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Solutions to Exercise Sheet no.8

Analysis for CS

(G 21)

a) We have that

$\langle u+v, v\rangle = \langle u, v\rangle + \langle v, v\rangle = \alpha + \gamma^2$,

$\langle u, 2u-3v\rangle = \langle u, 2u\rangle - \langle u, 3v\rangle = 2\langle u, u\rangle - 3\langle u, v\rangle = 2\beta^2 - 3\alpha$,

$$\begin{aligned} ||u-v|| &= \sqrt{\langle u-v, u-v\rangle} = \sqrt{\langle u-v, u\rangle - \langle u-v, v\rangle} = \sqrt{\langle u, u\rangle - \langle v, u\rangle - \langle u, v\rangle + \langle v, v\rangle} \\ &= \sqrt{\langle u, u\rangle - 2\langle v, u\rangle + \langle v, v\rangle} = \sqrt{\beta^2 - 2\alpha + \gamma^2}. \end{aligned}$$

b1) We have that

$\alpha = \langle(-1,2,3),(-2,1,-3)\rangle = (-1)(-2) + 2\cdot 1 + 3\cdot(-3) = 2+2-9 = -5$,

$\beta = \sqrt{\langle(-1,2,3),(-1,2,3)\rangle} = \sqrt{(-1)^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$,

$\gamma = \sqrt{\langle(-2,1,-3),(-2,1,-3)\rangle} = \sqrt{(-2)^2+1^2+(-3)^2} = \sqrt{4+1+9} = \sqrt{14}$.

b2) We have that

$$v \notin B(u,r) \iff ||u-v|| \geq r.$$

We use a) and b1) to get that $||u-v|| = \sqrt{\beta^2-2\alpha+\gamma^2} = \sqrt{14+10+14} = \sqrt{38}$. In conclusion, $r \in (0, \sqrt{38}]$.

b3) We have that

$$(1,-1,t) \in \overline{B}(u,5) \iff ||(1,-1,t)-u|| \leq 5$$

and

$$||(1,-1,t)-u|| = ||(1,-1,t)-(-1,2,3)|| = ||(2,-3,t-3)|| = \sqrt{4+9+(t-3)^2}.$$

Hence $||u-(1,-1,t)|| \leq 5 \Leftrightarrow \sqrt{13+(t-3)^2} \leq 5 \Leftrightarrow (t-3)^2 \leq 12 \Leftrightarrow t \in [3-2\sqrt{3}, 3+2\sqrt{3}]$.

(G 22)

a) Since the sequence $((-1)^k)_{k\in\mathbb{N}^*}$ is divergent, the sequence $(x^k)_{k\in\mathbb{N}^*}$ is divergent, too.

b) The equalities $\lim\limits_{k\to\infty} \dfrac{2^k}{k!} = 0$, $\lim\limits_{k\to\infty} \dfrac{1-4k^7}{k^7+12k} = -4$ and $\lim\limits_{k\to\infty} \dfrac{\sqrt{k}}{e^{3k}} = 0$ yield that the sequence $(x^k)_{k\in\mathbb{N}^*}$ converges to $(0,-4,0)$.

c) Since the sequence $(-k^3+k)_{k\in\mathbb{N}^*}$ is divergent, the sequence $(x^k)_{k\in\mathbb{N}^*}$ is divergent, too.

d) We have that

$$\lim_{k\to\infty} \frac{2^{2k}}{\left(2+\frac{1}{k}\right)^{2k}} = \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{2k}\right)^{2k}} = \frac{1}{e} \text{ and } \lim_{k\to\infty} \frac{1}{\sqrt[k]{k!}} = 0.$$

Denote by $a_k := (e^k + k)^{\frac{1}{k}}$, for $k \in \mathbb{N}^*$. Then $\ln a_k = \frac{\ln(e^k+k)}{k}$. Using L'Hospital's rules, we compute

$$\lim_{x\to\infty} \frac{\ln(e^x + x)}{x} = \lim_{x\to\infty} \frac{e^x+1}{e^x+x} = \lim_{x\to\infty} \frac{e^x}{e^x+1} = 1.$$

