Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

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Solutions to Exercise Sheet no.1

Analysis for CS

(G 4)

a) An element $x \in \mathbb{R}$ is not a lower bound of S if $\exists s \in S$ such that $s < x$. An element $x \in \mathbb{R}$ is not an upper bound of S if $\exists s \in S$ such that $s > x$.

b)

S	LB(S)	UB(S)	$\min S$	$\max S$	$\inf S$	$\sup S$
$\emptyset$	$\mathbb{R}$	$\mathbb{R}$	$\nexists$	$\nexists$	∞	$-\infty$
$(-5,3) \cup [4,+\infty)$	$(-\infty,-5]$	$\emptyset$	$\nexists$	$\nexists$	-5	∞
$(-2,4) \cup \{5\}$	$(-\infty,-2]$	$[5,\infty)$	$\nexists$	5	-2	5
$(-\infty,0] \cup \{1,2\}$	$\emptyset$	$[2,\infty)$	$\nexists$	2	$-\infty$	2
$(-2,3) \cap \mathbb{Z}$	$(-\infty,-1)$	$[2,\infty)$	-1	2	-1	2
$\mathbb{N}$	$(-\infty,0]$	$\emptyset$	0	$\nexists$	0	∞
$(-2,\sqrt{3}) \cap \mathbb{Q}$	$(-\infty,-2]$	$[\sqrt{3},\infty)$	$\nexists$	$\nexists$	-2	$\sqrt{3}$
$\{x \in \mathbb{R} \mid x^3 - x^2 - 6x \geq 0\}$	$(-\infty,-2]$	$\emptyset$	-2	$\nexists$	-2	∞

For the last set note that $x^3 - x^2 - 6x \geq 0 \Leftrightarrow x(x-3)(x+2) \geq 0 \Leftrightarrow x \in [-2,0] \cup [3,\infty)$.

c) Take for instance $S = (-\infty,-5] \cup (-2,-1)$.

(G 5)

Let $S \subseteq \mathbb{R}$.

a) If $x \in$UB(S) then $[x,\infty) \subseteq$UB(S), thus UB(S) contains infinitely many elements.

b) Since S has a greatest element, S is nonempty and bounded above, thus $\sup S \in \mathbb{R}$. As $\max S \in S$ and $\sup S \in UB(S)$ we have from the definition of the upper bound that

$$\max S \leq \sup S. \tag{1}$$

From the definition of the greatest element we have that $\max S \in S \cap UB(S)$. Since $\sup S$ is the least upper bound, and $\max S \in UB(S)$, it holds

$$\sup S \leq \max S. \tag{2}$$

Hence, from (1) and (2), we get the desired conclusion, i.e., $\max S = \sup S$.

c) When there is no greatest element, the statement is obvious. Let us now consider the case when $m_1 \in S$ and $m_2 \in S$ are such that they are both greatest elements of S.

From m_1 being the greatest element of S and $m_2 \in S$ we have

$$m_2 \leq m_1. \tag{3}$$

From m_2 being the greatest element of S and $m_1 \in S$ we have

$$m_1 \leq m_2. \tag{4}$$

Hence, from (3) and (4), we get the desired conclusion, i.e., $m_1 = m_2$.

d) If $S = \emptyset$, then, by definition, $-\infty$ is the supremum of S. If S is unbounded above, then ∞ is the supremum of S. In this case the supremum cannot be a real number, since that would imply the boundedness from above of S.

Suppose now that S is nonempty and bounded above. In this case the supremum of S cannot be ∞. Assume that a and b are reals and both suprema of S. Note that in particular both a and b are then upper bounds of S. Since a is a least upper bound of S and b is an upper bound of S, $a \leq b$. Similarly, since b is a least upper bound and a is an upper bound of S, $b \leq a$. Thus $a = b$.

In conclusion, the supremum of a set is unique.

HOMEWORK:

(H 5)

a)

A	LB(A)	UB(A)	$\min A$	$\max A$	$\inf A$	$\sup A$
$\mathbb{R}_+$	$(-\infty, 0]$	$\emptyset$	0	$\nexists$	0	∞
$\mathbb{Q}^*$	$\emptyset$	$\emptyset$	$\nexists$	$\nexists$	$-\infty$	∞
$[-2,1) \cup (2,\infty)$	$(-\infty, -2]$	$\emptyset$	-2	$\nexists$	-2	∞
$(-\infty,-1) \cup (2,3)$	$\emptyset$	$[3, +\infty]$	$\nexists$	$\nexists$	$-\infty$	3
$(-2,5) \cap \mathbb{N}$	$(-\infty, 0]$	$[4, +\infty)$	0	4	0	4
$\mathbb{Z}$	$\emptyset$	$\emptyset$	$\nexists$	$\nexists$	$-\infty$	∞
$(-\infty, 5] \cap \mathbb{Q}$	$\emptyset$	$[5, +\infty)$	$\nexists$	5	$-\infty$	5
$\left\{x \in \mathbb{R} \mid \frac{x+1}{x^2+1} < 1\right\}$	$\emptyset$	$\emptyset$	$\nexists$	$\nexists$	$-\infty$	∞

For the last set note that $\frac{x+1}{x^2+1} < 1 \Leftrightarrow (x+1) - (x^2+1) < 0 \Leftrightarrow x \in (-\infty, 0) \cup (1, +\infty)$.

b) Take for instance $S = (3,4] \cup (5, +\infty)$ or $S = (3, +\infty) \cap \mathbb{Q}$.

(H 6)

The proofs are almost similar to those done in (G 5).

(H 7)

C4: Let T be a subset of $\mathbb{R}$ which is bounded below and let S be a nonempty subset of T. Then S is also bounded below and the inequality $\inf T \leq \inf S$ does hold.

Proof: From $\emptyset \neq S \subseteq T \Rightarrow T \neq \emptyset$. As T is nonempty, from **Th3** in the first lecture we know that $\exists \inf T \in \mathbb{R}$. Since $\inf T \in \mathrm{LB}(T)$, we have that $\inf T \leq t, \forall t \in T$, and thus, due to the fact that $S \subseteq T$, we have $\inf T \leq s$ for all $s \in S$. This means that $\inf T \in \mathrm{LB}(S)$, implying that S is bounded below. As $S \neq \emptyset$, **Th3** in the first lecture yields that $\exists \inf S \in \mathbb{R}$. Since $\inf T \in \mathrm{LB}(S)$, we finally get $\inf T \leq \inf S$.