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Solutions to Exercise Sheet no.12

Analysis for CS

(G 29)

a) We have that $f(x) = \frac{1}{x\sqrt{1+x^2}} > 0$ for all $x \geq 1$. Due to the fact that $L = \lim\limits_{x\to\infty} x^2 f(x) = 1 < \infty$ and $p = 2 > 1$, it follows that f is improperly integrable on $[1, \infty)$.

b) The function f is positive on $[0, \frac{\pi}{2})$. By applying L'Hospital's rules, we get

$$L = \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} \left(\frac{\pi}{2} - x\right) f(x) = \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} \frac{\frac{\pi}{2} - x}{\cos x} = \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} \frac{1}{\sin x} = 1.$$

Since $p = 1 \leq 1$, we conclude that f is not improperly integrable on $[0, \frac{\pi}{2})$.

c) We have that $f(x) > 0$ for all $x > 0$. Due to the fact that

$$\lim_{\substack{x\to 0\\ x>0}} x^0 f(x) = \lim_{\substack{x\to 0\\ x>0}} \left(\frac{\operatorname{arctg} x}{x}\right)^2 = 1,$$

we conclude that f is improperly integrable on $(0, 1]$. The relations

$$L = \lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} x^2 \left(\frac{\operatorname{arctg} x}{x}\right)^2 = \frac{\pi^2}{4} < \infty$$

and the fact that $p = 2 > 1$ lead us to the conclusion that f is improperly integrable on $[1, \infty)$. Hence f is improperly integrabe on $(0, \infty)$.

d) The function f is positive on its domain. From

$$\lim_{\substack{x\to 1\\ x>1}} \sqrt{x-1} f(x) = \lim_{\substack{x\to 1\\ x<1}} \frac{\sqrt{x-1}\ln x}{x\sqrt{x^2-1}} = 0$$

and $p = \frac{1}{2} < 1$ it follows that f is improperly integrable on $(1, 2]$. Since

$$L = \lim_{x\to\infty} x^{\frac{3}{2}} f(x) = \lim_{x\to\infty} \frac{x^2}{x\sqrt{x^2-1}} \cdot \frac{\ln x}{\sqrt{x}} = 0$$

and $p = \frac{3}{2} > 1$, it follows that f is improperly integrable on $[2, \infty)$, thus f is improperly integrable on $(1, \infty)$.

e) Note that $-1 < a < 1$ implies that $1 - a^2x^2 > 0$ for all $x \in [0, 1]$. As well, $f(x) > 0$ for all $x \in [0, 1)$. From

$$L = \lim_{\substack{x\to 1\\ x<1}} \sqrt{1-x} f(x) = \lim_{\substack{x\to 1\\ x<1}} \frac{\sqrt{1-x}}{\sqrt{(1-x^2)(1-a^2x^2)}} = \frac{1}{\sqrt{2}(1-a^2)} < \infty$$

and the fact that $p = \frac{1}{2} < 1$ it follows that f is improperly integrable on $[0, 1)$.

(G 30)

a) We consider the function $f\colon [2,\infty) \to \mathbb{R}$, defined by $f(x) = \frac{1}{x(\ln x)^2}$, which is decreasing on its domain, and we notice that

$$\int \frac{1}{x(\ln x)^2} dx = \int (\ln x)^{-2} \cdot (\ln x)' dx = -\frac{1}{\ln x} + \mathcal{C}.$$

Thus $F\colon [2,\infty) \to \mathbb{R}$, defined by $F(x) = -\frac{1}{\ln x}$, is an antiderivative of f. Moreover, $\lim\limits_{x\to\infty} F(x) = \lim\limits_{x\to\infty} -\dfrac{1}{\ln x} = 0$. Thus (by the formula of Leibniz-Newton for improper integrals) f is improperly integrable on $[2,\infty)$. From the integral criterion it follows that the series $\sum\limits_{n\geq 2} \dfrac{1}{n(\ln n)^2}$ is convergent.

b) We consider the function $f\colon [2,\infty) \to \mathbb{R}$, defined by $f(x) = \frac{\ln x}{x^2}$, which is decreasing on its domain (this may be checked by observing that f' is negative on$[2,\infty)$), and we notice that

$$\int \frac{\ln x}{x^2} dx = \int \ln x \cdot \left(-\frac{1}{x}\right)' dx = -\frac{1}{x} \cdot \ln x + \int \frac{1}{x^2} dx = -\frac{1}{x} \cdot \ln x - \frac{1}{x} + \mathcal{C}.$$