Thus $\lim\limits_{k\to\infty} \ln a_k = 1$, so $\lim\limits_{k\to\infty} a_k = e$. Furthermore we have that

$$\lim_{k\to\infty} \frac{\alpha^k}{k} = \begin{cases} 0, \text{ if } \alpha \in [0,1] \\ \infty, \text{ if } \alpha > 1. \end{cases}$$

So, if $\alpha > 1$, the sequence $(x^k)_{k\in\mathbb{N}^*}$ is divergent, and, if $\alpha \in [0,1]$, the sequence $(x^k)_{k\in\mathbb{N}^*}$ converges to $\left(\frac{1}{e}, 0, e, 0\right)$.

(G 23)

Let $(x^k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ having a limit. We assume by contradiction that this sequence has two limits $x, y \in \mathbb{R}^n$, $x \neq y$. By **L1** in the lecture no. 8, there exist $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$ such that $U \cap V = \emptyset$. Using twice the definition of the limit of a sequence in terms of neighborhoods (given in the exercise-class), we have that there exist $k(U), k(V) \in \mathbb{N}$ such that $x^k \in U$, for every $k \geq k(U)$, and $x^k \in V$, for every $k \geq k(V)$. For $k := \max\{k(U), k(V)\}$ we then have that $x^k \in U \cap V$, a contradiction. Therefore, $(x^k)_{k\in\mathbb{N}}$ has exactly one limit.

HOMEWORK:

(H 20)

a) Note that $f(0) = 0$. For every $x \in \mathbb{R}$ we have that

$$f'(x) = e^{2x}(2\sin x + \cos x) \text{ and } f''(x) = e^{2x}(3\sin x + 4\cos x),$$

hence $f'(0) = 1$ and $f''(0) = 4$. It follows that $T_2(x, 0) = x + 2x^2$.

b) For every $x \in \mathbb{R}$ we have that $f^{(3)}(x) = e^{2x}(2\sin x + 11\cos x)$. By Taylor's formula there exists a point c strictly between x and 0 such that $R_2(x, 0) = \frac{f^{(3)}(c)}{3!}x^3$.

c) It follows easily (using, for instance, mathematical induction) that $(e^{2x})^{(n)} = 2^n e^{2x}$, for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

d) Applying the formula of Leibniz, we get that

$$f^{(n)}(x) = e^{2x} \sum_{k=0}^{n} C_n^k 2^{n-k} \sin\left(x + k\frac{\pi}{2}\right),$$

for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

(H 21)

a) It is easy to prove by mathematical induction that

$$f^{(n)}(x) = (-1)^n n! x^{-(n+1)} = \frac{(-1)^n n!}{x^{n+1}},$$

for all $n \in \mathbb{N}$ (even for $n = 0$) and for all $x > 0$.

b) Since $f^{(k)}(1) = (-1)^k k!$, we obtain that

$$T_n(x, 1) = \sum_{k=0}^{n} \frac{f^k(1)}{k!}(x-1)^k = \sum_{k=0}^{n} (-1)^k (x-1)^k = \sum_{k=0}^{n} (1-x)^k.$$

c) According to Taylor's formula, there exists c strictly between x and 1 such that

$$R_n(x,1) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1} = \frac{(-1)^{n+1}(x-1)^{n+1}}{c^{n+2}}.$$

(H 22)

We have that

$$||x+y||^2 = \langle x+y, x+y\rangle = ||x||^2 + \langle x,y\rangle + \langle y,x\rangle + ||y||^2$$

and

$$||x-y||^2 = \langle x-y, x-y\rangle = ||x||^2 - \langle x,y\rangle - \langle y,x\rangle + ||y||^2.$$

By adding up these two equalities we obtain the parallelogram identity.