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Secţia: Informatică engleză, Curs: Dynamical Systems, An: 2014/2015

Laboratory 5. Orbits of nonlinear planar systems

I. Hyperbolic equilibria.

1. We consider the planar nonlinear system $\dot{x} = 2x - x^2 - xy, \quad \dot{y} = -y + xy.$

a) Find all its equilibria.

b) For each equilibrium point, find the matrix of the linearized system around it. Find the eigenvalues of this matrix. Notice that each equilibrium point is hyperbolic (this means that there is no eigenvalue with 0 real part). Specify the type and the stability character of each equilibrium point. We just remind you that when both eigenvalues are real and have the same sign the equilibrium point is a *node*, while when both are real and of opposite sign we have a *saddle*, and when they are complex conjugate we have a *focus*. You have to remember the conditions for stability.

c) Represent the direction field of the system in the box $[-3, 3] \times [-3, 3]$ of the phase plane. Notice that this box contains all the equilibria. Localize them. Now focus the image on smaller boxes around each equilibrium, like $[-0.5, 0.5] \times [-0.5, 0.5]$, $[1.5, 2.5] \times [-0.5, 0.5]$ and, respectively, $[0.5, 1.5] \times [0.5, 1.5]$. Can you identify the shape of the orbits around each equilibrium point?

d) Using **DEplot** represent few orbits near each equilibrium point. For better results, first represent simultaneously only orbits near the same equilibrium point. For the variable time t take, for example, the interval $[0, 1]$ for various reasons: to identify better on the picture the initial and, respectively the future states (that is why we take $t \geq 0$) and, on the other hand, the interval is sufficiently small to not obtain errors. Try also on larger intervals like $[0, 20]$, sometimes it works. Choose the initial states to obtain a nice picture.

e) Gluing all the information obtained until now, sketch the phase portrait of this system in your notebook.

2. This exercise has the same requirements as the previous one for the planar system $\dot{x} = x - 2xy, \quad \dot{y} = x^2/2 - y$. Of course, at c) choose suitable boxes for this system.

II. Non-hyperbolic equilibria.

3. We consider the conservative (or undamped) pendulum system

$$\dot{x} = y, \quad \dot{y} = -4 \sin x.$$

a) Notice that $(0,0)$ is an equilibrium point and show that it is non-hyperbolic.

b) Remember that the differential equation of the orbits of this system is

$$\frac{dy}{dx} = -\frac{4 \sin x}{y}.$$

Find the general solution this equation. Do not forget that here y is a function of x, and in Maple you have to write $y(x)$.

c) Notice that the general solution found at b) can be written as

$$y^2 - 8 \cos x = c, \quad c \in \mathbb{R}.$$

From here deduce that the function $H : \mathbb{R}^2 \to \mathbb{R}$,

$$H(x,y) = y^2 - 8 \cos x$$

is a first integral of the pendulum system (that is the orbits of the pendulum system lie on the level curves of H). Represent the level curves of H in the box $[-5,5] \times [-5,5]$. Notice that the orbits which starts near the equilibrium point $(0,0)$ are closed curves. If you want to convince yourself of this, represent the level curves of H in smaller boxes. From here deduce that the equilibrium point $(0,0)$ is a center and that the oscillations of the pendulum with small initial data are periodic.

4. We consider the Lotka-Volterra system (also called the predator-prey system)

$$\dot{x} = x - xy, \quad \dot{y} = -0.3y + 0.3xy.$$

a) Notice that $(1,1)$ is an equilibrium point and show that it is non-hyperbolic.

b) Represent the level curves of the function $H : (0,\infty) \times (0,\infty) \to \mathbb{R}$, $H(x,y) = y - \ln y + 0.3(x - \ln x)$ in the box $[0.4, 2] \times [0.6, 1.5]$.

c) Knowing that this H is a first integral of the Lotka-Volterra system in the region $(0,\infty) \times (0,\infty)$, deduce that $(1,1)$ is an equilibrium point of center type.