COURSE 2

2.2. Lagrange interpolation

Let $[a,b] \subset \mathbb{R}$, $x_i \in [a,b]$, $i = 0, 1, ..., m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a,b] \to \mathbb{R}$.

The Lagrange interpolation polynomial is given by

$$(L_m f)(x) = \sum_{i=0}^{m} \ell_i(x) f(x_i). \tag{1}$$

We have

$$u(x) = \prod_{j=0}^{m} (x - x_j), \qquad u_i(x) = \frac{u(x)}{x - x_i}$$

and

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)}, \quad i = 0, 1, ..., m. \tag{2}$$

Example 1 *1) Consider the nodes x_0, x_1 and a function f to be interpolated.*

We have $m = 1$,

$$\begin{aligned} u(x) &= (x - x_0)(x - x_1) \\ u_0(x) &= x - x_1 \\ u_1(x) &= x - x_0 \end{aligned}$$

$$\begin{aligned} (L_1 f)(x) &= l_0(x) f(x_0) + l_1(x) f(x_1) \\ &= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1), \end{aligned}$$

which is the line passing through the given points $(x_0, f(x_0))$ *and* $(x_1, f(x_1))$.

2) Find the Lagrange polynomial that interpolates the data in the following table and find the approximative value of $f(-0.5)$.

x	-1	0	3
$f(x)$	8	-2	4

Sol. *We have* $m = 2$. *The Lagrange polynomial is*

$$(L_2 f)(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2).$$

$u(x) = (x+1)(x-0)(x-3)$ *and it follows*

$$l_0(x) = \frac{(x-0)(x-3)}{(-1-0)(-1-3)} = \frac{1}{4}x(x-3)$$
$$l_1(x) = \frac{(x+1)(x-3)}{(0+1)(0-3)} = -\frac{1}{3}(x+1)(x-3)$$
$$l_2(x) = \frac{(x+1)(x-0)}{(3+1)(3-0)} = \frac{1}{12}x(x+1),$$

The polynomial is

$$(L_2 f)(x) = 2x(x-3) + \frac{2}{3}(x+1)(x-3) + \frac{1}{3}x(x+1).$$

and $(L_2 f)(-0.5) = 2.25$.

Remark 2 *Disadvantages of the form (1) of Lagrange polynomial: requires many computations and if we add or substract a point we have to start with a complete new set of computations.*

Some calculations allow us to reduce the number of operations:

$$(L_m f)(x) = \frac{(L_m f)(x)}{1} = \frac{\sum\limits_{i=0}^{m} l_i(x) f(x_i)}{\sum\limits_{i=0}^{m} l_i(x)}.$$

Dividing the numerator and the denominator by

$$u(x) = \prod_{i=1}^{m} (x - x_i)$$

and denoting

$$A_i = \frac{1}{\prod\limits_{j=0, j\neq i}^{m} (x_i - x_j)} = \frac{1}{u_i(x_i)}$$

one obtains

$$(L_m f)(x) = \frac{\sum\limits_{i=0}^{m} \frac{A_i f(x_i)}{x - x_i}}{\sum\limits_{i=0}^{m} \frac{A_i}{x - x_i}}, \tag{3}$$

called **the barycentric form** *of Lagrange interpolation polynomial*.

Remark 3 *Formula (3) needs half of the number of arithmetic operations needed for (1) and it is easier to add or substract a point.*

The Lagrange polynomial generates **the Lagrange interpolation formula**

$$f = L_m f + R_m f,$$

where $R_m f$ denotes **the remainder** (**the error**).

Theorem 4 *Let* $\alpha = \min\{x, x_0, ..., x_m\}$ *and* $\beta = \max\{x, x_0, ..., x_m\}$. *If* $f \in C^m[\alpha, \beta]$ *and* $f^{(m)}$ *is derivable on* (α, β) *then* $\forall x \in (\alpha, \beta)$, *there exists* $\xi \in (\alpha, \beta)$ *such that*

$$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi). \tag{4}$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_m f)(z) \\ u(x) & (R_m f)(x) \end{vmatrix}.$$

From hypothesis it follows that $F \in C^m[\alpha, \beta]$ and there exists $F^{(m+1)}$ on (α, β).

We have

$$F(x) = 0, \; F(x_i) = 0, \qquad i = 0, 1, ..., m,$$

as

$$u(x_i) = \prod_{j=0}^{m} (x_i - x_j) = 0$$

and

$$(R_m f)(x_i) = f(x_i) - (L_m f)(x_i) = f(x_i) - f(x_i) = 0,$$

so F has $m+2$ distinct zeros in (α, β). Applying successively the Rolle theorem it follows that: F has $m+2$ zeros in $(\alpha, \beta) \Rightarrow F'$ has at least $m+1$ zeros in $(\alpha, \beta) \Rightarrow ... \Rightarrow F^{(m+1)}$ has at least one zero in (α, β)

So $F^{(m+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(m+1)}(\xi) = 0$.

