$$\Longleftrightarrow (c^2 - a^2)x^2 - a^2y^2 - a^2(c^2 - a^2) = 0.$$

By using the notation $c^2 - a^2 = b^2$ ($c > a$) we obtain the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \tag{1.2}$$

The equation (1.2) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}; \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

Therefore, the coordinate axes are axes of symmetry of the hyperbola and the origin is a center of symmetry equally called the *center of the hyperbola*.

Remark 1.7

To sketch the graph of the hyperbola, is it enough to represent the function

$$f:(-\infty,-a]\cup[a,\infty)\to\mathbb{R},\qquad f(x)=\frac{b}{a}\sqrt{x^2-a^2},$$

by taking into account that the hyperbola is symmetric with respect to the x-axis.

Since $\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=\dfrac{b}{a}$ and $\lim\limits_{x\to-\infty}\dfrac{f(x)}{x}=-\dfrac{b}{a}$, it follows that $y=\dfrac{b}{a}x$ and $y=-\dfrac{b}{a}x$ are asymptotes of f.

One has, also

$$f'(x) = \frac{b}{a}\frac{x}{\sqrt{x^2 - a^2}}, \qquad f''(x) = -\frac{ab}{(x^2 - a^2)\sqrt{x^2 - a^2}}.$$

x	$-\infty$				$-a$				a				∞
$f'(x)$	$-$	$-$	$-$	$-$	$\vert$	$/$	$/$	$/$	$\vert$	$+$	$+$	$+$	$+$
$f(x)$	∞		$\searrow$		$0\vert$	$/$	$/$	$/$	$\vert 0$		$\nearrow$		∞
$f''(x)$	$-$	$-$	$-$	$-$	$\vert$	$/$	$/$	$/$	$\vert$	$-$	$-$	$-$	$-$

- [AnTo] Andrica, D., Ţopan, L., Analytic geometry, Cluj University Press, 2004.
- [GaRa] Galbură Gh., Radó, F., Geometrie, Editura didactică şi pedagogică-Bucureşti, 1979.
- [Pi] Pintea, C. Geometrie. Elemente de geometrie analitică. Elemente de geometrie diferenţială a curbelor şi suprafeţelor, Presa Universitară Clujeană, 2001.
- [ROGV] Radó, F., Orban, B., Groze, V., Vasiu, A., Culegere de Probleme de Geometrie, Lit. Univ. "Babeş-Bolyai", Cluj-Napoca, 1979.