Geometry[1]
First Year, Computer science

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[1]These notes are not in a final form. They are continuously being improved

Content

Cartesian and affine reference systems

A basis of the direction $\vec{\pi}$ of the plane π, or for the vector space $\mathcal{V}$ is an ordered basis $[\vec{e}, \vec{f}]$ of π, or an ordered basis $[\vec{u}, \vec{v}, \vec{w}]$ a of $\mathcal{V}$.
If $b = [\vec{u}, \vec{v}, \vec{w}]$ is a basis of $\mathcal{V}$ and $\vec{x} \in \mathcal{V}$, recall that the column vector of $\vec{x}$ with respect to b is being denoted by $[\vec{x}]_b$. In other words

$$[\vec{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

whenever $\vec{x} = x_1\vec{u} + x_2\vec{v} + x_3\vec{w}$.

Definition 1.1

A cartesian reference system *of the space $\mathcal{P}$, is a system $R = (O, \vec{u}, \vec{v}, \vec{w})$ where O is a point from $\mathcal{P}$ called the* origin *of the reference system and $b = [\vec{u}, \vec{v}, \vec{w}]$ is a basis of the vector space $\mathcal{V}$.*

Denote by E_1, E_2, E_3 the points for which $\vec{u} = \overrightarrow{OE}_1,\ \vec{v} = \overrightarrow{OE}_2,\ \vec{w} = \overrightarrow{OE}_3$.

Definition 1.2

The system of points (O, E_1, E_2, E_3) is called the affine reference system associated to the cartesian reference system $R = (O, \vec{u}, \vec{v}, \vec{w})$.

The straight lines $OE_i,\ i \in \{1, 2, 3\}$, oriented from O to E_i are called *the coordinate axes*. The coordinates x, y, z of the position vector $\vec{r}_M = \overrightarrow{OM}$ with respect to the basis $[\vec{u}, \vec{v}, \vec{w}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y, z)$.

Also, for the column matrix of coordinates of the vector $\vec{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\vec{r}_M = x\vec{u} + y\vec{v} + z\vec{w}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Remark 1.3

If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are two points, then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= x_B\vec{u} + y_B\vec{v} + z_B\vec{w} - (x_A\vec{u} + y_A\vec{v} + z_A\vec{w}) \\ &= (x_B - x_A)\vec{u} + (y_B - y_A)\vec{v} + (z_B - z_A)\vec{w}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

The cartesian equations of the straight lines and planes

Let Δ be a straight line passing through the point $A_0(x_0, y_0, z_0)$ which is parallel to the vector $\vec{d}(p, q, r)$. Its vector equation is

$$\vec{r}_M = \vec{r}_{A_0} + \lambda\vec{d}. \tag{2.1}$$

Denoting by x, y, z the coordinates of the generic point M of the straight line Δ, its vector equation (2.1) is equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases}, \quad \lambda \in \mathbb{R} \tag{2.2}$$

The relations (2.2) are being called the *parametric equations* of the straight line Δ and they are equivalent to the following relations

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r} \tag{2.3}$$

If $r = 0$, for instance, the canonical equations of the straight line Δ are

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \ \wedge \ z = z_0.$$

If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are different pointsof the straight line Δ, then $\overrightarrow{AB}\,(x_B - x_A, y_B - y_A, z_B - z_A)$ is a director vector of Δ, its canonical equations having, in this case, the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}. \qquad (2.4)$$

Te cartesian equations of the planes

Let $A_0(x_0, y_0, z_0) \in \mathcal{P}$ and $\vec{d}_1(p_1, q_1, r_1), \vec{d}_2(p_2, q_2, r_2) \in \mathcal{V}$ be linearly independent vectors, that is

$$\text{rang}\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$$

The vector equation of the plane π passing through A_0 which is parallel to the vectors $\vec{d}_1(p_1, q_1, r_1), \vec{d}_2(p_2, q_2, r_2)$ is

$$\vec{r}_M = \vec{r}_{A_0} + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{2.5}$$

If we denote by x, y, z the coordinates of the generic point M of the plane π, then the vector equation (2.5) is the equivalent to the following system of relations

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases},\ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{2.6}$$

The relations (2.6) reprezint a characterization of the points of the points of the plane π called the *parametric equations* of the plane π. More precisely, the compatibility of the linear system (2.6) with the unknowns λ_1, λ_2 is a necessary and sufficient condition for the point $M(x, y, z)$ to be contained within the plane π. On the other hand the compatibility of the linear system (2.6) is equivalent to the relations

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ p_1 & q_1 & z_1 \\ p_2 & q_2 & z_2 \end{vmatrix} = 0. \tag{2.7}$$

and express the fact that the rank of the matrix of the system is equal to the rank of the extended matrix of the system. The condition (2.7) is a characterization of the points of the plane π expressed in terms of the cartesian coordinates of the generic point M and is called the *cartesian equation* of the plane π.