Thus $F\colon [2,\infty) \to \mathbb{R}$, defined by $F(x) = -\frac{1}{x} \cdot \ln x - \frac{1}{x}$, is an antiderivative of f. Moreover, by applying L'Hopital's rule we have

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} -\frac{\ln x + 1}{x} = -\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

Thus f is improperly integrable on $[2,\infty)$ (by the formula of Leibniz-Newton for improper integrals). From the integral criterion it follows that the series $\sum\limits_{n\geq 2} \dfrac{\ln n}{n^2}$ is convergent.

c) We consider the function $f\colon [1,\infty) \to \mathbb{R}$, defined by $f(x) = \frac{1}{\sqrt{1+e^x}}$, which is decreasing on its domain. Since

$$L = \lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} \sqrt{\frac{x^4}{1+e^x}} = 0$$

and $p = 2$, the second comparison criterion for improper integrals yields that f is improperly integrable on $[1,\infty)$. Thus we conclude from the integral criterion that the series $\sum\limits_{n\geq 1} \dfrac{1}{\sqrt{1+e^n}}$ is convergent.

Homework:

(H 31)

a) The function f is positive on its domain. Since

$$\lim_{\substack{x\to 0\\ x>0}} x^0 f(x) = \lim_{\substack{x\to 0\\ x>0}} \frac{\operatorname{arctg} x}{x(1+x^2)} = 1,$$

f is improperly integrable on $(0,1]$. The equalities

$$L = \lim_{x\to\infty} x^3 f(x) = \lim_{x\to\infty} x^3 \frac{\operatorname{arctg} x}{x(1+x^2)} = \frac{\pi}{2}$$

and the fact that $p = 3 > 1$ imply that f is improperly integrable on $[1,\infty)$ as well. Hence f is improperly integrable on $(0,\infty)$.

b) The function f is positive on its domain. Since

$$L=\lim_{x\to\infty}x^{\frac{3}{2}}f(x)=\lim_{x\to\infty}\frac{x^{\frac{3}{2}}}{\sqrt{x(x-a)(x-b)}}=1$$

and $p=\frac{3}{2}>1$, it follows that f is improperly integrable on pe (x_0,∞).

(H 32)

We have for all $(x,y)\in\mathbb{R}^*\times\mathbb{R}^*$ that $\frac{\partial f}{\partial x}(x,y)=y-\frac{1}{x^2}$ and $\frac{\partial f}{\partial y}(x,y)=x-\frac{1}{y^2}$. Thus the stationary points of f are the solutions of the system

$$\begin{cases} y=\frac{1}{x^2} \\ x=\frac{1}{y^2}.\end{cases}$$

It follows that $x^4=x$. Since $x\in\mathbb{R}^*$, we conclude that $x=1$. Hence $(1,1)$ is the only stationary point of f. We have for all $(x,y)\in\mathbb{R}^*\times\mathbb{R}^*$ that

$$H_f(x,y)=\begin{pmatrix}\frac{2}{x^3} & 1\\ 1 & \frac{2}{y^3}\end{pmatrix},$$

thus $H_f(1,1)$ is positive definite. We conclude that $(1,1)$ is a local minimum of f and that $f(1,1)=3$ is the corresponding extreme value.

(H 33)

Without any loss of generality we may assume that $a>0$.

1° Using the definition of the limit of a function at a point, there exists a real $c\geq a$ such that $x^pf(x)\leq L+1$, for all $x\geq c$. Thus

$$0\leq f(x)\leq\frac{L+1}{x^p},\forall\, x\geq c.$$

We know from the exercise-class no. 11 that the function $x\in[a,\infty)\mapsto\frac{L+1}{x^p}\in\mathbb{R}$ is improperly integrable on $[a,\infty)$, since $p>1$. Using assertion 1° of the first comparison criterion for improper integrals (see **Th1** in lecture no. 12), we conclude that f is improperly integrable on $[a,\infty)$.

2° Fix a real number r such that $0<r<L$. Using once again the definition of the limit of a function at a point, there exists a positive real $c\geq a$ such that $r\geq x^pf(x)$, for all $x\geq c$. Thus

$$0<\frac{r}{x^p}\leq f(x),\forall\, x\geq c.$$

We know from the exercise-class no. 11 that the function $x\in[a,\infty)\mapsto\frac{r}{x^p}\in\mathbb{R}$ is not improperly integrable on $[a,\infty)$, since $p\leq 1$. Using assertion 2° of the first comparison criterion for improper integrals (see **Th1** in lecture no. 12), we conclude that f is not improperly integrable on $[a,\infty)$.