We have

$$F^{(m+1)}(z) = \begin{vmatrix} u^{(m+1)}(z) & (R_m f)^{(m+1)}(z) \\ u(x) & (R_m f)(x) \end{vmatrix},$$

with

$$u(z) = \prod_{i=0}^{m} (z - z_i) \Rightarrow u^{(m+1)}(z) = (m+1)!,$$

and

$$\begin{aligned} (R_m f)^{(m+1)}(z) &= (f - (L_m f))^{(m+1)}(z) \\ &= f^{(m+1)}(z) - (L_m f)^{(m+1)}(z) = f^{(m+1)}(z) \end{aligned}$$

(as, $L_m f \in \mathbb{P}_m$).

We have $F^{(m+1)}(\xi) = 0$, for $\xi \in (\alpha, \beta)$, so

$$F^{(m+1)}(\xi) = \begin{vmatrix} (m+1)! & f^{(m+1)}(\xi) \\ u(x) & (R_m f)(x) \end{vmatrix} = 0,$$

i.e., $(m+1)!(R_m f)(x) = u(x) f^{(m+1)}(\xi)$, whence $(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi)$.
■

Corollary 5 *If $f \in C^{m+1}[a,b]$ then*

$$|(R_m f)(x)| \leq \frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty, \qquad x \in [a,b]$$

where $\|\cdot\|_\infty$ denotes the uniform norm, and $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$.

Example 6 *Which is the limit of the error for computing $\sqrt{115}$ using Lagrange interpolation formula for $f(x) = \sqrt{x}$ and $x_0 = 100$, $x_1 = 121$ and $x_2 = 144$? Find the approximative value of $\sqrt{115}$.*

Example 7 *If we know that $\lg 2 = 0.301$, $\lg 3 = 0.477$, $\lg 5 = 0.699$, find $\lg 76$. Study the approximation error.*

The Aitken's algorithm

Let $[a, b] \subset \mathbb{R}$, $x_i \in [a, b]$, $i = 0, 1, ..., m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a, b] \to \mathbb{R}$.

A practical method for computing the Lagrange polynomial is **the Aitken's algorithm.** This consists in generating the table:

$$
\begin{array}{c|c|c|cccc}
x_0 & f_{00} & & & & & \\
x_1 & f_{10} & f_{11} & & & & \\
x_2 & f_{20} & f_{21} & f_{22} & & & \\
x_3 & f_{30} & f_{31} & f_{32} & f_{33} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
x_m & f_{m0} & f_{m1} & f_{m2} & f_{m3} & \cdots & f_{mm}
\end{array}
$$

where

$$f_{i0} = f(x_i), \quad i = 0, 1, ..., m,$$

and

$$f_{i,j+1} = \frac{1}{x_i - x_j} \begin{vmatrix} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix}, \qquad i = 0, 1, ..., m; j = 0, ..., i - 1.$$

For example,

$$
\begin{aligned}
f_{11} &= \frac{1}{x_1 - x_0} \begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} \\
&= \frac{1}{x_1 - x_0} [f_{00}(x_1 - x) - f_{10}(x_0 - x)] \\
&= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) = (L_1 f)(x),
\end{aligned}
$$

so f_{11} is the value in x of Lagrange polynomial for the nodes x_0, x_1. We have

$$f_{ii} = (L_i f)(x),$$

$L_i f$ being Lagrange polynomial for the nodes $x_0, x_1, ..., x_i$.

So $f_{11}, f_{22}, ..., f_{ii}, ..., f_{mm}$ is a sequence of approximations of f.

If the interpolation procedure is convergent then the sequence is also convergent, i.e., $\lim\limits_{m \to \infty} f_{mm} = f(x)$. By Cauchy convergence criterion it follows

$$\lim_{i \to \infty} |f_{ii} - f_{i-1,i-1}| = 0.$$

This could be used as a stopping criterion, i.e.,

$$\left|f_{ii} - f_{i-1,i-1}\right| \leq \varepsilon, \quad \varepsilon = \text{error}.$$

Recommendation is to sort the nodes $x_0, x_1, ..., x_m$ with respect to the distance to x, such that

$$|x_i - x| \leq \left|x_j - x\right| \text{ if } i < j, \quad i, \quad j = 1, ..., m.$$

Example 8 *Approximate* $\sqrt{115}$ *with precision* $\varepsilon = 10^{-3}$, *using Aitken's algorithm.*