If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$, $C(x_C, y_C, z_C)$ are noncollinear points, then the plane (ABC) determined by the three points can be viewed as the plane passing through the point A which is parallel to the vectors $\vec{d}_1 = \overrightarrow{AB}$, $\vec{d}_2 = \overrightarrow{AC}$. The coordinates of the vectors $\vec{d}_1$ şi $\vec{d}_2$ are

$(x_B - x_A, y_B - y_A, z_B - z_A)$ and $(x_C - x_A, y_C - y_A, z_C - z_A)$ respectively.

Thus, the equation of the plane (ABC) is

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0, \qquad (2.8)$$

or, echivalently

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \qquad (2.9)$$

On can put the equation (2.7) in the form

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \text{ or} \tag{2.10}$$

$$Ax+By+Cz+D=0, \tag{2.11}$$

where the coefficients A, B, C satisfy the relation $A^2+B^2+C^2>0$. It is also easy to show that every equation of the form (2.11) reprezints the equation of a plane. Indeed, if $A \neq 0$, then the equation (2.11) is equivalent to

$$\begin{vmatrix} x+\frac{D}{A} & y & z \\ B & -A & 0 \\ C & 0 & -A \end{vmatrix} = 0.$$

We observe that one can put the equation (2.10) in the form

$$AX+BY+CZ=0 \tag{2.12}$$

where $X=x-x_0,\ Y=y-y_0,\ Z=z-z_0$ are the coordinates of the vector $\overrightarrow{A_0M}$.

Analytic conditions of parallelism

The equation $AX + BY + CZ = 0$ is a necessary and sufficient condition for the vector $\overrightarrow{A_0M}\,(X, Y, Z)$ to be contained within the direction of the plane

$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Thus the *equation of the director subspace* $\vec{\pi} = \{\overrightarrow{A_0M} \mid M \in \pi\}$ is $AX + BY + CZ = 0$.

Proposition 3.1

The straight line

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

is parallel to the plane $\pi : Ax + By + Cz + D = 0$ *iff*

$$Ap + Bq + Cr = 0 \tag{3.1}$$

Proposition 3.2

Consider the planes $(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0$, $(\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0$. *Then* $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) \in \{1, 2\}$ *and the following statemenets are equivalent*

1. $\pi_1 \| \pi_2$.
2. $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 2$, *i.e.* $\vec{\pi}_1 = \vec{\pi}_2$.
3. *rang* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1$.
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly dependent.*

Corollary 3.3

Consider the planes $(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0$, $(\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0$. *The following statements are equivalent*

1. $\pi_1 \not\parallel \pi_2$.
2. $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 1$.
3. *rang* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$.
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly independent.*

By using Proposition (3.1) we shall find a necessary and sufficient condition for a vector to be contained within the direction of a straight line which is given as the intersection of two planes.

Consider the planes $(\pi_1)\, A_1 x + B_1 y + C_1 z + D_1 = 0$, $(\pi_2)\, A_2 x + B_2 y + C_2 z + D_2 = 0$ such that

$$\operatorname{rang}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2,$$

alongside their intersectio9n straight line $\Delta = \pi_1 \cap \pi_2$ of equations

$$(\Delta)\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_1 = 0. \end{cases}$$

Thus, $\vec{\Delta} = \vec{\pi}_1 \cap \vec{\pi}_2$ and therefore, by means of Proposition (3.1), it follows that the equations of $\vec{\Delta}$ are

$$(\vec{\Delta})\begin{cases} A_1 X + B_1 Y + C_1 Z = 0 \\ A_2 X + B_2 Y + C_2 Z = 0. \end{cases} \tag{3.2}$$

By solvyng the system (3.2) one can therefore deduce that $\vec{d}(p,q,r) \in \vec{\Delta} \Leftrightarrow \ \exists \lambda \in \mathbb{R}$ such that

$$(p,q,r) = \lambda \left(\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right). \tag{3.3}$$

The relation is usually (3.3) written in the form

$$\frac{p}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{q}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{r}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \tag{3.4}$$

Let us mention that the chosen values for (p,q,r) are usually precisely

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \text{ şi } \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}.$$

Pencils of planes

Definition 4.1

The collection of all planes containing a given straight line

$$(\Delta)\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$$

is called the pencil of planes *through* Δ.

Proposition 4.2

The plane π belongs to the pencil of planes through the straight line Δ if and only if there exists $\lambda, \mu \in \mathbb{R}$ such that the equation of the plane π is

$$\lambda(A_1x+B_1y+C_1z+D_1)+\mu(A_2x+B_2y+C_2z+D_2)=0. \tag{4.1}$$

Remark 4.3

The family of planes

$$A_1x + B_1y + C_1z + D_1 + \lambda(A_2x + B_2y + C_2z + D_2) = 0,$$

where λ covers the whole real line $\mathbb{R}$, is the so called reduced pencil of planes *through Δ and it consists in all planes through Δ except the plane of equation $A_2x + B_2y + C_2z + D_2 = 0$.*